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KERNEL FUNCTIONS AND PARABOLIC LIMITS
FOR THE HEAT EQUATION

by

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0. Introduction

In the present work we introduce the notion of a kernel function for the heat equation in regions \( D \) of the \((x,t)\) plane whose lateral boundaries are given by functions satisfying a Lipschitz condition with exponent \( 1/2 \), and use these functions to study the boundary behavior of non-negative temperatures (solutions of the heat equation) in such regions. In Section 1 the existence and uniqueness of kernel functions are proven and a representation theorem is obtained for non-negative temperatures in finite regions satisfying the Lipschitz condition. Our approach parallels the one used by Hunt and Wheeden [10] to obtain similar results for non-negative harmonic functions in Lipschitz domains, although certain essential changes are required.

In particular, difficulties in applying the methods of Hunt and Wheeden arise because of the weakened form of Harnack's inequality in the case of the heat equation. To compare the values of a non-negative temperature in a compact subset of \( D \) with its value at a particular point, it is necessary that the set be bounded away from the point in the time coordinate (a so-called "time-lag"), and only a one-sided inequality is true, bounding the values of the function at earlier times by a constant multiple of its value at the later time. These problems are overcome by adjustments which are made in the course of the individual proofs. Some simplifications of the techniques of [10] are also achieved, notably in the proof of Theorem 1.6.

In Section 2, again following the lead of Hunt and Wheeden [9] and, previously, Carleson [3], we are able to apply the representation theorem of Section 1 to prove that, for non-negative temperatures in \( D \), finite parabolic limits exist at all points of the parabolic boundary except for a set of zero caloric measure. This generalizes the results of Hattemer [8] and Jones and Tu [12], who considered regions less general than those dealt with here.

It is of interest to note that the results of [12] on the existence of parabolic boundary values hold for regions which are similar to ours, but whose boundaries are given by functions satisfying a Lipschitz
condition with exponent $a$, where $a > 1/2$. Tu's techniques, however, do not seem to apply in the limiting case ($a = 1/2$), and an entirely different approach, similar to the one used by Hunt and Wheeden for harmonic functions, has been adopted in the present treatment.
1. **Existence and Uniqueness of Kernel Functions; a Representation Theorem**

Let $D$ be a domain in the $(x,t)$ plane bounded by curves $x = \eta_1(t)$, $t = 0$, and $x = \eta_2(t)$ which is contained in the upper half-plane. (See Figure 1.) We assume that $\eta_1(t) < \eta_2(t)$ for $t \geq 0$ and that each $\eta_i(t)$ satisfies a Lipschitz condition with exponent $1/2$ on any interval $[0,T]$ with $T < \infty$. By this we mean that, for each $T$, there is a constant $C > 0$ such that, for each $t$, $t_1 \in [0,T]$, we have $|\eta_i(t) - \eta_i(t_1)| \leq C|t - t_1|^{1/2}$.

For $T > 0$, $D_T$ will denote the intersection of $D$ with $\{(x,t): t \leq T\}$. For any set $\Omega$, $\partial_p \Omega$ will denote the parabolic boundary of $\Omega$. We have $\partial_p D = \partial D$ and $\partial_p D_T = \partial D \cap \{(x,t): t < T\}$.

We adopt the convention of representing points of $D$ by $(x,t)$ and points of $\partial D$ by $(y,s)$.

For a fixed point $(X,T)$ in $D$, we have the following: Definition. If $(y_o,s_o) \in \partial D$ with $s_o < T$, a function $K(x,t)$ defined in $D$ is a kernel function at $(y_o,s_o)$ for the heat equation in $D$ with respect to $(X,T)$ if

1. $K(x,t) \geq 0$ for $(x,t) \in D$
2. $K$ satisfies the heat equation, $K_{xx} = K_t$, in $D$
3. $\lim_{(x,t) \to (y,s)} K(x,t) = 0$ for $(y,s) \in \partial D \setminus \{(y_o,s_o)\}$
4. $K(X,T) = 1$ (normalization condition).

(If $s_o \geq T$, we shall take the kernel function at $(y_o,s_o)$ to be identically zero.)

Clearly, if $s_o < T$ and a function $K(x,t)$ satisfies (1)-(4) in $D_T$, then $K$ can be extended to a kernel function in $D$ by solving a Dirichlet problem in $D - D_{s_o + \varepsilon}$ with (continuous) boundary values

$$f(y,s) = \begin{cases} K(y,s) & \text{for } s = s_o + \varepsilon, \eta_1(s_o + \varepsilon) \leq y \leq \eta_2(s_o + \varepsilon) \\ 0 & \text{for } (y,s) \in \partial D \cap \{(x,t): t > s_o + \varepsilon\}. \end{cases}$$
Consequently, we shall often consider the kernel function only in a bounded set such as $D_T$, the extension being implied.

By the condition of Petrowski [13], every boundary point of $D$ is regular for the heat equation, so that the Dirichlet problem is solvable. In fact, the problem is solvable in the Wiener-Perron sense for any (Borel) integrable boundary values. This justifies the following:

**Definition.** If $(x,t) \in D$ and $Z \subset \partial D$ is a Borel measurable set, the **caloric measure in $D$ of $Z$ at $(x,t)$**, denoted $\omega^D_{x,t}(Z)$, is the value at $(x,t)$ of the unique (Wiener-Perron) solution of the Dirichlet problem in $D$ with boundary data given by the characteristic function of the set $Z$. (When there is no possibility of confusion, we shall suppress the subscript $D$, writing $\omega^{x,t}(Z)$.)

If $t \leq \inf\{s: \exists y \text{ with } (y,s) \in Z\}$, we must have

$$\omega^{x,t}(Z) = 0,$$

by the maximum principle. In particular,

$$\omega^{X,T}(\partial D - \partial D_T) = 0.$$  

We also note that if $(x,t) \in D$ and $t < T$,

by Harnack's inequality there is a constant $C > 0$ such that

$$\omega^{x,t}(Z) \leq C \omega^{X,T}(Z)$$

for any Borel set $Z \subset \partial D$. Thus, caloric measure at $(x,t)$ is absolutely continuous with respect to caloric measure at $(X,T)$. Then, by Besicovitch's general theory of differentiation [2], the Radon-Nikodym derivative $d\omega^{x,t}(\cdot)/d\omega^{X,T}(\cdot)$, which exists as an element of $L^1(\omega^{X,T}(\cdot))$, is given almost everywhere $\omega^{X,T}$ by

$$\frac{d\omega^{x,t}(Y,S)}{d\omega^{X,T}(Y)} = \lim_{\Delta_n \to (Y,S)} \frac{\omega^{x,t}(\Delta_n)}{\omega^{X,T}(\Delta_n)},$$

where $\{\Delta_n\}$ is any sequence of closed sets which contain $(Y,S)$ and which satisfy the condition

$$\inf_n \frac{\omega^{X,T}(\Delta_n)}{\omega^{X,T}(B_n)} > 0,$$

where $B_n$ is the intersection of $\partial D$ with the smallest disk centered at $(Y,S)$ and containing $\Delta_n$.

We shall first prove the existence of kernel functions at points of $\partial D$. There are essentially two cases. The first involves "bottom" boundary points $(y,0)$ with $\eta_1(0) < y < \eta_2(0)$. The second involves "side" boundary points $(\eta_1(s),s)$ with $s > 0$ and $i = 1$ or 2. The kernel
function at a point \((\eta_1(0),0)\) can be extended by zero to a rectangle \(R = \{(x,t): \eta_1(0) < x < \eta_2(0) \text{ and } t_0 < t \leq 0\}\) for any \(t_0 < 0\), and this extended function is a kernel function at a "side" boundary point of the domain \(D' = D \cup R\). Furthermore, a kernel function at \((\eta_1(0),0)\) in \(D'\), when restricted to \(D\), continues to be a kernel function in \(D\). Thus, such "corner" points of \(\partial D\) may be treated along with points of the second type above.

The proofs for the two basic cases are very similar. We shall concentrate our attention on the second case, pointing out the differences in technique required to prove the results for the first case without going into complete detail.

As mentioned previously, it is only necessary to establish the properties of a kernel function at a point \((y_o,s_o) \in \partial D\) in the bounded set \(D_n\) for \(T > s_o\). Recalling that there is a constant \(C > 0\) such that \(|\eta_i(t_1) - \eta_i(t_2)| \leq C|t_1 - t_2|^{1/2}\) for \(t_1, t_2 \in [0,T]\) and \(i = 1,2\), we fix a constant \(d > 2C\). We establish the following notation:

for a point \((y_o,s_o) = (\eta_1(s_o),s_o) \in \partial D\) with \(0 < s_o < T\), and for \(r > 0\), define \(\mathcal{W}((y_o,s_o),r) = D \cap \{(x,t): |x - y_o| < rd\text{ and }|t - s_o| < r^2\}\) and \(\Delta((y_o,s_o),r) = \partial D \cap \{(x,t): |t - s_o| < r^2\}\). (Of course, we can define these sets for other large numbers \(d\), and later we shall have occasion to consider a \(d\) which differs from the one we have fixed here.)

We begin with the following lemma:

**Lemma 1.1.** If \(\gamma \in (0,1)\), then there exists a constant \(C = C(\gamma) > 0\) such that, if \((y_o,s_o) \in \partial D\) with \(s_o \in (0,T)\), we have

\[
\omega(x,t)(\Delta((y_o,s_o),r)) \geq C \text{ for } (x,t) \in \mathcal{W}((y_o,s_o),\gamma r),
\]

as long as \(r\) is sufficiently small.

**Proof** (We assume that \(y_o = \eta_1(s_o)\).)

Let \(\Gamma((y_o,s_o),r) = \{(x,t): |t - s_o| < r^2 \text{ and } |x - y_o| < rd\}\) and let

\[
h(x,t) = \omega(x,t) \Gamma((x,t): x = y_o - rd \text{ and } |t - s_o| < r^2).\]

For small enough \(r\), \(\Gamma((y_o,s_o),r) \cap \{(x,t): x > \eta_1(t)\} \subseteq D\) and, applying the maximum principle in \(\mathcal{W}((y_o,s_o),r) = D \cap \Gamma((y_o,s_o),r)\),

\[
\omega(x,t)(\Delta((y_o,s_o),r)) \geq h(x,t) \text{ for } (x,t) \in \mathcal{W}((y_o,s_o),r).
\]

Clearly, \(\mathcal{W}((y_o,s_o),\gamma r)\) is contained in \(\Gamma((y_o,s_o),\gamma r)\), a compact
subset of \( \Gamma((y_o, s_o), r) \). Since \( C = \inf_{(x, t) \in \Gamma((y_o, s_o), t \nu)} h(x, t) > 0 \),

we have \( \omega((x, t), (\Delta((y_o, s_o), r))) \geq C > 0 \) for \((x, t) \in \nu((y_o, s_o), t \nu)\).

Note that \( C \) is independent of \((y_o, s_o)\) as long as \( r^2 \leq \min(s_o, T - s_o) \).

Q.E.D.

Now, for a constant \( \mu > 0 \), let \( \Lambda = (y_o, s_o) + (\nu + rd, s_o + r^2(1 + \mu)) \).

By our choice of \( d \), if \( \mu \) is small, then \( \Lambda(\eta_1(s_o), s_o) = r \in D \) for all sufficiently small \( r \) and for \( s_o \in (0, T) \).

Q.E.D.

**Lemma 1.2.** There is a constant \( C > 0 \) such that for \( r' \in (0, r) \)

and \((y_o, s_o) = (\eta_1(s_o), s_o)\), with \( s_o \in (0, T) \)

\[ (*) \quad \omega((x, t), (\Delta((y_o, s_o), r'))) \leq C \omega(\Lambda((y_o, s_o), r))(\Delta((y_o, s_o), r')) \]

holds for \((x, t) \in D - \nu((y_o, s_o), r)\) if \( r \) is sufficiently small.

**Proof.** (For convenience, let \( \Delta = (\Delta((y_o, s_o), r), \Delta'(\Delta((y_o, s_o), r'))) \).)

Since \( \omega((x, t), (\Delta')) = 0 \) for \((x, t) \in \partial D - \Delta\), it suffices, by the maximum principle, to prove \((*)\) for \((x, t) \in D \cap \partial \nu((y_o, s_o), r)\). We define

an auxiliary function \( h(x, t) \) as follows:

Let \( \tilde{D} = \{(x, t) : x > \eta_1 \text{ or } |t| > 1\} \cup \{(x, t) : x > -M|t - 4|^{1/2}\} \) and let \( \tilde{D}_n = \tilde{D} \cap \{(x, t) : |x| < n \text{ and } |t| < n\} \).

(See Figure 2.) Take \( h_n(x, t) \) to be

the caloric measure in \( \tilde{D}_n \) of that part of \( \partial \tilde{D}_n \) which lies on the boundary of

the removed strip, \( \{(x, t) : x \leq \eta_1, |t| \leq 1\} \).

By the maximum principle, \( h_n \) increases in \( \tilde{D} \) as \( n \) increases. Since \( h_n \leq 1 \) for each \( n \),

there exists a temperature in \( \tilde{D} \), \( h(x, t) = \lim_{n \to \infty} h_n(x, t) \).

Furthermore, since the \( h_n \)'s are uniformly bounded and vanish on a

common neighborhood of \((0, 4)\) on \( \partial D \), and since that point is regular,

we have

\[ \lim_{(x, t) \to (0, 4)} h(x, t) = 0. \]

Now, define sets \( \nu_n = \nu((y_o, s_o), 2^{-n} r') \) and corresponding points

\( A_n = A((y_o, s_o), 2^{-n} r') \) for \( 1 \leq n \leq n_o \), where \( 2^{-n_o} r' < r < 2^{-n_o} r' \).
By Harnack's inequality there is a constant $C_1 > 0$ such that

(a) $\omega^{A_n}(\Delta') \leq C_1 \omega^{A_{n+1}}(\Delta')$ for $1 \leq n \leq n_0$.

($C_1$ may be chosen independent of $n$ since the $A_n$'s are all contained in a fixed parabolic cone at $(y_o, s_o)$ and two different pairs, $(A_n, A_{n+1})$ and $(A_m, A_{m+1})$, can be made to coincide by a change of variables which preserves temperatures in the cone.)

Choose $L$ so that $h(x, L) \leq 1/C_1$ for $x \in (0, L)$. For each $n$, shrink $D$ parabolically by $2^{n-1}r'$ (use the transformation $x = 2^{n-2}r't$, $t = 2^{n-2}r', x'$) and translate the new region so that the point $(0, 2^{n}r')$, corresponding to $(0, L)$ in $\tilde{D}$, is moved to the point on $\partial D$ with $s = s_o + 2^{n}r'$.

