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Deformation Spaces of Period Matrix Domains
of Compact Kähler Surfaces

by

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Thesis Director's Signature:

R.D. Wells

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In Memory of

Joseph I. Davies
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Table of Contents

Introduction

Section I: Preliminaries

Section II: Complex Structure on $\Psi_X$

Section III: Structure of $\Psi_D$
   A: Preliminary construction
   B: Structure theorem

Section IV: Coherent Analytic Sheaves on $\Psi_D$
   A: Direct images
   B: Sheaf cohomology

Section V: Homogeneous Line Bundles on D.
Introduction

P.A. Griffiths [5] has introduced what are called period matrix domains in studying the problem of determining complex structures on differentiable manifolds. If the differentiable manifold can be given the structure of a compact Kähler surface, then its associated period matrix domain can be described as follows. Let

\[ Q = \begin{bmatrix} I_{2h} & 0 \\ 0 & -I_k \end{bmatrix} \]

(where \( I_j \) is the \( j \times j \) identity matrix) act as a real quadratic form on \( \mathbb{R}^b \subset \mathbb{C}^b \), where \( b = 2h + k \). The number \( b \) is determined from topological invariants of the manifold, and the numbers \( h \), \( k \), from complex-analytic invariants. The determination of these numbers is given in Griffiths [5] and will not be of concern to us here. \( Q \), then, can be extended linearly to act on \( \mathbb{C}^b \). Let \( G_{h,b}(\mathbb{C}) \) be the Grassmannian manifold of \( \mathbb{C} \)-linear, \( h \)-dimensional planes in \( \mathbb{C}^b \). If \( S \in G_{h,b}(\mathbb{C}) \) then we write

1) \( Q(S,S) = 0 \) if for any \( b \times h \) matrix, \( \Omega \), whose columns form a basis for the plane \( S \),

\[ \Omega^T Q \Omega = 0 \]

and

2) \( Q(S,\bar{S}) > 0 \) if for \( \Omega \) as above, \( \Omega^T Q \bar{\Omega} \) is positive definite.
The set
\[ D = \{ S \in G_{h,b}(c) : Q(S,S) = 0 \text{ and } Q(S,S) > 0 \} \]
is the **period matrix domain** associated to the given manifold.
Griffiths has shown that \( D \) is, in fact, a complex-analytic manifold.

We give, in Section I, a detailed survey of the properties of \( D \). Here, too, we construct the deformation space, \( \Psi_D \), associated to \( D \). \( \Psi_D \) is an analytic family of compact submanifolds of \( D \) and the purpose of this paper is to derive some of its properties. Also, in Section I we review, for the case \( k=1 \), the construction of a parameter space, \( M \), for the family of manifolds, \( \Psi_D \), (Wells [10]).

In Section II we show that \( \Psi_D \) is actually a complex analytic manifold and that there are holomorphic maps relating \( D, \Psi_D, \) and \( M \) as follows
\[
M \leftarrow \Psi_D \rightarrow D
\]

In Section III we examine the structure of \( \Psi_D \) and show that it is actually a product space \( Y \times M \), where \( Y \) is a fixed submanifold of \( D \).

In Section IV, we derive results which allow one to compute sheaf cohomology on \( \Psi_D \) in terms of sheaf cohomology on \( D \) and \( M \).

Finally in Section V we relate this work to questions posed by P.A. Griffiths in [5] concerning homogeneous line bundles on period matrix domains.
Section I

PRELIMINARIES

In this Section we will review some constructions of Griffiths [5] and Wells [10]. In parts A and B the construction will be valid for arbitrary $h$ and $k$ and in part C, for $k = 1$.

A. We begin with the period matrix domain

$$D = \{ S \in G_{h,b}(c): Q(S,S) = 0 \text{ and } Q(S,\bar{S}) > 0 \}. $$

This set is clearly an open subset of the algebraic subvariety

$$X = \{ S \in G_{h,b}(c): Q(S,S) = 0 \}. $$

Griffiths has shown that the complex Lie group,

$$\mathcal{G} = \{ g \in \text{SL}(b,c): \ ^t gQg = Q \}$$

is a transitive group of biholomorphic mappings of $X$ onto itself, hence $X$ is an algebraic manifold, and therefore so is $D$. In fact, Griffiths has also shown that the real Lie group,

$$G = \{ g \in \text{SL}(b,R): \ ^t gQg = Q \} = \mathcal{G} \cap \text{GL}(b,R)$$

is a transitive group of biholomorphic mappings of $D$ onto itself. The isotropy group of a point in $D$ is given by

$$H = U(h) \times SO(k)$$

where

$$\begin{bmatrix}
\text{Re}A & \text{Im}A & 0 \\
-\text{Im}A & \text{Re}A & 0 \\
0 & 0 & B
\end{bmatrix} \in G$$
So, $D$ is represented as a homogeneous space, $G/H$. And, we have according to Wells

**Proposition I.1** There is a bianalytic mapping, $\mu$, between the homogeneous space $G/H$, and $D$ given by

$$\mu(gH) = \langle Ag + iBg \rangle$$

where $g = [A_g \ B_g \ C_g]$, with $A_g, B_g, b \times h$ matrices and $C_g$, a $b \times k$ matrix, and $\langle \ast \rangle$ denotes the space spanned by the columns of $\ast$. This gives a complex structure to the homogeneous space, $G/H$.

Wells also states that the domain, $D$, consists of two components which we denote by $D^+$ and $D^-$, ($D^+$ being the component containing the identity coset of $G/H$). And that the transformation

$$T_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & I_{b-2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is in $G$ and is a biholomorphism of $D^+$ onto $D^-$.

B. We would now like to consider the maximal compact subgroup of $G$. Griffiths has shown this to be

$$K = SO(2h) \times SO(k)$$

where

$$(A,B) \rightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in G.$$ 

Hence we have,
And we have the compact, homogeneous, real-analytic submanifold of $G/H$, $K/H$. Wells has shown that

**Proposition I.2** The submanifold, $K/H$, of $G/H$ corresponds under the mapping, $\mu: G/H \to D$ to the connected complex submanifold, $Y$, of $D$ given by

$$Y = \{S \in D^+: Q(S, <C_1>) = 0\}$$

and if $g \in \tilde{G}$ then

$$gY \subset \{S \in X: Q(S, <C_g>) = 0\}.$$  

Moreover Griffiths shows that

$$Y = \{ \left[ \begin{array}{c} A \\ B \end{array} \right]: A^tA + B^tB = 0 \} \cap D^+$$

We would now like to consider the translates of $Y$ by elements, $g \in \tilde{G}$. To do this we define $\Psi_X$ to be the disjoint union of all the submanifolds of $X$ which are translates of $Y$ by elements of $\tilde{G}$, i.e., each submanifold in $\Psi_X$ is $gY$ for some $g \in \tilde{G}$. (Note that a given submanifold can be represented as a $gY$, by more than one $g \in \tilde{G}$.) Suppose we let

$$\tilde{K} = \{ g \in \tilde{G}: gY = Y \}$$
then $K$ is a closed subgroup of $G$ and the manifolds of $U_X$ are in one-to-one correspondence with the elements of $G/K$, i.e., $G/K$ serves as a parameter space for the submanifolds of $U_X$. We have then

**Theorem I.1** For $h \geq 2$

$$K = SO(2h, \mathbb{C}) \times SO(k, \mathbb{C})$$

the complexification of $K$.

