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Portfolio Selection and the Theory of Corporation Finance

by

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CHAPTER I

INTRODUCTION

The starting point of this dissertation is the well known article by Modigliani and Miller (henceforth MM), entitled, "The Cost of Capital, Corporation Finance, and the Theory of Investment." [24] In this, the authors begin by proving a theorem that is fundamental to the rest of the article, and which sharply conflicts with what they characterize as the "traditional view." This theorem (henceforth the MM theorem) states that the market value of any firm is independent of its capital structure; an immediate consequence of this is that the firm's average cost of capital as they define it is also independent of its capital structure. The "traditional view" that they oppose envisages a U-shaped cost of capital, so that the firm's managers have the task of selecting the particular debt-equity ratio that minimizes the cost of capital. (This is also the debt-equity ratio that maximizes the value of the firm under their assumptions.)

For their proof, MM make the following explicit assumptions:

1. There are no corporate taxes.
2. Retained earnings may be regarded as a fully subscribed, pre-emptive stock issue.
3. Individuals and corporations may both borrow at the riskless interest rate.
4. Firms may be grouped into risk classes, with at least two firms in each class, defined such that the returns on the shares of all firms in the same risk class are perfectly correlated.

MM then show how individuals may use "home made" leverage to increase or decrease the effective leverage of the firm. The effect of raising the debt-equity ratio is to magnify fluctuations in earnings per share, since the number of shares is reduced, while fluctuations in gross profits are not. (See Kalecki's "Principle of Increasing Risk," [17]) If the firm has a debt-equity ratio of one, for example, an investor who views with distaste the amount of risk involved in holding its stock may purchase a combination of stock and bonds instead, thus reducing the risk taken to an acceptable level. On the other hand, a risk-loving investor can raise both his risk and expected return by increasing his stock purchases by borrowing money "on margin" from his broker or banker. Given their assumptions, MM assert that this "home-made" leverage will be a perfect substitute for the firm's own leverage, so that investors are indifferent between two firms from the same risk class with different debt-equity ratios if and only if their market values are identical. It follows that the value of the firm must be independent of the capital structure.

They then extend this result by modifying the third assumption above, allowing the cost of debt to the firm to rise with increasing leverage as long as the cost of debt to the individual rises as the same function of his personal debt-equity ratio.
When taxes are introduced, it is admitted that the value of the firm can be increased by raising leverage, but MM suggest that the practical effect of this is very small. To support this contention, they perform an empirical test to determine the effect of leverage on the average cost of capital within a given risk class. Taking an industry to be equivalent to a risk class and measuring the average cost of capital as the ratio of actual net profits for a particular year to the value of the firm, they test two industries and find no statistically significant linear or curvilinear relationship.

A number of criticisms may be leveled at the theorem and at the empirical work supporting it. First, the identification of a risk class with an industry for the purposes of the test is unsatisfactory; the presence of a relatively wide spread of values for the average cost of capital may simply indicate the poorness of this approximation.

Secondly, the ratio between actual earnings and the value of the firm need not be a good approximation for the average cost of capital, which in MM's theoretical discussion is defined as the ratio between expected earnings and the value of the firm. Weston in [40] suggests that current earnings may significantly understate expected earnings in the case of rapidly growing firms, and so MM's estimate of the cost of capital for such firms may be too low. When he adjusts the figures to allow for the rate of growth, he finds a significant negative relationship between leverage and the cost of capital, which, he says, supports the traditional view that up to a point the cost of debt finance is less than that of equity finance.
Thirdly and crucially, MM's conclusions from their empirical test would not even be warranted if the difficulties just mentioned did not exist. They assert that, according to the traditional view, one might expect to observe a negative relationship between the cost of capital and leverage. But this would be true only if the firms involved failed to act according to MM's own prescribed standard of rationality. If the principle of market value maximization is followed, and if the sample contains a homogeneous group of firms, entrepreneurs will all choose the debt-equity ratio corresponding to the lowest part of the U-shaped cost of capital curve. We would then expect to find the observed debt-equity ratios closely clustered around a single point; deviations from this point could be accounted for by the accidental inclusion of firms from other risk classes (as strictly defined) or by errors of measurement, rather than by assuming they correspond to firms failing to maximize their market value. MM's hypothesis, however, would imply that any debt-equity ratio would be optimal, and therefore the observed combinations of cost of capital and degrees of leverage would be strung out at random along a horizontal line. Distinguishing between these two hypotheses, then, is likely to be difficult; as far as any indication can be inferred from MM's scatter diagrams, the evidence appears to support the traditional view rather than their own, since there seems to be a significant difference between the average degrees of leverage in the two industries considered. Furthermore, the points are much more tightly clustered in the sample of electrical utility companies which is likely to represent a more homogeneous group that the sample of oil companies.
At best, then, MM have failed to prove that their theorem fits the facts better than the traditional view. More criticism may be made of their theoretical model. I find objectionable their original definition of expected return, \( \bar{X} \), ([24] fn. 6, p. 265), which is the average of all profits created by the firm's present assets over all periods into the infinite future. There is no guarantee that this must be defined; if there is continuing inflation, profits may increase without bound, and the average will approach infinity. On the other hand, if the assets stop earning at some point, the average will go to zero. This variable also ignores the timing of fluctuations around the mean. MM note this, but then say that this is unimportant; however, its importance or lack of it surely depends on the level of interest rates. If interest rates are high, a (finite) declining sequence of profits may be significantly more attractive than an increasing sequence with the same average. Another source of dissatisfaction with MM's use of \( \bar{X} \) is an estimate of a population parameter, then it is to be expected that it will change over time as investors revise their expectations in response to past errors, giving rise to capital gains or losses in stock values.

Another major problem of the MM theorem is the need for the existence of risk classes containing at least two firms. Risk classes exist only in the mind of the investor; the existence of a single investor who considers unique the distribution of the earnings of a firm whose
stock he holds is sufficient to invalidate the theorem.

There are other theoretical difficulties present that I discuss in later chapters; especially weak is the extension of their original theorem to cover the case of a rising cost of debt and their neglect of the problems caused by the existence of limited liability.

Nevertheless, my intention in this dissertation is not so much to point out the errors in their approach as to develop a more general and comprehensive theory, built along essentially similar lines. Despite the numerous objections that may be leveled at it, their theorem has shown a remarkable degree of robustness, and it has not really been replaced by a satisfactory alternative theory to this day. Perhaps the reason for its durability is that the most significant contribution made by the article is methodological. It removes attention from individuals' risk preferences or attempts to appeal to a "market indifference function" between expected return and some measure of risk (e.g. Schwartz in [31]), and focuses instead upon the objective opportunities presented by the market. To get around the major problems already mentioned, I propose to begin by building a model of portfolio selection. This method of approach has been tried before, although with little impact on the state of economic knowledge. (See Hamada [14], Ben Shahar [6], Baldwin and Velk [3], Heins and Sprenkle [15].) The reason why these fail to make an important contribution is that they begin with models of portfolio selection that depend on very particular assumptions that are even more restrictive than those used by MM. Confirmation of the theorem, then, through the use of more restrictive assumptions than the original is of no significance, while attempts
to criticize MM's method of proof applied to a different set of assumptions (e.g. [3]) are absurd.\footnote{An exception to this is Mossin [29]. He uses a more general model of portfolio selection than the other references cited, similar to that used in the next chapter, and his confirmation of the MM theorem is a significant advance. However, his discussion is limited to the case I consider in Chapter IV, Section II.} Despite this, I believe the general direction of approach to be a fruitful one, as long as the theory of portfolio selection used does not introduce any more objectional assumptions than those eliminated. The principal gain from using this approach is that it enables me to dispense with the assumption that firms may be grouped in risk classes. Rather than using the income from the stock of one firm as a point of reference in order to compare the income stream that can be earned from a combination of securities from another firm in the same risk class, I start by stating the conditions for an investor's portfolio to be optimal and then examine the way in which he will adjust his portfolio in response to a change in the leverage of a particular firm in order to re-optimize his portfolio. An objectionable feature of the MM analysis is thus eliminated, while the loss of generality is small, since my model of portfolio selection is based on the very general axioms of the von Neumann and Morgenstern approach to decision-making in risky situations. The investor's objective is then the maximization of expected utility; his utility function is assumed to be concave and continuously differentiable, and has as its argument the investor's accrued one period dollar return on his portfolio, consisting of all cash receipts from holdings of financial assets, plus all capital gains or losses whether realized or accrued.
There are no problems with corner solutions, since I use the Kuhn-Tucker theorem to derive the first order optimizing conditions, and margin restrictions on short sales and brokers' loans may be easily handled.

This approach turns out to be highly flexible, enabling me to relax progressively a number of restrictive assumptions that are initially imposed and to follow the implications with respect to the effect of leverage on the value of the firm. Maximizing the value of the firm is taken to be the objective of the firm, on the grounds that it will simultaneously maximize the current value of the stockholders' wealth, rather than in order to minimize the cost of capital for investment purposes. (The conditions under which both objectives may be achieved simultaneously are considered in Chapter III.) This is because MM's assertion that the cost of capital is equal to the expected rate of return on unlevered equity makes sense only if all investors share the same expectations; by making this undesirable assumption, they are sweeping a number of thorny problems under the rug. Furthermore, their proposition even then is only true if the prospective investment is of a type that will shift the distribution of profits to the right by a simple multiplicative factor, and additionally if the firm is a perfect competitor in the capital market. It is not necessary to become embroiled in these problems, since there are adequate reasons for the firm to maximize its market value without appealing to the cost of capital.

It is my contention that my approach to the theory of corporation finance is based on more realistic and general assumptions than those used
(explicitly and implicitly) by MM. My conclusions, as might be expected, are somewhat less simple than theirs, though their theorem may easily be derived as a special case. After relaxing all the restrictive assumptions in my model, I conclude that the traditional view is in fact correct to the extent that there does exist an optimal debt-equity ratio. Furthermore, MM are mistaken in believing that only objective market opportunities need be considered, to the exclusion of all subjective elements, since investors' portfolio adjustments are seen to depend upon their subjective estimates of shifts in the distribution of profits in response to changes in leverage and the cost of legal constraints on financial institutions. However, MM are correct in pointing out that investors' attitudes towards risk are in most cases irrelevant, and that for this reason, traditional approaches to the problem are seriously in error.

The model of portfolio selection developed in the next chapter is primarily oriented towards providing tools for application in Chapter IV, although I believe it also makes a useful contribution in its own area. Furthermore, in Chapter III, I show how portfolio theory may be employed usefully to answer other questions about stability and the degree of competition in the capital market.
CHAPTER II

PORTFOLIO SELECTION

I Assumptions

1. Financial assets are perfectly divisible.

2. There are no frictional costs, including brokerage fees, the cost of time and trouble in giving orders to the broker, and the cost of acquiring information about stocks and stock prices, and there are also no external benefits, such as love of gambling. This greatly simplifies the analysis, although clearly these costs do play a considerable part in the real world in limiting the effectiveness of diversification to reduce risk. Furthermore, in assumption 3 below, I postulate the existence of a riskless interest rate which must be greater than zero; this in turn may be taken to imply the existence of transactions costs so that riskless assets may coexist with money in equilibrium. This possible source of inconsistency appears in other work on portfolio selection (e.g. Sharpe [32] and Lintner [20]), although it is not demonstrated. An attempt may perhaps be made to reconcile these two assumptions by letting the riskless interest rate simply equal a constant and certain rate of inflation of commodity prices.

3. There exist \((n + 1)\) classes of assets, a class being defined so
that all assets within a given class are always perfect substitutes, having identical rates of return. Typically, one class will include the stock certificates of one firm only; note also that the return I am considering is that yielded over a single period, and not an average of returns into the infinite future, and so my definition of an asset class differs clearly from that used by MM. At least one class is assumed to consist of riskless assets; this is numbered class zero, with a certain rate of return $r_0$ which is greater than zero. The rates of return on the other $n$ classes may be represented by a random vector $R = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$.

4. The investor's decision variable is a vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, where $x_j$ is the dollar amount invested in the $j$-th class of asset for the coming period. $x_0 = w - V'X$ is the amount invested in riskless assets where $w$ is the total value of his portfolio, and $V$ is the sum vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

5. The investor's utility function $U(I)$ is unique up to a positive linear transformation, (i.e. the scale and origin are arbitrary). The argument of the function is the investor's accrued one period
dollar income from his wealth, including both cash receipts and changes in net worth. The function is assumed to be strictly concave, with a continuous first derivative for all possible values of I. The first derivative, $U_I$, is assumed to be always positive, and the second derivative, $U_{II}$, negative (where it exists). Thus he is assumed to be everywhere risk-averse; to avoid situations like the St. Petersburg paradox, it will also be assumed that the function has finite upper and lower bounds. (See Arrow [2] p27)

6. The investor is a perfect competitor; i.e., he ignores any influence his purchases or sales may have on asset prices.

7. There is no taxation. The effect of allowing taxation is considered later.

8. All asset holdings must be non-negative. The reason for this assumption is that, although short sales are permitted on the stock market, there are rigid constraints imposed by the broker on the investor's freedom to invest the proceeds from the short sale. Generally, more than 100% of the proceeds must be held in the form of cash until the short position is covered; therefore the effective interest rate the investor is paying for "borrowing" the securities is higher than the rate on the shares sold short by the opportunity cost of the money sitting idle, and typically short sales will only be made if the expected rate of return on the share is negative. Tobin in [38]
pays lip service to non-negativity constraints, but fails to incorporate them adequately into his model. Lintner in [22] does allow for short sales, but only in the special case of a 100% margin requirement. I prefer to consider first the case in which short sales are ruled out, and then to show how they may be introduced into the model without any restrictions on the margin requirement.

II The Optimum Portfolio

Let us write the dollar return on the portfolio:

\[ I = \sum_{j=0}^{n} x_j r_j = R'X + (w - V'X)r_o \]

(1) \[ = wr_o + (R - Vr_o)'X \]

Note that this function is linear in \( X \); i.e., for any two arbitrary distinct vectors \( X^1 \) and \( X^2 \), and corresponding return \( I^1 \) and \( I^2 \):

\[ \lambda I^1 + (1 - \lambda)I^2 = \lambda (wr_o + [R - r_o V]'X^1) + (1 - \lambda)(wr_o + [R - r_o V]'X^2) \]

(2) \[ = wr_o + (R - r_o V)'(\lambda X^1 + [1 - \lambda]X^2) \]

Therefore, we may quickly show the utility function to be concave in \( X \), since, by the concavity in \( I \), we have:

(3) \[ \lambda U(I^1) + (1 - \lambda) U(I^2) \leq U(\lambda I^1 + [1 - \lambda]I^2) \]
and, using (2):

\[ (4) \quad \lambda U(I^1) + (1 - \lambda) U(I^2) \leq U(wr_o + [R - r_o V]' [\lambda X^1 + (1 - \lambda) X^2]) \]

which proves \( U(I) \) to be concave in \( X \). It follows immediately that the expected value of \( U(I) \) is also concave in \( X \).

We now wish to maximize \( E[U(I)] \) with respect to \( X \), where \( E[\quad] \) is the expected value operator, subject to the constraints that all asset holdings should be non-negative. Since \( x_o \) is not an explicit decision variable, it is necessary to introduce a separate constraint for it, i.e. \( X \succcurlyeq 0, \ w - V'X \succcurlyeq 0 \). To do this, first set up the Lagrangean function:

\[ (5) \quad L(X, \mu) = E[U(I)] + \mu(w - V'X) \]

The constraint is linear in \( X \) and therefore is also concave, and it also satisfies the constraint qualification that there should exist at least one feasible point that is not on the constraint boundary; i.e., that there exists a vector \( X \succcurlyeq 0 \) for which \( w - V'X \succcurlyeq 0 \). This may be assured by choosing \( 0 \leq x_j < \frac{1}{n} \quad (j = 1 \ldots n) \). The Kuhn-Tucker theorem \( [18] \) then states that \( X^* \) is the vector that maximizes \( E[U(I)] \), subject to the constraints, if and only if there is a \( \mu^* \succcurlyeq 0 \) such that \( (X^*, \mu^*) \) is a saddle point of \( L(X, \mu) \) in \( X, \mu \succcurlyeq 0 \).

