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WITH FADDEEV THEORY.

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THREE-ALPHA FINAL STATE INTERACTIONS
WITH FADDEEVO THEORY

by

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CHAPTER I

INTRODUCTION

Three-body systems occur frequently in both high energy and nuclear physics and in many cases two-body interactions can be studied only as subsystems of three particles. Until recently, understanding of these systems was limited because of the lack of an adequate theory. An important advance was made by Faddeev (1) in reducing the non-relativistic three-body problem to that of solving a set of coupled integral equations. However, these equations are difficult to solve and require complete knowledge of the off shell two-particle scattering amplitudes.

Most advances in the solution of the Faddeev equations have been made by assuming some simple form for the off shell two-particle amplitudes. The usual choice is to assume that amplitudes are separable in the initial and final momenta. This assumption leads to a significant reduction in the complexity of the Faddeev equations and has been given justification by Lovelace (2) for amplitudes dominated by bound states or resonances.

In addition to studying two-body systems which are available only as subsystems of three particles there are several reasons for
studying three-body problems. From experimental two-body elastic scattering, only the on shell matrix elements of the t-matrix are known. Many interaction models are able to reproduce the same on shell behavior and knowledge of the off shell matrix elements is required to distinguish between the models. Three-body calculations can be used to discriminate between the models because of the dependence on the off shell behavior. Many three-body systems occur at high energies where the need for a relativistic extension of the Faddeev theory would be required. This is not available; however, by studying non-relativistic systems in nuclear physics, insights can be gained which will be applicable in all regions.

The simplest three-body system would be one of identical spinless particles. There is such a physical system available, that of three alpha particles. The alpha particles meet the foregoing criteria, and because of the tight binding, an alpha particle can be considered as a single particle to a good approximation. The two-body interactions are dominated by resonances which are well known from alpha-alpha scattering and should be good candidates for approximation by a separable t-matrix.

Additional motivation for studying this problem is the availability of extensive data from the study of the reaction $^{11}$B$(p, \alpha) 2 \alpha$ to serve as a proving ground for the calculations.
These experiments detect two particles in coincidence and thus uniquely determine the kinematics of the reaction. The data from this reaction contain interesting structure such as interference minima and narrowing of two-body resonance widths. Thus, this problem seems to be a promising place to study final state interactions in three-particle states.

This problem is not completely ideal, however. It is complicated by the presence of Coulomb and hard core interactions; the interaction which initiates the breakup and the initial state wavefunction are not known; and the structure of the alpha particles cannot be included.

We have considered an exactly soluble model which describes the breakup of a parent particle into three identical, spinless, structureless daughter particles. Using this model the breakup of a $2^+$ particle with final state interactions dominated by two-body resonances has been studied. The two-body resonances were chosen to correspond to the ground and first excited states of $\text{Be}^6$ and the parent particle to correspond to the 18.37 MeV. state of $\text{C}^{12}$.

The model treats the interaction which initiates the breakup to first order and the strong final state interactions to all orders. The strong interactions are described by equations of the Faddeev type.
with separable two-body interactions in order to make the problem numerically tractable.

Several papers of general content are relevant to this problem. Amado and Noble (3), using a similar model, examined the breakup of a spinless particle into three identical spinless particles with separable S-wave interactions. They calculated the total decay rate as a function of the total energy and potential strength and found large enhancements and de-enhancements over all ranges of the potential strength as well as a striking dependence on the spatial extent of the bound state vertex. For a nearly point bound state vertex the strong interactions affected the final state wavefunction more than for a spread out vertex. The total energies and resonance positions were chosen so that two pairs were allowed to resonate simultaneously but no anomalies were observed in the total rate.

In calculating Dalitz plots a strong dependence of the size of the bound state vertex was again observed. For a nearly point vertex the final state interactions dominated and the resonance bands were prominent, whereas for a spread out vertex the contribution from direct decay was of the same order as the final state effects, and interference between the two effects occurred. The resonance bands no longer dominated and the Dalitz plots had a smeared appearance even though the strong interactions were the same in both cases. The
size of the initial vertex entered only the inhomogeneous term and not in the rescattering. Amado and Nobel observed regions of constructive and destructive interference in the region of overlapping resonance bands but did not see any effects of rescattering singularities.

They attempted to extract resonance parameters from the singles spectra and the difficulty in doing this accurately was demonstrated. Narrowing and broadening of the widths as well as increases and decreases in the resonance energy from the two-body input were found. Furthermore, no pattern for this change could be discovered. The extracted parameters were more in agreement with the input parameters when the size of the bound state vertex was small and the resonances far from the limits of phase space.

Schmid (4) used perturbation theoretic arguments to determine whether or not rescattering singularities could produce peaks on invariant mass plots that simulate resonances. The singularities of the triangle diagram were examined to find out if a pole was produced near the physical region. Even when such a pole was near the physical region and produced a rescattering band on the Dalitz plot, the projection of the Dalitz plot onto an invariant mass plot did not produce a peak.
Schmid also examined the interference of overlapping resonance bands and found no effect to first order in the width of the resonance to the diameter of the Dalitz plot. The interference rearranged events on the Dalitz plot but its projection showed no effect. Because the average of the interference terms over the overlap region is zero, an effect would only be expected when the overlap region is near the boundary of the Dalitz plot. In that case, some of the interference effects would be outside the physical region and would not contribute to the projection onto an invariant mass plot.

Chen-Chung and Sommerfeld (5) have examined three-particle final states in the framework of the Lee model to ascertain if the overlap of two-particle resonances could simulate a three-particle resonance. No effect was found other than one artificially constructed by ignoring interference effects.

Of the papers directly concerned with this reaction one of the most relevant is the calculation by Duck (6). He developed a phenomenological method to study this reaction by considering the relevant Feynman diagrams for sequential decay. The vertex functions were constructed using the Breit-Wigner resonance theory, the Watson theory (7), and the Phillips-Biedenharn-Griffy (8) theory. Most of the calculations were made with the Breit-Wigner theory where the two-body parameters were taken from alpha-alpha
scattering. A good fit to the data was obtained without an extensive parameter search. The observed interference minima were found and the fit to the overall envelope was good, although there were some unexplained peaks which did not appear in the data. The fit was improved by the inclusion of a hard sphere phase shift as prescribed by the Phillips-Biedenharn-Griffy theory. Alignment of the initial C^{12}_1 state was eliminated as an important factor in understanding the results.

Komarov and Salman (9) also have made a calculation using a sequential decay model with vertex functions constructed of Breit-Wigner resonances with Coulomb effects and potential scattering included. Rescattering effects were included only phenomenologically by the adjustment of parameters. The calculations were compared to the data for the decay of the J^{π} = 1^- state at 17.23 MeV. and the width of the Be^8 D-state was narrowed until the data were fitted. This seems to be in conflict with the findings of Amado and Noble who found both narrowing and broadening of the widths with no pattern for the change. There is no \textit{a priori} reason to narrow the widths. Instead, these effects should appear due to rescattering and interference. Because the results were presented at only one set of angles, the quality of fit over an entire range could not be determined.
This reaction has been examined using R-matrix theory by Goulard (10) and the results compared to the data for the decay of the 17.77 and 18.37 MeV. states; rescattering effects were not included. The interference effects narrowed the peak resulting from the overlap of two resonances and the data for the decay of the 17.77 MeV. state were well fitted. The data for this state do not exhibit the interference minima that are observed in the decay of the 18.37 MeV. state. The calculations for the 18.37 MeV. state did not reproduce the observed interference minima.

Calculating the binding energy of $^{12}C$ in this model has been the subject of several investigations. Harrington (11) and this author (12) have examined the problem in the absence of Coulomb effects using a variety of separable potentials and found the binding energy insensitive to the form of the potential. The D-state of $^{8}Be$ was found to make only a small contribution. Duck (13) used a separable off shell extension of a Breit-Wigner resonance and found a $^{12}C$ resonance but no bound state. Fulco and Wong (14) have accomplished the most complete treatment of this problem by expanding the t-matrix in a separable series of functions which are solutions of the homogeneous Lippmann-Schwinger equation. They included both Coulomb and hard core effects and found a binding energy considerably lower than the experimental value. A variational
calculation by Herndon and Tang (15) found a similar low value. These calculations point to the need to include nucleon exchange effects, that is, allow for structure of the alpha particles. It is thought that in the breakup problem the alpha clusters will be more widely separated and the need for including alpha structure will be reduced.

This reaction has been studied experimentally by Bronson, et al. (16), at a variety of proton energies and involving several resonant states of C\textsuperscript{12}. The kinematics of the reaction were completely determined by detecting two of the final particles in coincidence. The reaction was determined to be primarily sequential decay through the ground and first excited states of Be\textsuperscript{8} with strong interference effects involving the first excited state. These interference effects were attributed to the identity of the particles and ambiguity in the order of emission. Less than 5 percent of the cross section was attributed to direct three-body decay.

The reaction involving the J\textsuperscript{π} = 2\textsuperscript{−} state at 16.57 MeV. has been examined by Cockburn, et al. (17). The results indicated sequential decay through the first excited state of Be\textsuperscript{8}. Using a sequential model with Breit-Wigner resonances to interpret the data, they found the two-body resonance parameters to be virtually unchanged from the values obtained from two-body scattering.
Interference effects were observed on the Dalitz plot and a deviation of the model predictions from the data was attributed to rescattering.

In Chapter II the development of the integral equation, the partial wave decomposition and the reduction using the separable t-matrix approximation are given. Chapter III contains a discussion of the two-body interactions used to generate the separable t-matrix and in Chapter IV the form taken for the initial bound state breakup vertex is presented. Numerical methods for the solution of the integral equations are described in Chapter V; in Chapter VI, results of calculations and the comparison with the experimental findings are presented and discussed. Some reasonable extensions and improvements of the model are outlined in Chapter VII; Chapter VIII contains a summary and the conclusions. Details of the angular momentum decomposition are treated in Appendix A, and Appendix B contains the details of the laboratory coordinate system and the necessary kinematics.
CHAPTER II

DEVELOPMENT OF EQUATIONS

The amplitude for the breakup of a particle into three final particles is described by the matrix element

$$\langle \psi_3 | H_w | \phi \rangle = \langle \varphi_3 | U | \phi \rangle,$$

(2.1)

where $H_w$ is a supposedly weak residual interaction, not diagonalized by the bound and continuum states, which initiates the breakup; $U$ takes an initial state into three outgoing plane wave states including all final state interactions; $| \phi \rangle$ is the initial state and is discussed in Chapter IV; and the full scattering state $\langle \psi_3 |$ is given in terms of plane wave states $\langle \varphi_3 |$ by the Lippmann-Schwinger equation

$$\langle \psi_3 | = \langle \varphi_3 | + \langle \psi_3 | V G_0$$

$$\langle \psi_3 | = \langle \varphi_3 | + \langle \varphi_3 | T G_0.$$

In the foregoing $G_0$ is the free three-particle Green's function

$$G_0 = (S - H_0 + i\varepsilon)^{-1},$$

where $S$ is the energy, $H_0$ is the kinetic energy operator, and the small imaginary part provides for the boundary condition of
outgoing waves. The three-particle T-matrix satisfies

\[ T = V + V_G T. \]

It is assumed the potential acts only between pairs, that is,

\[ V = \sum_i V_i, \]

where the subscript labels the spectator particle. Now make a Faddeev decomposition of the T-matrix,

\[ T = \sum_i T_i, \]

where \( T_i \) is that part of the three-particle T-matrix with the pair \( i \) interacting last. With these substitutions the Lippmann-Schwinger equation separates to the Faddeev equations

\[ T_i = t_i + \sum_{j \neq i} t_i G_j T_j, \]

where \( t_i \) is the two-particle T-matrix in the three-particle Hilbert space.

