PICKEL, Paul Frederick, 1942-
ON THE ISOMORPHISM PROBLEM FOR FINITELY
GENERATED TORSION FREE NILPOTENT GROUPS.

Rice University, Ph.D., 1970
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan
RICE UNIVERSITY

ON THE ISOMORPHISM PROBLEM FOR
FINITELY GENERATED TORSION FREE NILPOTENT GROUPS

by
Paul Frederick Pickel

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's signature:

Houston, Texas
May, 1970
Table of Contents

Introduction

Notations

I. Finiteness of the class number
   1. Completions 1
   2. Lie algebras of torsion free finitely generated nilpotent groups 6
   3. Algebraic groups and automorphism groups 13
   4. Manna from Borel 15
   5. \(\mathbb{Z}_p\)-Groups 19
   6. Proof of the main theorem 27

II. Examples
   7. Preliminaries 33
   8. One-relator nilpotent groups 38
   9. H-Groups and R-Groups 42
   10. Stick-up groups 46

Bibliography 48
Introduction

The major result of this thesis arises in the attempt to solve the isomorphism problem for torsion free finitely generated nilpotent groups by means of their finite homomorphic images. This approach is suggested by the success of the techniques in the solution of the word problem and conjugacy problem for these groups. In all cases the procedure is quite similar. On the one hand, one has an effective procedure for listing out all words which belong to the relation subgroup (respectively, all words conjugate to a given word, respectively, all presentations which define groups isomorphic to that defined by the given presentation), so that if a word does belong to the relation subgroup (respectively, a word is conjugate the the given word, respectively, a presentation defines a group isomorphic to the given one), one may so decide in a finite number of steps by comparison with the list as it is generated. What is needed is an effective procedure which will reach a decision in a finite number of steps if the respective statement above does not hold. Since torsion free finitely generated nilpotent groups have the property that for an element g≠1 of the group, there is a finite quotient of the group in which this element is non-trivial as well as the property that given two non-conjugate elements, there is a finite quotient in which the two elements are non conjugate, a check against an effective list of all homomorphisms from the given group to finite groups will fulfill the above need.
What one would hope for is that the set of isomorphism classes of finite quotients \( \pi(G) \) of a torsion free finitely generated nilpotent group \( G \), would determine \( G \) up to isomorphism, that is \( \pi(G) = \pi(H) \) would imply \( G \) isomorphic with \( H \). Unhappily this is not the case. Remeslemnikov ([15]) has given an example of two torsion free nilpotent groups of class four on two generators which have the same finite images, and shortly thereafter Higman ([10]) gave examples which were worse in the sense that the two groups were not commensurable. (Groups \( G \) and \( H \) are said to be commensurable if there are subgroups \( G_1 \) and \( H_1 \) of finite index in \( G \) and \( H \) respectively such that \( G_1 \) is isomorphic to \( H_1 \)). A. Borel then, in a conversation with G. Baumslag and later in a letter to him ([5]), showed that the latter situation was not too bad in the sense that the torsion free finitely generated nilpotent groups having the same finite quotients as a given such group may lie in only finitely many commensurability classes. After this result and the examples of Higman, there was a short lived hope that commensurable groups of this type, having isomorphic finite quotients, must be isomorphic, which hope was quashed by the observation that, in fact, Remeslemnikov's original examples were commensurable. The study of generalizations of these examples (the R-groups and H-groups of section 9) brought out the close connection of the isomorphism problem for these groups with the problem of equivalence of quadratic forms and led to the formulation
and, in part, the method of proof of our result: the number of isomorphism classes of torsion free finitely generated nilpotent groups having the same finite homomorphic images as a given such group $G$ (the "class number" of $G$) is finite.

This result reduces the isomorphism problem to that of finding an algorithm which will list out precisely one presentation representing each isomorphism class of groups which have finite quotients isomorphic with those of the group defined by a given presentation, all groups in question being finitely generated torsion free nilpotent. If one were given such an algorithm, which must be finite by the above result, one could solve the isomorphism problem for finitely generated torsion free nilpotent groups as follows: Given two presentations $P$ and $Q$, apply the algorithm to $P$ to obtain $P = P_1, P_2, \ldots, P_n$. Now cyclically list out presentations isomorphic to the $P_1: P_{11}, P_{21}, \ldots, P_{n1}, P_{12}, \ldots, P_{n2}, P_{13} \ldots$. If $Q$ and $P$ have isomorphic finite quotients we may so decide after a finite number of steps by comparing $Q$ with this sequence and if not we may so decide in a finite number of steps by comparing finite quotients.

The examples of part II furnish an indication of the difficulty of the problem, as a solution for the isomorphism problem for finitely generated torsion free nilpotent groups would essentially yield solutions to many, as yet unsolved, diophantine problems. In particular, there seems to be a close relation between the isomorphism problem for $R$-groups
(section 9) and a weak form of Hilbert's 10th problem, which asks for an algorithm which will determine whether a given integral binary form represents a given integer. The recent demonstration of the non existence of an algorithm for solving an arbitrary diophantine equation ("strong" Hilbert's 10th problem) emphasizes the difficulties.

The proof of the main theorem is by translation and reduction of the group theoretic problem to a known result in the arithmetic theory of algebraic groups by means of the lie algebras of the groups, a technique first used in [1]. The proof is probably best understood by reading sections 4 and 6 and referring to the previous sections as needed. First it is shown that the isomorphism classes of groups having the same finite images as a given group G and commensurable with G are in one-to-one correspondence with certain double cosets, using the results of sections 1 and 4. Then using sections 2 and 3, these double cosets are shown to correspond for lattice nilpotent groups, to double cosets of an algebraic group, which are finite in number. Finally, using section 5, it is shown that this is sufficient as the result is unaffected if a group is replaced by a subgroup of finite index.

I would like to thank Gilbert Baumslag for his many words of encouragement and helpful suggestions and, in general, for believing everything I said. I would also like to thank John Ledlie for freely giving his time,
knowledge of the literature and most of all for his skepticism, which forced me to think through my wild ideas. Last, but not least, I would like to thank A. Borel for proving the theorems that made it all possible and Kathy for typing the manuscript.
I. **Finiteness of the class number**

1. **Completions**

We define in this section the p-adic completion of a finitely generated torsion free nilpotent group as well as its Mal'cev completion, which we will term "rationalization" to avoid confusion with the p-adic completion. All groups denoted by roman capitals will be finitely generated torsion free nilpotent groups except as otherwise indicated.

**Lemma 1.1** The following topologies on G are the same:

(a) The topology whose neighborhood basis at the identity is the set of all normal subgroups of finite p-power index.

(b) The topology generated by the neighborhood basis at the identity given by the subgroups \( G^p_i = \text{gp}\{x^p_i \mid x \in G\}. \)

**Proof:** The groups \( G^p_i \) are all normal and of p-power index and conversely if N is a normal subgroup of index \( p^k \) in G then N contains \( G^p_k \) so that the topologies are the same.

We will call this topology, which is Hausdorff by virtue of the fact that G is residually a finite p-group, the p-adic topology. We will denote the completion of G in this topology by \( \hat{G} \).

Recall that every torsion free nilpotent group may be embedded in a divisible nilpotent group of the same class. The smallest such divisible group is called the Mal'cev completion of the given group, is unique up to isomorphism.
and satisfies the additional property that given any element of the Mal'cev completion, some integral power of that element lies in the original group. For a torsion free finitely generated nilpotent group $G$, we will term the Mal'cev completion of $G$ the rationalization of $G$ and denote it by $QG$. For such groups $G$, we will denote the Mal'cev completion of $Z_pG$ by $Q_pG$. $G$ is canonically included in each of $QG$, $Z_pG$, and $Q_pG$ while each of $Z_pG$ and $QG$ is canonically included in $Q_pG$, the last because $Q_pG$ is a divisible group containing an isomorphic copy of $G$. We will identify any of these groups with their various canonical images.

By $x^\lambda$, $x \in Z_pG$ and $\lambda \in Z_p$, we will mean the element $\lim_{n \to \infty} x^{a_n}$ in $Z_pG$, where $\{a_n\}$ is a Cauchy sequence of integers converging to $\lambda$ in $Z_p$. $\{x^{a_n}\}$ is thus a Cauchy sequence in $G$ and the element $x^\lambda$ is uniquely determined by $x$ and $\lambda$. For $x \in Q_pG$, $\lambda \in Q_p$, we may define $x^\lambda$ in the following manner: there is an integer $n$ such that $x^n \in Z_pG$ and an integer $m$ such that $m\lambda \in Z_p$ so that $x^\lambda = ((x^n)^{m\lambda})^{1/mn}$ is a well defined element of $Q_pG$.

Recall that a normal basis for $G$ is an ordered set $(x_1, \ldots, x_m)$ of elements of $G$ such that $x_1$ is central modulo $gp[x_{i+1}, \ldots, x_m]$ and each element of $G$ may be written uniquely as $x_1^{n_1} \cdots x_m^{n_m}$ for integers $n_i$. Such normal bases exist for any torsion free finitely generated nilpotent group. We generalize this in
Lemma 1.2: Let \((x_1, \ldots, x_m)\) be a normal basis for a torsion free finitely generated nilpotent group \(G\), then

(a) every element of \(\mathbb{Z}_p G\) may be written uniquely as \(x_1^{r_1} \ldots x_m^{r_m}\) with \(r_i \in \mathbb{Q}_p\).

(b) every element of \(\mathbb{Q}G\) may be written uniquely as \(x_1^{r_1} \ldots x_m^{r_m}\) with \(r_i \in \mathbb{Q}\).

(c) every element of \(\mathbb{Q}_p G\) may be written uniquely as \(x_1^{r_1} \ldots x_m^{r_m}\) with \(r_i \in \mathbb{Q}_p\).

Proof: We first prove the

Sublemma: If \(G\) is a Hausdorff topological group with a neighborhood basis for the identity consisting of normal subgroups totally ordered by inclusion and \(N\) is a closed normal subgroup, then \(\hat{G}/\hat{N} \cong (G/N)^\wedge\), where \(^\wedge\) denotes completion in the given topology or the topology it induces on \(N\) or \(G/N\).

Proof of Sublemma: Let \(\mathcal{O}\) be the neighborhood basis of the hypothesis. Then \(\hat{G}\) is the subgroup of \(\prod_{0 \in \mathcal{O}} G/0\) consisting of elements \((x_0)_{0 \in \mathcal{O}}\) such that if \(0 \supset 0'\) then \(x_0/0 = x_0'.\)

We map \(\hat{G}\) to \((G/N)^\wedge\), which is the group of all elements of \(\prod_{0 \in \mathcal{O}} G/NO\) satisfying a similar compatibility relation, by \(\phi: (x_0)_{0 \in \mathcal{O}} \rightarrow (x_0 NO)_{0 \in \mathcal{O}}\). This is well defined for if \(NO = NO'\) then \(NO \supset 0'\) and \(NO \supset 0\) and say \(0 \supset 0'\) so that as \(x_0/0 = x_0\) by compatibility in \(\hat{G}\) we have \(x_0 NO = x_0 NO'.\)

The kernel of \(\phi\) is the group of elements \((x_0)_{0 \in \mathcal{O}}\) such that \(x_0 \in NO/0\) for all \(0\). Thus \((x_0) \in \prod_{0 \in \mathcal{O}} NO/0 = \prod_{0 \in \mathcal{O}} N/N\cap NO\).
To show compatibility in the latter product, suppose 
\( N \cap 0 \neq N \cap 0' \); then \( 0 \neq 0' \). Given \( x_0 \in NO/0 \) and \( x_0' \), in we apply the isomorphism \( NO/0 \cong N/N_0 \) and \( NO'/0' \cong 
N/N \cap 0' \) which are given as follows: pick representatives 
\( \tilde{x}_0 \) and \( \tilde{x}_0' \), of \( x_0 \) and \( x_0' \), respectively which lie in \( N \) then consider them in \( N/N_0 \) and \( N/N_0' \) respectively. By compati-
bility in \( G \), we have \( \tilde{x}_0^{-1} \tilde{x}_0' \in 0 \cap N \) so that we have 
compatibility of \( \ker \varphi \subseteq \pi_{0 \in 0} N/N \cap 0 \) so that \( \ker \varphi \cong \hat{N} \). 
Thus \( \hat{G}/\hat{N} \cong (G/N)^\wedge \).