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\(^2\) Equation (3) is written as a weak inequality although \( U(I) \) is strictly concave because it is possible for \( I^1 = I^2 \) even though \( X^1 \neq X^2 \).
The first order conditions for a saddle point are:

\[(1) \frac{\partial L}{\partial x} \leq 0, \ (ii) \frac{\partial L}{\partial \mu} \geq 0, \ (iii) \ x^* \frac{\partial L}{\partial x} = 0, \ (iv) \ \mu^* \frac{\partial L}{\partial \mu} = 0.\]

Since \(U\) has a continuous first derivative in \(I\) and hence in \(X\), we may differentiate (5):

\[(6) \quad \frac{\partial L}{\partial x} = E_\theta \left[ \frac{\partial U(I)}{\partial x} \right] - \mu^* \nu \leq 0\]

\[(7) \quad \therefore E_\theta \left[ U(I)(R - r_o \nu) \right] - \mu^* \nu \leq 0\]

Let us define \(\tilde{r} = r_o + \frac{\mu^*}{E_\theta [U(I)]}\). Then we may rewrite (7) as:

\[(8) \quad E_\theta \left[ U(I)(R - \tilde{r} \nu) \right] \leq 0\]

and the \(k\)-th inequality is:

\[(9) \quad E_\theta [U(I)(r_k - \tilde{r})] \leq 0.\]

This is the particular form of the portfolio optimization condition that will be used in the following two chapters. If we let \(\text{Cov} \ [U_I, r_k]\) be the covariance between \(U_I\) and \(r_k\), and \(\tilde{r}_k = E[r_k]\), we may rewrite (9) as follows:

\[\text{Cov} \ [U_I, r_k] + E[U_I] \ E[r_k - \tilde{r}] \leq 0\]

\[(10) \quad \therefore \tilde{r}_k \leq \tilde{r} - \frac{\text{Cov}[U_I, r_k]}{E[U_I]}\]
From (ii) above, we have:

(11) \[ \frac{\partial L}{\partial \mu} = w - \eta^t x^* = x^*_o > 0. \]

From (iv) above, we see that if the optimum portfolio contains some of the riskless asset \( x^*_o > 0 \), then \( \mu^* = 0 \) and \( \tilde{r} = r_o \); if \( \tilde{r} > r_o \), then it must be the case that \( x^*_o = 0 \). It is possible for \( x^*_o \) and \( \mu^* \) both to equal zero; for example, if there were another class of asset with a certain rate of return \( r_o \), the investor would be indifferent between these assets from class zero. If \( \mu^* = 0 \), the optimal portfolio could contain assets from either class.

For class \( k \) assets to be included in the optimum portfolio, (10) must hold as an equation; if \( \bar{r}_k \) should be less than the value on the right hand side for all positive values of \( x_k \), then, by (iii), \( x_k^* = 0 \).

If \( \bar{r}_k \) should be greater than the value on the right hand side, then the portfolio is not utility maximizing, since \( \frac{d}{dx_k} E[U(I)] > 0 \) and the investor should increase his holding of class \( k \) assets. We may then regard equation (10) as the generalization of the portfolio optimization conditions given by Lintner, Sharpe and Tobin. In the case that \( x^*_o > 0 \) and \( x^*_k > 0 \):

(12) \[ \bar{r}_k = r_o - \frac{\text{Cov} [U_I, r_k]}{E[U_I]} \]
We could have derived equation (12) by maximizing $E[U(I)]$ without the constraints; in that case it would have to hold as an equation rather than an inequality for all asset classes, whether the implied values for each component of $X^*$ were positive or negative. It would also permit $x_0 < 0$; i.e. the investor must be able to borrow unlimited amounts at the riskless interest rate.

The Lagrangean multiplier in the optimal portfolio, $\mu^*$, may be interpreted as the marginal value of relaxing the non-negativity constraint on class zero assets, provided there are no other effective constraints on $X^*$. (See Lancaster [19] p. 71.) If the non-negativity constraint on any component of $X^*$ is binding, then the value of relaxing the constraint on $x_0$ is reduced.

$\tilde{r}$ may be called the marginal opportunity cost of riskless return; it must be at least equal to the highest attainable rate of return on a riskless asset, since, for any riskless class $j$, (10) reduces to $\tilde{r}_j < \tilde{r}$. If $\tilde{r}$ should be greater than the highest attainable certain rate of return, then the investor will hold no riskless assets. The second term on the right hand side of (10) I shall call the marginal contribution of class $k$ assets to the riskiness of the total portfolio. This implies that the solution $X^*$ to the original problem is also the solution to the problem of maximizing $E[I] - \rho(I)$ subject to the same constraints, where $\rho(I)$ is defined so that $\frac{\partial \rho}{\partial x_k} = - \frac{\text{Cov} [U_I, r_k]}{E[U_I]}$. $\rho(I)$ is then a measure of risk in terms of dollars and cents, incorporating into itself the investor's own degree
of risk aversion.

We may now make some general remarks about the composition of the investor's portfolio. Most rates of return are heavily influenced by general market conditions, and so we might expect most or all of them to be positively correlated. This in turn implies that \( r_k \) is most likely to be positively correlated with \( I \), except in the unusual case in which \( r_k \) is strongly negatively correlated with other rates of return. Since \( U_I \) is assumed to be a decreasing function of \( I \), we can expect that \( r_k \) will then be negatively correlated with \( U_I \), and it will follow from equation (10) that, for the investor to include class \( k \) assets in his portfolio, \( \bar{r}_k \) must be greater than \( \bar{r} \), the excess representing a subjective risk premium. For example, if most commodities are normal goods, a fall in the level of national income can be expected to cause demand for each of them and hence profits to be reduced. If the level of national income is an important source of variation in profits, then the expected rate of return on the stock of the companies producing mainly normal goods must exceed the riskless interest rate to induce the investor to hold them. On the other hand, margarine manufacturers might stand to gain from a decrease in national income if margarine is an inferior good, and the rate of return on their stock might therefore be perceived to be negatively correlated with other rates of return. In that case, the investor might be induced to hold their stock even if the risk premium were negative, and the expected rate of return were less than the riskless interest rate, since the inclusion
of such assets into the portfolio would reduce rather than increase the riskiness of the total portfolio. Note that the size of the risk premium necessary to induce an investor to include an asset in his portfolio also depends in general on his utility function. In the limiting case, if the utility function is linear, \( U \) must be a constant, and so the covariance term is always zero, leaving \( \tilde{r}_k \leq \tilde{r} \) as \( x_k^* = 0 \). In this case, the investor is indifferent to risk, and \( \tilde{r} \) becomes the highest possible expected return; the investor will only hold assets which offer this rate of return.

III  Short Sales

It is quite simple to introduce the possibility of short sales into the model at this point. Under present law, making a short sale is very much the same as buying a security, since some of the investor's own wealth is tied up meeting the margin requirement. For the moment, then, let us double the number of components in the \( X \) vector, so that the component in position \( (n + j) \) will represent the amount of money tied up in meeting the margin requirement for a short sale of class \( j \) assets. The amount of class \( j \) asset sold short, then, will be \( \frac{x_{n+j}}{m_1} \), where \( m_1 \) is the percentage of the amount of the short sale that must be put up by the investor. The rate of return on the investor's money is \( \frac{r_1}{m_1} \); therefore, the optimality condition is:
\[
- \frac{\bar{r}_j}{m_1} < - \frac{\text{Cov} \left[ U_I, - \frac{r_j}{m_1} \right]}{E[U_I]}
\]

(13) \quad \therefore \quad \bar{r}_j \geq - m_1 r - \frac{\text{Cov} \left[ U_I, r_j \right]}{E[U_I]}

It follows that (13) must hold as an equation if class \( j \) assets are to be sold short; if \( \bar{r}_j \) should exceed this value, the asset will not be sold short and \( x_{n+1}^* = 0 \). Note that for the interval

\[
- \frac{m_1 r}{E[U_I]} - \frac{\text{Cov} \left[ U_I, r_j \right]}{E[U_I]} < \bar{r}_j < \frac{m_1 r}{E[U_I]} - \frac{\text{Cov} \left[ U_I, r_j \right]}{E[U_I]}
\]

it will neither be sold short nor included in the portfolio. The prediction of such an interval is reassuring, since investors typically do not hold short or long positions in every available asset. If we drop the non-negativity constraints, this is equivalent to having a margin requirement of -1 (not only is none of the investor's own wealth tied up in the margin requirement, but the full amount received from the short sale may be used by the investor to purchase other securities as he pleases); in this case, the interval above vanishes, and we may conclude that investors will have positive or negative holdings of assets from all classes, except for classes or class combinations that are perfect substitutes for each other.
We may drop the requirement that \( x_0 \geq c \) by simply maximizing (5) without the constraint. The term \( \mu^\ast \) would then disappear from all the maximizing conditions, and \( \tilde{r} \) could be replaced everywhere by \( r_0 \). This makes the rather objectionable assumption that investors can borrow without limit at the riskless interest rate; on the other hand, maintaining the constraint makes the equally objectionable assumption that borrowing is impossible. It is possible, however, to allow the investor to buy "on margin" by replacing the constraint \( w - V'X \geq c \) by \\
w(1 + m_2) - V'X \geq c, \) where \( m_2 \) is the maximum allowable amount of borrowed funds, expressed as a fraction of the investor's net worth. This relaxes the constraint somewhat, and it may be argued plausibly that it will be binding in this form on very few individuals, so that typically \( \mu^\ast = 0 \) and \( \tilde{r} = r_0 \). Furthermore, the implication that borrowing is done at the riskless rate is quite reasonable, since the loan is secured by the financial assets purchased with it, and the broker or banker who makes the loan reserves the right to sell the assets to recover his money if the ratio of the loan to the net value of the investor's portfolio should rise above the holding margin, which is somewhat greater than \( m_2 \). Historically, margin loans have usually been extended at little more than the prime lending rate, except in times of disequilibrium, like the speculative fever of 1929 before the crash, when margin loans were made at rates up to 20\%. 
It would not be difficult to construct a portfolio decision model in which the cost of margin loans is above the riskless interest rate. Class zero assets could be redefined as a riskless asset with certain rate of return $r_0$, involving two constraints: one restricting holdings to non-positive values and the other representing the margin constraint. This would be slightly more general than the present model, but it would add a further complication in the form of another Lagrangean multiplier, and I do not believe that it is worth it. On top of this, at the expense of some more complication, $r_0$ could be made to rise with the investor's personal debt-wealth ratio; the effect of this is to raise $r$ as more debt is acquired.

In the rest of the thesis, I shall generally ignore the possibility of short sales, on the assumption that they are essentially of a transitory nature, although I believe that it would be possible to introduce them without destroying any of my results. I shall also assume that the possibility of obtaining margin loans at the riskless interest rate exists, rather than introduce the further complications mentioned above.

IV Financial Institutions

By asserting that there exist corporate utility functions, derived partly from the preferences of stockholders and partly from those of the management, we may reduce the portfolio selection decisions of financial institutions to the same optimization conditions given in this chapter
for individual investors. The only major difference between institutions and individual investors is that insurance companies and mutual savings banks that do business in New York (which includes most of the large companies) are required by New York State law to hold a certain minimum fraction of their total reserves in the form of fixed interest securities. If such securities are considered riskless, they may all be lumped together into a single class with a rate of return $r_o$, and the constraint may simply be of the form $w(1 - m_3) - X^V \geq o$, where $m_3$ is the minimum required fraction of fixed interest securities. The optimization conditions, then, will be identical to those for individual investors, although the probability of an interior maximum occurring is much reduced, and it is most likely that the marginal cost of riskless return will be considerably above $r_o$.

If not all fixed-interest securities can be regarded as riskless, it is necessary to add an extra constraint of the form $\sum_{j \in \phi} x_j - wm_3 \geq o$, as well as the non-negativity constraint on class zero holdings, where $\phi$ is the set of fixed interest securities, with $\phi'$ as its complement. If we may assume that the non-negativity constraint on the riskless asset is not binding (to avoid multiplication of multipliers), the portfolio optimization conditions become:

(14) $E[U_I(r_k - r_o)] - \mu^* \leq o$ for $k \in \phi'$

(15) $E[U_I(r_j - r_o)] \leq o$ for $j \in \phi$
Thus, the marginal cost of riskless return is higher (assuming the constraint to be binding) when the purchase of stock is considered than when the purchase of bonds is considered, the difference representing the marginal cost of being constrained.

V  Substitutability and Complementarity in the Portfolio

Note that the optimum portfolio is not necessarily uniquely defined, since $E[U(I)]$ is not strictly concave in $X$. This in turn is because, under certain circumstances, two different vectors $X_1$ and $X^2$ may map into the same return, $T$. In such a case, the investor will obviously be indifferent between the two vectors, since they both imply the same expected utility, and, since I is linear in $X$, any feasible linear combination of $X_1$ and $X^2$ will be a perfect substitute for any other.

Consider first the case in which $r_1 = a r_2$ where $a < 1$. (If $a > 1$, then the classes may be renumbered so that this constraint holds.) If the investor includes class 2 assets in his optimum portfolio, then:

\[(16) \quad \bar{r}_2 = \bar{r} - \frac{\text{Cov} \left[ U_T, r_2 \right]}{E[U_T]} \]

\[\therefore \quad \bar{r}_1 = a \bar{r}_2 = \bar{r} - \frac{a \text{ Cov} \left[ U_T, r_2 \right]}{E[U_T]} \]

\[(17) \quad = \bar{r} - \frac{\text{Cov} \left[ U_T, r_1 \right]}{E[U_T]} \]
since Cov \([U_I, r_1]\) = a Cov \([U_I, r_2]\). Therefore, if \(a\) is less than one,

\[\bar{r}_1 < \bar{r} - \frac{\text{Cov} \ [U_I, r_1]}{E[U_I]},\]

and so no class 1 assets will be included in the optimum portfolio, while if \(a = 1\), the two rates of return are identical, and assets from the two classes may be substituted freely for one another without changing in any way the characteristics of the portfolio. In this latter case, the two assets could be lumped together into a single class and the optimum holding of assets from this composite class could be uniquely determined. It should be noted that, if

\[\bar{r}_2 < \bar{r} - \frac{\text{Cov} \ [U_I, r_2]}{E[U_I]},\]

then both \(x_2^*\) and \(x_1^*\) must be zero.

Note also that the assumption that \(r_o > 0\) is crucial, for if \(\bar{r} = 0\), then the investor would be indifferent between class 1 and 2 assets whatever the value of \(a\), and, if \(\bar{r} < 0\), the conclusion just given would be reversed. The reason for this is that, if \(a < 1\), \(r_2\) will only exceed \(r_1\) when they are both positive, the opposite being true when they are negative. It is not true, then, that \(r_2\) dominates \(r_1\), and so it is far from obvious that \(r_2\) should always be preferred to \(r_1\), particularly since it has a greater dispersion than \(r_1\).

A second case to consider is when \(r_1 = a + br_2\), where \(b > 0\) and \(a \leq \bar{r} (1 - b)\). Again, if the parameters \(a\) and \(b\) do not fit the second
constraint, the classes may be renumbered so that they will. If 

\[ x^*_2 > 0, \] 

then:

\[
\begin{align*}
(18) \quad \bar{r}_2 &= \bar{r} - \frac{\text{Cov} [U_1, r_2]}{E[U_1]} \\
\bar{r}_1 &= a + b \bar{r}_2 = a + b \bar{r} - b \frac{\text{Cov} [U_1, r_2]}{E[U_1]} \\
(19) \quad &= a + b \bar{r} - \frac{\text{Cov} [U_1, r_1]}{E[U_1]}
\end{align*}
\]

since \( \text{Cov} [U_1, r_1] = b \text{Cov} [U_1, r_2] \). Therefore, if \( a < \bar{r} (1 - b) \), then it follows that the expression in equation (19) is less than

\[ \bar{r} - \frac{\text{Cov} [U_1, r_1]}{E[U_1]}, \]

and so \( x^*_1 = 0 \). Now let us assume for the moment that \( \bar{r} \) corresponds to the rate of return on a riskless asset; if all riskless assets have the same rate of return then this assumption implies that \( \mu^*_1 = 0 \) and \( \bar{r} = r_0 \). If \( a = \bar{r} (1 - b) \) and \( b < 1 \), then the investor has the opportunity of holding a composite asset which consists of class 1 assets and riskless assets in the ratio \( \frac{1}{b} \) to \( 1 - \frac{1}{b} \); the rate of return on this composite asset is \( \frac{1}{b} r_1 + (1 - \frac{1}{b}) \bar{r} = \frac{1}{b} r_1 - \frac{a}{b} = r_2 \). He will then be indifferent between such a composite asset and class 2 assets. If \( b > 1 \), then the investor may combine class 2 assets and riskless
assets in the ratio \( b \) to \( 1 - b \), creating a composite asset with a rate of return 
\[
\tilde{r} = a + br_2 = r_1,
\]
which will serve as a perfect substitute for class 1 assets.