Returning to Equation (2.1) and substituting for the final state wavefunction \( \langle \psi_3 | \),

\[ \langle \psi_3 | U | \psi \rangle = \langle \psi_3 | H_W | \psi \rangle + \langle \psi_3 | T G H_W | \psi \rangle. \]

Leaving matrix elements understood.
\begin{align*}
U &= H_w + TG_0 H_w \\
U &= H_w + \sum_i T^i G_0 H_w.
\end{align*}

Making a decomposition of $U$ similar to the decomposition of $T$,

\begin{align*}
U &= H_w + \sum_i U_i.
\end{align*}

Therefore,

\begin{align*}
U_i &= T^i G_0 H_w \\
U_i &= \left[ t_i + \sum_j T^j G_0 T^j \right] G_0 H_w \\
U_i &= t_i G_0 H_w + \sum_j t_i G_0 U_j. \quad (2.2)
\end{align*}

The meaning of these terms is displayed in the diagrammatic representation shown in Figure 1.

We take for final states the equivalent set of center-of-mass plane wave states

\begin{align*}
\langle \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle &= \langle \vec{p}_1 \vec{k}_1 \rangle = \langle \vec{p}_2 \vec{k}_2 \rangle = \langle \vec{p}_3 \vec{k}_3 \rangle,
\end{align*}

where $\vec{p}_i$ is the momentum of particle 1 and

\begin{align*}
\vec{k}_1 &= \frac{1}{2} (\vec{p}_3 - \vec{p}_2)
\end{align*}

is the momentum of particle 3 relative to the center-of-mass of the
Figure 1

A diagrammatic representation of the integral equations.
2, 3 subsystem. By cyclic interchange of the subscripts \( \vec{k}_2 \) and \( \vec{k}_3 \) may be obtained.

Taking matrix elements of Equation (2.2) between plane wave final states and the initial state \( | \phi \rangle \) we obtain

\[
\langle \vec{p}_i', \vec{k}_i' | U_i | \phi \rangle = \langle \vec{p}_i' \vec{k}_i' | t_1 G_0 H_\omega | \phi \rangle + \sum_{j \neq i} \langle \vec{p}_i' \vec{k}_i' | t_1 G_0 U_j | \phi \rangle.
\]

The total angular momentum and its projection are constants of the motion so it is convenient to change to a representation in which \( J \) is diagonal. These states are given by

\[
\langle \vec{p}_i' \vec{k}_i' | = \sum_{\ell' = 0}^{\infty} \sum_{m_{\ell'}}^{\infty} Y_{\ell'}^{m_{\ell'}} (\vec{p}_i ') \ Y_{\ell'}^{m_{\ell'}} (\vec{k}_i ') \ \left( \begin{array}{ccc} m_{\ell'} & m_{\ell'} & m_{\ell'} \\ \ell' & \ell' & \ell' \end{array} \right) \langle p_i', k_i', L', J' m_{J'} |.
\]

Details of the partial wave decomposition are given in Appendix A.

Since the particles are identical the set of three coupled equations becomes a single integral equation. The partial wave matrix elements of \( U_i \) are related

\[
\langle p_i', k_i', L', J' m_{J'} | U_i | \phi \rangle = (-1)^{L'} \langle p_i', k_i', L', J' m_{J'} | U_i | \phi \rangle
\]

with the initial state \( | \phi \rangle \) properly symmetrized.

The partial wave equations in the representation \( \langle \vec{p}_i \vec{k}_i | \) are
\[ \langle \rho', l', k', L', J_{m_{\sigma'}} | U, | \Phi \rangle = \frac{2}{\pi} \int_0^\infty k_i^2 \, dk_i \]

\[ G_0(\rho_i^2, k_i^2, s) \langle k', L' | t_i(\frac{s-3\nu}{\rho_i^2}) | k, L' \rangle \]

\[ \langle \rho', l', k, L', J_{m_{\sigma'}} | H_\omega | \Phi \rangle \]

\[ + \frac{16\sqrt{\pi}}{\rho_i'} \int_0^\infty r \, dr \int_0^\infty \left. \frac{(p_i'^2 + n_i'^2)^2}{(p_i'^2 + n_i'^2)} \right. \]

\[ \langle k', L' | t_i(\frac{s-3\nu}{\rho_i'^2}) | k, L' \rangle \sum_{\delta L} \sqrt{2L+1} \]

\[ (-1)^{L'+L-\delta-\lambda} \sum_{\lambda} Y^\ast_{L'}(\Theta_{\rho_i'k_i}, 0) Y^\ast_{L'}(\Theta_{k_iL_i}, 0) \]

\[ \sum_{\lambda} Y_{\lambda}(\Theta_{\rho_i'k_i}, \pi) \left( \begin{array}{c} l' \ L' \ J' \\ n_2', n_1, -n_2 \end{array} \right) \left( \begin{array}{c} l \ L \ J \\ n_2, 0 \ -n_2 \end{array} \right) \]

\[ \langle \rho_2, l, k_1, L, J_{m_{\sigma'}} | U, | \Phi \rangle. \]

The inhomogeneous term gives what may be called the Born amplitude. Strictly speaking, this is an impulse approximation rather than a Born approximation but it is referred to as the Born amplitude because it is the first term in the iterated solution of the integral equations. This term corresponds to only one pair undergoing final state interactions after the breakup, with the third particle leaving the three-particle interaction volume.
The matrix element of $H_\mu$ between the initial state and an intermediate state will have to be parameterized. This is discussed in Chapter IV.

We now have a set of two-dimensional coupled integral equations, the straightforward numerical solution of which is beyond the capabilities of present day computers. To reduce the equations to a level suitable for numerical solution a $t$-matrix separable in the initial and final momenta is used. That is,

$$t_L(k', k, S) = \frac{\lambda_L v_L(k') v_L(k)}{D_L(S)},$$

where the meaning of these terms is discussed in Chapter III.

The use of a separable $t$-matrix reduces the equation to a one-dimensional integral equation. Defining

$$\langle p_1', l', k_1', L', J'm_{j'} | U_1 | \phi \rangle = \frac{v_{l'}(k'_1)}{D_L(S - 3/4 p_{l'}^2)} \langle p_1', l', L', J'm_{j'} | U_1 | \phi \rangle$$

the equation becomes

$$\left(p', l', L', J'm_{j'} | U_1 | \phi \right) = \frac{2}{\pi} \int_0^\infty k_i^2 dk_i \int G_\omega(p', k_i^2, S) v_{l', k_i^2} \left(p', l', k, L', J'm_{j'} | H_\omega | \phi \right)$$
finite imaginary part and moving the pole away from the contour of integration before solving the equations. After the solution is obtained at complex energies it is continued numerically to real, physical energies.
CHAPTER III

TWO-BODY INTERACTIONS

One way of obtaining a separable t-matrix is to use a separable potential of the form

$$V_\ell (k', k) = \lambda_\ell v_\ell (k') v_\ell (k),$$

where $k$, $k'$ are the relative momenta in initial and final states, $\ell$ is the relative angular momentum, and the form factor $v_\ell (k)$ and interaction strength $\lambda_\ell$ are chosen to fit the experimentally determined phase shifts. A series of separable terms could be used but here only a single term is used for simplicity in the calculations. This choice excludes treatment of the hard core interaction.

The Lippmann-Schwinger equation can be solved exactly for a separable potential where the scattering amplitude is given by

$$\langle k' \ell | t(S) | k \ell \rangle = \frac{\lambda_\ell v_\ell (k') v_\ell (k)}{D_\ell (S)},$$

where the denominator function

$$D_\ell (S) = 1 + \frac{2 \lambda_\ell}{\pi} \int_{0}^{\infty} \frac{q^2 dq}{q^2 - (S + i\epsilon)}.$$
The denominator function contains the phase of \( t(S) \) and vanishes at energies corresponding to bound states or resonances. On the energy shell, that is, \( k'^a = k^a = S \), the amplitude is

\[
\langle k' \ell | t(S) | k \ell \rangle = \frac{-e^{i \delta_\ell} \sin \delta_\ell}{k},
\]

where \( \delta_\ell \) is the phase shift for the partial wave with angular momentum \( \ell \).

A simple \( S \)-wave potential without Coulomb effects will not reproduce the extremely narrow \( \text{Be}^8 \) ground state resonance; hence, some modification is required. As indicated in Chapter VI the data for the three-body breakup cannot be fitted using amplitudes which only reproduce the two-body resonances. Because the relative pair energies are low the Coulomb force is important. Therefore, the next step is to examine a way to modify the simple form factors to include some Coulomb effects.

The form for a separable potential to represent the nuclear interaction in combination with a Coulomb potential has been given by Harrington (18). A sketch of the formalism is presented for completeness.

