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K-MATRIX STUDY OF LIGHT NUCLEI SCATTERING

by

Daniel Ruben Ovidio Bruno

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
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I. INTRODUCTION

In this paper we present a dynamical study of the \(d-He^3\) and \(p-\alpha\) systems. There is a large amount of data involving these two systems. In particular, the cross section data indicate the presence of the particle exchange mechanism and thus this scattering problem seems an appropriate testing ground for theoretical techniques which have proved useful in particle physics, such as approximating the amplitude by an amplitude which has the same analytical structure in a region close to the one under consideration. Furthermore, much of the data is on the polarization phenomena and so provides a rigid test on any theoretical model which is required to predict the correct phase as well as magnitude of elastic and reaction amplitudes.

We will discuss primarily the \(dh-dh\) (\(h=He^3\)) and \(dh-p\alpha\) reactions; we do not attempt to explain the \(p\alpha\) elastic reaction, though we calculate it phenomenologically. The reason for this is that our model is based on one particle exchange amplitudes, and while both \(dh-dh\) and \(dh-p\alpha\) can proceed via a one nucleon exchange mechanism the \(p\alpha-p\alpha\) reaction does not, the lightest particle exchange being a triton (t) exchange and, for a given value of coupling strength, its
contribution is weaker since its singularities be farther than the other particle exchanges (see Appendix A).

In contrast to the abundant experimental results, the theoretical results are quite unsatisfactory either for the lack of firm dynamical basis or for failing to explain the experimental results. It is quite obvious that the exact solution is not possible presently, even in the nonrelativistic region to which we confine ourselves. Thus the efforts are directed into finding reasonable approximations that illuminate different aspects of the problem.

We will restrict ourselves to low energies, deuteron lab energies between 6 and 10 MeV since this is the energy region best covered by experiments. Let us briefly describe the main features of the experimental data. In this region there appears to be a broad \( ^5 \text{Li} \) resonant state as is indicated by the cross section curves \(^8\). For conciseness, let us agree to call \( \sigma_1 \), \( \rho \), and \( A \) the differential cross section, proton polarization and asymmetry in \( dh-pc \), and \( \sigma_2, \rho_{1/2}, \rho_1 \) the differential cross section, \( h \) polarization, and deuteron's vector polarization in \( dh-dh \) (for the definitions of these quantities see section V). The most interesting features of the data are in the relations between \( \rho \) and \( A \) and between \( \rho_{1/2} \) and \( \rho_1/\sqrt{2} \). For some time experimental results seemed
to show an almost exact equality of $p$ and $-A^{3,5}$, but later experiments$^{4,7}$ have shown that there is only a qualitative comparison between them. Experimental results on dh-dh also indicate an approximate equality between $p_{1/2}$ and $p_1/\sqrt{2}$.

These relations do not have a kinematical explanation, see [9,10] and section V, and thus present a challenge to any theory for the scattering between these channels.

There is much previous theoretical work of deuteron scattering, principally those works based on deuteron stripping, which was done rigorously (based on the impulse approximation) by Chew and Goldberger$^{11}$. Butler's stripping theory$^{12,13}$ also falls within the class of stripping reactions considered by Chew and Goldberger with some additional assumptions that simplify the problem and modifications for lower energies (the impulse approximation can be justified in most cases only for high energies). Butler and Symonds$^{14}$ applied this model to dh-px only and for $S$ waves and obtained reasonable cross sections for $\theta < 70^\circ$ and $E_d(\text{lab}) = 10 \text{ MeV}$, but the applicability of the approximations is questionable. Several works are more phenomenological. Buck$^{15}$ did a preliminary kinematical analysis of dh-px and examined the constraints placed on the amplitudes by the condition $p = -A$ by doing a parameter fit to the data.
Researchers at Los Alamos \textsuperscript{16} are presently conducting a parametric search (with around 60 parameters) of the data. They consider a coupled channel system of dh, pα and allow for coupling of all kinematical channels with a given \textit{j}^P (for \textit{J} \leq 9/2) and the search is for best values of reactance matrix elements.

Other work was done by Tanifuji \textsuperscript{17}; he studied the relation between \textit{P} and -\textit{A} in direct reaction models with heavy particle stripping (i.e. in initial state had d and h interacting and in final state p and α interacting). He took several different combinations of central and tensor potentials, with and without a D state for the deuteron and finally concluded that although several combinations gave \textit{P} = -\textit{A} the most likely one is a tensor force for ph and a central pα potential. Presently, Tanifuji and Yazaki \textsuperscript{18} are working on the dh-dh, dh-pα reactions assuming potentials in each partial wave so as to obtain good agreement with the data.

We have done a coupled channel study of dh, pα, and bh, where b is the singlet deuteron, low energy $^1S_0$ np pair. The validity of this study basically depends on two assumptions. The first is that we can treat each channel as a two body system and that we can obtain the scattering amplitude from an approximate K matrix \textsuperscript{19}, and the second assumption is that
we can obtain this approximate K matrix from one nucleon exchange amplitudes with the three-particle vertices parametrized to fit the experimental data. (Except for the αhn vertex which we fitted to the dh-α results, the vertices were parametrized independently of dh-α and dh-dh) The assumptions are independent insofar as we could have used different amplitudes to construct an approximate K matrix and also, given the one nucleon exchange amplitudes we could have obtained different final unitarized amplitudes (the K matrix approach does give an unitary S matrix), e.g. by using a multichannel N/D method or else by calculating potentials from these amplitudes and then solving the Lippmann-Schwinger equation. Of course, the assumptions are not completely independent since in obtaining the final amplitude via the K matrix we are adding a series of ladder diagrams; thus one should not take an arbitrary input since then one could be adding twice certain diagrams.

We have calculated the d exchange contribution to dh-α; this is the only two-nucleon exchange allowed with dh and α (b exchange is not allowed since the dbα vertex vanishes identically), and we show the effect on the results although we do not include it in the final results in this paper.
This model, then, provides a dynamical analysis of the reactions, albeit not an exact one. One of its major a priori justifications is its simplicity. By using the helicity formalism the one nucleon exchange amplitudes are quite simple to calculate, and the K matrix approach allows us to obtain unitarized amplitudes through straightforward matrix operations, and thus there are no integral equations to be solved.

To calculate the one nucleon exchange amplitudes, we construct model field theories with interaction Hamiltonians that will give in second order the amplitudes for the processes considered. These Hamiltonians are given in terms of creation and destruction operators for the different particles. We neglect any internal structure of the particles, the non-elementarity of d, b, h, α will only affect the problem through the pole position and the only allowed interactions for these particles will be the ones given by the Hamiltonians. In this manner we obtain the vertices for the nucleon exchange diagrams. Note that these Hamiltonians can be used to calculate also the one particle exchange diagrams in other reactions such as dn-dn, dd-pt, etc.

With the exception of the dpn vertex, we constructed these interaction Hamiltonians via S wave couplings. This
was done mainly to avoid the introduction of other parameters into this work but it is an important shortcoming since then the only non-central interaction is the one present in the dpn vertex.

The general contents of our thesis are as follows:
In the following section, section II, we write the basic equations from scattering theory for our multichannel calculation, settle some basic notation and do the partial wave decomposition in the helicity basis\(^{21}\). Section III gives the necessary helicity formalism and there we do an analysis of the constraints placed on the helicity amplitudes for the three channel system (pa, dh, bh) by parity and time reversal invariance.

In section IV we calculate the dynamical input for the calculations, the one nucleon and deuteron exchange amplitudes. The main features of the analytic structure are given in the Appendix. Firstly we construct interaction Hamiltonians for the three-particle vertices and with these we calculate the exchange diagrams. In this section we also fix the values of the parameters in these interaction Hamiltonians (except for \(K_{\text{chn}}\) and \(K_{\text{dd\alpha}}\)) with reactions independent of the ones under consideration. The parameters in \(K_{\text{chn}}\) will be fitted to the dh-pa data.
In section V we study the spin observables for spin 1/2-spin 0 and spin 1-spin 1/2 systems, obtaining relations between these observables and then giving explicit equations for some of them in terms of the helicity amplitudes. This brings us to section VI where we give some computational details and there we present and interpret our results.
II. SCATTERING THEORY

Our approach is based on the K matrix theory\textsuperscript{19).} In this theory, the Lippmann-Schwinger equation for the scattering amplitude is replaced by the two Heitler equations, which are simpler to solve in approximate form while conserving certain desirable properties.

Let $T$ be the scattering operator, i.e. in terms of $S$, the scattering matrix, we have

$$S = \mathcal{I} - 2\pi i\delta(E-H_0)T$$  \hspace{1cm} (II.1)

where $H_0$ is the Hamiltonian without the interactions. The L-S equation for $T$ is

$$T = V + V(E-H_0 + i\epsilon)^{-1}T$$  \hspace{1cm} (II.2)

where $V = H-H_0$.

We then introduce the $K$ operator by

$$K = V + VG_S(E)K$$  \hspace{1cm} (II.3)

where

$$G_S(E) = \text{P.V.}(E-H_0)^{-1} = \frac{1}{2} \lim_{\epsilon \to 0} \left[ \frac{1}{E-H_0+i\epsilon} + \frac{1}{E-H_0-i\epsilon} \right]$$

Heitler's equations\textsuperscript{19) } give the relation between $T$ and $K$, \ldots
T(E) = K(E) - i\pi K(E) \delta(E-H_O) T(E) = K(E) - i\pi T(E) \delta(E-H_O) K(E)

(II.4)

Thus the L-S equation is replaced by 2 linear integral equations,

K(E) = V + V G_S(E) K(E)

T(E) = K(E) + (-i\pi)K(E) \delta(E-H_O) T(E)

but since in this second equation we have a \delta function after the angular decomposition we will have just one algebraic equation, if we assume 2 body intermediate states, thus we only need to solve one integral equation (II.3). An important advantage of this method is that it is easier to approximate the solution, since for any hermitian matrix K' that we use Equation (II.4) will give a T matrix that satisfies unitarity; this is easily seen since

T - T^\dagger = K' - i\pi K' \delta(E-H_O) T - K^\dagger - i\pi T^\dagger \delta(E-H_O) K'^\dagger

= -i\pi \left( K' \delta(E-H_O) T + T^\dagger \delta(E-H_O) K' \right)

but (II.4) gives

K' = T + i\pi K' \delta(E-H_O) T = T^\dagger - i\pi T^\dagger \delta(E-H_O) K'

and therefore
\[ T - T^\dagger = -i\pi \{ T^\dagger \delta (E-H_o)T - i\pi T^\dagger \delta (E-H_o)K'\delta (E-H_o)T + T^\dagger \delta (E-H_o)T \} + i\pi T^\dagger \delta (E-H_o)K'\delta (E-H_o)T \} \]

\[ = -2i\pi T^\dagger \delta (E-H_o)T \]  

(II.5)

Thus we can approximate the solution of the integral equation for \( K \) by any hermitian operator \( K' \) and then by a simple algebraic equation obtain a unitary \( T \) matrix. This is the advantage over trying to approximate the L-S equation, since simple approximations to it violate unitarity. This is in the spirit of considering that approximations to the left hand cut are less important than approximations in the physical region. In this study, we will take the simplest approximation to (II.3), i.e. we let

\[ K = V \]

where \( V \) will be the first Born approximation to the amplitude. Thus (II.4) gives

\[ T(E) = V - i\pi V \delta (E-H_o)T(E) \]

and comparing with the L-S equation we see that this approximation to \( K \) consists in replacing the propagator by a \( \delta \)
function, i.e. we neglect in the L-S equation the term

\[ V\{p.v. (E-H_0)^{-1}\}_T \]

Moreover, in our solution to (II.4) we will consider only two body intermediate states (although we will take several intermediate 2-body channel and thus improve on purely elastic unitarity for a given process).

Now we want to obtain Equations (II.3), (II.4), (II.5) for the matrix elements between CM helicity states and then further obtain the relations for the partial wave helicity amplitudes. First we need to settle some notational points. We only consider two particle channels in a ket \( c, c', \ldots \) refers to the channel, \( \lambda_1, \lambda_2 \) to the helicity of particles 1 and 2 (in the given channel), \( p_i \) refers to momentum of particle i and \( k_c, g_c \) (or \( g_c \)) to momentum with respect to CM. We have then states

\[ |c; p_1, p_2, \lambda_1, \lambda_2\rangle \]

which satisfy

\[ \langle c'; p_1', p_2', \lambda_1', \lambda_2' | c; p_1, p_2, \lambda_1, \lambda_2\rangle = (2\pi)^6 \delta(p_1-p_1') \delta(p_2-p_2') \delta(\lambda_1-\lambda_1') \delta(\lambda_2-\lambda_2') c, c' \]

(II.6)

The S operator satisfies
We want to obtain now the unitarity relation for the matrix elements in CM. Introducing a unit operator, (II.7) gives

$$\sum_{c''} \sum_{\lambda_1'' \lambda_2''} d^3 p'' \delta^3 (p_1'' - p_1) \delta^3 (p_2'' - p_2) \langle c''; p'' \lambda'' \lambda'' | S | c''; p'' \lambda'' \lambda'' \rangle$$

$$= (2\pi)^6 \delta^3 (p_1 - p_1') \delta^3 (p_2 - p_2') \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \delta_{c c'}$$

now let us change CM variables $p$, $k$.

$$d^3 p'' \delta^3 (p_1'' - p_1) = d^3 p \quad d^3 k$$

thus we can change the labelings in the states to $p$ and $k$.

Also to simplify notation for the moment let us write $\lambda$ instead of $\lambda_1, \lambda_2$; we then have

$$\sum_{c''} \sum_{\lambda''} d^3 p'' \delta^3 (p_1'' - p_1) \delta^3 (p_2'' - p_2) \langle c''; p'' \kappa'' \lambda'' | S | c''; p'' \kappa'' \lambda'' \rangle$$

$$= (2\pi)^6 \delta^3 (p - p') \delta (\kappa - \kappa') \delta_{\lambda, \lambda'} \delta_{c, c'}$$

but, in CM we can write

$$\langle c'; p' \kappa' \lambda' | S | c''; p'' \kappa'' \lambda'' \rangle = (2\pi)^3 \delta (p' - p') \delta_{\kappa', \kappa''} \delta_{\lambda', \lambda''}$$
Then, doing one integral and separating the CM $\delta$ functions we get

$$
\sum_{c''} \sum_{\lambda''} \int d^3 k'' \langle c'; k'\lambda' | S | c''; k''\lambda'' \rangle \langle c''; k''\lambda'' | S^\dagger | c; k\lambda \rangle
$$

$$
= (2\pi)^3 \delta(k-k') \delta_{\lambda, \lambda'} \delta_{c, c'}
$$

(II.8)

So (II.8) is the unitarity relation for $S$ matrix elements (in CM). Now let us consider the $T$ matrix.

$$
SS^\dagger = 1 \longrightarrow T-T^\dagger = -2\pi i T \delta(E-H_0) T^\dagger
$$

(II.9)

and (II.9), in the CM states gives

$$
\langle c'; k'\lambda' | T-T^\dagger | c; k\lambda \rangle = -2\pi i \sum_{c''} \sum_{\lambda''} \int d^3 k''
$$

$$
\cdot \langle c'; k'\lambda' | T | c''; k''\lambda'' \rangle \delta(E-E'') \langle c''; k''\lambda'' | T^\dagger | c; k\lambda \rangle
$$

(II.10)

but $k''^2 dk'' = \delta_{c'; c''} dE''$ where $\delta_{c'} = k''^2 \frac{dk''}{dE''} = \mu_{c'} k_{c''}$ with $\mu_{c'}$ the reduced mass in channel $c''$. Therefore the r.h.s. of (II.10) is

$$
-i \sum_{c''} \sum_{\lambda''} \int d^3 k'' \langle c'; k'\lambda' | T | c'' k\Omega''\lambda'' \rangle \delta_{c'} \langle c''; k\Omega''\lambda'' | T | c, k\Omega\lambda \rangle
$$

where $\Omega$ stands for the angles of $k$. So on-shell we have
\[ \langle c'; k_c, \Omega', \lambda' | T - T^\dagger | c; k_c \Omega \lambda \rangle = -i \sum_{c''} \sum_{\lambda''} d\Omega'' \]

\[ \cdot \langle c'; k_c', \Omega', \lambda' | T | c'' k_{c''} \Omega'' \lambda'' \rangle \delta_{c''}^{c''} \langle c''; k_{c''} \Omega'' \lambda'' | T^\dagger | c; k_c \Omega \lambda \rangle \]

where \( k_c, k_{c'}, k_{c''} \) are all fixed by being so that \( E_c = E_{c'} = E_{c''} \). Now we further introduce the states \( | c; \Omega, \lambda \rangle \) such that

\[ \langle c'; k_c, \Omega', \lambda' | T | c; k_c \Omega \lambda \rangle = -\frac{2(2\pi)^2}{\sqrt{\mu_c} \mu_{c'}} \langle c'; \Omega', \lambda' | T(E) | c; \Omega \lambda \rangle \]

and let \( T_{\lambda', \lambda}(E) = \langle c'; \Omega', \lambda' | T(E) | c; \Omega \lambda \rangle \)

and for these reduced matrix elements we have

\[ T_{\lambda', \lambda}(E) - T_{\lambda', \lambda}(E) = 2i \sum_{c''} \sum_{\lambda''} d\Omega'' T_{\lambda', \lambda''} k_{c''} T_{\lambda''} \]

\[ (II.11) \]

This is the basic unitarity equation; now let us introduce the partial wave helicity amplitudes, also we need to be more explicit with helicity labels:

\[ T_{\lambda', \lambda}(E) = \langle c'; \Omega', \lambda', \lambda' | T(E) | c; \Omega \lambda \lambda_2 \rangle \]

\[ = \sum_{J, M} \frac{2J+1}{4\pi} \prod_{J} (\Omega')^J M_{\lambda_1 \lambda_2} \prod_{J} T_J^{\lambda'}(E) \prod_{\lambda} T_J^{\lambda}(\Omega, \theta, \phi) \]

\[ (II.12) \]

where

\[ T_{\lambda', \lambda}(E) = \langle c'; \lambda', \lambda' | T_J | c; \lambda \lambda_2 \rangle ; \lambda = \lambda_1 - \lambda_2 , \]

\[ \lambda' = \lambda_1' - \lambda_2' \]
Inserting (II.12) in r.h.s. of (II.11) we have

\[ 2i \sum \sum d\Omega'' \sum \left\{ \frac{2J'+1}{4\pi} \frac{2J''+1}{4\pi} \left[ T^J_{\lambda',\lambda''} k_c T^J_{\lambda',\lambda''} \right] \right\} \]

\[ \times \mathcal{D}^{J*}_{M\lambda'}(\Omega') \mathcal{D}^J_{M\lambda''}(\Omega'') \mathcal{D}^{J'*}_{M'\lambda''}(\Omega'') \mathcal{D}^{J'}_{M'\lambda'}(\Omega) \]

but,

\[ \int d\Omega'' \mathcal{D}^J_{M\lambda''}(\Omega'') \mathcal{D}^{J'*}_{M'\lambda''}(\Omega'') = \frac{4\pi}{2J+1} \delta_{J,J'} \delta_{M,M'} \]

and therefore we get

\[ 2i \sum \sum \sum_{\lambda'\lambda''} \frac{2J'+1}{4\pi} T^J_{\lambda',\lambda''} k_c T^J_{\lambda',\lambda''} \mathcal{D}^{J*}_{M\lambda'}(\Omega') \mathcal{D}^J_{M\lambda}(\Omega) \]