(See Figure 3.) Let $D_n^*$ denote the transformed region and $h_n^*$ the temperature in $D_n^*$ corresponding to $h(x, t)$ in $\tilde{D}$.

Set $\beta_n = \{(x, t) \in \partial \Omega_n : x - 2^{n-1}r' < 2^{n-1}r' L \}$ with $a_n = \partial \Omega_n^* - \beta_n$. Harnack's inequality gives a constant $C_2 > 0$ such that

(b) $\omega(x, t)(\Delta') \leq C_2 \omega(A_n)(\Delta')$ for $(x, t) \in a_n$.

If $B_1 = (y_o + r'd, s_o)$, Lemma 3.1, with $\gamma = 1/2$ and the parameter in the definition of $\Delta$ and $\psi$ equal to $2d$, shows that there is another constant $C_3 > 0$ such that

$\omega^{B_1}(\Delta') \geq C_3' r'$ if $r'$ is sufficiently small.

By Harnack's inequality, again, there is another constant $C_3' > 0$ such that

$\omega^{A_1}(\Delta') \geq C_3' \omega(B_1)(\Delta')$.

If $C_3 = C_3', C_3''$, then $\omega^{A_1}(\Delta') \geq C_3$, and, for any $(x, t) \in D$,

$\omega(x, t)(\Delta') \leq 1 \leq \frac{1}{C_3} \omega^{A_1}(\Delta')$.

Let $C_4 = \max(C_2, 1/C_3)$. We have $\omega(x, t)(\Delta') \leq C_4 \omega^{A_1}(\Delta')$ for $(x, t) \in \partial \psi_1$ and, from (b), $\omega(x, t')(\Delta') \leq C_4 \omega^{A_2}(\Delta')$ for $(x, t) \in a_2$. Thus, to prove $\omega(x, t')(\Delta') \leq C_4 \omega^{A_2}(\Delta')$ in all of $D - \psi_2$, it suffices to prove it for $(x, t) \in \beta_2$. Figure 3.
First, however, we must have chosen \( M \) in the definition of \( D \) so that the cone in the complement of \( D^* \) lies in the complement of \( D \). This is possible since \( \partial D \) is given by a function satisfying a Lipschitz condition with exponent \( 1/2 \). We must also have chosen \( d_1 \) large enough that \( \mathcal{V}_1 \) is contained in the complement of \( D_1^* \). Then,

\[
\omega(x,t)(\Delta') \leq C_4 \omega^1(\Delta') \text{ for } (x,t) \in D \cap D_1^*,
\]

and the maximum principle in \( D \cap D_1^* \) implies that

\[
\omega(x,t)(\Delta') \leq C_4 \omega^1(\Delta') h_1(x,t) \text{ for } (x,t) \in D \cap D_1^*.
\]

Recalling (a), we have

\[
\omega(x,t)(\Delta') \leq C_4 C_1 \omega^2(\Delta') h_1(x,t) \text{ for } (x,t) \in D \cap D_1^*,
\]

but, for \( (x,t) \in \beta_2 \), \( h_1(x,t) \leq 1/C_1 \), so that

\[
\omega(x,t)(\Delta') \leq C_4 \omega^2(\Delta') \text{ for } (x,t) \in \beta_2.
\]

Hence, the same inequality holds in \( D - \mathcal{V}_2 \).

Continuing inductively,

\[
\omega(x,t)(\Delta') \leq C_4 \omega^{\text{no}-1}(\Delta') \text{ for } (x,t) \in D - \mathcal{V}_{\text{no}-1} \text{ and,}
\]

in particular, for \( (x,t) \in D - \mathcal{V}(y_0, s_0, r) \). Noting that

\[
(1+\mu)^{2} - (1+\mu)(2^{\text{no}-1} r^{2}) \geq 7/16 r^{2},
\]

by our choice of \( n_0 \), Harnack's inequality shows that there is a constant \( C > 0 \) such that

\[
\omega^{\text{no}-1}(\Delta') \leq C \omega(y_0, s_0, r)(\Delta').
\]

Thus,

\[
\omega(x,t)(\Delta') \leq C \omega(y_0, s_0, r)(\Delta') \text{ for } (x,t) \in D - \mathcal{V}(y_0, s_0, r).
\]

Q.E.D.

We next state a lemma of fundamental importance.

**Lemma 1.2.** For \( T > 0 \) there is a constant \( C > 0 \) such that, if

\[
(y_0, s_0) = (\eta_1(s_0), s_0), \text{ with } s_0 < T, \text{ and, for sufficiently small } r,
\]

\( N \subset \mathcal{V}(y_0, s_0, r/4) \) is a neighborhood of \( (y_0, s_0) \) in \( D_T \) such that

\( D - N \) is bounded by curves satisfying a Lipschitz condition with exponent \( 1/2 \), and \( u(x,t) \) is a non-negative temperature in \( D_T - N \) which is continuous in the closure of that set and satisfies

\( u(y,s) = 0 \) for \( (y,s) \in \partial D_T - N \), then
\[ u(x,t) \leq C u(A((y_o,s_o),r)) \omega(x,t)(\Delta((y_o,s_o),r)) \]

for \((x,t) \in D_T - \psi((y_o,s_o),(1+\mu)^{1/2}r/\lambda)\).

(The upper bound for \(r\) may depend on \((y_o,s_o)\), but \(C\) does not.)

**Proof.** Fix \(\lambda \in (0,1)\) and let \(\Delta' = \Delta((\bar{y},\bar{s}),r')\), where \((\bar{y},\bar{s}) \in \Delta((y_o,s_o),r)\)

and \(r' < \lambda r\). By Lemma 1.2, if \(r\) is sufficiently small,

\[ \omega(x,t)(\Delta') \leq C \omega^A(\Delta') \text{ for } (x,t) \in D - \psi((\bar{y},\bar{s}),\lambda r), \]

where

\[ A = (\bar{y} + \lambda rd, \bar{s} + \lambda^2 r^2(1 + \mu)). \]

\(C\) is independent of \((\bar{y},\bar{s}) \in \Delta((y_o,s_o),r)\).

Since \(\psi((\bar{y},\bar{s}),\lambda r) \subseteq \psi((y_o,s_o),(1+\lambda)r)\), we have

\[ \omega(x,t)(\Delta') \leq C \omega^A(\Delta') \text{ for } (x,t) \in D - \psi((y_o,s_o),(1+\lambda)r). \]

If \(\lambda\) is small, \(\bar{s} + \lambda^2 r^2(1+\mu) \leq s_o + r^2 + \lambda^2 r^2(1+\mu) < s_o + (1+\mu/2)r^2\),

so that Harnack's inequality applies to show that

\[ \omega^A(\Delta') \leq C \omega^{A}(\Delta'), \text{ where } A = A((y_o,s_o),r). \]

If we also require \(\lambda < (1+\mu)^{1/2} - 1\), we have

\[ \omega(x,t)(\Delta') \leq C \omega^{A}(\Delta') \text{ for } (x,t) \in D - \psi((y_o,s_o),(1+\mu)^{1/2}r). \]

Next, let \((\bar{x},\bar{t}) \in D_T - \psi((y_o,s_o),(1+\mu)^{1/2}r)\). By Besicovitch's results

on the general theory of differentiation,

\[ \frac{\partial \omega}{\partial \omega}(\bar{x},\bar{t}) (\bar{y},\bar{s}) = \lim_{\Delta' \to (\bar{y},\bar{s})} \frac{\omega(\bar{x},\bar{t})(\Delta')}{\omega^A(\Delta')} \text{ for a.e. } (\omega(X,T))(\bar{y},\bar{s}) \text{ in } \Delta((y_o,s_o),r). \]

Thus,

\[ \frac{\partial \omega}{\partial \omega}(\bar{x},\bar{t}) (\bar{y},\bar{s}) \leq C \frac{\partial \omega^A}{\partial \omega}(X,T) (\bar{y},\bar{s}) \text{ for a.e. } (\omega(X,T))(\bar{y},\bar{s}) \in \Delta((y_o,s_o),r), \]

where \(C\) is independent of \((\bar{y},\bar{s}) \in \Delta((y_o,s_o),r)\).

Now, if \(u\) is a non-negative temperature in \(D_T\) which is continuous in \(D_T\)

and satisfies \(u(y,s) = 0 \) for \((y,s) \in \partial D_T - \Delta((y_o,s_o),r)\), we have

\[ u(\bar{x},\bar{t}) = \int_{\Delta((y_o,s_o),r)} u(\bar{y},\bar{s}) \, d\omega(\bar{x},\bar{t})(\bar{y},\bar{s}). \]
Then,
\[ u(\bar{x}, \bar{t}) = \int \frac{\partial u(\bar{x}, \bar{t})}{\partial \omega(X, T)(\bar{y}, \bar{s})} \, d\omega(X, T)(\bar{y}, \bar{s}) \]
\[ \Delta((y_o, s_o), r) \]
\[ \leq C \int \frac{\partial u^A}{\partial \omega(X, T)(\bar{y}, \bar{s})} \, d\omega(X, T)(\bar{y}, \bar{s}) \]
\[ \Delta((y_o, s_o), r) \]
\[ = C \, u(A) \text{ for } (\bar{x}, \bar{t}) \in D_T - \Psi((y_o, s_o), (1+\mu)^{1/2}r). \]

Applying this argument to the function \( u \) given in the hypothesis and the region \( D_T - N \), with \( r \) replaced by \( r/4 \), we obtain
\[ u(x, t) \leq C \, u(A^*) \text{ for } (x, t) \in (D_T - N) - \Psi((y^*, s^*), (1+\mu)^{1/2}r/4), \]
where \((y^*, s^*) \in \partial_p N \) with \( s = s_o \) and \( A^* = (y + rd/4, s + (1+\mu)r^2/16). \)

By Harnack's inequality, \( u(A^*) \leq C \, u(A) \) and \( u(x, t) \leq C \, u(A) \) for \((x, t) \in \Psi((y^*, s^*), (1+\mu)^{1/2}r/4) - \Psi((y_o, s_o), (1+\mu)^{1/2}r/4), \) so that
\[ u(x, t) \leq C \, u(A) \text{ for } (x, t) \in D_T - \Psi((y_o, s_o), (1+\mu)^{1/2}r/4). \]

By Lemma 1.1, \( \omega(x, t) \{ \Delta((y_o, s_o), r) \} \geq C \) for \((x, t) \in \Psi((y_o, s_o), r/2), \)
so, applying the maximum principle in \( D_T - \Psi((y_o, s_o), (1+\mu)^{1/2}r/4), \)
we obtain
\[ u(x, t) \leq C \, u(A) \, \omega(x, t) \{ \Delta((y_o, s_o), r) \} \]
for \((x, t) \in D_T - \Psi((y_o, s_o), (1+\mu)^{1/2}r/4). \) \quad Q.E.D.

We will most often use the following form of Lemma 1.3:

**Lemma 1.4.** For \( T > 0 \) there is a constant \( C > 0 \) such that, if \((y_o, s_o) = (\Pi_1(s_o), s_o) \) with \( s_o < T \) and, for sufficiently small \( r, \)
\( u(x, t) \) is a non-negative temperature in \( D_T \) which vanishes on \( \partial_p D_T - \Delta((y_o, s_o), r/8), \) then
\[ u(x, t) \leq C \, u(A((y_o, s_o), r)) \, \omega(x, t) \{ \Delta((y_o, s_o), r) \} \]
for \((x, t) \in D_T - \Psi((y_o, s_o), (1+\mu)^{1/2}r/4). \)

**Proof.** For each \((y_o, s_o), \) take \( r < r_0, \) where \( r_0 \) satisfies the requirements of Lemma 1.3, and choose \( N \subset \Psi((y_o, s_o), r/4) \) such that
\[ \Delta((y_0, s_0), r/8) \subset N \cap \partial D \text{ and } \partial(D_T - N) \text{ is given by a function satisfying a Lipschitz condition with exponent } \gamma, \kappa. \] Then, any \( u \) which satisfies the hypotheses above also satisfies the hypotheses of Lemma 1.3. The result follows from that lemma. Q.E.D.

To prove the existence of a kernel function at a point \((y_0, s_0)\) in \(\partial D\) with respect to \((x, T)\), where \(0 < s_0 < T\), we consider a sequence of positive numbers \(r_n\) such that \(r_n\) tends to 0 as \(n\) tends to infinity. Let \(\Delta_n = \Delta((y_0, s_0), r_n)\) and set

\[ v_n(x, t) = \omega(x, t)(\Delta_n). \]

Each \(v_n\) is a non-negative temperature in \(D\) and \(v_n(y, s) = 0\) for \((y, s) \in \partial D - \Delta_n\). If \(r\) is sufficiently small for Lemma 1.4 to hold, there exists \(\Delta_n\) such that \(v_n\) satisfies the hypotheses of that lemma for \(n > n_0\). Therefore, if \(A = A((y_0, s_0), r)\) and \(\Delta = \Delta((y_0, s_0), r)\),

\[ (*) \quad v_n(x, t) \leq C v_n(A) \omega(x, t)(\Delta) \text{ for } (x, t) \in D_T - \psi((y_0, s_0), (1+\mu)^{1/2}/4). \]

Since each \(v_n\) is defined in all of \(D\) and has zero boundary values for \(s \geq T\), the inequality (*) holds in \(D - \psi((y_0, s_0), (1+\mu)^{1/2}/4)\) by the maximum principle. By Harnack's inequality, for small enough \(r\),

\[ v_n(X) < C. \]

Since \(\omega(x, t)(\Delta) \leq 1\), we have \(v_n(x, t) \leq C\) for \((x, t) \in D - \psi_o\), where \(\psi_o = \psi((y_0, s_0), (1+\mu)^{1/2}/4)\). Since any compact subset of \(D\) contained in \(D - \psi_o\) for sufficiently small \(r\), the Ascoli theorem assures the existence of a subsequence (which we also denote \(v_n\)) that converges uniformly on compact subsets of \(D\). (The functions \(v_n\) can easily be seen to form an equicontinuous family by showing 'na' the first derivatives are uniformly bounded in a neighborhood of each point of \(D\).) By Harnack's convergence theorem, the limit function, \(K(x, t)\), must also be a temperature in \(D\). Clearly, \(K(x, t) > 0\) and \(K(X, T) = 1\). Finally, to see that \(K\) vanishes on \(\partial D - \{y_0, s_0\}\),
we let $n$ tend to infinity in (*) to show that

$$K(x,t) \leq C \omega(x,t)(\Delta) \text{ for } (x,t) \in D - \psi, \text{ if } r \text{ is small.}$$

Thus, $K(x,t)$ is a kernel function at $(y_o,s_o)$ in $D$ with respect to $(X,T)$.

