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WHICH DO NOT CONTAIN PROJECTIVE PLANES

by

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In [12] F. Waldhausen proved the following theorem:
Let \( M \) and \( N \) be orientable 3-manifolds which are irreducible and boundary irreducible, let \( M \) be sufficiently large. Then every isomorphism \( \psi: \pi_1(N) \to \pi_1(M) \) which respects the peripheral structure can be realized by a homeomorphism \( f: N \to M \) (which induces \( \psi \)).

In this paper this theorem is extended to apply to non-orientable 3-manifolds (see corollary 5.5).

One could try to use Waldhausen's result and to prove his theorem by means of 2-sheeted orientable coverings. This fails for different reasons. For example, to prove the existence of a hierarchy one could find a system \( F \) of orientable incompressible 2-manifolds in a non-orientable 3-manifold \( M \) such that \( M - F \) would be orientable. The difficulty arises from the fact that \( F \) need not be 2-sided in \( M \). Trying to overcome this in taking a regular neighborhood \( U \) of \( F \) in \( M \) and considering \( \partial U \) which is 2-sided, we are confronted with the fact that \( \partial U \) may be completely compressible in \( M \). The main reason however that we get into trouble is that we don't have the following conjecture related to the loop theorem:
Let $F$ be a one-sided surface embedded in $M$ and let $k$ be a closed curve on $F$, $k \neq 0$ on $F$, but $k = 0$ in $M$. Then there exists a disc $D$, semi-linearly embedded in $M$, such that $D \cap F$ is a simple closed curve $k' \neq 0$ on $F$.

In [10] J. Stallings gives a counterexample in case $F$ is a one-sided (non-orientable) surface in an orientable 3-manifold. One might hope that the conjecture remains true for one-sided orientable surfaces in a (non-orientable) 3-manifold. But here we have the following counterexample:

Let $M = \mathbb{P}^2 \times S^1$, the product of a projective plane and the 1-sphere. Then the torus $F$ in the figure (which represents $\mathbb{P}^2 \times x$ for a point $x \in S^1$) is clearly 1-sided in $M$. If $m$ is a meridian of $F$, then $m^2 = a^2 = 0$ in $M$. But any simple closed curve $k$ on $T$ is homotopic to $ym + zt$, $(y,z) = 1$, where $t$ is a longitude of $F$. Thus $k \simeq ym + zt \simeq ya + zt$ (in $M$), where $t$ is a generator of $S^1$. But $ya + zt = 0$ iff $y \equiv 0 \pmod{2}$ and $z = 0$. 
In our proof of Waldhausen's theorems for non-orientable surfaces we employ the same techniques as Waldhausen and we use his arguments where they apply.

To solve the classification problem of 3-manifolds in terms of their fundamental groups we would need the following things. First, a proof (or disproof) of the Poincaré conjecture, second a classification of manifolds which are not sufficiently large. For the latter problem it suffices to classify manifolds with finite first homology group, because if the homology group is infinite, then the manifold is sufficiently large.

Furthermore we have to classify manifolds which are irreducible but not $\mathbb{P}^2$-irreducible. If these problems were solved, then we would be able to solve the classification problem (provided the Poincaré conjecture turns out to be true), by making use of the unique factorization of 3-manifolds into irreducible manifolds and investigating the factors.

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## Introduction

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§1 Preliminaries

We adopt the piecewise linear standpoint. A "manifold" is a compact 3-manifold (unless otherwise specified).

A surface $F$ in the manifold $M$ is properly embedded, $F \cap \partial M = \partial F$.

By a system of surfaces, finitely many surfaces are understood.

A surface $F$ in $M$ or $\partial M$ is "parallel" to a surface $G$ in $M$ or $\partial M$ iff there exists an embedding of $F \times I$ into $M$, $F \times 0 = F$ and $F \times I \cup \partial F \times I = G$.

**Definition 1.1:** A system $F$ of surfaces in a 3-manifold $M$, with $F \cap \partial M = \partial F$ or $F \subset \partial M$, is **compressible** in $M$ iff either

(a) there is a ball $E$ in $M$ such that $E \cap F = \partial E$, or

(b) there is a non-contractible simple closed curve in $F$ which bounds a disc $D$ in $M$, $\partial D \subset \partial M$, $D \cap F = \partial D = k$.

$F$ is **incompressible** in $M$, iff it is not compressible in $M$.

**Definition 1.2:** A 3-manifold $M$ is **irreducible**, iff every 2-sphere in $M$ is compressible (i.e., bounds a 3-ball).

**Definition 1.3:** $M$ is **boundary-irreducible**, iff $\partial M$ is incompressible.
In the following propositions let \( M \) be a 3-manifold and \( F \) a system of surfaces in \( M \) or \( \partial M \).

**Proposition 1.4:** \( F \) is incompressible in \( M \) iff every component of \( F \) is.

**Proof:** If every component is incompressible then it is trivial that \( F \) is incompressible. Suppose \( F' \) is a component of \( F \) and \( F' \) is compressible. We want to show that \( F \) is compressible:

(a) Suppose \( F' = \partial E \). If \( \partial E \cap F = \emptyset \) we are through. If \( \partial E \cap F \neq \emptyset \) then \( E \) contains closed surfaces of the system \( F \). But \( E \) does not contain closed incompressible surfaces.

(Proof: If all the surfaces in \( E \) are 2-spheres, then the Schoenflies-theorem gives an innermost one for which 1.1(a) holds. If \( F' \) is a component different from a 2-sphere, then by the generalized Alexander theorem [3], \( F' \) is compressible. Then the same proof as in (b) below shows that \( F \) is compressible). Hence \( F \) is compressible.

(b) There exists a disc \( D \) in \( M \), \( D \cap F' = \partial D \), \( \partial D \) does not bound a disc in \( F' \). After deforming \( D \) (if necessary), \( D \cap F \) consists of pairwise disjoint simple closed curves \( k_j \) \((j = 1, \ldots, u)\). If \( k_j \) bounds a disc in \( F \), then we take an innermost disc \( D' \) (i.e., such that \( \partial D \cap F = \emptyset \)) and replace that disc in \( D \) which is bounded by \( \partial D' \) by \( D' \) and push \( D' \) out of \( F \). Then the number of intersection curves \( k_j \) becomes smaller. At last no \( k_j \) bounds a disc
in \( F \), then taking an innermost of the \( k_1 \)'s in \( D \), we see that \( F \) is compressible.

**Proposition 1.5:** If \( F \) is connected, 2-sided and not a 2-sphere then \( F \) is incompressible iff \( \ker(\pi_1(F) - \pi_1(M)) = 0 \).

**Proof:** Suppose \( F \) is incompressible and \( \ker(\pi_1(F) - \pi_1(M)) \neq 0 \). Then there exists a closed curve \( k \) on \( F \), \( k \neq 0 \) on \( F \), \( k = 0 \) in \( M \). Since \( F \) is 2-sided the loop theorem [10] gives a disc \( D \) in \( M \) such that \( D \cap F = \partial D \), a simple closed curve on \( F \). It follows that \( F \) is compressible, a contradiction.

Suppose \( \ker(\pi_1(F) - \pi_1(M)) = 0 \). If \( F \) were compressible then there would exist a closed curve \( k \neq 0 \) on \( F \), \( k = 0 \) in \( M \).

**Proposition 1.6:** Let \( U(F) \) be a regular neighborhood of \( F \). If \( F \) is incompressible then

(a) \( \bar{M} - U(F) \) is irreducible iff \( M \) is

(b) \( \ker(\pi_1(M') - \pi_1(M)) = 0 \), for every component \( M' \) of \( \bar{M} - U(F) \).

**Proof:** (a) 1) Let \( \bar{M} = \bar{M} - U(F) \) be not irreducible. Then there exists a 2-sphere \( S^2 \subset \bar{M} \) which does not bound a 3-element in \( \bar{M} \). Suppose \( S^2 \) bounds a 3-element \( B^3 \) in \( M \). Then there would be a closed component of \( F \) in \( \partial B^3 \) (closed, because \( \partial B^3 = S^2 \) is disjoint from \( \bar{U}(F) \)). Since \( E \) does not contain (closed) incompressible surfaces, \( F \) would be compressible.

2) Let \( M \) be not irreducible. Let \( S^2 \) be a 2-sphere which does not bound a 3-element and such that \( S^2 \cap F \) consists
of a minimal number of disjoint simple closed curves. We want to show \( S^2 \cap F = \emptyset \) (because then certainly \( S^2 \) does not bound a 3-element in \( \mathcal{M} \cup (F) \)). Suppose \( S^2 \cap F \neq \emptyset \).

Then there exists an innermost intersection curve on \( S^2 \).

Since \( F \) is incompressible this curve bounds a disc on \( F \). Eventually we find a disc \( D \) in \( F \), \( D \cap S^2 = \partial D \subset S^2 \cap F \).

\( \partial D \) bounds 2 discs \( D_1 \) and \( D_2 \) in \( S^2 \). \( D \cup D_1 \) and \( D \cup D_2 \) are 2-spheres which have fewer intersections with \( F \) than \( S^2 \).

Hence they bound 3-elements \( E_1 \) and \( E_2 \) respectively. We have \( \partial E_1 \cap \partial E_2 = D \) and either \( \hat{E}_1 \cap \hat{E}_2 = \emptyset \) or \( E_2 \subset E_1 \), say.

Hence \( S^2 \) either bounds the 3-element \( E_1 \cup E_2 \) or \( \overline{E_1 - E_2} \), which is a contradiction.

(b) Let \( \alpha \in \pi_1(M') \) be represented by a simple closed curve \( k \in \hat{M}' \). Assume \( k \sim 0 \) in \( M \). We want to show:

\( k \sim 0 \) in \( M' \). There exists a homotopy \( f: I \times I \to M \) such that \( f(0 \times I) = f(1 \times I) = \hat{P} \in \hat{M}' \) (let \( P \) be the base point for \( M' \) and \( M \)), \( f(I \times 1) = P \) and \( f(I \times 0) = k \).

Make this homotopy in general position with respect to \( F \), i.e., there exists a singular disc \( D \subset \hat{M} \), \( \partial D = k \) and \( D \) intersects \( F \) in general position. By small isotopic deformations and cuts of \( D \), keeping \( k \) fixed we achieve that \( D \cap F \) consists of closed double curves only: We deform \( D \) such that \( F \) meets no triple points of \( D \). If \( F \) intersects a double curve of \( D \), we perform a cut.
A double arc \( \partial D \cap F \) would have its endpoints in \( \partial F \),
but \( D \in \hat{M} \) and \( D \) remains in \( \hat{M} \) all the time.