If we also introduce the partial wave decomposition on l.h.s. of (II.11) and equating coefficients of \( \mathcal{D}^{J*}\mathcal{D}^J \) for each \( J \) we get

\[ T^J_{\lambda',\lambda} - T^J_{\lambda',\lambda} = 2i \sum \sum \sum_{\lambda'\lambda''} T^J_{\lambda',\lambda''} k_c T^J_{\lambda',\lambda''} \]

(II.13)

Note that if we write \( T^J \) as a matrix both elements

\[ \langle c';\lambda'_{1}\lambda'_{2} | T^J | c\lambda_{1}\lambda_{2} \rangle \]

and if we let \( \rho \) be a diagonal matrix with elements \( k_c \), we can write (II.13) as

\[ T^J - T^J = 2i T^J \rho T^J \]

(II.14)

Proceeding in exactly analogous manner, we obtain Heitler's equation (II.4) for the partial wave amplitudes
\[ T_{\lambda',\lambda}^J(E) = K_{\lambda',\lambda}^J + i \sum_{\lambda''} \sum_{\lambda''' \lambda} T_{\lambda',\lambda}^J K_{\lambda''} K_{\lambda''}^{J*} \] (II.15)

To obtain \( T_{\lambda',\lambda}^J \) in terms of \( T_{\lambda',\lambda} \) we invert (II.11) using the orthogonality relation for the rotation matrices. First, since the initial momentum is parallel to z axis \( \theta, \varphi = 0 \) and as \( \mathcal{D}^J_{M\lambda}(0) = \delta_{M\lambda} \), we have

\[ T_{\lambda',\lambda}^J(E) = \sum_J \frac{2J+1}{4\pi} \mathcal{D}^{J*}_{\lambda\lambda'}(\Omega') T_{\lambda',\lambda}^J(E) \] (II.16)

Inverting this, and as scattering occurs in x-z plane \( (\varphi = 0) \),

\[ T_{\lambda',\lambda}^J = 2\pi \int d(\cos \theta) T_{\lambda',\lambda}^J(E, \theta) dJ_{\lambda\lambda'}(\theta) \]

but

\[ T_{\lambda',\lambda} = -\frac{\sqrt{\mu_c' \mu_c}}{8\pi^2} \langle c'; k_c, \Omega', \lambda' | T | c; k_c \Omega \rangle \]

and therefore

\[ T_{\lambda',\lambda}^J = -\frac{\sqrt{\mu_c' \mu_c}}{2\pi} \frac{1}{2} \int d(\cos \theta) \langle c'; k_c, \Omega', \lambda' | T | c; k_c \Omega \rangle dJ_{\lambda\lambda'}(\theta) \] (II.17)

To obtain cross sections, we begin from the basic equation

\[ \sigma_{\text{fi}} = \int d^3 p_f d^3 \lambda_f (2\pi)^4 \delta^4(p_f - \mathbf{P}_i) \frac{|T_{\text{fi}}|^2}{v_i} \]

where

\[ T_{\text{fi}} = \langle c_f, k_f, \lambda_f | T | c_i, k_i, \lambda_i \rangle \]

is an invariant (density) so
\[ \sigma_{fi} = \int \frac{d\Omega}{4\pi^2} \frac{\mu_f k_f dE}{v_i} \delta(E_i - E_f) \frac{|T_{fi}|^2}{v_i} \]

then

\[ \frac{d\sigma_{fi}}{d\Omega} = \frac{\mu_f k_f}{4\pi^2} \frac{|T_{fi}|^2}{v_i} \]

Now,

\[ v_i = v_{rel, i} = \frac{dE_i}{dk_i} = \frac{k_i}{\mu_i} \]

\[ \therefore \frac{d\sigma_{fi}}{d\Omega} = \frac{\mu_i u_f k_f}{4\pi^2} \frac{k_f}{k_i} |T_{fi}|^2 \quad (II.18) \]
III. HELICITY FORMALISM

In this section we will construct the helicity states and define the notation for the amplitudes, and also obtain the constraints on the amplitudes given by parity and time reversal invariance.

A. Helicity States

In our work we have the following two-particle channels,

\[ \begin{align*}
\text{p-\(a\)} & : & \lambda_p = \pm \frac{1}{2}, & \lambda_a = 0 \\
\text{d-He}\(^3\) & : & \lambda_d = 0, \pm 1, & \lambda\_{\text{He}}\(^3\) = \pm \frac{1}{2} \\
\text{b-He}\(^3\) & : & \lambda_b = 0, & \lambda\_{\text{He}}\(^3\) = \pm \frac{1}{2}
\end{align*} \]

We construct from single particle helicity states along \(g\) and \(-g\), where \(g\) is the momentum in CM frame, thus

\[ |g, \lambda_1^1\lambda_2^1\rangle = |g, \lambda_1^1\rangle_1 | -g, \lambda_2^2\rangle_2 \]

where 1 and 2 in the single particle kets emphasize the fact that there is an addition of a phase factor introduced in the kets along \(-g\) which is introduced as in Jacob and Wick\(^2\), so that in the static limit \(|g=0, \lambda_1^1\rangle_1 = | -g=0, -\lambda_1^2\rangle_2\).

Generally those subscripts will be left implicit as it will be clear from context which state is being considered.
First let us consider spin $1/2$ states. Now let $\tilde{\kappa}$ be along the $z$ axis and $\vec{g}$ in $x$-$z$ plane making an angle of $\theta$ with the $z$ axis. We have

$$|\tilde{\kappa}, +1/2\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\tilde{\kappa}, -1/2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(we are only writing the nonscalar part of the spinors), and to obtain spinors along $\vec{g}$ we use the rotation operator about the $y$ axis

$$|\vec{g}, \pm 1/2\rangle = e^{-i\theta \sigma_y/2} \ |\tilde{\kappa}, \pm 1/2\rangle$$

which gives

$$|\vec{g}, +1/2\rangle = \begin{bmatrix} \cos \theta/2 \\ \sin \theta/2 \end{bmatrix}, \quad |\vec{g}, -1/2\rangle = \begin{bmatrix} -\sin \theta/2 \\ \cos \theta/2 \end{bmatrix}$$

The spinors for the particle along $-\tilde{\kappa}$ and $-\vec{g}$ are defined including a factor $(-1)^{1/2-\lambda}$, i.e.

$$|{-\tilde{\kappa}}, \lambda_2 = \pm 1/2\rangle = (-1)^{1/2-\lambda_2} e^{-i\pi \sigma_y/2} \ |\tilde{\kappa}, \pm 1/2\rangle$$

and $$|{-\vec{g}}, \lambda_2 = \pm 1/2\rangle = e^{-i\theta \sigma_y/2} \ |\tilde{\kappa}, \pm 1/2\rangle$$

explicitly

$$|{-\tilde{\kappa}}, +1/2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |{-\tilde{\kappa}}, -1/2\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
\[ | -g, 1/2 \rangle = \begin{bmatrix} -\sin \theta/2 \\ \cos \theta/2 \end{bmatrix}, \quad | -g, -1/2 \rangle = \begin{bmatrix} \cos \theta/2 \\ \sin \theta/2 \end{bmatrix} \]

Now let us turn to spin 1 deuteron states. First we will examine the creation and destruction operators for deuterons. \( d \) is a pseudo vector, let \( \vec{d}^\dagger \) be the creation operator for a vector deuteron, i.e. its spherical components \( (\vec{d}^\dagger)^\pm_l, (\vec{d}^\dagger)_0 \) create deuterons with helicity \( \pm 1, 0 \) respectively, along the \( z \) axis; thus in Cartesian components

\[ d_{\pm 1} = \frac{-1}{\sqrt{2}} (d_x \pm id_y) \quad , \quad d_0 = d_z \]

Note that

\[ (\vec{d}^\dagger)^\pm_l = \frac{-1}{\sqrt{2}} (d^\dagger_x \pm id_y) = - (d_{\pm 1})^\dagger \]

\[ (\vec{d}^\dagger)_0 = d^\dagger_z = (d_0) \quad \text{(III.1)} \]

so

\[ d_{\pm 1} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix}, \quad d_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

in the Cartesian basis. We then have

\[ | k, \pm 1 \rangle = (\vec{d}^\dagger)^\pm_l | \emptyset \rangle \]

and

\[ | k, 0 \rangle = (\vec{d}^\dagger)_0 | \emptyset \rangle \]
where \( |\emptyset\rangle \) is the vacuum state (again we are only writing the spin part of the states). As for spin 1/2, we obtain the states along \( \bar{g} \) by a rotation, and thus

\[
|\bar{g}, \lambda \rangle = e^{-i\theta J^{(1)}} |\bar{k}, \lambda \rangle
\]

where \( e^{-i\theta J^{(1)}} \), in the basis of eigenstates of \( S_z \), i.e. the rotational basis, is

\[
\begin{bmatrix}
| -i\theta J^{(1)} \\
\lambda_1 \lambda_2
\end{bmatrix} = d^{(\lambda_1 \lambda_2)}(\theta) = \begin{pmatrix}
\cos^2 \theta/2 & -\frac{\sin \theta}{\sqrt{2}} & \sin^2 \theta/2 \\
\frac{\sin \theta}{\sqrt{2}} & \cos \theta & -\frac{\sin \theta}{\sqrt{2}} \\
\sin^2 \theta/2 & \frac{\sin \theta}{\sqrt{2}} & \cos^2 \theta/2
\end{pmatrix}
\]

(III.3)

Equation (III.2) gives for the creation operators,

\[
d^{(\bar{g}, \lambda)}(\theta) = e^{-i\theta J^{(1)}} d^{(\bar{k}, \lambda)}
\]

so, using (III.3), we have

\[
d^{(\bar{g}, \pm 1)} = \cos^2 \theta/2 d^{(\bar{k}, \pm 1)} + \frac{\sin \theta}{\sqrt{2}} d^{(\bar{k}, 0)} + \sin^2 \theta/2 d^{(\bar{k}, \pm 1)}
\]

\[
d^{(\bar{g}, 0)} = -\frac{\sin \theta}{\sqrt{2}} [d^{(\bar{k}, 1)} - d^{(\bar{k}, -1)}] + \cos \theta d^{(\bar{k}, 0)}
\]

(III.4)

To obtain deuteron states along \(-\bar{k}\) (i.e. -z) and \(-\bar{g}\) we proceed in the same way but introducing a phase factor
\[ S_d^{-\lambda} \] as we did for spinors, so
\[ |-\kappa,\lambda\rangle = (-1)^{1-\lambda} e^{-i\pi J_y^{(1)}} |\kappa,\lambda\rangle \]
and
\[ |\tilde{\gamma},\lambda\rangle = (-1)^{1-\lambda} e^{-i(\theta+\pi) J_y^{(1)}} |\tilde{\kappa},\lambda\rangle \]

B. Helicity Amplitudes

We want to define the various amplitudes for the reactions and find the constraints placed on them by parity and time reversal invariance. To do this we begin by constructing parity eigenstates.

Let \(|c;J,M,\lambda_1\lambda_2\rangle\) be a CM eigenstate for channel c, then, \(^{21}\)

\[ \mathcal{P}|c;JM,\lambda_1\lambda_2\rangle = \eta_1 \eta_2 (-1)^{J-S_1-S_2} |c;JM-\lambda_1-\lambda_2\rangle \] (III.5)

where \(\mathcal{P}\) is the parity operator and \(\eta_i, S_i\) the intrinsic parity and spin of particle i in channel c. We thus have,

\[ \mathcal{P} |1;JM,\lambda_1\lambda_2\rangle = (-1)^{J-1/2} |1;JM-\lambda_1-\lambda_2\rangle \]

\[ \mathcal{P} |2;JM,\lambda_1\lambda_2\rangle = (-1)^{J-3/2} |2;JM-\lambda_1-\lambda_2\rangle \]

\[ \mathcal{P} |3;JM,\lambda_1\lambda_2\rangle = (-1)^{J-1/2} |3;JM-\lambda_1-\lambda_2\rangle \]
where 1, 2, 3 stand for $\text{pHe}^3$, $\text{dHe}^3$, and $\text{bHe}^3$ respectively.

Using these equations we obtain the parity eigenstates by taking linear combinations and we get ($\pi$ is the parity),

$$|c;JM,\lambda, 1 \lambda, 2 \pi= \pm 1\rangle = \frac{1}{\sqrt{2}} \{ |c;JM,\lambda, 1 \lambda, 2\rangle \mp (-1)^{J-1/2-C} |c;JM,\lambda, -1 \lambda, 2\rangle \}$$

Invariance under parity and time reversal$^{21}$ of the scattering matrix gives the following two relations for the partial wave helicity amplitude

$$\langle c', \lambda, 3 \lambda, 4 | T^J | c; \lambda, 1 \lambda, 2 \rangle =$$

$$= \frac{\eta_1 \eta_2}{\eta_3 \eta_4} (-1)^{S_1+S_2-S_3-S_4} \langle c', -\lambda, 3 \lambda, 4 | T^J | c; -\lambda, 1 \lambda, 2 \rangle \quad (\text{III.6})$$

and

$$\langle c', \lambda, 3 \lambda, 4 | T^J | c; \lambda, 1 \lambda, 2 \rangle = \langle c; \lambda, 1 \lambda, 2 | T^J | c'; \lambda, 3 \lambda, 4 \rangle \quad (\text{III.7})$$

We have that the total number of helicity amplitudes for these processes is

$$[\frac{(2S_p+1)(2S_d+1)}{a} + \frac{(2S_d+1)(2S_h+1)}{b} + (2S_h+1)(2S_h+1)]^2 = 100$$

but the number of independent amplitudes for each $J^\pi$ is quite small, 15 in fact. This is easily seen by noticing that there are only 1, 3, and 1 independent parity eigenstates for channel 1, 2, 3 respectively; parity reduces the number
of amplitudes to $(1+3+1)^2 = 25$ for each $J^\pi$ and then time 
reversal further reduces the number to $5(5+1)/2 = 15$
independent amplitudes for each $J^\pi$.

Later we will need to exact relations between the 
full (i.e. not partial wave) helicity amplitudes, thus we 
will derive them now from Eqs. (III.6) and (III.7). Since 
a 10x10 matrix is somewhat unmanageable we will consider 
the different reactions separately.

From Eq. (II.16) we have for the amplitude

$$\langle c';\lambda_3\lambda_4 | T | c;\lambda_1\lambda_2 \rangle = \sum_J a_J^{cc'} d_J^{\lambda_i\lambda_f} (\theta) \langle c';\lambda_3\lambda_4 | T^J | c;\lambda_1\lambda_2 \rangle$$

where \( \lambda_i = \lambda_1 - \lambda_2 \), \( \lambda_f = \lambda_3 - \lambda_4 \)

Let \( \eta^a = \frac{\eta_1^{c'} \eta_2^c}{\eta_3^{c'} \eta_4^c} (-1)^{S_1+S_2-S_3-S_4} \), i.e. a indexes the 
reaction \( c\rightarrow c' \), then (III.6) implies

$$\langle c';\lambda_3\lambda_4 | T | c;\lambda_1\lambda_2 \rangle = \sum_J a_J^{cc'} d_J^{\lambda_i\lambda_f} (\theta) \eta^a \langle c';-\lambda_3-\lambda_4 | T | c;-\lambda_1-\lambda_2 \rangle$$

but \( d_{-\lambda,-\mu}^J = (-1)^{\lambda-\mu} d_{\lambda\mu}^J = (-1)^{\lambda-\mu} d_{\mu}^J \)

Thus

$$\langle c';\lambda_3\lambda_4 | T | c;\lambda_1\lambda_2 \rangle = \eta^a (-1)^{\lambda_i-\lambda_f} \langle c';-\lambda_3-\lambda_4 | T | c;-\lambda_1-\lambda_2 \rangle$$

(III.8)
and analogously, (III.7) gives

\[ \langle c'; \lambda_3 \lambda_4 | T | c; \lambda_1 \lambda_2 \rangle = (-1)^{\lambda_1 - \lambda_2} \langle c; \lambda_1 \lambda_2 | T | c'; \lambda_3 \lambda_4 \rangle \]

(III.9)

Now, armed with (III.8) and (III.9) we proceed to analyze the amplitudes

i) \( p\alpha \rightarrow p\alpha \)

\[ \eta^1 = 1 \times (-1)^{1/2 - 1/2} = 1 \]

and we have

<table>
<thead>
<tr>
<th>[ l; 1/2 \ 0 ]</th>
<th>[ l; -1/2 \ 0 ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ t'_1 ]</td>
<td>[ t'_2 ]</td>
</tr>
<tr>
<td>[ -t'_2 ]</td>
<td>[ t'_1 ]</td>
</tr>
</tbody>
</table>

since

\[ t'_1 = \langle l; 1/2 \ 0 | T | l; 1/2 \ 0 \rangle = (-1)^{1/2 - 1/2} \]

\[ \cdot \langle l; -1/2 \ 0 | T | l; -1/2 \ 0 \rangle \]

\[ t'_2 = \langle l; 1/2 \ 0 | T | l; -1/2 \ 0 \rangle = (-1)^{-1/2 - 1/2} \]

\[ \cdot \langle l; -1/2 \ 0 | T | l; 1/2 \ 0 \rangle \]

ii) \( d\hbar \rightarrow d\hbar \)

\[ \eta^2 = 1 \]