At this point we remark that Lemmas 1.1, 1.2, 1.3, and 1.4 can be similarly proven for points $(y_o,0)$, with $\eta_1(0) < y_o < \eta_2(0)$. For such a point, we would take $A((y_o,0),r) = (y_o,(1+\mu)r^2)$. The proof of Lemma 1.1 is unchanged. In Lemma 1.2 we would take $D = \{(x,t): |x| > 1 \text{ or } t > d_0\} \cap \{(x,t): t > -M|x-2|^2\}$. The shrunken sets $D^{n^+}$ are translated so that the point $(2^n r',0)$ corresponds to one of the points $(y,0)$ on $\partial D$ with $|y - y_o| = 2^n r'$. The remainder of the proof of that lemma follows as before. For Lemma 1.3 in the case of a boundary point $(y_o,0)$, taking $A^{0+}_{\lambda}(\bar{x},\bar{t}+\lambda^2 r^2 (1+\mu))$, the proof is essentially unchanged. Thus the existence of kernel functions at "bottom" points of $\partial D$ is also assured.

Our next goal is to prove the uniqueness of kernel functions. Before proceeding with the actual proof, however, we are able to simplify the form of the domain $D$ with which we must deal. (Of course, we continue to consider a kernel $\gamma$ at $(y_o,s_o) \in \partial D$ only i.e. with respect to a point $(X,T) \in D$ with $T > s_o$.)

We first show that it suffices to consider domains $D$ with $\eta_2(\cdot) = \text{constant} = \eta_2$. To see this, suppose that $u$ and $v$ are both kernel functions at $(y_o,s_o) \in \partial D - \{\eta_2(s,s), s \geq 0\}$ with respect to a point $(X,T) \in D$ with $T > s_o$. (For boundary points $(y_o,s_o)$ of the form $(\eta_2(s_o),s_o)$, we would consider domains with $\eta_1 = \text{constant}$.)

For $\delta$ small, the sphere $D' = D \cap \{ (x,t): |x-y_o| < \delta^{1/4}, |t-s_o| < \delta \}$ is bounded away from $\{\eta_2(s,s), s \geq 0\}$ and is a region of the simplified type. Let $B = \partial D \cap \partial D'$. $B$ is a closed subset of $\partial D$ and $(y_o,s_o)$ is an element of $B$. Define $P_u(x,\cdot)$ to be the solution of the Dirichlet problem for the heat equation in $D'$ with (continuous) boundary values equal to zero at points of $B$ and equal to $u$ at points of $\partial D' - B$. Then $u(x,t) - P_u(x,t)$ is a temperature in $D'$ and has boundary values equal to zero on $\partial D'$ except at $(y_o,s_o)$. 
However, we do have \( \liminf_{(x,t) \to (y_o,s_o)} [u(x,t) - P_u(x,t)] = \liminf_{(x,t) \to (y_o,s_o)} u(x,t), (x,t) \in D' \), and since \( u \) is non-negative, the maximum principle implies that \( u(x,t) \geq P_u(x,t) \) in \( D' \). If \((X',T') \in D' \) with \( T' > s_o \), and
\[
C_1 = \frac{1}{u(X',T') - P_u(X',T')}, \text{ then } u'(x,t) = C_1 [u(x,t) - P_u(x,t)] \]
is a kernel function at \((y_o,s_o)\) in \( D' \) with respect to \((X',T')\). (Clearly, \( u(X',T') \neq P_u(X',T') \). Otherwise, \( u = P_u \) in \( D' \) by the strong maximum principle, an obvious contradiction.)

Similarly, there is a constant \( C_2 > 0 \) such that \( \nu'(x,t) = C_2 [v(x,t) - P_v(x,t)] \) is also a kernel function at \((y_o,s_o)\) in \( D' \) with respect to \((X',T')\). If we assume uniqueness of kernel functions for regions with \( \eta_2 \equiv \text{constant} \), we must have \( u' = \nu' \) in \( D' \). Then, \( u - P_u = C(v - P_v) \) where \( C = C_2/C_1 \), so that
\[
u = C_1 [u(x,t) - P_u(x,t)], \text{ and } u - C\nu = P_u - C P_v \in D'.
\]
Therefore, \( u - C\nu \) must take on zero boundary values on \( B \) and, in particular, at \((y_o,s_o)\). Since \( u \) and \( v \) are kernel functions in \( D \), \( u - C\nu \) has zero boundary values on \( \partial D - \{ (y_o,s_o) \} \) also, so, by the maximum principle in \( D \), \( u = C\nu \). Because \( u(X,T) = \nu(X,T) = 1 \), we have \( u = v \) in \( D \). Thus, we obtain the uniqueness of kernel functions in the more general domain \( D \).

As was seen in the discussion of the existence of kernel functions, a "corner" point \((\eta_1(0),0)\) of \( \partial D \) may be treated as a side point of a new domain formed by extending \( D \) below \( t=0 \). This observation holds true for the uniqueness argument as well, so we need only consider the uniqueness problem for kernel functions at points of the form either \((y_o,0)\) with \( \eta_1(0) < y_o < \eta_2(0) \) or \((\eta_1(s_o),s_o)\) with \( s_o > 0 \). Again, it is sufficient to consider \( i = 1 \).

We next show that it is enough to prove our result for regions \( D_T \) having a still more restricted form.
Definition. A region \(D_T\) in the \((x,t)\) plane bounded by curves \(x=\eta_1(t)\), \(t=0\), \(x=\eta_2(t)\), and \(t=T\), with \(\eta_1(t)<\eta_2(t)\) and \(T>0\), where \(\eta_1(t)\) satisfies a Lipschitz condition with exponent \(1/2\), is \textit{parabolically starlike at} \((X,T)\), with \(\eta_1(T)<X<\eta_2(T)\), if, for each \((y,s)\) \(\in \partial D_T\)
there exists a finite parabolic ray with vertex \((y,s)\) and endpoint \((X,T)\) which is contained in \(D_T\). (We allow the degenerate case in which the parabolic arc becomes a verticle line segment.)

We also require the notion of a parabolic cone.

Definition. \(\Gamma\) is a \textit{parabolic cone} with vertex \((x_0,t_0)\) if one of the following equalities holds for a pair of constants \(C,C'>0\):

\[
\Gamma = \{(x,t) : C > x - x_0 > C' \ |t - t_0|^{1/2}\} \quad \text{or} \\
\Gamma = \{(x,t) : -C < x - x_0 < -C' \ |t - t_0|^{1/2}\} \quad \text{or} \\
\Gamma = \{(x,t) : C > t - t_0 > C' (x - x_0)^2\}.
\]

Since the functions \(\eta_1(t)\) in the definition of our general domain \(D\) satisfy a Lipschitz condition with exponent \(1/2\), it is clear that, for each \((y,s)\) \(\in \partial D\) which is not of the form \((\eta_1(0),0)\), there exists a parabolic cone \(\Gamma\) with vertex \((y,s)\) such that \(\Gamma \subset D\) and \(\Gamma \cap \partial D = \{(y,s)\}\). In fact, for a given \((y_0,s_0)\) \(\in \partial D\), with \((y_0,s_0)\) not of the form \((\eta_1(0),0)\), there is a neighborhood of \((y_0,s_0)\) in \(\partial D\) and a given size and orientation of parabolic cone such that each point in the neighborhood is the vertex of a parabolic cone of that type which lies in \(D\). Furthermore, the union of these cones, intersected with a small rectangle centered at \((y_0,s_0)\), is a parabolically starlike region. This is obvious if \((y_0,s_0) = (y,0)\) with \(\eta_1(0) < y < \eta_2(0)\). If \((y_0,s_0) = (\eta_1(s_0),s_0)\), with \(s_0 > 0\), first choose \(\delta < \min (s_0,T-s_0)\). We also require that

\[
\delta < \frac{1}{2}(\eta_2 - \sup_{|t-s_0|<\delta} \eta_1(t)).
\]

We also know that there is a constant \(C > 0\) such that

\[
|\eta_1(t) - \eta_1(t')| \leq C |t - t'|^{1/2} \quad \text{for} \quad t,t' \in [s_0-\delta/2,s_0+\delta/2].
\]

Let \(\Gamma(y,s) = \{(x,t) : x - y \geq 2C \ |t - s|^{1/2}\}\) and, for \(\delta_0 < \frac{\delta^2}{12 C^2}\),

\[
\text{set } D^* = \bigcup_{|s-s_0|<\delta_0} \Gamma(\eta_1(s),s) \cap \{(x,t) : x<y+\delta \text{ and } |t-s_0|<\delta_0\}.
\]
We have $D^* \subset D$ and $B = \{ (\eta_1(s), s) : |s-s_0| < \delta_0 \} \subset \partial D \cap \partial D^*$. (See Figure 4.)

We note next that $\cap \Gamma(\eta_1(s), s) \supset \cap \Gamma(\sup_{|t-s_0| < \delta_0} \eta_1(t), s)$, and that the intersection of $\{ (x,t) : t = s_0 + \delta_0 \}$ with the second set above contains $\{ (x,t) : t = s_0 + \delta_0 \} \cap \Gamma(\sup_{|t-s_0| < \delta_0} \eta_1(t), s_0 - \delta_0)$. Because we have

$$\sup_{|t-s_0| < \delta_0} \eta_1(t) + C (2\delta_0)^{1/2} \leq y_0 + C \delta_0^{1/2} + 2C \delta_0^{1/2} < y_c + \delta,$$

there is a point $(x_0, s_0 + \delta_0)$ which is in $\Gamma(\eta_1(s), s)$ for each $s \in [s_0 - \delta_0, s_0 + \delta_0]$. Clearly, we may then construct a parabolic arc with vertex $(\eta_1(s), s)$ and endpoint $(x_0, s_0 + \delta_0)$ which remains in $D^*$. For other points $(y, s)$ which are in $\partial P$ this observation is immediate. Thus, $D^*$ is parabolically starlike.

Repeating a previous argument, the kernel function at $(y_0, s_0)$ in $D$ is unique if the kernel function at $(y_0, s_0)$ in the (starlike) region $D^*$ is unique. (Note that this same argument reveals that uniqueness of the kernel function at a point $(y_0, 0)$, with $\eta_1(0) < y_0 < \eta_2$, follows from the uniqueness of the kernel function at the center bottom point of a rectangle.)

We now state the final lemma which is necessary for our uniqueness result.

**Lemma 1.5.** Let $D_T$ be parabolically starlike at $(X, T)$ with parabolic arcs from $(y_0, s_0) \in \partial D_T$ to $(X, T)$ given by $t-s_0 = \gamma_0 (x-y_0)^2$. Suppose that $\Gamma = \{ (\eta_1(s), s_0) : 0 \leq s \leq s_0 \} \in (0, T)$. Then there is a constant $C > 0$ such that, for $(y_0, s_0) \in \Gamma$, if $u(x, t)$ is any kernel function at $(y_0, s_0)$ in $D$ with respect to $(X, T)$, we have

$$u(x, t) \geq C K(x, t) \text{ for } (x, t) \in D,$$

where $K(x, t)$ is a kernel function at $(y_0, s_0)$ given by the limit of a convergent sequence

$$\frac{\omega(x, t)(\Delta_n)}{\omega(X, T)(\Delta_n)},$$

for $\Delta_n = \Delta((y_0, s_0), r_n)$ with $r_n \to 0$ as $n \to \infty$. 

Figure 4.
Proof. (Recall that the existence of $K(x,t)$ has been established. Also, we may assume that $\eta_2(t) = \eta_1$ in the definition of $D$.)

Since the result is trivially true for $t < s_o$, we may as well assume that $(y_o, s_o) = (\eta_1(0), 0)$ and, without loss of generality, we will suppose that $\eta_1(0) = 0$. Thus, we take $(y_0, s_0) = (0, 0)$.

If $r > 0$, define $\beta = \beta(r) = 1 - \frac{1}{2}z(3(1+\mu))^{1/2}y_o^{-1/2}r$ and set

$$u_r(x,t) = u(Q_r(x,t)) = u(\beta x + (1-\beta)y + Br^{1/2}(y^2 + y_o(1-\beta)^2)),$$

where $B$ is to be chosen later. Note that $u_r$ is a temperature in $Q_r^{-1}(D)$. Now, define $D^r = D_{x + (1-\beta)y + Br^{1/2}(y^2 + y_o(1-\beta)^2)} \subseteq D$.

We show first that, for small $r$, $u_r$ is defined in $D^r$; i.e., $Q_r(D^r) \subseteq D$.

It is easily seen that $Q_r(D^r) \subseteq \{(x,t): x < \eta_1(t), t > 0\}$. To prove that $Q_r(D^r) \subseteq \{(x,t): x > \eta_1(t)\}$, we must show

$$\beta \eta_1(s) + (1-\beta) + Br^{1/2} > \eta_1(y^2 + y_o(1-\beta)^2).$$

The Lipschitz condition on $\eta_1(t)$ implies that there is a constant $C > 0$ such that

$$\eta_1(s) - \eta_1(y^2 + y_o(1-\beta)^2) \geq -C(1-\beta)^2s + y_o(1-\beta)^2)^{1/2}.$$

Thus, it suffices to show

$$(\beta - 1)\eta_1(s) + (1-\beta) + Br^{1/2} - C[(1-\beta)^2s + y_o(1-\beta)^2]^{1/2} > 0.$$

Since $\Gamma$ is bounded away from $t = T$, there are constants $\gamma, \gamma' > 0$ such that $\gamma \leq y_o \leq \gamma'$ for $y_o$ corresponding to any $(y_o, s_o) \in \Gamma$.

For small $r$, we have

$$C[(1-\beta)^2s + y_o(1-\beta)^2]^{1/2} \leq C[2T(3(1+\mu)/\gamma)^{1/2}r + \gamma'(3(1+\mu)/\gamma)r^2]^{1/2} < C_1r^{1/2},$$

and

$$(1-\beta)\eta_1(s) < (3(1+\mu)/\gamma)^{1/2}Cr^{1/2} < C_2r^{1/2}$$

where $C_1$ and $C_2$ are independent of $(y_o, s_o) \in \Gamma$. Choosing $B$ sufficiently large, we then have $Q_r(D^r) \subseteq D$.

