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CONTENTS

CHAPTER I
1.1 Introduction
1.2 Outline of Thesis

CHAPTER II
2.1 Introduction and Notation
2.2 Problem Formulation
2.3 On Zeroing the Output of a Linear System
2.4 Solution of the Error System Problem
2.5 Concluding Remarks

CHAPTER III
3.1 Stabilizability of the Closed-Loop System
3.2 Output Feedback, Observers and Compensators

CHAPTER IV
4.1 Examples
4.2 Summary and Conclusions

REFERENCES
CHAPTER I

I.I INTRODUCTION

In 1960, Kalman [1] stated and solved the problem of driving the state of a linear dynamic system to zero, in an optimal manner. This problem, which came to be known as the linear regulator problem, has received a great deal of attention and in the past eleven years a voluminous literature has built up around it.

The regulator problem, however, is only a special case of the problem of making the outputs of a plant follow or track a desired trajectory, or command input. This problem, known as the servomechanism or tracking problem, is representative of most control situations. The regulator problem is simply the case where the desired state is zero.

Despite its fundamental nature only two previous attempts have been made toward the solution of the servomechanism problem. These attempts were by Kalman and Koepcke [2] in 1958 and by Kalman [3] in 1963. Both attempts, however, resulted in unrealizable control laws. In [4] a special class of tracking problems were demonstrated to be reducible to regulator problems.

The first satisfactory solution to the problem of optimal tracking was presented in 1969, [5]. The solution consisted of a time-varying controller which operating on available measurements only would make the outputs of a linear plant track a known class
of command signals optimally, in the sense of minimizing a finite-time quadratic performance index. Conceptually, this controller was merely a generalization of a controller proposed originally by Pearson [6] for the regulator problem.

While [5] completely solved the finite-time optimal tracking problem, the solution could not be extended to the infinite-time case in a straightforward manner. The difficulty involved was that a necessary condition for optimality in this case is that the steady-state tracking errors be zero, since otherwise the value of the performance index is infinite. The fundamental problem therefore, was that of determining conditions for the existence of a control law that would guarantee zero steady-state error for the class of command signals under consideration.

This problem was solved in [7] by introducing a dynamic system, called an "error system" whose state when driven to zero would result in zero steady-state error. A set of 5 matrix equations connected with the error system were derived in [7], and solved for the case of a single-output plant. This solution led to the design of a controller that would guarantee optimal tracking and stability of the plant for the infinite-time case.

In a recent book [8], the infinite-time servomechanism problem is treated at some length and conditions similar to those obtained in [7] for the tracking problem are obtained. However the question of stability is not considered and even though the problem is formulated as an optimal control problem, the resulting
system is not optimal since the control law is realized via state-estimators.

The algebraic technique of solution of the error system equations presented in [7] was however inadequate for handling the error systems for the general case of a multi-output plant tracking a desired class of signals and rejecting disturbances. These equations are derived in this thesis and necessary and sufficient conditions for the existence of stabilizable and controllable error systems are presented utilizing the geometric notions of a controllable subspace and a certain invariant subspace introduced in [9]. These conditions are therefore, the conditions for solvability of the multivariable tracking problem.

1.2 OUTLINE OF THESIS

In Chapter II, a problem formulation and solution assuming state-feedback is presented. In Chapter III a condition is derived that guarantees stability of the plant while tracking with zero steady-state error. The output-feedback problem is considered and it is shown how an observer or a compensator can be used for tracking. Conclusions and comments along with illustrative examples are presented in Chapter IV, and directions for future research are indicated.
2.1 **INTRODUCTION AND NOTATION**

The plant under consideration is described by

\[ \dot{x}(t) = A x(t) + B u(t) + D \tilde{y}(t) \]  
\[ y(t) = C x(t) + E \tilde{y}(t) \]  
\[ \xi'(t) = Q \xi(t) \]  
\[ t \geq 0 \quad x(0) = x_0, \quad \xi(0) = \xi_0 \]

and the class of command signals \( T(t) \) is generated by all possible initial conditions on the system

\[ \dot{z}(t) = M z(t) \]  
\[ r(t) = H z(t) \]  
\[ t \geq 0 \quad z(0) = z_0 \]

Here \( A, B, C, D, E, Q, M \) and \( H \) are linear time-invariant maps over the appropriate vector spaces, and are represented by real matrices of size \( n \times n, n \times r, m \times n, n \times s, m \times s, m \times q, q \times q \), and \( m \times q \) respectively. \( x, u \) and \( y \) are to be regarded respectively as elements of the \( n \)-dimensional state space, \( U \) the input space and \( Y \) the output space. \( \xi \in \mathcal{E}^5 \) is a disturbance, and \( z \in \mathcal{E}^q \) is the state of the input system (2.1.4-5).

If \( B \) is a linear map \( B \) or \( \mathcal{B} \) will denote the range of \( B \). If \( C \) is a linear map, \( \mathcal{N}(C) \) will denote the kernel (null-space) of \( C \). The symbol \( \{ A \mid \mathcal{B} \} \) is defined by \( \{ A \mid \mathcal{B} \} = \mathcal{B} + A \mathcal{B} + \cdots + A^n \mathcal{B} \) where \( A : \mathcal{E}^n \to \mathcal{E}^n \) and will be called the controllable space of the pair \( (A, B) \). The restriction of \( A \) to an \( A \)-invariant subspace \( \mathcal{V} \) is denoted by \( A \mid \mathcal{V} \). With (2.1.1-5) will be associated the system
\[ \dot{w} = \hat{A} w + \hat{B} u \]

where

\[
\begin{pmatrix}
\dot{x} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
A & D & 0 \\
0 & Q & 0 \\
0 & 0 & M
\end{pmatrix}
\begin{pmatrix}
x \\
\xi \\
z
\end{pmatrix}
\begin{pmatrix}
\dot{B} \\
0 \\
0
\end{pmatrix}
\]

Additional notation is introduced where appropriate.

2.2 PROBLEM FORMULATION

Let \( e(t) = y(t) - r(t) = \hat{C} w(t) \)

where \( \hat{C} = (c \ E \ -H) \)

and \( \eta(\hat{C}) = \hat{\eta} \).

Then the problem of interest is:

Under what conditions does there exist \( F = [F_1 \ F_2 \ F_3] \)

such that with \( u = F_1 x + F_2 \xi + F_3 z \), the following conditions hold

(a) If \( e(0) = 0 \), then \( e(t) = 0 \), for all \( t \).

(b) If \( e(0) \neq 0 \), then \( \lim_{t \to \infty} e(t) = 0 \)

for all \( x_0, \xi_0, z_0 \).

Condition (a) holds if and only if \((\hat{A} + \hat{B}F)\hat{\eta} = \hat{\eta}\).

In general there exists no such \( F \) and the largest subspace of \( \hat{\eta} \) which is \((\hat{A} + \hat{B}F)\)-invariant for some \( F \) is a proper subspace of \( \hat{\eta} \).

Therefore the following more general problem is formulated: When does there exist

\( T = [T_1 \ T_2 \ T_3] \) and \( F = [F_1 \ F_2 \ F_3] \) such that with \( u = F_1 x + F_2 \xi + F_3 z \) and \( \hat{e}(t) = T_1 x + T_2 \xi + T_3 z \) the following conditions are satisfied

(a') If \( \hat{e}(0) = 0 \), \( \hat{e}(t) = 0 \) for all \( t \)

(b') If \( \hat{e}(0) \neq 0 \), then \( \lim_{t \to \infty} \hat{e}(t) = 0 \) for all

\( x_0, \xi_0, z_0 \)

(c') \( \eta(T) \subset \hat{\eta} \)

The original problem (a), (b) is a special case of this problem,
which results when \( \eta(t) = \hat{\eta} \). Condition \((a')\) implies that \((\hat{A} + \hat{B}F) - \eta(t) < \eta(t)\), and therefore by \((c')\), \(\hat{e}(0) = 0\) guarantees that \(e(t) = 0\) for all \(t\). \((a')\) also implies that \(\hat{e}\) is the state of a dynamic system with \(\hat{e} = 0\) an equilibrium state, and \((b')\) implies that this dynamic system is asymptotically stable.

These considerations indicate the following approach to the problem. Let \(F_0 = \begin{bmatrix} F_0^1 & F_0^2 & F_0^3 \end{bmatrix}\) and set \(u = F_0^1 x + F_0^2 z + F_0^3 x + \nu\).