In order to define Coulomb phase shifts and partial wave amplitudes, we must suppose that the Coulomb potential is cut off
at some shielding radius $R$, which is large compared to the range of the separable nuclear potential. Define the following complete sets of states: the set of free particle plane wave states $|\vec{k}\rangle$, the outgoing and incoming Coulomb states

$$|\vec{k}(\pm)\rangle_c = |\vec{k}\rangle + [E \pm i\epsilon - H_0 - V_c]^{-1} V_c |\vec{k}\rangle$$

and the states including the full Coulomb and nuclear interaction

$$|\vec{k}(\pm)\rangle = |\vec{k}\rangle + [E \pm i\epsilon - H_0 - V_c - V_s]^{-1} (V_s + V_c) |\vec{k}\rangle$$

$$|\vec{k}(\pm)\rangle = |\vec{k}(\pm)_c\rangle + [E \pm i\epsilon - H_0 - V_c - V_s]^{-1} V_s |\vec{k}(\pm)_c\rangle,$$

where $V_c$ is the Coulomb potential and $V_s$ is the separable nuclear potential. The two-potential formalism is used with the Coulomb potential treated exactly. As described in detail by Goldberger and Watson (19), $S$-matrix satisfies

$$S(\vec{k}, \vec{k}') = \langle \vec{k}' (-) | \vec{k} (+) \rangle$$

$$S(\vec{k}, \vec{k}') = (2\pi)^3 \delta^3 (\vec{k}' - \vec{k}) - 2\pi i \delta (E' - E)$$

$$[T_{sc}(\vec{k}', \vec{k}, E) + T_c(\vec{k}', \vec{k}, E)],$$

where

$$T_c(\vec{k}', \vec{k}, E) = \langle \vec{k}' | V_c | \vec{k} (+) \rangle_c$$

$$T_{sc}(\vec{k}', \vec{k}, E) = \langle \vec{k}' (-) | V_s | \vec{k} (+) \rangle .$$
Here $T_c$ is the usual Coulomb scattering amplitude, and $T_{sc}$ satisfies the integral equation

$$T_{sc} (\mathbf{k}', \mathbf{k}, E) = c^2 \langle \mathbf{k}' (-) | V_s | \mathbf{k} (\pm) \rangle_c$$

$$+ \int d^3 k \quad c^2 \langle \mathbf{k}' (-) | V_s | \mathbf{k}'' (-) \rangle_c \frac{T_{sc} (\mathbf{k}'', \mathbf{k}, E)}{E + i \epsilon - k''^2}$$

which is reduced by partial wave analysis to a one-dimensional integral equation for the partial wave $t$-matrix

$$T_{sc, \ell} (k', k, E) = V_{sc, \ell} (k', k)$$

$$+ \frac{2}{\pi} \int_0^\infty k''^2 \, dk'' \quad V_{sc, \ell} (k', k'') \, T_{sc, \ell} (k'', k, E) \frac{E + i \epsilon - k''^2}{E + i \epsilon - k''^2}$$

where $V_{sc, \ell} (k', k)$ is the partial wave matrix element of

$$c^2 \langle \mathbf{k}' (-) | V_s | \mathbf{k} (\pm) \rangle_c .$$

On the energy shell the $t$-matrices are

$$T_{c, \ell} (k, k, k^2) = -e^{i \delta_{c, \ell}} \frac{\sin \delta_{c, \ell}}{k}$$

$$T_{c, \ell} (k, k, k^2) + T_{sc, \ell} (k, k, k^2) = -e^{i \delta_\ell} \frac{\sin \delta_\ell}{k} .$$

If a Coulomb corrected nuclear phase shift, $\delta_{sc, \ell}$, is defined such that
\[
\delta_{c, \ell} = \delta_{c, \ell} + \delta_{sc, \ell}
\]

then

\[
T_{sc, \ell}(k, k, k^2) = -e^{i} \delta_{c, \ell} \frac{e^{i \delta_{sc, \ell}} \sin \delta_{sc, \ell}}{k}.
\]

The potential \( V_{sc, \ell}^{\pm}(k', k) \) is a matrix element between Coulomb distorted initial and final state wavefunctions of the nuclear potential

\[
V_{sc, \ell}^{\pm}(k', k) = \left( \frac{1}{2 \pi^2} \right)^{\frac{1}{2}} \int_{0}^{\infty} q' q d q' \int_{0}^{\infty} q^2 d q \ c_{\ell}^* \langle \vec{k}(-) | \vec{q}' \rangle \langle \vec{q} | \vec{k}(\pm) \rangle_{c, \ell},
\]

where \( \langle \vec{q} | \vec{k}(\pm) \rangle_{c, \ell} \) is the partial wave momentum-space Coulomb wavefunction. When \( V_{s, \ell} \) is separable

\[
V_{s, \ell}(k', k) = \lambda_{\ell} \ g_{\ell}(k') \ g_{\ell}(k)
\]

then \( V_{sc, \ell}^{\pm} \) is also separable, that is,

\[
V_{sc, \ell}^{\pm}(k', k) = e^{i \delta_{c, \ell}(k')} \ g_{c, \ell}(k') \lambda_{\ell} \ g_{c, \ell}(k) \ e^{\pm i \delta_{c, \ell}(k')},
\]

where

\[
g_{c, \ell}(k) e^{\pm i \delta_{c, \ell}(k')} = \left( \frac{1}{2 \pi^2} \right)^{\frac{1}{2}} \int q^2 d q \ g_{\ell}(q) \langle \vec{q} | \vec{k}(\pm) \rangle_{c, \ell}.
\]
If $G_{\ell}(r)$ is the inverse Fourier transform of $g_{\ell}(k)$ then

$$g_{s,\ell}(k) = \frac{1}{4\pi k} \int_0^\infty r \, dr \, G_{\ell}(r) \, W_{\ell}(k, r),$$

where $W_{\ell}(k, r)$ is the solution of the Schrödinger equation with a cutoff Coulomb potential. However, for $R$ large and $k$ not too small this is approximately the true Coulomb solution

$$W_{\ell}(k, r) = F_{\ell}(kr),$$

where

$$F_{\ell}(kr) = (2i)^{-\ell-1} C_{\ell}(\eta) \, M_{\eta, \ell+1/2}(2i kr),$$

$$\eta = \frac{\mu e_1 e_2}{k}, \quad C_{\ell}(\eta) \text{ is the barrier penetration factor and}$$

$M_{\eta, \ell+1/2}(2i kr)$ is the confluent hypergeometric function. $F_{\ell}(kr)$ has the asymptotic form

$$F_{\ell}(kr) = \sin(kr - \eta \, \eta \, 2kr - \frac{1}{2} \ell \pi + \sigma_{\ell}),$$

where $\sigma_{\ell}$ is the Coulomb phase shift and is given by

$$\sigma_{\ell} = \text{ARG} \, \Gamma(\ell + 1 + i \eta).$$

$F_{\ell}(kr)$ takes on its asymptotic form at the shielding radius $R$ and
must join smoothly to the force free solution. We should therefore define a Coulomb phase shift

\[ \delta_{c, \ell} = \sigma_{\ell} - \eta \ln 2 \k R. \]

Because the potential is separable the solution as before is

\[ T_{sc, \ell}(k', k, E) = e^{i \delta_{c, \ell} (k')} \frac{g_{c, \ell}(k') \lambda_{\ell} g_{c, \ell}(k)}{D_{\ell}(E)} e^{i \delta_{c, \ell} (k)}. \]

Except for the Coulomb phase factors, \( T_{sc, \ell}(k', k, E) \) is obtained by replacing \( g_{\ell}(k) \) with \( g_{c, \ell}(k) \).

\( T_{sc, \ell}(E) \) contains the bound states and resonances, and at the poles for these states the pure Coulomb scattering amplitude makes no contribution. Therefore, in the neighborhood of these poles the total scattering amplitude should be well approximated by \( T_{sc, \ell} \).

The simplest separable potential is that given by Yamaguchi (20)

\[ g_{o}(k) = (k^2 + a^2)^{-1}. \]

With the Coulomb potential this potential becomes

\[ g_{c, o}(k) = g_{o}(k) C_{o}(\eta) e^{3\eta \tan^{-1}(k/a)}. \quad (3.1) \]

The complicated form of this potential prevents the analytic solution of the denominator function integral, and because of the singularity
of the Green's function the integral would have to be evaluated numerically at complex energies and continued to real energies. Since this procedure would reduce the accuracy of the final solution, we have chosen instead the compromise form

\[ v_L(k) = \frac{k^2}{(k^2 + a^2)^\lambda}. \]

This form factor has the correct low energy form for the \( L = 2 \) state without Coulomb effects but is incorrect for the \( L = 0 \) state. This form may be thought of as the usual S-wave form factor modified by a function due to Coulomb effects as appears in Equation (3.1). This factor simulates some of the effects of the barrier penetration factor but does not approach zero as rapidly as the correct penetration factor. Further, the choice of parameters includes some of the effects of the Coulomb potential since they are chosen to best fit the experimental results.

The two-body interactions are dominated by the \( \text{Be}^6 \) states of angular momentum \( 0^+ \) and \( 2^+ \) at \( 0.094 \text{ MeV.} \) and \( 2.99 \text{ MeV.} \)

For the \( L = 0 \) state the parameters \( a \) and \( \lambda \) were adjusted to give the experimentally determined position and width (21). The lack of a hard core is apparent in the slow decrease of the phase shift which experimentally decreases rapidly and goes negative at
about 10. MeV. The values of the parameters are

\[ a = 8.32 \, \text{F}^{-1} \]
\[ a^{-1} \lambda = -1150. \, \text{MeV}. \]

The parameters for the \( L = 2 \) state were also chosen to match the experimentally determined position and width. Again the phase shift shows the neglect of a hard core but the fit is considerably better than the \( L = 0 \) case. The parameters are

\[ b = 1.67 \, \text{F}^{-1} \]
\[ b^{-1} \lambda = -402. \, \text{MeV}. \]
CHAPTER IV

INITIAL STATE

The integral equation for the breakup amplitude, $\langle \varphi_3 | U | \bar{\varphi} \rangle$, contains a matrix element of the residual interaction $H_w$ which must be parameterized. We make the simplest possible choice, $H_w$ a constant. Thus,

$$\langle \vec{p}_1 \vec{k}_1 | H_w | \bar{\varphi} \rangle = H_w \langle \vec{p}_1 \vec{k}_1 | \bar{\varphi} \rangle.$$

The amplitude is homogeneous in the constant $H_w$ so that $H_w$ will be included in the normalization and only relative decay rates calculated. The matrix element

$$\langle \vec{p}_1 \vec{k}_1 | \varphi \rangle$$

is the Fourier transform of the initial state wavefunction.

A Gaussian form was chosen for the initial wavefunction for the three alpha particles. The Fourier transform of the normalized, symmetrized wavefunction is, for $J^\pi = 2^+$,
These states satisfy Bose statistics and have the correct low energy form. The parameter $R_0$ determines the spatial extent of the bare amplitude and is taken to be the same in $\psi_a$ and $\psi_b$. The parameters $a$ and $b$ determine the magnitude of the contributions from $\psi_a$ and $\psi_b$ and their relative phase. These parameters will be adjusted to best fit the experimental angular distributions.
CHAPTER V

NUMERICAL METHODS

With the Faddeev equations reduced to a set of one-dimensional integral equations, the two-body interactions and the initial states specified, the next step is the discussion of the numerical solution of the integral equations for the three alpha breakup amplitude.