Thus, $u_r$ is a temperature in $D^r$. Furthermore, $u_r$ is continuous in $D^r$ since the only discontinuity of $u_r$ in $D_{x + (1-\beta)y + Br^{1/2}(y^2 + y_o(1-\beta)^2)}$ occurs at the
point \( Q^{-1}_=(0,0) = (-1+\beta+Br^{1/2})/\beta, -\nu o(1-\beta)^2/\beta^2 \), which has negative t-coordinate. So, for \((x,t) \in D^r\),

\[
u(x,t) = \int_{D^r} u(y,s) \omega(x,t)(y,s) \] 

Since \( \Delta_r = \Delta \left((y_o,s_o), r\right) \subset D^r \), we have

\[
u(x,t) \geq \inf_{(y,s) \in \Delta_r} u(y,s) \cdot \omega(x,t) (\Delta_r).
\]

We next show that \( \inf_{(y,s) \in \Delta_r} u(y,s) \geq C u(A_r) \), where \( A_r = \Delta \left((y_o,s_o), r\right) \) and \( C \) is a constant independent of \((y_o,s_o) \in \Gamma\) and independent of \( r \), for \( r \) sufficiently small. We have

\[
\inf_{(y,s) \in \Delta_r} u(y,s) = \inf_{(x,t) \in Q_r(\Delta_r)} u(x,t),
\]

and, if \((y,s) \in \Delta_r\), \( Q_r(y,s) \) has t-coordinate equal to

\[
\beta^2 s + \nu o(1-\beta)^2 \geq \beta^2 r^2 + 3(1+\mu)r^2 \geq (2+3\mu)r^2.
\]

Recalling that the t-coordinate of \( A_r \) is \((1+\mu)r^2\), Harnack's inequality will give the desired result if the x-coordinate of \( Q_r(y,s) \) remains bounded away from \( \partial D \) by some fixed multiple of \( r \) as long as \((y,s) \in \Delta_r\). Since \( Q_r(y,s) = Q_r(\eta_1(s),s) = (\beta \eta_1(s) + (1-\beta) + Br^{1/2}, \beta^2 s + \nu o(1-\beta)^2) \), we require that

\[
\beta \eta_1(s) + (1-\beta) + Br^{1/2} - \eta_1(\beta^2 s + \nu o(1-\beta)^2) \geq C' r.
\]

By the Lipschitz condition on \( \eta_1(t) \),

\[
\eta_1(\beta^2 s + \nu o(1-\beta)^2) - \eta_1(s) \leq C |(\beta^2-1)s + \nu o(1-\beta)^2|^{1/2},
\]

so it suffices to show

\[
(\beta-1) \eta_1(s) + (1-\beta) + Br^{1/2} - C |(\beta^2-1)s + \nu o(1-\beta)^2|^{1/2} \geq C' r,
\]

which follows from our choice of \( B \) above. Thus, Harnack's inequality does apply to show

\[
\inf_{(y,s) \in \Delta_r} u(y,s) \geq C u(A_r), \text{ and it follows that } u(x,t) \geq C u(A_r) \omega(x,t)(\Delta_r) \text{ for } (x,t) \in D^r. (\star)
\]
By Lemma 1.4, there is a constant $C_0 > 0$ such that

$$1 = u(X,T) \leq C_0 u(A_r) \omega_D(X,T)(\Delta_r),$$

because $u$ is a kernel function at $(y_0, s_0)$. Combining this with (*), we have

$$u_r(x,t) \geq C \frac{\omega_D(x,t)(\Delta_r)}{\omega_D(X,T)(\Delta_r)} \text{ for } (x,t) \in D^r,$$

where $C$ is another constant. We know there is a sequence $r_n$ tending to 0 such that

$$K(x,t) = \lim_{n \to \infty} \frac{\omega_D(x,t)(\Delta_{r_n})}{\omega_D(X,T)(\Delta_{r_n})}$$

is a kernel function at $(y_0, s_0)$ in $D$ with respect to $(X,T)$. By the maximum principle,

$$\omega_D(x,t)(\Delta_r) \geq \omega_D(x,t)(\Delta_r) - \sup_{(z,u) \in \partial D \cap D} \omega_D(z,u)(\Delta_r) \text{ for } (x,t) \in D^r.$$ 

Therefore,

$$u_r(x,t) \geq C \frac{\omega_D(x,t)(\Delta_r)}{\omega_D(X,T)(\Delta_r)} - C \sup_{(z,u) \in \partial D \cap D} \frac{\omega_D(z,u)(\Delta_r)}{\omega_D(X,T)(\Delta_r)}$$

for $(x,t) \in D^r$.

As $n \to \infty$, $D^{r_n} \to D_T$ and $u_{r_n} \to u$, so to prove that $u \geq CK$ in $D_T$, it suffices to show

$$\sup_{(z,u) \in \partial D^{r_n} \cap D} \frac{\omega_D(z,u)(\Delta_{r_n})}{\omega_D(X,T)(\Delta_{r_n})} \to 0 \text{ as } n \to \infty.$$

However, by Lemma 1.4, we have

$$\frac{\omega_D(z,u)(\Delta_{r_n})}{\omega_D(X,T)(\Delta_{r_n})} \leq C_0 \omega_D(z,u)(\Delta_{r_0}) \text{ for } (z,u) \in D - \psi((y_0, s_0), (1+\mu)^{1/2}/4),$$

where $r_0 > 0$ is fixed and $n$ is sufficiently large that $r_n < r_0$. (Note that Harnack's inequality has been used to compare $A_{r_0}$ and $(X,T)$.)
Since $\partial D^n \cap D \subset D - \psi((y_o, s_o), (1+u)^{1/2})/4$ for large $n$ and

$$\sup_{(z,u) \in \partial D^n \cap D} \omega_D(z,u)(\Delta_{\partial D}) \to 0 \text{ as } n \to \infty,$$

we obtain the desired result in $D_T$. Then, by the maximum principle,

$$u(x,t) \geq C \, K(x,t) \text{ in all of } D.$$

Q.E.D.

We can now prove that kernel functions are unique.

**Theorem 1.6.** If $(y_o, s_o) = (\eta_1(s_o), s_o) \in \partial D$, with $0 < s_o < T$,
then there is a unique kernel function at $(y_o, s_o)$ in $D$ with respect to $(X,T)$, $\eta_1(T) < X < \eta_2(T)$.

**Proof.** (We have seen that it suffices to consider parabolically
starlike regions $D_T$ with $\eta_2(t) = \eta_2(\cdot)$.)

We know there is a kernel function $K(x,t)$ at $(y_o, s_o)$ in $D$
with respect to $(X,T)$ and, by Lemma 1.5, if $u(x,t)$ is any other
kernel function at $(y_o, s_o)$ in $D$ with respect to $(X,T)$, then
$u \geq CK$ in $D$, where $C$ is independent of $u$.

Let $C_o = \sup\{C: u \geq CK \text{ in } D \text{ for every kernel function } u \text{ at } (y_o, s_o)\}$. Obviously, $u \geq C_o \, K$ for every kernel function $u$ at $(y_o, s_o)$. It is
clear, too, that $C_o \leq 1$, since we may take $u = K$. If $C_o = 1$, and
$u$ is any kernel function at $(y_o, s_o)$ (always, with respect to $(X,T)$),
then $u = K$ by the strong maximum principle in $D$, as $u(X,T) = 1 = K(X,T)$.

Suppose $C_o < 1$. If $u$ is a kernel function at $(y_o, s_o)$, then

$$u' = \frac{u - C_o K}{1 - C_o}$$

is also a kernel function at $(y_o, s_o)$.

Thus, $u' \geq C_o \, K$ in $D$, so that $u \geq (2C_o - C_o^2)K$ in $D$. But, $2C_o - C_o^2 > C_o$,
contradicting our assumption that $C_o$ was maximal. Therefore, $C_o = 1$
and the kernel function must be unique. Q.E.D.

**Corollary 1.7.** The unique kernel function at $(y_o, s_o) = (\eta_1(s_o), s_o)$
with respect to $(X,T) \in D$, with $0 < s_o < T$, is

$$K(x,t,y_o,s_o) = \frac{d\omega(x,t)}{d\omega(X,T)(y_o,s_o)} = \lim_{r \to \infty} \frac{\omega(x,t)(\Delta((y_o,s_o),r))}{\omega(X,T)(\Delta((y_o,s_o),r))}.$$
Proof. Let \( V_n \) be any sequence of neighborhoods of \((y_0, s_0)\) in \(\partial D\) which converge to \((y_0, s_0)\) and satisfy
\[
\inf_{n} \frac{\omega(X, T)(V_n)}{\omega(X, T)(B_n)} > 0,
\]
where \(B_n = \partial D \cap [\text{the smallest disk containing } V_n]\).

As in the proof of the existence of kernel functions, Lemma 1.4 shows that, for some subsequence \( V'_n \) of \( V_n \), the corresponding functions,
\[
\frac{\omega(x, t)(y'_n)}{\omega(X, T)(y'_n)},
\]
converge to a kernel function at \((y_0, s_0)\). Since there is only one kernel function at \((y_0, s_0)\) with respect to \((X, T)\), any such convergent sequence must have the same limit, \(K(x, t, y_0, s_0)\). From Besicovitch's theory the limit of such a sequence is also equal to the Radon-Nikodym derivative of \(\omega(x, t)(\cdot)\) with respect to \(\omega(X, T)(\cdot)\) at a.e. \((x, t)(y_0, s_0) \in \partial D\). Redefining that derivative on a set of zero caloric measure, we have
\[
K(x, t, y_0, s_0) = \frac{d\omega(x, t)}{d\omega(X, T)(y_0, s_0)} \text{ for every } (y_0, s_0). \quad \text{Q.E.D.}
\]

Corollary 1.8. \(K(x, t, y, s)\) is a continuous function of \((y, s)\) for \((y, s) \in \partial D_T\).

Proof. Let \((y_0, s_0) \in \partial D_T\) and suppose \((y_n, s_n) \to (y_0, s_0)\) in \(\partial D\). For sufficiently small \(r > 0\), the kernel functions \(K(x, t, y_n, s_n)\) satisfy the hypothesis of Lemma 1.4 for large \(n\). As before, some subsequence converges to a kernel function at \((y_0, s_0)\). By Theorem 1.6, this limit is \(K(x, t, y_0, s_0)\). Since this is true for any sequence \((y_n, s_n)\) which approaches \((y_0, s_0)\), we have
\[
\lim_{(y, s) \to (y_0, s_0)} K(x, t, y, s) = K(x, t, y_0, s_0). \quad \text{Q.E.D.}
\]

Remarks. We have given details above only for points \((y_0, s_0) \in \partial D\) of the form \((\eta_1(s_0), s_0)\), with \(s_0 > 0\). Of course, the technique also applies to points \((\eta_2(s_0), s_0)\). For a point \((y_0, 0) \in \partial D\) with \(\eta_1(0) < y_0 < \eta_2(0)\), a simplified argument can be used. In this case we have already seen that it suffices to prove uniqueness.
for the center bottom point of a rectangular region $R$. The representation theorem of Hartman and Wintner [7] for non-negative temperatures in a rectangle leads immediately to uniqueness. We can also construct a proof of uniqueness in this case along lines similar to Lemma 1.5 by defining, for an arbitrary kernel function $u$ at $(y_o,0)$ in $R$ with respect to $(X,Y)$, approximating functions

$$u_r(x,t) = u(y_o + \beta(x-y_o), \beta^2 t + (1-\beta)^2),$$

where $\beta = \beta(r) = 1 - [3(1+\mu)]^{1/2} r$ for small $r$. $u_r$ is continuous in $\bar{R}$ and a solution of the heat equation in $R$, so that

$$u_r(x,t) = \int_{\partial R} u_r(y,s) \, d\omega_r(x,t)(y,s)$$

$$\geq \inf_{y \in \Delta} u_r(y,0) \, \omega_r(x,t)(\Delta)$$

$$\geq C \, u(A) \, \omega_r(x,t)(\Delta)$$

by Harnack's inequality,

where $\Delta = \Delta((y_o,0),r)$ and $A = A((y_o,0),r)$. Consider a sequence $\Delta_n = \Delta((y_o,0),r_n)$ with $r_n \to 0$ such that

$$\frac{\omega_r(x,t)(\Delta_n)}{\omega_r(x,t)(\Delta_n)}$$

converges in $R$. Applying Lemma 1.4 at the point $(X,T)$, and letting $n \to \infty$, we obtain $u \geq CK$, where $K$ is the kernel function at $(y_o,0)$ given by

$$\lim_{n \to \infty} \frac{\omega_r(x,t)(\Delta_n)}{\omega_r(x,t)(\Delta_n)}.$$

We can now prove a representation theorem for non-negative temperatures in our general region $D_T$.

**Theorem 1.9.** If $u$ is a non-negative temperature in $D_T$, $0 < T < \infty$, then there is a unique regular Borel measure $\mu$ on $\partial D_T$ such that

$$u(x,t) = \int_{\partial D_T} K(x,t,y,s) \, d\mu(y,s),$$

where $K(x,t,y,s)$ is the kernel function at $(y,s)$ in $D$ with respect to the fixed point $(X,T).$
Proof. (Recall that $D$ is bounded by curves $x = \eta_1(t)$, $t = 0$, and $x = \eta_2(t)$ for $t > 0$, where $\eta_1(t)$ satisfies a Lipschitz condition with exponent $1/2$ on any interval $[0, T]$, $0 < T < \infty$. Also, $D_T = D \cap \{(x, t): t < T\}$.)

For any relatively closed subset $B$ of $D_T$, define, for $(x, t) \in D$,

$$R_u^B(x, t) = \inf \{\tau(x, t): \tau \geq 0, \tau \text{ is supercaloric in } D_T \text{ and } \tau \geq u \text{ on } B\}.$$  

(A supercaloric function is a supersolution of the heat equation.) Because $u$ is a non-negative temperature in $D_T$, $R_u^B(x, t) = u(x, t)$ in $B$ and, in $D_T - B$, $R_u^B(x, t)$ is equal to the Wiener solution of the Dirichlet problem for the heat equation with boundary values $u$ on $\partial B \cap D_T$ and zero on $\partial p D_T - B$.

Next, for a closed subset $F$ of $\partial_D D_T$ and for $(x, t) \in D_T$, define

$$\overline{m}(x, t)(F) = \overline{m}(x, t)(F) = \inf \{R_u^D(x, t): U \text{ is open in } \mathbb{R}^2 \text{ and } F \subseteq U\}.$$  

For any sequence of open sets $U_i$ which decrease to $F$, we have

$$\overline{m}(x, t)(F) = \lim_{i \to \infty} R_u^{U_i \cap D}(x, t).$$

Thus, by Harnack's monotone convergence theorem, $\overline{m}(x, t)(F)$ is seen to be a non-negative temperature in $D_T$. If $F_1 \subseteq F_2$ are closed subsets of $\partial_D D_T$, then $\overline{m}(x, t)(F_1) \leq \overline{m}(x, t)(F_2)$. Now suppose that $F_i$ is a sequence of closed subsets of $\partial_D D_T$, and, for each $i$, let $U_i$ be an open set in $\mathbb{R}^2$ such that $F_i \subseteq U_i$. Since

$$R_u^{U_i \cap D}(x, t) \leq \sum_{j} R_u^{U_j \cap D}(x, t),$$

it follows that $\overline{m}(x, t)(\cdot)$ is subadditive on closed sets of $\partial_D D_T$.