Let
\[
\hat{e}(t) = \hat{A}\hat{e}(t) + \hat{B}\nu \tag{2.2.1}
\]
\[
e(t) = \hat{C}\hat{e}(t) \tag{2.2.2}
\]

Substitution of (2.2.1) and (2.2.2) in (2.1.1-5) yields the following equations

\[
\begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix} \begin{pmatrix} A + BF_0^1 & BF_0^2 + D & BF_0^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{A} \begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix} \tag{2.2.3}
\]

\[
\begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix} \begin{pmatrix} B \\ 0 \\ 0 \end{pmatrix} = \hat{B} \tag{2.2.4}
\]

\[
\hat{C} \begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix} = \begin{pmatrix} C & E & -H \end{pmatrix} \tag{2.2.5}
\]

Note that (2.2.3-5) may be rewritten

\[
T(\hat{A} + \hat{BF}_0) = \hat{A}T \tag{2.2.6}
\]

\[
T \hat{B} = \hat{B} \tag{2.2.7}
\]

\[
\hat{C}T = \hat{C} \tag{2.2.8}
\]

Condition \((a')\) is equivalent to the existence of \(F_0, T, \) and \(\hat{A}\) satisfying (2.2.6). Existence of \(\hat{C}\) is equivalent, by Lemma 3.1 [9] to condition \((c')\). Condition \((b')\) can be satisfied if and only if there exists \(F\), such that \(\hat{A} + \hat{BF}\) is stable.

The system described by (2.2.1) and (2.2.2) will be called an error system and the original problem \((a'), (b'), (c')\) can now be restated as the error system problem.
Error System Problem: Given \( \{ \hat{A}, \hat{B}, \hat{C} \} \) determine conditions for the existence of \( \{ F_o, T, \tilde{A}, \tilde{B}, \tilde{C} \} \) satisfying (2.2.6-8) such that \( (\tilde{A}, \tilde{B}) \) is a stabilizable (controllable) pair.

Stabilizability of \( (\tilde{A}, \tilde{B}) \) guarantees zero steady-state error; controllability of \( (\tilde{A}, \tilde{B}) \) guarantees, by Wonham [10], that an arbitrary set of exponents may be assigned to the rate of decay of \( \tilde{e}(t) \), thus assuring a "good" transient response.

The error system problem can be regarded as a special case of the problem of driving the state of a linear system into an invariant subspace. When the invariant subspace is contained in the null-space of the output map, the problem is one of zeroing the output. This problem was formulated and solved in [11]; some results obtained therein are presented below as a preliminary step toward solution of the error system problem.

2.3 On Zeroing the Output of a Linear System

Let

\[
\dot{x} = Ax + Bu \quad x(0) = x_0 \tag{2.3.1}
\]

where \( A: \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( B: \mathbb{R} \rightarrow \mathbb{R}^n \) and \( U \subset \mathbb{R}^n \) are given. Define \( \mathcal{F}(U) \) to be the class of all linear maps \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) with the property that \((A + BF)V \subset V\). Assume \( \mathcal{F}(U) \) is not empty. Then the problem to be considered is: Determine conditions for the existence of \( F \), such that with \( u = Fx \), the following is true

1) If \( x_0 \in U \), then \( x(t) \in U \) for all \( t \)

2) If \( x_0 \not\in U \), then \( \text{dist} (x(t), U) = \min \{ |x - x| : x \in U \} \rightarrow 0 \) as \( t \rightarrow \infty \)
These conditions are determined as follows. Let $F_o \in \mathcal{F}(\mathcal{U})$ and let $A_o = A + BF_o$. Let $\tilde{\mathcal{E}} = \mathcal{E}^n / \nu$ denote the factor space $\mathcal{E}^n \mod \nu$, $\tilde{x} \in \tilde{\mathcal{E}}$ the coset of $x \in \mathcal{E}^n$, $\tilde{\mathcal{E}} \to \mathcal{E}^n / \nu$ the canonical projection, and $A_o$, the map induced in $\tilde{\mathcal{E}}$, by $A_o$. Therefore $\tilde{x} = P \bar{x}$ and $P A_o = A_o P$. 1) is satisfied with $u = (F_o + F_i P) \bar{x}$ where $F_i : \tilde{\mathcal{E}} \to \mathcal{U}$ is arbitrary.

2) is satisfied if $\bar{x}(t) \to \bar{0}$, $t \to \infty$ for all motions of the system

$$\dot{\tilde{x}} = (A_o + PB F_i) \tilde{x} \quad (2.3.2)$$

Let $P B = \bar{B}$. Then 1) and 2) are satisfied if and only if the pair $(A_o, \bar{B})$ is stabilizable.

The following notation is introduced for convenience. Let $\alpha(\gamma)$ denote the minimal polynomial of $A$, and $\alpha = \alpha^+ \alpha^-$ a factorization of $\alpha$ with the zeros of $\alpha^+(\alpha^-)$ in the closed right-half (open left-half) complex plane. Define

$$\mathcal{E}^+ = \{ x \mid x \in \mathcal{E}^n, \alpha^+(A) x = 0 \}$$

and $\mathcal{R} = \{ [A] \in \mathcal{B} \}$. Then the main result of this section is

**Proposition 2.3**

The pair $(A_o, \bar{B})$ is

a) stabilizable, if and only if
$$\mathcal{E}^+ \subset \mathcal{R} + \nu \quad (2.3.3)$$

b) controllable, if and only if
$$\mathcal{E}^n = \mathcal{R} + \nu \quad (2.3.4)$$

The proposition will follow from a slightly more general result to be proved in the following theorem. For this let $\mathbb{C}$ denote the complex
plane, $C_1$, $C_2$, a partition of $C$ into symmetric subsets, i.e. $\lambda \in C_i$ implies $\lambda^* \in C_i$ ($i = 1, 2$), such that $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = \emptyset$. Let $\alpha = \alpha_1 \alpha_2$ be a factorization of the m.p. of $A$ such that the zeros of $\alpha_i$ belong to $C_i$ ($i = 1, 2$). If $\alpha$ has no zeros in $C_i$, set $\alpha_i = 1$ ($i = (1, 2)$). Let $E_i = E_i(A) = \text{ker}(\alpha)$ and let $\sigma(A)$ denote the complex spectrum of $A$ with complex eigenvalues occurring in conjugate pairs.

Theorem 2.3

With $\alpha_i$, $E_i$, $C_i$ ($i = 1, 2$) defined as above, there exists a map $F: E^n \rightarrow \mathcal{U}$, $F \in \mathcal{F}(U)$, such that
\[
\sigma(A + BF) \subset C_i
\]
if and only if
\[
E_2(A) \subset \{A \mid \sigma \} + U
\]
Clearly, the proposition follows by setting $\alpha_1 = \alpha^-$, $\alpha_2 = \alpha^+$, $E_1 = E^-$, $E_2 = E^+$ and $C_1 = C^-$, $C_2 = C^+$ where $C^+$ is the closed right-half and $C^-$ the open left-half of the complex plane.

Proof of Theorem

Lemma 2.3.1 There exists a map $F: E^n \rightarrow \mathcal{U}$ such that $\sigma(A + BF) \subset C_i$ if and only if $E_2(A) \subset \{A \mid \sigma \}$

Proof:-- A proof for the stability case was given in Wonham [10] and the general case is treated in exactly the same way.

Before stating Lemmas (2.3.2) and (2.3.3) the following assumptions and notation are introduced.

(C1): $A: E \rightarrow E$ has m.p. $\alpha(\lambda)$ and $\alpha_i(\lambda)$, $E_i$ ($i = 1, 2$) are defined as before.

(C2): $U \subset E$ and $AV \subset U$

(C3) $P: E \rightarrow E/V$ is the canonical projection,
\[ \bar{E} = E / \nu; \quad \hat{E}_i = (E_i + \nu) / \nu = P E_i \quad (i = 1, 2). \]
\[ \bar{A} \] is the map induced in \( \bar{E} \) by \( A \) and \( \bar{z}(A) \) is the m.p. of \( \bar{A} \).

Lemma 2.3.2 Assume that conditions \( C_1 - C_3 \) are satisfied. Then
\[ \bar{A} \hat{E}_i \subseteq \hat{E}_i \quad i = 1, 2 \]
If \( \bar{z} = \bar{z}_1, \bar{z}_2 \), where \( \bar{z}_i \mid \chi_i \) \( (i = 1, 2) \) then \( \bar{z}_i \) is the m.p. of \( \bar{A} \mid \hat{E}_i \).
Furthermore
\[ \hat{E}_i = \bar{E}_i = \text{Ker} \bar{z}_i(\bar{A}) \quad (i = 1, 2) \]

Proof
\[ \bar{A} \hat{E}_i = \bar{A} P E_i = P \bar{A} E_i \subseteq P E_i = \hat{E}_i \quad (i = 1, 2). \]
Let \( \hat{z}_i \) be the m.p. of \( \bar{A} \mid \hat{E}_i \) \( (i = 1, 2) \). Clearly \( \hat{z}_i \mid \chi_i \)
\( (i = 1, 2) \). Since \( \chi_1, \chi_2 \) are coprime, so are \( \hat{z}_1, \hat{z}_2 \);
hence the m.p. of \( \bar{A} \) is \( \hat{z}_1, \hat{z}_2 \). Thus
\[ \hat{z}_1 \hat{z}_2 = \bar{z} = \bar{z}_1, \bar{z}_2 \]
and since \( \bar{z}_i \mid \chi_i \) there results \( \hat{z}_i = \bar{z}_i \).