The proof of the lemma will be by induction on 
the torsion free rank \( m \) of \( G \), the case \( m=1 \) being obvious.
For arbitrary \( m \), we have \( \hat{G}/(x_m)^\wedge \cong (G/(x_m))^\wedge \) by the sub-
lemma. As \( G/(x_m) \) has torsion free rank \( <m \), and in fact 
the image \( (\bar{x}_1, \ldots, \bar{x}_{m-1}) \) of \( (x_1, \ldots, x_{m-1}) \) is a normal 
basis for \( G/(x_m) \), the induction hypothesis implies that 
each element of \( \hat{G}/(x_m)^\wedge \) may be uniquely written as 
\( \bar{x}_1^{r_1}, \ldots, \bar{x}_{m-1}^{r_{m-1}} \) for \( r_i \in \mathbb{Z}_p \). Thus each element of \( \hat{G}/(x_m)^\wedge \) 
has a unique coset representative of the form \( x_1^{r_1} \ldots x_{m-1}^{r_{m-1}} \) 
and as each element of \( (x_m)^\wedge \) may be written \( x_m^{r_m} \) the 
result follows for part (a).

For parts (b) and (c), let \( (x_m)^Q \) (resp. \( (x_m)^P \)) denote 
the set of all rational (p-adic rational) powers of \( x_m \) 
in \( QG \) (resp. \( Q_p G \)). As \( x_m \) is central in \( G \), \( (x_m)^Q \) (resp. \( (x_m)^P \)) 
is central in \( QG \) (resp. \( Q_p G \)). Consider \( QG/(x_m)^Q \) (resp. \( Q_p G/ 
(x_m)^P \)). For each element \( x \) in \( QG/(x_m)^Q \) (resp \( Q_p G/(x_m)^P \)), 
there is an integer \( n \) such that \( x^n \in G/(x_m) \) (resp \( Z_p G/(x_m)^\wedge \)) 
and \( QG/(x_m)^Q \) (resp \( Q_p G/(x_m)^P \)) is divisible so it must 
be the rationalization of \( G/(x_m) \) (resp \( Z_p G/(x_m)^\wedge \)).
Applying the induction hypothesis we have that each element of $QG/(x_m)^Q$ (resp. $P G/(x_m)^P$) can be written uniquely in the form $\bar{x}_1^r \ldots \bar{x}_{m-1}^r$ with $r \in Q$ (resp. $P$). Since each element of $(x_m)^Q$ (resp. $x_m^p$) is of the form $r_m$, $r_m \in Q$ (resp. $P$), the result follows for parts (b) and (c).

**Lemma 1.3:** Any isomorphism of $Z_G$ to $Z_H$ is continuous in the $p$-adic topology so that isomorphisms $Z_G \to Z_H$ (resp $Q_G \to Q_H$) are $Z$ (resp $Q$) isomorphisms in the sense that $\varphi(x^\lambda) = (\varphi(x))^\lambda$ for $\lambda \in Z$ (resp $Q$).

**Proof:** For $Z$ a neighborhood basis of the identity is given by the kernels of the homomorphisms $Z_G \to G/P_i$, which exist as $Z_G$ is the inverse limit of the groups $G/P_i$. The kernels of these maps are in fact the groups $(Z_G)^i$ which must be taken by any isomorphism onto the groups $(Z_H)^i$ so the isomorphism is obviously continuous. The second part of the statement follows immediately from the definition $\varphi(x^\lambda) = \varphi(\lim_n x_n^\lambda) = \lim_n \varphi(x_n^\lambda) = \lim_n \varphi(x_n^\lambda) = \varphi(x)^\lambda$.

**Lemma 1.4:** Any isomorphism of $G$ onto $H$ extends uniquely to an isomorphism of $Z_G$ onto $Z_H$, $QG$ onto $QH$ or $Q_G$ onto $Q_H$, and an isomorphism of $QG$ onto $QH$ or $Z_G$ onto $Z_H$ extends uniquely to an isomorphism of $Q_G$ onto $Q_H$. 
Proof: By lemma 1.3, any isomorphism of $QG$, $Z_pG$, or $Q_pG$ is respectively a $Q$-, $Z_p$- or $Q_p$-isomorphism, so that by lemma 1.2 any such isomorphism is uniquely determined by the images of a normal basis $(x_1, \ldots, x_m)$ of $G$, i.e.,

$$\varphi(x_1^{\lambda_1} \cdots x_m^{\lambda_m}) = \varphi(x_1)^{\lambda_1} \cdots \varphi(x_m)^{\lambda_m}.$$ 

As we may take any $Q$-, $Z_p$- or $Q_p$- power respectively, the isomorphisms in question do extend.

Lemma 1.5: If $\varphi$ maps $QG$ isomorphically onto $QH$, then the extension $\varphi_p$ of $\varphi$ to $Q_pG$ sends $Z_pG \cong Z_pH$ for all but a finite number of primes $p$.

Proof: $\varphi(G)$ is a subgroup of $QH$ commensurable with $H$ so that $\varphi(G) \cap H$ is of finite index in both $\varphi(G)$ and $H$, so we may as well assume $\varphi(G) \subseteq H$ as a subgroup of finite index. As there is an integer $n$ for each $x$ in $H \setminus G$ such that $x^n \in \varphi(G)$, there is some $n$ such that $H^n \subseteq \varphi(G)$.

On completing at $p$ we will have $(Z_pH)^n_p \subseteq \varphi_p(Z_pG) \subseteq Z_pH$, but if $p \nmid n$, $n$ is a unit in $Z_p$ so that $(Z_pH)^n_p = Z_pH$ and thus $Z_pH = \varphi_p(Z_pG)$ for all primes except for those dividing $n$, which is a finite set.

2. Lie algebras of torsion free finitely generated nilpotent groups

We recall first some facts about torsion free finitely
generated nilpotent groups and related lie algebras and
divisible groups. Again all groups denoted by roman
capitals will be torsion free finitely generated nilpotent.
We will use the notation $F(G)$ to indicate the group ring
of the group $G$ over the ring $F$. We have the following

**Theorem A:** (S.A. Jennings [11]) If $G$ is a finitely
generated torsion free nilpotent group and if $A(F,G) = A$
is the augmentation ideal of the group ring $F(G)$ of $G$
over a field $F$, then $\prod_{n=1}^{\infty} A^n = 0$. (For a short proof
see Formanek [8]).

This fact enables us to form the completion $\overline{Q(G)}$
of the rational group ring $Q(G)$ in the $A$-adic topology.
For an element $a$ of $\overline{A}$ we define $\exp(a) = 1 + a + a^2/2! + \ldots$
an element of $\overline{Q(G)}$. Also for $b \in 1 + A$ we define $\log (b) =
(b-1) - (b-1)^2/2 + (b-1)^3/3 - \ldots$ also an element of $\overline{Q(G)}$.
We have as usual $\log(\exp(a)) = a$ and $\exp(\log(b)) = b$
so that $\exp$ is a one-to-one map of $A$ onto $1 + A$ with
inverse given by $\log$. We then have.

**Theorem B:** ([2], Theorem 4.4) The $Q$ subspace $\mathcal{L}$ of $\overline{Q(G)}$
spanned by $\log G$ is a vector space of dimension $r$, the
torsion free rank of $G$. In addition, $\mathcal{L}$ is a nilpotent
lie subalgebra of the commutation lie algebra on $\overline{Q(G)}$. 
Recall that by using the Baker-Campbell-Hausdorff formula, we may define a multiplication \(*\) in any nilpotent lie algebra \(\wedge\) over a field \(F\) of characteristic 0 in such a way that \((\wedge,\ast)\) is a nilpotent \(F\)-group, where by \(F\)-group we mean that \((\wedge,\ast)\) admits parametric exponents from \(F\). If \([a,b]\) denotes the commutator in the group \((\wedge,\ast)\) and \((a,b)\) denotes the lie product in \(\wedge\), we have for any commutator

\[ [g_1, \ldots, g_n] = (g_1, \ldots, g_n) + \text{higher order lie products in which each } g_i \text{ occurs at least once.} \]

The product \(*\) in \(\wedge\) above is such that, for \(a, b \in \log G\), \(\exp(a\ast b) = \exp(a) \cdot \exp(b)\) so that \(\exp\) gives an isomorphism of \(G\) with \((\log G, \ast)\) and in fact an isomorphism of \((\wedge, \ast)\) with the rationalization \(QG\) of \(G\), and automorphisms of \(G\) are in one-to-one correspondence with automorphism of \(\wedge\) stabilizing \(\log G\). ([2], Chapter 4)

**Lemma 2.1:** An isomorphic set map \(\varphi: \wedge \to \Gamma\) of nilpotent lie algebras over a field \(F\) of characteristic 0 is an isomorphism of \(\wedge\) onto \(\Gamma\) as lie algebras if and only if it is an isomorphism of \((\wedge, \ast)\) onto \((\Gamma, \ast)\) as \(F\)-groups.

**Proof:** \(\Rightarrow\): This direction follows directly from the fact that the Baker-Campbell-Hausdorff formula is a rational linear combination of lie products and the parametric \(F\)
exponents are given by scalar multiplication, i.e., 
$\lambda \in \mathbb{R}$.

$\Leftrightarrow$: As $\varphi$ preserves parametric F exponents, $\varphi$ preserves
scalar multiplication by $\mathbb{F}$, so we have only to show that
$\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(a, b) = (\varphi(a), \varphi(b))$ for any
$a, b$ in $\Lambda$. This we do for $a \in \Lambda$ and $b \in Z_{i}(\Lambda)$, the i-th
center of $\Lambda$, by induction on i. By the remarks prior
to the lemma we have $b \in Z_{i}(\Lambda)$ implies that $b \in Z_{i}(\Lambda, \star)$,
for if $b$ occurs in a lie product containing $i + 1$ or a
greater number of terms, that lie product must be 0 so
that any commutator in $(\Lambda, \star)$ of length greater than $i$
containing $b$ must also be 0, which is the identity of
$(\Lambda, \star)$. Conversely $b \in Z_{i}(\Lambda, \star)$ implies $b \in Z_{i}(\Lambda)$ by an
induction on i using the same argument.

For $i = 1$, $b \in Z(\Lambda)$ so that $a \star b = a + b$. As $\varphi$ is an
isomorphism of $(\Lambda, \star)$, $\varphi(a \star b) = \varphi(a) \star \varphi(b)$. $b \in Z(\Lambda)$
implies $b \in Z(\Lambda, \star)$ so that $\varphi(b) \in Z(\Gamma, \star)$ and thus $\varphi(b) \in
Z(\Gamma)$ and $\varphi(a) \star \varphi(b) = \varphi(a) + \varphi(b)$. This yields
$\varphi(a + b) = \varphi(a) + \varphi(b)$. Also $(a, b) = 0$ and $(\varphi(a),
\varphi(b)) = 0$ so that $\varphi(a, b) = (\varphi(a), \varphi(b))$.