**Proof:** First of all, let for $\alpha = 1, \ldots, h-1$

$$I^\alpha = \begin{bmatrix}
1 & & & \\
& & & \\
& & \ddots & \\
& & & & 1 \\
& & & & 0 \end{bmatrix}
$$

an $h \times h$ matrix, i.e.,

$$I^\alpha_{ij} = \begin{cases}
0 & i \neq j \\
1 & i = j, j \neq \alpha, \alpha + 1 \\
-1 & i = j = \alpha, \alpha + 1
\end{cases}$$

then if
\[ \Omega^\alpha = \begin{cases} 
\begin{bmatrix} 
I \\
ii \\
0 
\end{bmatrix} & \alpha = 0 \\
\begin{bmatrix} 
I^\alpha \\
ii \\
0 
\end{bmatrix} & \alpha \geq 1 
\end{cases} \]

then \( \langle \Omega^\alpha \rangle \in Y \). This is true since

\[
\begin{bmatrix} 
I \\
ii \\
0 
\end{bmatrix} = \mu(IH) = \mu(H)
\]

and

\[
\begin{bmatrix} 
I^\alpha & 0 & 0 \\
0 & I_h & 0 \\
0 & 0 & I_k 
\end{bmatrix} \cdot \begin{bmatrix} 
I \\
ii \\
0 
\end{bmatrix} = \begin{bmatrix} 
I^\alpha \\
ii \\
0 
\end{bmatrix} = \Omega^\alpha
\]

with

\[
\begin{bmatrix} 
I^\alpha & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I 
\end{bmatrix} \in K.
\]

Now suppose \( g \in \tilde{G} \) and \( gY = Y \), then from Proposition I.2.

1) \( t^\Omega^\alpha QC_g = 0 \) for \( \alpha = 0, \ldots, h-1 \)

2) \( g\Omega^\alpha = \begin{bmatrix} 
A \\
B \\
0 
\end{bmatrix} \in Y \)
So if we let
\[
g = \begin{bmatrix}
  A_1 & B_1 & C_1 \\
  A_2 & B_2 & C_2 \\
  A_3 & B_3 & C_3
\end{bmatrix}
\]

we have from 1) for \( \alpha = 0, \ldots, h-1 \)
\[
(*) \quad C_1 I^\alpha + iC_2 = 0.
\]

and from 2) for \( \alpha = 0, \ldots, h-1 \)
\[
(*) \quad A_3 I^\alpha + iB_3 = 0
\]

then choosing appropriate parts of (*) we obtain,

for \( A_3 = [a_3^{\beta\gamma}] \)
\( B_3 = [b_3^{\beta\gamma}] \)

and \( C_j = [c_j^{\beta\gamma}] \) for \( j = 1,2 \).

\[
\begin{align*}
  c_1^{\beta\gamma} + i c_2^{\beta\gamma} &= 0 \\
  -c_1^{\beta\gamma} + i c_2^{\beta\gamma} &= 0
\end{align*}
\]

for all \( \beta, \gamma \)

\[
\begin{align*}
  a_3^{\beta\gamma} + i b_3^{\beta\gamma} &= 0 \\
  -a_3^{\beta\gamma} + i b_3^{\beta\gamma} &= 0
\end{align*}
\]
Hence

\[ A_3 = B_3 = C_1 = C_2 = 0. \]

So \( g \) is of the form

\[
\begin{bmatrix}
A & 0 \\
0 & C
\end{bmatrix}
\]

for \( A \), an \( 2h \times 2h \) matrix and \( C \), a \( k \times k \) matrix.

But \( g \in \widetilde{G} \) implies \( t_gQ_0 = Q \) or

\[
\begin{bmatrix}
t^A & 0 \\
0 & t^C
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & C
\end{bmatrix} =
\begin{bmatrix}
t^AA & 0 \\
0 & t^CC
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}
\]

So

\[ t^AA = I \]

and

\[ t^CC = I \]

hence, \( A \in O(2h, c) \) and \( C \in O(k, c) \)

Furthermore,

\[ 1 = \det g = \det A \cdot \det C \]

so

\[ \det A = \det C = \pm 1 \]

Recalling \( T_0 \), the biholomorphism of \( D^+ \) onto \( D^- \) we see that

\[ T_0K = T_0(SO(2h) \times SO(k)) \]
are mappings of \(Y\) into \(D^\dagger\). But matrices in \(T_0K\) are just those of the form
\[
\begin{bmatrix}
A & 0 \\
0 & C
\end{bmatrix}
\]
with \(\det A = \det C = -1\).

Hence the same is true for
\[T_0(SO(2h, c) \times SO(k, c))\]

So if \(gY = Y \subset D^\dagger\) then
\[g \in SO(2h, c) \times SO(k, c)\]
i.e., \(K \subset SO(2h, c) \times SO(k, c)\)

but clearly then
\[K = SO(2h, c) \times SO(k, c)\]

We define, now, maps
\[\tau_X: \psi_X \rightarrow X\]
by \(gY \rightarrow gY\)
i.e., the manifold \(gY\) is mapped into \(X\) by inclusion,
and
\[\pi_X: \psi_X \rightarrow \mathcal{G}/K\]
by \(gY \rightarrow gK\).

We will be primarily concerned with a subset of \(\psi_X\), namely, \(\psi_D\), which is the disjoint union of all those submanifolds, \(gY\) so that
\[ gY \subset D \]
which is the deformation space of the domain \( D \).

C. From now on we restrict ourselves to the case, \( h \geq 2gk = 1 \). The first restriction is not at all unreasonable; because for the case \( h = 1 \), the manifold, \( Y \), is a single point and the situation is relatively trivial. Removal of the second restriction will be the goal of papers following this one.