Next
\[ \hat{E}_1 + \hat{E}_2 = \bar{E} = \text{Ker} \bar{z}(\bar{A}) = \text{Ker} \bar{z}_1(\bar{A}) \oplus \text{Ker} \bar{z}_2(\bar{A}) \]
Since \( \hat{E}_i \subseteq \text{Ker} \bar{z}_i(\bar{A}) \), the result follows.

Lemma 2.3.3

Assume that \( C_1 - C_3 \) are satisfied. There exists a map \( \bar{F}: \bar{E} \rightarrow U \)
such that
\[ \sigma(\bar{A} + \bar{B}G) \subseteq C_1 \]
(2.3.5)
if and only if
\[ \bar{E}_2 (A) \subseteq \{ A \mid \beta \} + U \]
(2.3.6)
Proof: By the previous two lemmas there exists a map $F$ such that (2.3.5) is true if and only if 
$$\bar{e}_2(A) \subseteq \{\bar{A} \mid \bar{B}\} \quad (2.3.7)$$
Now, (2.3.6) implies
$$\bar{e}_2(A) = P\bar{e}_2(A) \subseteq P\{A \mid \bar{B}\} = \{\bar{A} \mid \bar{B}\}$$
Conversely (2.3.7) yields
$$P\bar{e}_2(A) \subseteq \{\bar{A} \mid \bar{B}\} = P\{A \mid \bar{B}\}$$
so that
$$\bar{e}_2(A) \subseteq \{A \mid \bar{B}\} + \langle V \rangle$$

Lemma 2.3.4

Let $\mathcal{S} \subseteq \bar{e}_2$ have the property that
$$\bar{e}_2(A) + A\mathcal{S} + \bar{B} \subseteq \mathcal{S} \quad (2.3.8)$$
Then for every map $F : \bar{e}_2 \rightarrow \mathcal{U}$, $\bar{e}_2(A + BF) \subseteq \mathcal{S}$ (2.3.9)

Proof

By (2.3.8),
$$(A + BF)\mathcal{S} \subseteq A\mathcal{S} + \bar{B} \subseteq \mathcal{S}$$
for all $F$.

Let $Q : \bar{e}_2 \rightarrow \bar{e}_2 / \mathcal{S}$ be the canonical projection and let $\overline{A}$ be the map induced in $\bar{e}_2 / \mathcal{S}$ by $A$; $\overline{A}$ is uniquely determined by the relation $\overline{A}Q = QA$.

Thus from
$$(\overline{A} + BF)Q = Q(A + BF) = QA$$
it follows that
$$(A + BF) = \overline{A}$$. By Lemma (2.3.2) \hfill \bar{e}_2(A + BF) = \bar{e}_2(\overline{A}) = Q \bar{e}_2(A) = \overline{\mathcal{S}}.$$
Therefore $\sigma(A + BF) \subseteq \mathcal{C}_1$. Since $\sigma(A + BF) = \sigma[(A + BF) \mid \mathcal{S}] \cup \sigma(\overline{A} + BF)$, there results
$$\sigma(A + BF) \cap \mathcal{C}_2 \subseteq \sigma[(A + BF) \mid \mathcal{S}]$$
and therefore
$$\bar{e}_2(A + BF) \subseteq \mathcal{S}$$
Proof of Theorem 2.3

Let \( \mathcal{L} = \{ A_{11} + \mathcal{V} \} \) and assume \( \mathcal{E}(A) \subseteq \mathcal{L} \). By Lemma (2.3.4) \( \mathcal{E}(A + BF) \subseteq \{ A_{11} + \mathcal{V} \} = \{ A + BF_{11} + \mathcal{V} \} \) for all \( F : \mathcal{E} \rightarrow \mathcal{U} \). Choosing \( F_0 \in \mathcal{F}(\mathcal{V}) \) it follows by Lemma (2.3.3) that for some \( \overline{F_1} : \overline{\mathcal{E}} \rightarrow \mathcal{U} \)

\[
\sigma(A + BF_0 + BF_1) \subseteq C.
\]

(2.3.10)

If \( F_1 = \overline{F_1} \) then \( F = F_0 + F_1 \in \mathcal{F}(\mathcal{V}) \) and (2.3.10) implies

\[
\sigma(A + BF) \subseteq C.
\]

(2.3.11)

Conversely, if there exists \( F \in \mathcal{F}(\mathcal{V}) \) such that (2.3.11) is true, then by Lemma (2.3.3)

\[
\mathcal{E}(A + BF) \subseteq \{ A + BF_{11} + \mathcal{V} \} = \{ A_{11} + \mathcal{V} \}
\]

By Lemma (2.3.4) applied to \( A + BF \) and with \( \mathcal{L} \) as before,

\[
\mathcal{E}(A) = \mathcal{E}(A_{11} + BF) \subseteq \{ A_{11} + \mathcal{V} \}
\]

as asserted before. The proof is complete.

2.4 Solution of the Error System Problem

The results of the previous section will be used to state conditions for the existence of stabilizable and controllable error systems. As before \( \hat{A}, \hat{B}, \hat{C} \) are defined by

\[
\hat{A} = \begin{pmatrix} A & D & 0 \\ 0 & \hat{A} & 0 \\ 0 & 0 & M \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} C & E & -H \end{pmatrix}
\]

and \( \hat{n} = \eta(\hat{C}) \). Let \( \hat{\mathcal{V}}_m \) denote the maximal subspace of \( \hat{n} \) that is invariant under \( \hat{A} + BF \) for some \( F \). The existence, uniqueness and a computational algorithm for determination of \( \hat{\mathcal{V}}_m \) was established in
Wonham and Morse 1970 [9]. Let \( \alpha \) be the m.p. of \( \hat{A} \) and let 
\[ \alpha = \alpha^+ \alpha^- \]
be a factorization of \( \alpha \) with the zeros of \( \alpha^+ (\alpha^-) \) in 
the closed right-half (open left-half) complex plane. Let 
\[ \mathcal{E} = \mathcal{E}^n \oplus \mathcal{E}^s \oplus \mathcal{E}^b \]
denote the state space of the system (2.1.6).

Define 
\[ \mathcal{E}^\pm = \ker \alpha^\pm (\hat{A}) = \{ \omega | \omega \in \mathcal{E}, \alpha^\pm (\hat{A}) \omega = 0 \} \]

Then, the solution to the error system problem is the following theorem.

**Theorem 2.4** Given \( \{\hat{A}, \hat{B}, \hat{C}\} \) there exists \( \{ F_0, T, \tilde{A}, \tilde{B}, \tilde{C}\} \)
satisfying (2.2.6-8) such that the pair \( (\tilde{A}, \tilde{B}) \) is

a) **stabilizable**, if and only if
\[ \mathcal{E}^+ \subset \{ \hat{A} | \hat{B} \} + \mathcal{U}_M \]  \hspace{1cm} (2.4.1)

b) **controllable** if and only if
\[ \mathcal{E} = \{ \hat{A} | \hat{B} \} + \mathcal{U}_M \]  \hspace{1cm} (2.4.2)

**Proof:**

**Sufficiency** Let \( F_0 \) be chosen such that \( (\hat{A} + BF_0) \mathcal{V}_M \subset \mathcal{V}_M \).
Such an \( F_0 \) exists, since by definition of \( \mathcal{V}_M, \mathcal{F}_M(\mathcal{V}_M) \) is not empty. Let 
\( p : \mathcal{E} \rightarrow \mathcal{E}/\mathcal{V}_M \) denote the canonical projection, 
\( \tilde{A} : \mathcal{E}/\mathcal{V}_M \rightarrow \mathcal{E}/\mathcal{V}_M \)
the map induced in \( \mathcal{E}/\mathcal{V}_M \) by \( \hat{A} + BF_0 \) and let

\[ \mathcal{P}: \mathcal{E} \rightarrow \mathcal{E}/\mathcal{V}_M \]  \hspace{1cm} (2.4.3)

Also,
\[ p (\hat{A} + BF_0) = \tilde{A} p \]  \hspace{1cm} (2.4.4)

and since \( \eta(p) = \mathcal{V}_M \subset \mathcal{N}_\mathcal{L} \), there exists \( \tilde{C} \) such that

\[ \mathcal{C}\mathcal{P} = \tilde{C} \]  \hspace{1cm} (2.4.5)

By Proposition 2.3 and (2.4.1(2)) it follows that \( (\tilde{A}, \tilde{B}) \) is
stabilizable (controllable). Let \( (\tilde{A}, \tilde{B}, T) \) be matrix representations
of \( (\tilde{A}, \tilde{B}, P) \). Then from the fact that equations (2.4.3), (2.4.4) and
(2.4.5) are identical to (2.2.7), (2.2.6) and (2.2.8), it follows that the
corresponding error system is stabilizable (controllable).

**Necessity** Assume there exist \( \{ F_0, T, \tilde{A}, \tilde{B}, \tilde{C}\} \) satisfying (2.2.6-8)
such that \( (\tilde{A}, \tilde{B}) \) is a stabilizable (controllable) pair. Let \( \eta(T) = \mathcal{V} \)
and \( \mathbf{u} \mathbf{v} = \left[ v_1, \ldots, v_n \right] \) be a matrix representation of \( \mathbf{v} \) with \( v_i, i = 1, \ldots, n \) forming a basis for \( \mathbf{v} \). Let \( \mathbf{R} = [r_1, \ldots, r_n] \) be chosen so that the columns of \( \mathbf{S} = [\mathbf{v} \ \mathbf{R}] \) form a basis for \( \mathbb{E} \).