If, however, \( \tilde{r} \) does not correspond to the rate of return on a riskless asset or asset combination, then this opportunity for switching assets in the portfolio without disturbing its optimality no longer exists. For example, if \( b < 1 \) and the investor attempts to substitute class 1 assets directly for class 2 assets, both the riskiness and the expected rate of return of the portfolio as a whole will be reduced. But as this happens, \( \text{Cov} [U_I, r_k] \) will tend to fall in absolute value and \( E[U_I] \) will most likely rise. Therefore the equality between \( \tilde{r}_k \) and 
\[
\tilde{r} = \frac{\text{Cov} [U_I, r_k]}{E[U_I]} \]
will no longer hold for any class \( k \) included in the optimal portfolio. If \( \tilde{r} \) should change, too, which is most likely, then \( a \) will no longer be equal to \( \tilde{r} (1 - b) \), and class 1 and 2 assets will not be substitutes under any circumstances.

To sum up: if \( a = \tilde{r} (1 - b) \) and there exists an asset or asset combination with a certain return \( \tilde{r} \), then composite assets may be constructed, including that asset and either class 1 (if \( b < 1 \)) or class 2 (if \( b > 1 \)) assets, that are perfect substitutes for the other class. If, however, \( \tilde{r} \) does not correspond to any such specific asset class return, this possibility does not exist, and both class 1 and 2 assets may be included in the optimum portfolio, despite the fact that their rates of
return are perfectly correlated. If \( a < \frac{\bar{r}}{1 - b} (1 - b) \), then class 2 assets will always be preferred to class 1.

Now consider the case in which \( r_1 = a + br_2 \), where \( b < 0 \); i.e., the two rates of return are perfectly negatively correlated. By selecting class 1 and class 2 assets in the ratio \( \frac{\frac{1}{1 - b}}{1 - b} \) to \( \frac{-b}{1 - b} \), the investor may hold a composite asset with a rate of return \( \frac{\frac{1}{1 - b} r_1 - \frac{b}{1 - b} r_2}{1 - b} = \frac{\frac{1}{1 - b} (a + br_2 - br_2)}{1 - b} = \frac{a}{1 - b} \). Thus, the rate of return on this composite asset has been reduced to a constant; all risk has been eliminated. This implies that it is impossible for \( a > \frac{\bar{r}}{1 - b} (1 - b) \) since \( \bar{r} \) is the marginal cost of riskless return which cannot be less than \( \frac{a}{1 - b} \), the certain rate of return on the composite asset.

It follows then that if \( a = \frac{\bar{r}}{1 - b} (1 - b) \) and \( \bar{r} \) corresponds to the certain rate of return on a specific asset or asset combination, the investor will be indifferent between holding this and the composite asset just described which has the same rate of return. On the other hand, if \( \bar{r} \) does not correspond to such an asset, then the investor will hold the composite asset with \( x_1^* = -bx_2^* \). If \( a < \frac{\bar{r}}{1 - b} (1 - b) \), then if he holds class 2 assets he will not hold class 1 assets and vice versa; it is not in general possible to say which asset he will choose, since, by renumbering the classes, the condition \( a < \frac{\bar{r}}{1 - b} (1 - b) \) will still hold. This choice depends
in general on the correlation of the assets with assets in other classes. Under no circumstances will he be indifferent between the two classes or between one class and an asset combination from the other class and a riskless class—the reason being that such a combination would call for a negative holding of the other class.

The foregoing cases may now be generalized. Suppose that we can write \( \sum_{i \in \Theta_1} a_i r_i = 0 \), where \( i \) ranges over some subset \( \Theta_1 \) of the \( n + 1 \) asset classes, and \( a_1 \neq 0 \). The constant term, if any, may be included in the coefficient of the riskless rate. \( \Theta_1 \) may be subdivided further into \( \Theta_j \) and \( \Theta_k \), which consist of all classes for which \( a_i > 0 \) and \( a_i < 0 \) respectively, so that \( \sum_{j \in \Theta_j} a_j r_j = - \sum_{k \in \Theta_k} a_k r_k \). Now the investor will be indifferent between a combination of the assets from \( \Theta_j \), weighted by the \( a_j \)'s, and a combination of the assets from \( \Theta_k \), weighted by the \(- a_k \)'s, if and only if \( \sum a_j = - \sum a_k \). He will prefer the first to the second if \( \sum a_j < - \sum a_k \), and the second to the first if \( \sum a_j > - \sum a_k \). The proof follows the same lines as before. Suppose that the investor holds the first combination, with a composite rate of return \( \frac{\sum_{j \in \Theta_j} a_j r_j}{\sum a_j} \)

\[ = - \frac{\sum_{k \in \Theta_k} a_k}{\sum a_j} \] . Therefore:
(20) \[ \tilde{\theta}_j = r - \frac{\text{Cov}(U_I, r_{\theta_j})}{E[U_I]} \]

\[ \Rightarrow \tilde{\theta}_k = -r \frac{\Sigma a_j}{\Sigma a_k} + \frac{\text{Cov}(U_I, r_{\theta_j})}{E[U_I]} \frac{\Sigma a_j}{\Sigma a_k} \]

(21) \[ = -r \frac{\Sigma a_j}{\Sigma a_k} - \frac{\text{Cov}(U_I, r_{\theta_k})}{E[U_I]} \frac{\Sigma a_j}{\Sigma a_k} \]

If \( \Sigma a_j = -\Sigma a_k \), then \( r_{\theta_j} = r_{\theta_k} \) and the two combinations are perfect substitutes. If \( \Sigma a_j < -\Sigma a_k \) then \( \tilde{\theta}_j < \tilde{\theta}_k < r - \frac{\text{Cov}(U_I, r_{\theta_k})}{E[U_I]} \frac{\Sigma a_j}{\Sigma a_k} \)

and the investor will not hold the assets from \( \theta_k \), while if \( \Sigma a_j > -\Sigma a_k \), the investor cannot hold the assets from \( \theta_j \), or the optimality condition would be violated for \( \tilde{\theta}_k \). Of course, the coefficients will not be unique; if they are not uniquely determined up to a scale factor, then there will be further opportunities for substitution within each asset combination and between combinations.

VI The Effect of Taxes

The effects of allowing personal taxation on the selection of the portfolio may now briefly be considered. If the tax is applied to total
investment income, then we may redefine $I$ as investment income after taxes, so that:

$$I = I_p - \int_0^I \tau_m(s) ds$$

where $I_p$ is pretax income and $\tau_m$ is the marginal tax rate. $I_p$ is linear in $X$, and so for $U(I)$ to be concave in $X$, we require that

$$- \int_0^I \tau_m(s) ds$$

be also concave, since the sum of two concave functions is concave. We shall also require that $\frac{dI}{dI_p}$ be always positive; otherwise the investor will be averse to increases in pretax income. Differentiating, we get:

$$\frac{dI}{dI_p} = 1 - \tau_m(I_p)$$

i.e. the function will be increasing if $\tau_m$ is always less than one, and concave in $X$ if $\tau_m$ is a nondecreasing function if $I_p$. Note that, under present tax laws, this concavity condition does not hold, since there is an implicit 100% marginal tax rate on welfare recipients' income. This means that for people with very low income, the utility function may very well be convex in $X$ over the relevant range, and such individuals will behave as risk lovers, since the government will effectively confiscate
any income they make unless it is enough to disqualify them from receiving welfare. They have nothing to lose from holding high risk assets and at least the possibility of gaining, while low risk assets carry a certain after tax yield of zero. Portfolio selection, however, is a problem that few welfare recipients are lucky enough to be confronted by, and for most other individuals, the utility function will be concave in X over the relevant range.

Present methods of taxation also depart from the case just considered in another important respect; taxes are not levied on total investment income in all cases. This principle is violated in the case of capital gains, which are not taxed until realized and then at a special rate if the asset has been held for more than six months, and also municipal bonds, the income from which is tax free. To see what effect this might have, let us separate each rate of return out into two components, so that \( R'X = R'_1X + R'_2X \), where \( R'_1X \) is total cash payments and \( R'_2X \) is long term capital appreciation. (Short term capital appreciation is treated as a cash payment.) Assume for simplicity that all capital gains are realized and that the return from riskless assets is entirely in the form of cash, and let \( C \) be total cash income, which is equal to \( w_{r_0} + (R'_1 - Vr_0)'X \), let \( G \) be capital appreciation, which is equal to \( R'_2X \), and let \( M \) be income from municipal bonds, which is equal to \( \Sigma_{j} r'_j \), where \( j \) ranges over the classes of municipal bonds. Total tax payments are then equal to twenty-five percent of capital appreciation,
plus taxes calculated at the regular rates on cash payments after de-
ducting income from municipal bonds. Therefore:

\[ I = I_p - \int_0^\infty \tau_m(s)ds - .25 G \]

\[ = C + .75 G - \int_0^\infty \tau_m(s)ds \]

(24)

\[ C \text{ and } .75 G \text{ are again linear (and therefore concave) functions of} \]

\[ X, \text{ and } - \int_0^\infty \tau_m(s)ds \text{ will be concave in } C - M \text{ and therefore in } X \text{ as long} \]

as \( \tau_m \) is non-decreasing. Therefore I is still concave in X.

The main effect of this method of taxation will be to make munici-
pal bonds and stocks with a high retention rate more attractive invest-
ments than would otherwise by the case; for example, if \( r_1 \) is the rate
of return on municipal bonds and \( r_2 \) is the rate on taxable bonds that are
identical in every other respect to municipal bonds, then, for the inves-
tor to be indifferent between the two, and assuming \( \tau_m \) to be constant
over the relevant range, it must be the case that \( r_1 = r_2 (1 - \tau_m) \),
which means that municipalities will be able to borrow at a lower rate
than institutions whose interest payments are taxable. As far as the
favorable treatment of capital gains is concerned, growing firms will be
expected to exploit these economies of internal over external finance by having a higher retention rate than they would otherwise have had. More will be said about this in Chapter IV.
CHAPTER III

THE CAPITAL MARKET AND COST OF CAPITAL

I Capital Market Equilibrium

The purpose of this chapter is to show how the theory of portfolio selection just developed may be used to answer some interesting questions about the capital market and the cost of capital. For the most part, I am not concerned with the way in which the supply of financial assets (or the demand for tangible assets) depends on the market determined rates of interest, and therefore I am dealing with essentially a short run model of asset price equilibrium, in which the supply of financial assets is assumed to be fixed. Since the rate of real investment is limited by the capacity of the capital goods industry, and the rate of disinvestment cannot exceed the rate of depreciation, changes in the relative supplies of financial assets affect the structure of market prices only very slowly. For example, if the rate of discount on high risk assets were reduced, it would be expected that the supply of those assets would gradually increase relative to the supply of other less risky assets as hitherto unprofitable risky projects become profitable, thus bringing about further changes in the structure of interest rates. I am, however, interested in the set of prices that equilibrate the capital market at any moment in time, rather than those prices that
have no tendency to change over time. When I refer to supply and demand, then, I am generally thinking in terms of stocks rather than flows, and the equilibrium prices are those prices that make investors desire to hold in their portfolios exactly the supply existing at that time. In flow terms, these are the prices that equate the number of second-hand securities offered for sale over some period with the demand to purchase securities, where the effect of the prices in inducing changes in the rate of saving or real investment is ignored. The equilibrium prices, then, are entirely demand determined; i.e. they represent the outcome of the portfolio choice decision only.

Each investor will calculate his optimum portfolio, given his wealth, his utility function, and the joint probability distribution he assigns to the rates of return on all assets. If \( X^i \) is the \( i \)-th investor's optimum portfolio, then \( \sum_i X^i \) is the aggregate demand for financial assets. Equilibrium in the capital market requires that this should be equal to aggregate supply, which may be represented as

\[
S = \begin{pmatrix}
S_1 \\
\vdots \\
S_n
\end{pmatrix}
\]

where \( S_j \) is the market value of all the financial assets in class \( j \), and \( \sum_{j=1}^{n} S_j + S_o = \sum_i w^i \). Both \( S_o \) and \( w^i \) are assumed to be net of brokers' loans. Since the number of assets in each class is fixed, equilibrium is achieved by adjusting the price of each asset;
if desired, a more conventional set of supply and demand curves may be derived simply by dividing both supply and demand by the price of each asset. Note that, while we have \((n + 1)\) asset classes, there are only at most \(n\) independent equations, since

\[
\sum_i x_i^* = \sum_j (w_j - \sum_k x_k^*) = \sum_j s_j + s_0 - \sum_j \sum_l x_l^j
\]

(1) \[= s_0 + \sum_j (s_j - \sum_l x_l^j) = s_0\]

This means that not all the \(s_j\) can be determined within the model. Before an equilibrium set of prices may be determined, we need an "anchor"; the price and rate of return on one class must be determined outside the model. This is the riskless asset, whose rate of return is determined by comparing it with alternative stores of value such as tangible assets (real capital or commodity wealth) or money, whose rate of return is fixed at zero, but which has the advantage of avoiding financial transactions costs. (See Baumol [4].) The model could be completed by including an extra set of equations relating the level of interest rates to real investment and hence commodity price levels, both of which will feed back into \(r_0\), the riskless rate.

Another source of indeterminacy in the model stems from the fact that the investor's optimum portfolio may not be uniquely determined. We may easily give the investor some ad hoc rule for choosing his portfolio—for example, if he is indifferent between two asset classes, he
might hold equal amounts from each—but such a rule would not necessarily lead to fixed equilibrium asset prices. If every investor considered the same two classes perfect substitutes and applied the rule just suggested, there would only be equilibrium in the capital market in the special case in which the two assets were supplied in equal quantities. If this were not the case, then investors faced with equal prices and rates of return would first pick equal quantities of the two assets and then, as the price of the more abundant one begins to fall, attempt to shift completely into this as it is now preferred to the other, thus tending to re-equate prices and rates of return. Strictly speaking, in these circumstances, no fixed equilibrium prices exist; however, in the real world, where transactions costs are important, it is impossible for one to readjust his portfolio to take advantage of an infinitesimal change in price, and so this theoretical source of indeterminacy is probably of little significance.

We may now consider the stability of the market. Let us write

\[ r_k = \frac{Y_k}{S_k} \]

where \( Y_k \) is all the income (dividends, interest payments and capital appreciation) that accrues to the holders of the securities belonging to class \( k \), and let us assume for the moment that the distribution of \( Y_k \) is unaffected by changes in \( S_k \). The very simplest case to analyze is where \( Y_k = aY_j \), \( j \neq k \), and where all investors, although they may disagree as to the exact distribution of the \( Y \)'s, at least agree that they have distributions that differ only by a scale
factor, \( a \), and that the two rates of return must therefore be perfectly correlated. It has already been shown in the previous chapter that investors in this case will always prefer class \( k \) to class \( j \) assets if and only if \( r_k > r_j \) when both are positive; therefore, for capital market equilibrium, we require that \( S_k = aS_j \). A rise in the price of class \( j \) securities will make \( S_k < aS_j \) and \( r_k > r_j \), and so investors will shift from class \( j \) into class \( k \) securities, thus re-equilibrating the market. The opposite is true for a fall in the price of class \( j \) securities; the market is then clearly seen to be stable. One critical assumption is that \( r_o \) and therefore \( \tilde{r} \) be greater than zero. If \( \tilde{r} = 0 \), there will be no determinate equilibrium.