For this case, Wells has shown that the parameter space \( \mathcal{G}/\mathcal{K} \) is biholomorphic to

\[ V = \{ z \in \mathbb{C}^b : t^z Q z = -1 \} \]

The biholomorphism is given by

\[ g = [A_g \ B_g \ C_g] \rightarrow C_g \in \mathbb{C}^b. \]

Wells has also shown that the set, \( \mathcal{U}_D \), is parameterized by the set

\[ M = \{ z \in V: \sum_{j=1}^{2h} x_j^2 < x_b^2 \} \cup \{ z \in V: \sum_{j=1}^{2h} y_j < y_b \} \]

where

\[ z = \begin{bmatrix} z_1 \\ \vdots \\ z_b \end{bmatrix} \quad \text{and} \quad z_j = x_j + iy_j. \]

Moreover, it is not difficult to see that \( M \) is an open subset of \( V \) and is, in fact, a Stein manifold. Note that \( M \) is not connected since \( D \) is not, but has two components corresponding to \( D^+ \) and \( D^- \). And we let \( n_X \), also denote the induced map.
\[ \pi_X : \psi_X \longrightarrow G/K \longrightarrow \psi^C. \]

Since the two components of $D$ are biholomorphically equivalent, it suffices to deal with only one of them. For convenience we choose $D^+$, the component containing the identity coset of $G/H$ (which is defined as a point in $G_{h,b}(C)$ by the matrix $\Omega^0$). Also for convenience of notation, we delete the plus sign and denote by $D$ the identity component, $D^+$; by $\psi_D$ the corresponding subset of $\psi_D$ and by $M$ the corresponding component of $M$. Furthermore we let

\[ \tau = \tau_X|_{\psi_D} \]

and

\[ \pi = \pi_X|_{\psi_D} \cdot \]

So, we will be dealing primarily with the members of the following diagram:

\[
\begin{array}{ccc}
V & \xleftarrow{\pi_X} & \psi_X \\
U & \xrightarrow{\tau_X} & X \\
M & \xleftarrow{\pi} & \psi_D \\
& & \xrightarrow{\tau} D
\end{array}
\]
Section II

COMPLEX STRUCTURE ON $\Psi_X$

It is possible to put a complex structure on $\Psi_X$ in such a way that the maps $\tau_X$ and $\pi_X$ are holomorphic and the structure induced on a given submanifold agrees with its original structure as a submanifold of $X$. In fact, it is possible to do this in a more general setting. Namely suppose $X$ is any homogeneous complex manifold with a transitive group, $\mathcal{G}$, of biholomorphic mappings of $X$ onto itself; and $Y$ is a complex submanifold of $X$. Let, as before,

$$\mathcal{K} = \{ g \in \mathcal{G} : gY = Y \}$$

and $\Psi_X$ be the disjoint union of the translates of $Y$ by elements in $\mathcal{G}$. Then, we have $\mathcal{K}$ is, again, a closed subgroup of $\mathcal{G}$ and the submanifolds of $\Psi_X$ are parametrized by the elements of $\mathcal{G}/\mathcal{K}$.

We define, as before, two maps

$$\tau_X : \Psi_X \to X$$

by $gY \mapsto gY$

and

$$\pi_X : \Psi_X \to \mathcal{G}/\mathcal{K}$$

by $gY \mapsto g\mathcal{K}$.

Now, in fact, $\mathcal{G}$ is a holomorphic principal bundle over $\mathcal{G}/\mathcal{K}$ (Hirzebruch [8] p. 43). Moreover, we can find an open
covering, \([U_\alpha]\) of \(\partial/\overline{\partial}\) such that for each \(\alpha\) there is a holomorphic bundle section,

\[\sigma_\alpha: U_\alpha \to \mathcal{G}\]

(Hirzebruch, [8] p. 43)

Using the sections, \([\sigma_\alpha]\), we can define a complex structure on \(\psi_X\) as follows. Define

\[\hat{\phi}_\alpha: Y \times U_\alpha \longrightarrow \pi_X^{-1}(U_\alpha) \subset \psi_X\]

by \(\hat{\phi}_\alpha(y, z) = \sigma_\alpha(z)y\) \((\epsilon \sigma_\alpha(z) Y)\)

then \(\hat{\phi}_\alpha\) is a bijection. Moreover, on \(U_\alpha \cap U_\beta\) we have

\[\hat{\phi}_\beta \circ \hat{\phi}_\alpha(y, z) = \hat{\phi}_\beta(\sigma_\alpha(z)y) = (\sigma_\beta(z)^{-1}\sigma_\alpha(z)y, z)\]

Hence \(\hat{\phi}_\beta^{-1} \circ \hat{\phi}_\alpha\) is a biholomorphism.

Now for each \(\alpha\) we induce on \(\pi^{-1}(U_\alpha)\) the topology making \(\hat{\phi}_\alpha\) a homeomorphism. Then on \(\pi^{-1}(U_\alpha \cap U_\beta)\), the topologies induced by \(\hat{\phi}_\alpha\) and \(\hat{\phi}_\beta\) are the same (since if \(\hat{\phi}_\alpha\) and \(\hat{\phi}_\beta^{-1} \circ \hat{\phi}_\alpha\) is a biholomorphism, this induces a complex structure on \(\psi_X\)).

So, \(\psi_X\) is a complex manifold and the maps \(\pi_X\) and \(\tau_X\) are related to the manifold structure as follows

a) \(\pi_X \circ \hat{\phi}_\alpha\) is the product projection of \(Y \times U_\alpha\) onto \(U_\alpha\).

b) \(\tau_X \circ \hat{\phi}_\alpha(y, z) = \sigma_\alpha(z)y\)

and so both are holomorphic. Moreover, the complex structure induced on \(\psi\) agrees with its original structure as a submanifold of \(X\).
Remark: If \( g, g' \) are in the same fiber of \( \mathcal{G} \to \mathcal{G}/\mathbb{K} \) then 
\( g^{-1}g' \in \mathbb{K} \) so the sections \( \{ \sigma_{\alpha} \} \) induce

\[
K_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{K}
\]

by \( K_{\beta\alpha}(z) = \sigma_{\beta}(z)^{-1} \sigma_{\alpha}(z) \)

then we have

\[
\hat{\phi}_{\beta}^{-1} \circ \hat{\phi}_{\alpha}(y, z) = (K_{\beta\alpha}(z)y, z)
\]

Theorem II.1 The complex structure on \( \psi_X \) is independent of the choice of sections.

Proof: Let \( \{ \sigma_{\alpha}, U_\alpha \}; \{ \sigma_{\alpha}', U_\alpha \} \) be a choice of sections (by a suitable refinement we can assume the sections are defined on the same covering) and \( \psi_X, \psi_X' \), resp. the set \( \psi_X \) with the induced complex structures. Then for all \( \alpha \) we have

\[
Y \times U_\alpha \xrightarrow{\hat{\phi}_{\alpha}} \psi_X \xrightarrow{\text{identity}} \psi_X' \xrightarrow{\hat{\phi}_{\alpha}'^{-1}} Y \times U_\alpha
\]

and the composition is just

\[
(y, z) \xrightarrow{(\sigma_{\alpha}'(z))^{-1}} (\sigma_{\alpha}(z)y, z)
\]

which is holomorphic. Hence the identity is a biholomorphism.
Section III

STRUCTURE OF $\psi_D$

In this section we give in part A, a preliminary construction that will be used in proving the structure theorem (Theorem III.1) which is proved in part B. Also in part B we give some consequences of the structure theorem.