Now we assume the statement is true for $i < n$ and
prove it for $i = n$, so $a \in \Lambda$ and $b \in Z_{n}(\Lambda)$. We have then
$a \star b = a + b + 1/2(a, b) + ... + n$-fold lie products of $a$ and $b$.
$\varphi(a \star b) = \varphi(a + b + 1/2(a, b) + ...)$
$= \varphi(a + b) + 1/2 \varphi(a, b) + ... + \varphi(\text{n-fold lie products})$. 

by the induction hypothesis since all lie products containing \( b \) are in \( Z_{n-1}(\wedge) \). Any \( n \)-fold commutator containing an element of \( Z_n(\wedge) \) is equal to the corresponding \( n \)-fold lie product as the lie products of higher order must all be trivial. Thus \([a, b, a, \ldots] = (a, b, a, \ldots)\) and as \( \varphi \) is a group homomorphism and \( \varphi(b) \in Z_n(\Gamma) \) we have

\[
\varphi(a, b, a, \ldots) = \varphi([a, b, a, \ldots]) \\
= [\varphi(a), \varphi(b), \varphi(a), \ldots] \\
= (\varphi(a), \varphi(b), \varphi(a), \ldots)
\]

so we see that \( \varphi \) preserves \( n \)-fold lie products in \( a \) and \( b \). Suppose we have shown that \( \varphi \) preserves \( j \)-fold lie products for \( j > i \) in \( a \) and \( b \) and let \((a, b, a, \ldots)\) be any \( i \)-fold lie product. We then have

\[
[a, b, a, \ldots] = (a, b, a, \ldots) + \text{higher order lie products in } a \text{ and } b
\]

so that

\[
\varphi[a, b, a, \ldots] = \varphi(a, b, a, \ldots) + \varphi's \text{ of higher lie products.}
\]

But we have

\[
\varphi[a, b, a, \ldots] = [\varphi(a), \varphi(b), \varphi(a), \ldots] \\
= (\varphi(a), \varphi(b), \varphi(a), \ldots) + \text{higher order lie products in } \varphi(a) \text{ and } \varphi(b).
\]

Since \( \varphi \) preserves higher order lie products, we may cancel off corresponding terms to obtain \( \varphi(a, b, a, \ldots) = \)
(\varphi(a), \varphi(b), \varphi(a), \ldots), that is that \varphi preserves i-fold lie products. Now a downward induction on i shows that \varphi preserves all lie products in a and b, so we have

\[ \varphi(a * b) = \varphi(a) * \varphi(b) = \varphi(a) + \varphi(b) + 1/2(\varphi(a), \varphi(b)) + \ldots \]
\[ = \varphi(a+b) + 1/2 \varphi(a,b) + \ldots \]

As \varphi preserves all lie products we may cancel them off to obtain \( \varphi(a) + \varphi(b) = \varphi(a + b) \) and the induction step has been accomplished, \textit{ged.}

Recall that a group \( G \) is lattice nilpotent if \( \log G \subset \mathfrak{g} \) is an additive lattice in \( \mathfrak{g} \). For the remainder of this section, we will assume that \( G \) is lattice nilpotent.

\textbf{Lemma 2.2:} The topology in \( G \) with neighborhood basis of the identity the sets \( \exp(p^i \log G) \) for all \( i \) is the p-adic topology in \( G \).

\textbf{Proof:} As \( \exp(p^i x) = (\exp(x))^p^i \) we have \( \exp(p^i \log G) \subset G^p^i \) we also have

\textbf{Theorem C:} (N. Blackburn [3]) For each prime \( p \) and each positive integer \( c \) there is an integer \( f(p, c) \) such that if \( G \) is nilpotent of class \( c \) then every product of \( p^n \)-th powers (with \( n \geq f(p, c) \)) of elements of \( G \) is a \( p^n-f(p, c) \)-th power.
Using this we see that $\mathbb{G}^1 \subset \exp(p^{i-f(p,c)} \log G)$ so that the two topologies are the same.

**Proposition 2.3:** Let $G$ be a lattice nilpotent group. If we let $\mathfrak{L}_p = Q_p \otimes \mathcal{L}$ and $Z_p \log G$ be the subset of $\mathfrak{L}_p = \{ z \otimes \log g \mid z \in Z_p, g \in G \}$ then:

(a) $\text{Aut } QG \cong \text{Aut } \mathfrak{L}$ and $\text{Aut } G \cong \text{stab } (\log G, \text{Aut } \mathcal{L})$

(b) $\text{Aut } Q_p G \cong \text{Aut } \mathfrak{L}_p$ and $\text{Aut } Z_p G \cong \text{stab}(Z_p \log G, \text{Aut } \mathfrak{L}_p)$

**Proof:** (a) Using the $\ast$ product in $\mathcal{L}$, we have that $QG$ and $(\mathfrak{L}, \ast)$ are isomorphic groups via $\exp$ and $\log$, which take $G$ isomorphically to $(\log G, \ast)$. Thus $\text{Aut } QG \cong \text{Aut}(\mathfrak{L}, \ast)$ and since $\text{Aut } G = \text{stab}(G, \text{Aut } QG)$ by lemma 1.4, by applying $\log$ we have $\text{Aut } G \cong \text{stab}(\log G, \text{Aut}(\mathfrak{L}, \ast))$. By lemma 2.1, $\text{Aut}(\mathfrak{L}, \ast) = \text{Aut } \mathcal{L}$ as lie algebra proving (a).

(b) We now put a $p$-adic topology on $\log G$ which is an abelian group by giving a basis for the neighborhoods of $0$ the sets $p^i \log G$, for each $i$. By lemma 2.2, $\exp$ and $\log$ give inverse isomorphisms (isomorphisms and homeomorphisms) of $(\log G, \ast)$ with $G$, therefore the respective completions must be isomorphic by an extension of $\exp$-$\log$. The completion of $(\log G, \ast)$ in this topology is $(Z_p \log G, \ast)$. Since in $\ast$ multiplication $x^n = nx$ if $x \in (Q_p \otimes \mathcal{L}, \ast)$, there is an integer $n$ with $nx$ in $Z_p \log G$ or $x^n \in (Z_p \log G, \ast)$. Also,
with the $\ast$ multiplication $(\mathcal{L}_p, \ast)$ is a divisible group so
$(\mathcal{L}_p, \ast)$ is isomorphic to $Q_p G$ by an extension of exp-log
which takes $(Z_p \log G, \ast)$ isomorphically to $Z_p G$. Thus we
have $\text{Aut } Q_p G \cong \text{Aut}(\mathcal{L}_p, \ast)$ and again by lemma 1.4 we have
$\text{Aut } Z_p G = \text{stab}(Z_p G, \text{Aut } Q_p G) \cong \text{stab}(Z_p \log G, \text{Aut}(\mathcal{L}_p, \ast))$.
By lemma 1.3 all automorphisms of $Q_p G$ are $Q_p$-automorphisms
so that all automorphisms of $(Q_p \otimes \mathcal{L}, \ast)$ are $Q_p$-automorphisms.
Thus by lemma 2.1, $\text{Aut}(\mathcal{L}_p, \ast) = \text{Aut}(\mathcal{L}_p)$ as a lie algebra
and (b) follows.

3. Algebraic groups and automorphism groups

Definition: An algebraic matrix group $H$ over a field $k$
of degree $n$ is given by an ideal $\mathfrak{a}$ of polynomials in
$k[X_{11}, \ldots, X_{nn}]$ such that for some (an hence every)
algebraic closure $\bar{k}$ of $k$ the set $H_{\bar{k}}$ of elements of $GL(n, \bar{k})$
whose coefficients annihilate $\mathfrak{a}$ is a group. If $B$ is a
subring of an overfield of $k$ then $H_B$ is the group of elements
of $GL(n, B)$ whose coefficients annihilate $\mathfrak{a}$.
Suppose now that $G$ is an algebraic matrix group over $\mathbb{Q}$ of degree $n$. Let $V$ be the set of all finite primes in $\mathbb{Z}$; then we define $G^\infty_A$ to be the subgroup of $\prod_{p \in V} G^\infty_{\mathbb{Q}_p}$ consisting of elements $x = (x_p)_{p \in V}$ such that, for all but a finite number of $p \in V$, $x_p \in G^\infty_{\mathbb{Z}_p}$. We note that $G^\infty_{\mathbb{Q}}$ may be diagonally embedded in $G^\infty_A$ by $x \mapsto (x_p)_{p \in V}$ where $x_p = x$ for all $p \in V$, since $x \in G^\infty_{\mathbb{Z}_p}$ for all but a finite number of primes $p$. We further define $G^{\infty \infty}_A$ to be the subgroup of $G^\infty_A$ consisting of $x = (x_p)_{p \in V}$ for which $x_p \in G^\infty_{\mathbb{Z}_p}$ for all $p \in V$. We now state a theorem of A. Borel ([4], Theorem 5.1).

**Theorem D:** Let $G$ be an algebraic matrix group over $\mathbb{Q}$, then the number $c(G)$ of distinct double coset $G^{\infty \infty}_A \cdot x \cdot G^\infty_{\mathbb{Q}}$ ($x \in G^\infty_A$) is finite.

We know that the automorphism group of a finite dimensional rational Lie algebra $\wedge$, with a given vector space basis is the set of rational points $G^\infty_{\mathbb{Q}}$ of an algebraic matrix group $G$ over $\mathbb{Q}$. In fact the ideal
$S$ of polynomials is generated by polynomials gotten from the constants of multiplication $\{\gamma_{ijk}\}$ with respect to the given basis by

$$\sum_r \sum_s X_{ir} X_{js} Y_{rst} = \sum_k \gamma_{ijk} X_{kt} \quad \text{(for all } i,j,t).$$

If $F$ is any overfield of $Q$, $G_{of} \cong \text{Aut}(F \otimes Q \wedge)$. If $\wedge = \mathcal{L}$ the lie algebra of our lattice nilpotent group $G$, and we take, as basis for $\mathcal{L}$, a $Z$-basis for the lattice $\log G$, then $G_{\mathcal{L}}$ will be stab $(\log G, \text{aut } \mathcal{L})$ and $G_{Z_p}$ will be stab $(Z_p \log G, \text{Aut } \mathcal{L}_p)$. Thus we have shown in consideration of Proposition 2.3.

**Proposition 3.1:** If $G$ is a lattice nilpotent group then there is an algebraic matric group $G$ such that

$$\text{Aut } G = G_{Z}, \text{ Aut } QG = G_{Q}, \text{ Aut } (Z_p G) = G_{Z_p} \text{ and Aut } (Q_p G) = G_{Q_p}. $$

4. **Manna from Borel**

This section contains some results due to A. Borel which were communicated to G. Baumslag in October, 1969. Two torsion free finitely generated nilpotent groups $G$ and $H$ are said to have isomorphic finite quotients if the sets $\mathcal{F}(G)$ and $\mathcal{F}(H)$ of isomorphism classes of finite homomorphic images of $G$ and $H$ respectively are equal.
Lemma 4.1  Two groups $G$ and $H$ have isomorphic finite quotients if and only if $Z_p G \cong Z_p H$ for all finite primes $p$.