From (2.2.6) it is clear that \( (\mathbf{A} + \mathbf{B} \mathbf{F}_0) \mathbf{v} \subset \mathbf{v} \). Let \( \mathbf{E}/\mathbf{v} \in \overline{\mathbb{E}} \) and let \( \overline{x} \in \overline{\mathbb{E}} \) denote the coset of \( x \in \mathbb{E} \). The cosets \( [\overline{r}_1, \ldots, \overline{r}_n] \) then form a basis for \( \overline{\mathbb{E}} \). In these bases the canonical projection \( \mathbf{P} : \mathbb{E} \rightarrow \overline{\mathbb{E}} \) has the matrix representation \( [a_{xy}, \mathbf{I}_{x,y}] \). In the new basis, the maps \( \mathbf{A} + \mathbf{B} \mathbf{F}_0 \) and \( \mathbf{B} \) have the representations

\[
\mathbf{S}^t (\mathbf{A} + \mathbf{B} \mathbf{F}_0) \mathbf{S} = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}
\]

\[
\mathbf{S}^{-1} \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}
\]

where \( A_1 \) is the matrix of \( \mathbf{A} + \mathbf{B} \mathbf{F}_0 \rceil \mathbf{v} \), \( A_2 \) is the matrix representation of the map \( \overline{\mathbf{A}} : \mathbf{E}/\mathbf{v} \rightarrow \mathbf{E}/\mathbf{v} \), induced in \( \mathbf{E}/\mathbf{v} \) by \( \mathbf{A} + \mathbf{B} \mathbf{F}_0 \) and \( B_2 \) is the matrix of \( \mathbf{P} \mathbf{B} = \overline{\mathbf{B}} \). Multiplying the above equation from the left by \( \mathbf{T} \mathbf{S} \) and using (2.2.6-7) yields

\[
\mathbf{\tilde{A}} = \mathbf{T} \mathbf{S} \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}
\]

\[
\mathbf{\tilde{B}} = \mathbf{T} \mathbf{S} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}
\]

But \( \mathbf{T} \mathbf{S} = \begin{pmatrix} \mathbf{TV} & \mathbf{TR} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{TR} \end{pmatrix} \). Then by definition of \( \mathbf{R} \)

\( \mathbf{L} = \mathbf{TR} \) has full column rank and consists of \( \mathbf{v} \) independent columns.

Now

\[
\mathbf{\tilde{A}} \mathbf{L} = \mathbf{L} A_2 \quad (2.4.6)
\]

\[
\mathbf{\tilde{B}} = \mathbf{L} B_2 \quad (2.4.7)
\]
Lemma 2.4.1

If \( L \) has full column rank, stabilizability of \((\tilde{A}, \tilde{B})\) implies stabilizability of \((A_2, B_2)\). Also controllability of \((\tilde{A}, \tilde{B})\) implies controllability of \((A_2, B_2)\).

Proof:

Let \( P_{A_2}, P_{A_2}^+ \) denote the m.p.'s of \( \tilde{A}, A_2 \) and let \( P_{A_2}^- = P_{A_2}^+ P_{A_2}^- \) and \( P_{A_2} = P_{A_2}^+ P_{A_2}^- \) be a factorization with the zeros of \( P_{A_2}^+ P_{A_2}^- \) in the closed right-half (open left-half) complex plane.

First, it will be shown that \( P_{A_2}^+ \left| P_{A_2}^+ \right. \). From (2.4.6) it follows that

\[
P_{A_2}^+ (\tilde{A}) L = L P_{A_2}^+ (A_2) = 0
\]

and therefore \( P_{A_2}^+ (A_2) = 0 \) since \( \text{Ker} \ L = 0 \). Therefore \( P_{A_2}^+ \) is an annihilating polynomial for \( A_2 \) and it follows that

\[
P_{A_2}^+ = P_{A_2}^+ / P_{A_2}^+.
\]

Now let \( A_1 : \tilde{E} \rightarrow \tilde{E} \) and \( A_2 : E_2 \rightarrow E_2 \) define the vector spaces over which \( \tilde{A}, A_2 \) are defined and let

\[
E_{A_2}^+ = \{ \chi \mid \chi \in \tilde{E}, \quad P_{A_2}^+ (\tilde{A}) \chi = 0 \}.
\]

\[
E_{A_2}^+ = \{ \gamma \mid \gamma \in E_2, \quad P_{A_2}^+ (A_2) \gamma = 0 \}.
\]

Assume \((\tilde{A}, \tilde{B})\) is stabilizable. Then

\[
E_{A_2}^+ \subset \{ \tilde{A} \mid \tilde{B} \}
\]

(2.4.8)

Now, \( \chi \in E_{A_2}^+ \) implies that \( P_{A_2}^+ (A_2) \chi = 0 \) and therefore

\[
L P_{A_2}^+ (A_2) \chi = 0.
\]

By (2.4.6), it follows that \( P_{A_2}^+ (\tilde{A}) L \chi = 0 \), and hence \( P_{A_2}^+ (\tilde{A}) L \chi = 0 \). Therefore \( L \chi \in E_{A_2}^+ \) and by the assumption (2.4.8) \( L \chi \in \{ \tilde{A} \mid \tilde{B} \} \). Since \( \{ \tilde{A} \mid \tilde{B} \} = \{ A_2 \mid B_2 \} \) and \( \text{Ker} \ L = 0 \), it follows that \( \chi \in \{ A_2 \mid B_2 \} \). Therefore \( E_{A_2}^+ \subset \{ A_2 \mid B_2 \} \) and the result follows. For controllability, note that \( d(\tilde{E}) \geq d(E_2) \) and therefore controllability of \((\tilde{A}, \tilde{B})\) implies controllability of \((A_2, B_2)\). □
To continue with the proof of necessity, assume now the \( (\hat{A}, \hat{B}) \) is stabilizable (controllable). By the foregoing Lemma then \( (A_2, B_2) \) is stabilizable (controllable). By Proposition 2.3,

a) stabilizability of \( (A_2, B_2) \) implies
\[
E^+ = \{\hat{A} | \hat{B} \} + \nu
\]

b) controllability of \( (A_2, B_2) \) implies
\[
E = \{\hat{A} | \hat{B} \} + \nu
\]
But equation (2.2.8) implies that \( \eta(T) = \nu \subset \hat{\nu} \) and since \( (\hat{A} + BF_0)\nu \subset \nu \) it follows that \( \nu \subset \hat{\nu}_M \). Therefore if \( (\hat{A}, \hat{B}) \) is stabilizable
\[
E^+ = \{\hat{A} | \hat{B} \} + \nu \]
and if \( (\hat{A}, \hat{B}) \) is controllable
\[
E = \{\hat{A} | \hat{B} \} + \hat{\nu}_M
\]

2.5 CONCLUDING REMARKS

Theorem 2.4 establishes conditions for the existence of stabilizable and controllable error systems. Feedback of the form \( \nu = F_1 \hat{e} \) then guarantees that \( \lim_{t \to \infty} \hat{e}(t) = 0 \) thereby solving the problem formulated in 2.2. The resulting control law is of the form
\[
u = (F_0^1 + F_1 T_1)x + (F_0^2 + F_2 T_2)\xi + (F_0^3 + F_1 T_3)\zeta
\]
(2.5.1)

and will guarantee that the steady state tracking errors in the presence of the disturbances \( \xi(\theta) \), will be zero. The feedback \( F_1 \) may be chosen to either place the eigenvalues of the error system at desirable locations, or to minimize an appropriate performance index.

The preceding theory has two main drawbacks:

a) even though the conditions of the theorem guarantee a stable error system, the resulting closed-loop system is not necessar-
ily stable, and

b) a control law of the form (2.5.1) is not realizable since the state vectors of the plant, disturbance and input systems will not be accessible for measurement.

These difficulties are discussed in Chapter III, and solutions are presented.
CHAPTER III

3.1 STABILIZABILITY OF THE CLOSED-LOOP SYSTEM

According to Theorem 2.4 of the previous chapter a servo-
mechanism can be designed for a given triple \((\hat{A}, \hat{B}, \hat{C})\), by setting
\[ u = F_1 x + F_2 \hat{z} + F_3 z = F w, \]
if there exists a subspace \( \hat{U} \subset \hat{N} \)
satisfying either \( \mathcal{E}^+ \subset \{ \hat{A} \mid \beta \} + \hat{U} \) or \( \mathcal{E} = \{ \hat{A} \mid \beta \} + \hat{U} \)
and \((\hat{A} + \hat{B} F) \hat{U} \subset \hat{U}\). In the resulting design the unstable modes of
\( \hat{A} + \hat{B} F \) are unobservable from the output \( \hat{C} \). In the absence of inputs
the response of the system is determined by the eigenvalues of \( A + BF_i \).
If some of the unstable unobservable modes of \( \hat{A} + \hat{B} F \), belong to \( A + BF_i \),
the resulting closed-loop system will be unstable. This is a conse-
quence of the fact that some of the unstable eigenvalues of \( \hat{A} + \hat{B} F \) that
are fixed for all \( F \in \mathcal{F}(\hat{U}) \) are eigenvalues of \( A + BF_i \). To avoid
this difficulty it is necessary to construct a subspace \( \mathcal{U}^* \) satisfying
either \( \mathcal{E}^+ \subset \{ \hat{A} \mid \beta \} + \mathcal{U}^* \) or \( \mathcal{E} = \{ \hat{A} \mid \beta \} + \mathcal{U}^* \) with the prop-
erty that the eigenvalues of \( A + BF_i \) that are fixed for all \( F \in \mathcal{F}(\mathcal{U}^*) \)
be stable. This is the problem that will be solved below.