A. From time to time we will need to consider for a point, $x \in X$, all the translates of $Y$ containing this point. This is, of course, the set $\tau^{-1}_X(x)$, which is parametrized by its projection into $V$, $\pi_X(\tau^{-1}_X(x))$. We would like now to give a characterization of these sets.

Since $X$ is homogeneous, it suffices to consider $x_0 \in D$, the point defined by the matrix

$$\Omega^0 = \begin{bmatrix} 1 & 0 \\ iI & 0 \end{bmatrix}$$

Since if $x \in X$ and $g \in \tilde{G}$ so that $gx_0 = x$ then

$$\tau^{-1}_X(x) = g \tau^{-1}_X(x_0)$$
and

\[ \pi_X(\tau_X^{-1}(x)) = g \pi_X(\tau_X^{-1}(x_0)) \]

where the actions of \( G \) on \( \Psi_X \) and \( V \) are those induced by its actions on \( D \). So, let

\[ T = \left\{ \begin{bmatrix} C \\ iC \\ 1 \end{bmatrix} : C \in \mathbb{C} \right\} \subset V \]

then we have.

Lemma III.1 \( T = \pi_X(\tau_X(x_0)) \)
Proof: Recall from Proposition I.2

\[ Y = \{ S \in D : \ Q(S, <C_1>) = 0 \} \]

and for \( g = [A_{g} B_{g} C_{g}] \in g \)

\[ gY = \{ S \in X : \ Q(S, <C_1>) = 0 \} \]

So for \( z \in c^b \) to be in \( \pi_X(\tau_X^{-1}(x_0)) \)

we must have

a) \[ t^z Qz = -1 \]

(since \( z \) must be in \( V \)) and

b) \[ t^n Qz = 0 \]

(since \( x_0 \in gY \) where \( g = [A_{g} B_{g} z] \)) i.e.

\[ a') \quad \sum_{j=1}^{2h} z^2_j - z^2_b = -1 \]

\[ b') \quad z_j + iz_{j+h} = 0 \quad \text{for } j = 1, \ldots, h. \]

Combining \( a' \) and \( b' \) we have

\[ z^2_b = 1 \]

so

\[ \pi_X(\tau_X^{-1}(x_0)) \subset \left\{ \begin{bmatrix} C \\ iC \\ \alpha \end{bmatrix} : \ C \in c^h \text{ and } \alpha = \pm 1 \right\} \subset V \]
So if

\[ T = \left\{ \begin{bmatrix} C \\ iC \\ 1 \end{bmatrix} : C \in \mathcal{C} \right\} \]

and

\[ T^{-1} = \left\{ \begin{bmatrix} C \\ iC \\ -1 \end{bmatrix} : C \in \mathcal{C} \right\} \]

we have

\[ \pi_X(\tau^{-1}(x_0)) \subset T \cup T^{-1} \]

Now define

\[ \sigma_1 : T \rightarrow M_b(\mathcal{C}) \quad (b \times b \text{ matrices}) \]

by

\[
\begin{pmatrix}
C \\
iC \\
1
\end{pmatrix} \quad \rightarrow \quad \begin{bmatrix}
I + \frac{1}{2} C^tC & \frac{1}{2} C^tC & C \\
\frac{1}{2} C^tC & I - \frac{1}{2} C^tC & iC \\
tC & i^tC & 1
\end{bmatrix}
\]
It is easy to check that

i) \( \sigma_1(z) \in \mathcal{G} \) for all \( z \in T \)

ii) \( \sigma_1(z)\mathcal{N}_0 = \mathcal{N}_0 \) for all \( z \in T \)

i.e. \( x_0 \in \sigma_1(z)Y \)

iii) \( \pi_X(\sigma_1(z)Y) = z \)

So each point of \( T \) corresponds to some manifold containing \( x_0 \), i.e.,

\[
T \subset \pi_X(\pi_X^{-1}(x_0)) \subset T \cup T^-
\]

Remark: \( \sigma_1 \) is actually a section of the principal bundle \( \mathcal{G} \overset{\pi}{\to} \mathcal{K} \) over \( T \).

Furthermore, suppose for

\[
\begin{bmatrix}
  c \\
  iC \\
  -1
\end{bmatrix}
\in T^-
\]

we attempt to find \( g \in \mathcal{G} \) so that

\[
\pi_X(gY) = \begin{bmatrix}
  c \\
  iC \\
  -1
\end{bmatrix}
\]
and

\[ x_0 \in gY. \]

If \( g \) exists then we can require that it satisfy three properties,

1) \( t_g \ Qg = Q \) \( \{ \text{i.e. } g \in \mathcal{G}. \} \)

2) \( \det g = 1 \)

3) \( g \Omega_0 = \Omega_0 \)

N.B. 3) is stronger than \( x_0 \in gY \) but if there does exist \( g \) so that \( x_0 \in gY \) there will be a \( k \in \mathbb{K} \) so that

\[ g^k \Omega_0 = \Omega_0 \]

Let, then

\[
g = \begin{bmatrix}
A_1 & B_1 & C \\
A_2 & B_2 & iC \\
t_{a3} & t_{b3} & -1 \\
h & h & h
\end{bmatrix}
\]

then \( g \) satisfying condition 1) states
(1) \( t_{A_1} A_1 + t_{A_2} A_2 - a_3 t_{a_3} = I \)

(2) \( t_{B_1} A_1 + t_{B_2} A_2 - b_3 t_{a_3} = 0 \)

(3) \( t_{C A_1} + i t_{C A_2} + t_{a_3} = 0 \)

(4) \( t_{B_1 B_1} + t_{B_2 B_2} - b_3 t_{b_3} = I \)

(5) \( t_{C B_1} + i t_{C B_2} + t_{b_3} = 0 \)

and \( g \) satisfying condition 3) states

(6) \( A_1 + i B_1 = I \)

(7) \( A_2 + i B_2 = I \)

(8) \( t_{a_3} + i t_{b_3} = 0. \)

From (6), (7), (8) we obtain

(6') \( A_1 = I - i B_1 \)

(7') \( A_2 = i(I - B_2) \)

(8') \( t_{a_3} = -i t_{b_3} \)

Substituting equations (6'), (7'), (8') into (2) we obtain

(2') \( B_1 + i B_2 = i I \)

Substituting (2') into (5) we obtain

(5') \( t_{b_3} = -i t_{C} \)

and then by appropriate substitutions we obtain
A_1 = I - iB_1
A_2 = B_1
B_2 = I + iB_1
t_α_3 = -t_C

So

\[
g = \begin{bmatrix}
I - iB_1 & B_1 & C \\
B_1 & I + iB_1 & iC \\
-t_C & -it_C & -1
\end{bmatrix}
\]

But then g is row equivalent to

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

So \( \det g = -1 \)

So then condition 2. cannot be satisfied by a matrix of this form satisfying 1. and 3. Hence vectors in \( T^{-} \) do not correspond to manifolds, \( gY \), in \( \psi_X \) with \( x_0 \in gY \).