If \( \ell \in D \cap F \) is an innermost closed double curve then
\( \ell = 0 \) in \( M \). Now since \( \ker(\pi_1(F) - \pi_1(M)) = 0 \) it follows
that \( \ell = 0 \) in \( F \), and hence, since \( \ell \) is a simple closed
curve, that \( \ell \) bounds a disc \( d \) (non-singular) in \( F \) (see
for example [2]). We replace the disc which \( \ell \) bounds in
\( D \) by \( d \) and push \( d \) out of \( F \) and obtain a disc \( D^* \), \( \partial D^* = k \),
\( D^* \cap F \) is smaller. Continuing in this way, we get a
disc \( D^\# \subset M \), \( \partial D^\# = k \), \( D^\# \cap F = \phi \), i.e., \( D^\# \subset M' \).

Proposition 1.7: Let \( M \) be irreducible. If \( M \) is orientable
and closed suppose \( \pi_1(M) \) is infinite. If \( M \) is non-orientable
suppose that \( M \) does not contain 2-sided projective planes
embedded in \( M \). Then \( M \) is aspherical (i.e., \( \pi_j(M) = 0 \), \( j \geq 2 \)).

Proof: If \( \pi_2(M) \neq 0 \) then the projective plane theorem
([1],thm. 1.1) asserts that there is a 2-sided essential
2-sphere or projective plane embedded in \( M \). By our hypotheses
on \( M \) this cannot occur. Let \( \hat{M} \) be the universal cover of
\( M \). We shall prove (theorem 3.3) that \( \pi_1(M) \) is infinite.
Hence \( \hat{M} \) is a non-compact space, \( \pi_1(\hat{M}) = 1, \pi_2(\hat{M}) = \pi_2(M) = 0 \), and by the Hurewicz-isomorphism \( H_2(\hat{M}) = 0 \). It follows that \( \pi_3(\hat{M}) = H_3(\hat{M}) = 0 \) (since \( \hat{M} \) is non-compact). By induction \( \pi_j(\hat{M}) = H_j(\hat{M}) = 0, j > 3 \), since \( \hat{M} \) is a 3-manifold. It follows that \( M \) is aspherical. In view of the proof of lemma 5.3 the following is of interest:

**Proposition 1.8:** Suppose a manifold \( M \) contains a 2-sided projective plane. Then \( M \) is not aspherical.

**Proof:** By lemma (6.3) of [1], \( P^2 \) is essential, hence \( \pi_2(M) \neq 0 \).

In this paper we are concerned with manifolds which are irreducible and do not contain 2-sided projective planes. The following proposition gives us certain manifolds which have this property.

**Proposition 1.9** Let \( M \) be a non-orientable irreducible manifold.

Let \( \tilde{M} \) be the 2-sided orientable cover of \( M \) and assume that \( \pi_1(\tilde{M}) \) is not infinite cyclic or a (nontrivial) free product. Then, if \( \partial M \) does not contain a projective plane (in particular if \( \partial M = \emptyset \)), \( M \) does not contain any 2-sided projective planes.

**Proof:** Suppose that \( \pi_2(M) = \pi_2(\tilde{M}) \neq 0 \). The sphere theorem gives a 2-sphere \( S^2 \subset \tilde{M} \), essentially embedded in \( \tilde{M} \).

a) If \( S^2 \) is not separating, there exists a simple closed curve \( k \) which intersects \( S^2 \) in one point. Let \( U \) be a regular
neighborhood of $S^2 \cup k$. Then $\pi_1(U) = \mathbb{Z}$ and $\partial U$ is a separating 2-sphere hence $\pi_1(\tilde{M}) = \mathbb{Z} \ast \pi_1(\tilde{M} - U)$, contrary to our hypothesis on $\tilde{M}$.

b) If $S^2$ is separating $\tilde{M}$ into 2 3-manifolds $\tilde{M}_1$, $\tilde{M}_2$, then, since $\pi_1(\tilde{M})$ is no free product, $\pi_1(\tilde{M}_1) = 1$, say, $\partial \tilde{M}_1 = S^2$, because any other boundary component $G$ of $\tilde{M}$ is different from $\mathbb{P}^2$ and $S^2$ and the image $\pi_1(G) \rightarrow \pi_1(\tilde{M}_1)$ would be non-trivial (see thm. 3.3). Hence $S^2$ is homologous to 0 in $\tilde{M}$ and since $\pi_1(\tilde{M}_1) = 1$, $S^2$ is contractible, a contradiction. It follows that $\pi_2(M) = \pi_2(\tilde{M}) = 0$. Now the same argument as in proposition 1.7 shows that $M$ is aspherical.

By proposition 1.8 $M$ contains no 2-sided projective planes.

Remark: If $\pi_1(\tilde{M}) = \mathbb{Z}$, then the conclusion of 1.9 does not hold, e.g., $M = \mathbb{P}^2 \times S^1$, $\tilde{M} = S^2 \times S^1$. 
§2 A Theorem on maps between surfaces

Let $G$ and $F$ be surfaces. Let $P^2$ denote the projective plane.

**Theorem 2.1:** Let $f : (G, \partial G) \to (F, \partial F)$ be a map such that

\[ \ker(f_* : \pi_1(G) \to \pi_1(F)) = 0. \]

Suppose $\pi_1(G) \neq 0$ and $F \neq P^2$.

Then there is a homotopy $f_t : (G, \partial G) \to (F, \partial F)$,

$t \in I$ of $f = f_0$, such that either (a) or (b) holds

(a) $f_1 : G \to F$ is a covering map

(b) $G$ is an annulus or moebius strip and $f_1(G) \subset \partial F$.

If $f|\partial G$ is locally homeomorphic, the homotopy may be chosen constant on $\partial G$.

**Remark:** For the special case that $G$ and $F$ are closed and $f_*$ is an isomorphism we obtain the theorem of J. Nielsen, that every isomorphism between the fundamental groups of two closed surfaces can be realized by a homeomorphism.

For the proof, we first need the following lemma.

**Lemma 2.2:** Let $f : G \to F$ be a map, let $F$ be not a 2-sphere or a projective plane. Let $k$ be a system of simple arcs and simple non-contractible 2-sided closed curves on $F$, $k \cap \partial F = \partial k$.

Then there exists a map $g \sim f$, transverse with respect to $k$ (i.e., there exists a regular neighborhood $U(k) = k \times I$, such that $g^{-1}U(k)$ is a line bundle and $g$ maps each fiber $I$ homeomorphically onto a fiber $I$), such that $g^{-1}(k)$ is a system of simple arcs and simple non-contractible 2-sided closed curves and $g^{-1}(k) \cap \partial G = \partial (g^{-1}k)$. 
If $f|\partial G$ has been transverse with respect to $k$, then the homotopy from $f$ to $g$ may be chosen constant on $\partial G$.

**Proof:** Choose a triangulation of $F$ and $G$ such that $f$ becomes a simplicial map. Since all components of $k$ are 2-sided, a regular nbd $U(k)$ in $F$ is homeomorphic to $k \times I$. Identify $U(k)$ with $k \times I$ and $k$ with $k \times \frac{1}{2}$. Choose $0 < \alpha < \beta < \delta < 1$ such that the inverse image under $f$ of $k \times [\alpha, \delta]$ does not contain a 0-simplex of the triangulation. Consider the map $\varphi$ defined by

$$f^{-1}(k \times [\alpha, \delta]) \xrightarrow{f} k \times [\alpha, \delta] \xrightarrow{p} [\alpha, \delta] \text{ (p = projection)}$$

Suppose $[\alpha, \delta]$ is contained in a 1-simplex $\tau$ of $F$. 

![Diagram](image-url)
The inverse image of \([\alpha, \delta]\) in the 1-skeleton \(G^{(1)}\)
of \(G\) consists (since \(f\) is simplicial) of lines in
the interior of 1-simplices. No three sides of a
2-simplex \(\sigma\) contain points \(\varphi^{-1}[\alpha, \delta]\), because if two
sides of \(\sigma\) contain points of \(\varphi^{-1}[\alpha, \delta]\) then the third
side is mapped onto a vertex of \(\tau\). Similarly, the
inverse image of \([\alpha, \delta]\) in \(G^{(2)}\) consists of bands,
which join the lines \(\varphi^{-1}[\alpha, \delta]\) on \(G^{(1)}\). Hence \(f^{-1}(k \times [\alpha, \delta])\) is a line bundle over \((pf)^{-1}(B)\). If
this line bundle were not a product line bundle, then
\(\varphi^{-1}(\alpha) \cup \varphi^{-1}(\delta)\) would be connected, which is im-
possible. Hence we may identify \(f^{-1}(k \times [\alpha, \delta])\) with
\(k^1 \times [\alpha, \delta]\), where \(k^1 = f^{-1}(k \times \alpha)\). \(k^1 \times \beta\) is a
system of arcs and simple 2-sided closed curves and
by small deformations of \(f\) we achieve that each
fiber of a product bundle-neighborhood \(k^1 \times I'\)
of \(k^1 \times \beta\) is mapped homeomorphically onto a
fiber of a product bundle-neighborhood \(k \times I'\) of
\(k \times \beta\). Since \(k^1 \times I'\) is a regular neighborhood
of \(k^1\), we have that \(k^1 \cap \partial G = \partial k^1\). Suppose the
closed curve \(\ell^1\) of \(k^1\) is contractible. \(\ell^1\) bounds
a (non-singular) disc \(e^2\) on \(G\) [2]. Let \(\partial e^2 = \ell^1 = \ell^{1 \times \frac{1}{2}}\)
and \(\ell^1 \times 0 \subset e^2, \bar{e}^2 = e^2 \cup (\ell^1 \times [\frac{1}{2}, 1])\), hence
\(\partial \bar{e}^2 = \ell^1 \times 1\). Since \(f(\ell^1)\) is an arc in \(k\), there is
a transversal homotopy of \(g\), such that \(\ell^1\) is after-
wards mapped onto a point \(p_1 \in k\) and \(\partial \bar{e}^2\) onto a
point $P_2 \in k \times 1$. Since $\pi_2(F) = 0$, $g$ is homotopic to a map which maps $\mathbb{S}^2$ onto $P_2$. After that, $g^{-1}(k)$ does not contain $t^1$ any more (and $g$ remains transversal with respect to $k$). If $f|\partial G$ has been transverse with respect to $k$ then we may deform $f$ by a homotopy which is constant on $\partial G$ such that for $(x,y) \in \partial G \times I$ we have that $f(x,y) = f(x,0)$, and apply our construction to $f| \mathring{G} - (\partial G \times I)$.

**Proof of 2.1**

**Definition:** A Hierarchy for $F = F_1$ is a system $\{k_j\}$, $j=1,\ldots,n$ of 2-sided arcs and 2-sided non-contractible closed curves

$$k_1 \subset F_1, k_2 \subset \overline{F_1 - U(k_1)} = F_2, \ldots, k_n \subset F_n = \overline{F_{n-1} - U(k_{n-1})}$$

such that $k_j \cap \partial F_j = \partial k_j$ and every component of $F_n$ is a disc.

We will use the following facts:

1. If $F$ is not the projective plane, then there exists a hierarchy for $F$. If $\partial F \neq \emptyset$ then we may assume that every $k_j$ is an arc.