The optimality condition for \( x_k^* > 0 \) becomes simply \( E[U_{t_k}] = 0 \), and so, if \( r_j \) is any multiple of \( r_k \), the investor will still be indifferent between the two classes, and so the price of the stock does not influence his decision. Furthermore, if \( \tilde{r} < 0 \), the market is unstable, since a rise \( S_j \) will actually increase the demand for class \( j \) securities. (See Chapter I, Section V)

For the more general case, in which there do not exist other securities with perfectly correlated rates of return, but we still assume \( Y_k \) to be unaffected by changes in \( S_k \), the market may again be proven stable. Let us assume that, after the investor has selected his optimum portfolio, the price of class \( k \) assets changes, so that the market
value of class \( k \) assets becomes \( \hat{S}_k \), and the new rate of return

\[
\hat{r}_k = \frac{\hat{y}_k}{\hat{S}_k} = r_k \frac{S_k}{\hat{S}_k}.
\]

With the change in capital value, the net worth of the investor will also change to \( \hat{w} = w + \frac{x_k^*}{S_k} (\hat{S}_k - S_k) \), and at the same time, the rate of return he receives on his new portfolio will have changed in the opposite direction, since his prospective dollar return has, by assumption, the same distribution as before. We may then obtain the expression for \( \frac{d}{dx_k} E[U(I)] \), assuming \( x_k^* > 0 \), and that he does not adjust his portfolio after price change, so that

\[
\hat{x}_k = x_k \frac{\hat{S}_k}{S_k}.
\]

(2) \[ E[U(I) (\hat{r}_k - \tilde{r})] = E[U(I) (r_k \frac{S_k}{\hat{S}_k} - \tilde{r})] \]

\[
= \frac{S_k}{\hat{S}_k} E[U_I r_k] - E[U_I] \tilde{r}
\]

\[
= \left( \frac{S_k}{\hat{S}_k} - 1 \right) E[U_I] \tilde{r}
\]

(3) Therefore, if \( \hat{S}_k > S_k \), \( \frac{d}{dx_k} E[U(I)] < 0 \), and the investor will
reduce his holding of class k securities to increase his utility; if he already holds no class k assets, he will continue to hold none. In the case that \( S_k < S_k' \), the investor will try to increase his holdings; if he previously held no class k stock, he may be induced to purchase some if the fall in the stock price is sufficient. In either case, the stock price must return to its former level before investors can have utility maximizing portfolios that are consistent with equilibrium in the capital market. Note that, as before, stability depends crucially upon \( \bar{r} \) being greater than zero, or the sign of (3) will be reversed and a price increase will give rise to increased demand. If \( \bar{r} = 0 \), then any price level is consistent with equilibrium in the capital market.

The conditions for stability for the case in which the distribution of \( Y_k \) is shifted by a change in \( S_k \) are somewhat more complicated. Since this shift is subjective, we must consider the conditions under which a single investor will act in a stabilizing manner; a sufficient, though not necessary condition for market stability is then that all investors should act in this way. For simplicity, let us consider a market consisting of only one risky and one riskless asset, and assume that the distribution of \( Y_1 \) is shifted by a multiplicative factor, so that

\[
\frac{1}{Y_1} \frac{dY_1}{dS_1} \text{ is constant for all } Y_1, \text{ and consequently } \frac{dr_1}{dS_1} = \frac{r_1}{\bar{r}_1} \frac{d\bar{r}_1}{dS_1}.
\]

The distribution of I will also shift, as the investor adjusts his
holdings of all assets classes in response to the change in $S_k$ as well as the change in his wealth. The rate of change of his wealth is equal to \(-\frac{dw}{ds_1} = \frac{x_1^*}{s_1^*}\); therefore:

\[
\frac{dl}{ds_1} = r_o \frac{dw}{ds_1} + \frac{dx_1^*}{ds_1} (r_1 - r_o) + x_1^* \frac{dr_1}{ds_1}
\]

\[
\frac{dx_1^*}{ds_1} (r_1 - r_o) + x_1^* \left( \frac{dr_1}{ds_1} + \frac{r_o}{s_1^*} \right)
\]

\(4\)

Therefore, assuming that $x_o^*$ and $x_1^*$ are both positive and that $U_{II}$ is everywhere defined, we may differentiate the portfolio optimization condition:

\[
0 = E[U_1] \frac{x_1^*}{r_1} \frac{dr_1}{ds_1} + U_{II} (r_1 - r_o) \left( \frac{dx_1^*}{ds_1} (r_1 - r_o) + x_1^* \left( \frac{dr_1}{ds_1} + \frac{r_o}{s_1^*} \right) \right)
\]

\(5\)

\[
E[U_1] \frac{x_1^*}{r_1} \frac{dr_1}{ds_1} + \frac{dx_1^*}{ds_1} E[U_{II}(r_1 - r_o)^2] + E[U_{II}(r_1 - r_o)x_1^* \left( \frac{dr_1}{ds_1} + \frac{r_o}{s_1^*} \right)]
\]

Writing $\eta_r(r_k, s_k)$ and $\eta(x_k^*, s_k)$ for the elasticities of $r_k$ and $x_k^*$ respectively with respect to $s_k$, we can solve for $(x_1^*, s_1)$ as follows:
\[
E[U_I] n(\tilde{r}_1, S_1) \frac{r_0}{\sum_{i=1}^k r_i} + E[U_{II}(r_1 - r_0)(r_1 n(\tilde{r}_1, S_1) + r_0)] \\
\eta(x_1^*, S_1) = \frac{-E[U_{II}(r_1 - r_0)^2]}{-E[U_{II}(r_1 - r_0)^2]}
\]

In order to explore this further, note that

\[
\frac{dI}{dw} = r_o + \frac{dx_1^*}{dw} (r_1 - r_o)
\]

Therefore, we may differentiate the portfolio optimization condition with respect to \( w \), to obtain:

\[
0 = E[U_{II}(r_1 - r_0) \left( r_o + \frac{dx_1^*}{dw} (r_1 - r_o) \right)]
\]

\[
= \frac{dx_1^*}{dx} E[U_{II}(r_1 - r_0)^2] + r_o E[U_{II}(r_1 - r_0)]
\]

\[
\Rightarrow \frac{dx_1^*}{dw} = - \frac{r_o E[U_{II}(r_1 - r_0)]}{E[U_{II}(r_1 - r_0)^2]}
\]

\[
\text{and } \frac{dx_1^*}{dw} = \frac{E[U_{II}(r_1 - r_0) r_1]}{E[U_{II}(r_1 - r_0)^2]}
\]

Substituting back into (6):

\[
\eta(x_1^*, S_1) = \eta(\tilde{r}_1, S_1) \left( \frac{r_0 E[U_I]}{-x_1^* E[U_{II}(r_1 - r_0)^2]} - 1 \right) + \frac{dx_1^*}{dw} (1 + \eta(\tilde{r}_1, S_1))
\]
We may place a significant interpretation upon the two terms of equation (11) as follows. Divide \( \frac{dx^*_1}{ds^*_1} \) into two parts, the first being a substitution effect in which the investor's asset holdings are compensated in such a way that his income is unaffected, and the second being the income effect from removing the compensation. The rate of change in income resulting from changes in \( S_1 \) is:

\[
(12) \quad \frac{dI}{dS_1} = x^*_1 \frac{dr^*_1}{dS_1} + \frac{x^*_1}{S_1} = \frac{x^*_1 r^*_1}{S_1} (1 + \eta(\tilde{r}^*_1, S_1))
\]

Therefore, in order to keep the distribution of \( I \) constant, it is necessary to compensate the investor by changing his holdings of class 1 assets at the rate \(-\frac{x^*_1}{S_1} (1 + \eta(\tilde{r}^*_1, S_1))\). When this compensation is removed, there will be an income effect on his demand for class 1 assets equal to \( \frac{dx^*_1}{dw} \frac{x^*_1}{S_1} (1 + \eta(\tilde{r}^*_1, S_1)) \). The net effect on his demand for class 1 assets is then:

\[
(13) \quad \frac{dx^*_1}{dS_1} = \frac{\partial x^*_1}{\partial S_1}|_{\text{income compensated}} + \frac{dx^*_1}{dw} \frac{x^*_1}{S_1} (1 + \eta(\tilde{r}^*_1, S_1))
\]

By comparing (13) with (11), we see that the two terms in (11)
are simply the substitution and income effects respectively, each multi-
mplied by $\frac{S_1}{x_1^*}$.

The condition for the investor to act in a stabilizing way is that
$\eta(x_1^*, S_1)$ should be less than one; i.e., his demand for the security
must rise (fall) by a smaller percentage than the rise (fall) in its
price. If this elasticity is negative, then stability is assured, since
the rise (fall) in price is accompanied by a fall (rise) in demand. The
cases already considered assumed that $Y_k$ was unaffected by the change in
price, and therefore $\eta(\tilde{r}_1, S_1) = -1$. In this case, (11) becomes:

\begin{equation}
\eta(x_1^*, S_1) = \frac{r_0 E[U_{1_1}]}{x_1^* E[U_{II}(r_1 - r_0)^2]} + 1
\end{equation}

which must be less than one as long as $r_0 > 0$; therefore the market is
stable if such expectations predominate.

Another interesting case is that in which the shift in the distrib-
ution of $Y_1$ exactly matches the change in the share price, so that
$\eta(\tilde{r}_1, S_1) = 0$.

\begin{equation}
\eta(x_1^*, S_1) = \frac{dx_1^*}{d\omega}
\end{equation}

This result is intuitively acceptable, since the change in price
changes the investor's wealth without affecting the opportunities for
investment in any way. The condition for stability is reduced to the condition that an increase in wealth must be at least partly used to increase holdings of other assets besides those in class 1.

An interesting, though somewhat puzzling, implication of (11) is that, in order that \( \eta(x_1^*, S_1) \) should be an increasing function of \( \eta(r_1, S_1) \), it must be the case that:

\[
(16) \quad \frac{r_o}{x_1} E[U_I] + E[U_{II}(r_1 - r_o)r_1] > 0
\]

It is usually presumed that highly volatile expectations lead to instability, and so (16) would have to hold; however, there does not appear to be any a priori reason why this should be the case. Note that it would be violated if \( r_o = 0 \).

As long as (16) holds, then we may note that \( \eta(r_1, S_1) < 0 \) and \( \frac{dx_1^*}{dw} < 1 \) are together a sufficient condition for the investor to behave in a stabilizing manner; if \( \eta(r_1, S_1) \) is positive, then \( \frac{dx_1^*}{dw} \) will have to fall short of one by a greater margin to insure stability.

It is possible, of course, that expectations will change in some way other than the simple multiplicative shift just considered; in such a case, the implications for stability or otherwise will depend on the precise way in which the probability distribution of \( Y_k \).
responds to a change in $S_k$.

II The Valuation of Uncertain Income Streams and the Cost of Capital

Without making stronger assumptions, it is not possible to say much more about the precise way in which the capital market will evaluate an uncertain income stream. However, by using a two parameter approach and assuming that all investors share the same expectations, a more explicit statement may be made. The two parameter approach to portfolio selection, used by Markowitz [23] and Tobin [38], may be justified on either of two grounds; either the investor's utility function may be assumed to be quadratic, or the joint probability distribution of returns must be multivariate normal. Neither assumption is particularly palatable; the first carries the implications that risky assets are inferior goods, so that an increase in wealth will actually reduce demand for them, and that the marginal utility of income eventually becomes negative, while the second possibility has been tested by Breen and Savage in [7] and the normality hypothesis rejected in 6 years out of the 16 tested, at a significance level of 1%. (The lognormality hypothesis fares somewhat better.) However, for the sake of the interesting implications that follow, the assumption of normality will be made, since it appears somewhat less objectionable than the other, and a new utility function may be constructed of the form $W(\bar{I}, \sigma_1)$, with continuous first derivatives, where $\bar{I}$ is the expected value, and $\sigma_1$ the standard deviation of the
dollar return on the portfolio. Following directly from the properties of $U(I)$, this function will be concave in its arguments, as proven by Tobin in [38], and $W_I > 0$, $W_o < 0$. It may also be quickly shown that $W(I, \sigma_I)$ is concave in $X$. First note that, from the definition of $I$, we have:

\[(17) \quad \bar{I} = w r_o + (\bar{R} - r_o V)'X\]

\[(18) \quad \sigma_I = (X'ZX)^{1/2}\]

where $Z$ is the variance-covariance matrix of rates of return. (17) is clearly a linear function, and (18) is convex, since its Hessian, which is

\[\frac{d^2\sigma_I}{dx^2} = \frac{1}{\sigma} \bar{Z}

must be non-negative definite in order to ensure that the variance is not negative. Let $\hat{X}$ and $\hat{X}$ be two arbitrary $n$-dimensional vectors, and let $\bar{I}$, $\sigma_I$ and $\bar{I}$, $\sigma_I$ be the corresponding expected values and standard deviations of $I$. Then, from the definitions of linearity, convexity and concavity, it follows that:

\[(19) \quad \lambda \bar{I} + (1 - \lambda) \bar{I} = w r_o + (\bar{R} - r_o V)' (\lambda \hat{X} + (1 - \lambda) \hat{X})\]

\[(20) \quad \lambda \sigma_I + (1 - \lambda) \sigma_I \geq (\lambda \hat{X} + (1 - \lambda) \hat{X})' Z (\lambda \hat{X} + (1 - \lambda) \hat{X}) \]
\[ (21) \quad \lambda \hat{W}(\hat{I}, \hat{\sigma}_I) + (1 - \lambda) \hat{W}(\hat{I}, \hat{\sigma}_I) \leq \hat{W}(\lambda \hat{I} + (1 - \lambda) \hat{I}, \lambda \hat{\sigma}_I + (1 - \lambda) \hat{\sigma}_I) \]

Since \( W \) is monotone decreasing in \( \sigma_I \), the expressions on the right hand sides of (19) and (20) may be substituted for the arguments of the function on the right hand side of (21) without disturbing the inequality, thus proving \( W \) to be concave in \( X \).

The utility maximizing portfolio may be obtained by maximizing \( W \) with respect to \( X \). Let us continue to impose non-negativity constraints on holdings of all \( n \) risky asset classes, but allow unlimited borrowing at the riskless interest rate. (The reason for this will appear later.) The conditions for a maximum in the non-negative orthant are then:

\[ (22) \quad W_{I}(\bar{R} - r_o V) + W_{o}ZX \frac{1}{\sigma_I} \leq 0 \]

Letting \( b = -\frac{W_{o}}{W_{I}} \) and \( C = X \frac{b}{\sigma} \) and rearranging terms:

\[ (23) \quad ZC \geq \bar{R} - r_o V \]

Assuming that \( Z^{-1} \) exists (i.e., the return on each asset class has at least one source of variation that is unique to that class), we may write:

\[ (24) \quad C \geq Z^{-1}(\bar{R} - r_o V) \]
If the \( j \)-th component of the vector on the right hand side is non-negative, then \( c_j \) takes on that value, while if it is negative, \( c_j \) becomes zero. Note that the vector \( C \) is only a function of the parameters \( r_0, Z \) and \( \bar{R} \), and not the investor’s utility function. Furthermore:

\[ (25) \quad \sigma_i = (X'ZX)^{1/2} = \frac{\sigma_i}{b} (C'ZC)^{1/2} \]

\[ (26) \quad \therefore b = (C'ZC)^{1/2} \]

This shows that the marginal rate of substitution between risk and expected return in the optimal portfolio is also a function only of \( r_0, Z \) and \( \bar{R} \). We may very simply demonstrate the so-called "separation theorem" at this point. The theorem states that the relative proportions of assets within the optimum portfolio of risky assets (excluding holdings of the riskless asset) are independent of the investor’s utility function.\(^3\) To obtain these proportions, \( X^0 \), divide by the total amount invested in risky assets:

\[ (27) \quad X^0 = X \cdot \frac{1}{X'V} = C \cdot \frac{\sigma_i}{b} \cdot \frac{1}{\sigma_i} \cdot \frac{1}{C'V} = C \cdot \frac{1}{C'V} \]

which, as previously stated, is dependent only upon \( r_0, Z \) and \( \bar{R} \). Note

\[^3\]This is not to say that the relative proportions of risky assets is uniquely determined, as Tobin erroneously asserts, since different types of risky assets may have been combined into a single class in order to ensure that \( Z^{-1} \) exists. The choice between such classes is essentially arbitrary.
that without the assumption of unlimited borrowing at the riskless interest rate, we should have to replace $r_o$ in equations (22), (23) and (24) by $\tilde{r}$, which is not independent of the investor's utility function, and the separation theorem would not be valid.

To derive the optimum portfolio from $X^O$, it is sufficient to find a value of $x_o$ such that, when combined with the optimum risky portfolio, the marginal rate of substitution between risk and expected return is equal to the value of $b$ obtained in (26).

If we now assume that all investors share the same expectations, then it follows that they will all demand risky assets in the same relative proportions, (although they may "water down" the risky portfolio with more or less of the riskless asset depending on their utility functions), and so a condition for capital market equilibrium is:

$$X^O = S \frac{1}{S'V}$$

i.e. risky assets must be supplied in the same relative proportions in which they are demanded.