Finally, we will show that $\overline{m}(x, t)(\cdot)$ is finitely additive on disjoint closed sets. Let $F_1$ and $F_2$ be disjoint closed subsets of $\partial_D D_T$, and suppose that $U_i$ and $V_i$ are sequences of open sets in $\mathbb{R}^2$ such that

$$\overline{m}(x, t)(F_1) = \lim_{i \to \infty} R_u^{U_i \cap D}(x, t),$$

$$\overline{m}(x, t)(F_2) = \lim_{i \to \infty} R_u^{V_i \cap D}(x, t),$$

and

$$\overline{m}(x, t)(F_1 \cup F_2) = \lim_{i \to \infty} R_u^{U_i \cup V_i \cap D}(x, t).$$
We can assume, by the monotonicity of \( \mu(x,t)(\cdot) \), that \( U_i \) decreases to \( F_1 \) and \( V_i \) decreases to \( F_2 \).

Let \( \varepsilon > 0 \). Since \( \overline{U_i \cap D}(x,t) \) is a Wiener solution for the heat equation in \( D_T - \overline{U_i} \) corresponding to boundary values which vanish on \( \partial D_T - \overline{U_i} \), and since each point of \( F_2 \) is regular, we must have

\[
\overline{R_{U_i \cap D}(x,t)} \leq \varepsilon \quad \text{for} \quad (x,t) \in V_i \quad \text{if} \quad i \text{ is large enough. Therefore,}
\]

\[
\overline{\mu}(x,t)(F_1) \leq \varepsilon \quad \text{for} \quad (x,t) \in V_i \cap D.
\]

Consider

\[
\tau_i(x,t) = R_{U_i \cap V_i \cap D}(x,t) - \overline{\mu}(x,t)(F_1) + \varepsilon.
\]

We have \( \tau_i(x,t) \geq R_{U_i \cap V_i \cap D}(x,t) \geq u(x,t) \) for \( (x,t) \in V_i \cap D \), and,

since \( R_{U_i \cap V_i \cap D}(x,t) \geq \overline{\mu}(x,t)(F_1) \), \( \tau_i > 0 \) in \( D_T \). Furthermore, \( \tau_i \) is a supercaloric function in \( D_T \), since \( R_{U_i \cap V_i \cap D}(x,t) \) is and \( \overline{\mu}(x,t)(F_1) \) is a temperature. Thus,

\[
\tau_i(x,t) \geq \overline{\mu}(x,t)(F_2), \text{ or}
\]

\[
R_{U_i \cap V_i \cap D}(x,t) \geq \overline{\mu}(x,t)(F_1) + \overline{\mu}(x,t)(F_2) - \varepsilon, \text{ for large } i.
\]

Letting \( i \to \infty \), we have

\[
\overline{\mu}(x,t)(F_1 \cup F_2) \geq \overline{\mu}(x,t)(F_1) + \overline{\mu}(x,t)(F_2) - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we can combine this with the subadditivity of \( \overline{\mu} \) to obtain

\[
\overline{\mu}(x,t)(F_1 \cup F_2) = \overline{\mu}(x,t)(F_1) + \overline{\mu}(x,t)(F_2).
\]

We have shown that \( \overline{\mu}(x,t)(\cdot) \) is a content on \( \partial D_T \) (a non-negative, monotone, subadditive function on the closed sets of \( \partial D_T \) which is additive on disjoint closed sets). Moreover, \( \overline{\mu}(x,t)(\cdot) \) is a regular content:

for \( F \) a closed subset of \( \partial D_T \),

\[
\overline{\mu}(x,t)(F) = \inf \left\{ \overline{\mu}(x,t)(G) : G \text{ is a closed subset of } \partial D_T, F \subseteq G^o \right\}.
\]
This follows immediately from the definition of \( \overline{\mu}(x,t)(\cdot) \). A known result (see Halmos [6]) now implies that \( \overline{\mu} \) can be extended to a regular Borel measure \( \mu(x,t)(\cdot) \) on \( \partial_p D_T \).

By Harnack's inequality, \( \mu(x,t) \) is absolutely continuous with respect to \( \mu(X,T) \) if \( (x,t) \in D_T \). Taking \( \Delta_r = \Delta((y,s),r) \) for \( (y,s) \) in \( \partial_p D_T \), it follows from Lemma 1.4 and the uniqueness of kernel functions that

\[
K(x,t,y,s) = \lim_{r \to 0} \frac{\mu(x,t)(\Delta_r)}{\mu(X,T)(\Delta_r)} = \frac{d\mu(x,t)}{d\mu(X,T)(y,s)}.
\]

Therefore,

\[
u(x,t) = \mu(x,t)(\partial_p D_T) = \int_{\partial_p D_T} d\mu(x,t)(y,s) = \int_{\partial_p D_T} K(x,t,y,s) \, d\mu(X,T)(y,s)
\]

and we have a representation for \( \nu \).

To see that this representation is unique, let \( \nu \) be another regular Borel measure on \( \partial_p D_T \) such that

\[
u(x,t) = \int_{\partial_p D_T} K(x,t,y,s) \, d\nu(y,s).
\]

For a closed subset \( F \) of \( \partial_p D_T \), choose a sequence \( G_k \) of open sets in \( \mathbb{R}^2 \) such that \( F = \cap G_k \) and

\[
u(X,T)(F) = \lim_{k \to \infty} R_{\nu}^{G_k \cap D}(X,T).
\]

Let \( \omega_k \) denote caloric measure in \( D - G_k \) and let \( H_k \) be the parabolic boundary of \( D_T - G_k \). For \( (x,t) \in D_T - G_k \),

\[
R_{\nu}^{G_k \cap D}(x,t) = \int_{D\cap H_k} u(y,s) \, d\omega_k(x,t)(y,s)
\]

\[
= \int_{D\cap H_k} [ \int_{\partial_p D_T} K(y,s,z,u) \, d\nu(z,u) ] \, d\omega_k(x,t)(y,s)
\]

\[
= \int_{\partial_p D_T} [ \int_{D\cap H_k} K(y,s,z,u) \, d\omega_k(x,t)(y,s) ] \, d\nu(z,u) \text{ (Fubini)}.
\]
If \((z,u) \in F\), \(K(y,s,z,u)\) is a temperature \((y,s)\) in \(D_T - \overline{G_k}\) which is continuous in the closure of that region. For such \((z,u)\),

\[
\int_{\partial D_T} \omega_k \d K(y,s,z,u)(x,t)(y,s) = K(x,t,z,u).
\]

If \((z,u) \in \partial D_T - F\),

\[
\lim_{k \to \infty} \int_{\partial D_T} \omega_k \d K(y,s,z,u)(x,t)(y,s) = 0,
\]

since \(K(y,s,z,u) \to 0\) as \((y,s) \to (y_0,s_0) \in \partial D - \{(z,u)\}\) and \((z,u) \notin \partial G_k\) for large \(k\).

By the maximum principle,

\[
\int_{\partial D_T} \omega_k \d K(y,s,z,u)(x,t)(y,s) \leq K(x,t,z,u).
\]

Therefore, by Lebesgue's theorem,

\[
\mu(X,T)(F) = \lim_{k \to \infty} \int_{\partial D_T} \omega_k \d K(y,s,z,u)(x,t) \d v(z,u)
\]

\[
= \lim_{k \to \infty} \int_{\partial D_T} \left[ \int_{\partial D_T} \omega_k \d K(y,s,z,u)(x,t)(y,s) \right] \d v(z,u)
\]

\[
= \int_{F} K(X,T,z,u) \d v(z,u)
\]

\[
= \nu(F).
\]

Since \(F\) was an arbitrary closed subset of \(\partial D_T\) and \(\mu(X,T)\), \(\nu\) are both regular Borel measures on \(\partial D_T\), we have \(\nu = \mu(X,T)\), and the uniqueness of the representation is assured.

Q.E.D.
2. **Existence of Parabolic Limits**

In this section we use the representation theorem of section 1 to obtain existence of parabolic limits almost everywhere \( (\omega(X,T)) \) on \( \partial_p D_T \) for non-negative temperatures in \( D_T \).

**Definition.** A function \( u(x,t) \) defined in \( D_T \) has *parabolic limit* \( L \) at a point \( (y,s) \in \partial_p D_T \) if, for each parabolic cone \( \Gamma \subset D \) with vertex \( (y,s) \) which opens away from \( \partial D \) and satisfies \( \Gamma \cap \partial_p D_T = \{ (y,s) \} \), we have

\[
\lim_{(x,t) \rightarrow (y,s)} u(x,t) = L.
\]

(For \( (y,s) = (\eta_1(s),s) \), \( \Gamma \) opens away from \( \partial D \) if \( \Gamma \subset \{ (x,t): x > \eta_1(s) \} \).

We first establish several lemmas.

**Lemma 2.1.** Let \( (y_o,s_o) \in \partial D \) and let \( \Delta = \Delta((y_o,s_o),r) \). If \( r \) is sufficiently small, then

\[
\sup_{(y,s) \in \partial D - \Delta} K(x,t,y,s) \rightarrow 0 \text{ as } (x,t) \rightarrow (y_o,s_o) \text{ in } D.
\]

**Proof.** (For the case \( (y_o,s_o) = (\eta_1(s_o),s_o), s_o > 0 \).)

Since \( K(x,t,y,s) = 0 \) for \( s \geq T \), we need only consider points \( (y_o,s_o) \) in \( \partial_p D_T - \Delta \). Let \( \Sigma = \{ (x,t): x \geq -M|t|^{1/2}, |x| < 1, |t| < 1 \} \), and let \( h(x,t) \) denote the caloric measure in \( \Sigma \) of the linear part of \( \partial_p \Sigma \). Shrink \( \Sigma \) parabolically to have height \( r/2 \) and translate it so that the origin is moved to the point \( (y_o,s_o) \). If \( M \) is large, the cone \( \{ (x,t): x \leq -M|t|^{1/2} \} \), after the shrinking and translation, lies in the complement of \( D \).

For \( s_o < s < T \), \( K(x,t,y,s) = 0 \) in a neighborhood of \( (y_o,s_o) \), so we need only prove the lemma for \( (y,s) \in \Omega \), where \( \Omega \subset \partial_p D_T - \Delta \) is bounded away from \( t = T \).

By Lemma 1.4, if \( \Delta' = \Delta((y',s'),r') \) for \( (y',s') \in \Omega \) and \( r' < r/4 \), we have

\[
\omega(x,t)(\Delta') \leq C \omega^A((y',s'),r)(\Delta') \text{ for } (x,t) \in D - \Psi((y',s'),(1+\mu)^{1/2}/4),
\]

where \( C \) is independent of \( (y',s') \in \Omega \).
By Harnack's inequality,

\[ \omega^A((y',s'),r)(\Delta') \leq C \omega^T(x,T)(\Delta'), \]

with \( C = C(r,\Omega,D) \). Hence,

\[ \omega(x,t)(\Delta') \leq C \omega^T(x,T)(\Delta') \text{ for } (x,t) \in \Omega - \psi((y',s'),(1+\mu)^{1/2}r/4). \]

If \( \Sigma_r \) denotes the transformed region \( \Sigma \), then \( D \cap \Sigma_r \subset D - \psi_r' \), where \( \psi_r' = \psi((y',s'),(1+\mu)^{1/2}r/4) \). The maximum principle in \( D \cap \Sigma_r \) then implies that

\[ \omega(x,t)(\Delta') \leq C \omega^T(x,T)(\Delta') h_r(x,t) \text{ for } (x,t) \in D \cap \Sigma_r, \]

where \( h_r \) is the caloric measure in \( \Sigma_r \) corresponding to \( h \) in \( \Sigma \).

Since \( h_r(x,t) \to 0 \) as \( (x,t) \to (y_o,s_o) \), we see that

\[ \frac{\omega(x,t)(\Delta')}{\omega^T(x,T)(\Delta')} \to 0 \text{ as } (x,t) \to (y_o,s_o) \text{ independent of } (y',s') \in \Omega. \]

Therefore,

\[ \sup_{(y,s) \in \Omega} K(x,t,y,s) \to 0 \text{ as } (x,t) \to (y_o,s_o). \quad \text{Q.E.D.} \]

**Lemma 2.2.** Let \( \Omega \subset \partial D_T \) be bounded away from \( \{(\eta_2(s),s) : 0 \leq s \leq T\} \). then

\[ \sup_{\{(x,t,y,s) : x=\eta_2(t)-\epsilon, t \leq T, (y,s) \in \Omega\}} K(x,t,y,s) \to 0 \]

as \( \epsilon \to 0 \).