2. If $k$ is a simple arc, $k \cap \partial F = \partial k$, or a simple closed non-contractible 2-sided closed curve, then

$$\ker \left( \pi_1(F - U(k)) \to \pi_1(F) \right) = 0.$$

3. Let $k,k'$ be 2 different boundary components of $F$. If a non-null multiple of $k$ is homotopic to a multiple of $k'$, then $F$ is an annulus. (Neither $F$ nor $k'$ need be compact.)
(Proof: If \( k' \) is compact, attach a disc to \( F \) along the boundary \( k' \).

This yields a surface \( \tilde{F} \), \( k' \sim 0 \) in \( \tilde{F} \). Now, since \( \pi_1(\tilde{F}) \) does not have elements of finite order \( k \sim 0 \) in \( \tilde{F} \). Hence \( k \) bounds a disc \( e^2 \) in \( \tilde{F} \), i.e., \( \tilde{F} \sim e^2 \) and we get \( F \) from \( \tilde{F} \) by removing a disc from \( \tilde{F} \).

If \( k' \) would not be compact, then \( k \sim 0 \) in \( F \) and \( F \) would be a disc.)

I) Proof of 2.1 in case \( \partial F \neq \emptyset \):

Let \( r \) be a boundary component of \( G \); \( r \neq 0 \) because \( \pi_1(G) \neq 0 \). Suppose \( r \) is mapped into \( s \in \partial F \). From the commutative diagram

\[
\begin{array}{ccc}
\pi_1(G) & \xrightarrow{f_*} & \pi_1(F) \\
\downarrow \quad i_* & & \downarrow \quad i_* \\
\pi_1(r) & \xrightarrow{(f|_r)_*} & \pi_1(s)
\end{array}
\]

it follows that \( (f|_r)_* \) is injective. Hence \( f: r \to s \) is homotopic to a covering map. Homotope \( f \) so that \( f|_{\partial G} \) is a covering map. Furthermore homotope \( f \) such that \( f^{-1}(\partial F) \sim \partial G \) (general position). From now on, \( f \) will not be altered on \( \partial G \), which will prove the last assertion of 1.2.1.

Choose a hierarchy for \( F \) (by (2)):

\[
F = F_1, \quad k_1 \subset F_1, \quad U(k_1) \subset F_1, \ldots, \quad F_{j+1} = F_j - U(k_j), \quad j = 1, \ldots, n
\]

such that every \( k_j \) is an arc.

Induction hypothesis: \( f|_{f^{-1}(\partial F \cup \cup_{m} U(k_j))} \) is a local homeomorphism.

If \( m = 1 \), we know that \( f|_{f^{-1}(\partial F)} \) is a covering.

Suppose we have proved the hypothesis for \( m \). Let \( G_m' \) be a component of \( f^{-1}(F_m) \) and let \( f' = f|_{G_m'} : (G_m', \partial G_m') \to (F_m, \partial F_m) \).
Since \( \partial F_m \subset \partial F \cup \bigcup_{j} U(k_j) \) it follows from the hypothesis that \( f'|_{\partial G_m} \) is locally homeomorphic. By the lemma, there is a homotopy, constant on \( \partial G_m \), such that afterwards \( f' \) is transversal with respect to \( k_m \) and such that \( f'^{-1}(k_m) \) contains no contractible closed curve. Now \( \text{ker}(f'_*) = 0 \) (Because from (2) it follows by induction that \( \text{ker}(\pi_1(F_m) \to \pi_1(F)) = 0 \) and by induction and by the lemma that \( \text{ker}(\pi_1(G'_m) \to \pi_1(G)) = 0 \). Then consider the commutative diagram

\[
\begin{array}{ccc}
\pi_1(G) & \xrightarrow{f'_*} & \pi_1(F) \\
\uparrow & & \uparrow \\
\pi_1(G'_m) & \xrightarrow{f'_*} & \pi_1(F_m)
\end{array}
\]

Hence \( \text{ker}(f'|_{\ell}) = 0 \) for every component \( \ell \in f'^{-1}(k_m) \)

(Because

\[
\begin{array}{ccc}
\pi_1(G'_m) & \xrightarrow{i_*} & \pi_1(F_m) \\
\uparrow i_* & & \uparrow j_* \\
\pi_1(\ell) & \xrightarrow{j_*} & \pi_1(k_m)
\end{array}
\]

and \( i_* \) and \( j_* \) are injective.) Since \( k_m \) is an arc each \( \ell \) is an arc and so we may assume:

\[
f'|_{\ell}: \ell \to k_m \]

is homotopic to a covering map by a homotopy which is constant on \( \partial \ell \), for all components \( \ell \) of \( f'^{-1}(k_m) \) and this is true for all components of \( f^{-1}(F_m) \). This proves the induction hypothesis for \( m + 1 \). Hence \( f|f^{-1}(\partial F \cup \bigcup_{j} U(k_j)) \) is locally homeomorphic. \( F_{n+1} \) consists of discs. Let \( F_{n+1}^* \) be a component of \( F_{n+1} \) and \( G^* \) a component of \( f^{-1}(F_{n+1}^*) \).

We have \( G^* \neq G \), because otherwise we would have \( \text{ker}(f_*: \pi_1(G) \to \pi_1(F)) \neq 0 \) since \( \pi_1(G) \neq 0 \).
$f |_{\partial G^*}$:

$\partial G^* \to \partial F^*_{n+1}$ is locally homeomorphic. $G^*$ is a disc:
let $t$ be a component of $\partial G^*$, then $t = 0$ in $G$, since
$\ker(f_*: \pi_1(G) \to \pi_1(F)) = 0$. By (2), $t = 0$ in $G^*$,
hence $t$ bounds a disc in $G^*$.

a) If $f|_{\partial G^*}$ is a homeomorphism, then we may deform $f$
(keeping $f|_{\partial G^*}$ fixed) such that $f: G^* \to F^*_{n+1}$ is
homeomorphic. This proves case (a) of 2.1.

b) If $f|_{\partial G^*}$ is a covering other than a homeomorphism then
exists an arc $t \in \partial G^*$ which maps locally homeomorphic onto
$\partial F^*$. We take a simple arc $t^* \in G^*$ near $t$,
$t^* \cap \partial G^* = \partial t^*$, $f(\partial t^*)$ is a point $p$, $f|_{t^*}: (t^*, \partial t^*) \to
(F^*_{n+1}, f(\partial t^*))$ contracts. If $\partial t^* \notin \partial G$, we join $p$
with a point in $\partial F$ by a simple arc in $\bigcup_{j \leq n} U(k_j)$ (which
is possible by our special choice of the hierarchy)
and lift this arc to $f^{-1}(\partial F \cup \bigcup_{j \leq n} U(k_j))$. Then:
there is a simple arc $k$ in $G$, $\partial k$ consists of
two points $P_1 \neq P_2 \in \partial G$, $f(P_1) = f(P_2)$ and $f|_k$
$(k, \partial k) \to (F, f(\partial k))$ contracts.

Let $s$ be that boundary curve in $F$, which contains $f(P_1)$.
Let $r_1$, $r_2$ be the boundary curves of $G$ which contain $P_1$
and $P_2$. Choose $P_1$ as base point for $\pi_1(G)$ and $f(P_1)$ as
basepoint for $\pi_1(F)$. $r_1$ and $r_2$ are coverings of $s$. 
a) Assume $r_1 \neq r_2$: $f_*[r_1] = \{s^m\}$

Then $f_*[k^{-1}r_2k] = \{s^n\}$ ([] means homotopy class). Then $f_*[r_1^{-n}(k^{-1}r_2k)^{m}] = 0$, hence it follows since $f_*$ is injective, that $r_1^n = k^{-1}r_2k$ (in $G$) and by (3), that $G$ is an annulus. Let $\tilde{G} \to F$, be the covering, which is associated to $i_\infty_1(s)$. There is a closed curve $\tilde{\gamma}$ covering $s$ (for which $\pi_1(\tilde{\gamma}) \to \pi_1(s)$ is an isomorphism). Lift $f: G \to \tilde{F}$ to $F: G \to \tilde{F}$ (this is possible since $f_1 \pi_1(G) = f_*\pi_1(r_1) \subset p_*i_\infty_1(s)$) such that $\tilde{F}(\partial G) \cap \tilde{\gamma} \neq \emptyset$.

Since $f|k: (k, \partial k) \to (F, \partial(k))$ contracts, $\tilde{F}(k)$ is a closed contractible curve and hence $F(\partial G) \subset \tilde{\gamma}$. Since $\pi_1(\tilde{F}) = Z$, $\tilde{F} \neq S^2$ or $\tilde{p}^2$ and hence $\tilde{F}$ (and $\tilde{\gamma}$) are aspherical. It follows that $\tilde{\gamma}$ is a Deformation retract of $\tilde{F}$. If $h_\gamma: \tilde{F} \to \tilde{F}$ denotes this deformation, we have $f = p \tilde{F} \sim \phi_1 F$ where $h_\gamma: \tilde{F} \to \gamma$ is a retraction, and $\pi_1(\tilde{F}(G) = s \subset \partial F$ which is part of case (b) of 2.1.

b) Assume $r_1 = r_2$ and $(k, \partial k) \to (G, r_1)$ does not contract onto $(r_1, r_1)$:

Let $\tilde{G} \to G$ be the covering associated to $i_\infty_1(r_1)$.

Let $r_1'$ be a closed curve over $r_1$ for which $\pi_1(r_1') \to \pi_1(r_1)$ is an isomorphism (1-1). Let $\tilde{k}$ be over $k$ and starting in $r_1'$. $\tilde{k}$ is an arc and the other endpoint of $\tilde{k}$ does not lie in $r_1'$, because $(k, \partial k) \to (G, r)$ does not contract onto $(r, r)$.

Therefore we conclude that $\tilde{G}$ is an annulus. It follows
that $\mathcal{G}$ is a two-sheeted covering of $G$, since both boundary curves lie over $r_1$ and one of them (namely $r'_1$) is a 1-sheeted covering. Therefore $G$ is a moebius band. As in α) we prove that $f(G) \sim \phi_1 F(G) \subset \mathcal{F}$, which is part of case (b) of 2.1.

α) Assume $r_1 = r_2$ and $(k, \partial k) = (G, r_1)$ contracts onto $(r_1, r_1)$:

$k$ is homotopic to an arc $\tilde{k}$ on $r_1$, $\partial \tilde{k} = \partial k$, and $f(\tilde{k})$ is a closed curve in $s$, $f(\tilde{k}) \neq 0$ in $s$, since $r_1 \sim s$ is a covering. On the other hand $f(k) \sim f(k) \sim 0$ in $F$. It follows that $F$ is a disc, a contradiction.