The equations are now in the form

\[ U_L' (\rho_1, s) = \varphi_L' (\rho_1, s) \]
\[ + \sum_L \int_0^\infty d\rho_2 \ K^{L'}_L (\rho_1, \rho_2, s) U_L (\rho_2, s) \]

where

\[ U_L' (\rho_1, s) = (\rho_1', l', L', J', m_J') | U_1(s) | \Phi > \]
\[ \varphi_L' (\rho_1, s) = (\rho_1', l', L', J', m_J') | t, G_0 H_{\nu} | \Phi > \]

and the kernel
\[ K^{L' L} (\rho_1', \rho_2', s) = 16 \sqrt{\pi} \frac{\rho_2'}{\rho_1'} \int \frac{(\rho_1' + \frac{i}{2} \rho_2')^2}{(\rho_1' - \frac{i}{2} \rho_2')^2} \, dk_1^2 \]

\[ G_0 (\rho_2^2, k_1^2, s) \sum_{\ell} \sqrt{2L + 1} (-1)^{L' + L - 2} \]

\[ \left( \begin{array}{c} l' \ L' \ \ell' \\ n_1' \ n_2' \ -n_3 \end{array} \right) \left( \begin{array}{c} L \ L' \ \ell' \\ n_1 \ n_2 \ -n_3 \end{array} \right) \gamma_{l'}^{n_1'} (\theta_{kk', 0}) \gamma_{l'}^{n_2'} (\theta_{kk', 0}) \gamma_{l'}^{n_3'} (\theta_{kk', 0}) \gamma_{l}^{n_1} (\theta_{kk', \ell}) \gamma_{l}^{n_2} (\theta_{kk', \ell}) \gamma_{l}^{n_3} (\theta_{kk', \ell}) \frac{\nu_{l'} (k_1^2) \nu_{l} (k_1^2)}{D_{l'} (s - 2 \rho_1^2)} \]

With the continuous variables approximated by discrete variables and the integrals approximated by finite sums, the set of coupled integral equations becomes a set of coupled algebraic equations

\[ U_{L' 1} (S) = \varphi_{L' 1} (S) + \sum_{j=1}^{N} \sum_{L=1}^{I} K^{L' L}_{1j} (S) U_{Lj} (S) W_j, \]

where

\[ U_{L' 1} (S) = U_L (p_{1i'}, S) \]
\[ \varphi_{L' 1} (S) = \varphi_L (p_{1i'}, S) \]
\[ K^{L' L}_{1j} (S) = K^L (p_{1i'}, p_{2j}, S) \]

\[ I = \text{number of angular momentum channels} \]
\[ N = \text{number of points necessary to approximate} \]
\[ \text{the integral over } \, dp_a \]
\[ W_j = \text{weight function for the numerical} \]
\[ \text{integration over } \, dp_a . \]

Now write the equations in matrix form

\[ \vec{U}(S) = \vec{\varphi}(S) + \mathbf{K}(S)\vec{U}(S) , \]

where

\[ \vec{U}(S) = \text{column matrix with elements } U(S) \]
\[ \{1 \, (j-1) \, + \, i \, l' \} \]
\[ \vec{\varphi}(S) = \text{column matrix with elements } \varphi(S) \]
\[ \{1 \, (j-1) \, + \, i \, l' \} \]
\[ \mathbf{K}(S) = \text{square matrix with elements } K^{l' \ell}(S) \]
\[ \{1 \, (j-1) \, + \, i \, l' \} \{1 \, (j-1) \, + \, i \, \ell \} . \]

This equation is solved by matrix inversion with the solution

\[ \vec{U}(S) = [\mathbb{I} - \mathbf{K}(S)]^{-1} \vec{\varphi}(S) . \]

The numerical solution of the integral equation as presented is complicated by the presence of poles in the Green's function near the real axis. This problem is usually avoided by rotating the contours of integration away from the singularities into the lower half momentum plane (22). Instead of this method we give the energy a finite imaginary part and move the pole away from the contour of integration.
After the solution is obtained at complex energies it is continued to real energies using a rational fraction method (23).

The rational fraction continuation procedure was applied to the solution of the Lippmann-Schwinger equation in order to gain knowledge about the accuracy of the method. In order to have an exact result available for comparison we used the separable potential already constructed to give a resonance at 2.99 MeV. Our interest is in $D_k^1(S)$ rather than the entire t-matrix, so $D_k^{-1}(S)$ was calculated at complex energies and continued to real energies for comparison with the exact result. Dependence on the distance from the real axis and upon the number of complex energy points was examined. The complex energy points were distributed equally about $S_0$ at intervals of 0.2 MeV. Continued and exact results as well as a unitarity check of the continued results are shown in Table 1. Inspection of these results indicates close agreement except when only a few points are used and the distances from the real axis are large. When either the real or imaginary part of the amplitude is much larger relative to the other, the results are better for the dominant part.

The propagation of errors was studied by introducing known errors into the denominator function and examining the effect on unitarity. The error function was introduced by letting

$$D_k'(S) = D_k(S) \{1 + f(S)\}.$$
Results are given in Table 2 for selected error functions, number of complex energy points, and distances from the real axis. The error in the continued results is of the same order as the error in the complex energy solution.

To test the accuracy in the solution of the three-body equations a bound state vertex of the form

\[
\langle \hat{\mathbf{p}}_1 \hat{\mathbf{k}}_1 \mid \psi \rangle = (k_1^3 + \frac{3}{4} p_1^3 + \alpha^3)^{-2} \{ \frac{3}{4} p_1^3 Y_2^{n_j} (\hat{p}_1) Y_0^o (k_1)
\]

was chosen in order to obtain the inhomogeneous term of the integral equations analytically. The Born term was calculated at complex energies with points distributed about \( S_0 \) at intervals of 1. MeV. and continued to real energy. The results of the calculations are compared in Table 3 for several values of the imaginary part of the energy and number of complex energy points. In order to conserve computer time it was desirable to use a small number of points, and at the same time, to obtain an accurate result. Continued results again agree closely with exact results, as would be expected, since without the two-body denominator functions the Born term is a smooth function of energy. If the two-body denominators were not factored from the integral equations, the presence of two-body resonance poles near the physical region would affect the convergence of the continuation procedure.
Using the same bound state vertex the full amplitude was calculated at imaginary parts of the energy of 1., 3. and 5. MeV. using three and five continuation points. The results given in Table 4 are consistent using only three points and the use of five points further reduces the difference. When either the real or imaginary part of the amplitude is dominant the results are again better for the dominant part. It is desirable to make calculations far from the real axis where the solution of the integral equations is stable yet it is also desirable to be close to the real axis so the results can be continued accurately. As a compromise the final calculations are made with an imaginary part of 3. MeV.
<table>
<thead>
<tr>
<th>$S_0$</th>
<th>IM (S)</th>
<th>N</th>
<th>$D_{\text{CONTINUED}}^2 (S_0)$</th>
<th>$D_{\text{EXACT}}^1 (S_0)$</th>
<th>Unitarity</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>1.0</td>
<td>11</td>
<td>0.198E-02  -i.283E00</td>
<td>0.198E-02  -i.283E00</td>
<td>1.000</td>
</tr>
<tr>
<td>3.0</td>
<td>1.0</td>
<td>5</td>
<td>0.198E-02  -i.283E00</td>
<td>0.198E-02  -i.283E00</td>
<td>1.000</td>
</tr>
<tr>
<td>3.0</td>
<td>1.0</td>
<td>3</td>
<td>0.506E-03  -i.283E00</td>
<td>0.198E-02  -i.283E00</td>
<td>1.000</td>
</tr>
<tr>
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<td>3.0</td>
<td>11</td>
<td>0.204E-02  -i.283E00</td>
<td>0.198E-02  -i.283E00</td>
<td>1.000</td>
</tr>
<tr>
<td>3.0</td>
<td>3.0</td>
<td>5</td>
<td>0.154E-02  -i.283E00</td>
<td>0.198E-02  -i.283E00</td>
<td>0.998</td>
</tr>
<tr>
<td>3.0</td>
<td>3.0</td>
<td>3</td>
<td>0.150E-02  -i.269E00</td>
<td>0.198E-02  -i.283E00</td>
<td>1.049</td>
</tr>
<tr>
<td>3.0</td>
<td>5.0</td>
<td>11</td>
<td>0.441E-02  -i.281E00</td>
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<td>1.008</td>
</tr>
<tr>
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<td>5.0</td>
<td>5</td>
<td>0.183E-02  -i.285E00</td>
<td>0.198E-02  -i.283E00</td>
<td>0.992</td>
</tr>
<tr>
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<td>5.0</td>
<td>3</td>
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<td>0.198E-02  -i.283E00</td>
<td>1.171</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0</td>
<td>5</td>
<td>0.214E00   -i.746E-01</td>
<td>0.214E00   -i.746E-01</td>
<td>0.999</td>
</tr>
<tr>
<td>2.0</td>
<td>3.0</td>
<td>5</td>
<td>0.212E00   -i.739E-01</td>
<td>0.214E00   -i.746E-01</td>
<td>0.998</td>
</tr>
<tr>
<td>2.0</td>
<td>5.0</td>
<td>5</td>
<td>0.209E00   -i.738E-01</td>
<td>0.214E00   -i.746E-01</td>
<td>1.036</td>
</tr>
</tbody>
</table>
Table 2

Unitarity check for denominator functions with known errors:

The error functions are $f_1(s) = 0.05$, $f_3(s) = 0.05 \left( \frac{S_0 - S}{S_0} \right)$, $f_3(s) = 0.05 \left[ 1 - \left( \frac{S_0 - S}{S_0} \right)^2 \right]$. 
<table>
<thead>
<tr>
<th>$S_0$</th>
<th>IM (S)</th>
<th>N</th>
<th>$f_1$ (s)</th>
<th>$f_2$ (s)</th>
<th>$f_3$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
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<td>11</td>
<td>1.05</td>
<td>1.00</td>
<td>1.04</td>
</tr>
<tr>
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<td>1.0</td>
<td>5</td>
<td>1.05</td>
<td>1.00</td>
<td>1.04</td>
</tr>
<tr>
<td>3.0</td>
<td>3.0</td>
<td>11</td>
<td>1.05</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>3.0</td>
<td>3.0</td>
<td>5</td>
<td>1.05</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>3.0</td>
<td>5.0</td>
<td>11</td>
<td>1.05</td>
<td>1.00</td>
<td>0.91</td>
</tr>
<tr>
<td>3.0</td>
<td>5.0</td>
<td>5</td>
<td>1.05</td>
<td>1.01</td>
<td>0.91</td>
</tr>
</tbody>
</table>
Table 3

A comparison of the exact and continued values of the Born term, \( \varphi \), at \( S_0 = 11.1 \) MeV, as a function of \( p_1^2/2 \), the number of complex energy points, and the distance from the real axis.
<table>
<thead>
<tr>
<th>$p_0 / 2$</th>
<th>$\Phi_{\text{exact}}$</th>
<th>$\Phi_{\text{cont}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N = 3$ IM(S) = 3.</td>
<td>$N = 5$ IM(S) = 3.</td>
</tr>
<tr>
<td>0.000</td>
<td>-2.56E03 -1.458E03</td>
<td>-2.55E03 -1.456E03</td>
</tr>
<tr>
<td>0.174</td>
<td>-2.73E03 -1.437E03</td>
<td>-2.72E03 -1.437E03</td>
</tr>
<tr>
<td>0.692</td>
<td>-3.08E03 -1.376E03</td>
<td>-3.06E03 -1.376E03</td>
</tr>
<tr>
<td>1.556</td>
<td>-3.36E03 -1.286E03</td>
<td>-3.36E03 -1.286E03</td>
</tr>
<tr>
<td>2.756</td>
<td>-3.55E03 -1.176E03</td>
<td>-3.55E03 -1.176E03</td>
</tr>
<tr>
<td>4.340</td>
<td>-2.92E03 -1.705E02</td>
<td>-2.92E03 -1.719E02</td>
</tr>
<tr>
<td>6.246</td>
<td>-2.13E03 -1.740E01</td>
<td>-2.13E03 -1.739E01</td>
</tr>
<tr>
<td>8.497</td>
<td>-1.46E03 1.103E00</td>
<td>-1.46E03 1.308E-01</td>
</tr>
<tr>
<td>11.064</td>
<td>-1.13E03 1.464E-01</td>
<td>-1.13E03 1.523E-04</td>
</tr>
<tr>
<td>16.365</td>
<td>-7.71E02 1.531E-02</td>
<td>-7.71E02 1.524E-04</td>
</tr>
<tr>
<td>22.269</td>
<td>-5.56E02 1.781E-02</td>
<td>-5.56E02 1.516E-02</td>
</tr>
<tr>
<td>29.783</td>
<td>-4.15E02 1.297E-03</td>
<td>-4.14E02 1.201E-03</td>
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</table>
Table 4