**Proof.** Let \( D^{\epsilon_0} = D_T \cap \{(x,t) : x < \eta_2(t) - \epsilon_0\} \), for

\[ 0 < \epsilon_0 < \inf_{(y,s) \in \Omega} (\eta_2(s) - y). \]

Since \( \sup_{(y,s) \in \Omega} K(z,u,y,s) \to 0 \) as \( (z,u) \to (0,\eta_2(0)-\epsilon_0) \) (Lemma 2.1),

Harnack's inequality can be applied to obtain

\[ \sup_{(z,u) \in D_T \cap \partial D^{\epsilon_0}} K(z,u,y,s) \leq C K(\chi_{T+\epsilon_0},y,s) \text{ for } (y,s) \in \Omega. \]

Furthermore, by the continuity of \( K(\chi_{T+\epsilon_0},y,s) \) as a function of
\((y,s) \in \partial_p D_T\), there exists a constant \(M > 0\) such that

\[
\sup_{(y,s) \in \Omega} K(x, t+\varepsilon, y, s) \leq M.
\]

Thus,

\[
\sup_{(z,u) \in D_T \cap \partial_p D^\varepsilon} K(z,u,y,s) \leq CM.
\]

We have already seen that, for \((y,s) \in \Omega, K(z,u,y,s) \to 0\) as \((z,u) \to \partial_p D_T \cap \partial_p D^\varepsilon\). Consequently, the maximum principle implies that

\[
K(z,u,y,s) \leq M(z,u) \text{ for } (z,u) \in D_T \cap \partial_p D^\varepsilon,
\]

where \(M(z,u)\) is the unique bounded temperature in \(D_T \cap \partial_p D^\varepsilon\) with boundary values

\[
M(z,u) = \begin{cases} 
CM & \text{for } (z,u) \in D_T \cap \partial_p D^\varepsilon \\
0 & \text{for } (z,u) \in \partial_p D_T \cap \partial_p D^\varepsilon.
\end{cases}
\]

Clearly,

\[
\sup_{(z,u) \in D_T \cap \partial_p D^\varepsilon} M(z,u) \to 0 \text{ as } \varepsilon \to 0.
\]

Hence,

\[
\sup_{(z,u) \in D_T \cap \partial_p D^\varepsilon} K(z,u,y,s) \to 0 \text{ as } \varepsilon \to 0. \quad \text{Q.E.D.}
\]

**Lemma 2.2.** Let \(\Omega_0 \subset \partial_p D_T\) be bounded away from \(\{(\eta_2(s), s) : 0 \leq s \leq T\}\) and let \(D_r = D \cap \{(x,t) : x < \eta_2(t) - r\}\). Then there is a constant \(C > 0\) such that, if \(r\) is sufficiently small,

\[
\omega_{D_r}(\Omega) \geq C \omega_{D}(\Omega) \quad \text{for each measurable set } \Omega \subset \Omega_0.
\]

**Proof.** For sufficiently small \(r\), \((X,T) \in D_r\), and we have

\[
\omega_{D}(\Omega) - \omega_{D_r}(\Omega) = \int_{D \cap \partial_p D_r} \omega_D(x,y) \ d\omega_{D_r}(x,y) \\
= \int_{D \cap \partial_p D_r} \left[ \int_{D \cap \partial_p D_r} K(z,u,y,s) \ d\omega_D(x,y) \right] \ d\omega_{D_r}(x,y) \\
= \int_{D \cap \partial_p D_r} \left[ \int_{D \cap \partial_p D_r} K(z,u,y,s) \ d\omega_D(x,y) \right] \ d\omega_{D_r}(x,y) \\
= \int_{D \cap \partial_p D_r} \omega_{D_r}(y) \ d\omega_D(x,y) \\
= \int_{D \cap \partial_p D_r} \omega_{D_r}(y) \ d\omega_D(x,y) \\
= \int_{D \cap \partial_p D_r} \omega_{D_r}(y) \ d\omega_D(x,y) \\
= \int_{D \cap \partial_p D_r} \omega_{D_r}(y) \ d\omega_D(x,y) \\
= \int_{D \cap \partial_p D_r} \omega_{D_r}(y) \ d\omega_D(x,y) \\
= \int_{D \cap \partial_p D_r} \omega_{D_r}(y) \ d\omega_D(x,y)
\]

Since \(\omega_{D_r}(\Omega) \geq C \omega_{D}(\Omega)\) for each measurable set \(\Omega \subset \Omega_0\).
\[ \begin{align*}
= & \int \left[ \int_{\Omega} K(z,u,y,s) \, \omega_{Dr}^{(X,T)}(z,u) \right] \, \omega_D^{(X,T)}(y,s) \quad \text{(Fubini)} \\
= & \int_{\Omega} \left[ 1 - \theta_r(y,s) \right] \, \omega_D^{(X,T)}(y,s) \\
= & \omega_D^{(X,T)}(\Omega) - \int_{\Omega} \theta_r(y,s) \, \omega_D^{(X,T)}(y,s), \quad \text{where } \theta_r \text{ is given by} \\
\theta_r(y,s) = & 1 - \int_{D_r \cap \omega_p dr} K(z,u,y,s) \, \omega_{D_r}^{(X,T)}(z,u). \\
\text{Therefore,} \\
\omega_{D_r}^{(X,T)}(\Omega) = & \int_{\Omega} \theta_r(y,s) \, \omega_D^{(X,T)}(y,s). \\
\text{To complete the proof, we will show that } \theta_r(y,s) \geq C > 0 \text{ for} \\
r \text{ sufficiently small and for } (y,s) \in \Omega. \\
\text{For } (y_0,s_0) \in \Omega, \text{ we have} \\
\theta_r(y_0,s_0) = & 1 - \int_{D_r \cap \omega_p dr} K(z,u,y_0,s_0) \, \omega_{D_r}^{(X,T)}(z,u) \\
\geq & 1 - \sup_{(z,u) \in D_r \cap \omega_p dr, (y,s) \in \Omega} K(z,u,y,s). \\
\text{By Lemma 2.2, there is an } r_0 > 0 \text{ such that} \\
\sup_{(z,u) \in D_r \cap \omega_p dr, (y,s) \in \Omega} K(z,u,y,s) \leq 1/2 \quad \text{if } r < r_0. \\
\text{Thus, for } (y_0,s_0) \in \Omega \text{ and } r < r_0, \text{ we obtain} \\
\omega_{D_r}^{(X,T)}(\Omega) \geq & 1/2 \quad \omega_D^{(X,T)}(\Omega). \quad \text{Q.E.D.}
\end{align*} \]

Lemma 2.4. Suppose $D_T$ is parabolically starlike at $(X,T)$. Then there is a constant $C > 0$ such that, if $(y_0,s_0) = (\eta_1(s_0),s_0)$ with $0 < s_0 < T$, and $r$ is sufficiently small, we have

\[ K(A,y,s) \leq \frac{C}{\omega^{(X,T)}(\Delta((y_0,s_0),r))} \text{ for } (y,s) \in \Delta((y_0,s_0),r), \]

where $A = A((y_0,s_0),r)$. 
Proof. From equation (*) in the proof of Lemma 1.5,

$$K_r(x,t,y,s) \geq C K(A,y,s) \omega_D^{(x,t)}(\Delta((y_o,s_o),r))$$

for $(x,t) \in D^r$, where the "$r$" notation corresponds to that lemma. However, each set $D^r$ (in the notation of Lemma 1.5) is equal to a set $D^{r'}$ (in the notation of Lemma 2.2) and $r, r'$ tend to zero together.

Setting $(x,t) = (X,T)$ in the inequality above and applying Lemma 2.3 with $\Omega_o = \{ (\eta_1(s), s) : 0 < s < T \}$, we have, for small $r$,

$$K(A,y,s) \leq C K_r(X,T,y,s) / \omega_D^{(X,T)}(\Delta((y_o,s_o),r)).$$

By an argument found in the proof of Lemma 2.2, there is a constant $M > 0$ such that, for small $r$,

$$\sup_{(y,s) \in \Omega_o} K_r(X,T,y,s) \leq M.$$  

Q.E.D.

Remark. We can prove Lemma 2.4 for points $(y_o,0)$ at the bottom of a rectangle by considering $K_r(x,t,y,s) = K(x,t+(2+2)u_r^2,y,s)$. By the maximum principle and Harnack's inequality,

$$K_r(X,T,y,s) \geq C K(A,y,s) \omega^{(X,T)}(\Delta((y_o,0),r))$$

for small $r$.

We can now prove the final essential lemma.

Lemma 2.5. Suppose $D_T$ is parabolically starlike at $(X,T)$. Let $(y_o,s_o) = (\eta_1(s_o), s_o)$ for $0 < s_o < T$ and let $\Gamma$ be a parabolic cone in $D_T$ with vertex $(y_o,s_o)$ which opens away from $\partial D$. Let $(x',t') \in \Gamma$ satisfying $x' - y_o = rd$, where $d$ is the fixed parameter in the definition of $\psi((y_o,s_o),r)$, and let $\Delta_j = \Delta((y_o,s_o), 2^j r)$. If $R_o = \Delta_o$ and $R_j = \Delta_j - \Delta_{j-1}$ for $j=1,2,\ldots$ for sufficiently small $r$, we have

$$\sup_{(y,s) \in R_j} K(x',t',y,s) \leq \frac{C_{\Delta_j}}{\omega^{(X,T)}(\Delta_j)}, \quad j = 0,1,\ldots,N,$$

where $C = C(\Gamma,D)$ and $\sum_{j=0}^{\infty} C_j \leq C' = C'(D)$.

Proof. Let $A_j = A((y_o,s_o), 2^j r)$, $j = 0,1,\ldots,N$. By Harnack's inequality, there is a constant $C = C(\Gamma,D)$ such that

$$u(x',t') \leq C u(A_o)$$

for any non-negative temperature $u(x,t)$ in $D_T$. 

In particular,
\[ K(x',t',y,s) \leq C K(A_0,y,s) \text{ for } (y,s) \in \partial_p D_T. \]

By Lemma 2.4,
\[ \sup_{(y,s) \in \Delta_j} K(A_j, y, s) \leq C/\omega(X,T)(\Delta_j). \]

Moreover, for \( j = 0,1,2,3,4 \), say, there are constants \( C_j = C_j(D) \) such that \( K(A_0, y, s) \leq C_j K(A_j, y, s) \), by Harnack's inequality.

Thus,
\[ \sup_{(y,s) \in R_j} K(A_0, y, s) \leq \frac{CC_j}{\omega(X,T)(\Delta_j)} \text{ for } j = 0,1,2,3,4. \]

Now, let \((y_j,s_j) \in R_j\) for \( j > 4 \) and suppose \( \Delta \subset \Delta'_j = \Delta((y_j,s_j), r_j) \), where \( r_j = 2^{j-4}r \). Let \( A'_j = A((y_j,s_j), r_j) \). By Lemma 1.4,
\[ K(x,t,y_j,s_j) \leq C K(A'_j, y_j, s_j) \text{ for } (x,t) \in D - \Psi_j, \]
where \( \Psi_j = \Psi((y_j,s_j), (1+\mu)^{-1/2}r_j/4) \). Since
\[ (2^{j+1}r)^2(1+\mu) - [(2^j r)^2 + (2^{j-4}r)^2(1+\mu)] \geq r^2(2^{2j+2} - 2^{2j+1}) = 2^{2j+1}r^2, \]

Harnack's inequality can be applied to show
\[ K(A'_j, y_j, s_j) \leq C K(A_{j+1}, y_j, s_j). \]

For \((x,t) \in D - \Psi_j\), combining this with Lemma 2.4, we have
\[ K(x,t,y_j,s_j) \leq C K(A_{j+1}, y_j, s_j) \leq C/\omega(X,T)(\Delta_{j+1}) \leq C/\omega(X,T)(\Delta_j). \]

Now, let \( \Sigma = \{(x,t) : |x - y_o| < 1, |t - s_o| < 1\} \cap \{(x,t) : x - y_o > B |t - s_o|^{1/2}\} \),
where \( B \) is chosen so that \( \{(x,t) : x - y_o \leq -B |t - s_o|^{1/2}\} \) is contained in the complement of \( D \). Let \( \Sigma_j \) be the region formed by parabolically shrinking \( \Sigma \) by the factor \( 2^{j-3}r \) and let \( h_j \) denote the caloric measure in \( \Sigma_j \) of that part of \( \partial_p \Sigma_j \) which does not lie on the cone \( x - y_o = -B |t - s_o|^{1/2} \).

Since \( (2^{j-3}r)^2 < (2^{j-4}r)^2 + (1+\mu)(2^{j-4}r)^2 \), we have \( D \Sigma_j \subset D - \Psi_j \).

(See Figure 5.) The maximum principle in \( D \Sigma_j \) then shows that
\[ K(x,t,y_j,s_j) \leq C h_j(x,t)/\omega(X,T)(\Delta_j) \text{ for } (x,t) \in D \Sigma_j. \]

Setting \((x,t) = A_0\),
\[ K(A_0, y_j, s_j) \leq C h_j(A_0)/\omega(X,T)(\Delta_j). \]
Because $K(x', t', y_j, s_j) \leq C K(A_o, y_j, s_j)$, to complete the proof it remains only to show that
\begin{equation*}
\sum_{j=5}^{\infty} h_j(A_o) < \infty.
\end{equation*}
Now, $h_j(A_o) = h_j(y_o + rd, s_o + (1+\mu)r^2)
\begin{equation*}
= h(y_o + 2^{j-3}d, s_o + 2^{6-2j}(1+\mu)),
\end{equation*}
where $h$ is the appropriate caloric measure in $\Sigma$.

If $m_o = \max \{ h(x, t) : (x, t) \in \Sigma, |x-y_o| < \Omega/2, |t-s_o| < \Omega/4 \}$, it is clear that $0 < m_o < 1$. By the maximum principle,
\begin{equation*}
h(x/2, t/4) \leq m_o h(x, t) \text{ for } (x, t) \in \Sigma.
\end{equation*}
Taking $(x, t) = (2^{3-j}d, 2^{6-2j}(1+\mu))$,
\begin{equation*}
\frac{h(y_o + 2^{3-(j+1)}d, s_o + 2^{6-2(j+1)}(1+\mu))}{h(y_o + 2^{3-j}d, s_o + 2^{6-2j}(1+\mu))} \leq m_o,
\end{equation*}
so, by the ratio test,
\begin{equation*}
\sum_{j=5}^{\infty} h_j(A_o) < \infty.
\end{equation*}
Q.E.D.

Remark. As usual, the proof for a bottom point $(y_o, 0)$ of a rectangle is similar. Here, $\Gamma$ would actually be a parabola with axis parallel to the $t$-axis and vertex $(y_o, 0)$. We would take $r^2 = t'$ for $(x', t') \in \Gamma$. Letting $A_o = (y_o, (1+\mu)r^2)$, $\Delta_j = \Delta((y_o, 0), 2^j r)$, and $\Sigma = \{(x, t) : |x - y_o| < 1, |t| < 1, x-y_o > -B t^2 \}$, the proof may be repeated.

We can now prove the existence of parabolic limits.

Theorem 2.6. Let $u(x, t)$ be a non-negative temperature in $D_T$. Then $u(x, t)$ has finite parabolic limits at points $(y, s)$ of $\partial_p D_T$, except for a set of zero caloric measure.

Proof. We shall first prove the theorem in certain simple cases, then combine these cases for the general result.
Case 1a. If $D_T$ is parabolically starlike, then $u(x,t)$ has finite parabolic limits along that portion of $\partial_p D_T$ given by $x = \eta_1(t)$ except for a set of zero caloric measure.

By Theorem 1.9, there is a unique regular Borel measure $\nu$ on $\partial_p D_T$ such that $u$ has the representation

$$ u(x,t) = \int_{\partial_p D_T} K(x,t,y,s) \, d\nu(y,s). $$

Now,

$$ d\nu(y,s) = f(y,s) \, d\omega(X,T)(y,s) + d\sigma(y,s), $$

where $f(y,s) \in L^1(\omega(X,T))$ and $\sigma$ is singular with respect to $\omega(X,T)$.