Using (III.8) and (III.9) we get
Table III.1

|                  | $|2;1\ 1/2\rangle$ | $|2;0\ 1/2\rangle$ | $|2;-1\ 1/2\rangle$ | $|2;1-1/2\rangle$ | $|2;0-1/2\rangle$ | $|2;-1-1/2\rangle$ |
|------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $\langle 2;1\ 1/2 | t_1^2 \quad t_2^2 \quad t_3^2 \quad t_4^2 \quad t_5^2 \quad t_6^2$ |
| $\langle 2;0\ 1/2 | -t_2^2 \quad t_7^2 \quad t_8^2 \quad t_9^2 \quad t_{10}^2 \quad t_5^2$ |
| $\langle 2;-1\ 1/2 | t_3^2 \quad -t_8^2 \quad t_{11}^2 \quad t_{12}^2 \quad t_9^2 \quad t_4^2$ |
| $\langle 2;1-1/2 | -t_4^2 \quad t_9^2 \quad -t_{12}^2 \quad t_{11}^2 \quad t_8^2 \quad t_3^2$ |
| $\langle 2;0-1/2 | t_5^2 \quad -t_{10}^2 \quad t_9^2 \quad -t_8^2 \quad t_7^2 \quad t_2^2$ |
| $\langle 2;-1-1/2 | -t_6^2 \quad t_5^2 \quad -t_4^2 \quad t_3^2 \quad -t_2^2 \quad t_1^1$ |
where we have that in the order chosen for the basis states, Eqs. (III.8) and (III.9) give, respectively

\[ t_{i,j}^2 = (-1)^{i+j} t_{7-i,7-j}^2 \]  \hspace{1cm} (III.10)

\[ t_{i,j}^2 = (-1)^{i+j} t_{j,i}^2 \]  \hspace{1cm} (III.11)

This particular order for the basis states was chosen so that later the expressions for the spin matrices in spin 1 spin 1/2 space are simplified. As we had expected (since there are only 3 independent parity eigenstates for each J) there are only 12 independent amplitudes for this reaction, \( t_{1}^2 \) through \( t_{12}^2 \).

iii) \hspace{1cm} \text{dh-\(p\alpha\)} \hspace{1cm} \eta^3 = -1

Here we can only use parity invariance (time reversal just gives the amplitudes for the inverse reaction, \( p\alpha\rightarrow\text{dh} \)), and we get
<table>
<thead>
<tr>
<th>2; 1 1/2</th>
<th>2; 1 1/2</th>
<th>2; 1 1/2</th>
<th>2; 1 1/2</th>
<th>2; 1 1/2</th>
<th>2; 1 1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>t₁</td>
<td>t₃</td>
<td>t₅</td>
<td>t₆</td>
<td>t₄</td>
<td>t₃</td>
</tr>
<tr>
<td>t₂</td>
<td>t₃</td>
<td>t₄</td>
<td>t₃</td>
<td>t₃</td>
<td>t₃</td>
</tr>
<tr>
<td>t₆</td>
<td>t₄</td>
<td>t₃</td>
<td>t₂</td>
<td>t₄</td>
<td>t₂</td>
</tr>
<tr>
<td>+ t₆</td>
<td>+ t₅</td>
<td>+ t₄</td>
<td>+ t₃</td>
<td>+ t₂</td>
<td>- t₁</td>
</tr>
</tbody>
</table>

Table III.2
since now, as \( \eta^3 = -1 \),

\[
\begin{align*}
t^3_{i,j} &= -(1)^i+j \ t^3_{3-i,7-j} \quad i=1,2 ; \ j=1,2,\cdots,6
\end{align*}
\]

and we have 6 independent amplitudes.

iv) \( bh-pa \) \[ \eta^4 = 1 \]

\( bh-dh \) \[ \eta^5 = 1 \]

\( bh-bh \) \[ \eta^6 = 1 \]

We have the following amplitude matrix for these reactions

<table>
<thead>
<tr>
<th>Table III.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle 1;1/2\ 0 \rangle )</td>
</tr>
<tr>
<td>( \langle 1;-1/2\ 0 \rangle )</td>
</tr>
<tr>
<td>( \langle 2;1\ 1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 2;0\ 1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 2;-1\ 1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 2;1-1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 2;0-1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 2;-1-1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 3;0\ 1/2 \rangle )</td>
</tr>
<tr>
<td>( \langle 3;0-1/2 \rangle )</td>
</tr>
</tbody>
</table>

And thus there are 10 independent amplitudes with initial state \( bh \). Adding this number to the ones obtained before
we have the 30 independent amplitudes necessary to specify the full scattering matrix for the three channels.
IV. INPUT AMPLITUDES

In this section we calculate the basic input to our K matrix equations. We proceed by forming model field theories with interactions between suitably defined elementary particle fields.\textsuperscript{23,24} Of course, we have the bona fide elementary fields p,n but we introduce also free fields corresponding to the deuteron (d), the pole in the $^1S_0$ np amplitude (b), and also for He$^4$ (α) and He$^3$ (h). We then construct interaction Hamiltonians for the following interactions, dpn, bpn, hd$p$, hbp, and αhn. With the exception of dpn, these interactions will be spherically symmetric and will give rise only to S wave interactions. In the dpn vertex we allow for a more general coupling. The different parameters appearing in the Hamiltonians will be fixed by comparing the amplitudes obtained in the second order of each Hamiltonian with experimental results or by comparison with wavefunctions (in the case of hd$p$, hbp). There is an arbitrary choice of form for the momentum dependence of the coupling, and we generally choose a Yamaguchi form.

We also construct the Hamiltonian corresponding to a ddα (S-wave) interaction, $\mathcal{H}_{dd\alpha}$. This Hamiltonian is needed to calculate the deuteron exchange contribution to dh-αα.
We again take for the momentum dependence at the vertex a Yamaguchi form with two parameters, related to the strength and range of the dd interaction. The d exchange amplitude (also calculated in IV.C) will not be included in our final results although we will examine its effects with reasonable values of the parameters.

To summarize, first we construct the Hamiltonians for the three-particle couplings (except for ddn), then we calculate the one nucleon exchange amplitudes. We then construct the ddn vertex and deuteron exchange amplitudes. All these form the input to the K matrix formalism. Lastly we fix the values of the parameters (except for the ones in the ddH and ddn Hamiltonians). We first consider dpn.

A. Interaction Hamiltonians

Let $g_i$, $\lambda_i$ be the momentum and $z$ projection of spin of particle $i$. The simplest coupling between $d$, $p$, and $n$ is then

$$\mathcal{K}_{dpn} = \sum_{\lambda_p} \int d^3 q_p d^3 q_n \langle 1/2 1/2 1 | \lambda_p \lambda_n \lambda_d \rangle \cdot g'_o$$

$$\cdot \left\{ d^+ (q_p + q_n, \lambda) p (q, \lambda) p_n (q_n, \lambda_n) + h.c. \right\} \quad (IV.1)$$

where $g'_o$ is a coupling constant, $d^+ (g, \lambda) [d(g, \lambda)]$ is the creation [destruction] operator for a deuteron with momentum
\( q \) and \( z \) projection \( \lambda \) and analogously for the other particles.

For the moment let us leave the momenta label implied, then

\[
\sum \langle 1/2 \ 1/2 \ 1 | \lambda_p \lambda_d \lambda_d \rangle d^\dagger (\lambda_d) p(\lambda_p) n(\lambda_n) = \\
\lambda_p \lambda_n \\
\frac{1}{\sqrt{2}} \ p(1/2) \{d^\dagger (0)n(-1/2) + \sqrt{2} \ d^\dagger (1)n(1/2)\} + \\
\frac{1}{\sqrt{2}} \ p(-1/2) \{d^\dagger (0)n(1/2) + \sqrt{2} \ d^\dagger (-1)n(-1/2)\}
\]

Now let \( \sigma_0, \sigma_{\pm 1} \) be the spherical components of \( q \), then

\[
\tilde{q}^\dagger \cdot q = d^\dagger (0) \sigma_0 - d^\dagger (+1) \sigma_{-1} - d^\dagger (-1) \sigma_{+1} \quad (IV.2)
\]

and therefore

\[
(\tilde{q} \cdot q) n = \begin{cases} 
\{d^\dagger (0)n(1/2) + \sqrt{2} \ d^\dagger (-1)n(-1/2)\} \\
-d^\dagger (0)n(-1/2) - \sqrt{2} \ d^\dagger (1)n(1/2)\}
\end{cases}
\]

and so if we define \( \tilde{p} = [p(-1/2), -p(1/2)] = (i \sigma_y p)^T \quad (IV.3) \)

we can write (with no reference to the axis of quantization)

\[
\mathcal{K}_{dpn} = \int d^3 q_p d^3 q_n \sigma_0 \{\tilde{\sigma}(q_p) \tilde{\sigma}^\dagger (q_p + q_n) \cdot \sigma n(q_n) + h.c.\}
\]

This interaction Hamiltonian corresponds to the diagrams
With the interaction defined by $\mathcal{K}_{\text{dnp}}$, the only allowed process is d-\(\bar{p}\)n, and if we let $g_o$ be a constant or depend only on $|\vec{k}|$, with $k = 1/2(g_p - g_n)$ it allows only S wave interactions (it can only depend on $k$ since it is the only Galilean invariant of momenta), but non S wave interactions can be introduced by writing\(^{25}\)

$$\mathcal{K}_{\text{dnp}} = \int d^3q_p d^3q_n g_o \{p(p)\slashed{d}^\dagger (q_p + q_n) \cdot \theta(k) n(q_n) + h.c.\} \quad (\text{IV.4})$$

then $\theta(k) = a(k^2)\gamma + b(k^2)\gamma \cdot k \kappa \quad (\text{IV.5})$

is the most general Galilean invariant pseudo-vector. Then we will take (IV.4) and (IV.5) to be the interaction Hamiltonian for dnp, and $b \neq 0$ allows for a non S wave interaction.

Let us calculate with this Hamiltonian the Born approximation, $B(\lambda)$, for pn scattering (Figure 2) in the CM, for scattering from an initial state of momentum $q_i$, helicities $\lambda_p, \lambda_n$ to a final state of momentum $q_f$, helicities $\lambda'_p, \lambda'_n$.  

\[ \text{Figure IV.1} \]
The amplitude is

\[ B(\lambda) = \sum_d \frac{\langle p f n_f | \tilde{\xi}_{dpn} | d \rangle \langle d | \tilde{\xi}_{dpn} | p_i n_i \rangle}{E - E_d + i\epsilon} \]

and the only nonvanishing part is

\[ B_{fi}(\lambda) = g_0^2 \int \frac{d^3 q'' d^3 q'' d^3 q' d^3 q'}{2p_p 2n_p 2p_2 2n_2} \frac{\delta^3(q'' - q'' - q' - q')}{E - E_d + i\epsilon} \]

\[ \cdot \langle p f n_f | \{ \tilde{\sigma}_i(p_{p_i}^0) \theta_i n_i(q''_{n_i}) \} \delta_{ij} \{ \tilde{\sigma}_j(p_{p_i}^0) \theta_j n_j(q''_{n_j}) \} | p_i n_i \rangle \]

where we reduced the intermediate states in terms of creation operators acting on the vacuum and used the commutation relation,

\[ [a(q, \lambda), a^\dagger(q', \lambda')] = (2\pi)^3 \delta^3(q - q')\delta_{\lambda\lambda'} . \]

Repeating this procedure for the initial and final states we finally get

\[ B_{fi}(\lambda) = \frac{g_0^2}{E - E_d + i\epsilon} \frac{n_i^\dagger (-q_i, \lambda_i) \tilde{\sigma}_p^\dagger (q_f, \lambda'_f) \cdot p(q_i, \lambda_i) \theta_n(-q_i, \lambda_i)} {2f_n} \]

(IV.6)
Here, by the symbols \( p, n \), we just mean spinors as defined on page (18).

Now some kinematics: \( E = \frac{q_i^2}{2 \mu} = q_i^2 = q_f^2 = q^2 \) and
\( E_d = -\alpha_o^2 \) where \( \alpha_o^2 \) is the binding energy of the unrenormalized d field. Now we calculate \( B_{fi} \) for the \( ^3S_1 \) np state.

In terms of helicity amplitudes we have

\[
| ^3S_1 \rangle = \sqrt{1/3} | J=1, 1/2 \ 1/2, \pi=+1 \rangle + \sqrt{2/3} | J=1, 1/2-1/2, \pi=1 \rangle
\]

(IV.7)

where the kets on the right are definite parity np helicity states, i.e.

\[
| J=1, \lambda_1 \lambda_2, \pi=+1 \rangle = \sqrt{1/2} \left( | J=1, \lambda_1 \lambda_2 \rangle + | J=1, -\lambda_1 -\lambda_2 \rangle \right)
\]

then evaluating (IV.6) for these values of the helicity and integrating over \( \theta \), we get

\[
t^{ ^3S_1}_{1} = -\frac{g_o^2}{2\pi} \frac{1}{q^2 + \alpha_o^2} \left\{ (a + 1/3 \ b q^2) + (2/\sqrt{3}) b q^2 \right\}
\]

(IV.8)

where we have introduced a factor \( -\frac{\mu_{pn}}{2\pi} = -\frac{1}{4\pi} \) so that

\[
t^j = \frac{\mu_{en}}{q} \left( a + 1/3 \ b q^2 \right)
\]

Equation (IV.8) gives the \( ^3S_1 \) amplitude for np scattering for the diagram in Figure 2. Here \( g_o \) and \( \alpha_o^2 \) are the coupling constant and binding energy of the free quasi-particle. Duck\textsuperscript{25} shows that if we consider the
sum of all bubbles (Figure IV.3)

\[ \begin{array}{c}
+ \\
- \\
\end{array} \]

Figure IV.3

we obtain a renormalized amplitude which has the same form

but \( g_o^2 \) is renormalized to \( g^2(q^2) = g_{dpn}^2 S(q^2) \) and \( \alpha_o^2 \)

becomes \( \alpha_d^2 \) the actual b.e. of d. \( S(q^2) \) includes the off

shell corrections to the deuteron propagator and is normalized to 1 at \( q^2 = -\alpha_d^2 \). Thus the renormalized \( ^3S_1 \) amplitude

is\(^25\)

\[
\begin{align*}
^3S_1 \rightarrow & - \frac{g_{dpn}^2}{2\pi} \frac{S(q^2)}{q^2 + \alpha^2} \left\{ (a+1/3bq^2) + (2/\sqrt{3} \ b q^2) \right\} \\
& \text{(IV.9)}
\end{align*}
\]

Thus our renormalized Hamiltonian can be taken to be the one
given in Equation (IV.4) but with \( g_o = g_{dpn} \) and \( a(q^2) \) and

\( b(q^2) \) such that (IV.9) agrees with a realistic np \( ^3S_1 \) ampli-
tude. In section (IV.D) we obtain values for \( g_{dpn}, a, \) and b

in this manner.
The bpn interaction is treated similarly. We take only an S wave interaction,

\[ \kappa_{\text{bpn}} = \sum_{\lambda_p \lambda_n} \int \frac{d^3 q_p}{2 \pi} \frac{d^3 q_n}{2 \pi} \frac{1}{2} \frac{1}{2} 0 | \lambda_p \lambda_n \rangle g_{\text{bpn}} \]

\[ \{ b^\dagger (q_p + q_n) p(q_p, \lambda_p) n(q_n, \lambda_n) + \text{h.c.} \} \quad (\text{IV.10}) \]

and proceeding as before we obtain

\[ \kappa_{\text{bpn}} = \int \frac{d^3 q_p}{2 \pi} \frac{d^3 q_n}{2 \pi} g_{\text{bpn}} (k^2) \{ p(q_p) n(q_n) b(q_p + q_n) + \text{h.c.} \} \quad (\text{IV.11a}) \]

Here we allow \( g \) to have the momentum dependence. We take \( \alpha_p = 0 \) and for the vertex form factor we take

\[ g(k^2) = \frac{g_{\text{bpn}}}{(k + \beta_3^2)} \quad (\text{IV.11b}) \]

This is the form one obtains by comparing the Yamaguchi\textsuperscript{26} n-p \( \frac{1}{2}S_0 \) amplitude calculated with separable potentials. The parameters \( g_{\text{bpn}} \) and \( \beta_3^2 \) are fixed in section (IV.D).

Now let us consider the hd\( p \) vertex. If we take h,d,p to be independent fields of spin 1/2, 1, 1/2 respectively, the simplest coupling is, in terms of creation and destruction operators,
\[ \chi = \sum_{\lambda_p, \lambda_d} \int d^3 q_p d^3 q_d \langle 1/2 1/2 1/2 | \lambda_p, \lambda_d, \lambda_n \rangle g^i \]

\[ \left\{ p^+ (q_p, \lambda_p) d^+ (q_d, \lambda_d) h(q_p + q_d, \lambda_h) + h.c. \right\} \quad (IV.12) \]

but

\[ \sum_{\lambda_p, \lambda_d} \langle 1/2 1/2 1/2 | \lambda_p, \lambda_d, \lambda_n \rangle p^+ (\lambda_p) d^+ (\lambda_d) h(\lambda_h) = \frac{1}{\sqrt{3}} \left\{ p^+ d^+ (0) \sigma_q h - p^+ (1) \sigma_{-1} h - p^+ (1) \sigma_{+1} h \right\} = \frac{1}{\sqrt{3}} \left\{ p^+ d^+ \cdot \sigma h \right\} \]

Thus, absorbing the \(1/\sqrt{3}\) factor into \(g\), we get

\[ \chi_{hdp} = \int d^3 q_p d^3 q_d g(k_{dp}) \left\{ p^+ (q_p) d^+ (q_d) \cdot \sigma h(q_p + q_d) + h.c. \right\} \quad (IV.13) \]

where \(k_{dp}\) is the pd relative momentum. At this vertex we take

\[ g(k_{dp}) = \frac{g_{hdp}}{(k_{dp}^2 + \beta_4^2)^2} \quad (IV.14) \]

where \(g_{hdp}\) is a coupling constant and \(\beta_4\) is related to the range of the interaction (see section IV.D).