The main theorem on stability will be approached via three lemmas.
The first two of these are stated and proved in [12]. Before stating
these, the concept of a controllability subspace and some of its associ-
ated properties are needed. These definitions and properties were
stated and proved in [9]. A subspace \( \mathcal{R} \subset \mathcal{E} \) is called a controllabil-
ity subspace (c.s.) of the pair \((A, B)\) if \( \mathcal{F}(\mathcal{R}) \neq \phi \) and there exists
\( F \) such that
\[
\mathcal{R} = \left\{ A + BF \mid B \in \mathcal{R} \right\}
\]  
(3.1.1)
\( \mathcal{R} \) is uniquely determined by the requirement that \( F \in \mathcal{F}(\mathcal{R}) \)
regardless of the particular \( F \in \mathcal{F}(\mathcal{R}) \) that appears in (3.1.1).
If $\mathcal{U}_m$ is the maximal subspace contained in a given subspace $\mathcal{U}$ that is $A + BF$ invariant for some $F$, then the (unique) maximal controllability subspace contained in $\mathcal{U}$ is given by

$$\mathcal{R}_m = \left\{ A + BF \mid \mathcal{G} \cap \mathcal{U}_m \right\} F \in \mathcal{F}(\mathcal{U}_m)$$ (3.1.2)

It is now possible to state Lemmas 3.1 and 3.2 as they appear in [12].

**Lemma 3.1**

Let $\hat{\mathcal{R}}_m \subset \hat{\mathcal{U}}_m$ be the maximal controllability subspace of $(\hat{A}, \hat{B})$ contained in $\hat{\mathcal{U}}$. Let $A_F \equiv \hat{A} + BF$, $F \in \mathcal{F}(\hat{\mathcal{U}}_m)$ and define

$$\overline{A}_F : \frac{\hat{\mathcal{V}}_m}{\hat{\mathcal{R}}_m} \to \frac{\hat{\mathcal{V}}_m}{\hat{\mathcal{R}}_m}$$ as follows: if $\overline{x}$ is the coset of $x$ in $\frac{\hat{\mathcal{V}}_m}{\hat{\mathcal{R}}_m}$ then

$$\overline{A}_F \overline{x} = \overline{A_F x}.$$ Then $\hat{\mathcal{R}}_m$ and $\overline{A}_F$ are fixed with respect to $F \in \mathcal{F}(\hat{\mathcal{U}}_m)$. The characteristic polynomial (ch.p.) of $A_F | \hat{\mathcal{R}}_m$ has the form $\Pi(\alpha) \Pi_F(\alpha)$ where $\Pi$ is the ch.p. of $\overline{A}_F$ and is fixed for all $F \in \mathcal{F}(\hat{\mathcal{U}}_m)$. $\Pi_F$ is the ch.p. of $A_F | \hat{\mathcal{R}}_m$ and the roots of $\Pi_F$ can be assigned arbitrarily by suitable choice of $F \in \mathcal{F}(\hat{\mathcal{U}}_m)$. \hfill \Box

**Lemma 3.2**

Under the conditions of Lemma 3.1, let $\alpha(\alpha)$ be the m.p. of $\overline{A}_F$, and factor $\alpha(\alpha) = \alpha_g(\alpha)\alpha_b(\alpha)$ where the polynomials $\alpha_g, \alpha_b$ are coprime. Then

$$\hat{\mathcal{V}}_m = \hat{\mathcal{R}}_m \oplus \mathcal{R}_g \oplus \mathcal{R}_b$$ (3.1.3)

where

$$\hat{\mathcal{V}}_g = \hat{\mathcal{R}}_m \oplus \mathcal{R}_g = \left\{ x \mid x \in \hat{\mathcal{V}}_m , \alpha_g(\overline{A}_F) \overline{x} = \delta \right\}$$ (3.1.4)

and similarly for $\hat{\mathcal{V}}_b = \hat{\mathcal{R}}_m \oplus \mathcal{R}_b$. The subspaces $\hat{\mathcal{V}}_g$ and $\hat{\mathcal{V}}_b$ are fixed with respect to $F \in \mathcal{F}(\hat{\mathcal{U}}_m)$. \hfill \Box
Before stating Lemma 3.3 the following notation is introduced.

In the map \( \hat{A} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & M \end{pmatrix} \)

de note the sub-map \( \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \)

by \( N \)

and let \( \alpha_N \) be its minimal polynomial. Let the m.p. of \( \bar{A}_F \) be \( \alpha_{\bar{A}_F} \)
and \( \sigma(N) \) the spectrum of \( N \), \( C^+(C^-) \) the closed right half (open left half) of the complex plane.

Lemma 3.3

If

\[
\mathcal{E}^+(\hat{A}) \subset \{ \hat{A} | \mathcal{B} \} + \hat{\mu}_m
\]

(3.1.5)

then

\[
\sigma(N) \cap C^+ \subset \sigma(\bar{A}_F)
\]

(3.1.6)

Similarly, if

\[
\mathcal{E} = \{ \hat{A} | \mathcal{B} \} + \hat{\mu}_m
\]

(3.1.7)

then

\[
\sigma(N) \subset \sigma(\bar{A}_F)
\]

(3.1.8)

Proof:

Write \( \mathcal{E} = \{ \hat{A} | \mathcal{B} \} + \hat{\mu}_m \) and \( \hat{A} + BF = A_F \).

Then \( A_F \mathcal{E} \subset \mathcal{E} \)

for all \( F \). Let \( F \in \mathcal{E}(\hat{\mu}_m) \) and let \( \bar{A}_F \) denote the map induced in \( \mathcal{E}/\mathcal{E} \)

by \( A_F \). \( \bar{A}_F \) is defined as follows: if \( \bar{z} \in \mathcal{E}/\mathcal{E} \) denotes the coset of \( z \in \mathcal{E} \) then \( \bar{A}_F \bar{z} = \bar{A}_F z \). \( \bar{A}_F \) is defined as before.

Then

\[
\sigma(A_F) = \sigma(A_F | \mathcal{E}) \cup \sigma(\bar{A}_F)
\]

\[
= \sigma(A_F | \hat{\mu}_m) \cup \sigma(\bar{A}_F) \cup \sigma(\bar{A}_F) \cup \sigma(\bar{A}_F)
\]

By (3.1.5) and Lemma 2.3.4 \( \mathcal{E}^+(A_F) \subset \mathcal{E} \) for all \( F \). Hence
\[ \sigma(\overline{A}_F) \cap \mathcal{C}^+ = \phi. \] Therefore \( \sigma(A_F) \cap \mathcal{C}^+ \subseteq \sigma(A_F/\mathcal{V}) \)
and in particular \( \sigma(N) \cap \mathcal{C}^+ \subseteq \sigma(A_F/\mathcal{V}) \). By Lemma 3.1,
\( \sigma(A_F/\hat{\mathcal{V}}_m) \) is arbitrary and by Prop. 2.3 with \( \bar{e}_2(A), \mathcal{V}, \mathcal{U} \)
replaced by \( \beta, \mathcal{C}, \hat{\mathcal{V}}_m \) it follows that \( \sigma(\overline{A}_F) \) is arbitrary.
Hence \( \sigma(N) \cap \mathcal{C}^+ \subseteq \sigma(\overline{A}_F) \).

If \( \mathcal{E} = \mathcal{V} \) it follows that \( \text{dim.} \mathcal{E}/\mathcal{V} = 0 \) and hence \( \sigma(N) \subseteq \sigma(A_F/\mathcal{V}) \).

By an argument similar to the one above, it follows that \( \sigma(N) \subseteq \sigma(\overline{A}_F) \).