II) $\mathcal{F} = \emptyset$. Then $F$ and $G$ are closed surfaces. Choose a 2-sided, simple closed curve $k$, noncontractible on $F$, make $f$ transverse with respect to $k$ (Lemma) and choose the homotopy such that the number of components of $f^{-1}(k)$ is as small as possible and cannot be made smaller if we take a copy $k'$ of $k$ in $U(k) = k \times I$, instead of $k$. Since $\mathcal{G} = \emptyset$ every component $t \in f^{-1}(k)$ is a closed noncontractible curve. It follows from

$$
\begin{array}{c}
\pi_1(G) \rightarrow \pi_1(F) \\
\downarrow \quad \downarrow \\
\pi_1(t) \rightarrow \pi_1(k)
\end{array}
$$

that $\ker (f|t)_* = 0$. 
Hence \( f \mid \ell \) is homotopic to a covering map. Choose a regular neighborhood \( U(k) \) such that \( f \mid f^{-1}U(k) \) is a covering map for each component \( f^{-1}U(k) \). Let \( G^* \) be a component of \( f^{-1}(F^*) \), \( F^* = F - \dot{U}(k) \), and \( f^* = f | G^* : G^* \to F^* \). From (2) and the diagram

\[
\begin{array}{c}
\pi_1(G) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
\]

it follows that \( \ker f^* = 0 \). Since \( f | \partial G^* \) is locally homeomorphic it follows from I) that there is a homotopy of \( f^* \), constant on \( \partial G^* \), such that (a) or (b) of 2.1 holds for \( f^* : G^* \to F^* \). If (a) holds, the proof of 2.1 is completed. We show that case (b) can not apply. Suppose (b) holds. Then \( G^* \) is an annulus or a moebius band and \( f(G^*) \) contracts into \( \partial F^* \). Suppose \( G^* \) is a moebius band.
Then $\Delta G^* = \tau_0 \in f^{-1}(k \times 0)$ and $\ell \in f^{-1}(k)$ bounds a moebius band $G^{**} = U(G^*)$ in $G$. Take two copies $k', k''$ of $k$ on different sides of $k$, then $\ell' \in f^{-1}k'$, $\ell'' \in f^{-1}k''$ are on different sides of $\ell$ and $\ell' \in G^{**}$, say. Homotop $f: G = F$, constant on $G = U(G^*)$, such that $G^{**} \not\approx k$.

Then the inverse images of $k'$ outside of $G^{**}$ have not changed, but $\ell' \in f^{-1}(k')$ has vanished. This contradicts our minimality condition on $f$. An analogous (simpler) construction can be made if $G^*$ is an annulus.

Lemma 2.3 In case (b) of (2.1) there exists a simple arc $\ell$ in $G$, $\ell \cap \partial G = \partial \ell$, $f(\partial \ell)$ is one point and $f|_\ell: (\ell, \partial \ell) \to (F, f(\partial \ell))$ contracts.

Proof: 1) Let $G$ be an annulus. Let $s_1, s_2$ be the two boundary curves of $G$. If $f(s_1)$ wraps $n$ times around the boundary curve $T \in \partial F$, then so does $f(s_2)$, because $s_1 \approx s_2$ are homotopic to the centercurve of $G$ which generates $\pi_1(G)$. Clearly $n \geq 1$. Pick points $P_1, \ldots, P_n$ on $s_1$ and $Q_1, \ldots, Q_n$ on $s_2$ such that $f(P_1) = f(Q_1) = P \in T$ and such that the arcs $P_{i-1}P_i$ and $Q_{i-1}Q_i$ are mapped essentially once around $T$. Let $c_1$ be a simple arc which joins $P_1$ and $Q_1$ and suppose $c_1$ is mapped $k_1$ times essentially around $T$. 

![Diagram of an annulus with labeled arcs and points]
If \( k_1 = 0 \) we are through. If not, we find simple arcs \( c_1 \) which are mapped \( k_1 - 1 + 1 \) times around \( T \). (s. figure).

Without loss of generality we may assume that \( 0 < k_1 < n-1 \). Then in the sequence \( c_1, \ldots, c_n \) we find an arc which is mapped 0 times around \( T \).

2) Let \( G \) be a moebious strip.

Since \( \ker(\pi_1(G) \to \pi_1(F)) = 0 \), the centerline \( t \) is mapped \( n \geq 1 \) times around \( T \). Then the boundary of \( G \) is mapped \( 2n \) times around \( T \), because \( \partial G = 2t \). Pick \( 2n \) points \( P_i \in \partial G \) such that \( f(P_i) = P \in T \) \( (i = 1, \ldots, 2n) \). Orient \( \partial G \) and number \( P_i \) on \( \partial G \) according to this orientation. Then \( P_n \) and \( P_{2n} \) determine 2 arcs which are mapped \( n \) times onto \( T \). Join \( P_n \) and \( P_{2n} \) by a simple arc \( C \) (as shown in the figure) which maps \( k \) times around \( T \). We claim that \( k = 0 \).

Suppose the arc \( c_1 \) of the figure is mapped \( k_1 \) times around \( T \).

\[ \begin{array}{cc}
\text{\( P_n \)} & \text{\( t_1 \)} \\
\text{\( t \)} & \text{\( C \)} \\
\text{\( t_1 \)} & \text{\( P_{2n} \)} \\
\end{array} \]

Then it follows from \( c_1 c^{-1} = t \) and \( c_1 = t_1 \) that \( k_1 - k = n \), \( k_1 = n \) hence that \( k = 0 \) and so the arc \( C \) satisfies the conclusion of the lemma.
§3 Existence of Hierarchies

In this chapter we want to show that for certain 3-manifolds there exists a hierarchy in the following sense.

Definition: Let $M$ be an irreducible manifold. A hierarchy for $M$ (of length $n$) is a finite sequence of manifolds $M_j \subset M$ and 2-sided incompressible surfaces $F_j$ in $M_j$ (proper embedded):

$$M_j, \ F_j \subset M_j, \ U(F_j) \subset M_j; \ M_{j+1} = \overline{M_j - U(F_j)}, \ 1 \leq j \leq n,$$

where $U(F_j)$ is a regular neighborhood in $M_j$, and each component of $M_{n+1}$ is a ball.

First we need a lemma on transverse maps.

Let $M$ be an $n$-manifold with a given triangulation, $F \subset M$ a subcomplex such that $F \cap \partial M = \partial F$. $F$ is 2-sided in $M$ if a regular neighborhood $U(F)$ in $M$ is homeomorphic to a product bundle $F \times I$. Thus in this case we may give $U(F)$ a product line bundle structure such that $F = F \times \frac{1}{2}$, $(F \times I) \cap \partial M = \partial F \times I$.

Call a map $f: N \to M$ ($N$ a manifold) transverse with respect to $F$, if there exists $U(F) \approx F \times I$ such that $f^{-1}F = G$ is two-sided in $N$, $f^{-1}(U) \approx G \times I$ and $f$ maps each fibre $I$ of $G \times I$ homeomorphically onto a fibre $I$ of $F \times I$.

Lemma 3.1 Let $M$ be an $n$-manifold, $F \subset M$ a 2-sided subcomplex of a triangulation of $M$. Let $\ker(\pi_j(F) \to \pi_j(M)) = 0$, $j = 1, 2$ and $\pi_2(M-F) = 0$, $\pi_3(M) = 0$. Let $N$ be a 3-manifold
which does not contain 2-sided projective planes and assume \( \pi_2(N) \) does not contain elements of finite order. Let \( f: N \to M \) be a map. Then there exists a map \( g: N \to M \), homotopic to \( f \), such that \( g^{-1}(F) = G \) is a system of 2-sided surfaces (proper embedded) in \( N \), \( g \) is transverse with respect to \( F \) and \( \ker(\pi_j(G) - \pi_j(N)) = 0 \), \( j = 1,2 \). If \( f|_{\partial N} \) was transverse with respect to \( F \) then the homotopy from \( f \) to \( g \) may be chosen constant on \( \partial N \).

Remark: If \( N \) is closed or/and orientable, then \( \pi_2(N) \) does not contain elements of finite order [9].

Proof: Exactly as in the proof of lemma 2.2 it follows that we may assume that \( f^{-1}(F) \) is a system of 2-sided surfaces \( G \) and that \( f \) is transverse with respect to \( F \). Furthermore, the last statement of 3.1 can be satisfied.

If \( \ker(\pi_1(G) - \pi_1(N)) \neq 0 \) for a component \( G \) of \( f^{-1}(F) \) then it follows from the loop theorem [10], noting that \( G \) is 2-sided in \( N \), that there exist a disc \( D \) in \( N \), \( D \cap G = \partial D \neq 0 \) on \( G \). If \( D \) meets a component \( G' \in f^{-1}(F) \), \( G' \neq G \), then (after small deformations of \( D \)) \( G' \cap D \) consists of simple closed curves. If these are homotopic to 0 in \( G' \), then we may deform \( D \), such that \( D \cap G' = \emptyset \).

If \( 0 \neq \ell \in D \cap G' \), then \( \ell \) bounds a disc \( D' \subset D \). By taking an innermost disc, we may assume that \( D \cap G = \partial D = f^{-1}(F) \cap D \). After small deformations of \( D \) we get \( D \subset N \) and \( D \cap (G \times I) = D* = \partial D \times (I/2) \). (\( D* \) is an annulus).
Let $E$ be a 3-ball, which is a neighborhood of $D$ in $\mathbb{N}$, such that $\mathring{E} = E \cap (G \times I) = \mathring{E}$ is a 3-ball, $E \cap G$ is an annulus neighborhood of $\partial D$ in $G$ and $E \cap (G \times I) = E \cap (F^{-1}(F) \times I)$. Let $\tilde{D} = D - D^\circ$. We want to cut $G$ along $\partial D$. Then we get 2 copies of $\partial D$ and we attach two discs (which are copies of $D$) to $G$ along $\partial D$. We show that this can be done by a map $g$ which is homotopic to $f$: From $\ker(\pi_1(F) \to \pi_1(M)) = 0$ it follows that $f(\partial D) = 0$ on $F$. Hence there exists a map $g$, homotopic to $f$, such that $g(E \cap G)$ is a point $P \in F$.