To construct the hbp interaction we proceed in the same way. It is simpler since \(b\) is a scalar. We take
\[ \kappa_{\text{hbp}} = \int d^3 q_p d^3 g_b g_k \{ p^+ (g_p) h (g_p + g_b) b^+ (g_b) + h.c. \} \]

(IV.15)

and

\[ g(k_{\text{bp}}) = g(k_{\text{bp}}^2) = \frac{g_{\text{hbp}}}{(k_{\text{bp}}^2 + \beta_5^2)^2} \]

Finally we get to the \( \alpha h n \) vertex. We write for a spherically symmetric interaction between \( \alpha, h, \) and \( p \)

\[ \kappa = \sum_{\lambda_h \lambda_n} \int d^3 q_h d^3 q_n \langle 1/2 \ 1/2 \ 0 | \lambda_n \lambda_h \ 0 \rangle g'(k_{hn}^2) \times \]

\[ \cdot \{ n (q_n, \lambda_n) h (q_n, \lambda_h) \alpha^+ (q_n + g_h) + h.c. \} \]

\[ = \int d^3 q_h d^3 q_n g_{\alpha h n} (k_{hn}^2) \{ \tilde{n} (q_n, h (q_n) \alpha^+ (q_n + g_h) + h.c. \} \]

(IV.16)

where as before \( \tilde{n} = (i \sigma_{1})_{y}^{T} \) and we choose

\[ g_{\alpha h n} (k_{hn}^2) = \frac{g_{\alpha h n}}{(k_{hn}^2 + \beta_1^2)} \]

These parameters will be free in our calculation.

B. One Nucleon Exchange Amplitudes

Now that we have the interaction Hamiltonians for the processes \( d p n, b p n, h d p, h b p, \) and \( \alpha h n \) we can calculate the
diagrams in Figure IV.4.

Let us consider an arbitrary one nucleon exchange amplitude, with initial particles $a$, $b$ and final particles $c$, $d$.

\[
\begin{array}{c}
\text{a, } p_a \\
\downarrow \\
\text{b, } p_b \\
\downarrow \\
\text{c, } p_c \\
\downarrow \\
\text{d, } p_d
\end{array}
\]

Then we have

\[
E_i = E^2 = \frac{p^2_a}{2m_a} - \alpha_a^2 + \frac{p^2_b}{2m_b} - \alpha_b^2
\]

\[
= E_f = \frac{p^2_c}{2m_c} - \alpha_c^2 + \frac{p^2_d}{2m_d} - \alpha_d^2
\]

and

\[
E_{\text{int}} = \text{total energy in intermediate state}
\]

\[
= \frac{p^2_c}{2m_c} - \alpha_c^2 + \frac{p^2_d}{2m_d} - \alpha_d^2
\]

where $\alpha_i^2$ is the binding energy of particle $i$. We always work in the CM system and let $m_N = 1$. We have three kinematical channels, $p\alpha$, $dh$, and $bh$, labelled 1, 2, and 3 and let $q_j$ ($q'_j$) be the corresponding momentum of $p$, $d$ or $b$ in initial (final) state. Also let $\mu_j$ be the reduced mass of channel $j$. Then the total energy of channel $j$, which is independent of $j$, is
<table>
<thead>
<tr>
<th></th>
<th>1 $p - \alpha$</th>
<th>2 $d - \text{He}^3$</th>
<th>3 $b - \text{He}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p - \alpha$</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>$d - \text{He}^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$b - \text{He}^3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- $p, n$ - He$^3$
- $d$ - $\alpha$
- $b$
\[ E = \frac{q_i^2}{2\mu_j} - (\alpha_{j1}^2 + \alpha_{j2}^2) \]

with
\[ \mu_1 = \frac{m_p m_a}{m_p + m_a} = \frac{4}{5} \]
\[ \mu_2 = \frac{m_d m_h}{m_d + m_h} = \frac{6}{5} \]
\[ \mu_3 = \frac{m_b m_h}{m_b + m_h} = \frac{6}{5} \]

and the only nonzero binding energies are
\[ \alpha_{\alpha}^2 = \frac{28}{938} \]
\[ \alpha_{\alpha}^2 = 8.47/938 \]
\[ \alpha_{d}^2 = 2.22/938 \]

Also the relative momentum
\[ k_{ab} = (m_{b} p_{a} - m_{a} p_{b})/(m_{a} + m_{b}) \]

Let us commence with dh-dh. We have
\[ \mathcal{K}_{hdp} = \int d^3q_p d^3q_d d^3q_h (2\pi)^3 \delta^3(q_h - q_d - q_p) \cdot g_{hdp} \{ h^+(q_h) d(q_d) g_p(q_p) + h.c. \} \quad (IV.17) \]

and we want to calculate the following amplitude
The amplitude for this process is

\[ T(\lambda_{d_f}^f, \lambda_{h_f}^f; \lambda_{d_i}^i, \lambda_{h_i}^i) = \sum_n \langle q_{d_f}^f q_{h_f}^f \lambda_{d_i}^i \lambda_{h_i}^i | \Sigma_{\text{hdp}} | n \rangle \cdot \frac{1}{E_n - E_n} \langle n | \Sigma_{\text{hdp}} | q_{d_i}^i q_{h_i}^i \lambda_{d_i}^i \lambda_{h_i}^i \rangle \quad (IV.18) \]

where

\[ |n\rangle = |q_{d_n}^n \lambda_{d_n}^n q_{p_n}^n \lambda_{p_n}^n q_{d_n}^n \lambda_{d_n}^n \rangle \]

and

\[ E_n = \frac{n}{2m_d^d} + \frac{n}{2m_d^d} + \frac{p_n}{2m_d^d} - 2a_d^2 \]

We can simplify the equation by writing

\[ |\Sigma_{\text{hdp}} | m \rangle \langle n | \Sigma_{\text{hdp}} = \delta_{\lambda_p^p, \lambda_{p'}^{p'}} \int d_\Sigma (2\pi)^3 \delta(q_{d'} - q_{d'} - q_{p'}) \]

\[ \cdot g_{\text{hdp}}^2 \{ h_{d'} \cdot g_{p}^p, p, d_{p'}^{p'}, h_{p'}^{p'} \} \quad (IV.19) \]
where \( d \) stands for integration over all six momenta, and
\[ p' = p(q'_i), \text{ etc., and we have that} \]
\[ E_m = E_{\text{int}} = q^2_d/m_d + q^2_p/2m_p - 2\alpha_d^2 \]

Now, with (IV.19) we can simplify (IV.18) using the commutation rules and get
\[ T(\lambda) = \frac{2g_{\text{hdp}}(\kappa_d^2)}{E-E_{\text{int}}} \sum \hbar^\dagger (-g_2', \lambda_{h_f}) \sigma(\lambda_d^i) p(k, \lambda_p) p^\dagger (k, \lambda_p) \]
\[ \sigma(\lambda_d^i) h(-g_2', \lambda_{h_i}) \]  \hspace{1cm} (IV.20)

where \( \sigma(\lambda_{\alpha_i}) \) is the component of \( \sim \) along \( d(\lambda_{\alpha_i}) \)
\( \{d^\dagger(\lambda_{\alpha_f})\} \), and here \( h, p \) are spinors with the indicated helicity. The kinematics corresponds to Figure 2, that is
\[ k = -g_2-g_2' \]
\[ k_{dp} = (g_2'-2\kappa)/3 = g_2'+2/3 \]
\[ E = q_2^2/2\mu_2 - (\alpha_d^2 + \alpha_h^2) \]
\[ E_{\text{int}} = q_2^2(5/2 + 2\cos\theta) - 2\alpha_d^2 \]

Now, \( \sum_{\lambda_p} p(\lambda_p) p^\dagger (\lambda_p) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

So finally (IV.20) gives
\[ T(\lambda) = \frac{g_{hdp}(k^2_{dp})}{E-E_{\text{int}}} h^\dagger(-q_2', \lambda_f) \sigma(\lambda_{d_i}) \sigma(\lambda_{d_f}) h(-q_2, \lambda_i) \]  

(IV.21)

Now,

\[ d \sigma = d_0 \sigma_0 - d_{+1} \sigma_{-1} - d_{-1} \sigma_{+1} \]

but, as we saw before, \((d^\dagger)_{+1} = - (d^-)_{+1} \), \((d^\dagger)_o = (d^\dagger)_o \)

and as the creation operator for a \( d \) with helicity \( \lambda_d \) is

\[ d^\dagger(\lambda_d) = (d^\dagger)_{\lambda_d} \]

then

\[ d(\lambda_d = \pm 1) = \{d^\dagger(\pm 1)\}^\dagger = \{-d^-_{+1}\} \]

and thus

\[ \tilde{d} \sigma = d(0)\sigma_0 + d(1)\sigma_1 + d(-1)\sigma_{-1} \]  

(IV.22a)

\[ \tilde{d}^\dagger \sigma = d^\dagger(0)\sigma_0 - d^\dagger(1)\sigma_{-1} - d^\dagger(-1)\sigma_{+1} \]  

(IV.22b)

so, from the way we defined \( \sigma(\lambda_{d_i}) \) and Equation (IV.22) we get

\[ \sigma(\lambda_{d_i} = \pm 1) = \sigma_{\pm 1} , \quad \sigma(\lambda_{d_i} = 0) = \sigma_0 \]  

(IV.23)

And, for \( \sigma(d_{f}) \) we have to consider that \( d_{f} \) makes an angle of \( \theta \) with respect to \( z \) axis, then using the transformation properties (page 22) and Equation (IV.22b) we get that
\[ \sigma(\lambda_{df} = \pm 1) = -\cos^2 \theta/2 \sigma_{\pm 1} \pm \frac{\sin \theta}{\sqrt{2}} \sigma_0 - \sin^2 \theta/2 \sigma_{\pm 1} \]

\[ \sigma(\lambda_{df} = 0) = \frac{\sin \theta}{\sqrt{2}} [\sigma_{-1} - \sigma_{+1}] + \cos \theta \sigma_0 \]  

(IV.24)

Using these and the expression for the spinors (section III) we can finally calculate the amplitudes, let

\[ B(\lambda) = h^\dagger (-g'_2, \lambda_2) \sigma(\lambda_{di}) \sigma(\lambda_{df}) h(-g'_2, \lambda_2) \]

then

\[ B_1(\lambda) \equiv B(1 \ 1/2; 1 \ 1/2) \]

\[ = (-\sin \theta/2, \cos \theta/2) \sigma_{+1} \left\{ -\cos^2 \theta/2 \sigma_{-1} + \frac{\sin \theta}{\sqrt{2}} \sigma_0 - \sin^2 \theta/2 \sigma_{+1} \right\} \]

\[ = (-\sin \theta/2, \cos \theta/2) \sin \theta \ (1 \ 0) = -2 \sin^2 \theta/2 \cos \theta/2 \]

and analogously,

\[ B_2(\lambda) = B(1 \ 1/2; 0 \ 1/2) = \sqrt{2} \cos \theta/2 \cos \theta \]

\[ B_3(\lambda) = B(1 \ 1/2; -1 \ 1/2) = -2 \sin \theta/2 \cos^2 \theta/2 \]

\[ B_4(\lambda) = B(1 \ 1/2; 1-1/2) = 2 \sin^2 \theta/2 \cos \theta/2 \]

\[ B_5(\lambda) = B(1 \ 1/2; 0-1/2) = \sqrt{2} \sin \theta/2 \cos \theta \]

\[ B_6(\lambda) = B(1 \ 1/2; -1-1/2) = 2 \sin \theta/2 \cos^2 \theta/2 \]

\[ B_7(\lambda) = B(0 \ 1/2; 0 \ 1/2) = \cos^3 \theta/2 - 3 \cos \theta/2 \sin^2 \theta/2 \]

\[ B_8(\lambda) = B(0 \ 1/2; -1 \ 1/2) = 2\sqrt{2} \sin \theta/2 \cos^2 \theta/2 \]

\[ B_9(\lambda) = B(0 \ 1/2; 1-1/2) = 2\sqrt{2} \sin^2 \theta/2 \cos \theta/2 \]

\[ B_{10}(\lambda) = B(0 \ 1/2; 0-1/2) = -\sin^3 \theta/2 + 3 \sin \theta/2 \cos^2 \theta/2 \]

\[ B_{11}(\lambda) = B(-1 \ 1/2; -1 \ 1/2) = 2 \cos^3 \theta/2 \]

\[ B_{12}(\lambda) = B(-1 \ 1/2; 1-1/2) = 2 \sin^3 \theta/2 \]  

(IV.25)
and then,

\[ t_{i,B}^{(2)} = \frac{g_{hdp}^2 (k_{dp}^2)}{E-E_{\text{int}}} B_i(\lambda) \]

where the superscript indicates the reaction as in section III. Now we proceed to dh-\( \alpha \)n. By the analysis in section III we have six independent amplitudes. The interaction Hamiltonians for dpn and \( \alpha \)hn are

\[ \kappa_{\text{dpn}} = \int d^3q_d d^3q_P d^3q_n (2\pi)^3 \delta(q_d - q_p - q_n) \{ \bar{\psi}(q_p) d^\dagger(q_d) \]

\[ \cdot \Theta \ n(q_n) + h.c. \} g_{\text{dpn}} \]

\[ \kappa_{\text{ahn}} = \int d^3q_\alpha d^3q_h d^3q_n (2\pi)^3 \delta(q_\alpha - q_h - q_n) \]

\[ \{ \bar{n}(q_n) h(q_h) \alpha^\dagger(q_\alpha) + h.c. \} g_{\text{ahn}} \]

where

\[ \Theta = a(q_{\text{pn}}^2) \bar{\psi} + b(q_{\text{pn}}^2) \bar{\psi} \cdot q_{\text{pn}}^2 q_{\text{pn}} \]

(IV.26)

We want to calculate the following amplitude

\[ \begin{array}{c}
\begin{array}{c}
\gamma_1, \lambda_d \\
\gamma_2, \lambda_h
\end{array}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
\gamma_3, \lambda_p \\
\gamma', \lambda_p
\end{array}
\end{array}
\]

\[ -\bar{\gamma}_2, \lambda_h \]

\[ -\bar{\gamma}_3, \lambda_p \]

Figure IV.6
\[ k = q_1 - q_2 \]
\[ k_{pn} = \frac{q_1 - (-q_2)}{2} = q_1 - q_2/2 \]
\[ k_{hn} = \frac{-q_2 + 3q_1}{4} = -q_2 + 3/4 q_1 \]

and
\[ E = q_1^2/2\mu_1 - c_0^2 \]
\[ E_{int} = q_1^2 + 2/3 q_2^2 - q_1 q_2 \cos \theta - c_0^2 \]

Proceeding as for dh-dh we obtain the amplitude,
\[ T(\lambda) = \frac{g_{dpn} (k^2) g_{ahn} (k^2)}{E - E_{int}} \]
\[ \cdot \sum \left\{ \tilde{n}(k_\lambda, \lambda_n) h(-q_2, \lambda_h) \right\} \left\{ \tilde{p}(q_1^i, \lambda_p) \theta(\lambda_{d_i}) n(k_\lambda, \lambda_n) \right\}^\dagger \]

(IV.27)

Let \( B(\lambda) \) be the sum
\[ B(\lambda) = \sum \left\{ n(k_\lambda, \lambda_n) h(-q_2, \lambda_h) \right\} \left\{ n^\dagger(k_\lambda, \lambda_n) \theta(\lambda_{d_i}) \tilde{p}^\dagger(q_1^i, \lambda_p) \right\} \]
\[ = \sum h^T(-q_2, \lambda_h) \tilde{n}^T(k_\lambda, \lambda_n) n^\dagger(k_\lambda, \lambda_n) \theta(\lambda_{d_i}) \tilde{p}^\dagger(q_1^i, \lambda_p) \]
\[ = \sum \tilde{p}^*(q_1^i, \lambda_p) \theta^T(\lambda_{d_i}) \left\{ n^*(k_\lambda, \lambda_n) \tilde{n}(k_\lambda, \lambda_n) \right\} h(-q_2, \lambda_h) \]

but,
\[ \sum_{\lambda_n} n^* \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = -i\sigma_y \]
and therefore,

\[ B(\lambda) = \widetilde{p}^\ast(q_1', \lambda_p) \theta^T(\lambda_d) \{-i \sigma_y \} \ h(-q_2, \lambda_h) \]

\[ \quad = -p^\dagger(q_1', \lambda_p) \{ \sigma_y \theta^T \sigma_y \} \ h(-q_2, \lambda_h) \]

\[ = p^\dagger(q_1', \lambda_p) \theta(\lambda_d) \ h(-q_2, \lambda_h) \]  

(IV.28)

since

\[ \sigma_y \theta^T \sigma_y = -0 \]  

(IV.29)

As for Equation (IV.23),

\[ \theta(\lambda_d = \pm 1) = \theta_{\pm 1} \ ; \ \theta(\lambda_d = 0) = \theta_0 \]

and, Equation (IV.26) gives

\[ \theta_{\pm 1} = a \sigma_{\pm 1} + b/\sqrt{2} \left\{ k_1^2 \sigma_x + k_1 k_1 \sigma_z \right\} \]

\[ \theta_0 = a \sigma_0 + b \left\{ k_1 k_1 \sigma_x + k_1^2 \sigma_z \right\} \]  

(IV.30)

where

\[ k_1 \equiv k_{\text{pn}} \]

Now we can calculate the amplitudes explicitly,

\[ B_1(\lambda) = B(1/2 \ 0; 1 \ 1/2) = p^\dagger(q_1', 1/2) \theta(\lambda_d = 1) h(-q_2, 1/2) \]

\[ = (\cos \theta/2, \sin \theta/2) \left\{ a \sigma_x - b/\sqrt{2} \left[ k_1^2 \sigma_x + k_1 k_1 \sigma_z \right] \right\} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ = (\cos \theta/2, \sin \theta/2) \left\{ \begin{pmatrix} -\sqrt{2} a \\ 0 \end{pmatrix} - \frac{b}{\sqrt{2}} \begin{pmatrix} k_1^2 \ 0 \\ k_1 k_1 \ 1 \end{pmatrix} \right\} \]

\[ = -(\sqrt{2} a + b/\sqrt{2} k_1^2) \cos \theta/2 + b/\sqrt{2} k_1 k_1 \sin \theta/2 \]
and analogously,

\[ B_2(\lambda) = B(1/2 \ 0; 0 \ 1/2) = -a \sin\theta/2 + b\{k_x^2 \sin\theta/2 - k_z k_x \cos\theta/2\} \]

\[ B_3(\lambda) = B(1/2 \ 0; -1 \ 1/2) = b/\sqrt{2}\{k_x^2 \cos\theta/2 - k_z k_x \sin\theta/2\} \]

\[ B_4(\lambda) = B(1/2 \ 0; 1-1/2) = -b/\sqrt{2}\{k_x^2 \sin\theta/2 + k_z k_x \cos\theta/2\} \]

\[ B_5(\lambda) = B(1/2 \ 0; 0-1/2) = a \cos\theta/2 + b\{k_x^2 \cos\theta/2 + k_z k_x \sin\theta/2\} \]

\[ B_6(\lambda) = B(1/2 \ 0; -1-1/2) = \sqrt{2} a \sin\theta/2 + b/\sqrt{2}\{k_x^2 \sin\theta/2 + k_z k_x \cos\theta/2\} \]

\[(IV.31)\]

and then

\[ t_{i,B}^{(3)} = \frac{g_{d\eta n}(k_x^2)g_{\alpha h n}(k_{hn}^2)}{E - E_{int}} B_i(\lambda) \]