The full amplitude continued from complex energies of 1, 3, and 5 MeV using three and five continuation points.
<table>
<thead>
<tr>
<th>( p^{3/2} )</th>
<th>( N = 3 \ IM(S) = 1 )</th>
<th>( N = 3 \ IM(S) = 3 )</th>
<th>( N = 5 \ IM(S) = 3 )</th>
<th>( N = 3 \ IM(S) = 5 )</th>
<th>( N = 5 \ IM(S) = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>(-.161E03) (-1.640E03)</td>
<td>(-.738E01) (-1.612E03)</td>
<td>(.254E02) (-1.610E03)</td>
<td>(.130E02) (-1.635E03)</td>
<td>(.195E02) (-1.628E03)</td>
</tr>
<tr>
<td>0.174</td>
<td>(-.117E03) (-1.587E03)</td>
<td>(-.116E02) (-1.598E03)</td>
<td>(.835E01) (-1.601E03)</td>
<td>(-.697E01) (-1.612E03)</td>
<td>(-.111E02) (-1.603E03)</td>
</tr>
<tr>
<td>0.692</td>
<td>(-.768E02) (-1.526E03)</td>
<td>(-.686E02) (-1.540E03)</td>
<td>(-.662E02) (-1.536E03)</td>
<td>(-.631E02) (-1.545E03)</td>
<td>(-.599E02) (-1.543E03)</td>
</tr>
<tr>
<td>1.556</td>
<td>(-.124E03) (-1.430E03)</td>
<td>(-.121E03) (-1.434E03)</td>
<td>(-.121E03) (-1.431E03)</td>
<td>(-.115E03) (-1.436E03)</td>
<td>(-.117E03) (-1.432E03)</td>
</tr>
<tr>
<td>2.766</td>
<td>(-.149E03) (-1.302E03)</td>
<td>(-.137E03) (-1.309E03)</td>
<td>(-.137E03) (-1.306E03)</td>
<td>(-.134E03) (-1.316E03)</td>
<td>(-.131E03) (-1.312E03)</td>
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<tr>
<td>4.340</td>
<td>(-.130E03) (-1.179E03)</td>
<td>(-.137E03) (-1.202E03)</td>
<td>(-.125E03) (-1.195E03)</td>
<td>(-.121E03) (-1.212E03)</td>
<td>(-.118E03) (-1.208E03)</td>
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<tr>
<td>6.246</td>
<td>(-.484E02) (-1.103E03)</td>
<td>(-.787E02) (-1.107E03)</td>
<td>(-.743E02) (-1.113E03)</td>
<td>(-.985E02) (-1.140E03)</td>
<td>(-.870E02) (-1.127E03)</td>
</tr>
<tr>
<td>8.497</td>
<td>(-.282E02) (-1.843E02)</td>
<td>(-.288E02) (-1.847E02)</td>
<td>(-.309E02) (-1.853E02)</td>
<td>(-.405E02) (-1.768E02)</td>
<td>(-.418E02) (-1.843E02)</td>
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<tr>
<td>11.094</td>
<td>(-.114E02) (-1.707E02)</td>
<td>(-.117E02) (-1.728E02)</td>
<td>(-.121E02) (-1.732E02)</td>
<td>(-.127E02) (-1.690E02)</td>
<td>(-.140E02) (-1.707E02)</td>
</tr>
<tr>
<td>16.365</td>
<td>(.675E01) (-1.394E02)</td>
<td>(.645E01) (-1.415E02)</td>
<td>(.676E01) (-1.412E02)</td>
<td>(.757E01) (-1.414E02)</td>
<td>(.740E01) (-1.411E02)</td>
</tr>
<tr>
<td>22.289</td>
<td>(.814E01) (-1.228E02)</td>
<td>(.756E01) (-1.299E02)</td>
<td>(.785E01) (-1.298E02)</td>
<td>(.813E01) (-1.301E02)</td>
<td>(.791E01) (-1.307E02)</td>
</tr>
<tr>
<td>29.783</td>
<td>(.621E01) (-1.215E02)</td>
<td>(.553E01) (-1.233E02)</td>
<td>(.576E01) (-1.232E02)</td>
<td>(.591E01) (-1.235E02)</td>
<td>(.598E01) (-1.232E02)</td>
</tr>
</tbody>
</table>
The results of the calculations are compared with the experimental findings at $\Theta_1 = 120^\circ$ and $\Theta_2$ varying from $30^\circ$ to $100^\circ$.

At complex energies three-body resonances appear as minima in the absolute value of the determinant

$$\text{DET} \left[ \mathbb{I} - \mathbf{K}(s) \right].$$

Therefore, such a minimum at the energy of the 18.37 MeV. state should be found. Because of neglect of such repulsive interactions as the Coulomb and hard core potentials this state appears at lower energy. No attempt was made to determine the exact location of the minimum but it was found to be in the neighborhood of 15. MeV.

The peaks corresponding to the ground state of Be$^9$ are not shown in any of the calculated distributions because the finite experimental resolution is much larger than the width of the ground state. The effects of finite resolution have not been examined but should be of little consequence for the broader first excited state.

In fitting the data, no attempt was made to use a least squares or similar technique. Instead, a best fit was chosen by inspection with the fit required to, first, reproduce the observed interference minima occurring between $\Theta_2 = 35^\circ$ and $\Theta_2 = 60^\circ$, and second, to fit the general envelope of the data.
The results for \( b = 0 \), that is, a pure \( C^{13} \) initial state, are shown in Figure 2. The results with the full amplitude are given by the solid line and with the first rescattering or Born approximation by the dashed line. As stated in Chapter II, this is not the Born approximation but the impulse approximation. Figure 2 gives the results with \( R_0 = 3. \) F. and \( R_0 = 2. \) F. The full solutions are normalized to the experimental value. For \( R_0 = 3. \) the bound state vertex is spread out with direct decay being of the same order as the Born term and there is interference between the effects. The rescattering increases the importance of the final state interactions and accounts for the difference in the distributions. For \( R_0 = 2. \) the bound state vertex is more compact and the final state effects play a larger role. These effects dominate and direct decay makes a smaller contribution. The only difference in the two distributions is the size of the initial vertex; the strong interactions are the same in both cases. The size of the initial vertex does not enter the homogeneous term and does not affect rescattering. In cases where direct decay makes a contribution of the same order as the final state effects the results might be misleading and because of the interference the two effects would be hard to disentangle.

The foregoing calculations demonstrate the need to include all rescattering effects.
Figure 2

A comparison of the calculated and experimental distributions for $\theta_1 = 120^\circ$, $\theta_2 = 50^\circ$. The solid lines indicate the calculated distribution with the full amplitude and the dashed lines with the Born approximation.
A pure \( C^{12} \) initial state does not reproduce the observed minima in the distributions. The possibility that the initial state is composed of a mixture of states is now examined. The interference between the \( \ell = 0 \) and \( \ell = 2 \) alpha particles is included by adding the state \( \psi_b \) discussed in Chapter IV. The decay rate now contains a relative phase and magnitude which can be adjusted in an attempt to fit the data.

The results of the calculations made with a mixed initial state are shown in Figure 3. The results are a compromise between fitting the interference minima and the general envelope of the data. An extensive parameter search was not made on \( R_o \) because each value of \( R_o \) required a solution of the integral equations. The fit was obtained with

\[
R_o = 3. \ F. \\
a = .86 \\
b = -.09 \ - i.51.
\]

The state \( \psi_b \) makes a contribution of about 27 percent. The fit does show the interference minima as required but the calculated minima are not as low as the experimentally observed values and the fit to the general envelope is poor. The magnitude of the cross section is too large outside a small region. Peaks corresponding to the first
Figure 3

A comparison of the calculated and experimental angular distributions.
rescattering and no new solution of the integral equations is necessary. The results depend on the value taken for the shielding radius. Some of this dependence would be removed if the correct barrier penetration factors were included because they approach zero rapidly and would reduce the contributions from terms where the energy is low and the Coulomb phase shift the most troublesome.

Figure 4 shows the results of the calculations with the Coulomb phase shift included. The shielding radius was taken to be \( R = 10.9 \) F. The results are stable against variations in the shielding radius of a factor of 10. The other parameters are

\[
\begin{align*}
R_0 & = 3. \text{ F.} \\
as & = 0.85 \\
b & = -0.17 - 0.51i.
\end{align*}
\]

The contribution of \( \gamma_b \) in this case is 29 percent. Again an exhaustive parameter search on \( R_0 \) was not made. The fit to the interference minima and to the overall envelope is improved over the results obtained without the Coulomb phase shift. The fit at large values of \( \Theta_2 \) is still not good. Peaks which correspond to the first excited states are again observed at large values of \( \Theta_2 \). This Coulomb phase shift is not a complete Coulomb correction; it is only used to gain an idea of the importance of the Coulomb potential.
Figure 4

A comparison of the calculated and experimental angular distributions including a Coulomb phase shift.
The broad $L = 4$ state of $\text{Be}^8$ at 11.7 MeV. has been omitted. Since this state is very broad and the relative pair energy in the neighborhood of this resonance is slowly varying it should contribute only a background and should not have an important influence on the distributions.
CHAPTER VII

POSSIBLE EXTENSIONS

Some reasonable extensions and improvements of the model are explored in this chapter.

A first improvement might be to include the Coulomb effects as outlined in Chapter III. If the Coulomb transform integrals could not be performed, at least the correct barrier penetration factors could be included. The inclusion of these factors would necessitate obtaining the two-particle denominator functions numerically and thus possibly reduce the accuracy of the final solution because these functions would also have to be continued numerically whereas now they are obtained analytically. The improved potentials should provide the correct low energy form and reduce the contributions from low energy terms where the Coulomb phase shift is the most troublesome.

To reproduce the hard core scattering effects a second term must be added to the potential. This is necessary in the S-wave potential but the D-wave potential already provides a sufficiently good fit to the experimental phase shifts.
The \( L = 4 \) state of \( \text{Be}^6 \) at 11.4 MeV. could be added. It might be possible to include this only in the Born terms since we would expect the rescattering effects to be small. This would have to be checked by calculations, however. We would not expect large changes to result from the inclusion of this state because of its large width and the slow variation of the pair energy near this state.

In principle, the above improvements could be added easily. The complication is the necessity of using larger matrices to solve the integral equations as well as the use of numerical methods to obtain additional quantities.