If \( \Pi_{\overline{A}_F}, \Pi_N = \Pi_N^+ \Pi_N^- \) denote respectively the characteristic polynomials of \( \overline{A}_F \) and \( N \) with the zeroes of \( \Pi_N^+ (\Pi_N^-) \)
in \( \mathcal{C}^+ (\mathcal{C}^-) \), then by Lemma 3.3 \( \Pi_{\overline{A}_F} = \Pi_N^+ \sigma^- \) or \( \Pi_{\overline{A}_F} = \Pi_N^- \sigma^+ \)
if (3.1.5) or (3.1.7) hold. In either case factor \( \sigma = \sigma^+ \sigma^- \)
where the roots of \( \sigma^+ (\sigma^-) \) lie in \( \mathcal{C}^+ (\mathcal{C}^-) \). Factor \( \alpha_{\overline{A}_F} = \alpha_g \alpha_b \)
where \( \alpha_g \mid \Pi_N^+ \sigma^- \) if (3.1.5) is true and \( \alpha_g \mid \Pi_N^- \sigma^+ \) if (3.1.7)
is true, and \( \alpha_b / \sigma^+ \). Notice that the only zeroes of \( \alpha_g \) that lie
in \( \mathcal{C}^+ \) are zeroes of \( \Pi_N^+ \). The following theorem is the main re-
sult on stabilizability.

**Theorem 3.1**

Let \( \alpha_g, \alpha_b, \sigma^+, \sigma^- \) be defined as above and let \( \hat{\mathcal{V}}_g \) be
defined as in (3.1.4). Assume
\[ 1) \quad (\Pi_N^+, \sigma^+) \quad \text{are coprime} \quad (3.1.9) \]
and \[ 2) \quad \{ \hat{A} \mid \hat{B} \} = \mathcal{C}^2 \quad \text{or equivalently the plant is}
\text{controllable.} \quad (3.1.10) \]

Then \[ \mathcal{E}^+ \subseteq \{ \hat{A} \mid \hat{B} \} \mathcal{V}_m + \hat{\mathcal{V}}_m \quad (3.1.11) \]
if and only if
\[ \mathcal{E}^+ = \{ \hat{A} | \hat{B} \} + \hat{V}_g \]  
(3.1.12)
and
\[ \mathcal{E} = \{ \hat{A} | \hat{B} \} + \hat{V}_m \]  
(3.1.13)
if and only if
\[ \mathcal{E} = \{ \hat{A} | \hat{B} \} + \hat{V}_g \]  
(3.1.14)

Proof:-

By (3.1.11), \( \mathcal{E} \in \mathcal{E}^+ \) implies \( \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \) with
\( \mathcal{E}_1 \in \{ \hat{A} | \hat{B} \} \), \( \mathcal{E}_2 \in \hat{V}_m \). By 1), \( \alpha_N, \alpha_A \) are coprime and by Lemma 3.2, \( \mathcal{E}_2 = y_m + y_g + y_b \) where \( y_m \in \hat{V}_m, y_g \in \hat{V}_g, y_b \in \hat{V}_b \). Since \( \alpha_N \) the m.p. of \( N \) divides \( \Pi_N \) it follows that \( \alpha_N, \alpha_A \) are coprime. Now \( y_b \in \hat{V}_b \) implies \( \alpha_A (A_F) y_b \in \hat{V}_m \leq \mathcal{E}_2 \). Partition \( y_b = \begin{pmatrix} y_{b_1} \\ y_{b_2} \end{pmatrix} \) with \( \begin{pmatrix} y_{b_1} \\ 0 \end{pmatrix} \in \mathcal{E}_2 \).

Then it follows from the structure of \( A_F \) that \( \alpha_A (N) y_{b_2} = 0 \)

Since \( \alpha_A, \alpha_N \) are coprime \( y_{b_2} = 0 \) and hence \( y_b \in \mathcal{E}_2 \). By 2) therefore \( y_b \in \{ \hat{A} | \hat{B} \} \) and hence \( \mathcal{E}_2 \in \{ \hat{A} | \hat{B} \} + \hat{V}_g \)

Therefore
\[ \mathcal{E}^+ = \{ \hat{A} | \hat{B} \} + \hat{V}_g \]
and the converse is obvious.

A similar argument establishes the equivalence of (3.1.13) and (3.1.14).

Corollary 3.1

Under the conditions of theorem 3.1, there exists a stabilizable (controllable) error system and the eigenvalues of \( A + BF \) that are fixed
for all $F \in \mathcal{F}(\hat{\mathcal{W}})$ by the requirement that $(\hat{A} + BF)\hat{\mathcal{W}} < \hat{\mathcal{W}}$ are stable.

Proof:-

Clearly (3.1.12) ((3.1.14)) guarantee the existence of a stabilizable (controllable) error system.

From the coprimeness of $\alpha_g', \alpha_b'$ it follows that

$$\hat{\mathcal{W}}_m/\hat{\mathcal{W}}_m = \hat{\mathcal{W}}_g/\hat{\mathcal{W}}_m \oplus \hat{\mathcal{W}}_b/\hat{\mathcal{W}}_m$$

Write $R_g = \hat{\mathcal{W}}_g/\hat{\mathcal{W}}_m$, $R_b = \hat{\mathcal{W}}_b/\hat{\mathcal{W}}_m$. Then $\alpha_g', \alpha_b'$ are the m.p.'s of $R_g$, $R_b$ respectively. Let $AF/R_g = L_1$ and let $L_2$ denote the map induced in $\mathcal{W}_g'$ by $AF$. Now

$$\sigma(AF) = \sigma(AF/R_g) \cup \sigma(L_2)$$

$$= \sigma(AF/R_m) \cup \sigma(L_1) \cup \sigma(L_2)$$

Also,

$$\sigma(AF) = \sigma(A + BF_1) \cup \sigma(N)$$

and by Lemma 3.1, there exists $F \in \mathcal{F}(\hat{\mathcal{W}})$ such that $\sigma(AF/R_m) \subseteq \bar{\zeta}$. By stabilizability (controllability) of the error system $\sigma(L_2) \subseteq \bar{\zeta}$.

By coprimeness of $\alpha_g', \alpha_b', \sigma(L_1)$ consists of the roots of $\pi_N^+ \bar{\zeta}'$.

Let $\sigma^-$ denote the roots of $\bar{\zeta}'$. Then

$$\sigma(A + BF_1) \subseteq \sigma(AF/R_m) \cup \sigma(L_2) \cup \sigma^-$$

and hence the eigenvalues of $A + BF_1$ are stable.

Theorem 3.1 and its corollary demonstrates that conditions 1 and 2 of the theorem are sufficient to guarantee a stable closed-loop system, which also tracks with zero steady-state error.

3.2 OUTPUT FEEDBACK, OBSERVERS AND COMPENSATORS

The tracking control law $u = Fw = F_1x + F_2\dot{x} + F_3\dot{x}$ is not dir-
ectly realizable because of inaccessibility of the entire state vector \( \mathbf{w} \). In practice some linear combinations of these state-variables will be available for measurement. Let \( \mathbf{\theta} = \mathbf{L w} \) denote the measurable outputs. Then, it is well known [14] that if the pair \((\hat{A}, \mathbf{L})\) is observable a dynamic system called an observer, may be designed, such that when driven by \( \mathbf{\theta} \) and \( \mathbf{u} \) the output of the observer \( \hat{\mathbf{w}} = \mathbf{w} + \mathbf{e} \) where \( \mathbf{e}(t) \to 0, a.s. t \to \infty \). The tracking control law \( \mathbf{u} = \mathbf{F w} \) may be then approximately realized by setting \( \mathbf{u} = \mathbf{F w} \) and choosing the eigenvalues of the observer to ensure that the rate of decay of \( \mathbf{e}(t) \) is fast compared with the dynamic response of the tracking system.

An alternative method of designing a system with output-feedback is the so-called compensator method [13]. This consists of augmenting the system

\[
\dot{x} = \mathbf{Ax} + \mathbf{Bu}, \quad y = \mathbf{Cx}
\]

with

\[
\mathbf{u}_1 = \mathbf{u}_2 \\
\vdots \\
\mathbf{u}_p = \mathbf{v}
\]

and showing that any state-feedback control-law in the augmented system is realizable by output feedback, provided the pair \((\mathbf{A}, \mathbf{C})\) is observable and \( p \geq \tau (\gamma - 1) \) where \( \tau \) is the number of inputs and \( \gamma \) the observability index [14] of the pair \((\mathbf{A}, \mathbf{C})\).

To demonstrate the applicability of the compensator method to the servomechanism problem, write

\[
\dot{\mathbf{w}} = \hat{\mathbf{A}} \mathbf{w} + \hat{\mathbf{B}} \mathbf{u} \\
\mathbf{e} = \mathbf{C w} \\
\mathbf{\theta} = \mathbf{L w}
\]
and let 
\[ u = u_1, \]
\[ u_1 = u_2, \]
\[ \vdots \]
\[ u_p = m \]
\[(3.2.1)\]

Write
\[ \phi = \begin{pmatrix} 1 \\ u_1 \\ \vdots \\ u_p \end{pmatrix}, \quad \hat{A}_p = \begin{pmatrix} \hat{A} & \hat{B} & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \end{pmatrix}, \quad \hat{B}_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \]
\[ \hat{C}_p = \begin{pmatrix} \hat{C} \\ 0 \end{pmatrix} \]

Then
\[ \hat{\phi} = \hat{A}_p \phi + \hat{B}_p m \]
\[(3.2.2a)\]

and
\[ e = \hat{C}_p \phi \]
\[(3.2.2b)\]

Let \( \hat{\mathcal{E}}_p \) denote the state space of \((3.2.2)\) and \( \mathcal{E}_p \) the state space of \((3.2.1)\). Then \( \hat{\mathcal{E}}_p = \mathcal{C} \oplus \mathcal{E}_p \). Now let \( \hat{\mathcal{V}}_m \) denote the maximal \( (\hat{A}_p + \hat{B}_F) \) invariant subspace contained in \( \hat{\mathcal{V}}_p = \mathcal{N}(\hat{C}_p) \).