In particular, $f(y,s) < \infty$ for $(y,s) \in \partial_p D_T$ except on a set of zero $(\omega(X,T))$ measure. We will show that $u(x,t)$ has parabolic limit equal to $f(y,s)$ a.e. $(\omega(X,T))$ on $\{(\eta_1(t),0): 0 < s < T\}$.

Let $(y_o,s_o) = (\eta_1(s_o),s_o)$ be a point such that $f(y_o,s_o) < \infty$.

Then,

$$ u(x,t) - f(y_o,s_o) = \int_{\partial_p D_T} K(x,t,y,s) \left[ f(y,s) d\omega(X,T)(y,s) + d\sigma(y,s) \right] $$

$$ - f(y_o,s_o) $$

$$ = \int_{\partial_p D_T} K(x,t,y,s) \left[ (f(y,s) - f(y_o,s_o)) d\omega(X,T)(y,s) + d\sigma(y,s) \right]. $$

Let $\Gamma$ be a parabolic cone in $D_T$ with vertex $(y_o,s_o)$ which opens away from $\partial D$, and let $(x,t) \in \Gamma$. Define $\Delta_j, R_j$, and $A_j$ as in Lemma 2.5 for $j=0,1,\ldots,N$, where $\Delta_{N-1} \subset \Delta = \Delta((y_o,s_o),r_o) \subset \Delta_N$ for a positive number $r_o$, small. Then,

$$ |u(x,t) - f(y_o,s_o)| \leq $$

$$ | \int_{\partial_p D_T - \Delta} K(x,t,y,s) \left[ (f(y,s) - f(y_o,s_o)) d\omega(X,T)(y,s) + d\sigma(y,s) \right] | $$

$$ + \sum_{j=0}^{N-1} \int_{R_j} K(x,t,y,s) \left[ |f(y,s) - f(y_o,s_o)| d\omega(X,T)(y,s) + d\sigma(y,s) \right]. $$

(*)
The second term on the right above is dominated by

\[ N \sum_{j=0}^{\infty} \sup_{(y,s) \in B_j} K(x,t,y,s) \left[ \int_{\Delta_j} |f(y,s) - f(y_o,s_o)| + \sigma(\Delta_j) \right]. \]

From Besicovitch's general theory of differentiation of measures,

\[ \int_{\Delta_j} |f(y,s) - f(y_o,s_o)| d\omega(X,T)(y,s) = o(\omega(X,T)(\Delta_j)) \]

and \( \sigma(\Delta_j) = o(\omega(X,T)(\Delta_j)) \) as \( \Delta_j \to \{(y_o,s_o)\} \)

for almost every \( (\omega(X,T)(y_o,s_o) \in \partial P_T) \).

Since

\[ \sup_{(y,s) \in B_j} K(x,t,y,s) \leq \frac{C_j}{\omega(X,T)(\Delta_j)} \]

and

\[ \sum_{j=0}^{\infty} C_j < \infty \]

by the previous lemma, the final term in (*) can be made small uniformly for \( (x,t) \in \Gamma \) if \( \Delta \) (and, hence, \( \Delta_j, j=0, \ldots, N \)) is small enough.

We can dominate the first term in (*) by

\[ \sup_{(y,s) \in \partial P_T - \Delta} K(x,t,y,s) (|v| + f(y_o,s_o)). \]

Since Lemma 2.1 implies that

\[ \sup_{(y,s) \in \partial P_T - \Delta} K(x,t,y,s) \to 0 \text{ as } (x,t) \to (y_o,s_o), \]

the proof is complete in this case.

Case lb. If \( P_T \) is a rectangle, then \( u(x,t) \) has finite parabolic limits along that portion of \( \partial P_T \) given by \( t = 0 \) except for a set of zero caloric measure.

Here, we repeat the proof of Case la, taking into account the remarks we have previously made concerning rectangles. (This result also follows from the representation theorem of Hartman and Wintner [7].)
Case 2. (The general form of the theorem.)

Let \((y_i, s_i)\) be a countable dense subset of \(\partial_p D_T\). For each \(i\), there is a neighborhood \(N_i\) of \((y_i, s_i)\) in \(\partial_p D_T\) with the following property:

(a) if \((y_i, s_i) = (y_i, 0)\) with \(\eta_1(0) < y_i < \eta_2(0)\), then 
\[N_i\] can be taken to be the base of a rectangle \(R_i \subset D_T\).

(b) if \((y_i, s_i) = (\eta_1(s_i), s_i)\) or \((\eta_2(s_i), s_i)\) with \(0 < s_i < T\),
then there is a parabolic cone \(\Gamma_i\) of fixed size and
and orientation such that
\[R_i = \bigcup_{(y,s) \in N_i} \Gamma_i(y,s) \cap \{ (x,t) : t > \inf_{s} \} \]
is a parabolically starlike region, where \(\Gamma_i(y,s)\)
indicates the parabolic cone \(\Gamma_i\) with vertex at \((y,s)\).

For each \(i\), \(N_i \subset \partial_p R_i\), and we can choose another neighborhood
\(N'_i\) of \((y_i, s_i)\) in \(\partial_p D_T\) with \(N'_i \subset N_i\). In either (a) or (b)
u(x,t) has parabolic limits on \(N'_i\) except for a set of zero caloric
measure in \(R_i\), from Cases 1a and 1b. To complete the theorem it
suffices to show that, if \(Z_i\) is the set of zero caloric measure
in \(R_i\) where \(u\) fails to have finite parabolic limit, then \(Z_i\) also
has zero caloric measure in \(D_T\). To see this, note first that,
for \((x,t) \in R_i\),
\[\omega_{R_i}^q (x,t) = \omega_D(x,t) (Z_i) - P_{Z_i} (x,t),\]
where \(P_{Z_i} (x,t)\) is the solution of the Dirichlet problem in \(R_i\)
with (continuous) boundary values
\[P_{Z_i}^q (z,u) = \begin{cases} \omega_D(z,u) (Z_i) & \text{for } (z,u) \in \partial_p R_i - \partial_p D_T, \\ 0 & \text{for } (z,u) \in \partial_p R_i \cap \partial_p D_T. \end{cases}\]
If \(\omega_{R_i}^q (x,t) (Z_i) = 0\), then \(\omega_D (x,t) (Z_i) = P_{Z_i} (x,t)\), so that
\[\lim_{(x,t)-(y,s)} \omega_D (x,t) (Z_i) = 0 \text{ for } (y,s) \in N_i, \text{ since } N_i \subset \partial_p R_i \cap \partial_p D_T.\]
For \((y, s) \in N_i\), we then have
\[
\lim_{(x, t) \to (y, s)} \omega_{D}(x, t)(Z_1) = 0.
\]
Because \(Z_i \subset N_i' \subset N_i\), we also have
\[
\lim_{(x, t) \to (y, s)} \omega_{D}(x, t)(Z_1) = 0 \text{ for } (y, s) \in \partial_p D_T - N_i.
\]
Therefore, by the maximum principle, \(\omega_{D}(x, t)(Z_1) = 0\) for \((x, t) \in D\).

Q.E.D.

We have the following corollary of the proof:

**Corollary 2.7.**

(a) If \(f(y, s) \in L_1(\partial_p D_T)\) with respect to \(\omega(X, T)(\cdot)\), and \(f \geq 0\),
then the non-negative temperature \(u(x, t) = \int_{\partial_p D_T} f(y, s) \omega(x, t)(y, s)\)
in \(D_T\) has parabolic limit equal to \(f(y, s)\) for a.e. \(\omega(X, T)\)(\(y, s\))
in \(\partial_p D_T\).

(b) \(\omega(x, t)(E)\) has parabolic limit equal to 1 a.e. \(\omega(X, T)\) on \(E\),
if \(E\) is a measurable subset of \(\partial_p D_T\).

**Proof.**
Since \(u(x, t) = \int_{\partial_p D_T} K(x, t, y, s) f(y, s) \omega(X, T)(y, s)\),
the proof of the theorem clearly shows that \(u\) has parabolic limit
equal to \(f(y, s)\) except on a set of zero caloric measure. This
establishes part (a). Part (b) is a special case.

Q.E.D.

We will now extend Theorem 2.6 to a somewhat more general
situation. Once again, we must first verify a couple of lemmas.

**Lemma 2.8.** Let \(F \subset \partial_p D_T\) and suppose that there is a parabolic cone
\(\Gamma\) of fixed size and orientation such that, if \(\Gamma(y, s)\) denotes the
cone \(\Gamma\) with vertex at \((y, s)\), then \(\Gamma(y, s) \subset D_T\) for every \((y, s) \in F\).
Set \(\Omega = \cap \Gamma(y, s)\). Then, if \(E \subset F\) is a set of zero caloric measure
\((y, s) \in F\)
with respect to \(\Omega\), it is also of zero caloric measure with respect
to \(D_T\).
Proof. For \((x', t') \in \partial_p \Omega \cap D_T\), let \(\Gamma(x', t')\) denote the parabolic cone of the same size as \(\Gamma\) with vertex at \((x', t')\) but oriented away from \(\Omega\). If \((x', t')\) is sufficiently close to \(F\), \(\Gamma(x', t')\) intersects \(\partial_p D_T\) in \(F^c\), the complement of \(F\) in \(\partial_p D_T\). In fact, this intersection contains a disk \(\Delta'\) centered at a point \((y, s) \in \partial_p D_T\) with \(s = t'\) and having radius equal to \(|t_o - t'|^{1/2}\), where \(t_o\) denotes the \(t\)-coordinate of \((x_o, t_o)\), one of the points of intersection of \(\partial \Gamma(x', t')\) and \(\partial_p D_T\). If \(\partial \Gamma\) is given by \(x = \gamma t^{1/2}\), then

\[ |x_o - x'| = \gamma |t_o - t'|^{1/2}. \]

Since \(\partial_p D_T\) is determined by a function \(\eta(t)\) satisfying a Lipschitz condition with exponent \(1/2\), we have

\[ |y - x'| \leq C |t' - t_o|^{1/2}. \]

Therefore,

\[ |y - x'| \leq |y - x_o| + |x_o - x'| \leq (C + \gamma)|t' - t_o|^{1/2}. \]

Setting \(r' = |t' - t_o|^{1/2}\), \((x', t') \in \Psi((y, s), r'/2)\) if the constant \(d\) in the definition of \(\Psi\) is chosen such that \(d \geq 2(C + \gamma)\). By Lemma 1.1, there is a constant \(C_o > 0\) such that

\[ \omega_D(x', t')(\Delta') \geq C_o. \]

Thus, for points \((x', t') \in \partial_p \Omega \cap D_T\) which are sufficiently close to \(F\),

\[ \omega_D(x', t')(E) = 1 - \omega_D(x', t')(E^c) < 1 - \omega_D(x', t')(\Delta') \leq 1 - C_o < 1. \]

Furthermore, if \((x', t') \in \partial_p \Omega \cap D_T\) is bounded away from \(F\), then

\[ \omega_D(x', t')(E^c) > C_1 > 0, \]

since, by the fact that each point of \(F\) is the vertex of a parabolic cone of fixed size and orientation, \(F^c\) must contain some interval of \(\partial_p D_T\) along \(t = 0\). Thus, for each \((x', t')\) of this type, there is a constant \(C_1'\) such that \(\omega_D(x', t')(E) \leq C_1 < 1\). Let \(C = \max(1 - C_o, C_1')\).

Suppose that \(\Theta(x, t)\) is a lower function for \(\omega_D(x, t')(E)\); i.e., \(\Theta(x, t)\) is a subcaloric function with limit superior less than or equal to the characteristic function of \(E\) on \(\partial D_T\). Then \(\Theta(x, t) - C\) is a lower function for \(\omega_p(x, t')(E)\). Since \(\omega_p(x, t')(E)\) is assumed to be zero, we must have \(\Theta(x, t) \leq C\) for \((x, t) \in \Omega\). Thus, \(\omega_D(x, t')(E) \leq C\), \((x, t) \in \Omega\), making it impossible for \(\omega_D(x, t')(E)\) to have parabolic limit \(1\) at any point of \(E\). By Corollary 2.7, part (b), we must have \(\omega_D(x, t')(E) = 0\).

Q.E.D.
Lemma 2.9. Suppose $F \subset \partial_{p}D_T$ satisfies the hypothesis of the previous lemma. Again take $\Omega = \bigcup (y,s) \in F$. Then for almost every $(\omega^{(x,T)}_D, y,s) \in F$ point $(y,s) \in F$, if $S(y,s)$ is a parabolic cone at $(y,s)$ in $D_T$, then there is a neighborhood $N(y,s)$ of $(y,s)$ in the plane such that $N(y,s) \cap S(y,s)$ contains a parabolic cone at $(y,s)$ in $\Omega$.

Proof. Let $F'$ be the subset of $F$ on which the conclusion fails to hold. For each $(y_o,s_o) \in F'$, there is a parabolic cone $S(y_o,s_o)$ at $(y_o,s_o)$ in $D_T$ and a sequence of points $(x_n^o,t_n^o)$ in $S(y_o,s_o) - \Omega$ such that $(x_n^o,t_n^o) \rightarrow (y_o,s_o)$. Arguing as in the proof of the preceding lemma, the cone of the same size as $\Gamma$ with vertex $(x_n^o,t_n^o)$ but oriented away from $\Omega$ must intersect $\partial_{p}D_T$ in $F'$. Lemma 1.1 then implies there is a constant $C_0 > 0$ such that

$$\omega^{(x_n^o,t_n^o)}_D(F') \geq C_0.$$ 

It follows that $\omega^{(x_n^o,t_n^o)}_D(F') \leq 1 - C_0 < 1$, so that the parabolic limit of $\omega^{(x,t)}_D(F')$ at $(y_o,s_o)$ cannot be equal to $1$. Since $(y_o,s_o)$ was an arbitrary point in $F'$, Corollary 2.7 implies that

$$\omega^{(x,T)}_D(F') = 0.$$

Q.E.D.

We can now state the general theorem.

Theorem 2.10. Suppose that $u(x,t)$ is a temperature in $D_T$ and suppose, for each point $(y,s)$ of a set $F \subset \partial_{p}D_T$, there is a parabolic cone $\Gamma(y,s)$ contained in $D_T$, having vertex $(y,s)$ and opening away from $\partial D$, in which $u(x,t)$ is bounded either from above or below. Then $u(x,t)$ has finite parabolic limit at each $(y,s)$ in $F$ except for a set of zero caloric measure in $D_T$.