We still need \( \eta - \alpha \), \( \eta - \delta h \), and \( \eta - \eta h \). For \( \eta - \alpha \) we can use Figure 3 and thus

\[ k = q_1 - q_3 \ ; \ \ k_{\eta n} = -q_1 - q_3/2 \ ; \ \ k_{hn} = 3/4 q_1 - q_3 \]

\[ E = q_3^2/2\mu_3 - (\alpha_h^2 + \alpha_h^2) \ ; \ \ E_{int} = q_1^2/2 + q_3^2/6 + k^2/2 - \alpha_h^2 \]

and then

\[ T(\lambda) = \frac{g_{d\eta n}(k_{\eta n}^2)g_{\alpha h n}(k_{hn}^2)}{E - E_{int}} B(\lambda) \]

where
\[ B(\lambda) = \sum_n \langle \tilde{p} n \rightharpoonup \sigma_n \rightharpoonup \rangle = \rho(g_1^\dagger, \lambda_p) h(-g_3, \lambda_n) \]

so finally,

\[ t_{1, B}^{(4)} = B_{bpn}(k_{pn}^2) g_{ahn}(k_{hn}^2) \sin \theta / 2 \left( E - E_{\text{int}} \right) \]

\[ t_{2, B}^{(4)} = g_{bpn}(k_{pn}^2) g_{ahn}(k_{hn}^2) \cos \theta / 2 \left( E - E_{\text{int}} \right) \quad \text{(IV.32)} \]

For bh-dh we want to calculate the following diagram,

\[ \begin{array}{c}
\tilde{g}_3 \\
-\tilde{g}_2, \lambda_\alpha' \\
\tilde{g}_2', \lambda_\alpha \\
\tilde{g}_3, \lambda_\lambda \\
\end{array} \]

with

\[ k = -g_3 - g_2^i \]

\[ k_{bp} = g_3 + 2/3 \ g_2' \]

\[ k_{dp} = g_2^i + 2/3 \ g_3 \]

\[ E = g_3^2 / 2 \mu_3 - (\alpha_b^2 + \alpha_h^2) \]

\[ E_{\text{int}} = 3/4 \ (g_2^2 + g_3^2) - 2g_2g_3 \cos \theta - (\alpha_b^2 + \alpha_d^2) \]

and therefore
\[ T(\lambda) = \frac{g_{hbp}(k^2_{dp}) g_{hdp}(k^2_{dp})}{E - E_{\text{int}}} B(\lambda) \]

where

\[ B(\lambda) = h^\dagger(-q^i_2, \lambda \lambda') \sigma(\lambda \lambda') h(-q^i_3, \lambda \lambda') \]

and using Equations (IV.24) we get

\[ B_1(\lambda) = B(1 1/2; 0 1/2) = -\sqrt{2} \sin \theta / 2 \]
\[ B_2(\lambda) = B(0 1/2; 0 1/2) = -\cos \theta / 2 \]
\[ B_3(\lambda) = B(-1 1/2; 0 1/2) = 0 \]
\[ B_4(\lambda) = B(1-1/2; 0 1/2) = 0 \]
\[ B_5(\lambda) = B(0-1/2; 0 1/2) = \sin \theta / 2 \]
\[ B_6(\lambda) = B(-1-1/2; 0+1/2) = \sqrt{2} \cos \theta / 2 \]

Finally, for bh-bh we can use the same kinematics as in dh-dh, replacing d's by b's and 2's by 3's. The amplitude is

\[ T(\lambda) = \frac{g_{hbp}(k^2_{bp})}{E - E_{\text{int}}} h^\dagger(-q^i_3, \lambda \lambda') h(-q^i_3, \lambda \lambda') \]

where

\[ E = q_3^2/2 \mu_3 - (\alpha_b^2 + \alpha_h^2) \]
\[ E_{\text{int}} = q_3^2 (5/2 + 2 \cos \theta) \]

and
Using then the expressions for the spinors, we get

\[
\begin{align*}
    t_{1,B}^{(6)} &= \left[ g_{h\beta}^2 (k_{\beta p}^2) \cos \theta/2 \right] \left[ E - E_{\text{int}} \right]^{-1} \\
    t_{2,B}^{(6)} &= \left[ g_{h\beta}^2 (k_{\beta p}^2) \sin \theta/2 \right] \left[ E - E_{\text{int}} \right]^{-1}
\end{align*}
\]

C. Deuteron Exchange

The deuteron exchange amplitudes for the dh-p\alpha reaction are also easily calculated. They correspond to the following diagram:

Here we need an additional interaction Hamiltonian, the one corresponding to the d\alpha vertex. We calculate it, for S wave couplings, as in section IV.A.

\[
\mathcal{K}_{d\alpha} \sum_{\lambda_1, \lambda_2} \langle 1 1 0 | \lambda_1, \lambda_2 \rangle \langle \lambda_1, \lambda_2 | d \rangle^\dagger \langle g_{d1}, \lambda_1 \rangle^\dagger \langle g_{d2}, \lambda_2 \rangle \langle \omega, \lambda \rangle^\dagger + \text{h.c.}
\]

We will include all normalization constants in the coupling strength. Performing the sum over the deuterons' spin projections, we obtain
$$\kappa_{dd\alpha} = - \int d\mathbf{q}_d \int d\mathbf{q}_{d_1} \int d\mathbf{q}_{d_2} g_{ddd} \left\{ \mathbf{d}^\dagger (\mathbf{q}_{d_1}) \cdot \mathbf{d}^\dagger (\mathbf{q}_{d_2}) \alpha (\mathbf{q}_{d_1} + \mathbf{q}_{d_2}) + \text{h.c.} \right\}$$

From IV.A we had for the hdp vertex,

$$\kappa_{hdp} = \int d^3 q_p d^3 q_d g_{hdp} \left\{ \mathbf{h}^\dagger (\mathbf{q}_h) \mathbf{d}(\mathbf{q}_d) \cdot \mathbf{g} \mathbf{p}(\mathbf{q}_p) + \text{h.c.} \right\}$$

Before proceeding with the amplitude let us do the kinematics of this reaction, corresponding to Figure IV.7. The notation is the same as in section IV.A.

$$E_{\text{initial}} = E_i = \frac{q_2^2}{2\mu_2} - (\alpha_d^2 + \alpha_h^2)$$

$$E_{\text{final}} = E_f = \frac{q_1^2}{2\mu_1} - \alpha^2$$

$$E_n = \frac{q_2^2}{2m_d} + \frac{k^2}{2m_d} + \frac{q_1^2}{2} - 2\alpha_d^2$$

Therefore,

$$E-E_n = - \frac{3}{4} q_1^2 - \frac{1}{12} q_2^2 - \frac{1}{2} q_1 q_2 \cos \theta + (\alpha_d^2 - \alpha_h^2)$$

since \( k = q_1^1 + q_2^2 \). We also need the relative momenta for both vertices, \( k_{dd} \) and \( k_{pd} \). They are

$$k_{dd} = \frac{1}{2} (q_2 - k) = q_1^1/2$$

$$k_{pd} = \frac{1}{3} (2q_1^1 - k) = (q_1^1 - q_2^2)/3$$
We can now proceed with the amplitude. The calculation is very straightforward, one proceeds exactly as before and one obtains

\[ T(\lambda) = - \frac{g_{\bar{d}d\bar{c}}(k^2_e)g_{hp\bar{c}}(k^2_p)}{E-E_n} \left\{ \alpha^+(q_1')d_i(q_2)d_i(q) \right\} \]

\[ \left\{ p^+(q_1)d^+(q)\sigma_jh(-q_2) \right\} \]

\[ = - \frac{g_{\bar{d}d\bar{c}}(k^2_e)g_{hp\bar{c}}(k^2_p)}{E-E_n} \cdot B(\lambda) \]

where

\[ B(\lambda) = p^+(q_1,\lambda)\sigma(\lambda_d)h(-q_2,\lambda_h) \]

and where \( \sigma(+1) = \sigma_+ \), \( \sigma(-1) = \sigma_- \), and \( \sigma(0) = \sigma_0 \).

This \( B(\lambda) \) is the same as the amplitude for neutron exchange (Equations IV.28, 30, 31 with \( a = 1 \) and \( b = 0 \)). Thus

\[ B(1/2, 0; 1/2) = -\sqrt{2} \cos\theta/2 \]

\[ B(1/2, 0; 0, 1/2) = -\sin\theta/2 \]

\[ B(1/2, 0; -1, 1/2) = 0 \]

\[ B(1/2, 0; 1-1, 2) = 0 \]

\[ B(1/2, 0; 0-1, 2) = \cos\theta/2 \]

\[ B(1/2, 0; -1-1, 2) = \sqrt{2} \sin\theta/2 \]

For \( g_{\bar{d}d\bar{c}}(k^2_e) \) we will use the same form as for \( g_{\alpha\bar{n}n} \), i.e.
\[ g_{\alpha\alpha}(k^2_{dd}) = \frac{g_{\alpha\alpha}}{(k^2_{dd} + \beta^2_{dd})} \]

The values of these two parameters, \( g_{\alpha\alpha} \) and \( \beta_{dd} \), will not be determined in this paper although in section VI we discuss reasonable guesses.

D. Vertex Parameters

We now proceed to fix the parameters in the interaction Hamiltonians \( \mathcal{H}_{\text{dpn}} \), \( \mathcal{H}_{\text{bpn}} \), and \( \mathcal{H}_{\text{hbp}} \) (Equations (IV.19), (IV.26), (IV.30), and IV.31). The vertex parameters for \( \alpha_{\text{hnn}} \) will be free parameters in the calculation and their best result will be given in section VI.

For \( \text{dpn} \) we already have a renormalized \( ^3S_1 \) amplitude which has the pole at the experimental deuteron binding energy (Equation (IV.24)). We can compare this amplitude to Yamaguchi's \( ^3S_1 \) amplitude which is parametrized so as to fit the low energy \(^3S_1\) scattering data (as well as the binding energy). Yamaguchi uses a separable potential with central and tensor parts, i.e.

\[ \langle g | V_{\text{np}} | g' \rangle = -\lambda g(g)g(g') \]

with

\[ g(g) = \frac{1}{g^2 + \beta^2} - \frac{t}{\sqrt{6}} \frac{q^2}{(g^2 + \gamma^2)^2} \left\{ \frac{3}{2} \left( q^p \cdot q^o \cdot g - q^p \cdot q^m \right) - \frac{3}{2} \right\} \]
and obtains the same form for $t^{3S_1}$ as in Equation (IV.24) if we let

$$\frac{g_{dpn}^2}{4\pi} = \left\{ \frac{1}{a_d^3 (a_d + \beta)^3} + t^2 \frac{5a_d \gamma}{8\gamma (a_d + \gamma)} \right\}^{-1}$$

$$a(q^2) = \frac{1}{q^2 + \beta^2} - \frac{t}{\sqrt{2}} \frac{q^2}{(q^2 + \gamma^2)^2}$$

$$b(q^2) = \frac{3}{\sqrt{2}} \frac{t}{(q^2 + \gamma^2)^2}$$

where $\beta$, $\gamma$ are range parameters and $t$ measures the strength of the contribution of the tensor part; Yamaguchi gives as best values

$$a = 4.868 \times 10^{-2}$$

$$\beta = (5.759)(4.868 \times 10^{-2})$$

$$\gamma = (6.771)(4.868 \times 10^{-2})$$

$$t = 1.784$$

This then defines our dpn interaction Hamiltonian.

The bpn vertex cannot be fixed so simply because we have no $b$ bound state wavefunction to use for normalization (strictly speaking, the $b$ pole is not even in the physical sheet; it lies, though, very close to the physical sheet).

Take an arbitrary three body amplitude with two of the final particles being a $p$ and an $n$ bound in $^1S_0$, i.e. we consider
The amplitude for this process is \( A \),

\[
A = T \ p_{\sigma}(k^2) g_{\beta\rho\nu}(k^2) = TF
\]

where \( p_{\sigma}(k^2) \) is the propagator, \( k \) is the relative np momentum and for \( g_{\beta\rho\nu}(k^2) \) we again use Yamaguchi's form,

\[
g_{\beta\rho\nu}(k^2) = \frac{g_{\beta\rho\nu}}{(k^2 + \beta_3^2)}
\]

Now, Yamaguchi's \( ^1S_0 \) np amplitude is \( t_0(k^2) \),

\[
\left[ t_0(k^2) \right]^{-1} = \frac{1}{(2\pi)^2 \lambda} (k - i\beta_3)^2(k - i\alpha)(k - i\gamma) \quad (IV.33)
\]

with \( \alpha = \sqrt{\frac{\lambda}{2\beta_3}} - \beta_3, \quad \gamma = -\sqrt{\frac{\lambda}{2\beta_3}} - \beta_3 \)

where \( \lambda \) is the potential strength, given in terms of the scattering length and \( \beta_3 \) by

\[
\pi^2 \lambda = \beta_3^3 \left( 1 - \frac{2}{\beta_3 a_s} \right)^{-1} \quad (IV.34)
\]

Now, we write
\[ F = p_b(k^2)g_{b_{pn}}(k^2) = t_o(k^2)/g_{b_{pn}}(k^2) \]

and we require that the unitarity sum over the two particle state with \( b \) be the same as that over the three particle state with a pair in the continuum state \( b \). That is

\[ \int |F|^2 \, d^3k = 1 \]  \hspace{1cm} (IV.35)

in order that the final state in \( A \) be a properly normalized 2 body system. A problem immediately arises, the upper limit in the integral in (IV.35) depends on the other momenta.

Equation (IV.35) gives

\[ \frac{g_{b_{pn}}^2}{4\pi} = \left(\frac{2\pi^2}{\lambda^2}\right)^2 \int \frac{k_m}{2\pi} \frac{k^2 \, dk}{(k^2 + \alpha^2)(k^2 + \gamma^2)} \]

\[ = \frac{4(2\pi^2\lambda^2)}{\pi\beta_3} \left( \frac{\arctan(k_m/\gamma) - \alpha}{\gamma - \alpha} \arctan(k_m/\alpha) \right) \]

Now, reference (26) gives

\[ \beta_3 = 0.2474 \quad ; \quad a_s = -23.78 \, \text{fm}. \]

and from reference (27)

\[ \alpha^2 = 70 \, \text{KeV} \quad ; \quad \gamma^2 = 229 \, \text{MeV} \]

We used \( k_{\text{max}}^2 = 1 \, \text{MeV} \) which gives

\[ g_{b_{pn}} = 0.0168 \]
\[ \beta_4 = 0.4205 \]

and therefore

\[ g_{\text{hdp}} = 0.038 \]

From B-P's t wavefunction, which contains an nd cluster and an nb cluster we can obtain the relative strength of the tdn and tbn vertices, and together with the above value of \( g_{\text{hdp}} \) we get \( g_{\text{hbp}} \). We also use their value for \( \beta_5 \). Thus we have

\[ \beta_5 = 0.556 \]

\[ g_{\text{hbp}} = -0.23 \]
There are three independent $J = 1/2^+$ amplitudes but near the $t$ pole we can consider just the $^2S_{1/2}^0$ elastic amplitude, call it $m_2$. Then, using the Hamiltonian (Equation (IV.13)) we get

$$m_2 = -\frac{g_{\text{had}}^2(k^2)}{4\pi} \frac{8\mu}{k^2 + 2\mu(\alpha_h^2 - \alpha_d^2)}$$

where $\mu = \frac{m_p m_d}{m_p + m_d} = \frac{2}{3}$ and $k^2$ is the square of the $p$ (or $d$) momentum (in CM) and also a factor of $-\frac{\mu}{2\pi}$ has been introduced for normalization. By choosing $\alpha_h^2$ to be the experimental B.E. of $h$ we have the pole at the correct position.

Let

$$\gamma_h^2 = 2\mu(\alpha_h^2 - \alpha_d^2)$$

then

$$\frac{g_{\text{had}}^2}{4\pi} (k^2 = -\gamma_h^2) = -\frac{1}{8\mu} \text{RES}[m_e, k^2 = -\gamma_h^2]$$

and

$$g_{\text{had}} = (\beta_4^2 - \gamma_h^2)^2 \left( g(k = -\gamma_h^2) \right)^{1/2}$$

Locher's value is

$$\text{RES}[m_2, k^2 = -\gamma_h^2] = -2 \times 0.382$$

and B-P (ref. 25) give
Though this is somewhat arbitrary choice the results were not very sensitive to it (and also its influence was included in the determination of $g_{\text{ahn}}$ which was free in our calculations).

As we noted in Equation (VI.14) we chose a vertex factor for the hdp interaction that contains two parameters, $g_{\text{hdp}}$ and $\beta_4$. We wrote

$$g_{\text{hdp}}(k^2) = \frac{g_{\text{hdp}}}{(k^2 + \beta_4^2)^2}$$

The extra factor of $(k^2 + \beta_4^2)$ in the denominator with respect to Yamaguchi's vertex factor was used to agree with the form used by Barbour and Phillips\textsuperscript{28} in their t($H^3$) wavefunction and we used their value for $\beta_4$. We did not use the vertex strength derived from their wavefunction because of their undefined normalization made it ambiguous. Furthermore, the strength of the vertex could be obtained more directly from Locher's work\textsuperscript{29}. Locher calculated the residue of the nd $^2S_{1/2}$ amplitude at the t pole by fitting forward dispersion relations for the nd amplitude. We can use the t position and residue since as h and t form an isodoublet $g_{\text{hdp}} = g_{\text{tdn}}$. 
V. OBSERVABLES

In this section we want to obtain the explicit expression for the observables, i.e. cross sections and spin expectation values for the reactions $p_d-p_q$. To do this we want to obtain the density matrices for various initial and final states, and we begin by obtaining the basic spin matrices.