Clearly there exists a control law of the form \( m = F \phi \) such that a controllable error system may be written for \((3.3.2)\) if and only if
\[ \hat{\mathcal{E}}_p = \{ \hat{A}_p | \hat{B}_p \} + \hat{\mathcal{V}}_m \]
\[(3.2.3)\]

To this end, the main result of this section follows, and shows when \((3.2.3)\) holds.

**Theorem 3.2**

For every integer \( p \geq 0 \)
\[ \hat{\mathcal{E}}_p = \{ \hat{A}_p | \hat{B}_p \} + \hat{\mathcal{V}}_m \]
\[(3.2.4)\]
if and only if
\[ \mathcal{E} = \{ \hat{A} | \hat{B} \} + \hat{V}_m \]  \hspace{1cm} (3.2.5)

Proof:-

Let \( V_p, W \) be matrix representations of \( \hat{V}_m^p \) and \( \hat{V}_m^p \) and partition
\[ V_p = \begin{bmatrix} V_1 \\ \vdots \\ V_{p+1} \end{bmatrix} \]  \hspace{1cm} (3.2.6)

Then by Lemma 3.2 [9]
\[ \hat{A}_p \hat{V}_m^p \subset \hat{V}_m^p + \hat{B}_p, \hat{V}_m^p \subset \hat{V}_p \]
and
\[ \hat{A} \hat{V}_m \subset \hat{B} + \hat{V}_m, \hat{V}_m \subset \hat{V} \].

In matrix terms, this means that
\[
\begin{pmatrix}
\hat{A} & \hat{B} & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
0 & 0 & 0 & \cdots & I \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{bmatrix}
V_1 \\
\vdots \\
V_{p+1}
\end{bmatrix}
= 
\begin{bmatrix}
V_1 \\
\vdots \\
V_{p+1}
\end{bmatrix} G_1 + 
\begin{bmatrix}
0 \\
\vdots \\
I
\end{bmatrix} G_2
\]  \hspace{1cm} (3.2.7a)

\[ \hat{C} \hat{V}_1 = 0 \]  \hspace{1cm} (3.2.7b)

and
\[ \hat{A} W = \hat{W} \hat{G}_1 + \hat{B} \hat{G}_2 \]  \hspace{1cm} (3.2.8a)

\[ \hat{C} W = 0 \]  \hspace{1cm} (3.2.8b)

for some \( G_1, G_2, G_3, G_4 \).

By (3.2.4) and the structure of \( \hat{A}_p, \hat{B}_p \), the rows of the matrix
\[
\begin{bmatrix}
\hat{B} & \hat{A} \hat{B} & \cdots & \hat{A}^N \hat{V}_1
\end{bmatrix}, \text{ with } N = \dim \hat{E}_p,
\]
are independent. By (3.2.7a)
\[ \hat{A} \hat{V}_1 = \hat{V}_1 \hat{G}_1 - \hat{B} \hat{V}_2 \]
and
\[ \hat{V}_1 = 0; \]
therefore by maximality of \( \hat{V}_m \), \( \{ W \} \subseteq \{ V_i \} \).

Hence
\[ \hat{E} = \left[ \hat{A} / \hat{B} \right] + \hat{D}_m \]
Conversely, assume (3.2.5) is true.

From (3.2.8) and (3.2.7)
\[
\begin{pmatrix}
\hat{A} & 0 & \cdots & 0 \\
0 & \hat{B} & \cdots & 0 \\
0 & 0 & \ddots & \cdots \\
0 & \cdots & 0 & \hat{I}
\end{pmatrix}
\begin{pmatrix}
W \\
-S_2 \\
-S_2 \sigma_1 \\
\vdots
\end{pmatrix}
= \begin{pmatrix}
W \\
-S_2 \\
-S_2 \sigma_1 \\
\vdots
\end{pmatrix} S_1 + \begin{pmatrix}
0 \\
0 \\
S_2 \sigma_1 \\
\vdots
\end{pmatrix} S_2 \leq \hat{P}
\]
\[ \hat{C} \hat{W} = 0 \]

Therefore
\[ \bar{\omega} = \begin{pmatrix}
-W \\
-S_2 \\
-S_2 \sigma_1 \\
\vdots
\end{pmatrix} \subseteq \begin{pmatrix}
V_1 \\
\vdots \\
V_{P+1}
\end{pmatrix}. \text{ Now } \hat{E}_p = \left[ \hat{A}_p / \hat{B}_p \right] + \bar{\omega} \]

Hence
\[ \hat{E}_p = \left[ \hat{A}_p / \hat{B}_p \right] + \bar{\omega} \]

By Theorem 3.2 a controllable error system can be written for (3.2.2) and choosing \( P \geq \tau(v-1) \) it follows by Theorem 1 [13] that the corresponding control law is realizable by output feedback if the pair \((\hat{A}, \hat{L})\) is observable, with observability index \( \gamma \).

The next chapter presents examples, a summary and conclusions.
CHAPTER IV

4.1  **EXAMPLES**

The theory of the preceding two chapters will be illustrated in the following examples.

**Example 1**

Consider the plant

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where \( \dot{x} = Ax + Bu \), \( y = Cx \)

and the input system

\[
M = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad
H = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

where \( \dot{z} = Mx \), \( z = Hx \)

Form the augmented system

\[
\hat{A} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
\hat{B} = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad
\hat{C} = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0
\end{bmatrix}
\]

(4.1.1)

The maximal \( \hat{A} + BF \) invariant subspace contained in \( \mathcal{N}(\hat{C}) \) computed by the algorithm of [9] is,

\[
\hat{V}_m = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} = \{v_1, v_2, v_3\}
\]

(4.1.2)
Since,
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
it follows that
\[
\begin{bmatrix}
\hat{A} \\
\hat{B}
\end{bmatrix} + \hat{V}_m = C \hat{x}
\]
and therefore a controllable error system of order 5, \( d(\hat{V}_m) = 2 \) exists.

To determine whether the closed-loop system is stable in the absence of inputs, with a second-order error system, choose \( F_0 \in \mathcal{F}(\hat{V}_m) \). An easy calculation shows that for all \( F_0 \in \mathcal{F}(\hat{V}_m) \)
\[
\hat{A} + \hat{B}F_0 | \hat{V}_m = \begin{pmatrix}
0 & -2 & 2 \\
0 & 2 & -2 \\
0 & 0 & 1
\end{pmatrix}
\]  \hspace{1cm} (4.1.3)
in the basis for \( \hat{V}_m \) displayed in (4.1.2). The characteristic and minimal polynomial of \( \hat{A} + \hat{B}F | \hat{V}_m = \lambda(\lambda-1)(\lambda-2) \). The eigenvalue \( \lambda = 2 \) which is fixed for all \( F \in \mathcal{F}(\hat{V}_m) \), would therefore cause the closed loop system to be unstable. To avoid this instability a subspace \( \hat{V}_g \) satisfying the conditions of Th. 3.1, is sought.

Since \( \hat{B} \cap \hat{V}_m = 0 \) the maximal controllability subspace \( \hat{R}_m \) contained in \( \hat{V}_m \) is \( O \). Therefore denoting by \( \bar{x} \) the coset of \( x \) in \( E^5/\hat{R}_m \) and \( \bar{A}_F \) the map induced in \( \hat{V}_m/\hat{R}_m \) it follows that the matrix of \( \bar{A}_F \) in the basis \( \{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \} \) is identical to (4.1.3). Hence the minimal polynomial of \( \bar{A}_F \), \( \alpha_{\bar{A}_F} = \lambda(\lambda-1)(\lambda-2) \). Let \( \alpha_2 = \lambda(\lambda-1) \) and \( \alpha_1 = (\lambda-2) \). Let \( \hat{A} + \hat{B}F = A_F \) and define
\[
\hat{V}_g = \{ \omega \mid \omega \in \hat{V}_m, \alpha_g (A_F) \omega = 0 \} \hspace{1cm} (4.1.4)
\]
\( \hat{V}_g \) is uniquely defined regardless of the \( F \in \mathcal{F}(\hat{V}_m) \) which appears
in (4.1.4). Choose

\[ F_0 = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \]

Then

\[ A_{F_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \]

The error system is

\[ \begin{align*}
\dot{e}_1^0 &= 3 \dot{e}_1 + v_1 \\
\dot{e}_2^0 &= 2 \dot{e}_2 + v_1 \\
\dot{e}_3^0 &= v_2
\end{align*} \]

and

\[ \begin{align*}
\dot{\tilde{e}}_1 &= (x_3 - z_1) \\
\dot{\tilde{e}}_2 &= (x_2 - 2z_1) \\
\dot{\tilde{e}}_3 &= (x_1 - x_2)
\end{align*} \]

Feedback of the form

\[ \nu = \begin{pmatrix} f_1 & f_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}\begin{pmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \\ \dot{\tilde{e}}_3 \end{pmatrix} \]

places all the poles of the error system. Choosing these pole-locations to be -5, -5, -5 yields

\[ f_1 = -64, \quad f_2 = 49, \quad g_3 = -5 \]

The closed-loop system therefore has poles at -5.