So

\[
T^{-} \cap \pi_X(\tau^{-1}_X(x_0)) = \emptyset
\]

and we have established \( \emptyset \).
Remark: The method used above to construct $g$ is precisely the same method used to construct

$$\sigma_1: T \to \mathcal{G}.$$ 

It was found in this case

$$\sigma_1 \left( \begin{bmatrix} C \\ iC \\ 1 \end{bmatrix} \right) = 
\begin{bmatrix}
I - iB_1 & B_1 & C \\
B_1 & I + iB_1 & iC \\
tC & iC & 1
\end{bmatrix}$$

and that it sufficed to let

$$B_1 = \frac{i}{2} c^TC$$

\textbf{Corollary III.1} \hspace{1cm} \pi(\tau^{-1}(x_0)) = \left\{ \begin{bmatrix} C \\ iC \\ 1 \end{bmatrix} : C \in \mathcal{G}^h \text{ and } tCC < 1 \right\}

\textbf{Proof:} \hspace{1cm} \pi(\tau^{-1}(x_0)) = \pi(\tau^{-1}(x_0)) \cap M

= \pi_X(\tau^{-1}(x_0)) \cap \left\{ z \in V : \Sigma_{j=1}^{2k} x_j^2 < x_b^2 \text{ or } \Sigma_{j=1}^{2k} y_j^2 < y_b^2 \text{ for } z_j = x_j + iy_j \right\}

But if \( z = \begin{bmatrix} C \\ iC \\ 1 \end{bmatrix} \) then

$$\sum_{j=1}^{2h} (\text{Re } z_j)^2 = \frac{h}{2} \left| c_j \right|^2 = \sum_{j=1}^{2h} (\text{Im } z_j)^2$$
$\pi(\tau^{-1}(x_0)) = \left\{ \begin{bmatrix} C \\ iC \\ 1 \end{bmatrix} : C \in \mathcal{C} \text{ and } \tau C C < 1 \right\}$

**Corollary III.2** \( \tau^{-1}(x_0) \cong \pi(\tau^{-1}(x_0)) \).

**Proof:** The biholomorphism is

\[ \Sigma: \pi(\tau^{-1}(x_0)) \longrightarrow \tau^{-1}(x_0) \]

by \( \Sigma(z) = \sigma_1(z)x_0 \) \((=x_0 \in \sigma_1(z)Y)\).

B. We are now ready to prove what we call the **structure theorem** for \( \mathcal{U}_D \), namely,

**Theorem III.1** \( \mathcal{U}_D \cong Y \times M \).

In order to prove this theorem we need the following two lemmas.

**Lemma III.2** Let \( W = \{ z \in \mathcal{C}^b : z_b = -1 \} \) then

\[ W \cap M = \emptyset. \]

**Proof:** First of all, it is clear that \( M \cap M^- = \emptyset \) (where \( M^- \) corresponds to \( D^- \)). We will show that

\[ (\ast) \quad W \cap (M^- \cup M) = W \cap M^- \]

Hence \( W \cap M = \emptyset \).

Recall, we have
\[ T_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & I_{b-2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G \]

the map from \( D \) to \( D^{-} \) and

\[ \pi_X(T_0Y) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} = C_{T_0} \]

hence \( C_{T_0} \in M^- \)

Suppose \( z \in W \cap (M \cup M^-) \), i.e.,

i) \[ z = \begin{bmatrix} z_1 \\ \vdots \\ z_{2h} \\ -1 \end{bmatrix} \]

ii) \[ \sum_{j=1}^{2h} x_j^2 < 1 \quad (z_j = x_j + iy_j). \]

Then we can define a path from \( z \) to \( C_{T_0} \), namely

\[ \gamma: [0,1] \rightarrow W \cap (M \cup M^-) \]

by

\[ \gamma(t) = \begin{bmatrix} t z_1 \\ \vdots \\ t z_{2h} \\ -1 \end{bmatrix} \]
then $\gamma(t) \in W \cap (M \cup M^-)$ for all $t \in [0,1]$ since

$$\sum_{j=1}^{2h} (tx_j)^2 = t^2 \left( \sum_{j=1}^{2h} x_j^2 \right) < t^2 < 1.$$ 

Moreover,

$$\gamma(0) = c_{T_0}$$

and

$$\gamma(1) = z.$$ 

So, $W \cap (M \cup M^-)$ is path connected; hence must be contained in one of the connected sets $M$ and $M^-$. But $T_0 \in M^-$ implies

$$W \cap (M \cup M^-) \subset M^-$$

So

$$(*) W \cap (M \cup M^-) = W \cap M^-$$

Let

$$T^* = \left\{ \begin{bmatrix} I & D \\ D & 1 \end{bmatrix} : D \in \mathfrak{h} \right\}$$

It is easy to see that $T^* \subset V$. Define now

$$\sigma_2 : T^* \longrightarrow M_b(\mathbb{C})$$
by

\[
\begin{bmatrix}
1D \\
D \\
1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 - \frac{1}{2} D^t D & \frac{1}{2} D^t D & iD \\
\frac{1}{2} D^t D & I + \frac{1}{2} D^t D & iD \\
iD & iD & 1
\end{bmatrix}
\]

Then, it is easy to check that

a) \( \sigma_2(z) \in \mathcal{C} \)

b) \( \pi_X(\sigma_2(z), Y) = z \)

and that \( \sigma_2 \) is, in fact, a section of the principal bundle \( \mathcal{G} \to \mathcal{G}/\mathcal{K} \) over \( T^* \).

Define, now the map

\[
\varphi: \mathcal{C}^{2h} \to \mathcal{C}^b
\]

by

\[
\begin{bmatrix}
C \\
D
\end{bmatrix}
\to
\sigma_2\begin{bmatrix}
1D \\
D \\
1
\end{bmatrix}
\begin{bmatrix}
C \\
1C
\end{bmatrix}
= \begin{bmatrix}
C - D^t D C + iD \\
IC + ID^t DC + C \\
2i D^t DC + 1
\end{bmatrix}
\]

for \( C, D, \in \mathcal{C}^h \). By its construction it is clear that

Image \( \varphi \subset V \).