Since $f(\partial D) = 0$ on $F$, we may choose this homotopy transverse with respect to $F$. Hence $g(\partial \mathring{D})$ is a point $P_0 \in F \times 0$ and $g|_{\mathring{D}} : (\mathring{D}, \partial \mathring{D}) \to (M - \mathring{F}, P_0)$ is a singular 2-sphere. Since $\pi_2(M - \mathring{F}) = 0$, we may deform $g$, constant on $g^{-1}(F \times I)$ such that $\mathring{E}$ is mapped onto $P_0$. 
The homotopies were transversal with respect to $F$, hence $g|_{E}$ is a map $g|_{E}: E \to F$, $g(E \cap G) = \frac{1}{2}$, $g(E) = 0$, $g(E \cap (G \times 1)) = 1$. The graph $\Gamma(g|_{E})$ is an embedding of $E$ into $E \times I \subset E \times \mathbb{R}^+$. We replace $\Gamma(g|_{E})$ by $\delta(C(E) \cap E \times I) - \Gamma(E)$ in the cone $C(E)$ over $\Gamma(g|_{E})$ from a point $\alpha \times \beta$, $\alpha \in \hat{E}$, $\beta$ sufficiently large in $\mathbb{R}^+$, and we get a map $g'|_{E}: E \to I$, homotopic to $g$, which is the same as $g$ on $\partial E$. Then $(g'|_{E})^{-1}(\frac{1}{2})$ consists of the two 2-elements $\delta(E \cup (G \times 0) \times \frac{1}{2}) - E \cap G$. Since the homotopy was constant on $\partial E$, we may extend it to a homotopy of $N$. Finally we make $g'$ transverse with respect to the new 2-elements. By this process, the Euler characteristic is lowered by 2. After a finite number of steps we either get $\ker((\pi_1(G) - \pi_1(N))) = 0$ or a projective plane or 2-sphere. The first case was excluded by hypothesis. If
ker \((\pi_2(G) \to \pi_2(N)) \neq 0\) for a component \(G\) of \(f^{-1}(F)\), then \(G\) is a 2-sphere \(S^2\). By our hypothesis, on \(\pi_2(N)\), \(S^2\) is contractible. Hence there exists a homotopy 3-ball \(B \subset N\), \(\partial B = S^2\). (Because a lifting of \(S^2\) in the universal cover bounds). Enlarging \(B\), we get a homotopy 3-ball \(B'\), \(S^2 \subset B'\) and \(\partial B' \subset S^2 \times 0\), say.

Since \(\ker(\pi_2(F) \to \pi_2(M)) = 0\), there is a transversal homotopy of \(g\), such that afterwards \(G\) is mapped into a point \(P_1 \in M\). Hence \(g(\partial B')\) is a point \(P_0\) in \(F \times 0\). Now deform \(g\) such that \(B'\) is mapped into \(P_0\); this is possible because \(\pi_3(M) = 0\). Then \(g^{-1}(F)\) does not contain \(G\) and \(g\) is transverse with respect to \(F\).

**Corollary 3.2:** Let \(M\) and \(N\) be irreducible 3-manifolds which do not contain a 2-sided projective plane. Let \(F\) be a system of 2-sided incompressible surfaces in \(M\). Let \(f: N \to M\) be a map. Then there exists a map \(g\), homotopic to \(f\), which is transverse with respect to \(F\) and such that \(G = g^{-1}(F)\) is a system of 2-sided, incompressible surfaces in \(N\).
If $f \mid \partial N$ was transverse with respect to $F$, then the homotopy from $f$ to $g$ may be chosen constant on $\partial N$.

**Proof:** By hypotheses and by (1.5) $\ker(\pi_j(F) - \pi_j(M)) = 0$, $j = 1, 2$. Hence $\pi_1(M)$ is not finite and so by (1.7)

$\pi_2(M) = \pi_3(M) = 0$. By (1.6) $M - UF$ is irreducible. If a component $M'$ of $M - \overline{J(F)}$ is not a ball, then $\partial M'$ contains no 2-spheres or projective planes and $\pi_1(M')$ is infinite (see the first part of the proof of theorem (3.3)). Hence it follows from (1.7) that $\pi_2(M') = 0$. If $N$ is nonorientable, then again by (1.7) $\pi_2(N) = 0$. Hence $M$ and $N$ satisfy the conditions of lemma (3.1).

**Theorem 3.3:** Let $M$ be an irreducible 3-manifold. If $M$ is non-orientable, suppose $M$ does not contain 2-sided $P^2$s. If $M$ is orientable, suppose $\partial M \neq \emptyset$ and $M \neq B^3$ (3-ball). Then there exists in $M$ a 2-sided, non-separating surface $F$, $F \cap \partial M = \partial F$, incompressible in $M$.

**Proof:** (1) If $M$ is a 3-manifold, $\partial M \neq \emptyset$, which does not contain $S^2$s or $P^2$s in the boundary, then $H_1(M)$ contains a factor $Z$ (infinite cyclic group):

If $X$ denotes the Euler characteristic, we have $2X(M) = X(\partial M)$.

Now $X(M) = -\rho_0 + \rho_1 - \rho_2 + \rho_3$, where $\rho_i$ denotes the $i$-th Betti number. If $\partial M$ contains $r$ orientable surfaces of genus $h_1, \ldots, h_r$ respectively and $s$ non-orientable surfaces of genus $k_1, \ldots, k_s$, then

$$X(\partial M) = \sum_{i=1}^{r} 2(h_i - 1) + \sum_{j=1}^{s} (k_j - 2).$$

Hence it follows that
\[ \rho_1 = \rho_0 + \rho_2 - \rho_3 + \frac{r}{s} \sum_{i=1}^{r} (h_i - 1) + \frac{1}{2} \sum_{j=1}^{s} (k_j - 2) \]

Now \( \rho_3 = 0 \), since there are no closed oriented 3-chains in \( M \), because \( \partial M \neq \emptyset \). Since \( M \) is connected, \( \rho_0 = 1 \).

If \( r > 0 \), then \( \rho_2 \geq r-1 \), since \( r-1 \) of the \( r \) orientable surfaces do not bound an oriented 3-chain in \( M \).

Furthermore, by assumption, each \( k_j \geq 2 \) and therefore

\[ \frac{1}{2} \sum_{j=1}^{s} (k_j - 2) \geq 0. \]

Hence, \( \rho_1 = 1 + \rho_2 + \sum_{i=1}^{r} h_i - r + \)

\[ \frac{1}{2} \sum_{j=1}^{s} (k_j - 2) \geq \sum_{i=1}^{r} h_i + \rho_2 - r + 1. \]

If \( r = 0 \), we have

If \( r \neq 0 \), we have \( \rho_1 \geq \sum h_i > 0 \).

\[ \rho_1 \geq 1 + \rho_2 > 0. \]

(2) If \( M \) is non-orientable and \( \partial M = \emptyset \), then \( \rho_1 > 0 \)

(Because \( X(M) = 0 = -\rho_0 + \rho_1 - \rho_2 + \rho_3 = -1 + \rho_1 - \rho_2 \)).

(3) Now we can map \( \bar{\pi}_1(M) \xrightarrow{\pi} H_1(M) \xrightarrow{\varphi} Z \approx \pi_1(S^1) \) nontrivially.

It follows, since \( S^1 \) is aspherical, that there exist a map \( f: M \to S^1 \) which induces \( \varphi \). Let \( x \in S^1 \) be a point.

By Corollary 3.2 we can homotope \( f \) in such a way as to have \( G = f^{-1}(x) \) a system of incompressible 2-sided surfaces in \( M \), \( G \cap \partial M = \partial G \). Since \( \varphi \) is onto, we find an oriented closed curve \( k \) in \( M \), \( k \neq 0 \) in \( M \), which maps non trivially onto a generator \( t \) of \( \pi_1(S^1) \). \( k \) is not homotopic to a closed curve of \( G' \), a component of \( G \), since \( f(G') = x \in S^1 \). We deform \( k \), such that \( k \) is a simple closed curve and \( k \cap G' \) consists of points only.

Starting with a point \( P_0 \in k \cap G' \) and running along \( k \) until we come to the next point \( P_1 \in k \cap G' \), we have
an arc \( t \), \( \partial t = P_0 \cup P_1 \) (\( P_0 = P_1 \) is possible). If \( t \) is not mapped onto \( t \), we join \( P_0 \) and \( P_1 \) by an arc \( t' \) in \( G' \) and obtain a simple closed curve \( k = (k \cup t') - t \) which maps onto \( t \). Eventually we find a simple closed curve \( k^* \) in \( M \) which intersects \( G' \) in just one point and which maps onto \( t \). It follows that \( G' \) does not separate \( M \).

**Remark:** 1) If \( M \) is orientable and \( \partial M \neq \emptyset \) then it is easily proved that the image of \( H_1(\partial M) \to H_1(M) \) is infinite. Therefore we may choose the homomorphism \( H_1(M) \to \pi_1(S^1) \) such that the composition \( H_1(\partial M) \to H_1(M) \to \pi_1(S^1) \) is non-trivial. It follows that we always can find an \( F \) such that \( \partial F \neq \emptyset \). This would simplify the proof of theorems (3.4) and (5.1).

If \( M \) is non-orientable then the image of \( H_1(\partial M) \to H_1(M) \) may be finite even if \( M \) is irreducible. The following example is due to J. Hempel:

Consider a solid Kleinbottle \( M \) whose boundary is generated by two curves \( a \) and \( b \), \( a^2 \sim 0 \) (on the boundary). Remove a regular neighborhood of a simple closed curve in \( M \) homotopic to \( b^2 \) and attach the boundary of a solid torus to the boundary of this neighborhood such that \( b^2 \) is killed. Then the 2-sheeted orientable cover \( \tilde{M} \) looks as follows:

![Diagram of a solid Kleinbottle](image-url)
The curves $t_1, t_2$ of the two tori are killed and $x$ and $y$ correspond to longitudes of the attached two solid tori. Clearly

$$\pi_1(\tilde{M}) = \{x, y, t_1 : t_1 x t_1^{-1} = x, (t_1 x) y (t_1 x)^{-1} = y, \text{(killing): } t_1 = 1, t_1 x = 1 \}$$

$= \mathbb{Z}(y)$. Now $\tilde{M}$ can be embedded in $\mathbb{R}^3$ and so does not contain any fake 3-cells. Since furthermore $\partial \tilde{M}$ is connected $\tilde{M}$ is irreducible. (It follows from a theorem of J. Stallings (Topology of 3-manifolds, Prentice Hall 1962) that $\tilde{M}$ is a solid torus.) Hence $M$ is irreducible, but the image of $H_1(\partial M) \rightarrow H_1(M)$ is not finite (since $H_1(\partial M) = \{a, b : a^2 = 1\}$ and $b^2 = 0$ in $M$).

2) If $M$ is closed and orientable there need not exist incompressible surfaces. According to F. Waldhausen, we call $M$ sufficiently large iff there exists an incompressible surface in $M$.

**Theorem 3.4:** Let $M$ be an irreducible manifold. If $M$ is orientable, suppose $\partial M \neq \emptyset$. If $M$ is non-orientable suppose $M$ does not contain 2-sided projective planes.

Then there exists a hierarchy for $M = M_1, M_j, F_j \subset M_j, U(F_j) \subset M_j, M_{j+1} = M_j - U(F_j), j = 1, \ldots, n, M_{n+1}$ is a ball. We may choose each $F_j$ to be non-separating in $M_j$.

This follows from Haken's theory of normal surfaces [4], [7]. Our proof is analogous to Waldhausen's proof for orientable manifolds [12].
We take a cell decomposition of $M$ and define a **handle decomposition** $\Sigma$ of $M$ as follows:

We associate to every cell (of dimension 0, 1, 2, 3) a 3-cell of $\Sigma$ by first taking pairwise disjoint ball-neighborhoods of the vertices, (the collection of these neighborhoods is denoted by $N^0$), then taking pairwise disjoint beam-neighborhoods (which are small with respect to $N^0$) of the 1-cells (their collection is denoted by $N^1$), and finally taking plate neighborhoods (small with respect to $N^0 \cup N^1$) of the 2-cells (denoted by $N^{II}$). Now $M - N^0 \cup N^1 \cup N^{II}$ consists of a collection $N^{III}$ of 3-balls.