Let us rewrite the portfolio optimality condition as follows:

$$\underline{R} = r_o V + \frac{b}{(X'ZX)^{1/2}} ZX$$

All investors must hold risky assets from all classes in order to clear the capital market, and so the inequality has been dropped. We
may now combine the conditions for capital market equilibrium and portfolio optimization. Combining (27) and (28), we have

\[ X = S \frac{X'V}{S'V} \text{ substituting this into (29):} \]

\[ R = r_o V + \frac{b}{(S'ZS)^{1/2}} S'V ZS \frac{X'V}{S'V} \]

\[ = r_o V + \frac{b}{(S'ZS)^{1/2}} ZS \]

(30)  

Isolating the k-th equation, and using \( r_k = \frac{Y_k}{S_k} \):

\[ r_k = r_o + \frac{bE_i Cov [r_i, r_k]}{(\Sigma E S_i S_j Cov [r_i, r_j])^{1/2}} \]

\[ \frac{\bar{V}_k}{S_k} = r_o + \frac{b \Sigma \frac{1}{S_k} Cov [Y_i, Y_k]}{(\Sigma Cov [Y_i, Y_j])^{1/2}} \]

(31)

\[ \frac{\Sigma}{S_k} = (\bar{V}_k - \frac{b}{\sigma_Y} \Sigma_{i=1}^n Cov [Y_i, Y_k]) \frac{1}{r_o} \]

(32)  

where \( \sigma_Y \) is the standard deviation of \( \Sigma_{i=1}^n Y_i \). This equation then shows how the market will evaluate uncertain income streams under the assumptions stated, imposing the requirements that investors are all optimizing and the capital market is exactly cleared. The only unknown is the
parameter b, which may be called the market price of risk, following Lintner in [20], and which depends on individual investors' wealth and utility functions.

Equation (32) divides conveniently into two parts, the first being the discounted value of the expected dollar return, capitalized at the riskless interest rate, and the second being a deduction for risk. We may use it to prove several interesting propositions. First, if two firms merge so that the earnings\(^4\) of the new firm is equal to the sum of two original firms' earnings (i.e., if they fail to achieve any economies from merging), the value of the new firm's equity is equal to the sum of the equity values of the original firms. If the merging firms are numbered 1 and 2, we may write the equity value of the new firm:

\[
S_{1+2} = \left( \bar{Y}_1 + \bar{Y}_2 \right) - b \frac{n}{\sigma} \sum_{i=3}^{n} \text{Cov} \left( (Y_1 + Y_2), Y_i \right) - \frac{b}{\sigma} \text{Cov} \left( (Y_1 + Y_2), (Y_1 + Y_2) \right) \frac{1}{r_0}
\]

But \(\text{Cov} \left( (Y_1 + Y_2), (Y_1 + Y_2) \right) = \sum_{i=1}^{2} \text{Cov} \left[ Y_1, Y_i \right] + \sum_{i=1}^{2} \text{Cov} \left[ Y_2, Y_i \right]\)

\(^4\)I am using the term "earnings" here as synonymous with "the total income accruing to the corporation's stockholders." It should be remembered that this includes capital gains as well as dividends, and so I am using the term in a somewhat special sense.
\[ S_{1+2} = (\bar{Y}_1 - \frac{b}{\sigma_Y} \sum_{i=1}^{n} \text{Cov} [Y_{1i}, Y_{1i}]) \frac{1}{r_0} + (\bar{Y}_2 - \frac{b}{\sigma_Y} \sum_{i=1}^{n} \text{Cov}[Y_{2i}, Y_{1i}]) \frac{1}{r_0} \]

\[ = S_1 + S_2 \]

(34)

The argument applies equally in reverse; that is, if a firm splits into two firms without changing the probability distribution of its total earnings, the total value of its equity will remain the same.

Second, if the earnings of one firm are a scalar multiple of another firm's earnings, the equity value of the first will be the same scalar multiple of the value of the second. Let \( Y_1 = aY_2 \). Then:

\[ S_2 = (\bar{Y}_2 - \frac{b}{\sigma_Y} \sum_{i=1}^{n} \text{Cov} [Y_{2i}, Y_{1i}]) \frac{1}{r_0} \]

\[ S_1 = (a\bar{Y}_2 - \frac{b}{\sigma_Y} \sum_{i=1}^{n} a \text{Cov} [Y_{2i}, Y_{1i}]) \frac{1}{r_0} \]

(35) \[ = aS_2 \]

Third, if a firm increases its earnings by a scalar multiple, so that \( \hat{Y}_k = aY_k \), where \( \hat{Y}_k \) represents the new earnings and \( a > 1 \), the equity value of the firm will increase by less than the same scalar multiple. Let us first assume that the market is large enough so that the effect of the increase on the market parameters \( b \) and \( \sigma_Y \) is negligible. Then the new equity value is:
\[ S_k = (a \bar{Y}_k - \frac{b}{\sigma_Y} \sum_{i \neq k} \text{Cov} [Y_k, Y_i] - \frac{b}{\sigma_Y} a^2 \text{Cov} [Y_k, Y_k]) \frac{1}{r_o} \]

(36) \hspace{1cm} = aS_k - \frac{ba}{\sigma_Y r_o} \text{Cov} [Y_k, Y_k] (a - 1)

The reason is that, while all the covariances with returns on other stocks increase by a factor of \( a \), the variance of the return on the \( k \)-th stock increases by a factor of \( a^2 \), thus increasing the risk deduction by more than a factor of \( a \). If \( a < 1 \), then the equity value will be reduced by a smaller fraction than the reduction in earnings.

The significance of this effect clearly depends on the relative importance of the variance term in the total risk deduction as well as the importance of the total risk deduction in the equity value of the firm. If the \( k \)-th firm's earnings are highly correlated with the earnings of other firms, the variance term will be swamped by the others, and, to all intents and purposes, the firm will double its equity value if it doubles its earnings. On the other hand, if the firm's earnings are risky and they are not closely correlated with those of other firms, this effect may be significant.

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5. The effect of the increase in supply of risky assets on \( b \) and \( \sigma_Y \) is likely to work in the opposite direction, but I find it difficult to believe that it could be of a significant order of magnitude.
Using (36), we may calculate the elasticity of demand for the firm's stock with respect to its price. Let \( q_k \) be the number of share certificates and let \( p_k \) be the price per share. Then, in order to keep the product homogenous, assume that the shift in earnings is exactly matched by an increase in the number of share certificates issued, so that \( \hat{q}_k = a q_k \), and the percentage change in quantity is \((a - 1)\). The probability distribution of earnings per share is therefore unchanged.

The new share price is \( \hat{p}_k = \frac{\hat{S}_k}{q_k} \) and so the percentage change in price is:

\[
\frac{\hat{p}_k - p_k}{p_k} = \frac{\frac{\hat{S}_k}{q_k}}{q_k} - 1
\]

\[
= \frac{\hat{S}_k}{S_k} \frac{1}{a} - 1
\]

(37)

Therefore the elasticity of demand is, using (36):

\[
\eta(q_k, S_k) = \frac{(a - 1)}{\frac{S_k}{S_k} \frac{1}{a} - 1}
\]

\[
= \frac{-S_k}{b \sigma_{Y^0} \text{Cov}[Y_k, Y_k]}
\]
It may be noted that this is also the reciprocal of the elasticity of the price/earning ratio, $\frac{S_k}{\bar{Y}_k}$, with respect to expected earnings, $\bar{Y}_k$. Since the price/earnings ratio for an unlevered firm is the reciprocal of the cost of capital, the elasticity of the cost of capital with respect to expected earnings, which may be written $\xi$, will be equal to $-\frac{1}{\eta(q_k^d, S_k)}$. If the variance of $Y_k$ has an insignificant effect on the value of the equity, then $\eta(q_k^d, S_k)$ will approach $-\infty$ and $\xi$ will approach zero, implying that the market for the firm’s stock is more or less perfect. However, for a firm whose earnings are uncorrelated with the earnings of other firms, so that the variance of its earnings represents a significant part of the risk deduction, any increase in scale is likely to be accompanied by an increase in the cost of capital, as the stock price must fall to induce investors to hold the additional stock. An obvious parallel may be made with the conventional microeconomic analysis of product market structures. The greater the covariance of a firm’s earnings with those of another firm, the greater the extent to which the shares of the two firms may be regarded as substitutes for each other, and the greater the interdependence between the demands for the shares of each. If there are several such firms with highly correlated earnings, then the market for their shares approaches perfection. On the other hand, the stock of a firm whose earnings are risky but poorly correlated with the earnings of other firms has no close substitutes;
in fact, it may be used very effectively to complement other stocks in the investor's portfolio in order to reduce risk. Therefore, because of the unique characteristics of the firm's earnings, the market for its stock is no longer perfect, and the firm is, in a sense, a monopolist in the capital market. Such a company is also likely to be a monopolist in the product market, as firms in the same industry will tend to share certain industry-wide sources of variations in earnings, stemming from such factors as technological change or shifts in tastes.
CHAPTER IV

OPTIMAL PORTFOLIO ADJUSTMENT TO CHANGES IN LEVERAGE

I Introduction

In this chapter, I am concerned with the way in which the value of the firm must adjust in order that a particular investor may reoptimize his portfolio in response to a change in leverage without contributing to either excess demand for or supply of the firm's stock. If all investors are identical, then this is the manner in which the value of the firm will adjust to its new equilibrium level. More generally, if different investors react in different ways, the new equilibrium value of the firm will be somewhere between the values required by the different investors, and there will be a redistribution of stock between investors' portfolios. If we then assume that the objective of the firm is the maximization of its market value, an optimum debt-equity ratio may be selected.

Before proceeding, let us examine the rationality of the firm's proposed objective in terms of the interests of its stockholders and management. First, let us assume the ultimate goal to be the maximization of the wealth of the stockholders, by adjusting financial leverage until the price of the stock is maximized. Assume also that the adjustment of leverage is achieved by the firm selling bonds and using the
proceeds to buy up its own stock or vice versa. Under what circumstances, then, will the maximization of market value coincide with the maximization of stock prices?

Consider the following situation in which the coincidence may readily be established. Let the firm be 100% equity financed to start with, and assume that the owners make a once and for all adjustment in leverage by selling a block of bonds at a single price, using the proceeds to repurchase some of their own stock. Let \( q_s \) and \( q_d \) be the number of shares and bonds in the hands of the public (excluding shares repurchased by the firm), with \( p_s \) and \( p_d \) as their respective prices, and let \( q_{so} \) be the original number of shares before leverage is applied. We may write the number of shares outstanding after leverage as:

\[
(1) \quad q_s = q_{so} - \frac{p_d}{p_s} q_d
\]

and so we may write \( V \), the value of the firm:

\[
V = p_s q_s + p_d q_d
\]

\[
(2) \quad = p_s q_{so}
\]

Since \( q_{so} \) is a constant (ignoring the possibility of stock splits, of course), maximizing \( V \) will be the same as maximizing \( p_s \) which will in turn maximize the value of each stockholder's wealth. Note that the
personal risk preferences of each current stockholder do not enter the calculation (except insofar as they influence the way in which the market as a whole discounts risk), and therefore there is no conflict of interest between investors. The debt-equity ratio that maximizes \( V \) will be optimal for everybody; if one investor has a high preference for security, he will modify his portfolio to compensate for the leverage, but his interests will still be best served by the market value maximizing degree of leverage.

Now consider the case in which the firm already has debt outstanding before the adjustment in leverage. Let \( q_{so}, q_{do} \) be the number of shares and bonds outstanding before the adjustment. We must replace equations (1) and (2) by:

\[
(3) \quad q_s = q_{so} - \frac{p_d}{p_s} (q_d - q_{do})
\]

\[\therefore \quad V = q_{so} - \frac{p_d}{p_s} (q_d - q_{do}) \quad p_s + q_dp_d \]

\[
(4) \quad = p_s q_{so} + p_d q_{do}
\]

\( q_{so} \) and \( q_{do} \) are still constant, but maximizing \( V \) will only coincide with maximizing \( p_s \) if either \( q_{do} = 0 \) (the case already considered) or \( p_d \) is constant. If \( p_d \) is a decreasing function of leverage, then the first order condition for maximizing \( V \), assuming the function to be
continuously differentiable, may be written:

\[
(5) \quad \frac{dV}{dq_d} = q_{so} \frac{dp_s}{dq_d} + q_{do} \frac{dp_d}{dq_d} = 0
\]

But if \( \frac{dp_d}{dq_d} < 0 \) and \( q_{do} > 0 \), then it must be the case that \( \frac{dp_s}{dq_d} > 0 \);

i.e. a further increase in leverage, although it would reduce \( V \), would increase \( p_s \) and therefore would be in the stockholders' interests. The reason for this is that the original bondholders who hold the \( q_{do} \) bonds issued previously stand to lose from an increase in leverage, which would more than offset the gain to stockholders after the point defined in (5). The stockholders, however, would be acting in a rather underhand way by increasing leverage at all, since they sold the original block of bonds to people who expected the degree of leverage and therefore the price of the bonds to remain constant; if they had expected the further increase in leverage, of course, they would have paid a lower price for the bonds in the first place, and the subsequent change in leverage would not have affected \( p_d \) at all. The most profitable, as well as the most underhand course of action for the stockholders to take might be to "skim" the market, selling a few bonds at a time to bondholders who would later suffer unexpected capital losses as more bonds were sold. This strategy, however, can only work if unanticipated by buyers of early bond issues, and, even if it does work, it is likely to endanger further fund-raising for legitimate expansion later. Since it
is probably unrealistic to suppose that it is in the stockholders' best 
long run interest to dupe their creditors, we may safely assume, I think, 
that they will fully inform prospective bond purchasers of the eventual 
target debt-equity ratio, so that $p_d$ will immediately attain its appro-
 priate long run level. In this case, $p_d$ ceases to be a function of the 
actual leverage at any moment in time before the target debt-equity ratio 
is reached, and it follows as before that maximizing $V$ is identical to 
maximizing $p_s$. In the rest of this thesis, then, I shall assume that 
the firm is initially unlevered, and that a single issue of debt is con-
sidered, so that these problems do not arise. If the firm later wishes 
to expand, and this expansion has the effect of shifting the distribution 
of profits by a multiplicative factor, then this may be optimally fin-
anced by issuing debt and equity in the same proportions as the currently 
existing debt-equity ratio, without penalty to bondholders. The case in 
which the expansion affects the distribution of profits in some other way 
than this may be treated by assuming that the nature of the expansion is 
made known to the original bondholders from the beginning, so that they 
do not receive unexpected capital gains or losses when the expansion 
takes place, and market value maximization is again appropriate.

It may be noted that the debt-equity ratio that maximizes the value 
of the firm is not necessarily the same one that minimizes the average 
cost of capital if we allow the distribution of profits to be affected 
by leverage.
It is possible to criticize the assumption that the firm is attempting to maximize its stockholders' present wealth for a number of reasons. Expectational effects might well lead the board of directors to act otherwise. For example, if they anticipate a change in the relation between the rate of return on bonds and that on stock, they might take advantage of this perception by changing the leverage. If they expect bond prices to fall but stock prices to remain the same, they might issue a large amount of debt now, planning to redeem it later at the lower price, and thereby to give stockholders a capital gain in a later period. Similarly, if they think the market has underestimated the company's growth prospects, they might rely heavily on debt finance so that, when the market later revalues the firm, the stockholders will receive a greater windfall gain.

Other considerations may also affect the financial structure decision. A closely held company may rely heavily on debt finance in order to prevent diffusion of control, under the assumption, perhaps, that the principal stockholder will be compensated for a low \( p_s \) now by superior profits later, resulting from his superior management. Conversely, a board of directors might prefer equity finance for the purpose of making it more difficult for any individual or other company to take over the firm and possibly threaten their jobs. Not only does debt finance increase the danger of such a takeover, but it also increases the danger of bankruptcy. These last points will be considered again later, since they imply that high leverage is likely to be against the interests of
the directors, and that profits may be affected if they are to be com-
pensated for this.

After making these points, all of which are likely to have some
validity, I wish to reassert the assumption made originally, that the
objective of the firm is the maximization of the market value of its
securities. The reason for ignoring these other effects is that the
expectational effects are not likely to be systematic, and the other
effects, insofar as they are systematic, may be superimposed upon a
theoretical structure based on market value maximization.

II Optimal Leverage — The MM Case

I shall now consider various sets of assumptions under which the
portfolio adjustment may occur. The first and most restrictive case
assumes the following:

1. No investor's borrowing constraint is binding
2. The rate of return on corporate debt is equal to the riskless
   interest rate
3. There is no limited liability
4. Stock prices are unchanging over time
5. There are no corporate taxes
6. Operating profits are independent of leverage
7. There are no retained earnings

These assumptions will be relaxed one by one and the implications
vis-a-vis optimal leverage will be noted.