So let \( U = \varphi^{-1}(M) \) then we have

Lemma III.3 \( \varphi \) is a biholomorphism of \( U \) onto \( M \).

Proof: Clearly \( \varphi \) is holomorphic, so,

a) \( \varphi \) is injective

Suppose
\[
\forall \begin{pmatrix} C \\ D \end{pmatrix} = \forall \begin{pmatrix} C' \\ D' \end{pmatrix}
\]

then, first of all,

\[
2i \ t_{DC} + 1 = 2i \ t_{D'C'} + 1
\]

So

\[
t_{DC} = t_{D'C'}
\]

Let \( q = i^{-t_{DC}} \in C \)

then we must also have

\[
C - D t_{DC} + iD = C + qD = C' + qD'
\]

and

\[
iC - iD t_{DC} + D = i(C-qD) = i(C'-qD')
\]

But this reduces to \( n \) pairs of equations of the form

\[
c + qd = c' + qd'
\]

\[
c - qd = c' - qd'
\]

for \( c, d, c', d' \in C \) and these have the unique solution

\[
c = c'
\]

\[
d = d'
\]

for \( q \neq 0 \). But if \( q = 0 \) then \( t_{DC} = i \) hence
\[ \Psi(\begin{bmatrix} C \\ D \end{bmatrix}) = \begin{bmatrix} C \\ iC \\ -1 \end{bmatrix} \]

and from Lemma III.2

\[ \Psi(\begin{bmatrix} C \\ D \end{bmatrix}) \notin M. \]

So on U, \( \Psi \) is injective

b) \( \Psi \) is surjective.

Let \( z \in M \) then from Lemma III.2 we have

\[ z_b \neq -1 \]

moreover,

\[ \bigotimes \sum_{j=1}^{2h} z_j^2 = z_b^2 - 1 \]

So let for \( j = 1, \ldots, h \)

\[ c_j = z_j - iz_{j+h} \]

\[ d_j = -i(z_j + iz_{j+h}) \]

and

\[ C = \begin{bmatrix} c_1 \\ \vdots \\ c_h \end{bmatrix} \quad ; \quad D = \begin{bmatrix} d_1 \\ \vdots \\ d_h \end{bmatrix} \]

If we let
\[ Z_1 = \begin{bmatrix} z_1 \\ \vdots \\ z_h \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} z_{h+1} \\ \vdots \\ z_{2h} \end{bmatrix} \]

then \( \otimes \) becomes

\[ t_{Z_1}Z_1 + t_{Z_2}Z_2 = z_b^2 - 1 \]

and \( \odot \odot \) becomes

\[
\begin{align*}
C &= 1/2 \left( Z_1 - iZ_2 \right) \\
D &= -i/z_{b+1} \left( Z_1 + iZ_2 \right)
\end{align*}
\]

So, first of all, we compute

\[
^{t}DC = \frac{-i}{z(z_{b}+1)} \left( t_{Z_1}Z_1 + i t_{Z_2}Z_2 \right) (Z_1 - iZ_2)
\]

\[ = \frac{-i}{2(z_{b}+1)} (t_{Z_1}Z_1 + t_{Z_2}Z_2) = \frac{-i}{2} (z_b - 1). \]

Then we have

a.) \( C - D^{t}DC + iD = C + (i - D^{t}C)D \)

\[ = 1/2(Z_1 - iZ_2) + (i + \frac{i(-z_b - 1)(-i)}{2(z_b + 1)})(Z_1 + iZ_2) \]

\[ = Z_1 \]

b.) \( iC + iD^{t}DC + D = iC + (iD^{t}C + 1)D \)

\[ = i/2 \left( Z_1 - iZ_2 \right) + (i\left( \frac{-i}{2} \right) + 1)\left( \frac{-i}{z_b+1} \right)(Z_1 + iZ_2) \]

\[ = Z_2 \]
\[ 2i^D + 1 = \frac{2i(-i)(z_b - 1)}{2} + 1 = z_b \]

So \( \Psi([C][D]) = z \) and \( \Psi \) is surjective.

Proof of Theorem III.1

By the construction of the complex structure on \( \Psi_X \) given in Section II, it suffices to prove this theorem to show that there is a section of the principal bundle \( \mathcal{G} \rightarrow \mathcal{G}/K \) over \( M \). This is done as follows. For \( z \in M \) we have from Lemma III.3

\[ z = \Psi([C][D]) \]

for some \( C, D \in \mathbb{C}^h \); then define

\[ \sigma: M \rightarrow \mathcal{G} \]

by

\[ \sigma(z) = \sigma_2 \begin{bmatrix} iD \\ D \\ 1 \end{bmatrix} \sigma_1 \begin{bmatrix} iC \\ 1 \end{bmatrix} \]

This, then, is well-defined and holomorphic, moreover, is itself a section since \( \sigma_1 \) and \( \sigma_2 \) are sections and the projection of \( \sigma(z) \) back into \( \mathcal{G}/K \) is just \( z \).

Recall, then, that the mapping

\[ \xi: Y \times M \rightarrow \Psi_D \]
by \((y,z) \rightarrow \sigma(z)y \quad (\epsilon \sigma(z)Y)\)

is required biholomorphism.

---

**Corollary III.3** \(\pi: \Psi_n \rightarrow M\) is proper.

**Proof:** The inverse image of a compact set \(K \subset M\), is equivalent to \(Y \times K\).

---

**Corollary III.4** \(\Psi_D\) is holomorphically convex

**Proof:** Since \(M\) is Stein and \(\pi\) is proper this follows from

**Lemma III.4** If \(M\) and \(N\) are complex manifolds and

\[f: M \rightarrow N\]

is proper holomorphic surjection then if \(N\) is holomorphically convex so is \(M\).

**Proof:** Let \(K\) be a compact subset of \(M\), then \(f(K)\) is compact in \(N\) and hence so is

\[\widehat{f(K)} = \{y \in N: |g(y)| \leq \|g\|_f(K)\text{ for all }g \in \Theta(N)\}\]

But since \(f\) is proper, \(f^{-1}(\widehat{f(K)})\) is compact. Moreover, since \(f\) is a surjection

\[\|g\|_f(K) = \|g \circ f\|_K\]

for all \(g \in \Theta(N)\). Hence

\[f^{-1}(\widehat{f(K)}) = \{x: |g \circ f(x)| \leq \|g \circ f\|_K\text{ for all }g \in \Theta(N)\}\]

So

\[f^{-1}(\widehat{f(K)}) \supset \hat{K}\]

and so \(\hat{K}\) is compact and \(M\) is holomorphically convex.
Corollary III.5 If $\text{dim} Y = q$ then $\psi_D$ is q-complete (Andreotti-Grauert [1])

Proof: Recall that q-completeness means there exists a $C^\infty$ function

$$\varphi: \psi_D \rightarrow \mathbb{R}$$

so that the Levi-form

$$L(\varphi) = \begin{bmatrix}
\frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_j}
\end{bmatrix}$$

has at most q non-positive eigenvalues. But now $M$ is Stein so there is a $C^\infty$-function

$$\varphi: M \rightarrow \mathbb{R}$$

So that $L(\varphi)$ is positive definite. If we extend $\varphi$ to $Y \times M$ by

$$\varphi(y, z) = \varphi(z)$$

then clearly $Y \times M$ is q-complete.
Section IV

COHERENT ANALYTIC SHEAVES ON $\mathcal{U}_D$

In this section we derive formulas relating the cohomology on $\mathcal{U}_D$ with coefficients in a coherent analytic sheaf to cohomology on $M$ and $D$.