Proof: For a positive integer $m$ let $G^m = \text{gp}\{x^m | x \in G\}$ which is the smallest normal subgroup of $G$ containing the elements $x^m$ for all $x \in G$, and let $\Gamma_m = G/G^m$. The group $\Gamma_m$ has exponent dividing $m$ and every quotient of $G$ with exponent dividing $m$ is a quotient of $\Gamma_m$. In fact $\Gamma_m$ is finite and has exponent exactly $m$. If $m$ divides $n$ then $G^m \supseteq G^n$ so there is a canonical epimorphism $\gamma_{n,m}: \Gamma_n \twoheadrightarrow \Gamma_m$. Similarly we have $H^m$, $\Theta_m$ and $\Theta_{m,n}$. If $G$ and $H$ have isomorphic finite quotients, since $\Gamma_m$ and $\Theta_m$ are the largest quotients of exponent dividing $m$ of $G$ and $H$ respectively, we must have $\Gamma_m \cong \Theta_m$ for all integers $m$.

We now let $\{m_j\}$ be a sequence of integers such that $m_j$ divides $m_{j+1}$ for all $j$. We claim there must exist isomorphisms $f_j: \Gamma_{m_{j-1}} \cong \Gamma_{m_j}$ such that the following diagram with horizontal arrows the $\gamma_{m_{j+1},m_j}$ or $\Theta_{m_{j+1},m_j}$ is commutative:

$$
\begin{array}{cccccc}
\Gamma_{m_1} & \Gamma_{m_2} & \Gamma_{m_3} & \cdots & \Gamma_{m_j} & \Gamma_{m_{j+1}} & \cdots \\
| & | & | & | & | & | \\
f_1 & f_2 & f_3 & \cdots & f_j & f_{j+1} & \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \\
\Theta_{m_1} & \Theta_{m_2} & \Theta_{m_3} & \cdots & \Theta_{m_j} & \Theta_{m_{j+1}} & \cdots 
\end{array}
$$
Let \( f_i \) be an isomorphism, \( f_i: \Gamma_{m_i} \rightarrow \mathbb{Z}_{m_i} \). We will say that \( f_i \) extends to \( j > i \) if there is a commutative diagram:

\[
\begin{array}{ccc}
\Gamma_{m_i} & \xrightarrow{\gamma_{m_i, m_i}} & \Gamma_{m_j} \\
\downarrow f_i & & \downarrow f_j \\
\mathbb{Z}_{m_i} & \xleftarrow{\mathbb{Z}_{m_j}} & \mathbb{Z}_{m_j}
\end{array}
\]

and that \( f_i \) is indefinitely extendable if \( f_i \) extends to \( j \) for each \( j > i \).

For \( i \leq j \) any isomorphism \( f_j: \Gamma_{m_j} \rightarrow \mathbb{Z}_{m_j} \) is an extension of some \( f_i \) as \( \Gamma_{m_k} \) (resp. \( \mathbb{Z}_{m_i} \)) is the largest quotient of \( \Gamma_{m_j} \) (resp. \( \mathbb{Z}_{m_j} \)) of exponent dividing \( m_i \). Thus for any \( j \), some isomorphism \( f_i: \Gamma_{m_i} \rightarrow \mathbb{Z}_{m_i} \) extends to \( j \) and since \( \text{Iso}(\Gamma_{m_i}, \mathbb{Z}_{m_i}) \) is finite it follows that there is at least one indefinitely extendable \( f_i \) and given such an \( f_i \) there is an indefinitely extendable \( f_j \) which extends it for each \( j > i \). We therefore have

\[
\lim (\Gamma_{m_j}, \gamma_{m_j, m_j}) \cong \lim ((\mathbb{Z}_{m_j}, \mathbb{Z}_{m_j})).
\]

If we let \( m_j = p^j \) for any prime \( p \), we will have

\[
\mathbb{Z}_p \cong \mathbb{Z}_H.
\]

Conversely, since every quotient of exponent \( m \) of \( G \) is a quotient of \( \Gamma_m \) and similarly for \( H \), it suffices to show \( \Gamma_m \cong \mathbb{Z}_m \) for all integers \( m \). Since every finite
nilpotent group is the direct product of its $p$-Sylow sugbroups for $p$ dividing the exponent of $m$ and since if $p^i$ exactly divides $m$ the $p$-Sylow subgroup of $\Gamma_m$ must be the largest quotient of $\Gamma_m$ of exponent $p^i$, i.e., $\Gamma_p$, we have $\Gamma_m = \Gamma_{p^1} \times \Gamma_{p^2} \times \ldots \times \Gamma_{p^k}$. If $Z_p G \cong Z_{p^i}$, then we have

$$\Gamma_p = G/G_p^i \cong Z_p G / (Z_p G)^{p^i} \cong Z_p H / (Z_p H)^{p^i} \cong H/H_p^i$$

Thus $Z_p G \cong Z_p H$ for all primes $p$ implies $\Gamma_m \cong \bigodot_m$ for all integers $m$ and consequently that $\mathcal{F}(G) = \mathcal{F}(H)$.

**Theorem 4.2:** If $G$ is a finitely generated torsion free nilpotent group, the finitely generated torsion free nilpotent groups having finite quotients isomorphic with those of $G$, lie in only finitely many commensurability classes.

**Proof:** $G$ is commensurable with $H$ is and only if $QG \cong QH$. If $G$ and $H$ have isomorphic finite quotients, then $Z_p G \cong Z_p H$ and thus $Q_p G \cong Q_p H$ for all primes $p$. If we let $\mathfrak{g}$ and $\mathfrak{m}$ be the lie algebras of $G$ and $H$ respectively, since $(\mathfrak{g}, *) \cong QG$ and $(\mathfrak{g}_p, *) \cong Q_p G$, we have, by lemma 2.1, $QG \cong QH$ if and only if $\mathfrak{g} \cong \mathfrak{m}$ and $Q_p G \cong Q_p H$ if and only if $Q_p \otimes \mathfrak{g} = \mathfrak{g}_p \cong \mathfrak{m}_p = Q_p \otimes \mathfrak{m}$. Therefore we see that the
set of commensurability classes which contain groups having finite quotients isomorphic with those of $G$ is in one-to-one correspondence with a subset of the set of isomorphism classes of rational lie algebras which are isomorphic with $L$ over $Q_p$ for all primes $p$. By a theorem of Borel-Serre ([6], Theorem 7.11) the number of such isomorphism classes is finite.

Remark: The theorem of Borel-Serre actually states that if $P$ is any finite set of primes then the number of isomorphism classes of finite dimensional algebras over $Q$ which become isomorphic over $Q_p$ for all $p$ in $V-P$ is finite. Thus Theorem 2 could be strengthened slightly to say: If $P$ is a finite set of primes and $G$ is a finitely generated torsion free nilpotent group, the finitely generated torsion free nilpotent groups $H$, which satisfy $Z_pG \cong Z_pH$ for all $p$ in $V-P$, lie in only finitely many commensurability classes. For the main theorem of this part, however, we will need all primes. For an example, see part II.

5. $Z_p$-Groups

In this section, we study groups admitting parametric exponents in $Z_p$ (see [12]) though the results will surely go through for a far more general class of rings.
We define a $\mathbb{Z}_p$-group $G$ to be a group which admits $\mathbb{Z}_p$ as parametric exponents, i.e., given any $g \in G$ and $\mu \in \mathbb{Z}_p$, there is a unique element $g^\mu$ in $G$ which is defined in such a way that $g^\mu \cdot g^\lambda = g^{\mu + \lambda}$, $(g^\mu)^\lambda = g^{\mu \lambda}$ and $h^{-1}g^\mu h = (h^{-1}gh)^\mu$ for all $g, h \in G$ and $\lambda, \mu \in \mathbb{Z}_p$.

We say a $\mathbb{Z}_p$-group $G$ is $\mathbb{Z}_p$-generated by $\{x_1, \ldots, x_k, \ldots\}$ if every element of $G$ can be written in at least one way as $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_k}^{\lambda_k}$ for some $\lambda_i \in \mathbb{Z}_p$. A $\mathbb{Z}_p$-subgroup $H$ of $G$ is a subgroup $H$ of $G$ which admits the action of $\mathbb{Z}_p$. We now fix a $\mathbb{Z}_p$-subgroup $H$ of finite index in a $\mathbb{Z}_p$-group $G$, which is $\mathbb{Z}_p$-generated by $x_1, \ldots, x_k$.

**Lemma 5.1:** Any right coset of $H$ in $G$ contains an element of $\mathbb{g}p[x_1, \ldots, x_k]$, i.e. an element of the form $x_{i_1}^{a_1} \ldots x_{i_j}^{a_j}$ with $a_k \in \mathbb{Z}$.

**Proof:** Suppose $x_{i_1}^{\lambda}$ is in some coset for some $i$ and $\lambda$. There can be only a finite number of cosets of the form $Hx_{i_1}^{\mu}$, so that there must be an ideal $I$ of $\mathbb{Z}_p$ such that $x_{i_1}^{\mu} \in H \forall \mu \in I = p^k\mathbb{Z}_p$. Then $x_{i_1}^{j} j = 0, 1, 2, \ldots, p^{k-1}$ represent all cosets containing any $x_{i_1}^{\lambda}$. Now suppose that it is known that if a coset contains an element of the form $x_{i_1}^{\lambda_1} \ldots x_{i_j}^{\lambda_j}$ for $j < m$, then that coset contains an element of the form $x_{i_1}^{\lambda_1} \ldots x_{i_j}^{\lambda_j}$ for $a_k \in \mathbb{Z}$ and suppose $x_{j_1}^{\lambda_1} \ldots x_{j_m}^{\lambda_m}$ is in some coset. Then the coset
of \(x_{j_1}^{\lambda_1} \ldots x_{j_{m-1}}^{\lambda_{m-1}}\) contains an element \(y = x_{j_1}^{a_1} \ldots x_{j_{m-1}}^{a_{m-1}}\) with \(a_i \in \mathbb{Z}\) so that \(H x_{j_1}^{\lambda_1} \ldots x_{j_{m-1}}^{\lambda_{m-1}} x_{j_m}^{\lambda_m} = H y x_{j_m}^{\lambda_m}\). Now let \(w = y x_{j_m}^{\lambda_m} y^{-1}\) and as in the case of \(x_i\), there is an integer \(t > 0\) such that the cosets \(H_w^f\) for \(f = 0,1,2,\ldots,p^t-1\) are all the cosets containing \(w^\mu\) for \(\mu \in \mathbb{Z}_p\), so there is an integer \(a_m\) such that \(H w^m = H w^a_m\). Thus \(H y x_{j_m}^{\lambda_m} y^{-1}\) and \(H y x_{j_m}^{a_m} y^{-1}\) are \(H y x_{j_m}^{\lambda_m} y^{-1}\) so that \(H y x_{j_m}^{a_m} = H y x_{j_m}^{\lambda_m} y^{-1}\).

Recall that a set \(\{g_1, \ldots, g_n\}\) of words in the generators \(\{x_1, \ldots, x_k\}\) of \(G\) is a Schreier system for \(G\) with respect to \(H\) if each \(g_i = x_{i_1}^{\epsilon_1} \ldots x_{i_m}^{\epsilon_m}\) is reduced as written, \(x_{i_1}^{\epsilon_1} \ldots x_{i_{m-1}}^{\epsilon_{m-1}}\) is one of the \(g_j\) and the elements of \(G\) these words define form a set of transversals of \(G\) with respect to \(H\).