Let \( \pi_k \) be the operating profit earned by the \( k \)-th firm over the
coming period, for which the investor has a subjective probability dis-
tribution. Since stock prices are assumed to be unchanging over time, it
must also be assumed that this distribution is unresponsive to past
errors. If \( D_k \) is the market value of the firm's debt, assumed for con-
venience to be in the form of perpetual bonds, to avoid the possibility
of expected capital gains or losses with maturity, we may write the bond
interest charge \( C_k = D_k r_o \). With assumption 4 above, the entire return
to stockholders must be in the form of dividends rather than capital
gains; therefore \( Y_k = \pi_k - C_k \), and:

\[
(6) \quad r_k = \frac{\pi_k - C_k}{S_k} = \frac{\pi_k - r_o D_k - r_o S_k}{S_k} + r_o
\]

\[
(7) \quad r_k - r_o = \frac{1}{S_k} (\pi_k - r_o V_k)
\]

Let the value of the unlevered firm be \( V_{Uk} \) and its rate of return
on equity \( r_{Uk} \), which may be written:

\[
(8) \quad r_{Uk} = \frac{\pi_k}{V_{Uk}}
\]

Now let the \( k \)-th firm issue \( D_k \) of debt, so that the new values of
the firm and its stock that are required for the investor to reoptimize
without disequilibrating the market are \( \hat{V}_{Lk} \) and \( \hat{S}_k \) respectively, and the
rate of return on equity becomes \( \hat{r}_{Lk} \). We may therefore write the
difference between this required rate of return on equity and the risk-
less interest rate:

\[(9) \quad \hat{r}_{Lk} - r_o = \frac{1}{\hat{S}_k} (\pi_k - r_o \hat{V}_{Lk})\]

Combining (8) and (9) to eliminate \(\pi_k\):

\[(10) \quad \hat{r}_{Lk} - r_o = \frac{V_{Uk}}{\hat{S}_k} (r_{Uk} - r_o) - \frac{r_o}{\hat{S}_k} (\hat{V}_{Lk} - V_{Uk})\]

Let us now assume that the investor adopts the market-clearing portfolio; i.e. he reduces his holdings of the firm's stock by the same fraction that \(\hat{S}_k\) is reduced, so that \(\hat{x}_k = x_k \frac{\hat{S}_k}{V_{Uk}}\), and that this portfolio adjustment is accomplished by compensating changes in bond holdings. Therefore the investor's total investment income must be the same as before, since his income from the stock will be reduced by exactly the same amount as his bond income is increased. This may be proven formally as follows. Assume the following sequence of events. First, the firm announces its intention of issuing debt; second, the price of each share moves to its new level and the investor receives a capital gain or loss; third, the adjustment in leverage takes place. The change in the value of the stock in the second phase is equal to \(\hat{V}_{Lk} - V_{Uk}\); therefore the investor's capital gain or loss is equal to \(\frac{x_k}{V_{Uk}} (\hat{V}_{Lk} - V_{Uk})\), and:
(11) \[ \hat{w} = w + \frac{x_k^*}{\hat{V}_{Uk}} (\hat{V}_{Lk} - \hat{V}_{Uk}) \]

Therefore, using (10), his new investment income is:

\[ \hat{\gamma} = \hat{w}r_o + \sum_{j \neq k} x_j^* (r_j - r_o) + x_k^* (\hat{r}_{Lk} - r_o) \]

\[ = I + \frac{r_o x_k^*}{\hat{V}_{Uk}} (\hat{V}_{Lk} - \hat{V}_{Uk}) + x_k^* \frac{S_k}{\hat{V}_{Uk}} \left( \frac{V_{Uk}}{S_k} (r_{Uk} - r_o) \right) \]

\[ - \frac{r_o}{S_k} (\hat{V}_{Lk} - \hat{V}_{Uk}) - x_k^* (r_{Uk} - r_o) \]

(12) \[ = I \]

According to assumption 1 above, no investor's borrowing constraint is binding, so that \( \hat{r} = r_o \) for everyone. The expression for \( \frac{d}{dx_k} E[U(I)] \), given that the investor adjusts his portfolio in the manner suggested is:

\[ E[U_I(\hat{r}_{Lk} - r_o)] = E[U_I(r_{Uk} - r_o)] \frac{V_{Uk}}{S_k} + E[U_I] \frac{r_o}{S_k} (V_{Uk} - \hat{V}_{Lk}) \]

Since the original portfolio was optimal, the first term on the right hand side of (13) drops out, leaving:

\[ E[U_I^*(\hat{r}_{Lk} - r_o)] = E[U_I] \frac{r_o}{S_k} (V_{Uk} - \hat{V}_{Lk}) \]

(14)
The other \((n - 1)\) optimality conditions are completely unaffected by the leverage, since the other rates of return are unchanged and \(\hat{r}\) is equal to \(r\). Therefore, in order to ensure that the market-clearing portfolio is also optimal, it must be the case that \(V_{Lk} = V_{Uk}\). If \(V_{Lk} > V_{Uk}\), then \(\frac{d}{dx_k} E[U(I)] < 0\) and the investor will attempt to reduce his holdings of the \(k\)-th firm's stock. But since all investors will be doing the same, this attempt will simply cause a situation of excess supply on the market and the price will fall until \(V_{Lk} = V_{Uk}\). The opposite is true for \(V_{Lk} < V_{Uk}\); investors will try to increase their holding, causing excess demand and driving the price up until \(V_{Lk} = V_{Uk}\). Note that an essential condition for stable adjustment is that \(r_o > 0\).

I conclude, then, that under the assumptions stated, the total market value of the securities of a firm will be independent of the firm's financial structure, and the Modigliani Miller theorem will be valid.

III Optimal Leverage with Some Binding Borrowing Constraints

The case in which some investors' borrowing constraints are binding is a little different, since \(\tilde{r}\), the marginal cost of riskless return, is greater than the riskless interest rate. It cannot therefore be optimal for such an investor to adjust his portfolio in the manner just described since his optimal portfolio cannot contain bonds. Let us first assume that \(\tilde{r}\) represents the riskless rate of return that can be earned on some
other asset or asset combination. Now let him adjust his portfolio in such a way that he continues to hold the same fraction of the k-th firm's stock after the change in leverage as before and \( x_k^* \frac{\hat{S}_k}{V_{uk}} \), but let him use compensating changes in holdings of this other riskless asset or asset combination. For convenience and without loss of generality, let this asset be from class 1. Then, since he will hold no class zero assets, we may rewrite his original investment income:

\[
(15) \quad I = \tilde{r}w + \sum_{j=2}^{n} x_j^* (r_j - \tilde{r})
\]

We may write the excess of the required k-th rate of return after leverage over the marginal cost of riskless return:

\[
(16) \quad \hat{r}_{Lk} - \tilde{r} = \frac{\hat{\pi}_k - \hat{r}_D}{\hat{S}_k} - \tilde{r}
\]

Combining this with equation (8):

\[
(17) \quad \hat{r}_{Lk} - \tilde{r} = \frac{V_{uk}}{\hat{S}_k} (r_{uk} - \tilde{r}) + \frac{r_D}{\hat{S}_k} + \frac{\tilde{r}}{\hat{S}_k} (V_{uk} - \hat{S}_k)
\]

His new investment income is:
\[ \hat{I} = I + \frac{r^*}{V_{Uk}} (\hat{V}_{Lk} - V_{Uk}) + \frac{x_k^*}{S_k} \frac{V_{Uk}}{S_k} (\hat{r}_{Uk} - \tilde{r}) - \frac{D_k}{S_k} \]

\[ + \frac{\tilde{r}}{S_k} (V_{Uk} - S_k) - x_k^* (r_{Uk} - \tilde{r}) \]

(18) \[ = I + \frac{x_k D_k}{V_{Uk}} (\tilde{r} - r) \]

The investor then receives a net gain in income, since his stock income has been reduced by less than the increase in his income from class 1 assets. Let us tax away part of his holdings of class 1 assets, so that \( \hat{I} \) is once again equal to \( I \). We may then write the new expression for \( \frac{d}{dx_k} E[U(I)] \):

\[ E[U_I^*(\hat{r}_{Lk} - \tilde{r})] = E[U_I^*(r_{Uk} - \tilde{r})] \frac{V_{Uk}}{S_k} - E[U_I] \frac{D_k}{S_k} \frac{\tilde{r}}{S_k} (V_{Uk} - S_k) \]

(19) \[ = E[U_I] \left( \frac{\tilde{r}}{S_k} (V_{Uk} - \hat{V}_{Lk}) + \frac{\tilde{r} - r}{S_k} D_k \right) \]

This will only be consistent with portfolio optimization if:

(20) \[ \hat{V}_{Lk} = V_{Uk} + \frac{\tilde{r} - r}{\tilde{r}} D_k \]

As before, if \( V_{Lk} \) is higher than this, then the investor will reduce his
demand for the firm's stock; the reverse is true for $V_{L_k}$ being too low.

Now eliminate the tax on the investor's wealth. As long as the $k$-th firm’s stock is a normal rather than an inferior good, this effect will be to strengthen further the result just obtained. His wealth rises, which will further increase his demand for the $k$-th firm's stock, and raise $\hat{V}_{L_k}$ somewhat above the level given in (20). However, if the $k$-th firm's stock represents only a small part of the investor's portfolio, this wealth effect will be negligible, as it will be spread over all other securities in the portfolio, and so equation (20) will closely approximate the general condition for the investor to optimize his portfolio while adjusting it in such a way as to contribute to neither excess demand for nor excess supply of the $k$-th firm's stock. There will be other small second order effects, since the investor has increased his demand for class 1 assets while the supply of class 0 assets has increased; this may tend to bring $r_o$ and $\tilde{r}$ closer together, but will not obviously have a very significant effect on the relationship between $V_{L_k}$ and $V_{U_k}$.

The case in which $\tilde{r}$ does not correspond to a riskless rate of return that may be earned on some specific asset or asset combination is essentially the same. Assume that the leverage takes place, and that the investor reduces his stock holdings by the same proportion as the reduction in $S_k$. At the same time, let us assume that the margin requirement for brokers' loans is increased, so that all of the cash raised by
selling the k-th firm's stock is required to repay part of the investor's loan, and therefore has an implicit rate of return of \( r_o \). Therefore, the investor's new investment income is exactly the same as before, since his stock income has been reduced by exactly the same amount as the reduction in interest on his loan. If the investor is a financial institution, the same result may be obtained by increasing the proportion of reserves that must be held in the form of fixed interest securities.

Now that the investor's income is compensated so that \( \hat{I} = I \), the condition for portfolio optimization is once again equation (20), which shows that \( V_{Lk} \) must be greater than \( V_{Uk} \) in order that the investor should be content to hold \( x_k^* \frac{S_k}{V_{Uk}} \) of the firm's stock. The second order effect arises when the compensating change in the constraint is relaxed; investors will be free to increase their borrowing and thus to bid up the value of the firm even higher.

If there exist such investors, then, with binding borrowing constraints, there will be some tendency for \( V_{Lk} \) to rise above \( V_{Uk} \) as leverage is increased. If such investors manage a small proportion of the economy's wealth, their effect on \( V_{Lk} \) will be extremely small, since, as soon as it rises above \( V_{Uk} \), all investors without binding borrowing constraints will begin to reduce their holdings of the k-th firm's stock, thus preventing \( V_{Lk} \) from rising much above \( V_{Uk} \).
It may be argued, however, that there is enough wealth being managed by investors with binding constraints for the effect to be significant, since this constraint is always effective on certain financial intermediaries—for example, insurance companies and mutual savings banks that are licensed to do business in New York. It is likely then that, if these institutions determine their portfolios according to the model suggested in Chapter 2, the marginal cost of riskless return for them is considerably above the riskless interest rate, and so they would eagerly increase their relative holdings of the k-th firm's stock in the event of rise in leverage as long as \( V_{Lk} \) is less than the value indicated by equation (17). We would then expect to see an increase in leverage accompanied by a slight rise in \( V_k \), and a definite shift of the stock out of individual investors' portfolios and into the portfolios of financial institutions on whom constraints on minimum holdings of riskless assets are binding. As far as the firm is concerned, any increase in leverage is desirable, and optimal leverage is attained where it is impossible to increase the amount of debt any more.

**IV Optimal Leverage with Rising Cost of Corporate Debt**

Let us now assume that the cost of corporate debt is no longer always equal to the riskless interest rate, but instead is an increasing
function of the debt-equity ratio. MM contend that their theorem can be extended to this case; it may be shown, however, that according to the model presented here, this is not so. Furthermore, as shown in Section V, the assumptions underlying this case are in a sense inconsistent.

Let \( r_{dk} \) be the coupon rate of return on the debt of the \( k \)-th corporation, so that \( C_k = r_{dk} D_k \), and assume initially that no borrowing constraints are binding, so that \( \hat{r} = r_o \) for every investor. Then:

\[
(21) \quad r_k = \frac{\pi_k - r_{dk} D_k}{S_k} = r_o + \frac{1}{S_k} (\pi_k - r_o V_k) - (r_{dk} - r_o) \frac{D_k}{S_k}
\]

\[
\hat{r}_{Lk} - r_o = \frac{1}{S_k} (\pi_k - r_o \hat{V}_{Lk}) - \frac{D_k}{S_k} (r_{dk} - r_o)
\]

\[
(22) \quad \frac{V_{Uk}}{S_k} (r_{Uk} - r_o) - \frac{r_o}{S_k} (\hat{V}_{Lk} - V_{Uk}) - \frac{D_k}{S_k} (r_{dk} - r_o)
\]

If the investor adjusts his portfolio through his riskless asset holdings so as not to disturb market equilibrium for the firm's stock his new income is:
\[ \hat{I} = I + \frac{r^*_o}{\hat{V}_{Uk}} (\hat{V}_{Lk} - \hat{V}_{Uk}) + x^*_k \frac{S_k}{\hat{V}_{Uk}} \left( \frac{V_{Uk}}{S_k} (r_{Uk} - r_o) - \frac{r_o}{S_k} (\hat{V}_{Lk} - \hat{V}_{Uk}) \right) - \frac{r_o}{S_k} (\hat{V}_{Lk} - \hat{V}_{Uk}) - \frac{D_k}{r_o} (r_{dk} - r_o) \]

(23) \[ = I - \frac{D_k}{V_{Uk}} (r_{dk} - r_o) \]

Since his income has been reduced, let us again compensate him via his holdings of riskless assets in order to leave his income the same as before. The new expression for \( \frac{d}{dx_k} E[U(I)] \) may be written:

\[ E[U_I(r_{Lk} - r_o)] = E[U_I(r_{Uk} - r_o)] \frac{V_{Uk}}{S_k} + E[U_I] \left( \frac{r_o}{S_k} (\hat{V}_{Uk} - \hat{V}_{Lk}) - (r_{dk} - r_o) \frac{D_k}{S_k} \right) \]

(24) \[ = E[U_I] \frac{r_o}{S_k} \left( V_{Uk} - \hat{V}_{Lk} - \frac{r_{dk} - r_o}{r_o} D_k \right) \]

This will only be consistent with portfolio optimization if:

(25) \[ \hat{V}_{Lk} = V_{Uk} - \frac{r_{dk} - r_o}{r_o} D_k \]

i.e. \( \hat{V}_{Lk} \) must be less than \( V_{Uk} \) by the amount shown in (25). Furthermore,
if we then eliminate the subsidy, the investor will further reduce his demand for the \( k \)-th firm's stock, and \( \hat{V}_{Lk} \) will be depressed even further.

If we ignore this last effect on the grounds that it will be of a much smaller order of magnitude than the effect expressed in (25), then we may make a very strong statement indeed about the precise way in which the market will evaluate the firm at various degrees of leverage, since all investors will react in the same way:

\[
(26) \quad V_{Lk} = V_{Uk} - \left( \frac{r_{dk} - r_o}{r_o} \right) D_k
\]

The value of the firm will then decrease as leverage is applied as long as \( r_{dk} > r_o \); if \( r_{dk} \) is also an increasing function of leverage, then this effect will be further amplified. Optimal leverage in this case is zero; the value of the firm may be maximized by eliminating debt entirely and financing 100% with equity.