A. If we have a coherent analytic sheaf, $\mathcal{F}$, over $\mathcal{U}_D$, we can induce analytic sheaves over $M$ and $D$ by taking direct images, i.e.,

\[
\begin{array}{c}
\pi^n_q \longrightarrow \mathcal{F} \longrightarrow \tau^n_q \\
\downarrow \quad \downarrow \quad \downarrow \\
M \quad \mathcal{U}_D \quad D
\end{array}
\]

where $\pi^n_q (\tau^n_q)$ is the $q$-direct image of $\mathcal{F}$ induced by $\pi(\tau)$. Recalling Gruaert's theorem on direct images (Grauert [4]), since $\pi$ is proper (Corollary III.3), $\pi^n_q$ is a coherent analytic sheaf on $M$. However, $\tau$ is not proper so $\tau^n_q$ is not necessarily coherent, but

Theorem IV.1 If $\mathcal{F}$ is a coherent analytic sheaf over $\mathcal{U}_D$ then for $q > 0$

\[\tau^n_q \mathcal{F} = 0.\]

Proof: Since $\mathcal{U}_D$ is homogeneous, it suffices to show that for $x_0 \in Y \subset D$

\[\tau^n_q x_0 = 0.\]
but

$$\tau_q \xi_0 = \text{dir} \lim_{U \text{ open}_x} H^q(\tau^{-1}(U), \mathfrak{g})$$

So the theorem follows from Cartan's theorem B and

**Lemma IV.1** $x_0 \in D$ has a fundamental system of neighborhoods $\{U_\alpha\}$ so that $\tau^{-1}(U_\alpha)$ is Stein.

**Proof:** a) Let $U$ be a neighborhood of $x_0$, chosen so that

1. $U$ is Stein
2. there is a section over $U$ of the principal bundle, $\widetilde{G} \to X$; call it,

$$\sigma: U \to \widetilde{G}$$

then since $\sigma$ is a section

$$\sigma(x) x_0 = x$$

Let

$$T = \left\{ \begin{bmatrix} c \\ ic \\ 1 \end{bmatrix} : c \in c^h \right\} = \pi^X(\tau^{-1}(x_0))$$

define
\( \Phi^* : U \times T \to \tau_X^{-1}(U) \)

by

\[(x, z) \to \Phi(x) \sigma_1(z)x_0 \ (\text{for } \sigma \in \Phi(x) \sigma_1(z)Y) \]

then \( \Phi^* \) is a biholomorphism, i.e.,

i) \( \Phi^* \) is injective.

Suppose \( \Phi(x^1) \sigma_1(z^1)x_0 = \Phi(x) \sigma_1(z)x_0 \in gY \)

for some \( g \in \mathcal{G} \) then

\[
\Phi(x^1) \sigma_1(z^1)x_2 = \Phi(x^1)x_0 = x^1
\]

""

\[
\Phi(x) \sigma_1(z)x_0 = \Phi(x)x_0 = x.
\]

So \( \sigma_1(z^1)x_0 = \sigma_1(z)x_0 \in \sigma(x)^{-1}gY \)

but this means \( \sigma_1(z^1)x_0 \) and \( \sigma_1(z)x_0 \) lie in the same manifold over \( V \), namely \( \Phi(x)^{-1}gY \) hence

\[ z^1 = \pi_X(\sigma_1(z^1)Y) = \pi_X(\sigma_1(z)Y) = z \]

ii) \( \Phi^* \) is surjective

Let \( x \in \tau_X^{-1}(U) \) then \( x \in gY \) for some \( g \in \mathcal{G} \) and \( x \in U \). We have then

\[ x_0 \in \Phi(x)^{-1}gY \]

so let
\[ z = \pi_X(c^*(x)^{-1}g_Y) \]

then \( z \in T \), since \( T \) parametrize all manifolds, \( g_Y \), containing \( x_0 \). So

\[ \sigma_1(z) Y = \sigma(x)^{-1}g_Y \]

i.e.,

\[ \sigma(x) \sigma_1(z) Y = g_Y \]

and \( \sigma(x) \sigma_1(z) x_0 = \sigma(x) x_0 = x \)

So

\[ \sigma(x, z) = x \in g_Y. \]

Now \( U \) is Stein, and \( T \) is Stein, so \( U \times T \) is Stein hence \( \tau_X^{-1}(U) \) is Stein.

b) We have from Corollary III.4.

\[ \psi_D = \pi^{-1}(M) \text{ is holomorphically convex. So} \]

\[ \tau^{-1}(U) = \psi_D \cap \tau_X^{-1}(U) \]

is holomorphically convex. But then \( \tau^{-1}(U) \) is a holomorphically convex open subset of the Stein manifold, \( \tau_X^{-1}(U) \), so is itself Stein.
Now let \( \{U'_{\alpha}\} \) be a fundamental system of Stein neighborhoods of \( x_0 \) in \( D \) and \( U \) be a neighborhood satisfying 1. and 2. above, then

\[
U_{\alpha} = U'_{\alpha} \cap U
\]

also satisfies 1. and 2. so \( \iota^{-1}(U_{\alpha}) \) is Stein. Furthermore \( \{U_{\alpha}\} \) is a fundamental neighborhood system of \( x_0 \) in \( D \).

\[\textcircled{8}\]

B. The relationship of cohomology on \( \mathcal{U}_D \) to cohomology on \( D \) and \( M \) is given by

**Theorem IV.2** If \( \mathfrak{u} \) is a coherent analytic sheaf on \( \mathcal{U}_D \) then for \( r \geq 0 \)

\[
H^0(M, u^r \mathfrak{u}) = H^r(\mathcal{U}_D, \mathfrak{u}) = H^r(D, \gamma_0 \mathfrak{u}).
\]