The control law is

\[ u = \left[ \begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -64 & 49 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix} \right] \begin{pmatrix} x \\ z \end{pmatrix} \]
\[ u = \begin{pmatrix} 0 & 49 & -64 & -34 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \]

Example 2

The plant is specified by

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \]

Consider the input system

\[ M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

then,

\[ \hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \end{pmatrix} \]

and

\[ \hat{V}_m = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \]

The m.p. of \( \hat{A} \) is \( \lambda^3 \) and \( \xi^+ = \text{ker} \hat{A}^3 \). Therefore

\[ \xi^+ = \xi \not\subset \{ \hat{A} \mid \hat{B} \} + \hat{V}_m \]

and there exists no stabilizable error system. This is a consequence of the fact that the plant outputs are related by a differential equation whereas the outputs of the input system are independent. Consider
now the input system

\[
M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] (4.1.6)

where the command signals are related by the same differential equation as the plant outputs. In this case

\[
\begin{pmatrix} \dot{v}_{m} \\ \dot{\nu}_{m} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_{m} \\ \dot{\nu}_{m} \end{pmatrix} + \begin{pmatrix} \hat{A} \end{pmatrix} \begin{pmatrix} \nu_{m} \end{pmatrix} = \xi_{e} \xi_{u}
\]

Therefore a 2nd order error system may be written. To examine stabilizability, choose \( F_{o} \in \mathcal{F}(\dot{\nu}_{m}) \)

\[
F_{o} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & f_{o} & 0 & 0 \end{pmatrix}
\]

Then

\[
\begin{pmatrix} \hat{A} + BF_{o} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{o} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Now

\[
\begin{pmatrix} \hat{B}_{n} \dot{\nu}_{m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\]
and the maximal controllability subspace contained in $\hat{n}$ is given by $\hat{\mathcal{R}}_m = \{ \hat{A} + \hat{B} F_0 \} \hat{\mathcal{R}}_m$

$\hat{\mathcal{R}}_m = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The matrix of $\hat{A} + \hat{B} F_0 \big| \hat{\mathcal{V}}_m$ is

$\begin{pmatrix} f_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

where $\begin{pmatrix} f_0 \end{pmatrix}$ is the matrix of $\hat{A} + \hat{B} F_0 \big| \hat{\mathcal{R}}_m$
and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the matrix of the $\hat{A}_{F_0}$ induced in $\hat{\mathcal{V}}_m$ by $\hat{A} + \hat{B} F_0$

The m.p. of $\hat{A}_{F_0}$, $\alpha_{\hat{A}_{F_0}} = n^2$. Write $\alpha_g (n) = n^2$
and $\alpha_h (n) = 1$ and define

$\hat{\mathcal{V}}_g = \{ x \mid x \in \hat{\mathcal{V}}_m, \alpha_g (\hat{A} + \hat{B} F_0) x \in \hat{\mathcal{R}}_m \}$

Then

$\hat{\mathcal{V}}_g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Choose

$\mathcal{S} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$
as a basis for $\xi^5$. In these coordinates the matrix of $\hat{A}_{F_0}$ is

$$
\begin{pmatrix}
F_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

and hence the matrix of the error system is

$$
\hat{A} = \begin{pmatrix} 0 & 0 \\ \hat{B} & 0 \end{pmatrix}
$$

By a simple calculation $\hat{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Note that the error system has only one input because $\hat{B} \hat{V}_m$ is one dimensional.

Now

$$
\tilde{e}_1 = x_2 - x_2, \quad \tilde{e}_2 = x_1 - x_1
$$

and a control law of the form

$$
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix} \quad v_2 = 0
$$

places the eigenvalues of the error system at the zeros of

$$
\lambda^2 - \lambda \bar{F}_1 + \bar{F}_2 \cdot \bar{F}_1. \quad \text{The control law is of the form}
$$

$$
\begin{pmatrix} F_2 \\ \bar{F}_1 \\ -1 \\ -\bar{F}_2 \\ -\bar{F}_1 \end{pmatrix} \begin{pmatrix} v_1 \\ \bar{F}_0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}
$$

and clearly the system can be stabilized by choice of $F_0, \bar{F}_1, \bar{F}_2$.

**Example 3**

Consider the plant

$$
\begin{pmatrix} \hat{x}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \quad \begin{pmatrix} y \\ \hat{x}_1 \end{pmatrix}
$$

and the input system

$$
\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}, \quad \begin{pmatrix} \lambda \\ \hat{z}_2 \end{pmatrix}
$$

Let $c = y - r = x_1 - x_1$ and define

$$
\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
$$
Then if  
\[
\mathbf{w} = \begin{pmatrix} x_1 \\ x_2 \\ u_1 \\ z_1 \\ z_2 \\ . \end{pmatrix}
\]
\[
\mathbf{w} = \mathbf{A}\mathbf{w} + \mathbf{B}\mathbf{u} \quad \mathbf{e} = \mathbf{C}\mathbf{w}
\]

\[
\hat{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} 
\]
\[
\hat{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} 
\]
\[
\hat{\mathbf{C}} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \end{pmatrix}
\]

and

\[
\hat{\mathbf{Q}}_m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and therefore
\[
\{ \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{F}_0 \} \hat{\mathbf{Q}}_m = \mathbf{e}^5
\]

Choose \( \mathbf{F}_0 \in \mathcal{F}(\hat{\mathbf{Q}}_m) \) to be \( \mathbf{F}_0 = \mathbf{0} \).

The matrix of \( \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{F}_0 \) is
\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

and its m.p. = \( \lambda^2 \). The error system is
\[
\begin{align*}
\hat{\mathbf{e}}_1 &= \mathbf{u}_2 \\
\hat{\mathbf{e}}_2 &= \hat{\mathbf{e}}_1 \\
\hat{\mathbf{e}}_3 &= \hat{\mathbf{e}}_2 \\
\end{align*}
\]

where \( \hat{\mathbf{e}}_1 = \mathbf{u}_1 \), \( \hat{\mathbf{e}}_2 = x_2 - z_2 \), \( \hat{\mathbf{e}}_3 = x_1 - z_1 \)

Feedback of the form \( \mathbf{u}_2 = \overline{f}_1 \hat{\mathbf{e}}_1 + \overline{f}_2 \hat{\mathbf{e}}_1 + \overline{f}_3 \hat{\mathbf{e}}_2 \) achieves arbitrary pole-placement in the error system. The control law is of the form
\[
\mathbf{u}_2 = \overline{f}_1 \mathbf{u}_1 + \overline{f}_2 x_2 - \overline{f}_2 z_2 + \overline{f}_3 x_1 - \overline{f}_3 z_1
\]

and therefore the compensator is
\[
\mathbf{u}_1 - \overline{f}_1 \mathbf{u}_1 = \overline{f}_2 \mathbf{y} + \overline{f}_3 \mathbf{y} - \overline{f}_2 \mathbf{z} - \overline{f}_3 \mathbf{z}
\]
This compensator places the poles of the closed-loop system at the same locations as the poles of the error system.

4.2 SUMMARY AND CONCLUSIONS

In Chapter II the problem of finding a feedback control law that makes the outputs of a linear system track command signals from a known class, in the presence of disturbances, is formulated and solved. The solution consists of choosing the feedback to first write an "error system" that can be stabilized by additional feedback. The eigenvalues of the error system determine the transient response of the system. Necessary and sufficient conditions for the existence of stabilizable and controllable error systems are presented. Stabilizability of the error system guarantees zero steady-state error; controllability guarantees an arbitrary transient response.

The control law that results can cause the closed-loop system to be unstable. This problem is discussed in Chapter III and a condition that suffices to ensure stability of the closed-loop system is derived. A simple argument demonstrates that this condition is also necessary for the single-input single-output case. It is shown how the control law can be realized by output feedback through observers or compensators.

Further investigation of the problem of stabilizing the closed loop system is required and necessary conditions for such stabilizability should be found. The question of determining whether dynamic compensation can achieve stabilizability needs to be answered. If such compensation exists the minimum order and structure of the compensator should
be specifiable.

Since the solution depends on the class of command signals and disturbances under consideration the sensitivity of the solution (control-law) to the parameters describing these classes should be determined and minimized if possible. The sensitivity of any computational algorithms, to these parameters should also be found.

The geometric conditions for the existence of a solution, that are presented here, should be interpreted in terms of algebraic properties of the given system maps, or transfer functions. Such a characterization could yield greater insight into the structural properties of multi-variable systems.
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