Another approach would be to abandon the use of separable potentials and examine ways of constructing separable approximations to the scattering amplitudes resulting from local potentials. As mentioned in the introduction, Fulco and Wong (14) have expanded the scattering amplitude in a series of functions which are solutions of the homogeneous Lippmann-Schwinger equation. A disadvantage is that this method cannot be used above the three-particle breakup threshold and requires a large number of terms in the separable expansion.

Brayshaw (24) has proposed a separable approximation to the two-body amplitude which contains the correct behavior at bound state and resonance poles and has the entire right hand cut. The method was tested by calculating the binding energy of three spinless
particles interacting through Yukawa and exponential potentials. The results for a two-term approximation compared favorably with the results from more direct calculations and the accuracy was increased by increasing the number of separable terms.

The most direct method of solving the Faddeev equations is to improve the numerical techniques so that double integrals can be evaluated accurately with a small mesh. The three Boson binding energy has been calculated with this method for Yukawa and exponential potentials by Osborn (25). The results were within the limits established by variational calculations but a large matrix was required and this method is not a practical scheme of making calculations at present.
CHAPTER VIII

SUMMARY AND CONCLUSIONS

An exactly soluble model of the breakup of a $J^\pi = 2^+$ particle into three identical spinless daughter particles with final state interactions has been examined. The interaction which initiates the breakup was treated to first order while the strong interactions were included to all orders and were described by equations of the Faddeev type. The final state interactions are dominated by resonances of relative angular momentum 0 and 2. The two-body resonances were chosen to correspond to the ground and first excited states of Be$^8$ and the parent particle to the 18.37 MeV. state of C$^{12}$. The results of the calculations were compared to the findings of the experimental study of the reaction $^8B(p, \alpha) 2\alpha$.

Differential cross sections

$$\frac{d^3 \sigma}{dT_1 d\Omega_1 d\Omega_2}$$

were calculated for a variety of angles and energies of the final particles.

It was necessary to include all the rescattering effects. The spatial extent of the initial bound state wavefunction was found to play
a significant part with direct decay making a larger contribution for a spread out initial state. When direct decay makes a significant contribution interference between direct decay and final state effects was found making the results hard to disentangle.

With a pure $^{13}$C initial state it was found that the experimentally observed interference effects were absent. With the inclusion of an additional component to the initial state wavefunction the observed interference effects were found when the magnitude and the relative phase of the second component were properly adjusted. It was not possible to obtain a fit at all angles with these parameters. If the parameters were adjusted to fit the general envelope of the data the interference minima vanished. The contribution of the second component was about 27 percent.

With the inclusion of an ad hoc Coulomb phase shift the fit to the envelope of the data and to the interference minima was improved. Results were dependent on the cutoff taken for the Coulomb potential. With this Coulomb correction the contribution of the second component of the wavefunction increased to 29 percent.

It would be naïve to presume that the simple model we have constructed would give a perfect fit to the data. However, this problem does offer a way to test the accuracy of approximation methods for the two-body amplitudes as well as the numerical methods for the
solution of the Faddeev equations. The simplicity offered in the spinless particles and the large amount of data constitute a suitable test for the calculations. The calculation of the binding energy is not as useful a test since the model neglects the structure of the alpha particles. This omission should be less important in the breakup problem where the clusters are more widely separated. To test approximations in a binding energy calculation an obvious choice is the three-nucleon problem. In this case it is not necessary to neglect internal structure and the Coulomb potential is absent in the $^3\text{H}$ case. However, the problem is complicated by spin and non-central forces.
APPENDIX A

ANGULAR MOMENTUM DECOMPOSITION

In this appendix the angular momentum decomposition of the integral equations is shown explicitly.

Recalling Equation (2.3),

\[
\langle \vec{p}'_i \vec{r}'_i | U_i | \Phi \rangle = \langle \vec{p}'_i \vec{r}'_i | t_i \ G_0 \ H_\omega | \Phi \rangle

+ \sum \langle \vec{p}'_i \vec{r}'_i | t_i \ G_0 \ U_j | \Phi \rangle.
\]

(A.1)

Because total angular momentum is conserved it is convenient to change to a representation in which the total angular momentum is diagonal. The transformation to these states is given by

\[
\langle \vec{p}'_i \vec{r}'_i | = \sum \left( \begin{array}{ccc} m_{\ell}' & m_{\ell} & m_{\tau}' \end{array} \right) \ Y_{\ell'}^{m_{\ell}'}(\vec{p}'_i) \ 

\ Y_{L'}^{m_{\ell}'}(\vec{r}'_i) \ 

\langle \vec{p}'_i \vec{r}'_i \rangle \ 

\left( \begin{array}{ccc} m_{\ell}' & m_{\ell} & m_{\tau}' \end{array} \right) \ 

\left( \begin{array}{ccc} m_{\ell}' & m_{\ell} & m_{\tau}' \end{array} \right) \ 

\text{where } \left( \begin{array}{ccc} m_{\ell}' & m_{\ell} & m_{\tau}' \end{array} \right) \ 

\text{is a Clebsch-Gordan coefficient.}
\]

A1
Applying this change of representation to Equation (A.1)

\[
\langle \phi | \{ \chi, \chi' \} | \theta \rangle = \langle \phi | \{ \chi, \chi' \} | \theta \rangle + \sum_{j \neq i} \langle \phi | \{ \chi, \chi' \} | \theta \rangle.
\]

(A.3)

**INHOMOGENEOUS TERM**

Concentrate for the moment on the inhomogeneous term of the equation. Using the inverse transformation of Equation (A.2), the inhomogeneous term becomes

\[
\langle \phi | \{ \chi, \chi' \} | \theta \rangle = \sum_{m_k, m_{\chi'}} \int d\Omega_{\chi} \int d\Omega_{\chi'} \left( \langle m_{\chi'} | m_{\chi' \chi'} \rangle \right)^{-1}
\]

\[
\langle \phi | \{ \chi, \chi' \} | \theta \rangle = \sum_{m_k, m_{\chi'}} \int d\Omega_{\chi} \int d\Omega_{\chi'} \left( \langle m_{\chi'} | m_{\chi' \chi'} \rangle \right)^{-1}
\]

Inserting a complete set of states of the free Hamiltonian
\[ \langle p_i', l_i', k_i', l_i', J_{m_i'} | t_i | G_0 | H_\omega | \Phi \rangle = \]
\[ \sum_{m_{i'}, m_{i''}} \int d^3 p_i \, d^3 k_i \, d \Omega_{p_i} \, d \Omega_{k_i} \]
\[ (m_{i'}, m_{i''} | m_{i''} \rangle \langle \pi_{i'}^* (\pi_i) \gamma_{L_i}^* (k_i') \rangle \]
\[ \langle \pi_{i'}^* (\pi_i) | t_i | \pi_{i'}^* (\pi_i) \rangle \]
\[ = \langle \pi_{i'}^* (\pi_i) | \pi_{i'}^* (\pi_i) \rangle \]
\[ \langle \pi_{i'}^* (\pi_i) | H_\omega | \Phi \rangle. \]

The two-particle t-matrix in the three-particle Hilbert space is given by

\[ \langle \pi_{i'} (\pi_i) | t_i (s) | \pi_{i'} (\pi_i) \rangle = (2\pi)^3 \delta^3 (\vec{p}_{i'} - \vec{p}_i) \]
\[ \langle \pi_{i'} (\pi_i) | t_i (s - 3/4 p_i^2) | \pi_{i'} (\pi_i) \rangle \]

where \( \langle \pi_{i'} | t_i (S - 3/4 p_i^2) | \pi_i \rangle \) is the off-shell two-particle t-matrix. Making a partial wave expansion, Equation (A.6) becomes
\[
\left< \hat{p}_i \cdot \hat{k}_i' \mid t_i (s) \mid \hat{p}_i \cdot \hat{k}_i \right> = (2\pi)^3 \delta^3 (\hat{p}_i' - \hat{p}_i)
\]
\[
(4\pi)^2 \sum_{\lambda \mu} \chi^\mu_\lambda (\hat{k}_i') \chi^{*\mu}_\lambda (\hat{k}_i) \tag{A.7}
\]
\[
\left< k_i' \lambda \mid t_i (s - 3\alpha \hat{p}_i^2) \mid k_i \lambda \right>.
\]

Applying this to Equation (A.5) and expanding \( \left< \hat{p}_i, \hat{k}_i \right> \)
\[
\left< \hat{p}_i', l', k_i', l', J', m_J' \mid t_i \mid G_0 \mid H_\omega \mid \Phi \right> = \sum_{m_l', m_l} \int d^3 \hat{p}_i \cdot \hat{k}_i \cdot d\Omega \hat{p}_i \cdot d\Omega \hat{k}_i' \left( \begin{array}{c|c}
 m_l' & m_l' \\
 l' & l'
\end{array} \right)
\]
\[
\chi^{*m_l'}_{l'} (\hat{p}_i') \chi^{*m_l'}_{l'} (\hat{k}_i') (2\pi)^3 \delta^3 (\hat{p}_i' - \hat{p}_i)
\]
\[
(4\pi)^2 \sum_{\lambda \mu} \chi^\mu_\lambda (\hat{k}_i') \chi^{*\mu}_\lambda (\hat{k}_i) \tag{A.8}
\]
\[
\left< k_i \lambda \mid t_i (s - 3\alpha \hat{p}_i^2) \mid k_i \lambda \right> G_0 (\hat{p}_i^2, \hat{k}_i^2, s)
\]
\[
\sum_{m_J, l, m_l} \left( \begin{array}{c|c}
 m_J & m_J \\
 l & l
\end{array} \right) \chi^m_\lambda (\hat{p}_i) \chi^{m_l}_\lambda (\hat{k}_i')
\]
\[
\left< \hat{p}_i \cdot l, \hat{k}_i \cdot l, J, m_J \mid H_\omega \mid \Phi \right>.
\]
Using the delta function to evaluate the integral over $d^3 p_1$ 
and the orthogonality properties of the spherical harmonics to 
inTEGRATE over $d \Omega_{p_1}$ and $d \Omega_{k_1}$,

\[
\begin{align*}
&\langle p'_i, l'_i, k'_i \mid t_i \mid G_o \mid \bar{H}_\omega \mid \Phi \rangle = (4\pi)^2 \sum_{m'_l m'_k} \int d^3 k_i \\
&\quad \left( \begin{array}{c} m'_l \\ l'_i \\ k'_i \end{array} \right) \left( \begin{array}{c} m'_k \\ l'_i \\ k'_i \end{array} \right) \quad (A.9)
\end{align*}
\]

\[
\begin{align*}
&\langle k'_i, l'_i \mid t_i \mid (s - \frac{3}{2} \vec{\rho} \cdot \vec{\rho}^{''}) \mid k_i, l_i \rangle \quad G_o \left( \frac{\rho_{i,j}^2}{\rho_{i,j}^2} \right) \\
&\quad \langle p'_i, l'_i, k'_i \mid \bar{H}_\omega \mid \Phi \rangle.
\end{align*}
\]

Using the unitary properties of the Clebsch-Gordon coefficients 
[Edmonds (26), 3.5.4]

\[
\begin{align*}
&\langle p'_i, l'_i, k'_i \mid t_i \mid G_o \mid \bar{H}_\omega \mid \Phi \rangle = \frac{2}{\pi} \int k_i^2 d k_i \\
&\quad \langle k'_i, l'_i \mid t_i \mid (s - \frac{3}{2} \vec{\rho} \cdot \vec{\rho}^{''}) \mid k_i, l_i \rangle \quad G_o \left( \frac{\rho_{i,j}^2}{\rho_{i,j}^2} \right) \quad (A.10)
\end{align*}
\]

\[
\begin{align*}
&\quad \langle p'_i, l'_i, k'_i \mid \bar{H}_\omega \mid \Phi \rangle.
\end{align*}
\]
HOMOGENEOUS TERM

Now return to the decomposition of the homogeneous term.