**Proof:** Associated to a map

\[
f: A \to B
\]

and a sheaf \( \mathfrak{u} - \to A \) we have the Leray spectral sequence \( \{E^{pq}_r, d_r\} \) where

\[
E^{pq}_2 = H^p(B, f_q \mathfrak{u})
\]

and

\[
E^{\infty}_\infty = H^r(A, \gamma_0 \mathfrak{u}).
\]

Moreover,

\[
H^p(B, f_q \mathfrak{u}) = H^{p+q}(A, \mathfrak{u})
\]

So consider

a) the Leray spectral sequence associated to

\[ \pi \quad \Downarrow \quad \Psi_D \quad \pi \quad \rightarrow \quad M \]

Since M is a Stein manifold and \( \pi_q \) is coherent it follows from Cartan's theorem B that for \( q \neq 0 \)

\[ E_2^{pq} = H^p(M, \pi_q \mathfrak{F}) = 0 \]

so we have the following diagram

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & \cdots \\
0 & E_2^{00} & 0 & \cdots \\
0 & d_1 & E_2^{10} & 0 & \cdots \\
0 & d_2 & E_2^{20} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & E_3^{30} & 0 & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

So since

\[ E_3^{pq} = \ker \left( E_2^{pq} \xrightarrow{d_2} E_2^{p+2, q-1} \right) / \text{im} \left( E_2^{p-2, q+1} \xrightarrow{d_2} E_2^{pq} \right) \]
we have for all $p, q$

$$E_3^{pq} = E_2^{pq}$$

Similarly for all $p, q$ and $r \geq 2$

$$E_r^{pq} = E_2^{pq}$$

So for all $p, q$

$$E_\infty^{pq} = \text{dir lim } E_r^{pq} = E_2^{pq}$$

i.e., the spectral sequence degenerates in the second place.

So we have that for $r \geq 0$

$$H^r(\mathfrak{g}_D, \mathfrak{g}) = \bigoplus_{p+q=r} E_2^{pq} = \bigoplus_{p+q=r} H^p(M, \pi_q \mathfrak{g})$$

$$= H^0(M, \pi_r \mathfrak{g})$$

(b) Moreover, it follows from Theorem IV.1 that for $q > 0$

$$H^p(D, \tau_q \mathfrak{g}) = 0$$

So considering the Leray spectral sequence for
we obtain, in a manner analogous to a), that for \( r \geq 0 \)

\[ H^r(\psi_D, \pi) = H^r(D, r_0 \pi). \]

Moreover, because \( \psi_D \) is q-complete (Corollary III.5) we have from the Andreotti-Grauert extension of Theorem B ([1] p. 250).

**Theorem IV.3** If \( r > \dim Y \) then

\[ H^r(\psi_D, \pi) = 0. \]
Section V

HOMOGENEOUS LINE BUNDLES ON D

The work which motivated this paper is a study by P.A. Griffiths of homogeneous line bundles on the period matrix domain, $D$, (Griffiths [5]). If we have a homogeneous line bundle, $E$ over $D$; then $\mathcal{O}(E)$, the sheaf of holomorphic sections of $E$ is a coherent (in fact, locally free) analytic sheaf. Schmid has shown that for $p \neq 0$ or $q = \dim Y$.

$$H^p(D, \mathcal{O}(E)) = 0$$

Griffiths also introduces the diagram,

$$M \xleftarrow{\pi} \Psi_D \xrightarrow{\tau} D$$

and points out that the sheaf, $\mathcal{O}(E)$, over $D$ induces a coherent analytic sheaf, $\tilde{\mathcal{O}}$, over $M$. In fact, $\tilde{\mathcal{O}} = \pi_q \tau^* \mathcal{O}(E)$, where $\tau^* \mathcal{O}(E)$ is the inverse image of $\mathcal{O}(E)$ by $\tau$ (a coherent sheaf on $\Psi_D$, in fact, it is the sheaf of sections of the pullback of $E$ over $\Psi_D$) i.e.,

$$\pi_q \tau^* \mathcal{O}(E) \leftarrow \tau^* \mathcal{O}(E) \leftarrow \mathcal{O}(E)$$

This induces a map on cohomology

$$H^q(D, \mathcal{O}(E)) \longrightarrow H^0(M, \tilde{\mathcal{O}}).$$
The question he asks is whether or not these two vector spaces are isomorphic. If they were then the cohomology of $E$ could be studied by considering the sections of $\pi$ (which Griffiths states is actually the sheaf of sections of a vector bundle on $M$).

We have shown in Theorem IV.2 that

$$H^0(M, \mathfrak{a}) = H^0(D, \tau_0^{-1*}\mathfrak{a}(E))$$

So we have reduced the question to that of considering the relation between $\mathfrak{a}(E)$ and $\tau_0^{-1*}\mathfrak{a}(E)$. In general we can say the following; if $\mathcal{F}$ is an analytic sheaf on $D$ then for $x \in D$

$$\tau_0^{-1*}\mathcal{F}_x \cong \mathcal{F}_x \otimes H^0(\tau^{-1}(x), \mathcal{O}_D)$$

where $\mathcal{O}_D$ is the sheaf of germs of holomorphic functions on $\psi_D$. So $\tau_0^{-1*} \not\cong \mathcal{O}_D$. But we have the natural injection.

$$0 \rightarrow \mathcal{O}_x \rightarrow H^0(\tau^{-1}(x), \mathcal{O}_D).$$

Hence, letting $Q_x$ be the quotient, the exact sequence

$$0 \rightarrow \mathfrak{a}(E)_x \rightarrow \mathfrak{a}(E)_x \otimes H^0(\tau^{-1}(x), \mathcal{O}_D)$$

$$\rightarrow \mathfrak{a}(E)_x \otimes Q_x \rightarrow 0$$

This induces an exact sequence of sheaves

$$0 \rightarrow \mathfrak{a}(E) \rightarrow \tau_0^{-1*}\mathfrak{a}(E) \rightarrow \mathcal{O} \rightarrow 0$$
And from the long exact cohomology sequence we obtain

\[ 0 \rightarrow H^{q-1}(D, \tau_0 \tau^* \mathcal{O}(E)) \rightarrow H^{q-1}(D, \mathcal{O}) \rightarrow H^q(D, \mathcal{O}(E)) \]

\[ \quad \rightarrow H^q(D, \tau_0 \tau^* \mathcal{O}(E)) \rightarrow H^q(D, \mathcal{O}) \rightarrow 0 \]

So the vector spaces

\[ H^{q-1}(D, \mathcal{O}) / H^{q-1}(D, \tau_0 \tau^* \mathcal{O}(E)) \]

and

\[ H^q(D, \mathcal{O}) \]

serve as obstructions to an isomorphism between \( H^q(D, \mathcal{O}(E)) \) and \( H^0(M, \mathcal{O}) \). This then provides a partial answer to Griffiths' question.
Bibliography