1. Spin $1/2$.

Here we take the usual Pauli matrices $\sigma^1, \sigma^2, \sigma^3$ and $\sigma^0 = 1$. (Note we will use $1$ to mean the identity in any $n \times n$ space, dimension will be clear from context.) These satisfy

$$[\sigma^i, \sigma^j] = 2i \varepsilon^{ijk} \sigma^k$$

$i, j, k = 1, 2, 3$

and

$$\text{Tr}(\sigma^\mu \sigma^\nu) = 2 \delta^{\mu \nu}$$

$\mu, \nu = 0, 1, 2, 3$

(V.1)

Using as basis states the eigenstates of $\sigma^3$ we have explicitly,

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
2. Spin 1.

The spin 1 rotation generators satisfy

$$[S^i, S^j] = i\epsilon^{ijk} S^k$$

and

$$\text{Tr}(S^i S^j) = 2\delta^{i,j}$$

$i, j, k = 1, 2, 3$

In the representation where $S^3$ is diagonal we have

$$S' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We want to have a basis of operators for the spin 1/2 and spin 1 spaces, for spin 1/2 $\{\sigma^\mu, \mu = 0, 1, 2, 3\}$ is such a basis and we can express any 2x2 matrix $M_2$ as

$$M_2 = \sum_\mu m^\mu \sigma^\mu$$

where, using (V.1), $m^\mu = 1/2 \text{Tr}(M_2 \sigma^\mu)$. Now we want such a set for spin 1 space. Call the nine basis operators $\lambda^\alpha$, $\alpha = 0, \cdots, 8$. We already have $\lambda^0 = 1$ and the $S^i$, $i = 1, 2, 3$.

The 9 operators have to be linearly independent and for convenience we introduce a normalization factor $= (2S+1) = 3$, i.e. we want $\{\lambda^\alpha\}$ to satisfy

$$\text{Tr}(\lambda^\alpha \lambda^\beta) = 3\delta^\alpha\beta$$

(V.2)
To choose the other operators we take the traceless symmetric second rank tensor, $\delta_{i}^{j} S^{i} S^{j} + \delta_{i}^{j} S^{i} S^{j} - \delta_{i}^{j} (S^{i} + S^{2} + S^{3})$ and we then have the following 9 hermitian matrices that satisfy (V.2)

\[ \lambda^{0} = 1 \]
\[ \lambda^{1} = \sqrt{3/2} S^{1} \]
\[ \lambda^{2} = \sqrt{3/2} S^{2} \]
\[ \lambda^{3} = \sqrt{3/2} S^{3} \]
\[ \lambda^{4} = \sqrt{3/2} (S^{1} S^{2} + S^{2} S^{1}) \]
\[ \lambda^{5} = \sqrt{3/2} (S^{2} S^{3} + S^{3} S^{2}) \]
\[ \lambda^{6} = \sqrt{3/2} (S^{3} S^{1} + S^{1} S^{3}) \]
\[ \lambda^{7} = \sqrt{3/2} [2/\sqrt{3} - \sqrt{3}(S^{3})^{2} - (S^{2})^{2} + (S^{1})^{2}] \]
\[ \lambda^{8} = \sqrt{3/2} [2/\sqrt{3} - \sqrt{3}(S^{3})^{2} + (S^{2})^{2} - (S^{1})^{2}] \]

where for $\lambda^{7}$ and $\lambda^{8}$ the appropriate linear combination was chosen. Table V.1 (pg. 68) gives the explicit form of the $\lambda^{\alpha}$.

Note that instead of the $\lambda^{\alpha}$ defined above we could have taken other sets of 3x3 operators that span the space.

A very convenient choice when one has to perform coordinate rotations is the set of non-hermitian operators $T_{jm}, j = 0, 1, 2$ and $m = j, j-1, \cdots, -j$. These operators satisfy

\[ \text{Tr}(T_{jm} T_{j'm'}) = 3 \delta_{j,j'} \delta_{m,m'} \; ; \; T_{jm}^{\dagger} = (-1)^{m} T_{j,-m} \]
Table V.1

\[ \lambda^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \lambda^1 = \frac{\sqrt{3}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ \lambda^2 = \frac{\sqrt{3}}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \]

\[ \lambda^3 = \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

\[ \lambda^4 = \frac{3}{2} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \]

\[ \lambda^5 = \frac{\sqrt{3}}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{bmatrix} \]

\[ \lambda^6 = \frac{\sqrt{3}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \]

\[ \lambda^7 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & -1 \end{bmatrix} \]

\[ \lambda^8 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & -\sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & -1 \end{bmatrix} \]
Their convenience is due to the fact that their transform under a notation $\mathcal{R}$ is

$$T'_{j'm'} = \sum_m \mathcal{D}^j_{mm'}(\mathcal{R}) T_{jm}$$

i.e. they transform under the $2J+1$ representation of the rotation group.

In this work we are interested in spin 1/2-spin 0 and spin 1-spin 1/2 systems. For spin 1/2-spin 0 the density matrix is 2x2 and thus can be expressed as linear combinations of the $\sigma^\mu$. The spin 1-spin 1/2 system requires a (3x3) · (2x2) dimensional space; to construct it we use the $\lambda^\alpha$ and $\sigma^\mu$ where $\alpha = 0, \cdots, 8$, $\mu = 0, \cdots, 3$.

We then define $\Omega_{(\alpha,\mu)} = \lambda^\alpha \otimes \sigma^\mu$ \hspace{1cm} (V.3)

and they satisfy

\[
\text{Tr}(\Omega_{(\alpha,\mu)}\Omega_{(\beta,\nu)}) = \text{Tr}(\lambda^\alpha \lambda^\beta \otimes \sigma^\mu \sigma^\nu) \\
= \text{Tr}(\lambda^\alpha \lambda^\beta) \text{Tr}(\sigma^\mu \sigma^\nu) = 6 \delta_{\alpha,\beta} \delta_{\mu,\nu} \hspace{1cm} (V.4)
\]

Thus the $\{\Omega_{(\alpha,\mu)}\}$ form a set of l.i. matrices and we can write any matrix $M$ in $1 \otimes 1/2$ space by

$$M = \sum_{(\alpha,\mu)} m_{(\alpha,\mu)} \Omega_{(\alpha,\mu)}$$
with \[ m(\alpha, \mu) = \frac{1}{6} \text{Tr}(M(\alpha, \mu)) \]

To obtain the explicit 6x6 form of the \( \Omega(\alpha, \mu) \) we must consider the ordering of the basis vectors in the definition of the amplitude, and we must be careful to note that since in our 2-particle helicity state the spin 1/2 has spin component measured opposite to the momentum. Thus a state with \( \lambda_{\text{He}^3} = \pm 1/2 \) has a \( (\text{He}^3) \) spin component \( \mp 1/2 \) in the direction of motion. Thus we must interchange all rows and columns with \( \lambda_{\text{He}^3} = 1/2 \) with those labeled \( \lambda_{\text{He}^3} = -1/2 \). Thus we have

\[
\Omega(\alpha, \mu) = \lambda^\alpha \otimes \sigma^\mu = \begin{pmatrix}
(\lambda^\alpha)(\sigma^\mu)_{22} & (\lambda^\alpha)(\sigma^\mu)_{21} \\
(\lambda^\alpha)(\sigma^\mu)_{12} & (\lambda^\alpha)(\sigma^\mu)_{11}
\end{pmatrix}
\]

For a given ensemble, the density matrix (normalized) is such that the expectation value of any operator, \( \hat{0} \), is given by\(^{28} \)

\[ \langle 0 \rangle = \text{Tr}(\rho 0) \]

Now we will consider the three reactions \( p\alpha-p\alpha \), \( dh-dh \), \( dh-p\alpha \) separately.
a) \( \text{dh-dh} \)

Let \( \mu \) be the (6x6) scattering matrix, \( M = -\frac{\mu}{2\pi} T \), \( T \) defined on pg. (27) and \( \rho^i \) (\( \rho^f \)) is the initial (final) density matrix. In terms of the \( \Omega_{(\alpha,\mu)} \) defined before we have

\[
\rho = \sum_{(\alpha,\mu)} a_{(\alpha,\mu)} \Omega_{(\alpha,\mu)} = \frac{1}{6} \sum \text{Tr}(\rho \Omega_{(\alpha,\mu)} \Omega_{(\alpha,\mu)}) = \frac{1}{6} \sum \langle \Omega_{(\alpha,\mu)} \rangle \Omega_{(\alpha,\mu)}
\]

The density matrix in the final state, \( \rho^f \), is given by

\[
\rho^f = M \rho^i M^\dagger / \{ \text{Tr}[M \rho^i M^\dagger] \}
\tag{V.5}
\]

and then the expectation value of any operator in the initial or final states is given by

\[
\langle \Omega_{(\alpha,\mu)} \rangle^i,f = \text{Tr}(\Omega_{(\alpha,\mu)} \rho^i,f)
\tag{V.6}
\]

We can obtain the various initial states by specifying which \( \langle \Omega_{(\alpha,\mu)} \rangle^i \neq 0 \), e.g. if we have an unpolarized initial system then \( \langle \lambda^\alpha \rangle^i = \langle \sigma^\mu \rangle = 0 \) for \( \alpha,\mu \neq 0 \) and thus \( \rho^i = \frac{1}{6} \). If \( \text{d} \) is polarized but \( \text{He}^3 \) is not then \( \langle \lambda^\alpha \rangle^i \neq 0 \) and

\[
\langle \sigma^\mu \rangle^i = 0 \text{ for } \mu \neq 0 \text{ and therefore } \rho^i = \frac{1}{6} \sum_{\alpha=0}^8 \langle \lambda^\alpha \rangle^i \lambda^\alpha \phi^i \text{,}
\]

and so on.
We will consider an initial unpolarized state. There are 36 expectation values but not all are independent. The symmetries of $M$ and the form of the spin operators constrain 18 expectation values to be zero.

To prove this we define $A$,

$$A = MM^\dagger = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where $A_1, \ldots, A_4$ are $3 \times 3$ hermitian matrices.

Let $i, j, \ldots = 1, 2, \ldots, 6$

$a, b, \ldots = 1, 2, 3$

Now we will prove that

$$A_{a,b}^2 = (-1)^{a+b+1} A_{4-b,4-a}^3 \quad (V.7)$$

$$A_{a,b}^1 = (-1)^{a+b} A_{4-a,4-b}^4 \quad (V.8)$$

To prove these we use the invariance of $M$ under parity transformations, i.e. Equation (III.10)

$$m_{i,j} = (-1)^{i+j} m_{7-i,7-j}$$

then

$$A_{i,j} = m_{i,k} m_{k}^* = (-1)^{i+k} m_{7-i,7-k} m_{7-j,7-k}^* (-1)^{j+k}$$

$$= (-1)^{i+j} m_{7-i,k} m_{7-j,k}^* = (-1)^{i+j} A_{7-i,7-j} \quad (V.9)$$
Equation (V.9) immediately gives (V.7), (V.8) when we let

\[ i = 1, 2, 3 \text{ and } j = 4, 5, 6 \text{ or } 1, 2, 3. \]

Now, using Equations (V.3) and the definition of \( A^1, \ldots, A^4 \) we have

\[
\text{Tr}(A^0_\alpha (\alpha, 0)) = \text{Tr}(A^1_\alpha + A^4_\alpha),
\]

\[
\text{Tr}(A^0_\alpha (\alpha, 1)) = \text{Tr}(A^2_\alpha + A^3_\alpha),
\]

\[
\text{Tr}(A^0_\alpha (\alpha, 2)) = (-i) \text{ Tr}(A^2_\alpha - A^3_\alpha),
\]

\[
\text{Tr}(A^0_\alpha (\alpha, 3)) = -\text{ Tr}(A^1_\alpha - A^4_\alpha) \quad (V.10)
\]

Now, let \( \Lambda \) be a 3x3 matrix. Define \( \Lambda \) to be even (odd), and write it \( \Lambda^+(\Lambda^-) \) if \( \Lambda^\pm_{ab} = \pm \Lambda^\pm_{4-a, 4-b} \) (note that

\( \lambda, \lambda^0, \cdots, \lambda^8 \)

are even while the rest of the \( \lambda^\alpha \) are odd).

It is easy to see that this definition gives the usual properties of even and odd quantities.

So now, using Equations (V.7) and (V.8) we have

\[
\text{Tr}(A^3_\Lambda^\pm) = \sum (-1)^{a+b+1} A^2_{4-b, 4-a} \Lambda^\pm_{a, b} = \sum (-1)^{a+b} A^2_{ba} \Lambda^\pm_{ba}
\]

and

\[
\text{Tr}(A^4_\Lambda^\pm) = \sum (-1)^{a+b} A^1_{a, b} \Lambda^\pm_{b, a}
\]

and therefore
\[ \text{Tr}(A \Omega^\pm_{(a,0)}) = \sum \lambda^\pm_{a,b} \lambda^\pm_{b,a} [1 \pm (-1)^{a+b}] \]

\[ \text{Tr}(A \Omega^\pm_{(a,1)}) = \sum \lambda^\pm_{a,b} [\lambda^\pm_{b,z} \pm (-1)^{a+b} \lambda^\pm_{a,b}] \]

\[ \text{Tr}(A \Omega_{(a,2)}) = \sum (-i) \lambda^\pm_{a,b} [\lambda^\pm_{b,a} \pm (-1)^{a+b} \lambda^\pm_{a,b}] \]

\[ \text{Tr}(A \Omega_{(a,3)}) = \sum (-1) \lambda^\pm_{a,b} \lambda^\pm_{b,a} [1 \mp (-1)^{a+b}] \]

(V.11)

and we see that there are at most 18 independent (real) numbers, \( \text{Tr} (A^1 \lambda^\alpha) \) and \( \text{Tr} (A^2 \lambda^\alpha) \). In table V.2 we give the \( \text{Tr}[M M^\dagger \Omega_{(a,\mu)}] \) in terms of these 18 quantities. Note that we cannot determine \( M \) completely by measuring final states for an initially unpolarized system since we know that \( M \) has twelve independent amplitudes, i.e. for each energy and angle there are 23 independent numbers (since we can multiply all of the amplitudes by a phase factor).
Table V.2

<table>
<thead>
<tr>
<th>$\alpha^\mu$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2 \text{Tr}(A^{1,0}_\lambda)$</td>
<td>0</td>
<td>-2i $\text{Tr}(A^{2,0}_\lambda)$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2 $\text{Tr}(A^{2,1}_\lambda)$</td>
<td>0</td>
<td>-2 $\text{Tr}(A^{1,1}_\lambda)$</td>
</tr>
<tr>
<td>2</td>
<td>2 $\text{Tr}(A^{1,2}_\lambda)$</td>
<td>2 $\text{Tr}(A^{2,2}_\lambda)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2 $\text{Tr}(A^{2,3}_\lambda)$</td>
<td>0</td>
<td>-2 $\text{Tr}(A^{1,3}_\lambda)$</td>
</tr>
<tr>
<td>4</td>
<td>2 $\text{Tr}(A^{1,4}_\lambda)$</td>
<td>2 $\text{Tr}(A^{2,4}_\lambda)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>-2i $\text{Tr}(A^{2,5}_\lambda)$</td>
<td>-2 $\text{Tr}(A^{1,5}_\lambda)$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>-2i $\text{Tr}(A^{2,6}_\lambda)$</td>
<td>-2 $\text{Tr}(A^{1,6}_\lambda)$</td>
</tr>
<tr>
<td>7</td>
<td>2 $\text{Tr}(A^{1,7}_\lambda)$</td>
<td>0</td>
<td>-2i $\text{Tr}(A^{2,7}_\lambda)$</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>2 $\text{Tr}(A^{1,8}_\lambda)$</td>
<td>0</td>
<td>-2i $\text{Tr}(A^{2,8}_\lambda)$</td>
<td>0</td>
</tr>
</tbody>
</table>

We now calculate some of the more common observables, the unpolarized $d\sigma/d\Omega$ and the final vector polarizations of the $d$ and He$^3$ (which by table 2 lie along the $y$ axis, i.e. normal to the scattering plane).