**Lemma 5.2:** In the situation of Lemma 5.1, we may choose a set of Schreier transversals for \(G\) with respect to \(H\).
**Proof:** As in the finitely generated case, define the length of a coset to be the length of the shortest word in the coset. As each coset has a representative of the form \( x_{j_1}^{a_1} \cdots x_{j_m}^{a_m} \), the length of each coset is finite. For each coset of length 1 choose any representative of length 1. Suppose now that representatives have been chosen for all coset of length less than \( m \), then in a coset of length \( m \) pick a word of length \( m \), \( x_{j_1}^{e_1} \cdots x_{j_m}^{e_m} \) with \( e_1 = \pm 1 \) and the word reduced as written. The word \( x_{j_1}^{e_1} \cdots x_{j_m}^{e_m-1} \) must lie in a coset of length \( m-1 \) for if not the length of the given coset would be less than \( m \). A representative \( x_{k_1}^{\delta_1} \cdots x_{k_m}^{\delta_m-1} \) has already been chosen for the coset containing \( x_{j_1}^{e_1} \cdots x_{j_m}^{e_m-1} \). For the representative of the given coset we choose \( x_{k_1}^{\delta_1} \cdots x_{k_{m-1}}^{\delta_{m-1}} x_{j_m}^{e_m} \). This word must be reduced as written for if it were not the coset in question would have length less than \( m \). This choice of transversals satisfies the requirements of a Schreier system.

Now we let \( \{ g_1, g_2, \ldots, g_n \} \) be a set of right coset representative for \( G \) with respect to \( H \), and let \( \phi \) be the function which assigns to each \( g \in G \), its coset representative among \( g_1 \).
Lemma 5.3: \( \varphi(g_k x_j^\lambda) \) for \( i = 1, \ldots, n \) and \( \lambda \in \mathbb{Z}_p \) may be determined from the permutation of the \( g_i \) given by \( \pi(g_i) = \varphi(g_i x_j^\lambda) \).

Proof: As before the set of \( \lambda \in \mathbb{Z}_p \) such that \( (g_i x_j^\lambda)^{-1} \in H \) form an ideal \( p^k \mathbb{Z}_p \) in \( \mathbb{Z}_p \). Thus if \( \lambda = m + p^k \lambda' \) we have

\[
\varphi(g_i x_j^\lambda) = \varphi(g_i x_j^m) \text{ since }
\]

\[
H g_i x_j^\lambda = H g_i x_j^m g_i^{-1} \forall g_i = H (g_i x_j^m g_i^{-1}) \forall g_i = H g_i x_j^m g_i^{-1} \forall g_i =
\]

so that \( \varphi(g_i x_j^\lambda) = \varphi(g_i x_j^m) \).

Lemma 5.4: In the \( \mathbb{Z}_p \)-group \( G \), the set of elements \( g s^\varepsilon \varphi(g s^\varepsilon)^{-1} \) are \( \mathbb{Z}_p \)-generators of the subgroup \( H \), where \( g \) runs through a set of coset representatives of \( G \) with respect to \( H \), \( s \) runs through a set of \( \mathbb{Z}_p \)-generators of \( G \) and \( \varepsilon = \pm 1 \), with \( \varphi \) as above.

Proof: If \( \varepsilon \) were allowed to be any element of \( \mathbb{Z}_p \), we would be done as follows. Any element \( f \) of \( G \) may be written

\[
f = x_1^\lambda_1 \ldots x_j^\lambda_j \text{ so that if we let }
\]

\[
h_\ell^\lambda = \varphi(x_1^\lambda_1 \ldots x_\ell^\lambda_\ell) \text{ then } f h_j^{-1} = h x_1^\lambda_1 k_1^\lambda_1 \cdot h_1 x_2^\lambda_2 h_2^{-1} \ldots h_{j-1} x_j^\lambda_j h_j^{-1}.
\]

If \( f \) is in \( H \) then \( h_j = \varphi(f) = 1 \) so \( f h_j^{-1} = f \), but \( h_\ell x_\ell^\lambda_\ell h_\ell^{-1} \) has the form \( g s^\varepsilon \varphi(g s^\varepsilon) \) with \( g \) a coset representative, \( s \) a \( \mathbb{Z}_p \)-generator of \( G \) and \( \varepsilon \) an element of \( \mathbb{Z}_p \). Thus if \( \varepsilon \) were
allowed to be anything in $Z_p$, we would be done. Again
the set of $\lambda$ such that $(h_i x_j h_i^{-1})^\lambda \in H$ is an ideal $p^r Z_p$
in $Z_p$, so that $\varphi((h_i x_j h_i^{-1})^{p^r}) = 1$ and $\varphi(h_i x_j^{p^r}) = $

$$
\varphi(h_i x_j^{p^r} h_i^{-1} h_i) = \varphi((h_i x_j h_i^{-1})^{p^r} h_i) = h_i. \text{ We have then that }
$$

$$(h_i x_j h_i^{-1})^{p^r} = h_i x_j^{p^r} h_i^{-1} = h_i x_j^{p^r} \varphi(h_i x_j^{p^r})^{-1}$$

$$= h_i x_j \varphi(h_i x_j)^{-1} \varphi(h_i x_j) x_j \varphi(h_i x_j)^2^{-1} \ldots
$$

$$\varphi(h_i x_j^{p^r-1}) x_j \varphi(h_i x_j^{p^r})^{-1}$$

which is an element of the subgroup of $G$ generated by
$g s^\varepsilon \varphi(g s^\varepsilon)$ with $g$ and $s$ as above and $\varepsilon = \pm 1$. Now let
$\lambda \in Z_p$ satisfy $\lambda = m + p^r \lambda'$ with $\lambda' \in Z_p$ and $0 \leq m < p^r$
in $Z$. Then $h_i x_j^\lambda \varphi(h_i x_j)^{-1} =$

$$(h_i x_j h_i^{-1})^{h_i \varphi((h_i x_j h_i^{-1})^{h_i})^{-1}} = (h_i x_j h_i^{-1})^{p^r \lambda'} (h_i x_j h_i^{-1})^{m h_i}
$$

$$\varphi((h_i x_j h_i^{-1})^{m h_i})^{-1}.$$  

The first factor is a $Z_p$-power of $(h_i x_j h_i^{-1})^{p^r}$ which is
in the group generated by the $g s^\varepsilon \varphi(g s^\varepsilon)^{-1}$ and $(h_i x_j h_i^{-1})^m$
h_i \varphi((h_i x_j h_i^{-1})^{m h_i})^{-1} = h_i x_j^m \varphi(h_i x_j^m)^{-1}$ is in the group
generated by the $g s^\varepsilon \varphi(g s^\varepsilon)^{-1}$ by a computation similar
to (a) above. Thus the elements of the form $g s^\varepsilon \varphi(g s^\varepsilon)^{-1}$
for $e$ in $\mathbb{Z}_p$ are in the $\mathbb{Z}_p$-group generated by the $g^e\varphi(g^e)^{-1}$ for $e = \pm 1$, so by the remark at the beginning of the proof that latter group must be all of $H$.

**Corollary 5.5:** Giving coset representatives $\{g_1, \ldots, g_n\}$ for $G$ with respect to $H$ and the permutations of the $g_i$ given by $\pi_j(g_i) = \varphi(g_i x_j)$ for all $j$ completely determines the subgroup $H$ as the $\mathbb{Z}_p$-subgroup generated by $g^e\varphi(g^e)^{-1}$ for $g \in \{g_1, \ldots, g_n\}$ and $s \in \{x_1, \ldots, x_k\}$ a $\mathbb{Z}_p$ generating set for $G$.

**Corollary 5.6:** If $H$ is a $\mathbb{Z}_p$-subgroup of finite index in a finitely $\mathbb{Z}_p$-generated $\mathbb{Z}_p$-group $G$, then $H$ is finitely $\mathbb{Z}_p$-generated.

**Proposition 5.7:** If $G$ is a finitely $\mathbb{Z}_p$-generated group there can be only a finite number of distinct $\mathbb{Z}_p$-subgroups of $G$ of a given finite index.

**Proof:** There can be only a finite number of sets of Schreier transversals of a given finite cardinality on a given finite number of generators and there can be only a finite number of sets of permutations of these transversals of a given cardinality so that the number of sets of $\{g_1 \ldots g_n\}$ and the number of sets of permutations $\{\pi_j\}$ are finite.

No doubt one could prove other statements about $\mathbb{Z}_p$-groups,
by mimicking the proofs in the finitely generated case as above, but at this point we return to nilpotent groups to prove

**Proposition 5.8:** If $H$ is a subgroup of finite index in a torsion-free finitely generated nilpotent group $G$, then $Z_p H$ is of finite index in $Z_p G$ and $\text{Aut}(Z_p G)$ is commensurable with $\text{Aut}(Z_p H)$.

**Proof:** For any normal basis $(x_1, x_2, \ldots, x_m)$ of $G$ we may choose a normal basis $(y_1, \ldots, y_m)$ of $H$ in such a way that for each $i$, $y_i \equiv x_i^{n_i} \pmod{p^n}[x_{i+1}, \ldots, x_m]$.

In this case the set $\{x_1^{j_1} \ldots x_m^{j_m}\}$ with $0 \leq j_i < n_i$ for each $i)$ is a complete set of coset representatives of $G$ with respect to $H$. If $p^{j_i}$ exactly divides $n_i$, $n_i/p^{j_i}$ will be a unit in $Z_p$ so the elements $x_1^{i_1} \ldots x_m^{i_m}$ with $0 \leq i_k < p^{j_k}$ form a complete set of coset representatives for $Z_p G$ with respect to $Z_p H$ so $Z_p H$ is of finite index in $Z_p G$.

**Lemma 5.9:** Let $G$ be a finitely $Z_p$-generated $Z_p$-group and let $H$ be a $Z_p$-subgroup of finite index in $G$ and let $X$ be a group of $Z_p$-automorphisms of $G$, then $\text{stab}(H,X)$ is of finite index in $X$. 
Proof: There are only finitely many \( \mathbb{Z}_p \)-subgroups of index \((G:H)\) and \(X\) permutes them giving a homomorphism of \(X\) into a finite group of permutations whose kernel is thus of finite index in \(X\) and obviously is contained in \(\text{stab}(H,X)\).

Continuation of the proof of Proposition 5.8: Let \(\hat{H}_p = \text{gp}\{x \in Q_p G | x^k \in z^2 \} \text{ where } k = (Z_p G : Z_p H)\). Every automorphism \(\alpha\) of \(\mathbb{Z}_p H\) extends uniquely to an automorphism \(\tilde{\alpha}\) of \(\hat{H}_p\) so if \(S = \text{stab}(\mathbb{Z}_p H, \text{Aut} \mathbb{Z}_p H)\) and \(\rho\) is map which restricts an element of \(S\) to an automorphism of \(\mathbb{Z}_p H\), we have \(\rho(s) = \text{Aut}(\mathbb{Z}_p H)\) and \(\rho\) is an isomorphism. By lemma 5.9 there is a subgroup \(T\) of \(S\) of finite index which stabilizes \(\mathbb{Z}_p G\) so \(\rho(T)\) is of finite index in \(\text{Aut} \mathbb{Z}_p H\). Thus if \(U\) is the stabilizer of \(\mathbb{Z}_p H\) in \(\text{Aut} \mathbb{Z}_p G\) and \(\sigma\) is restriction homomorphism of \(U\) into \(\text{aut} \mathbb{Z}_p H\), then \(\sigma(U) \geq \rho(T)\) so \(\sigma(U)\) is of finite index in \(\text{Aut} \mathbb{Z}_p H\). \(U\) is of finite index in \(\text{Aut} \mathbb{Z}_p G\) and \(\sigma\) is a monomorphism since every automorphism of \(\mathbb{Z}_p H\) extends uniquely to \(Q_p G\) by lemma 1.4. Thus \(\text{Aut} \mathbb{Z}_p G\) and \(\text{Aut} \mathbb{Z}_p H\) are commensurable.