The effect of binding borrowing constraints will naturally be in the opposite direction. Ignoring all second order effects (which act only to reinforce the first order ones), the condition for an investor to optimize his portfolio without contributing to market disequilibrium becomes:

\[
(27) \quad \hat{V}_{Lk} = V_{Uk} - \frac{r_{dk} - \tilde{r}}{r} D_k
\]
We may then write the rate at which $V_k$ must change when leverage is adjusted in order to satisfy this condition for any one investor as:

\[
\frac{d\hat{V}_{Lk}}{dD_k} = -\frac{r_{dk} - r}{r} - \frac{D_k}{r} \frac{dr_{dk}}{dD_k}
\]

(28)

The second derivative is:

\[
\frac{d^2\hat{V}_{Lk}}{dD_k^2} = -\frac{2}{r} \frac{dr_{dk}}{dD_k} - \frac{D_k}{r} \frac{d^2r_{dk}}{dD_k^2}
\]

(29)

which will be negative as long as $r_{dk}$ is increasing at at least a constant rate. That is, the more leverage is increased, the more likely it is that each investor will act in such a way as to depress the value of the firm. When leverage is low, it is likely that equation (28) for a large financial institution acting under a binding constraint will be positive, since $\bar{r}$ may well be considerably above $r_{dk}$. It is possible then that the influence of many such institutions might be sufficient to reverse the effects of all investors whose constraints are not binding, and so the firm could increase its market value by increasing its leverage, with an accompanying transfer of stock out of the hands of individual investors and into the portfolios of the institutions. However, as leverage is increased, $r_{dk}$ will rise and $\bar{r}$ for the institutions may even tend to fall (since the cost of the constraint is reduced by
the acquisition of highly levered stocks—this effect will be small and has been ignored in (28) and (29), and so eventually a point will be reached when the value of the firm will begin to decline; just before this happens is the point of optimal leverage.

We may note how MM's symmetry proposition may be interpreted in this model. They suggest that, if the cost of borrowing is an increasing function of the investor's personal debt equity ratio, and if it increases in exactly the same way as the firm's cost of debt, then the value of the firm will still be independent of its financial structure. We may make a similar statement; if $r_{dk}$ and $r$ are always equal for a given investor, then he will react to a change in leverage in such a way as not to disturb the market for the firm's stock if and only if the value of the firm remains the same. However, if $r_{dk}$ is rising with leverage, this proposition has little significance. Most individual investors probably do not have binding borrowing constraints, and so the opportunity cost of riskless return is simply equal to the riskless interest rate, while institutions whose constraints are binding are likely to find $r$ constant or even decreasing as leverage is increased. In such a case, the symmetry proposition no longer becomes applicable.

Even in MM's world of arbitrage, it is difficult to see how it can work, since the cost of borrowing is a function of the investor's total debt-equity ratio, taking into consideration his total asset holdings rather than simply his holdings of one particular stock. If we reject MM's claim that their theorem holds under the assumption of
increasing bond rates, we are spared the pain of trying to account for the startling implication that the rate of return on shares will rise with increasing leverage only up to a point, beyond which further increases may actually cause the rate of return to decline. MM require share prices to rise to compensate for the falling bond prices in order to preserve the value of the firm. It would be difficult indeed to see how such a phenomenon could be incorporated into a competitive model of the capital market.

V Optimal Leverage With Limited Liability

The preceding case was mainly analyzed in order to show that MM's symmetry proposition is not very useful. A powerful objection, however, may be levelled at the internal consistency of the assumptions behind it. Why should the rate of return on bonds be an increasing function of leverage? Presumably because leverage increases the risk of the firm being unable to meet its obligations to bondholders. But as the risk of default changes, so does the probability distribution of return to the stockholders, and this aspect of the problem has so far been ignored. Limited liability is essentially a device designed to protect stockholders at the expense of bondholders by limiting the rate of return on equity to a minimum of -100%; up to now, however, no such constraint has been imposed.
A further constraint that has also been ignored is that dividends cannot be negative. However, this presents few problems, since there are several ways in which a deficit may be handled. This arises when profits over the period are insufficient to cover the debt service charge; it may be covered quite simply by paying a zero dividend and liquidating some of the firm's assets to pay for the debt service or possibly by raising a short term loan from another source. Such a procedure is a kind of partial dividend smoothing; the negative dividend will appear in the form of a capital loss to stockholders. As long as we can ignore the effects of different tax rates on capital gains and losses versus dividends, nothing is changed. But since the value of the stock cannot become negative, the greatest possible capital loss is one hundred percent of the stockholders' equity. We may then redefine $Y_k$ as:

$$
Y_k = \begin{cases} 
\pi_k - C_k & \text{for } \pi_k > C_k - S_k \\
- S_k & \text{for } \pi_k \leq C_k - S_k 
\end{cases}
$$

If the firm is forced into bankruptcy, it will be placed into receivership while the financial structure is reorganized, and then resold to new owners. The proceeds minus the negative profits will go to the bondholders as repayment of capital, plus return on capital. If we assume that there are no costs involved in going bankrupt, so that the firm will be worth as much after reorganization as before, we may write
$Y_{dk}$, the dollar return to bondholders.

\[
Y_{dk} = \begin{cases} 
C_k & \text{for } \pi_k > C_k - S_k \\
V_k + \pi_k - D_k = \pi_k + S_k & \text{for } \pi_k \leq C_k - S_k
\end{cases}
\]

Let us assume that the probability of bankruptcy is zero for an unlevered firm, so that $x_k^* > 0$ implies:

\[
E[U(t\frac{\pi_k}{V_{Uk}} - r)] = 0
\]

This may be solved for $V_{Uk}$:

\[
V_{Uk} = \frac{E[U(t\pi_k)]}{rE[U_t]}
\]

When the firm issues debt, let the investor reduce his holdings of the firm's stock by the proportional decrease in $S_k$, so that $x_k^* S_k \hat{x}_k = x_k^* \frac{S_k}{V_{Uk}}$, and at the same time, let him compensate by buying the firm's bonds. The amount he can buy is equal to the reduction in his stock holding, plus any capital appreciation; i.e. $x_k^* \hat{x}_k + \frac{x_k^* (V_{Lk} - V_{Uk})}{V_{Uk}} = x_k^* \frac{D_k}{V_{Uk}}$. The investor now holds the same fraction of the firm's outstanding stock as he holds of the firm's bonds, and so
his combined income from both will equal \( \frac{x_k^*}{U_k} \pi_k \), irrespective of whether bankruptcy occurs or not. This is exactly the same as before the issue of debt; therefore \( \hat{I} = I \).

Some new notation may now be introduced. Let us divide the expected value of a function of \( \pi_k \) into two parts. The first, written \( E_L[\quad] \), places a lower limit of \( C_k - S_k \) on the integration over \( \pi_k \), and the second, \( E^U[\quad] \), places an upper limit of \( C_k - S_k \) on the integration over \( \pi_k \), so that \( E_L[\quad] + E^U[\quad] = E[\quad] \). Therefore, the condition for the investor to hold stock in the new optimal portfolio is:

\[
0 = E[U_I\left(\frac{\tilde{Y}_k}{S_k} - \tilde{r}\right)] = E_L[U_I\left(\frac{\pi_k - C_k}{S_k} - \tilde{r}\right)] - E^U[U_I\left(1 + \tilde{r}\right)] \\
= E[U_I\left(\frac{\pi_k - C_k}{S_k} - \tilde{r}\right)] - E^U[U_I\left(\frac{\pi_k - C_k + \hat{S}_k}{S_k}\right)]
\]

(34)

If the investor is to hold some of the \( k \)-th firm's bonds:

\[
0 = E[U_I\left(\frac{\tilde{Y}_{dk}}{D_k} - \tilde{r}\right)] = E_L[U_I\left(\frac{C_k}{D_k} - \tilde{r}\right)] + E^U[U_I\left(\frac{\pi_k + \hat{S}_k}{D_k} - \tilde{r}\right)] \\
= E[U_I\left(\frac{C_k}{D_k} - \tilde{r}\right)] + E^U[U_I\left(\frac{\pi_k + \hat{S}_k - C_k}{D_k}\right)]
\]

(35)

Weighting (34) and (35) by \( \frac{S_k}{V_{Lk}} \) and \( \frac{D_k}{V_{Lk}} \) respectively and summing,
we see that a necessary condition for the investor to hold both stocks and bonds in his new optimal portfolio is:

\[(36)\quad 0 = \text{E}[\frac{\pi_k}{V_{Lk}} - \bar{r}]\]

This may be solved for \(\hat{V}_{Lk}\):

\[(37)\quad \hat{V}_{Lk} = \frac{\text{E}[U_{I} \pi_k]}{\bar{r} \text{E}[U_I]}\]

which is the same as the value for \(V_{uk}\) given in (33); therefore \(\hat{V}_{Lk} = V_{uk}\). An investor will only hold a market-clearing portfolio if the value of the firm is unaffected by the degree of leverage; however, the reverse implication is unfortunately not true. If two different original shareholders have different estimates of the probability of bankruptcy after leverage, it will not be optimal for them both to make market-clearing portfolio adjustments even if the value of the firm is unchanged, since they will disagree over the relative merits of the firm's stock and bonds. There will be some redistribution of the firms securities among portfolios, and the new equilibrium value of the firm will no longer necessarily equal the unlevered value. However, it appears reasonable to suppose that the value of the firm will not change much, since the original stockholders will probably have fairly similar estimates of the distribution of \(\pi_k\) or they would not all have bought
the stock. There will be a tendency for the effects of the redistribution among portfolios to cancel out, so limited liability per se is not likely to have any significant or systematic influence on the value of the firm.

The existence of constrained financial institutions modifies this conclusion, since the proposed portfolio adjustment will not be optimal for them if their constraints are binding. Consider one such institution that holds some of the firm's stock before leverage is applied. As the firm issues the debt, let us simultaneously tighten the constraint, to such an extent that the institution finds it necessary to increase its holdings of fixed interest securities by exactly

\[ x_k^* \frac{D_k}{V_{uk}} \]

so that it may adopt a market-clearing portfolio without necessarily violating the optimality conditions. As before, the distribution of return will be unchanged if the institution does this. As seen in Chapter II, Section IV, the effect of the constraint in a world where bonds are risky is to drive a wedge in between the marginal cost of riskless return in the optimality condition for holding stock and in the condition for holding bonds. If we assume for convenience that \( x_o^* > o \), and let \( r \) incorporate the marginal cost of the constraint, the conditions for the institution to hold both stock and bonds in its optimal portfolio are:
(38) \[ 0 = E[U_i\left(\frac{\pi_k - \hat{C}_k}{\hat{S}_k} - r\right)] - E[U_i\left(\frac{\pi_k - \hat{C}_k + \hat{S}_k}{\hat{S}_k}\right)] \]

(39) \[ 0 = E[U_i\left(\frac{\hat{C}_k}{D_k} - r_o\right)] + E[U_i\left(\frac{\pi_k + \hat{S}_k - C_k}{D_k}\right)] \]

Weighting (38) and (39) by \[ \frac{\hat{S}_k}{V_{Lk}} \] and \[ \frac{D_k}{V_{Lk}} \] respectively and summing and rearranging:

\[ \hat{V}_{Lk} = \frac{E[U_i\pi_k]}{rE[U_i]} + \frac{\hat{r} - r_o}{r} D_k \]

(40) \[ \hat{V}_{Lk} = V_{Uk} + \frac{\hat{r} - r_o}{r} D_k \]

There will then be a second order effect as we relax the constraint again and the institution shifts from fixed interest securities into common stock. This will have a negligible effect on the value of any single firm; in any case, it is not clear in what direction it would be, since it affects the value of the stocks and bonds in opposite directions. Again, this is a necessary and not sufficient condition for the market-clearing portfolio to be optimal; however, the existence of constrained institutions that control a significant proportion of the economy's wealth will certainly tend to impart an upward bias to \( V_{Lk} \) as leverage is increased. At the same time, there will be a redistribution of stock among portfolios, principally out of
those of private investors and into those of financial institutions.

It may be noted that (40) is identical to (20), which represents the necessary adjustment in $V_{Lk}$ when bonds are riskless but the investor's borrowing constraint is binding. Note that (40) no longer applies to private investors, since the introduction of limited liability means that bonds are no longer riskless, and therefore holding bonds is no longer equivalent to paying off part of the broker's loan.

I conclude, then, that, under present assumptions, limited liability and therefore the rising cost of corporate debt will not have a significant or systematic effect upon the value of the firm, since it means simply that a claim for $\tau_k - C_k + S_k$ is transferred from the stockholders to the bondholders in the event of bankruptcy. The introduction of constrained financial institutions, however, does introduce a definite systematic bias in favor of high debt-equity ratios.

VI Optimal Leverage With Uncertain Capital Values

A weakness of the analysis so far is that stock prices are assumed to be fixed, so that the entire return to the investor comes in the form of dividends (except in the special case in which the dividend would be negative, when it takes the form of a temporary capital loss). This may however be simply remedied. Ignoring limited liability for the moment, we may rewrite equation (7) as:
\[ \hat{r}_{Lk} - r_o = \frac{1}{S_k} (\pi_k - r_o \hat{V}_{lk} + \Delta V_{Lk}) \]

\[ = \frac{V_{uk}}{S_k} (r_{uk} - r_o) - \frac{r_o}{S_k} (\hat{V}_{Lk} - V_{uk}) + \frac{1}{S_k} (\Delta V_{Lk} - \Delta V_{uk}) \]

(41)

It follows immediately that if changes in the value of the firm are independent of leverage, the last term in (41) drops out and the conditions for optimal portfolio adjustment are unchanged.

More generally, it may be argued that proportional changes in the value of the firm should be independent of leverage, so that

\[ \hat{V}_{Lk} = \frac{V_{Lk}}{V_{uk}}. \]

Substituting this into (41) yields:

\[ \hat{r}_{Lk} - r_o = \frac{V_{uk}}{S_k} (r_{uk} - r_o) - \frac{r_o}{S_k} (\hat{V}_{Lk} - V_{uk}) + \frac{1}{S_k} \frac{\Delta V_{uk}}{V_{uk}} (\hat{V}_{Lk} - V_{uk}) \]

(42)

However, in this case too, it may be seen immediately that

\[ \hat{V}_{Lk} = V_{uk} \]

will reduce (42) to:

\[ \hat{r}_{Lk} - r_o = \frac{V_{uk}}{S_k} (r_{uk} - r_o) \]

(43)

which ensures that the market-clearing portfolio is also optimal.

The introduction of limited liability brings no further problems; it is simply necessary to redefine (30) and (31) as follows:
If we then assume that proportional changes in the value of the firm are independent of leverage, it again follows that, in the absence of financial institutions, the value of the firm is more or less independent of leverage. The proof follows the same lines as before.

VII Optimal Leverage With Corporation Taxes

Let us assume that no borrowing constraints are binding, that bonds are riskless and that there are no capital gains or losses. Now introduce corporation taxes, imposed at a constant rate $\tau$, upon profits after the deduction of interest payments to bondholders. The income accruing to the firm's stockholders is then:

(46) $y_k = (\pi_k - c_k)(1 - \tau)$

(47) $\therefore r_k - r_o = \frac{\pi_k}{s_k} (1 - \tau) - \frac{r_o}{s_k} (v_k - \tau d_k)$

(48) $\therefore \hat{r}_{lk} - r_o = \frac{v_{uk}}{s_k} (r_{uk} - r_o) - \frac{r_o}{s_k} \left( (v_{lk} - v_{uk}) - \tau d_k \right)$
We may then write the investor's income after the change in leverage, assuming he adopts a market-clearing portfolio:

\[
\hat{I} = I + \frac{x_k^*}{\bar{V}_{U_k}} (\hat{V}_{L_k} - V_{U_k}) + x_k^* \frac{S_k}{\bar{V}_{U_k}} (\frac{V_{U_k}}{S_k} (r_{U_k} - r_o)
\]

\[
- \frac{r_o}{\bar{S}_k} \left( (\hat{V}_{L_k} - V_{U_k}) - \tau D_k \right) - x_k^* (r_{U_k} - r_o)
\]

(49) \[\hat{I} = I + \frac{x_k^*}{\bar{S}_k} \tau \frac{D_k}{r_o} \]

Let us compensate him for the change in leverage by reducing his holdings of riskless assets by \( \frac{x_k^*}{\bar{S}_k} \tau \frac{D_k}{r_o} \) so that \( \hat{I} = I \). We may now examine the new portfolio optimization conditions:

(50) \[E[U_I(\hat{V}_{L_k} - r_o)] = E[U_I(r_{U_k} - r_o)] \frac{V_{U_k}}{\bar{S}_k} - E[U_I] \bar{S}_k \frac{r_o}{\bar{S}_k} \left( (\hat{V}_{L_k} - V_{U_k}) - \tau D_k \right) \]

In order for this portfolio to be utility maximizing, it must be the case that:

(51) \[\hat{V}_{L_k} = V_{U_k} + \tau D_k \]

If we now eliminate the wealth compensation, this effect is merely amplified; the investor's wealth is increased and therefore likewise
his demand for all securities. Since all investors will adjust their portfolios in the same manner if we neglect the second order effect, the new equilibrium value of the firm will be \( V_{Lk} = V_{Uk} + \tau D_k \).