We demand the following properties:

1. $N^{III} \cap (N^0 \cup N^1 \cup N^{II}) = \partial N^{III}$, hence $\partial M \subset N^0 \cup N^1 \cup N^{II}$. 
(2) If we present each beam as $I \times D$, we have $I \times D \cap N^c = \partial I \times D$ and $I \times D \cap N^{II} = I \times d$, where $d$ is a collection of arcs in $\partial D$ ($d$ may be empty).

(3) If we present each plate as $D \times I$, we have

\[
D \times I \cap N^c = e_1 \times I \\
D \times I \cap N^I = e_2 \times I
\]

where $e_1$ and $e_2$ are collections of arcs in $\partial D$ ($e_1 \neq \emptyset \neq e_2$) such that $e_1 \cup e_2 = \partial D$. We consider surfaces which have nice intersections with $N^c$, $N^I$, $N^{II}$ and $N^{III}$. We call a 2-sided surface $F$ in $M$ to be normal, if the following holds:

(1) $F \cap N^{III} = \emptyset$

(II) For any plate, $D \times I \cap F = D \times r$, where $r$ is a collection of points in $I$. 
(iii) For any beam, \( I \times D \cap F = I \times k \), where \( k \) is a collection of arcs in \( D \), \( k \cap \partial D = \partial k \). Furthermore, if \( k_1 \in k \), then \( \partial k_1 \) is not contained in one component of \( d \) (cf(2)) or in one component of \( \partial D - d \), and \( \partial k_1 \) is not contained in adjacent components of \( d \) and \( \partial D - d \).

(iv) \( F \cap B \) consists of discs, for each ball \( B \in N^0 \).

Let \( \Sigma \) be a handle decomposition. The complexity of \( \Sigma \) is a triple of non-negative numbers \( (\chi, \eta, \xi) \) defined as follows:

If \( \delta \) is the number of components of \( d \) (in(2)) and \( \delta'' = \max(\delta - 2, 0), \delta' = \max(\delta - 1, 0) \), then \( \chi = \Sigma \delta'' \)
\( \eta = \Sigma \delta' \), where the sum is over all beams of \( \Sigma \). If \( B \) is a ball \( B \in N^0 \) and \( \varepsilon \) the number of components of \( B \cap (N^I \cup N^{II}) \), then \( \xi = \Sigma (\varepsilon - 1) \), where the sum is over all balls \( B \in N^0 \).

**Lemma 1**

If \( \Sigma \) is a handle decomposition of \( M, \partial M \neq \emptyset \) with complexity \( (\chi, \eta, \xi) \) and if \( G \) is a non-separating normal surface in \( M, \partial G \neq \emptyset \), then \( M' = M - U(G) \) has a handle decomposition \( \Sigma' \) of complexity \( (\chi', \eta', \xi') \) \( < (\chi, \eta, \xi) \) in lexicographical ordering.

**Proof:** Choose \( U(G) \) small with respect to \( \Sigma \). The Balls, Plates and Beams of \( M' \) are defined to be the components of \( N - U(G) \), where \( N \) is a Ball, Beam or Plate of \( M \). This defines a handle decomposition for \( M' \). \( U(G) \) intersects the disc \( D \) of a beam \( I \times D \) in a system of curves \( k \), which satisfies condition (iii). We have \( k \cap \partial D = \partial k \) and \( k \cap \partial D = \emptyset \). \( D - U(k) \) consists of discs \( D_1, \ldots, D_n \). Let \( \delta_j \) be the number of components of \( d \cap D_j \) \( (j = 0, 1, \ldots, n) \), where \( D_0 = D \). Let \( \delta_j'' = \max(\delta_j - 2, 0) \),
\( \delta_j' = \max (\delta_j -1, 0) \). Then \( \chi' = \Sigma \delta_j'', \eta' = \Sigma \delta_j' \). Now assume \( k \) consists of one arc \( k_1 \). It is easily seen by distinguishing cases (e.g. Figure), that if \( \partial k_1 \) is not contained in the same component of \( d \), then always \( \delta'' = \delta_0' = \Sigma \delta_j' \). Moreover,

(a) if \( \partial k_1 \) is contained in non-adjacent components of \( d \) and \( \partial D - d \), then \( \delta'' > \Sigma \delta_j' \)

(b) if \( \partial k_1 \) is contained in different components of \( \partial D - d \) then \( \delta' > \Sigma \delta_j' \)

Now we remove \( U(k) \) by removing one component of \( U(k) \) after the other. Then we never come across an arc \( k_1 \) such that \( \partial k_1 \) is contained in one component of \( d \). Hence we always
have \( \delta'' = \Sigma \delta_j'' \). We consider 3 cases:

a) \( G \cap N^{II} \neq \emptyset \).

Since by hypothesis \( G \cap \partial M \neq \emptyset \), there exists at least one beam \( I \times D \) and at least one arc \( k_1 \in k \), such that \( \partial k_1 \) is contained in non-adjacent components of \( d \) and \( \partial D - d \) (by iii) and hence we have \( \delta' > \Sigma \delta_j'' \) by (a). Hence \( \chi' < \chi \).

b) \( G \cap N^{II} = \emptyset \), but \( G \cap N^{I} \neq \emptyset \).

There exists at least one beam and one arc \( k_1 \in k \) such that \( \partial k \subset \partial D - d \), hence by (b) : \( \delta' < \Sigma \delta_j' \). Hence \( \chi' < \chi \) and \( \eta' < \eta \).

c) \( G \cap N^I = \emptyset \). Then \( \chi' = \chi, \eta' = \eta \). Since \( G \) does not intersect beams and plates, \( G \) must be a disc by (i), whose boundary is contained in \( \partial M \). Thus we have \( \xi' \leq \xi \). If \( \xi = \xi' \) then the disc \( G \) would be contained in a ball \( B \in N^o \) and would be parallel to a disc in \( \partial M \). Since \( G \) was non-separating, this case can not occur.

**Lemma 2:** If \( \Sigma \) is a handle decomposition of \( M \) and \( M \) is irreducible and if \( F \) is an incompressible, non-separating surface in \( M \), then there exists an incompressible, non-separating surface \( G \) in \( M \), which is normal. If \( \partial F \neq \emptyset \), then \( \partial G \neq \emptyset \).

**Proof:** If we call a surface \( F \) in \( M \) characteristic if \( F \) is incompressible and non-separating, then by [4] or [7,2.2] the lemma will be proved if we show that the results of the two following operations (\( \rho \)) and (\( \sigma \)) contain again characteristic surfaces.
(ρ) If $D$ is a disc in $\dot{M}$ such that $D \cap F = \partial D$, then we replace a neighborhood of $D \cap F$ in $F$ by 2 copies $D', D''$ of $D$. Since $F$ is incompressible, we obtain a 2-sphere and an incompressible surface $F'$, $\partial F' = \partial F$. Since $M$ is irreducible it follows that $F'$ is characteristic.

(σ) If $D$ is a disc in $M$, $D \cap (F \cup \partial M) = \partial D$, such that each of $D \cap F$ and $D \cap \partial M$ is one arc, we replace a neighborhood of $D \cap F$ in $F$ by two copies $D', D''$ of $D$.

The result $F'$ is clearly incompressible. Since $F$ has been non-separating, there exists a simple closed curve $k \in M$ such that the intersection number $S(k,F) = 1 \pmod{2}$. We may assume $k \cap \partial D = \phi$. Let $S(k,D) = m$, then $S(k,F') = 2m+1 = 1$. Hence $F'$ (or if $F$ has been disconnected, one component of $F'$) is non-separating and therefore is characteristic (and $\partial F' \neq \phi$).
Proof of 3.4

If $M$ contains 2-sided, non-separating, incompressible surfaces $F_j$ which are closed, then by the finiteness theorem on incompressible surfaces [6], there exists only a finite number of the $F_j$'s, $F_1, \ldots, F_k$ say, which are not parallel in $M$ ($k$ may be 0). If $M_1 = M$ is a ball, there is nothing to prove. If $M_1$ is not a ball then no $F_j$ is a 2-sphere. Hence by (1.6a) and induction $M_{k+1}$ satisfies the conditions of theorem 3.3 and therefore there exists in $M_{k+1}$ a 2-sided, non-separating surface $F_{k+1}$.

Case 1) If $F_{k+1}$ is closed in $M_{k+1}$, then $F_{k+1}$ is closed in $M$ and so by the finiteness theorem $M_{k+2} = M_{k+1} - U(F_{k+1})$ is homeomorphic to $F_{k+1} \times I$. Since $F_{k+1}$ was different from a 2-sphere or projective plane, we find a hierarchy $H$ for $F_{k+1}$ and $H \times I$ is a hierarchy for $M_{k+2}$, and $F_1, \ldots, F_{k+1}$, $H \times I$ a hierarchy for $M$.

Case 2) $\exists F_{k+1} \neq \phi$.
We take a handle decomposition $\Sigma_{k+1}$ of $M_{k+1}$ and we normalize $F_{k+1}$ by isotopic deformations and by the $(\rho)$ and $(\sigma)$ operations (lemma 2). By lemma 1, $M_{k+2} = M_{k+1} - U(F_{k+1})$ has a handle decomposition of complexity $\chi_{k+2}(k+2) \zeta_{k+2} < \chi_{k+1}(\eta_{k+1}, \zeta_{k+1})$. Now we may proceed by induction (using proposition 1.6), since we may assume that every incompressible, non-separating, 2-sided surface $F_{k+j}$ ($j \geq 1$) has nonempty boundary in $M_{k+j}$. (Because otherwise $F_{k+j}$ would be closed in $M$ and we could apply case 1). We get a sequence of triples, such that
$(x_\ell, \eta_\ell, \zeta_\ell) > (x_{\ell+1}, \eta_{\ell+1}, \zeta_{\ell+1}) \geq (0,0,0)$. Since such a sequence is of finite length, we eventually come across a $M_{k+m}$ which does not contain an incompressible 2-sided non-separating surface. By theorem 3.3, $M_{k+m}$ is a ball.
§4 Line bundles over surfaces

Definition 4.1 Let $M = F \times I$ be the product line bundle over the surface $F$, $p : M \to F$ the projection onto $F$. A subspace $X$ of $M$ is vertical iff $X = p^{-1}(pX)$. A homeomorphism $h : M \to M$ is level-preserving iff it can be written as $h(x, y) = (f_y(x), y)$, $x \in F$, $y \in I$. An isotopy $h_t$ $(0 \leq t \leq 1)$ is level-preserving iff each homeomorphism $h_t$ is.