6. Proof of the Main Theorem:

Suppose now that \(G\) is a torsion free finitely generated nilpotent group. We define the Group \(Q_A\) to be the subgroup of \(\Pi_p \in V \text{ Aut}(Q_p G)\) consisting of elements \((\alpha_p), p \in V\), such that, for all but a finite number of \(p\),
\[ \alpha_p \] sends \( Z_p G \) isomorphically onto \( Z_p G \), i.e., \( \alpha_p \in \text{stab} (Z_p G, \text{Aut}(Q_p G)) \). We define the group \( Q^A \) to be the subgroup of \( Q_A \) consisting of all \((\alpha_p)_{p \in V}\) for which \( \alpha_p \in \text{stab}(Z_p G, \text{Aut}(Q_p G)) \) for all \( p \in V \). If we denote \( \text{Aut}(QG) \) by \( Q \), we have \( Q \) embedded as a subgroup of \( Q_A \) by \((\alpha) \rightarrow (\alpha_p)_{p \in V}\) with \( \alpha_p \) being the unique extension of \( \alpha \) to an automorphism of \( Q_p G \). By lemma 1.5, \( \alpha_p \in \text{stab}(Z_p G, \text{Aut}(Q_p G)) \) for all but a finite number of \( p \in V \) so \((\alpha_p)_{p \in V} \in Q_A \).

**Proposition 6.1:** The isomorphism classes of finitely generated torsion free nilpotent groups which are commensurable with \( G \) and have finite quotients isomorphic to those of \( G \("\text{classes in the genus of } G\)\) are in one-to-one correspondence with a subset of the set of double cosets \( G^A \backslash G_A / Q_A \).

**Proof:** Suppose then that \( H \) is in the genus of \( G \) so that by lemma 4.1 for each \( p \in V \) there are isomorphism \( \varphi_p : G_p \rightarrow H_p \) and there is an isomorphism \( \psi : QG \rightarrow QH \).

The \( \varphi_p \) extend to maps \( \tilde{\varphi}_p : Q_p G \cong Q_p H \) and \( \psi \) extends to \( \tilde{\psi}_p : Q_p G \cong Q_p H \) for all \( p \) and \( \tilde{\psi}_p : Z_p G \rightarrow Z_p H \) for almost all \( p \in V \) by lemma 1.5, so that \( \tilde{\psi}_p (\tilde{\varphi}_p^{-1} \psi_p) \) is an element of \( Q_A \). We now show that the corresponding
double coset in $Q_A^\circ \backslash Q_A/Q$ is independent of several choices. Suppose $\theta_p$ were a different choice of isomorphism $Z_pG \to Z_pH$, so that since $\theta_p = \varphi_p \circ \varphi_p^{-1} \circ \theta_p$ and $\varphi_p$ is an automorphism of $Z_pG$ for each $p$, we have $\pi_p \in V(\varphi_p^{-1} \circ \varphi_p) \in Q_A^\circ$ so that $\pi_p \in V(\varphi_p^{-1} \circ \psi_p)$ and $\pi_p \in V(\varphi_p^{-1} \circ \psi_p)$ lie in the same double coset $Q_A^\circ \backslash Q_A^\circ$. Similarly if $\gamma$ were a different choice $\gamma: QG \to QH$ then $\gamma = \varphi_p \circ \varphi_p^{-1} \circ \gamma$ and $\varphi_p^{-1} \circ \gamma \in \text{Aut } QG = Q$ so that $\pi_p \in V(\varphi_p^{-1} \circ \gamma_p)$ is the same double coset as $\pi_p \in V(\varphi_p^{-1} \circ \gamma_p)$. Suppose $\tilde{H}$ were a group isomorphic to $H$ by an isomorphism $\sigma: H \to \tilde{H}$. Then $\sigma$ extends to $\sigma_p: Z_pH \to Z_p\tilde{H}$ and $\tilde{\sigma}: QH \to Q\tilde{H}$ each of which extends to the same isomorphism $\tilde{\sigma}_p: Q_pH \to Q_p\tilde{H}$. We have then $\sigma_p \circ \varphi_p: Z_pG \to Z_pH$ and $\tilde{\sigma} \circ \gamma: QG \to Q\tilde{H}$. The element of the double coset corresponding to these isomorphisms is $\pi_p \in V(\varphi_p^{-1} \circ \tilde{\sigma}_p \circ \varphi_p \circ \gamma_p) = \pi_p \in V(\varphi_p^{-1} \circ \gamma_p)$. Thus the map from classes in the genus of $G$ to the double cosets $Q_A^\circ \backslash Q_A^\circ / Q$ is well defined.

Now suppose $H$ and $K$ are two groups in the genus of $G$, whose respective isomorphism classes go to the same double coset in $Q_A^\circ \backslash Q_A/Q$. Let $\varphi_p: Z_pG \to Z_pH$ $\gamma: QG \to QH$, $\sigma_p: Z_pG \to Z_pK$ and $\tau: QG \to QK$ be isomorphisms.
By modifying \( \phi_p \) or \( \psi \) by automorphisms of \( Z_p G \) or \( QG \) respectively, we may choose \( \phi_p, \psi, \sigma_p \) and \( \tau \) so that

\[
\pi_p \circ \psi \circ (\phi_p^{-1} \circ \psi)_p = \pi_p \circ (\sigma_p^{-1} \circ \tau)_p
\]

in \( Q_A \). This means for each \( p \in \mathcal{V} \)

\[
\tilde{\phi}_p^{-1} \circ \psi_p = \tilde{\sigma}_p \circ \tau_p
\]

as automorphisms of \( Q_p G \).

Consider the maps \( \theta_p = \tilde{\sigma}_p \circ \tilde{\phi}_p^{-1} : Q_p H \to Q_p K \) and

\[
\gamma_p = \tau_p \circ \psi_p^{-1} : Q_p H \to Q_p K.
\]

The fact that \( \tilde{\phi}_p^{-1} \circ \psi_p = \tilde{\sigma}_p^{-1} \circ \tau_p \)

in \( Q_G \) implies that \( \theta_p = \gamma_p \) for all \( p \). We have, however,

\[
\theta_p : Z_p H \to Z_p K
\]

isomorphically and \( \gamma_p : QH \to QK \) isomorphically

for each \( p \in \mathcal{V} \). Thus \( \gamma = \tau \circ \psi^{-1} \) must send \( H \subset QH \) to

\[
\gamma(H) \subset QK \cap \bigcap_{p \in \mathcal{V}} Z_p = Z,
\]

so that \( \gamma(H) = K \). So we have \( \gamma(H) \subset K \) and similarly \( \gamma^{-1}(K) \subset H \) so that \( \gamma \) restricts to an isomorphism of \( H \) with \( K \).

This implies that the map from classes in the genus of \( G \) to double cosets \( Q_A^\infty \setminus Q_A / Q \) is one-to-one.

We now show that the number of double cosets \( Q_A^\infty \setminus Q_A / Q \) is finite.

Suppose first that \( G \) is a lattice nilpotent group.

Then by proposition 3.1 we have that there is an algebraic matrix group \( G \) such that \( Q_A^\infty = G_A^\infty \), \( Q_A = G_A \) and \( Q = G \).

By Theorem D of section 3, we have that the number of double cosets \( G_A^\infty \setminus G_A / G \) is finite so that the number of double cosets \( Q_A^\infty \setminus Q_A / Q \) is finite.
Now suppose that $G$ is any torsion free finitely generated nilpotent group. Then by a result of C. Moore [14] we have that there is a lattice nilpotent group $H$ which contains $G$ as a subgroup of finite index. Thus we have by lemma 1.5 that $Z_p G \cong Z_p H$ for all but a finite number of $p \in V$ and for that finite set of $p \in V$ we have $\text{Aut}(Z_p G)$ and $\text{Aut}(Z_p H)$ are commensurable. Now since $QG \cong QH$ we have $Q \cong Q$ and since $\text{Aut}(Z_p G) \cong \text{Aut}(Z_p H)$ for all but a finite number of $p$, $Q_A \cong H_A$ since if $(\alpha_p) \in Q_A$ we have $\alpha_p \in \text{Aut}(Z_p G)$ for all but a finite number of $p \in V$, and this implies that $\alpha_p \in \text{Aut}(Z_p G)$ for all but that finite number of $p$ and perhaps the additional finite number of $p$ where $\text{Aut}(Z_p G) \neq \text{Aut}(Z_p H)$, and conversely for $\alpha_p \in H_A$. Since $\text{Aut}(Z_p G) \cong \text{Aut}(Z_p H)$ for all but a finite number of $p \in V$ at which $\text{Aut}(Z_p G)$ and $\text{Aut}(Z_p H)$ are commensurable, we have that $Q_A \cong H_A$ and $H_A$ are so there are subgroups $K \subset Q_A$ and $L \subset H_A$ each of finite index with $K \cong L$ by the isomorphism of $Q_A$ with $H_A$. We then have $|Q_A \backslash Q_A / Q_Q| \leq |K \backslash Q_A / Q_Q| = |L \backslash H_A / H_Q|$ which is finite as $L$ is of finite index in $H_A$ and $|H_A \backslash H_A / H_Q|$ is finite. The above computation together with Proposition 6.1 prove

**Theorem 6.2:** The number of classes in the genus of a finitely generated torsion free nilpotent group $G$ is finite.

This result together with Theorem 4.2 of Borel yields
Theorem 6.3: Given a finitely generated torsion free nilpotent group $G$, the finitely generated torsion free nilpotent groups, which have the same finite homomorphic images of $G$, lie in only finitely many isomorphism classes. (The class number of $G$ is finite).
II. Examples:

7. Preliminaries

We will state here several theorem and definitions which will be used in the development of the examples.

Theorem E: ([13], Theorem 5.3) Let $a, b, c$ be elements of a group $G$ and let $k, m, n$ be integers such that $a \in \gamma_k G$, $b \in \gamma_m G$, $c \in \gamma_n G$, then

1. $a \cdot b = b \cdot a \mod \gamma_{k+m} G.$
2. $[a, b \cdot c] = [a, b] \cdot [a, c] \mod \gamma_{k+m+n} G$
3. $[a \cdot b, c] = [a, c] \cdot [b, c] \mod \gamma_{k+m+n} G$
4. $[a, b, c] \cdot [b, c, a] \cdot [c, a, b] = 1 \mod \gamma_{k+m+n+1} G.$

Theorem F: ([13], Theorem 5.4) Let the group $G$ be generated by $a_1, a_2, \ldots, a_r$, then $\gamma_n G / \gamma_{n+1} G$ is abelian and generated by the cosets of the left normed commutators:

$[a_{\rho_1}, a_{\rho_2}, \ldots, a_{\rho_n}]$ with $\rho_i \in \{1, 2, \ldots, r\}$.

Theorem G: ([7], Theorem 1, §5) If $A$ is a principal ideal ring, $E$ is a free $A$-module of dimension $n$ and $\phi$ is an antisymmetric bilinear form on $E$, then there is a basis $(e_i)$, $1 \leq i \leq n$, of $E$ and an even integer $2r \leq n$ such that
1° \( \phi(e_1, e_2) = \alpha_1, \phi(e_3, e_4) = \alpha_2, \ldots, \phi(e_{2r-1}, e_{2r}) = \alpha_r \)

where \( \alpha_i \) are non zero elements of \( A \) and \( \alpha_i \) divides \( \alpha_{i+1} \) for all \( i, 1 \leq i \leq r-1 \).