This may be rationalized as follows. For every dollar paid as interest on the firm's debt, the stockholders are saving an amount \( \tau \) that would otherwise have been paid in taxes to the government. The total saving is then equal to \( \tau C_k \); when discounted at the riskless interest rate, this becomes \( \tau D_k \), which is added to the unlevered value of the firm.

In order to get some idea of the significance of this effect, taken by itself, assume that the corporation tax rate is 50% and that the unlevered firm issues debt amounting to 45% of the new value of the firm. According to (51), this would increase the value of the firm by nearly 30% of its unlevered value. MM state that the effect of corporation taxes on the firm's cost of capital is "remarkably small"; however, the gains to stockholders in switching from equity to debt finance appear to be remarkably large, and any corporation that neglected them would be certainly failing to fulfill its obligations to stockholders.

VIII Optimal Leverage When Profits Are Not Independent of Debt

So far, if we can assume that the effect of a rising cost of debt is more or less offset by the existence of limited liability, there
appear to be two effects tending to make the value of the firm increase with leverage—the existence of constrained financial institutions and corporation taxes. These next two sections consider two forces that are likely to act in the opposite direction.

In this section, I relax the assumption that leverage does not in any way affect the distribution of \( \pi_k \). This assumption is made implicitly by MM, and accepted implicitly by their critics, although there appear to be several good reasons why it should be rejected.

Perhaps most important are the indirect effects of leverage on \( \pi_k \) through the upper-level managers of the firm, since life for them is made more difficult in many respects by increases in debt. First, since it is generally accepted that dividend smoothing is a desirable goal, this task is rendered more difficult as leverage is increased. The standard deviation of earnings per share is equal to the standard deviation of profits, divided by the number of shares; therefore this rises as more shares are repurchased by the firm, and dividend smoothing can only be achieved at the expense of wider fluctuations in retained earnings or compensating changes in the firm's holdings of liquid assets.

Second, managers are more likely to come under fire from their stockholders as leverage is increased. This follows from a plausible hypothesis about stockholders' reactions to changes in share prices. If share prices rise, stockholders are content to leave the management alone and simply compliment themselves on their own perspicacity in
making such a wise investment. If they remain the same or decline slightly, stockholders will begin to grumble, but if share prices drop more than a certain threshold amount, they will blame the directors and are likely to take a more careful look at the management of the firm. If things are serious enough, some of the directors may be fired in an effort to stimulate increased efficiency and improve profits or simply perhaps because they are convenient scapegoats. Greater leverage magnifies fluctuations in share prices and therefore increases the probability that this threshold will be attained, at which point the stockholders begin to interfere actively with the running of the company.

Thirdly, even if the directors are not fired by irate stockholders dissatisfied with their firm's performance, they are always in danger of being fired in the event that the company is taken over and re-organized by another firm. The danger of a take-over is greatly increased by leverage, since less capital is required by the acquiring firm to gain control.

Fourthly, leverage increases the probability that profits will fall below $C_k - S_k$, at which point the firm will be forced into bankruptcy. This again poses a threat to the jobs of everyone in the company, especially the upper level management.

All these points will be taken into consideration by a rational man seeking a managerial job in a firm, and it is to be expected that
he will require compensation for the additional risks and disutilities he takes on if the firm is highly levered. This in turn implies that the distribution of $\pi_k$ shifts to the left as leverage is increased as managers' salaries have to be raised.

Finally, the effect of leverage on the danger of bankruptcy has other, more direct, effects on the firm's profits. Other firms will be reluctant to deal with a company that is likely to go bankrupt, and so demand for its product may be reduced. The legal and other costs of financial reorganization are considerable, and so in reality bankruptcy does not simply represent a transfer of losses from the stockholders to the bondholders in the way suggested in Section V. The stockholders will benefit by less than the total loss to bondholders, the difference representing these costs of bankruptcy. We should then redefine $\pi_k$ by deducting these costs from the operating profit when the latter falls below $C_k - S_k$; the effect is to split off part of the distribution of $\pi_k$ a certain distance to the left. As leverage is increased, a larger and larger proportion of the distribution is split off, thus reducing the expected value. Furthermore there are two reasons why this effect should tend to accelerate. Each additional dollar of debt raises $C_k$ by an amount $r_k + D_k \frac{dr}{dk}$, which rises with leverage as the risk borne by bondholders increases. On top of this, as $C_k - S_k$ moves up the left
hand tail of the distribution of \( \pi_k \), the probability that \( \pi_k \) will take on the value of \( C_k - S_k \) increases, and \( \pi_k \) is reduced at an increasing rate (in absolute value). At some point, it will become impossible to issue any more debt, however high the promised coupon return to bondholders, since increasing \( r_{dk} \) merely further increases the probability of bankruptcy to such an extent that it would be impossible for the firm to provide an adequate return to bondholders.

This may be regarded as the point at which \( \frac{dr_{dk}}{dD_k} \) becomes infinite; a kind of credit rationing occurs similar to that described by Hodgman in [16].

As the distribution of \( \pi_k \) shifts to the left, investors will of course lower their demand for the firm's stock. Unfortunately, this is difficult to incorporate into a rigorous model of portfolio adjustment, since there is no general way of compensating an investor for a change of leverage in such a way as to leave the distribution of \( I \) unchanged. One way of solving this would be to make the somewhat objectionable assumption that the reduction in \( \pi_k \) is a simple uniform shift of the form \( \pi_{Lk} = \pi_{UK} - f(D_k) \), where \( \pi_{UK} \) is operating profit in the absence of leverage and \( f(D_k) \) is a non-stochastic term representing the amount of shift for each degree of leverage, which is a positive, increasing and accelerating function of \( D_k \). We may therefore write:
\begin{equation}
(52) \quad r_k - r_o = \frac{\pi_{Uk} - f(D_k) - r_o V_k}{S_k}
\end{equation}

The cost of debt has been assumed to be \( r_o \) rather than \( r_{dk} \); this is to avoid having to constrain \( \pi_k \) to values greater than \( C_k + S_k \). Therefore:

\begin{equation}
(53) \quad \hat{r}_{Lk} - r_o = (r_{Uk} - r_o) \frac{V_{Uk}}{S_k} - \frac{1}{S_k} f(D_k) + r_o (V_{Lk} - V_{Uk})
\end{equation}

If the investor adopts a market clearing portfolio, his income will have fallen by \( \frac{x_k}{V_{Uk}} f(D_k) \); therefore, let us compensate him by raising his holdings of riskless asset. The portfolio optimization condition is:

\begin{equation}
(54) \quad E[U_T(\hat{r}_{Lk} - r_o)] = E[U_T(r_{Uk} - r_o)] \frac{V_{Uk}}{S_k} - \frac{E[U_T]}{S_k} f(D_k) + r_o (V_{Lk} - V_{Uk})
\end{equation}

which will equal zero if and only if:

\begin{equation}
(55) \quad \hat{V}_{Lk} = V_{Uk} - \frac{f(D_k)}{r_o}
\end{equation}

In words, the value of the firm must be reduced by the amount of the shift in profits, discounted at \( r_o \), in order for the new portfolio
to be optimal. Removal of the compensation further amplifies this effect. The same results may be obtained, at the expense of a little more complicated mathematics, by explicitly introducing limited liability and using the same method as used in Section V.

If the reduction in $\pi_k$ does not follow a simple pattern like this, it will be impossible to compensate investors for their change in income, although the theorem will still follow approximately as long as the dispersion of $\pi_k$ is not radically reduced. There is in fact reason to suppose that the dispersion will actually be increased; therefore it may safely be asserted that the costs mentioned in this section will tend to lower the value of the firm as leverage is increased.

IX  Optimal Leverage With Retained Earnings

Up to now, it has been assumed that all the firm's earnings have been distributed to stockholders, while in reality, retained earnings provide a very important source of funds for investment purposes. In [24], MM suggest that retained earnings may be considered as equivalent to a fully subscribed, pre-emptive issue of common stock. In other words, the effect would be exactly the same if the earnings were actually distributed, but then immediately returned by the stockholders in exchange for a new issue of equity, to be used for expanding the firm. Under present tax laws, however, this comparison is not strictly true, since personal income taxes would be paid on the earnings if they were distributed, while, if the earnings are merely reflected in an accrued capital gain to stockholders, no tax liability is incurred.
until the stock is sold. Furthermore, if the sale occurs over six months after the original purchase of the stock, the capital gain is taxed at a flat rate of 25%, as opposed to the investor's marginal personal rate. Since virtually all investors' marginal personal tax rates are above 25%, it will be in their interest to take the capital gain resulting from the retained earnings rather than the dividend; this effect is further strengthened by the effective interest free loan on the tax liability from the accrued capital gain thanks to the ability of the investor to postpone payment as long as he wishes to hold the stock.

There is yet another reason why retentions represent a cheaper source of funds than new issues of equity, and that is the existence of other restrictions and costs associated with new issues. These include underwriting costs, the costs of publishing and circulating information about the company, and statutory delays before the issue may be made; all of them may be avoided by financing expansion through retained earnings.

Against these advantages of retained earnings there are no real disadvantages. The investor who relies on the return from his stock holdings as a source of income for immediate consumption purposes will have to sell stock in order to spend his capital gains; this involves transactions costs and may be awkward if there are indivisibilities present (i.e. if the price of a single share is very high.) Neither objection carries much weight; the financial transactions costs involved are of a smaller order of magnitude than the
costs associated with a new issue, and the indivisibility problem may be solved by making infrequent sales of the stock in question, reinvesting the proceeds in a highly divisible asset such as a savings and loan account, and liquidating this gradually as the need arises. In any case, such investors might be better off investing in relatively stagnant firms, while relatively dynamic firms would attract investors who are not concerned with these problems.

The cheapness of retentions vis-a-vis new issues means that the optimum debt-equity ratio should, strictly speaking, be determined simultaneously with the amount of investment, since each decision affects the other. The financial structure decision affects the cost of capital since it determines the amount of earnings that may be retained before the costs of a new issue need be incurred. This affects the investment decision, which in turn feeds back into the financial structure decision. For simplicity's sake, let us assume that the investment decision may be made logically prior to the other. If \( J_k \) is the amount of investment decided on, then the firm can finance this entirely with retained earnings as long as \( \pi_k > J_k + C_k \).

For every dollar, however, that \( J_k \) exceeds \( \pi_k - C_k \), the firm must incur the excess costs associated with making a new issue, and the greater the leverage, the greater the probability that this will be necessary. (It is possible that it might be cheaper simply to postpone part of the investment; in that case, the cost of the postponement should be considered instead.)
In order to allow for this we must modify the distribution of \( \pi_k \). Values less than \( J_k + C_k \) must be shifted to the left; the further below \( J_k + C_k \), the higher the excess costs and the further the shift. As leverage is increased, more of the distribution is shifted away; the effect, then, is to reduce the expected value and increase the dispersion, so that, as seen in the last section, the value of the firm will tend to be reduced.

Furthermore, the higher the intended rate of growth the more pronounced will be this effect, up to a point after which it becomes certain that an additional dollar of investment will always require a new issue. After this point, a one dollar increase in debt will reduce retained earnings by \( r_k + D_k \frac{dr}{dD_k} + D_k \frac{df(D_k)}{dD_k} \), where the last term represents the decline in profits resulting from the increased danger of bankruptcy. If this is assumed to be non-stochastic, then the value of the firm will be reduced by this amount, multiplied by the excess cost of a dollar raised externally over the cost of internal finance and discounted at the riskless rate. If the projected rate of growth of the firm is less than this, the effect of increasing debt on the value of the firm will be correspondingly reduced, since there is some possibility that the excess cost will not be incurred.
CHAPTER V

SUMMARY AND CONCLUSIONS

In order to predict the way in which the market will evaluate a change in leverage, the managers of a firm must consider the precise way in which investors will respond to it in order to reoptimize their portfolios. The condition for reoptimization depends on the change in the value of the firm; therefore the new equilibrium value of the firm must be that level for which the sum of all investors' optimum adjustments in demand for each security is equal to the corresponding changes in supply. In the preceding chapter, I found two reasons why certain types of investor might respond to an increase in leverage by raising their demand for the firm's stock, thereby increasing the value of the firm. First, financial institutions that are constrained to hold a certain minimum fraction of their portfolios in the form of fixed interest securities regard increased leverage as identical to a relaxation of their constraint. They are therefore prepared to pay more for highly levered stocks, and will tend to raise the value of such firms. Secondly, each dollar of debt reduces the firm's tax obligation, since corporate taxes are calculated on earnings after deduction of interest charges. This induces all investors to raise their demand for the firm's stock until the value of the firm has risen by
the amount of the tax saving, capitalized at the riskless interest rate.

These effects may now be combined into a marginal benefit of leverage curve, showing the increase in the value of the firm resulting from a one dollar increase in debt if no other effects are operating. The second element is probably the more important; its marginal effect is a constant \( \$T \) per extra dollar of debt. The marginal effect of institutional constraints increases up to a maximum which is attained when no stock remains in any private investors' portfolios; after that, the effect will be more or less constant as long as different institutions have similar marginal costs of riskless return. It is likely, then, that the combined marginal benefit of leverage curve will be upward sloping at a decreasing rate.

Working in the opposite direction is the effect of leverage on the profits and profit expectations of the firm. Greater leverage increases the risks borne by the employees and managers of the firm as well as by other firms dealing with it. It also reduces the amount of retainable earnings, thus robbing a dynamic firm of a cheap source of funds for reinvestment. For these reasons, profits will be adversely affected by growing leverage, and investors will tend to reduce their demand for the firm's stock, thus depressing its market value. Furthermore, these effects tend to accelerate, since each successive dollar of debt necessitates raising the interest charge by ever increasing amounts. At some point, no matter how high the promised return, it becomes impossible to
issue any more debt, because raising $r_{dk}$ merely increases the danger of bankruptcy even further so that the return to bondholders is not increased. The marginal cost of leverage, then, becomes undefined at this point.

We may now combine all the ingredients just mentioned in order to come up with a theory to predict a firm's optimal financial structure. As long as the slope of the marginal benefit of leverage is diminishing and that of the marginal cost curve is increasing, there will be a unique optimum capital structure defined at the point where the marginal cost curve intersects the other from below. If we assume that the firm is a perfect competitor in the capital market, then changes in scale will shift both curves to the right or left by the same amount, leaving the optimal debt-equity ratio unaffected.

Figure 1
There are two important variables that will have a significant effect on the optimal debt-equity ratio. The first is the amount of business risk the firm faces, which influences the way in which debt increases the danger of bankruptcy. If business risk is low, then, for each level of debt, the probability of bankruptcy is low and the depressing effect on profits correspondingly small. High business risk makes the likelihood of bankruptcy more probable for each level of debt; this feeds back by raising $r_{dk}$ and lowering profits, thus further compounding the danger.

If firms j and k are of the same size, but j has more business risk than k, the marginal cost of leverage curve for j will be to the left of that for k. The marginal benefit curves will be more or less the same, and so the point of intersection of the two curves for firm j will be to the left of that for firm k, and therefore the former will have a lower optimal debt-equity ratio than the latter.

The second important variable that influences the optimal debt-equity ratio is the rate of growth of the firm. The faster the intended rate of growth, the greater the additional excess costs from having to rely on external finance, per extra dollar of debt. This raises the marginal cost of leverage without affecting the benefits, thus shifting the point of intersection to the left and lowering the optimal debt-equity ratio.

To sum up; if corporation managers calculate their firms' optimum financial structures in the manner described in this thesis, there should be an inverse correlation between debt-equity ratios and both business risk and rate of growth. There exists some evidence tending to confirm
this hypothesis; the debt-equity ratios of public utility companies are on average significantly higher than those of manufacturing companies which face considerably higher business risks. In [40], Weston presents the results of a regression of the cost of capital on the degree of leverage and on the rate of growth of earnings per share among a group of electric utilities, showing highly significant negative coefficients that explain about one half of the observed variation. Weston considers this as evidence in support of the traditional hypothesis that the cost of capital is a decreasing function of leverage; this interpretation, however, implies that the firms considered are not maximizing their market value. His results may be turned around and used to support my hypothesis by considering leverage to be the dependent variable, and the other two to be independent variables. It is likely that if the regression were performed on the variables in that manner, the results would be substantially the same. By interpreting the dependent variable as the \textit{optimum} degree of leverage and the cost of capital as a proxy for the degree of business risk, my hypothesis is confirmed.
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