Let

$$\sigma^\mu(\theta) = \frac{d\sigma}{d\Omega}(\theta) \text{ unpolarized} = \frac{1}{6} \text{Tr}(MM^\dagger \Omega_{(0,0)})$$

$$P_{1/2}(\theta) = \frac{1}{6\sigma^\mu(\theta)} \text{Tr}(MM^\dagger \Omega_{(0,2)})$$

$$P_1(\theta) = \frac{1}{6\sigma^\mu(\theta)} \text{Tr}(MM^\dagger \Omega_{(2,0)})$$
i) $\sigma^u(\theta)$

$$
\sigma^u(\theta) = \frac{1}{6} \text{Tr}(MM^\dagger \Omega_{(0,0)}) = \frac{1}{3} \text{Tr}(A^1_\lambda^0)
$$

$$
= \frac{\mu^2}{4\pi^2} \sum_{i=1}^{12} \left| t_i(2) \right|^2
$$

where the $t_i^2$, $i = 1, \ldots, 12$ are as defined on pg. 27

ii) $P_{1/2}(\theta)$

$$
\sigma^u_{P_{1/2}} = \frac{1}{6} \text{Tr}(MM^\dagger \Omega_{(0,2)}) = -\frac{i}{3} \text{Tr}(A^2_\lambda^0)
$$

$$
= \frac{\mu^2}{4\pi^2} \left( -\frac{2}{3} \right) \text{Im} \left\{ (t_2(2) - t_8(2)) (t_5(2) - t_9(2))^* + (t_6(2) + t_{11}(2)) t_4(2) + t_3(2) t_5(2) + t_3(2) t_7(2) t_{10}(2) \right\}
$$

and,

iii) $P_1(\theta)$

$$
\sigma^u_{P_1} = \frac{1}{6} \text{Tr}(MM^\dagger \Omega_{(2,0)}) = \frac{1}{3} \text{Tr}(A^1_\lambda^2)
$$

$$
= \frac{\mu^2}{4\pi^2} \left( \frac{1}{\sqrt{3}} \right) \text{Im} \left\{ (t_4(2) + t_{10}(2)) (t_5(2) - t_9(2))^* + (t_2(2) + t_8(2)) (t_3(2) - t_7(2))^* + t_1(2) t_2(2)^* + t_5(2) t_6(2)^* - t_8(2) t_{11}(2)^* - t_9(2) t_{12}(2)^* \right\}
$$
b) \( p\alpha - p\alpha \)

Let \( M \) be the amplitude for this reaction defined by

\[
M = - \frac{\mu_1}{2\pi} T
\]

(T defined on pg. 26)

We can obtain the expressions for the observables by considering an initially unpolarized system, then

\[
\rho^i = 1/2
\]

and

\[
\rho^f = \frac{M_0^i M^i}{\text{Tr}(M_0^i M^i)} = \frac{M M^\dagger}{\text{Tr}(M M^\dagger)}
\]

The unpolarized cross section is given by

\[
\sigma^\mu(\theta) = \frac{d\sigma^\mu}{d\Omega} = 1/2 \text{ Tr}(M M^\dagger) = \frac{\mu_1^2}{4\pi} \left[ |t_1^{(1)}|^2 + |t_2^{(1)}|^2 \right]
\]

and

\[
P_{1/2}^f(\theta) = \text{Tr}(\rho^f \sigma^2)
\]

so

\[
\sigma_{1/2}^\mu = 1/2 \text{ Tr}(M M^\dagger \sigma^2) = \frac{\mu_1^2}{4\pi} 2 \text{ Im} \left[ t_1^{(1)} t_2^{(1)*} \right]
\]

c) \( d\alpha - p\alpha \)

Here \( M \) is the 2x6 amplitude matrix, in terms of the \( T \) defined on pg. (25), \( M = - \frac{1}{2\pi} \sqrt{\frac{q_1}{q_2}} \frac{\mu_1 \mu_2}{\mu_2} T \). Firstly, let us calculate the observables for an initially unpolarized system.
\[ \rho^i = 1/6 \]

then

\[
\sigma^\mu(\theta) = \text{Tr}(M^\dagger M^i) = 1/6 \text{ Tr}(MM^\dagger)
\]

\[
= 1/6 \sum_{i=1,2} \sum_{j=1,\ldots,6} |m_{ij}^{(3)}|^2 = \frac{q_1^1 \mu_1^1 \mu_2^1}{q_2} \frac{1}{4\pi^2} \sum_{i=1}^{6} \sum_{i=1}^{6} |t_i^{(3)}|^2
\]

and

\[
P_{1/2}(0) = \frac{\text{Tr}(M^\dagger M^i \sigma^2)}{\text{Tr}(M^\dagger M^i)}
\]

so

\[
\sigma^\mu_{1/2} = 1/6 \text{ Tr}(MM^\dagger \sigma^2) = 1/6 \sum_{i,k=1,2} m_{ij} m^*_{kj} \sigma_{ki} \sum_{j=1,\ldots,6}
\]

\[
= -1/3 \sum_{j=1,\ldots,6} \text{IM}(m_{ij} m^*_{2j})
\]

\[
= 2/3 \frac{q_1^1 \mu_1^1 \mu_2^1}{q_2} \frac{1}{4\pi^2} \text{IM}\left\{t_1^{(3)} t_6^{(3)*} - t_2^{(3)*} t_5^{(3)} + t_3^{(3)} t_4^{(3)*}\right\}
\]

We also will calculate the expressions for the proton polarization when either in the initial state the $\text{He}^3$ is vector polarized, call it $P_{1/2}^a$, or when the d is vector polarized, call it $P_{1/2}^b$. (We always consider the initial state totally polarized along the y axis, with initial polarizations $p_{1/2,i}$ and $p_{1,i}$). Thus
\[ \rho^i, a = \frac{1}{6} \otimes \{ I + P_{1/2}, i \sigma^2 \} \]
\[ \rho^i, b = \frac{1}{6} \{ I + P_{1}, i \lambda^2 \} \otimes I \]

and
\[ p_{1/2}^a[b] = \text{Tr}\{ M_p^i, a[b] M^\dagger \sigma^2 \} \]

so
\[ \sigma_{p_{1/2}}^a = \sigma_{p_{1/2}^+ p_{1/2}, i}^a \times \frac{1}{6} \text{Tr}\{ M_1 \otimes \sigma^2 M^\dagger \sigma^2 \} \]
\[ \sigma_{p_{1/2}}^b = \sigma_{p_{1/2}^+ p_{1}, i}^b \times \frac{1}{6} \text{Tr}\{ M_1 \otimes \sigma^2 M^\dagger \sigma^2 \} \]

calculating the traces one finally gets
\[ \sigma_{p_{1/2}}^a = \sigma_{p_{1/2}}^+ \frac{q_1^{1/2} q_2^{1/2}}{q_2 4\pi^2} \frac{1}{3} p_{1/2, i} \]
\[ \left\{ |t_{(3)}^2|^2 + |t_{(3)}^5|^2 - 2 \text{Re}\left[ |t_{(3)}^1 t_{(3)}^3|^2 + t_{(3)}^4 t_{(3)}^6 \right] \right\} \]
\[ \sigma_{p_{1/2}}^b = \sigma_{p_{1/2}}^+ \frac{q_1^{1/2} q_2^{1/2}}{q_2 4\pi^2} \frac{1}{\sqrt{3}} p_{1, i} \]
\[ \left\{ \text{Re}\left[ t_{(3)}^6 t_{(3)}^4 - t_{(3)}^5 t_{(3)}^6 + t_{(3)}^4 t_{(3)}^6 \right] \right\} \]

An observable usually measured is the proton asymmetry \[ A(p) \]
which is the difference in the cross sections for initial state polarized along \( \hat{q}_i \times \hat{q}_f \). Thus
\[ A_{1/2}(\theta) = \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} \]

and

\[ \sigma_+ = \text{Tr}(M \sigma^i M^\dagger) = \frac{1}{6} \text{Tr}(MM^\dagger) + \frac{1}{6} \text{Tr}(M \otimes \sigma^2 M^\dagger) \]

\[ \sigma_- = \text{Tr}(M \sigma^i M^\dagger) = \frac{1}{6} \text{Tr}(MM^\dagger) - \frac{1}{6} \text{Tr}(M \otimes \sigma^2 M^\dagger) \]

and therefore

\[ \sigma_\mu A_{1/2} = \frac{1}{6} \text{Tr}(M \otimes \sigma^2 M^\dagger) \]

\[ = \frac{q_1^\mu q_2^\mu}{q_2^2 4\pi^2} \left( - \frac{2}{3} \text{IM} \{ t_1^{(3)} t_4^{(3)*} + t_2^{(3)} t_5^{(3)*} + t_3^{(3)} t_6^{(3)*} \} \right) \]
VI. RESULTS AND CONCLUSIONS

In this section we will describe the computational processes and then present and discuss the results.

In this paper we only consider the results for a fixed energy, $E_d^\text{lab} = 6$ MeV. This corresponds to a proton lab energy of about 26.5 MeV in the $p\alpha$ channel. In addition, let us number the vertices on the exchange diagrams so that $\alpha n \sim 1$, $dpn \sim 2$, $bpn \sim 3$, $hdp \sim 4$, $hbp \sim 5$, and $dd\alpha \sim 6$.

A. Computational Details

The general structure of our calculation is given in Figure VI.1,

\[
\begin{align*}
&i \quad B \\
&ii \quad B^J \\
&iii \quad T^{JP} = B^{JP} \left[ 1 - i\rho B^{JP} \right]^{-1} \\
&iv \quad T^{JP}_c = C^T T^{JP} C \\
&v \quad \delta T = \sum_f a^J d^{J\mu}_{\lambda}(\theta) \left[ T^{JP}_c - B^J \right] \\
&vi \quad T = B + \delta T + T_c
\end{align*}
\]
i) $B$ is our input amplitude. It consists of the diagrams in Figure IV.4 plus the phenomenological $p\alpha-p\alpha$ input. This $p\alpha$ input consists of real phase shifts such that after unitarization we would have the real part of the phase shift agreeing with the experimental values for $J < 5/2$. In this manner, of course, we reproduced the experimental $p\alpha-p\alpha$ cross section and proton polarization. For the final choice of parameters for the other reactions, the input phase shifts, and corresponding output and experimental values, are given in table VI.1, together with the inelasticities $\eta$, $\eta = \exp(-2\text{Im}\delta)$.

<table>
<thead>
<tr>
<th></th>
<th>Input</th>
<th>Output</th>
<th>Experiment $^{30}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{1/2}$</td>
<td>$(70^\circ, \eta=1)$</td>
<td>$(78^\circ, 0.98)$</td>
<td>$(77^\circ, 1.)$</td>
</tr>
<tr>
<td>$P_{1/2}$</td>
<td>$(52^\circ, \eta=1)$</td>
<td>$(51^\circ, 0.89)$</td>
<td>$(53^\circ, 0.99)$</td>
</tr>
<tr>
<td>$P_{3/2}$</td>
<td>$(35^\circ, \eta=1)$</td>
<td>$(86^\circ, 0.99)$</td>
<td>$(85^\circ, 0.82)$</td>
</tr>
<tr>
<td>$D_{3/2}$</td>
<td>$(88^\circ, \eta=1)$</td>
<td>$(13^\circ, 0.95)$</td>
<td>$(14^\circ, 0.82)$</td>
</tr>
</tbody>
</table>

Table VI.1

ii) $B^J$ is obtained by integrating over $\theta$ as in Equation (II.16). In the numerical calculation, this was the only step where numerical non-roundoff errors were introduced. The integration was done with Simpson's rule with 21 points. By increasing this number, we estimated that the
final error (e.g. in cross sections) caused by this approximation is less than 1%.

iii) Here we unitarize the partial wave definite parity amplitudes (see Equation (II.14)).

iv) In this level we introduce the Coulomb correction factors. These Coulomb phase shifts depend only on L, and thus we had to transform the (diagonal) matrices in the L,S representation to the helicity representation.

The transformation matrices are obtained by considering the transformations for the states which are

\[ |JM;\lambda_1 \lambda_2 \pi\rangle = \sum_{L,S} \langle JMLS|JM\lambda_1 \lambda_2 \pi\rangle|JM;LS\rangle \]

where

\[ \langle JMLS|JM\lambda_1 \lambda_2 \pi\rangle = \sqrt{\frac{2(2L+1)}{2J+1}} \langle LS|0,\lambda_1 \lambda_2 \rangle \langle S_1 S_2 S|\lambda_1 \lambda_2 \rangle \delta_{\pi,(-1)L} \]

Thus for each J\pi we have an orthogonal matrix \( U^{J\pi} \) such that

\[ |JM(\lambda_1 \lambda_2 i;\pi)\rangle = U^{J\pi}_{ij} |JM;(LS)\rangle \]

and therefore

\[ T^{J\pi}_{LS} = [U^{J\pi}]^T \lambda \lambda T^{J\pi}_{\lambda \lambda} [U^{J\pi}] \]

Now for simplicity let \( T^{J\pi}_{LS}(L_f S_f;L_i S_i) = T^{J\pi}_{L_f L_i} \).
The Coulomb corrected amplitude [see reference 16, pg. 265] is then given by

\[ \exp(i\gamma^C_{L_f}) T_{L_f, L_i} \exp(i\gamma^C_{L_i}) \]

with

\[ \exp(i\gamma^C_{L_i}) = \left[ \frac{\Gamma(L+1+\xi^C_c)}{\Gamma(L+1-\xi^C_c)} \right]^{1/2} \]

\[ \xi^C_c = \mu \frac{e^1 e^2}{q_c} \]

and thus, finally

\[ C^J^\pi = U^{J^\pi} e^{i\gamma^C_{L}} [U^{J^\pi}]^T \]

In level (v) of Figure VI.1 we compute the correction to the full input amplitude from the unitarization and Coulomb corrections, using Equation (II.15) for the difference between \( T^J_c \) and \( B^J \). Then, finally, the full amplitude is obtained in level (vi), where one also adds \( T^J_c \) which is the Coulomb amplitude for elastic scattering, i.e.

\[ T_c^J(\lambda) = \langle a'; \lambda'^{1} \lambda'^{2} | \delta_{a, a'} f^{c}_c, a(\theta) | a; \lambda \lambda \rangle \]

where

\[ f^{c}_c, a(\theta) = -\frac{\xi^a}{2q^a} (\sin\theta/2)^{-2} \exp\{2i\gamma^{a}_{L=0} -2i\xi^a a \ln(\sin\theta/2)\} \]

These Coulomb corrections are valid for \( \theta >> (qR)^{-1} \), R being a shielding radius.
B. Results

The results are exhibited in Figure VI.2 through Figure VI.16, and for comparison we give the experimental results in Figures VI.15 and VI.16. These results, with the exception of Figure VI.9, do not include the deuteron exchange contribution to dh-pα; we did not include this exchange for although a priori it would seem to be a large effect due to the proximity of its singularities to the physical region (especially as compared to the one nucleon exchange, see the Appendix) we expect that the vertex strength, \( g_{dd\alpha} \), to be quite smaller than the corresponding factor in the one nucleon exchange \( g_{\eta\alpha} \), since the experimental cross section (Figure VI.16) does indicate a predominant one nucleon exchange mechanism (the d exchange has a backward peak). In addition, the necessary parameters, \( g_{dd\alpha} \) and \( \beta_6 \), are indeterminate and although in principle they could be determined independently of dh-pα (e.g. from dd scattering or an α wavefunction) it does not appear practical to do so and we did not want to introduce further free parameters into our calculations.

Before entering into the description of results let us briefly list them. Firstly we will have the dh-dh and dh-pα (\( \sigma_1 \) and \( \sigma_2 \), respectively) due to the nonunitarized
exchange amplitudes. In this approximation all the processes are uncoupled (since all coupling comes from the non-linear unitarity condition imposed on the partial wave amplitude) and polarizations are identically zero since the amplitudes are real. In the next few graphs we show $\sigma_1$ and $\sigma_2$ computed with the unitarity corrections for a variety of inputs and showing the effect of the various parameters in the problem, both free and externally fixed. Here is where we examine the different contributions, i.e. tensor forces, one nucleon exchanges, bh channel, deuteron exchange, etc. We also examine the different spin averages and show the need for additional noncentral interactions, especially for the dh-dh D waves. This additional input was determined by its effect on the cross section and polarizations. Then, finally, we present our final results and the experimental ones.

Now let us turn to the actual results, beginning with the Born cross sections, Figure VI.2. We have $\sigma_1(\sigma_{\text{dh-pa}})$ for the Born amplitudes (i.e. they are calculated by using an amplitude obtained by bypassing levels (ii) through (v) in Figure VI.1). The upper graph has curves (a), (b) for $\sigma_2$ without and with Coulomb effects, respectively. In the lower graph we have curves for $\sigma_1$ showing the dependence
Figure VI.2. Born Cross Sections for dh·dh ($\sigma_2$) and dh·pa ($\sigma_1$). Curve b, upper graph, includes Coulomb interaction. Lower graph gives dependence on $\beta_1$,
a) $\beta_1 = 0.40$, b) $\beta_1 = 0.45$, and
c) $\beta_1 = 0.5$, with $g_1 = 0.9$. 
upon $\beta_1$ for a fixed $g_1$, (a) $\beta_1^2 = 0.45$, (b) $\beta_1^2 = 0.50$, and (c) $\beta_1^2 = 0.55$. The dependence upon $g_1$ (not plotted) is simple, $\sigma_1 \propto g_1^2$. The dependence of $\sigma_1$ (Born) upon $t$ (the factor in non-central part of dpm vertex, see Section IV.C) also not plotted, was very small ($\approx 4\%$) at $\theta = 0$ and fairly large ($\approx 25\%$) at $\theta = \pi$.

These Born cross sections are very large, especially $\sigma_1$. They exhibit the usual shape of particle exchange cross sections and have no other structure. Now we will proceed to examine the cross sections computed with unitarized amplitudes. Also, from now on unless we state otherwise, we are including the $p\alpha-p\alpha$ input given in table VI.1.

We unitarized the partial waves for $J \leq 7/2$ since the unitarized amplitudes converged rapidly to the Born amplitudes (which were rather small for large $J$). In fact, after $J = 3/2$ the change in the cross section from unitarizing another partial wave was slight, on the order of a few percent.

Figure VI.3 shows $\sigma_2$ for different inputs. In (a), we have $g_1 = 0$ (i.e. no coupling to $p\alpha$) and no Coulomb effects, (c) is as (a) plus Coulomb effects. At $\pi/2$ these cross sections are approximately half of the experimental cross section and moreover have a minimum there. Coupling
Figure VI.3. Cross Sections for dh-dh.

a) One particle exchanges within dh and bh channels.

b) Additional $^4D$-wave input, $R_D$.

c) As a) but including Coulomb effects.

d) All of the above input.
the $p\alpha$ channel (Figure VI.4b) did not improve this noticeably. Furthermore, $P_{1/2}$ and $P_1$ were extremely small in these cases ($|P_{1/2}| < 0.01$, $|P_1| < 0.05$). The problem with the polarizations was to be expected. So far the only non-central interaction is in the $dpn$ vertex and in the $p\alpha$ elastic, but these non-central forces affect $dh$-$dh$ through $dh$-$p\alpha$ and so have little effect as can be seen from the above quoted values of $P_{1/2}$ and $P_1$ and from Figure VI.6b, where the tensor part of $dpn$ is set equal to zero. The cross section data show another shortcoming: The experimental data indicate strong D waves while in VI.3c (or VI.4b) the D waves either interfere destructively at $\pi/2$ or else they are small. We checked the amplitudes and noticed that they were very small. We thus decided to add an additional D wave input, call it $R_D$, which was to be determined by its effect on $P_{1/2}$ and $P_1$. $R_D$ was introduced in level ii of Figure VI.1 and later we will see how it was determined. Its effect on $\sigma_2$ is seen in Figure VI.3b where it is the only input and in VI.3d where we have the exchange diagrams, Coulomb effects and $R_D$. As can be seen agreement with experiment has improved noticeably.

Figure VI.4 shows both $\sigma_2$ (upper graph) and $\sigma_1$ (lower graph) for different inputs, a) only $g_1 \neq 0$, b) all $g_1 \neq 0$,  

Figure VI.4. Cross Sections for $dh\cdot dh$ ($\sigma_2$) and $dh\cdot pa$ ($\sigma_1$).

a) One-particle-exchange in $dh\cdot pa$.

b) All nucleon-exchange interactions.

c) As a) but including Coulomb effects.

d) As a) but including $R_D$. 
c) (a) plus Coulomb effects, and (d) gives (a) plus $R_D$.

Notice that the elastic dh channel has strong effect on dh-px but not the other way around. The plots of $\sigma_1$ here have values of $g_{12}^2 = 1.0$ and $\beta_{12}^2 = 0.5$ which are close to the best values which were determined to be (by fitting $\sigma_1$ to $\sigma_1^{(exp)}$) $g_{12}^2 = 0.9$ and $\beta_{12}^2 = 0.45$ (these are the values used in Figure VI.15). Now we examine the influence of the different parameters on the cross sections.