Lemma 4.2 Let $M = F \times I$, $F \neq S^2, P^2$. Let $G$ either be a system of discs in $M$ each component of which intersects $\partial F \times I$ in two vertical arcs, or a system of 2-sided incompressible annuli in $M$ each component of which has one boundary curve in $F \times 0$ and the other in $F \times 1$.

Then there exists an isotopy, constant on $(F \times 0) \cup (\partial F \times I)$, which makes $G$ vertical. This isotopy may be composed of isotopies which are either constant on $\partial M$ or level-preserving and constant on $(F \times 0) \cup (\partial F \times I)$.

Remark: If $F \neq S^2, P^2$, then $M = F \times I$ is irreducible.

Proof:

Let $G_1$ be a component of $G$. Let $k_1 = G_1 \cap F \times 0$, $k_2 = G_1 \cap F \times 1$.

Let $k'_1 = k_1 \times I \cap F \times I$, then $k'_1 \cap \partial(F \times 1) = k_2 \cap \partial(F \times 1) = \partial k'_1 = \partial k_2$. Now $k_2$ and $k'_1$ are homotopic, by a homotopy (namely the projection of $G_1$ to $F \times 1$) which is constant on $\partial k_2$. It follows by Baer's theorem [2] that there exists an isotopy, constant on $\partial(F \times 1)$, which takes $k_2$ to $k'_1$. We extend this isotopy to a level-preserving isotopy of $M$, which is constant on $F \times 0 \cup \partial F \times I$. 
Case 1: \( G_1 \) is a disc

Let \( G'_1 = p^{-1}(k_1) \), where \( p: M \to F \) is the projection. After small deformations of \( G \), constant on \( \partial M \), \( G_1 \cap G'_1 \) consists of their common boundary \( k_1 \cup k'_1 \) and a number of simple closed curves with transversal intersection.

Since \( M \) is irreducible we may remove the intersection curves by isotopic deformations (constant on \( \partial M \)). Then \( G_1 \cup G'_1 \) bounds a ball and there is a deformation (constant on \( \partial M \)), which takes \( G_1 \) to \( G'_1 \). Now split \( M \) at \( G_1 \) and proceed inductively.

Case 2: \( G_1 \) is an annulus:

\( k_1 \) is a non-contractible 2-sided curve in \( F \times 0 \). If \( \partial F \neq \emptyset \) then there exists a simple arc \( k \subset F \times 0 \), \( k \cap \partial(F \times 0) = \partial k \), such that \( k \cap k_1 \) consists of one or two points, and cannot be made smaller by an isotopy of \( k \). We claim that if \( F \) is closed, there exists a simple closed 2-sided curve \( k \) with that property. Because if \( k_1 \) is non-separating, there exists a curve \( k' \) which intersects \( k_1 \) in one point. If \( k' \) is 2-sided, we take \( k \) to be \( k' \). If \( k' \) is 1-sided we take \( k \) to be the boundary of a regular neighborhood of \( k' \). If \( k \sim 0 \) in \( F \), then \( F \) would be a projective plane, which is impossible; clearly \( k_1 \cap k \) cannot be made empty by isotopic deformations of \( k \). If \( k_1 \) separates \( F \) into \( F' \), \( F'' \) then there is a curve \( k' \) which intersects \( k_1 \) in 2 points, and such that \( k_1 \cap k' \) is minimal. If \( k' \) is not 2-sided, then at least one component of \( F - \text{U}(k_1) \), \( F' \) say, contains a moebiusband in its interior. By tracing \( k' \) once around the
center of this moebiusband, we get a closed curve \( k \) which is 2-sided and such that \( k_1 \cap k \) consists of 2 points and is minimal.

Now let \( H = p^{-1}(k) \). After small deformations of \( G \) constant on \( \partial M \), \( G_1 \cap H \) consists of simple closed curves and arcs. If \( G_1 \cap H \) contains an arc \( \ell \) which has both end points in \( F \times 0 \), then \( \partial \ell = \partial H \cap k_1 \). Now \( \ell \) splits a disc in \( H \) and a disc \( \tilde{D} \) in \( G_1 \) (since \( \partial \ell \) is contained in one boundary component of \( G_1 \)). But \( \partial \tilde{D} = \ell \cup k_1' \), where \( k_1' \) is an arc in \( k_1 \). So, if we project \( D \) onto \( F \times 0 \), we see that we can make \( \partial H \cap k_1 \) empty, contrary to the definition of \( H \). Now \( \partial H \cap k_1 \) and \( \partial H \cap k_2 \) have the same number of points, because \( k_2 = (k_1 \times I) \cap F \times 1 \), hence any arc in \( G_1 \cap H \) intersects both \( F \times 0 \) and \( F \times 1 \). It follows, that any closed curve in \( G_1 \cap H \) is contractible in both \( G_1 \) and \( H \), and so these can be removed, since \( M \) is irreducible. Hence we have the result that there is a deformation of \( G \) composed of one which is level-preserving and constant on \( F \times 0 \cup \partial F \times I \), and one which is constant on \( \partial M \), such that afterwards \( G_1 \cap H \) consists of one or two vertical arcs \( p^{-1}(k_1 \cap k) \). By splitting \( M \) at \( H \), we get a manifold \( \tilde{M} = \tilde{F} \times I \), where \( \tilde{F} = F - U(k) \), and a system of discs \( \tilde{G} = G_1 \cap \tilde{M} \). By case 1, there exists a deformation, composed of one which is level-preserving and constant on \( \tilde{F} \times 0 \cup \partial \tilde{F} \times I \), and one which is constant on \( \partial M \), which makes \( \tilde{G} \) vertical. Hence there is a deformation of \( M \) of this sort, which makes \( G_1 \) vertical. Now we may split \( M \) at \( G_1 \) and proceed by induction.
Lemma 4.3: Let $F$ be a closed surface, $M = F \times I$ and let $G$ be a 2-sided incompressible annulus in $M$ such that $G \cap (F \times 0) \neq \emptyset \neq G \cap (F \times 1)$. Let $S^1 \times I$ be a fiberering of $G$. Then there exists a fiberering $F \times I$ of $M$ which is consistent with the fiberering of $G$.

Proof:

By lemma 4.2 there is an isotopy of $M$, which makes $G$ vertical. We use the inverse isotopy to deform the fiberering of $M$ and have an induced fiberering on $G$. Thus, there is an isotopy $g_t: G \rightarrow G$ which carries the fiberering $S^1 \times I$ of $G$ into that induced by $M$. We expand this isotopy to an isotopy $h_t: U(G) \rightarrow U(G)$, where $U(G) \approx G \times I$, by

$$
hs(x, \tau) = \begin{cases} 
(g_s(x), \tau) & , x \in G \times 0, \tau \leq \tau \\
(g_s(x), \tau) & , x \in G \times 0, \tau \geq \tau 
\end{cases}
$$

In particular, we have $h_0(x, \tau) = (g_0(x), \tau), \ h_1(x, \tau) = (g_1(x), 1)$ where $h_0(x, 0) = (g_0(x), 0)$ is the original fiberering of $G$ transferred to $G \times 0$ and $h_1(x, 1) = (g_1(x), 1)$ is the fiberering of $G$ which is induced by $M$, transferred to $G \times 1$. By our definition of $h_\tau$, we see that we may extend this isotopy to an isotopy of $M$, which proves (4.3).

Proposition 4.4: Let $F$ be a closed surface different from $S^2$ and $P^2$. Let $M = F \times I$ be a two-sheeted cover of a 3-manifold $N$. Then $N$ is homeomorphic to a line bundle over a closed surface.
Proof: Let \( f: M \to N \) be the covering map and \( g: M \to M \) the covering translation. Suppose we have constructed a fibering of \( M \) (as a line bundle over \( F \)) which is invariant under \( g \) (i.e. \( g \) maps each fiber homeomorphically onto a fiber). Then, since \( g \) has no fixed points, for any fiber \( I \) we have \( g(I) \cap I = \emptyset \). Hence \( f \) maps each fiber \( I \) homeomorphically into \( N \). If \( g \) does not interchange \( F \times 0 \) and \( F \times 1 \) both \( F \times 0 \) and \( F \times 1 \) are 2-sheeted coverings of \( \partial N \) and it follows that \( N \cong G \times I \), where \( G = f(F \times 0) \). If \( g \) interchanges \( F \times 0 \) and \( F \times 1 \), then each of \( F \times 0 \) and \( F \times 1 \) maps homeomorphically onto \( \partial N \) and hence \( N \) is a twisted line bundle over a closed surface. Construction of a fibering, which is invariant under \( g \):

(i) There exists a 2-sided incompressible annulus \( G_1 \) in \( M \) such that \( G_1 \cap (F \times 0) \not= \emptyset \not= G_1 \cap (F \times 1) \) and either \( G_1 \cap g(G_1) = \emptyset \) or \( G_1 = g(G_1) \).

Proof: By our hypothesis on \( F \), there exists a 2-sided vertical incompressible annulus \( G \) in \( M \). Since \( f(\partial G) \subset \partial N \) and since \( f|G: G \to N \) has no local singularities we may isotopically deform \( G \) such that \( f|G \) will consist of simple closed double curves and simple double arcs only.

We consider 4 cases:

(a) There is a disc \( D \) in \( G \), such that \( D \cap g(G) = \partial D \).

Since \( G \) and \( g(G) \) are incompressible, there is a disc \( D' \) in \( g(G) \), \( \partial D' = \partial D \). Then \( g(D) \) and \( g(D') \), too, are discs
which intersect (transversally) along the closed curve \( g(\partial D) \).
Hence if we replace \( D' \) in \( g(G) \) by a disc \( D'' \) near \( D \), at
the other side, and replace \( g(D') \) in \( G \) by the corresponding
disc \( g(D'') \) near \( gD \), at least one intersection curve has
vanished. So we assume, there are no intersection curves
of this type.

(b) There is a disc \( D \) in \( G \), such that \( D \cap (\partial G \cup g(G)) = \partial D \)
and \( D \cap g(G) \) is one arc \( k \). Since the boundary \( \partial G \) of \( G \) is
mapped (under \( g \)) onto \( \partial (gG) \), it follows that the boundary of
\( k \) is contained in one boundary component of \( g(G) \) and hence
that \( k \) cuts off a disc \( D' \) in \( g(G) \). Hence, replacing \( D' \) in
\( g(G) \) by a disc near \( D \), at the other side, and doing the
corresponding change at \( G \), at least one intersection curve
will vanish.

So we assume, there are no intersection curves of this type.
Hence we only have to consider closed intersection curves,
which are not contractible in \( G \) (in this case there are no
double arcs, since \( G \cap g(G) \) does not have triple points),
and double arcs which intersect both \( G \times 0 \) and \( G \times 1 \) (in
this case there are no noncontractible double curves).

c) \( G \cap g(G) \) consists of closed curves only, each of which
is parallel in \( G \) to a boundary curve of \( G \). Let \( U(f(G)) \) be
a regular neighborhood in \( N \) and let \( V = f^{-1}(U(f(G))) \).
Then \( (\partial V - \partial M) \) is non-singular and contains four annuli,
which are incompressible. We claim that if \( F \) is not a

"torus or Kleinbottle, then there exists among those four
annuli at least one annulus \( G_1 \) which intersects both \( F \times 0 \)
and \( F \times 1 \). Then it follows that either \( g(G_1) \cap G_1 = \emptyset \) or \( g(G_1) = G_1 \) and we are through.