Applying the inverse of Equation (A.2) and inserting a complete set of states of the free Hamiltonian

\[ \langle i | \hat{p}_i \hat{k}_i | t_i G_0 U_j | \Phi \rangle = \sum_{m_e, m_e'} \int d\rho_j \, d^3 k_j \, \int d\Omega_{\rho_e} \, d\Omega_{k_e} \, \left( \frac{m_e'}{l_e'} \, \frac{m_{j}}{l_{j}} \right) \frac{Y^*_m(r_i)}{l_i} (\rho_e) \]

\[ \times \, Y^m_{l_e}(k_e) \, \langle i | \hat{p}_i \hat{k}_i' | t_i | \hat{p}_j \hat{k}_j \rangle G_0 (\rho_j, k_j, s) \]

\[ = \sum_{m_e, m_e'} \int d\rho_j \, d^3 k_j \, \int d\Omega_{\rho_e} \, d\Omega_{k_e} \, \left( \frac{m_e'}{l_e'} \, \frac{m_{j}}{l_{j}} \right) \frac{Y^*_m(r_i)}{l_i} (\rho_e) \]

\[ \times \, Y^m_{l_e}(k_e) \, \langle i | \hat{p}_i \hat{k}_i' | t_i | \hat{p}_j \hat{k}_j \rangle G_0 (\rho_j, k_j, s) \]

Since our representations are equivalent

\[ \langle i | \hat{p}_i \hat{k}_i' | t_i | \hat{p}_j \hat{k}_j \rangle = \sum_{m_e, m_e'} \int d\rho_j \, d^3 k_j \, \int d\Omega_{\rho_e} \, d\Omega_{k_e} \, \left( \frac{m_e'}{l_e'} \, \frac{m_{j}}{l_{j}} \right) \frac{Y^*_m(r_i)}{l_i} (\rho_e) \]

\[ \times \, Y^m_{l_e}(k_e) \, \langle i | \hat{p}_i \hat{k}_i' | t_i | \hat{p}_j \hat{k}_j \rangle \cdot \]

Making a decomposition of \( \langle \hat{p}_i \hat{k}_i \mid t_i \mid \hat{p}_j \hat{k}_j \rangle \) using Equation (A.7) and \( \langle \hat{p}_i \hat{k}_j \rangle \) using Equation (A.2), Equation (A.11) becomes
\[ \langle \rho_c' , l' , k_c' , l_c' | t_c \ G_o \ U_j \ | \bar{\Phi} \rangle = \sum_{m_{\rho_{c}'}, m_{l_c'}} \int d^3 p_{l'} \ d^3 k_{l_c'} \ \bar{G}_o \left( \rho_{c}' , k_c' , l_c' , m_{\rho_{c}'} , m_{l_c'} , m_{l_c'} \right) \]

\[ \sum_{\lambda , \mu} \ Y_{\lambda}^{m_{\lambda}} (\hat{\rho_{c}'}) \ Y_{\lambda}^{m_{\lambda}} (\hat{k_{c}'}) \ Y_{\lambda}^{m_{\lambda}} (\hat{k_{c}}) \ \langle k_c' , \lambda | t_c (s - \Omega_{\rho_{c}'} - \Omega_{k_{c}'}) | k_c \lambda \rangle \]

\[ \sum_{m_{\rho_{c}'}, m_{l_c'}} \left( \begin{array}{cc} m_{\rho_{c}'} & m_{l_c'} \\ m_{l_c'} & \end{array} \right) \ \bar{G}_o \left( \rho_{c}' , k_{c}' , l_{c}' , m_{\rho_{c}'} \right) \]

\[ \langle \rho_{c}' , k_{c}' , l_{c}' , j_{c}' | \bar{G}_o | U_j \ | \bar{\Phi} \rangle . \]

The delta function

\[ (2 \pi)^3 \ \delta^3 (\hat{\rho_{c}'} - \hat{\rho}) = (2 \pi)^3 \ \frac{2}{p_{l}} \ \delta (p_{l}^{\prime 2} - p_{l}^2) \ \delta (\Omega_{\rho_{c}'} - \Omega_{\rho_{l}}) . \]

Equation (2.13)

The delta function is used to integrate over \( d\Omega_{\rho_{c}'} \) and the orthogonality of the spherical harmonics is used to integrate over \( d\Omega_{k_{c}'} \). Equation (A.12) now becomes
\[ \langle \rho', \ell', k', \ell', \mathcal{J}^m \mathcal{J} \mid t_c G \mid \mathcal{U}_j \mid \Phi \rangle = \sum_{m_l', m_l} 8(2\pi)^5 \int d^2 \rho_i \]

\[ \left( \frac{m_l'}{l'} \frac{m_l}{l} \right) Y_{m_l'}^{m_l'}(\hat{\rho}_i) \left( \frac{-m_l^2}{l^2} \right) \left( \frac{m_l^2}{l^2} \right) \]

\[ \sum_{s_l m_j} \left( \begin{array}{ccc} m_s & m_l & m_j \\ l & l & \mathcal{J} \end{array} \right) Y_s^{m_s}(\hat{\rho}_j) Y_j^{m_j}(\hat{k}_j) \]

\[ \langle \rho, \ell, k, \ell, \mathcal{J}^m \mathcal{J} \mid \mathcal{U}_j \mid \Phi \rangle. \]

Rotation operators are used to express the spherical harmonics in terms of different angles. Choosing \( \hat{k}_j \) as the axis of quantization

\[ Y_{\ell}^{m}(\theta, \phi) = \sum_{\alpha} D_{\alpha}^{\ell} (\alpha, \theta_{k_j}, \phi_{k_j}) Y_{\ell}^{m}(\theta_{k_j}, \phi_{k_j}) \quad (A.15) \]

\( D_{\alpha}^{\ell} (\alpha, \theta_{k_j}, \phi_{k_j}) \) are the rotation matrices and \( \alpha, \theta_{k_j}, \phi_{k_j} \) are the Euler angles defining the solid angle \( \omega \). This corresponds to a rotation from a coordinate system with Z-axis parallel to \( \hat{k}_j \) to the space fixed coordinate system. The plane \( \varphi_{k_j} = \pi \) is defined by the cross product \( \hat{k}_j \times \hat{p}_j \) and the plane \( \varphi_{k_j} = 0 \) by the cross products \( \hat{k}_j \times \hat{p}_i \) and \( \hat{k}_j \times \hat{k}_i \). We have to integrate over \( d\Omega_{k_j} d\Omega_{p_j} \) in Equation (A.14) but use instead the equivalent set of variables \( d\omega d\varphi_{p_j} k_j \). Equation (A.14) now becomes
\begin{align}
\langle \rho_i \mid l', k_i L', J \rho_j \mid t_c \mid G_0 \mid U_j \mid \Phi \rangle = \sum_{m_j} \frac{1}{m_i} \int k_j^2 \, dk_j
\end{align}

\begin{align}
p_i^2 \, dp_j \, d\omega \, d\mu_{k_j \rho_j} \frac{1}{p_i} \delta \left(p_i^2 - p_i^2\right)
\end{align}

\begin{align}
\langle k_i L \mid t_c \left(s - 2 \rho_i \rho_j^* \right) \mid k_i L' \rangle
\end{align}

\begin{align}
\sum_{L m_L} \left( \begin{array}{ccc} m_i' & m_i' & m_j' \end{array} \right) \left( \begin{array}{ccc} m_k & m_k & m_j \end{array} \right) \sum_{m_k} \frac{D^{x}_{m_k m_k}}{m_i m_i} (\omega)
\end{align}

\begin{align}
\sum_{g} \left( \begin{array}{ccc} 2 \rho_i \rho_j & 0 \end{array} \right) \left( \begin{array}{ccc} n_i & n_i & n_j \end{array} \right) \left( \begin{array}{ccc} \rho_k \rho_k & \rho_k \rho_k & \rho_k \rho_k \end{array} \right)
\end{align}

\begin{align}
D_{n_i m_k}^x (\omega) Y_{l}^{n_i} (\rho_i \rho_j, \rho_j) D_{n_j m_j}^c (\omega) Y_{l}^{n_j} (0, 0)
\end{align}

\begin{align}
G_0 \left( \rho_i^2, k_j^2, s \right) \langle \rho_i l \mid k_j L, J \rho_j \mid U_j \mid \Phi \rangle.
\end{align}

Since

\begin{align}
Y_{l} \eta_L (0, 0) = \delta_{\eta_L} \left( \frac{2}{4} \frac{L + 1}{\pi} \right)^{1/2}
\end{align}

the summation over \( \eta_L \) can be performed.