2° \( \phi(e_i, e_j) = 0 \) for all other pairs \( i, j \) with \( i \leq j \).

The ideals \( A\alpha_i \) are uniquely determined by the above conditions.

**Proposition 7.1:** Suppose \( F \) is a free nilpotent group of class \( c \) on \( n \) generators and that \( N_1 \) and \( N_2 \) are isolated normal subgroups of \( F \) which are contained in \( \gamma_2F \). If we let \( G_1 = F/N_1 \) and \( G_2 = F/N_2 \), we have \( G_1 \) isomorphic to \( G_2 \) (resp. \( Z_pG_1 \cong Z_pG_2 \), resp \( QG_1 \cong QG_2 \)) if and only if there is an automorphism of \( F \) (resp. \( Z_pF \), resp \( QF \)) which takes \( N_1 \) to \( N_2 \) (resp \( Z_pN_1 \) to \( Z_pN_2 \), resp \( QN_1 \) to \( QN_2 \)).

**Proof:** We first demonstrate

**Lemma 7.2:** Suppose that \( \phi \) is an endomorphism of \( F \) (resp. \( Z_pF \), resp. \( QF \)); then \( \phi \) is an automorphism of \( F \) (resp. \( Z_pF \), resp. \( QF \)) if and only if the induced endomorphism \( \bar{\phi} \) of \( F/\gamma_2F \) (resp. \( Z_pF/\gamma_2F \), resp \( QF/\gamma_2F \)) is an automorphism.

**Proof of Lemma:** The "only if" part of the lemma is obvious so it remains to prove the "if" part, which is also obvious
in case the class $c$ of $F$ is 1. Suppose then that the lemma holds for all free nilpotent groups of class $c$ less than $n$, and suppose $\varphi$ is an endomorphism of $F$ which induces an automorphism of $F/\gamma_n F$. By the inductive hypothesis $\varphi$ induces an automorphism $\bar{\varphi}$ of $F/\gamma_n F$ with inverse $\bar{\psi}$ induced by some endomorphism $\psi$ of $F$. We have $\psi(\varphi(x_i)) = x_i c_i$ for each generator $x_i$ of $F$ and some element $c_i$ of $\gamma_n F$ which is also the center of $F$. If we let $\delta$ be the automorphism of $F$ which takes $x_i$ to $x_i c_i^{-1}$, we will have $\delta \psi \varphi = \text{id}$. So $\delta \psi$ is a left inverse for $\varphi$. Similarly, $\varphi$ has a right inverse so that $\varphi$ is an automorphism of $F$. The proof for $Z_p F$ and $\text{QF}$ is exactly the same since the homomorphism is determined in each case by the images of generators of $F$.

To prove the proposition, we first observe that the "if" part of the proof is immediate. Suppose then that $\varphi: G_1 \to G_2$ is an isomorphism and consider the diagram:

\[
\begin{array}{c}
F \\ \downarrow \pi_{N_1} \\
\uparrow \pi_{N_1} \\
F/N_1 \\ \downarrow \varphi \\
F/N_2 \\
\end{array}
\]

where $\pi_{N_1}$ and $\pi_{N_2}$ are the canonical epimorphisms and $\psi$ is given on generators $x_i$ of $F$ by $\psi(x_i) = \pi_{N_2}^{-1} \varphi \pi_{N_1}(x_i)$, where by $\pi_{N_2}^{-1}$ we mean: pick a representative of $\varphi \pi_{N_1}(x_i)$ in $F$. The set map extends to an endomorphism of $F$ as $F$ is a free group in the variety $\text{N}_c$. Since $N_1$
are contained in $\gamma_2^F$, we have $\pi_{N_1}(\gamma_2^F) = \gamma_2 G$ and that

$$\overline{\pi}_{N_1} : F/\gamma_2^F \to G/\gamma_2 G$$

is the identity. As $\overline{\phi}$ is an isomorphism, $\overline{\gamma}$ must be an automorphism of $F$, thus $\gamma$ is an automorphism of $F$ which by construction takes $N_1$ onto $N_2$. Again the proof is the same for $Z_1G$ and $QG$.

Remark: The same proof yields the above result for quotients $G_1$ and $G_2$ of a free nilpotent of class $c$ and metabelian group $F$ by isolated normal subgroups $N_1$ and $N_2$ each contained in $\gamma_2^F$.

Recall ([13], pp. 288-9) that a bracket arrangement $\beta^n$ of weight $n$, $n = 1, 2, 3, \ldots$ is defined recursively as a sequence of asterisks and brackets as follows. There is one bracket arrangement of length one, $\beta_1 = [\ast]$, and a bracket arrangement of weight $n > 1$ is obtained by choosing bracket arrangements $\beta^k$ and $\beta^l$ of lengths $k$ and $l$ respectively so that $k + l = n$ and setting $\beta^n = [\beta^k, \beta^l]$. In practice the brackets about a single asterisk in such an expression will be suppressed.

Lemma 7.3: Suppose that $F$ is nilpotent of class $c$ and that $\beta$ is any bracket arrangement of length $c$; then the function $\beta : F/\gamma_2^FX \ldots \times F/\gamma_2^F$ (c copies) $\to \gamma_c^F$ defined by $\beta(\bar{a}_1, \ldots, \bar{a}_c)$, which is obtained by replacing the $c$ asterisks in $\beta$ by representatives $a_1, \ldots, a_c$ of $\bar{a}_1, \ldots, \bar{a}_c$ in order to obtain a commutator of length $c_1$ is well defined and $c$-linear.
That is any commutator of length c in F is a multilinear function of its entries and depends only on the coset modulo $\gamma_2^F$ of those entries.

**Proof:** $\beta$ defines a function from $F \times \ldots \times F$ to $\gamma_c^F$ so that if we show that this function is multilinear, we will have that the function $\beta$ above is well defined and multilinear since if any entry in a commutator of length c is contained in $\gamma_2^F$, the commutator lies in $\gamma_{c+1}^F = 1$. The multilinearity is obvious if the class c of F is one. Suppose then that the lemma is known for all groups of class $k < c$, that F is of class c and that $\beta^c = \beta = [\beta^k, \beta^l]$. Suppose then that $a_1, \ldots, a_c, b_j$ are elements of F and suppose for definiteness that $j \leq k$; then we have

$$\beta^c(a_1, \ldots, a_j b_j, \ldots, a_c)$$

$$= [\beta^k(a_1, \ldots, a_j b_j, \ldots, a_k), \beta^l(a_{k+1}, \ldots, a_c)]$$

$$= [\beta^k(a_1, \ldots, a_j, \ldots, a_k), \beta^k(a_1, \ldots, b_j, \ldots, a_k) \cdot \gamma, \beta^l(a_{k+1}, \ldots, a_c)]$$

where $\gamma$ is an element of $\gamma_{k+1}^F$, by applying the inductive hypothesis to $F/\gamma_{k+1}^F$ and $\beta^k$. Using Theorem E, part (3) and the fact that $[\gamma, \beta^l(a_{k+1}, \ldots, a_c)]$ is an element of $\gamma_{c+1}^F = 1$ we have

$$\beta^c(a_1, \ldots, a_j b_j, \ldots, a_c) = [\beta^k(a_1, \ldots, a_j, \ldots, a_k), \beta^l(a_{k+1}, \ldots, a_c)] \cdot$$

$$[\beta^k(a_1, \ldots, b_j, \ldots, a_k), \beta^l(a_{k+1}, \ldots, a_c)]$$

$$= \beta^c(a_1, \ldots, a_j, \ldots, a_c) \cdot \beta^c(a_1, \ldots, b_j, \ldots, a_c).$$
Corollary 7.4: The image of an element of $\gamma_cF$, under an endomorphism $\varphi$ of $F$, a nilpotent group of class $c$, depends only on the induced endomorphism $\tilde{\varphi}$ of $F/\gamma_2F$.

8. One-relator nilpotent groups

We now study one-relator nilpotent groups of class $c$, that is groups, $G_x$, which are quotients of free nilpotent groups $F$ of class $c$ on $n$-generators by the cyclic subgroup generated by an element $x$ of $\gamma_cF$ which is not a proper power. By proposition 7.1 and Corollary 7.4, $G_x$ is isomorphic to $G_y$ (resp. $Z_pG_x \cong Z_pG_y$, resp. $\Omega G_x \cong \Omega G_y$) if and only if there is an element $\varphi$ of $GL(n,Z)$ (resp. $GL(n,Z_p)$, resp. $GL(n,Q)$) which takes $x$ to $y^u$ where $u$ is $\pm 1$ (resp. a $p$-adic unit, resp. a non-zero rational number) where, by $\varphi$ takes $x$ to $y$, we mean that the automorphism of $F$ given on a set of generators $x_1, \ldots, x_n$, by $x_i \rightarrow \prod_{j}^{\varphi_{ij}} x_j$ with $\varphi_{ij}$ the $(i,j)$-entry in the matrix $\varphi$, sends $x$ to $y$.

If we choose a basis $w_\alpha$ for $\gamma_cF$ and if an element $x$ of $\gamma_cF$ can be written $x = \prod_{\alpha} a_\alpha^j$, we may define a homomorphism $\varphi$ from $\gamma_cF$ to the integers by $\varphi(w_\alpha) = j_\alpha$. If $\beta$ is the left-normed bracket arrangement of length $c$ and $\beta$ also denotes the corresponding homomorphism we have that the map $f = \varphi \circ \beta: F/F' \times \ldots \times F/F' \rightarrow Z$ is a $c$-multilinear form, which we will say corresponds to $x$. As the image of $\beta$ contains a generating set for $\gamma_cF$ by Theorem F, each such
f corresponds to a unique $x \in \gamma_c F$. The forms $f$ have
certain symmetry properties determined by relations among
commutators, for instance

$$f(a_1, a_2, \ldots, a_c) = -f(a_2, a_1, \ldots, a_c)$$

and

$$f(a_1, a_2, a_3, \ldots, a_c) \cdot f(a_2, a_3, a_1, \ldots, a_c) \cdot f(a_3, a_1, a_2, \ldots, a_c) = 1$$

the latter given by theorem E, part (4). We will term such
forms C-forms.

Two such forms are said to be equivalent (resp. $p$-
adically equivalent, resp. rationally equivalent) if there
is an automorphism $\varphi$ of $F/F'$ (resp. $\mathbb{Z}_p (F/F')$, resp. $Q(F/F')$)
such that $f = g \circ (\varphi x \ldots x \varphi)$. It is clear that an automorphism
$\varphi$ of $F/F'$, which takes an element $x$ in $\gamma_c F$ to $y$ in $\gamma_c F$,
also gives an equivalence of the G-form $f$ corresponding
to $x$ with the C-form $g$ corresponding to $y$ and conversely,
so we have

**Proposition 8.1:** Two one relator groups $G_x$ and $G_y$ are
isomorphic (resp. $p$-adically isomorphic, resp. commensurable)
if and only if the C-form $f$ corresponding to $x$ is equivalent
(resp. $p$-adically equivalent, resp. rationally equivalent)
to $\mu g$, where $g$ is the C-form corresponding to $y$ and $\mu$ is
$\pm 1$ (resp. a $p$-adic unit, resp. a non-zero rational).