In Figure VI.5 we exhibit the dependence of $\sigma_1$ on $\beta_{12}$ and $g_{12}$ while everything else is "on", i.e. all $g_i \neq 0$, and Coulomb effects and $R_D$ are included. The upper graph shows influence of $\beta_{12}$ for fixed $g_{12}^2 = 0.9$, (a) $\beta_{12}^2 = 0.45$, (b) $\beta_{12}^2 = 0.40$, and (c) $\beta_{12}^2 = 0.50$. In the lower graph, $\beta_{12}^2 = 0.45$ and $g_{12}^2$ varies, (a) $g_{12}^2 = 0.9$, (b) $g_{12}^2 = 1.0$, and (c) $g_{12}^2 = 0.8$.

Figure VI.6 shows the influence of $t$ (tensor parameter in dpn vertex) on $\sigma_2$ (upper graph) and on $\sigma_1$ (lower graph). Curves (a) have $t$ as given on pg. 49, and in curves (b) $t = 0$. As we had mentioned previously, the tensor part of the dpn vertex affects dh-dh very slightly and mainly near $\theta = \pi$. The inelastic cross section $\sigma_1$ has a more striking dependence on $t$. 
Figure VI.5. Dependence of dh-\( p \alpha \) cross section on n\( \hbar \alpha \) vertex parameters, with other nucleon exchanges, Coulomb effects, and \( R_D \) fixed.

Upper graph: \( g_1^2 = 0.9 \)
- a) \( \beta_1 = 0.45 \)
- b) \( \beta_1 = 0.40 \)
- c) \( \beta_1 = 0.50 \)

Lower graph: \( \beta_1 = 0.45 \)
- a) \( g_1^2 = 0.9 \)
- b) \( g_1^2 = 1.0 \)
- c) \( g_1^2 = 0.8 \)
Figure VI.6. Dependence of cross sections upon $t$, tensor parameter in dpn vertex.

a) $t = 1.784$

b) $t = 0$
In Figure VI.7 we exhibit the effect of the bh channel on the cross sections. Curves (a) have the full input and in curves (b) \( g_3 = g_5 = 0 \). They seem to indicate that we could have observed the effect of this channel into the \( \alpha \eta \) vertex parameters, i.e. \( g_1 \) and \( \beta_1 \).

We have also studied the effects of particle breakup in channels 2 (dh) and 3 (bh). These effects were simulated by introducing inelasticity in our unitarity equation (II.15) by changing \( \rho[ij] = q_i \cdot (\delta[ij]) \) to \( \rho'[ij] = q_i \cdot (1+\gamma_i) \cdot (\delta[ij]) \) for \( j = 2,3 \). In Figure VI.8 we show the effect of this change for channel 2, curve (a) has \( \gamma_2 = 0 \), and curve (b) \( \gamma_2 = 0.25 \). The effect is noticeable especially for \( \theta = 0,\pi \). \( \gamma_3 \neq 0 \) (not plotted) changed curve (a) very slightly.

In Figure VI.9 we have the graphs which include the deuteron exchange. Upper and middle graphs are for \( \sigma_2 \) calculated with the unitarized amplitudes corresponding to all our input, varying \( g_{d\bar{d}\alpha} \) (upper graphs) and \( \beta_6 \) (middle graphs) and leaving all other parameters fixed. Given this restriction, the best fit to the data was as we expected, a deuteron exchange interaction weaker but longer ranged (\( g_{d\bar{d}\alpha} = 0.1 \), \( \beta_6 = 0.3 \)) than the nucleon exchange (in the lower graph we show \( \sigma_2 \)(Born) corresponding to these two exchanges). The
Figure VI.7. Dependence of cross sections upon bh channel.

a) With bh channel

b) Without bh channel
Figure VI.8. Dependence of cross sections upon inelasticity (see text, pg. 90)

a) \( \gamma_2 = 0 \)

b) \( \gamma_2 = 0.25 \)
Figure VI.9. Dependence of \( \sigma_1 \) on deuteron exchange parameters.

Upper graph: \( \beta_6 = 0.3 \)
   a) \( g_{d\alpha}^2 = 0.01 \)
   b) \( g_{d\alpha}^2 = 0.1 \)
   c) \( g_{d\alpha}^2 = 0.5 \)

Middle graph: \( g_{d\alpha}^2 = 0.1 \)
   a) \( \beta_6 = 0.6 \)
   b) \( \beta_6 = 0.3 \)
   c) \( \beta_6 = 0.1 \)

Lower graph: Born Cross Sections,
   a) nucleon exchange \( (g_1^2 = 0.9, \beta_1 = 0.45) \)
   b) deuteron exchange \( (g_{d\alpha}^2 = 0.1, \beta_6 = 0.3) \)
deuteron exchange affects the back angle scattering considerably and in the right direction. In fact, for the above value of $g_{d d \alpha}$ and $\beta_6$ the fit to the experimental cross section is extremely good for $\theta \geq \pi/2$.

Now we will determine $R_0$, the additional D wave input. As the experimental data indicate a D wave resonance we first examine the effect of adding D wave resonances to our model, especially the effects on $P_{1/2}$ and $P_1$, and this is shown in Figures VI.10 through 13, where the solid curves give $P_{1/2}$ and the dashed curves $P_1$. Tombrello et al. 8) discarded the possibility of the resonance being in $^4D_{1/2}$ or $^4D_{3/2}$ and thus there are four possibilities to test, $^2D_{3/2}$, $^2D_{5/2}$, $^4D_{5/2}$, and $^4D_{7/2}$. To simulate a resonance, we added the appropriate $K$ matrix element with phase shift near 90°. In Figure VI.10 we see the effect of the doublet resonances, $^2D_{3/2}$ (upper graph) and $^2D_{5/2}$ (lower graph). In both the sign of $P_{1/2}$ is the opposite of that of $P_1$ and also there is no symmetry about $\pi/2$ and thus disagree completely with the data; for the quartet resonances ($^4D_{5/2}$, $^4D_{7/2}$), Figure VI.11, we do have both approximate equality of $P_{1/2}$ and $P_1$ and some symmetry about $\pi/2$. In Figure VI.12 we show the effect of adding to a fixed $^4D_{5/2}$ resonant input increments of $^4D_{7/2}$, upper graph has $\delta(^4D_{7/2}) \approx 15^\circ$, middle graph
Figure VI.10. Polarizations in dh-dh

\[ \frac{P_{1/2}}{P_1} \]

Upper graphs: \( ^2D_{3/2} \) resonance input
Lower graphs: \( ^2D_{5/2} \) resonance input
Figure VI.11. Polarizations in dh-dh

\[ P_{1/2} \]

\[ P_1 \]

Upper graphs: \( ^4P_{5/2} \) resonance input

Lower graphs: \( ^4P_{7/2} \) resonance input
Figure VI.12. Polarizations in dh-đh, with a $^4D_{5/2}$ resonance input

Upper graphs: $\delta(^4D_{7/2}) \approx 15^\circ$

Middle graphs: $\delta(^4D_{7/2}) \approx 30^\circ$

Lower graphs: $\delta(^4D_{7/2}) \approx 45^\circ$
\( \delta(4^D_{7/2}) \approx 30^\circ \), and lower graph \( \delta(4^D_{7/2}) \approx 45^\circ \). It is thus seen that the middle graph gives the best choice (as compared to experimental data) and this choice is what we call \( R_D \). In Figure VI.13 we further show that an increase of doublet amplitudes, \( 2^D_{3/2} \) (middle graph) and \( 2^D_{5/2} \) (lower graph) as small as \( \delta(2^D) \approx 15^\circ \) destroys the qualities of \( R_D \) (upper graph), i.e. similarity of \( P_{1/2} \) and \( P_1 \) and symmetry about \( \pi/2 \). Thus we can say that, at least within our model, the interference of the \( 4^D_{5/2} \) nearly resonant amplitude with the one nucleon exchange amplitude causes the observed polarization phenomena and cross section shape near \( \pi/2 \).

This additional input had little effect on the spin observables in dh-p\( \alpha \). In Figure VI.14 (solid curve = (-P), dashed curve A) we show the effect of adding D wave resonances to the elastic channel on dh-p\( \alpha \). The upper graph gives -P and A for no additional D wave input, and the middle and lower graphs show the effect of \( 2^D_{3/2} \) and \( 2^D_{5/2} \), respectively. The effect of the quartet resonances (or \( R_D \) as determined above) is not plotted because it was very slight, as was to be expected because in dh-p\( \alpha \) the doublet-quartet amplitudes are small, especially for \( J = 5/2^+ \), \( J = 7/2^+ \).
Figure VI.13. Polarizations in dh•dh, with a

\[ {}_4^D_{5/2} \text{ resonance plus } \delta({}_4^D_{7/2}) = \]

30° (\(\equiv R_D\)) input

Upper graphs: No additional \( {}_2^D \)

input

Middle graphs: \( R_D \text{ plus } \delta({}_2^D_{3/2}) \approx 15° \)

Lower graphs: \( R_D \text{ plus } \delta({}_2^D_{3/2}) \approx 15° \)
Figure VI.14. Polarizations in dh-pα

__________ (-F)

__________ A

Upper graphs: No D wave input in dh-dh
Middle graphs: $^2\text{D}_{3/2}$ elastic resonance input
Lower graphs: $^2\text{D}_{5/2}$ elastic resonance input
Finally, in Figures VI.15 and VI.16 we show our final results, together with the experimental data points, for dh-dh and dh-pα, respectively.

C. Summary, Conclusions

As a summary of our results, we note reasonable agreement with the experimental cross sections, and our results for the spin observables do exhibit qualitatively the properties seen in the experimental data, especially in dh-dh where $p_{1/2}$ and $P_1$ are very similar to their experimental curves though there is a fairly large discrepancy in the magnitude of $P_1$. The inelastic channels -P,A are related as indicated by the experiments although the agreement is not quite as good as for dh elastic. Of course, the above are the results with the introduction of $R_D$, which was not accounted for at all in our initial model. As a by-product, we were able to identify the resonant D wave amplitude, i.e. the $^4D_{5/2}$.

It is quite clear that our initial model, based solely on the one nucleon exchange amplitudes as calculated in section IV, does not reflect well the complexities of the problem, especially insofar as having very weak non-central forces (the only non-central interaction was the one present in the dpn vertex, but this vertex fitted to give up
Figure VI.15. Cross Section and Polarizations, dh-dh

Upper graph: ______ calculated cross section

0 0 experimental points

Lower graph: ______ $P_{1/2}$ (theory)

------ $P_1$ (theory)

$\# \# P_{1/2}$ (experiment)

0 0 $P_1$ (experiment)
Figure VI.16. Cross Sections and Polarizations,
\[ dh-p \]

Upper graph: \( \text{calculated cross section} \)
\[ 0 \quad 0 \quad \text{experimental points}^5 \]

Lower graph: \( (-P) \) (theory)
\[ \text{----} \quad A \quad \text{(theory)} \]
\[ \text{\textcircled{\textcolor{red}{\textbullet}}} \quad \text{\textcircled{\textcolor{red}{\textbullet}}} \quad (-P) \quad \text{(exp.)}^6 \]
\[ \text{\textcircled{\textcolor{red}{\textbullet}}} \quad \text{\textcircled{\textcolor{red}{\textbullet}}} \quad A \quad \text{(exp.)}^4 \]
scattering through a deuteron is fixed so as to give approximately 5% D-wave mixing). In addition, although it is evident that the one nucleon exchange amplitudes do have a primary importance in describing the system, we have seen that the deuteron exchange is not to be neglected to obtain a better agreement with experiment. Another shortcoming, although seemingly not crucial, is the lack of breakup amplitudes. As can be seen in Figures VI.8(a) and (b), one can simulate this effect by an artificial inelasticity and decrease the cross sections at forward and backward angles. It is not clear what causes the discrepancy in $\sigma_1$ for $30^\circ < \theta < 60^\circ$. The minimum there in the calculated cross section is due to the presence of a strong $dh$-$dh$ system (compare Figures VI.4(a) and 4(b)) and varying $g_1$ and $\beta_1$ near to their best values did not decrease its presence as can be seen in Figure VI.5.

There are many possible paths that can be taken to improve on the present discussion of the $p\alpha$, $dh$ system, and they fall in two general classes. Firstly we have to consider improvements in our input interactions and secondly improvements on the unitarization procedure. We will merely mention some of the more obvious improvements.
From our previous discussion, it seems clear that we must include non-S-wave couplings in our Hamiltonians. This could be done in the manner we generalized Amado's S wave dpn Hamiltonian, or else by introducing more complex vertex functions of the relative momentum. Either way, additional parameters will be introduced though probably one could determine them independently of the dh, pa system. Another possible improvement is the inclusion of two nucleon exchange diagrams. Apart from calculational difficulties, this should be straightforward. Still another path we can take\(^{34}\) is to use DWBA wavefunctions instead of plane waves to calculate the one nucleon exchange amplitudes.

The unitarization procedure could be improved by using the N/D method instead of the K matrix approach. This, to say the least, would not be straightforward since the analytic structure of these amplitudes is quite complex, and secondly it would require solving integral equations.
APPENDIX

In this Appendix we will exhibit the main features of the analytic structure of the partial wave amplitudes, both the contributions of the one particle exchanges and the branch points due to elastic unitarity in the direct channel.

We do this by calculating the pole terms due to the particle exchange diagrams in the nonrelativistic limit. The exchange diagrams the exchange with the $4\varpi$ momenta labels are

We begin with the exchange poles in $dh-p\alpha$ and then proceed to contribution from the triton pole to $p\alpha-p\alpha$.

Referring to the above diagrams and taking nonrelativistic limits we have

\[ P_1 = (E_1, \tilde{P}) \]
\[ P_2 = (E_2, -\tilde{P}) \]
\[ P_3 = (E_3, \tilde{P}') \]
\[ P_4 = (E_4, -\tilde{P}') \]
with \((\epsilon_i \) is the binding energy of particle \(i\))

\[
E_1 = \sqrt{p^2 + M_d^2} \approx 2m - \epsilon_d + p^2 / 4m
\]

\[
E_2 = \sqrt{p^2 + M_t^2} \approx 3m - \epsilon_t + p^2 / 6m
\]

\[
E_3 = \sqrt{p'^2 + M_\alpha^2} \approx m + p'^2 / 2m
\]

\[
E_4 = \sqrt{p'^2 + M_\alpha^2} \approx 4m - \epsilon_\alpha + p'^2 / 8m
\]

The above diagrams contribute to the full helicity amplitude \((t - m^2)^{-l}\) and \((u - M_d^2)^{-l}\) respectively, where \(t\) and \(u\) are the usual Mandelstam variables,

\[
t = (p_1 - p_3)^2
\]

\[
u = (p_1 - p_4)^2
\]

To find the branch points in the partial wave amplitudes we do not need to explicitly integrate the amplitudes, we can obtain their position simply by calculating the values of \(p^2 \) for which the pole integrand meets the integration contour; since this contour can be varied the position of the cut is arbitrary but not so the branch points which are given when the pole in the integrand lies at the contour end points. Thus, assuming that the other factors of the amplitude are not zero at the end points, we obtain the
branch points by finding the values of $p^2$ (or $p'^2$) for which $(t-m^2)_{x=\pm 1} = 0$ and $(u-M^2_d)_{x=\pm 1} = 0$, where $x = \cos \theta$.

We thus have (in NR limit)

$$t-m^2 = (p_1 - p_3)^2 - m^2 \simeq (2m - \epsilon_d)^2 + m^2 - 4m^2 - 2p^2 + 2m \epsilon_d - p^2 / 2 + 2p \cdot p' - m^2$$

$$\simeq - [2m \epsilon_d + p^2 / 2 + 2p'^2] + 2pp' \cos \theta$$

$$u-M^2_d = (p_1 - p_4)^2 - M^2_d \simeq -[4m(\epsilon_a - 2\epsilon_d) + 2p^2 + p'^2 / 2] - 2pp' \cos \theta$$

Now, since $E_1 + E_2 = E_3 + E_4$,

$$p'^2 = 2p^2 / 3 + 8m(\epsilon_a - \epsilon_h - \epsilon_d) / 5$$

and therefore

$$t-m^2 \simeq - [11p^2 / 6 + \frac{16}{5} m(\epsilon_a - \epsilon_h - 3\epsilon_d / 8)] + 2pp' \cos \theta$$

$$u-M^2_d \simeq - [7p^2 / 3 + \frac{4}{5} m(6\epsilon_a - \epsilon_h - 11\epsilon_d)] - 2pp' \cos \theta$$

which are, of course, the same results we obtained in section IV. It is now convenient to define the variable $\nu_1$, $\nu_1 = p'^2 / 2\mu$ and then the solutions of $t-m^2 = 0$ and $u-M^2_d = 0$ are, respectively, $\nu_{1\pm}^{(n)}$ and $\nu_{1\pm}^{(d)}$, where

$$\nu_{1-}^{(n)} \simeq -29.2 \text{ MeV} \quad \nu_{1-}^{(d)} \simeq -0.2 \text{ MeV}$$

$$\nu_{1+}^{(n)} \simeq -2.4 \text{ MeV} \quad \nu_{1+}^{(d)} \simeq -24.6 \text{ MeV}$$
These values of $\nu_{1}^{(n)}$ ($\nu_{1}^{(d)}$) give the approximate position of the branch points due to the exchange of a neutron (deuteron).

We also want to consider the singularities in the $p\alpha$ elastic partial wave amplitudes due to one particle exchange. The only possibility in our few nucleon model is a $t$ ($H^3$) exchange:

Here the singularities are caused by the factor $(t-M_e^2)^{-1}$ (it is clear that $t$ stands for both the kinematic variable and for the triton). We can easily calculate the position $\nu_{1\pm}^{(t)}$ of the branch points due to this term proceeding as before. We obtain,

$$\nu_{1-}^{(t)} \simeq -33.4 \text{ MeV} \quad \nu_{1+}^{(t)} \simeq -12 \text{ MeV}$$

In Figure A1 we exhibit the above branch cuts together with the ones due to S-channel unitarity. Note that these latter branch points are square root types while the particle exchange branch points are logarithmic.
Figure A.1. One Particle Exchange Branch Points

\[ \nu^\text{(n)}_{1-}, \nu^\text{(n)}_{1+} \quad \text{Nucleon Exchange} \]

\[ \nu^\text{(d)}_{1-}, \nu^\text{(d)}_{1+} \quad \text{Deuteron Exchange} \]

\[ \nu^\text{(t)}_{1-}, \nu^\text{(t)}_{1+} \quad \text{Triton Exchange} \]
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