For convenience, call
A_{\ell n_3} = \sum_{\{m_3\}} \left( \begin{array}{c c c c} m_{l'} & m_{L'} & n_{J'} \end{array} \right) \left( \begin{array}{c c} m_\ell & m_L \end{array} \right) \int d\omega \ D_{n_\ell' m_\ell'}^{L'}(\omega) \ D_{n_L' m_L'}^{L'}(\omega) \ D_{m_3 m_0}^{\ell}(\omega) \ D_o^{\ell}(\omega) \ (A.18)

Using relations between the rotation operator (Edmonds, 4.3.2)

A_{\ell n_3} = \sum_{\{m_3\}} \left( \begin{array}{c c c c} m_{l'} & m_{L'} & n_{J'} \end{array} \right) \left( \begin{array}{c c} m_\ell & m_L \end{array} \right) \int d\omega \ \sum_{j'\mu'} (2j'+1) \left( \begin{array}{c c c c} l' & L' & j' \end{array} \right) \left( \begin{array}{c c} j' \mu' & n_{l'} & n_{L'} & \mu' \end{array} \right) \ D_{n_{l'} m_{l'}}^{j'}(\omega) \left( \begin{array}{c c c c} l' & L' & j' \end{array} \right) \left( \begin{array}{c c} j \mu \end{array} \right) \left( \begin{array}{c c c c} l & L & j \end{array} \right) \left( \begin{array}{c c} L & j \end{array} \right) D_{\ell' m_{\ell'}}(\omega) \ D_{m_3 m_0}^{\ell}(\omega) \ D_o^{\ell}(\omega) \ (A.19)
Using another relation (Edmonds, 4.6.1) the integration over \( d\omega \) can be performed. This integration produces delta functions which may be used to perform some of the summations. The resulting expression is

\[
A_{\ell n j} = 8\pi^2 \sum_{\xi \eta \lambda} \left( \begin{array}{ccc} m_{\lambda'} & m_{\lambda} & m_{\lambda''} \\ \ell' & \ell & \ell'' \end{array} \right) \left( \begin{array}{ccc} m_{\nu} & m_{\mu} & m_{\nu''} \\ \ell & \ell' & \ell'' \end{array} \right)
\sum_{j' \mu' \nu'} (2j'+1) \left( \begin{array}{ccc} \ell' & \ell & j' \\ \eta' & \eta & \mu' \end{array} \right) \left( \begin{array}{ccc} \ell' & \ell' & j' \\ \eta' & \eta' & \nu' \end{array} \right) (A.20)
\left( \begin{array}{ccc} \ell & \ell & j \\ \eta & \eta' & \nu \end{array} \right)
\left( \begin{array}{ccc} \ell & \ell & j' \\ \eta & \eta' & \nu' \end{array} \right).
\]

The Clebsch-Gordon coefficients are expressed in terms of the 3-j coefficients and relations between the 3-j coefficients (Edmonds, 3.7.8, 3.7.6) are used to reduce Equation (A.20) to

\[
A_{\ell n j} = 8\pi^2 (-1)^{\ell'+\ell-\ell'-\ell} \left( \begin{array}{ccc} \ell' & \ell & j' \\ \eta' & \eta & -\eta \end{array} \right)
\left( \begin{array}{ccc} \ell & \ell & j \\ \eta & \eta & -\eta \end{array} \right) (A.21)
\left( \begin{array}{ccc} \ell & \ell & j \\ \eta & \eta' & -\eta' \end{array} \right).
\]
Using this in Equation (A.16)

\[
\langle p_e', l', k_i, L', J, m_j \mid t_i, G_o(U_j, l) \Phi \rangle = \frac{32 \pi}{p_e'} \int \frac{k_j^2 d k_j}{p_e'}
\]

\[
b_i^2 d p_i, d \mu_{ij|k_j} \delta \left( p_i^2 - p_e^2 \right) G_o \left( p_j^2, k_j^2, s \right)
\]

\[
\langle k_i L' \mid t_i \left( s - 2 k_i^2 \right) \mid k_i L \rangle \quad (A.22)
\]

\[
\sum_{L} \left( \frac{2 L + 1}{4 \pi} \right)^{1/2} \chi^*_{n_i} \left( \theta_{l' k_j, o} \right) \chi_{n_i} \left( \theta_{k_i l', o} \right)
\]

\[
\chi_{n_i} \left( \theta_{l' k_j, n} \right) \left( \begin{array}{ccc} l' & L' & J' \\ n_i & n_i & -n_e \end{array} \right) \left( \begin{array}{ccc} l & L & J \\ n_i & 0 & -n_e \end{array} \right)
\]

\[
(-1)^{L' + L - l - l'} \langle p_j, l, k_j, L, J, m_j \mid U_j \mid \Phi \rangle.
\]

The delta function is used to integrate over \( d \mu_{ij|k_j} \). From the kinematics \( p_i^2 = k_j^2 + \frac{1}{4} p_j^2 - k_j p_j \mu_{k_j p_j} = p_i'^2 \). (A.23)

Therefore,

\[
\delta \left( p_i'^2 - p_i^2 \right) = \delta \left( p_i'^2 - k_j^2 - \frac{1}{4} p_j^2 - k_j p_j \mu_{k_j p_j} \right)
\]

\[
\delta \left( p_i^2 - p_i^2 \right) = \frac{1}{2 k_j p_j} \delta \left( \frac{2 p_i'^2}{2 k_j} - \frac{2 k_j^2}{2 k_j} p_j^2 - \mu_{k_j p_j} \right).
\]
This is true if

\[ -1 \leq \frac{2 \, p_i^a - 2 \, k_i^a - \frac{1}{2} \, p_i^a}{2 \, k_j \, p_j} \leq 1. \quad (A.25) \]

It follows that

\[ (p_i' - \frac{1}{2} \, p_j)^a \leq k_i^a \leq (p_i' + \frac{1}{2} \, p_j)^a. \quad (A.24) \]

Equation (A.16) now becomes

\[ \langle \rho_i' | L, k_i' | L', T' \rangle | t_i \rangle \, G_0 \, U_j | \Phi \rangle = \frac{8 \sqrt{\pi}}{\rho_i'} \int_0^{\infty} \frac{d \rho_j}{\rho_j} \int \frac{d k_j^2}{(k_i'^a - k_j)^2} \frac{G_0}{\rho_j^2, k_j^2, s} \]

\[ \sum_{\ell \in L \setminus \{n_3\}} \sqrt{2L+1} \, \sum_{m, n} \hat{Y}^{*\ell} \left( \Theta_{\ell k_j, o} \right) \hat{Y}^{*\ell'} \left( \Theta_{k_i', k_j', o} \right) \]

\[ \hat{Y}^{\ell} \left( \Theta_{\ell k_j, T'} \right) \left( \rho_i', L', T' \right) \left( n_1, n_1', -n_2 \right) \left( n_3 \odot -n_2 \right) \]

\[ (-1)^{L'+L-\ell' - \ell} \langle \rho_i' | L, k_j, L' | T' \rangle | U_j | \Phi \rangle. \]
Applying the results of Equation (A.10) and Equation (A.27) to Equation (A.3) yields

\[ \langle p_i', l', k_i', L', J', m_J' | U_i | \Phi \rangle = \frac{2}{\pi} \int k_i^2 \, dk_i \]

\[ G_0 (p_i^2, k_i^2, s) \langle k_i' l' | \xi_i (s - 2 \gamma p_i^2) | k_i L' \rangle \]

\[ \langle p_i', l', k_i, L', J', m_J | H_u | \Phi \rangle \]

\[ + \sum_{j \neq i} \frac{8 \sqrt{\pi}}{p_i} \int_0^\infty p_j \, dp_j \int \frac{(p_i + \gamma p_j)^2}{(p_i - \gamma p_j)^2} \, dk_j^2 \]

\[ G_0 (p_j^2, k_j^2, s) \langle k_i' l' | \xi_i (s - 2 \gamma p_j^2) | k_i L' \rangle \]

\[ \sum_{kL} \sqrt{2L + 1} \sum_{\{h\}} Y_{l'}^* (\Omega_{p_i' k_i}, 0) Y_{l'}^* (\Omega_{k_i l_i}, 0) \]

\[ Y_{l}^{n_i} (\Omega_{p_i k_i}, \pi) \begin{pmatrix} l' & L' & J' \\ \eta_2 & \eta_2 & -\eta_2 \end{pmatrix} \begin{pmatrix} l & L & J \\ \eta_2 & 0 & -\eta_2 \end{pmatrix} \]

\[ (-1)^{L' + L - \gamma - \ell} \langle p_j, l, k_j, L, J', m_J' | U_j | \Phi \rangle. \]
APPENDIX B

KINETICS

The conservation of energy and momentum requires that

\[ \vec{P}_0 = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 \]
\[ T_0 + Q = T_1 + T_2 + T_3, \]  \hspace{1cm} (B.1)

where the subscripts 0, 1, 2, 3 refer to the incident particle and the three final particles. The laboratory momenta are given by

\[ \vec{P}_1 = \frac{\vec{P}_{cm}}{3} + \vec{p}_1. \]

Specification of five independent quantities together with Equations (B.1) completely specify the kinematics of the reaction. The variables used by Bronson, et al. (16), in the experimental study of this reaction are employed here. The direction of two of the particles and the energy of one of the particles are specified.

The Z-axis of the coordinate system was chosen in the direction of the incident beam. This coordinate system is shown in Figure B1. Since we calculate only in the plane determined by \( \vec{P}_1 \) and \( \hat{Z} \), the azimuthal angles will be \( \hat{\phi}_1 = 0 \) and \( \hat{\phi}_2 = \pi \).
Figure B1

Laboratory coordinate system: \( Z \) is along the direction of the incident beam and \( P_1 \) and \( P_2 \) are in the \( X-Z \) plane.
The calculations for the complete specification of the kinematics have been carried out by Bronson and are presented here only for completeness.

Using Equations (B.1) $P_2$ is a quadratic function of $P_1$ and is given by

$$P_2^\pm = \frac{1}{2} (B \pm \sqrt{B^2 - 2C}),$$

where

$$B = P_0 \cos \Theta_2 - P_1 \cos \Delta_{12}$$
$$C = 2 P_1^2 - 3 P_0^2 - 2 P_0 P_1 \cos \Theta_1 - 2Q$$

and $\Delta_{12}$ is the angle between $\vec{P}_1$ and $\vec{P}_2$. The range of values of $P_1$ are determined by $B^2 - 2C = 0$. When $P_2$ becomes negative it will be taken to be opposite to the specified direction of $\vec{P}_2$. Similarly, $P_3$ may be obtained from Equations (B.1).

The azimuthal angles in the center-of-mass system are the same as in the laboratory system.

The law of cosines is used to determine the center-of-mass velocities

$$v_1 = (V_1^2 + V_{cm}^2 - 2V_1 V_{cm} \cos \Theta_1)^{1/2}$$

and the center-of-mass polar angles

$$\cos \Theta_1 = (V_1 \cos \Theta_1 - V_{cm}) / v_1.$$
The Jacobians are required for the transformation of the cross section from the center-of-mass system to the laboratory system, that is,

\[ \sigma_{\text{LAB}} = J_L \sigma_{\text{CM}} , \]

where

\[ \sigma_{\text{CM}} = \frac{d^3 \sigma}{dt_1 d \Omega_{1(\text{CM})} d \Omega_{2(\text{CM})}} \]

\[ \sigma_{\text{LAB}} = \frac{d^3 \sigma}{dT_1 d \Omega_{1(\text{LAB})} d \Omega_{2(\text{LAB})}} . \]

This Jacobian is given by

\[ J_L = J \left( \frac{t_1, \mu_1, \varphi_1, \mu_2, \varphi_2}{T_1, M_1, \xi_1, M_2, \xi_2} \right) , \]

where \( \mu_i, M_i \) are the cosines of the polar angles in the center-of-mass and laboratory system. Since the azimuthal angles are the same in both systems

\[ J_L = J \left( \frac{t_1, \mu_1, \mu_2}{T_1, M_1, M_2} \right) . \]

t_1 and \( \mu_1 \) are not functions of \( M_2 \) so the determinant becomes

\[ J_L = \frac{\partial \mu_2}{\partial M_2} J \left( \frac{t_1, \mu_1}{T_1, M_1} \right) . \]
Using Equations (B.2) and (B.3) to evaluate these terms

\[
J_L = \frac{P_1 P_2^3}{P_1 P_2^2} \left[ \frac{2 P_2 + P_1 M_{1e}}{2 P_2 + P_1 M_{1e} - P_0 M_2} \right].
\]
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REFERENCES