**Example (8.2):** The class of one-relator, class two nilpotent
groups. If $F$ is free nilpotent of class two on $n$-generators
\(x_1, \ldots, x_n\) the elements \([x_i, x_j] \mid 1 \leq i < j \leq n\) form a basis for \(\gamma_2^F\) so that any element \(x\) of \(\gamma_2^F\) has the form \(\sum_{i<j}^\alpha_{ij}[x_i, x_j]\).

The form \(f\) corresponding to \(x\) is an antisymmetric bilinear form on \(F/F'\), whose matrix in the basis \(\bar{x}_1, \ldots, \bar{x}_n\) is \((\alpha_{ij})\) where \(\alpha_{ji} = -\alpha_{ij}\) for \(i < j\). Thus, since by Theorem G we may classify bilinear antisymmetric forms with integer coefficients, we have

**Proposition 8.3:** The isomorphism problem for one relator, class two nilpotent groups is solvable.

If \((\alpha_1, \ldots, \alpha_r)\) is a sequence of integers as given in theorem G for a form on \(F/F'\), the same sequence satisfies the hypotheses of theorem G when \(\mathfrak{r}\) is considered as a form on \(Z_p(F/F')\). Thus if \((\beta_1, \ldots, \beta_r)\) is a sequence of integers given for the form \(\mathfrak{r}\) and the forms \(\mathfrak{r}\) and \(\mathfrak{v}\) are \(p\)-adically equivalent for all primes \(p\), we have by the uniqueness statement of theorem G that for each prime \(p\) there are \(p\)-adic units \(\mu_{p, i}\) such that \(\alpha_i = \mu_{p, i} \beta_i\).

However, for this to be true for all primes, we must have \(\alpha_i = \pm \beta_i\) for each \(i\), and the forms are equivalent, so we have shown:

**Proposition 8.4:** If \(G_x\) and \(G_y\) are one relator class two nilpotent groups with isomorphic finite quotients then \(G_x\) is isomorphic to \(G_y\).
Example 8.5: One relator class three nilpotent groups on three generators. The elements

\[ w_{11} = [y, z, x] \quad w_{12} = [y, z, y] \quad w_{13} = [y, z, z] \]

\[ w_{21} = [z, x, x] \quad w_{22} = [z, x, y] \quad w_{23} = [z, x, z] \]

\[ w_{31} = [x, y, x] \quad w_{32} = [x, y, y] \quad w_{33} = [x, y, z] \]

generate \( \gamma^3_3 F \) subject to the single relation \( w_{11} w_{22} w_{33} = 1 \), where \( F \) is a free nilpotent group of class three on generators \( x, y, z \). Thus any element of \( \gamma^3_3 F \) may be uniquely expressed as \( \prod a_{ij} \) with \( a_{11} + a_{22} + a_{33} = 0 \) and the \( a_{ij} \) may thus be arrayed as a matrix \( (a_{ij}) \) of trace zero. A tedious computation shows that the matrix \( (m_{ij}) \) in \( GL(3, \mathbb{Z}) \) takes \( \prod a_{ij} \) to \( \prod b_{ij} \) where \( b_{ij} \) is given by

\[
(b_{ij}) = (m_{ij})^{-1} (a_{ij}) (m_{ij}) \det(m_{ij})
\]

and "takes" is in the sense of the first paragraph of this section. Thus we have

Proposition 8.6: The isomorphism problem for one relator class three nilpotent groups on three generators is equivalent to the problem of deciding whether a given three-by-three integral matrix of trace zero is conjugate to \( \pm 1 \) times another matrix of the same type.
9. **H-Groups and R-Groups**

**Lemma 9.1**: If $F$ is a free metabelian class $c$ nilpotent group and $\beta = [[[a_1, a_2], a_3], \ldots, a_c]$ is any left normed commutator of length $c$, then $\beta$ is symmetric in the last $(c-2)$ variables, that is

$$[a_1, a_2, a_3, \ldots, a_c] = [a_1, a_2, a_\sigma(3), \ldots, a_\sigma(c)]$$

where $\sigma$ is any permutation of $(3, \ldots, c)$.

**Proof**: As $F$ is metabelian, the group ring $Z(F/F')$ acts on $F'$ and we may write $[a_1, a_2, a_3, \ldots, a_c]$ as

$$\ldots([a_1, a_2](-1+a_3)(-1+a_4)\ldots(-1+a_c')$$

where exponentiation denotes the action of the elements of $Z(F/F')$. Since $Z(F/F')$ is commutative, we have

$$(1+a_3)(-1+a_4)\ldots(-1+a_c) = (-1+a_\sigma(3))(-1+a_\sigma(4))\ldots(-1+a_\sigma(c))$$

and the result follows.

We now define R-groups and H-groups, two classes of one relator, metabelian, nilpotent groups. The R-groups are just the one relator, metabelian, nilpotent groups on two generators $x$ and $y$. An H-group is the quotient of a free metabelian nilpotent group $F$ of class $c$ on generators $x, y, z_1, \ldots, z_n$ by an element of the subgroup $K$ of $\gamma_c F$ generated by
the elements \([x, y, z_1, \ldots, z_c]\).

For a free metabelian nilpotent group \(F\) of class \(c\)
on two generators \(x\) and \(y\), the elements \(w_i=[x, y, x \ldots x, y, \ldots, y]\),that is commutators with \(x\) followed by \(y\) followed
by \(i\) \(x\)'s followed by \((c-i-2)\) \(y\)'s, form a basis of \(\gamma_c F\),
for \(0 \leq i \leq c-2\). Therefore the function corresponding
to an element \(x = \prod w_i^{a_i}\), when restricted to the last \((c-2)\)
factors \(F/F'\) gives a symmetric multilinear function of
two variables which in turn is determined by a form of
degree \((c-2)\) in variables \(x\) and \(y\) namely \(\sum_{i=0}^{c-2} a_i x^i y^{c-2-i}\).

If \(f\) is a form corresponding to \(x\) and \(g\) is the
form corresponding to \(y\) and if \(\phi\) is an element of
\(GL(2, \mathbb{Z})\) which takes \(x\) to \(y\) then \(\phi\) takes \(f\) to \((\det \phi) g\).
Thus we have

**Proposition 9.2:** Two \(R\)-groups \(G_x\) and \(G_y\) of class \(c\)
with corresponding forms \(f\) and \(g\) are isomorphic (resp.
p-adically isomorphic, resp. commensurable), if and only if
\(f\) is equivalent (resp. \(p\)-adically equivalent, resp. ration-
ally equivalent) to \(\mu g\), where \(\mu = \pm 1\) (resp. \(\mu\) is a \(p\)-adic
unit, resp. \(\mu\) is a non-zero rational).

Remeslemnikov's original example consisted of two
\(R\)-groups of class 4, corresponding to the quadratic forms
\(2x^2 + xy + 3y^2\) and \(x^2 + xy + 6y^2\) which are not equivalent
(and \(2x^2+xy+3y^2\) is not equivalent to \(-x^2-xy-6y^2\) either)
but which are \(p\)-adically equivalent for all \(p\). They are
also rationally equivalent as they must be since two
forms on 2 or 3 variables which are p-adically equivalent
for all p must have the same genus and thus be rationally
equivalent ([16], p. 73).

For a free metabelian nilpotent group $F$ of class $c$,
on generators $x, y, z_1, \ldots, z_n$, the elements $w_{i_1, i_2, \ldots, i_n} = [x, y, z_1, \ldots, z_1, z_2, \ldots, z_2, z_3, \ldots, z_n]$ with

$i_j$'s for each $j$ and $i_1 + i_2 + \ldots + i_n = c-2$ form a
basis of the subgroup $K$ of $\gamma_cF$ defined above. For any
element $x$ of $K$ the form corresponding to $x$, when restricted
to the last $(c-2)$ factors $F/F'$ and to the subgroup of $F/F'$
generated by $\tilde{z}_1, \ldots, \tilde{z}_n$ in each factor give a $(c-2)$-linear
function in $n$-variables which determines $x$ and which is
determined by a form of degree $c-2$ in the variables
$z_1, \ldots, z_n$, namely if $x = \prod w_{i_1, \ldots, i_n}^a$ then the corresponding
form is $\sum a_{i_1, \ldots, i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}$. If an automorphism $\varphi$
of $F/F'$ takes an element $x$ of $K$ to an element $y$ of $K$,
the matrix of $\varphi$ in the basis $x, y, z_1, \ldots, z_n$ must have
the form:

$$(\varphi_{ij}) = \begin{pmatrix} M' & R \\ 0 & M'' \end{pmatrix},$$

where $M'$ is $2 \times 2$, $M''$ is a $(c-2) \times (c-2)$ matrix which
takes the form corresponding to $x$ onto $(\det M'')$ times
the form corresponding to $y$. Thus we have
Proposition 9.3: If $G_x$ and $G_y$ are two $H$-groups on $n + 2$ generators of class $c$, and $f$ and $g$ respectively are the forms of degree $c$ in $n$ variables corresponding to $x$ and $y$, then $G_x$ and $G_y$ are isomorphic (resp. $p$-adically isomorphic, resp. commensurable) if and only if $f$ is equivalent (resp. $p$-adically equivalent, resp. rationally equivalent) to $\mu g$ where $\mu$ is $\pm 1$ (resp. a $p$-adic unit, resp. a non-zero rational).

Higman's original example consisted of two $H$-groups of class four on seven generators, corresponding to the forms $z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2$ and $z_1^2 + z_2^2 + z_3^2 + z_4^2 - z_5^2$ which are $p$-adically equivalent for each prime $p$, but as they have different signatures the first can not be rationally equivalent to any rational multiple of the second, so that the corresponding groups have isomorphic finite quotients but are not commensurable.

Proposition 9.4: The isomorphism problem for $H$-groups of class $c$ on $n + 2$ generators is equivalent to the problem of determining whether a given form of degree $c-2$ on $n$ generators is equivalent to $\pm 1$ times another such form also given.

If we restrict to $H$-groups of class 4 on $n+2$ generators the forms are $n$-ary quadratic forms for which there is a rich theory. It was the finiteness of the class number of a quadratic form which motivated Theorem 6.3. In one proof of the finiteness of the class number,
a crude bound is computed using reduction theory ([16], pp. 19-20). For a given number of variables this upper bound increases rapidly with the discriminant of the form, which indicates that for torsion free nilpotent groups it is highly unlikely that a bound on the class number exists which depends only on the class of nilpotency and number of generators or relations. As we do not know of any examples of quadratic forms with arbitrarily large class number, no definite statement can be made though some such a result must be in the literature if true.

The properties of quadratic forms also hint at another conjecture concerning a "Hasse principle" for a torsion free finitely generated nilpotent groups. One might guess that if two such groups had isomorphic finite quotients, and the real lie groups obtained in the usual manner were isomorphic, then the groups must have been commensurable. The H-groups of class greater than four may be a good place to find counterexamples.

10. Stick-up groups

Let $G_n$ be the nilpotent group of class two group on three generators $a, b, c$ which satisfies the single additional relation $c_p^n = [a, b]$ for some fixed prime $p$. Since $p^n$ is a $q$-adic unit if $q$ is a prime other than $p$, we have for such $q$, $Z_{q^n} G_n \cong Z_{q^m} G_m \cong Z_{q^n}$ for all $m$ and $n$, where $F$
is a free nilpotent group of class two on two generators. Also as \( G_n/\gamma_2 G_n \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/(p^n\mathbb{Z}) \), we have \( G_n \neq G_m \) for \( m \neq n \). Thus we have an infinite set of non-isomorphic groups, which are all \( q \)-adically isomorphic for all primes \( q \) but one, so that the remark after theorem 4.2 does not hold in the context of theorem 6.3.
Bibliography