Proof. We first adjust the sizes of the cones $\Gamma(y,s)$, $(y,s) \in F$, so that there are only countably many types of cones occuring. It then suffices to prove the theorem when the cones $\Gamma(y,s)$ all have the same size and orientation. We can also decompose $F$ in the following manner:

$$F = \bigcup_{i=1}^{\infty} E_i \bigcup_{i=1}^{\infty} G_i,$$

where $E_i = \{(y,s) \in F: u \leq i \text{ in } \Gamma(y,s)\}$ and $G_i = \{(y,s) \in F: u \geq -i \text{ in } \Gamma(y,s)\}$. 


Thus, it suffices to consider sets $F$ for which $u(x,t)$ is uniformly bounded from above or below on $\bigcup_{(y,s) \in F} \Gamma(y,s)$. Let $\Omega = \bigcup_{(y,s) \in F} \Gamma(y,s)$.

Finally, it is enough to consider sets $F$ of sufficiently small diameter that $\Omega \cap R$ is a region of the type we have considered throughout (bounded by curves $x=\eta_1(t), t=$ constant, $x=\eta_2(t)$, etc.), for some rectangle $R$ containing $F$.

Theorem 2.6 then applies in the region $\Omega \cap R$, showing that $u(x,t)$ has finite parabolic limits (in $\Omega \cap R$) on $F$ except for a set $F_o$ with $\omega_{\Omega \cap R}(F_o) = 0$.

It is easily seen that the proof of Lemma 2.8 is unchanged if $\Omega$ is replaced by $\Omega \cap R$. Therefore, $\omega_D(x,t)(F_o) = 0$.

Now, by Lemma 2.9, except for a set $F_1 \subset F$ with $\omega_D^{(x,t)}(F_1) = 0$, parabolic cones at points $(y,s) \in F$ in $D_T$ are, in a neighborhood of $F$, parabolic cones at $(y,s)$ in $\Omega$ (or $\Omega \cap R$). Hence, $u(x,t)$ has finite parabolic limits (as a function in $D_T$) on $F - (F_o \cup F_1)$.

Since $F_o \cup F_1$ has zero caloric measure in $D$.

Q.E.D.
3. Extension of Results; Examples

Here we extend the results of sections 1 and 2 to domains

\[ D = \{(x,t): t > 0 \text{ and } x > \eta(t)\} \],

where \( \eta(t) \) satisfies a Lipschitz condition with exponent 1/2 on any interval \([0,T]\). (See Figure 6.)

Again we have \( D_T = D \cap \{(x,t): t < T\} \), and we distinguish a point \((X,T)\) with \( \eta(T) < X < \infty \).

We first show that the kernel functions for these one-sided regions are unique. Suppose \( K_1(x,t) \) and \( K_2(x,t) \) are both kernel functions in \( D \) at \((y_o,s_o) \in \partial D \) with respect to \((X,T)\), where \( s_o < T \). Choose \( L = \max \{ y_o, X, \sup_{0 \leq t \leq T} \eta(t) \} \) and define

\[ D^L = D_T \cap \{(x,t): x < L\}. \]

For \( i = 1,2 \), let \( p_i(x,t) \) be the solution of the Dirichlet problem for the heat equation in \( D^L \) with (continuous) boundary values

\[ p_i(z,u) = \begin{cases} K_i(z,u) & \text{for } (z,u) \in D \cap \partial_p D^L, \\ 0 & \text{for } (z,u) \in \partial D \cap \partial_p D^L. \end{cases} \]

Then there are constants \( C_1, C_2 > 0 \) such that

\[ K_i'(x,t) = C_i \left[ K_i(x,t) - p_i(x,t) \right], \quad i = 1,2, \]

is a kernel function at \((y_o,s_o) \) in \( D^L \) with respect to \((X,T)\). By Theorem 1.6, \( K_1' = K_2' \) in \( D^L \). Thus, for \( C = C_2/C_1 \),

\[ K_1 - CK_2 = p_1 - CP_2 \text{ in } D^L. \]

Since \( K_1 \) and \( K_2 \) vanish on \( \partial D - \{(y_o,s_o)\} \), this shows that \( K_1 - CK_2 \) must vanish on all of \( \partial D \). Since each \( K_i \) is non-negative and bounded on \( \{(L,t): 0 \leq t \leq T\} \), vanishing on \( \{(x,0): x \geq L\} \), a form of the maximum principle (see Friedman [5]) implies that each \( K_i \) is bounded at infinity in \( D_T \). Therefore, \( K_1 - CK_2 \) is also bounded at infinity in \( D_T \). Since \( K_1 - CK_2 = 0 \) on \( \partial_p D_T \),
another form of the maximum principle (again see Friedman) implies that \( K_1 - CK_2 = 0 \) in \( D_T \). Since \( K_1(X,T) = K_2(X,T) = 1 \), we must have \( K_1 = K_2 \) in \( D_T \). The maximum principle extends the equality to all of \( D \), proving that kernel functions in \( D \) are uniquely determined.

To settle the question of existence of a kernel function in \( D \) at \((y_0,s_0)\) \( \in \partial D \) with respect to \((X,T)\), with \( s_0 < T \), we again consider the regions \( D^L \) for \( L \geq L_o > \max\{y_0, X, \sup_{0 \leq t \leq T} \eta(t)\} \). If \( \forall ((y_0,s_0),r) \) is contained in \( D^{L_o} \), which is true for small \( r \), a careful examination of the proof of Lemma 1.3 reveals that the constant occurring in the conclusion of that lemma can be taken to be independent of \( L \) for \( L \geq L_o \). That is, if \( K_L(x,t) \) is the kernel function at \((y_0,s_0)\) in \( D^L \) with respect to \((X,T)\), we have

\[
K_L(x,t) \leq C K_L(A) \omega_{D^L}(\Delta) \text{ for } (x,t) \in D^L - \gamma((y_0,s_0),(1+\mu)^{1/2} r/4),
\]

where \( \Delta = \Delta((y_0,s_0),r) \) and \( A = A((y_0,s_0),r) \). By Harnack's inequality in \( D^{L_o} \),

\[
K_L(A) \leq C K_L(X,T) = C \text{ for each } L \geq L_o,
\]

so we obtain

\[
K_L(x,t) \leq C \text{ for } (x,t) \in D^L - \gamma((y_0,s_0),(1+\mu)^{1/2} r/4).
\]

The Ascoli theorem then allows the selection of a subsequence \( K_{L_n} \) which converges uniformly on compact subsets of \( D \). By the Harnack convergence theorem, \( K(x,t) = \lim_{n \to \infty} K_{L_n}(x,t) \) is a temperature in \( D \).

Clearly, \( K(x,t) \geq 0 \) and \( K(X,T) = 1 \). Furthermore, since the \( K_L \)'s have a uniform bound away from \((y_0,s_0)\) and each \( K_{L_n} \) has zero boundary values on \( \partial_p D^{L_n} - \{(y_0,s_0)\} \), we see that \( \lim_{(x,t) \to (y,s)} K(x,t) = 0 \) for \( (x,t) \in \partial \cap (y,s) \in \partial D - \{(y_0,s_0)\} \), recalling that each such boundary point is regular. Thus, \( K(x,t) \) is a kernel function at \((y_0,s_0)\) in \( D \) with respect to \((X,T)\).

The representation of non-negative temperatures in \( D_T \) by an integral of the kernel function with respect to some regular Borel measure on \( \partial D_T \) follows from the uniqueness of kernel functions, as
in section 1. Finally, we can show that a non-negative temperature \( u \) in \( D_T \) must have finite parabolic limits at all points of \( \partial_D T \) except for a set of zero caloric measure. If not, then \( u \) fails to have parabolic limits on a set \( F \) of positive \( (\omega_{D}(x,T))^{\text{measure}} \) which is contained in \( D_T \cap \{ (x,t) : x < N \} \) for some integer \( N \). For sufficiently large \( L \),

\[
\omega_{D}^{(L,t)}(F) \leq (1/2) \omega_{D}^{(x,T)}(F) \quad \text{for} \quad 0 \leq t \leq T, \quad \text{so,}
\]

by the maximum principle in \( D_{L}^{l} \), we have

\[
\omega_{D}^{(x,t)}(F) - \omega_{D}^{(x,t)}(F) \leq (1/2) \omega_{D}^{(x,T)}(F) \quad \text{for} \quad (x,t) \in D_{L}^{l}.
\]

In particular,

\[
\omega_{D}^{(x,T)}(F) \geq (1/2) \omega_{D}^{(x,T)}(F).
\]

Hence, the restriction of \( u \) to \( D_{L}^{l} \) fails to have finite parabolic limit on a set of positive \( (\omega_{D}^{(x,T)})^{\text{measure}} \). Since this contradicts the conclusion of Theorem 2.6, \( u \) must have parabolic limits a.e. \( (\omega_{D}^{(x,T)})^{\text{measure}} \) on \( \partial_D T \).

An interesting example of such one-sided domains are the regions

\[
D_{a} = \{(x,t) : x > at^{1/2}, \ t > 0\},
\]

where \( a \) is any real number. (See Figure 7.)

Let \( K_{a}(x,t) \) be the kernel function at \((0,0)\)
in \( D_{a} \) with respect to a point \((x,T)\) in \( D_{a} \).

For any \( \beta > 0 \), the function \( K_{a}^{\beta}(x,t) = K_{a}(\beta x, \beta^{2} t) \)
is a non-negative temperature in \( D_{a} \) which vanishes on \( \partial D_{a} = [(0,0)] \). Thus, with \( f(\beta) = 1/K_{a}(\beta x, \beta^{2} t) \),

the function \( f(\beta)K_{a}(x,t) \) is a kernel function at \((0,0)\) in \( D_{a} \) with respect to \((x,T)\). By uniqueness, \( K_{a}(x,t) = f(\beta)K_{a}(\beta x, \beta^{2} t) \) for any \( \beta > 0 \).

Letting \( \beta = t^{-1/2} \), \( K_{a}(x,t) = f(t^{-1/2})K_{a}(xt^{-1/2}, 1) \). For ease of notation, we rewrite this: \( K_{a}(x,t) = f(t)g(xt^{-1/2}) \), where, of course, \( f \) and \( g \) depend on \( a \). Since \( K_{a} \) is a temperature in \( D_{a} \), we have

\[
t^{-1}f(t)g''(xt^{-1/2}) = -2xt^{-3/2}f(t)g'(xt^{-1/2}) + f'(t)g(xt^{-1/2}).
\]
Thus,
\[
\frac{tf'(t)}{f(t)} = \frac{g''(y) + (y/2)g'(y)}{g(y)} = -\lambda \quad (a \text{ constant}),
\]
where we have made the substitution \( y = xt^{-1/2} \). Therefore,
\[
lf'(t) + \lambda f(t) = 0 \quad \text{and} \quad g''(y) + (y/2)g'(y) + \lambda g(y) = 0.
\]
Clearly, \( f(t) = t^{-\lambda} \), so that \( K_a(x,t) = t^{-\lambda}g(xt^{-1/2}) \) for some \( \lambda \),
where \( g(y) \) solves the boundary value problem:
\[
g''(y) + (y/2)g'(y) + \lambda g(y) = 0 \quad \text{for} \: a < y < \infty,
\]
\[
g(a) = 0, \quad \text{and limit} \quad g(y) = 0.
\]
\( y \to -\infty \)
Letting \( g(y) = h(y/2) \exp(-y^2/4) \), \( h \) must satisfy
\[
h''(z) - 2zh'(z) + 4(\lambda - 1/2)h(z) = 0 \quad \text{(the Hermite equation)}
\]
with conditions
\[
h(a/2) = 0 \quad \text{and} \quad \lim_{z \to -\infty} h(z) = 0.
\]
If we take \( \lambda_a \) to be the minimum eigenvalue for this problem,
Sturm-Liouville theory (see, e.g., Coddington and Levinson [4])
shows that the eigenfunction \( h_a \) corresponding to \( \lambda_a \)
in the interval \((a/2, \infty)\) and that limit \( h_a(z) = 0 \). Thus,
\[
K_a(x,t) = C_a t^{-\lambda} a h_a(xt^{-1/2}) \exp(-x^2/4t),
\]
where \( C_a \) is chosen to assure \( K_a(x,t) = 1 \).

Specifically, if \( \alpha_n/2 \) is the largest root of the Hermite
polynomial \( P_n \), we have \( 4(\lambda_{\alpha_n} - 1/2) = 2n \), so that \( \lambda_{\alpha_n} = (n+1)/2 \)
and
\[
K_{\alpha_n}(x,t) = C_{\alpha_n} t^{-(n+1)/2} P_n(xt^{-1/2}) \exp(-x^2/4t).
\]
As \( n \) increases, the domains \( D_{\alpha_n} \) shrink and the order of the pole
of \( K_{\alpha_n} \) at the origin along curves \( x = \nu t^{1/2}, \nu > \alpha_n \), increases.

It is worth noting that the singularities of kernel functions
in general are not limited to poles. To see this, let \( K(x,t) \) be
the kernel function at the origin for the region \( D = \{(x,t): 0 < x/t < \pi, t > 0\} \).

(See Figure 8.) As this region unfortunately fails to be of the type we have studied throughout \( \eta_1(0) = \eta_2(0) \), here, a different method of establishing the existence and uniqueness of a kernel function at \((0,0)\) is required. Since the image of \( D \) under the change of variables \((x,t) \rightarrow (x/t, -1/t)\) is the region \( D' = \{(x,t): t < 0, 0 < x < \pi\} \), and since the Appell transform,

\[
u(x,t) = \frac{t^{-1/2}}{\exp(-x^2/4t)} u(x/t, -1/t)\]

preserves temperatures (see Appell[1]), the existence and uniqueness questions for a kernel function at \((0,0)\) in \( D \) are equivalent to proving that there is a non-trivial, non-negative temperature \( u(x,t) \) in \( D' \) satisfying \( u(0,t) = u(\pi,t) = 0 \), and that \( u(x,t) \) is unique up to constant multiples. This problem has been solved by Jones [11], who showed that \( u(x,t) = \exp(-t)\sin x \). Thus,

\[
K(x,t) = C \frac{t^{-1/2}}{\exp(-x^2/4t)} \exp(1/t) \sin (x/t)
\]

is the kernel function at \((0,0)\) in \( D \), where \( C \) is a normalizing constant. Approaching the origin along lines \( x = \gamma t \), we see that \( K(x,t) \) has a singularity on the order of \( \exp(1/t) \), considerably stronger than the poles of \( K_q(x,t) \) above.
References

[1] P. Appell: *Sur l'équation $\frac{d^2}{dx^2} - \frac{\partial z}{\partial t} = 0$ et la Théorie de la Chaleur*, J. de Mathématique Pures et Appliquées 8 (1892) 187-216.