If \( \partial V - \partial M \) would not contain such an annulus \( G_1 \), then there would be a component of \( g(G) - (G \cap g(G)) \), which would meet \( G \) from different sides. Consider the map \( h : H \to M \) which maps an annulus \( H \) onto \( g(G) \). By homotoping \( h \), we may assume that \( h \) maps one boundary curve of \( H \) onto \( g(G) \cap (F \times 1) \) and the other one onto a (simple closed) curve in \( F \times 0 \) which lies below \( g(G) \cap (F \times 1) \) (i.e. which is \( ((g(G) \cap F \times 1) \times 1) \cap (F \times 0) \)).

Then if \( p : F \times I \to F \times 0 \) is the projection, \( ph(a) = g(G) \cap F \times 0 \), and \( ph(b) \) is mapped onto a closed curve which is essential in \( F \times 0 \), where \( b \) is any arc in \( H \) such that \( h(\partial b) \)
consists of a point in \( g(G) \cap (F \times 1) \) and of a point lying below this point in \( F \times 0 \). Hence \( \phi(H) \) determines a singular torus in \( F \times 0 \). Thus we may find a map \( t: T \to F \), from the torus \( T \) onto \( F \), such that \( t \) maps the longitude and meridian of \( T \) onto non-contractible curves in \( F \). Since \( F \neq P^2 \), \( \pi_1(F) \) does not contain elements of finite order and therefore it follows that \( t_*: \pi_1(T) \to \pi_1(F) \) is injective. From theorem 2.1 it follows that \( T \) is a covering of \( F \) and hence that \( F \) is the Kleinbottle or the torus. Now suppose that in this case \( \delta V - \delta M \) does not contain an annulus which intersects both \( F \times 0 \) and \( F \times 1 \).

\[
\begin{array}{c}
\text{F} \times 1 \\
\vdots \\
\vdots \\
\vdots \\
\text{F} \times 0
\end{array}
\]

Then performing cuts ("Umschaltung") along closed double curves of \( g \) and \( g(G) \), we get an annulus \( G' \), such that after lifting a regular neighborhood \( U(f(G')) \) to \( M \) the system \( \delta V - \delta M \) (where \( V = f^{-1}U(f(G')) \)) will contain at least 2
2 annuli each of which intersects both $F \times 0$ and $F \times 1$. If $G_1$ is one of the latter, we again have $G_1 \cap g(G_1) = \emptyset$ or $G_1 = g(G_1)$.

d) $G \cap g(G)$ consists of arcs only, each of which intersects both $F \times 0$ and $F \times 1$:

$f(G)$ is a singular annulus or moebiusband. If $t$ is a double arc of $f(G)$, then there are two possibilities to do a cut at $t$. We claim that we can always obtain a singular annulus or moebiusband with non-contractible boundary.

Proof: Let $f(G)$ be an annulus. Then the inverse image of $t$ in $G$ will consist of 2 arcs $t'$, $t''$.

The first cut gives an annulus which has as boundary curves $\sigma \varphi^{-1}$ and $\nu \tau^{-1}$. The second cut gives two moebius strips with boundary curves $\sigma \nu$ and $\varphi \tau$ respectively. Suppose, $\sigma \nu = 0$ and $\varphi \tau = 0$. Then $0 \neq (\tau \sigma)(\nu \varphi) = \tau \varphi = 0$ in $N$, a contradiction. So at least one of the boundary curves is non-contractible.
One cut gives two annuli with boundaries $\sigma, \nu$ and $\tau, \varphi$. The other gives one annulus with boundary $\sigma \tau^{-1}, \nu \varphi^{-1}$. Again we see that at least one annulus has non-contractible boundary in $N$.

c) $f(G)$ is a moebiusband:

One cut gives a moebiusband with boundary $\sigma \tau^{-1}, \nu \varphi^{-1}$ the other one gives a moebiusband with boundary $\sigma \nu$ and an annulus with boundary $\tau$ and $\varphi$.

Note all of $\varphi, \sigma \nu$ and $\sigma \tau^{-1}, \nu \varphi^{-1}$ are $\sim 0$.

d)
The other cut gives an annulus with boundary \( \sigma, \nu \) and a moebiusband with boundary \( \varphi_T \). At least one boundary curve is not contractible in \( \mathbb{N} \).

Performing cuts so that we always get a singular annulus or moebiusband with non-contractible boundary we finally get a non-singular annulus or a non-singular moebiusband \( \mathbb{H} \). \( f^{-1}(\mathbb{H}) \) consists of two or one incompressible annuli, which satisfy condition (i).

(ii) Let \( G_1 \) be an annulus with properties (i). Let \( G_2 = g(G_1) \). Suppose \( G_1 \cap g(G_1) = \emptyset \). By lemma 4.2 there exists an isotopy of \( M \), which makes \( G_1 \cup G_2 \) vertical. Using the inverse isotopy, we may deform the fibering of \( M \), keeping \( G_1, G_2 \) fixed. Then we get induced fiberings on \( G_1 \) and \( G_2 \). Deforming \( M \) near \( G_2 \) such that the fibering of \( G_2 \) induced by the fibering in \( M \) is carried onto the fibering of \( G_2 \) which is defined by \( G_2 = g(G_1) \), we make \( g|_{G_1} \) fiber-preserving. Extending this deformation to suitable neighborhoods we find vertical neighborhoods \( U(G_1) \) and \( U(G_2) \), such that \( g(U(G_1)) = U(G_2) \) and \( g|U(G_1) \) is fiber preserving. If \( G_1 = g(G_1) \), \( G_1 \) covers a (2-sided) moebius-strip \( f(G_1) \) or a 2-sided annulus. Hence there exists a fibering \( S^1 \times I \) of \( G_1 \), such that \( g(I) \cap I = \emptyset \), for each fiber \( I \). By lemma 4.3 there exists a fibering of \( M \) as \( F \times I \) consistent with that fibering of \( G_1 \). Noting that \( f(G_1) \) is 2-sided in \( N \), we find a fibered vertical neighborhood \( U(G_1) \) such that \( g(U(G_1)) = U(G_1) \) and \( g \) maps each fiber \( I \) homeomorphic onto a fiber \( g(I) \), \( I \cap g(I) = \emptyset \).
(iii) If $G_1 \neq G_2$ let $M'$ be a component of $\overline{M - U(G_1) \cup U(G_2)}$ and assume $g(M') = M'$. If $G_1 = G_2$, let $M'$ be a component of $\overline{M - U(G_1)}$. Then $g(M') = M'$.

If $\partial M' \neq S^2$, there is a vertical disc $D$ in $M'$, such that $\partial D$ is not contractible in $\partial M'$, and such that $g(\gamma_i) \cap \gamma_j = \emptyset$, $i, j = 1, 2$ where $\gamma_1$ and $\gamma_2$ are the arcs $D \cap (U(G_1) \cup U(G_2))$, respectively the arcs $D \cap U(G_1)$.

By small isotopic deformations constant on $U(G) \cup U(gG)$, the singularities of $f|D$ will consist of simple double arcs and closed simple double curves only, so we can eliminate these by the usual arguments (step (i) (a) and (b)).

So we find a disc $D'$ in $M'$, $\partial D'$ not contractible in $\partial M'$

$$(D' \cup g(D')) \cap (U(G_1) \cup U(G_2)) = (D \cup g(D)) \cap (U(G_1) \cup U(G_2)),$$

or respectively: $(D' \cup g(D')) \cap U(G_1) = (D \cup g(D)) \cap U(G_1)$, and $g(D') \cap D' = \emptyset$.

By lemma 4.2 we make $D' \cup g(D')$ vertical by a deformation of the fibering of $M'$, constant on $\overline{\partial M' \cap \overline{M}}$. 

By further deformations of the fibering of $M'$, constant on $\partial M' \cap M$, we achieve vertical regular neighborhoods $U(D')$, and $U(gD')$ such that $g(U(D')) = U(gD')$ and such that $g|U(D')$ is fiber preserving.

(iv) If $\partial M'' \neq S^2$, $M''$ a component of $\overline{M' - U(D') \cup g(U(D'))}$, then we repeat step (iii) for $M''$. Continuing in this way and noting that $M', M'', \ldots$ are submanifolds in $M$, we finally construct a maximal submanifold $M^*$ in $M$, which is the union of all the neighborhoods $U(D'), U(D''), \ldots$, such that $M^*$ is vertical $g(M^*) = M^*$ and $g|M^*$ is fiber preserving. Then we have: If $\tilde{M}$ is any component of $M-M^*$, then either $g(\tilde{M}) \neq \tilde{M}$ or there is no disc $\tilde{D}$ with $\partial \tilde{D}$ noncontractible in $\partial \tilde{M}$. In the latter case $\tilde{M}$ is a ball and $g(\tilde{M}) \neq \tilde{M}$, because $g$ has no fixpoint. Since $g(\tilde{M}) \neq \tilde{M}$ we may, if necessary, define a new fibering of $g(\tilde{M})$, such that $g|M'$ is fiber preserving. Hence we have constructed a fibering of $M$ which is invariant under $g$.

**Proposition 4.5:** Let $M$ be an irreducible manifold, let $F$ and $F'$ be different incompressible boundary components of $M$.

(Neither $M$ nor $F'$ need be compact). Suppose that for any closed curve $k$ on $F$, some non-null multiple of $k$ is homotopic to a curve in $F'$. Then $M$ is homeomorphic to $F \times I$.

**Proof:** If $M$ is orientable, this is lemma 5.1 of [11]. If $M$ is non-orientable, let $\tilde{M}$ be the orientable (2-sheeted) cover of $M$, $\tilde{F}$ and $\tilde{F}'$ liftings of $F$ and $F'$. We have $\tilde{F} \neq \tilde{F}'$ and $\tilde{F}, \tilde{F}' \subset \partial \tilde{M}$. Furthermore $\tilde{F}$ and $\tilde{F}'$ are incompressible. (This follows from the commutative diagram
Let \( p: \tilde{M} \to M \) be the covering map. Let \( \tilde{k} \) in \( \tilde{F} \) be any closed

curve, then a non-null multiple of \( p(\tilde{k}) = k \) is homotopic to

curve \( k' \) in \( F' \), \( k' = k^\alpha \), say. But \( \tilde{k}^\alpha \) covers \( k^\alpha \) and thus

is homotopic (in \( \tilde{M} \)) to a lifting \( \tilde{k}' \) of \( k' \) which lies in \( \tilde{F}' \).

Hence the hypothesis of our lemma applies to \( \tilde{M}, \tilde{F}, \tilde{F}' \) and

from lemma 4.5 for orientable manifolds it follows that

\( \tilde{M} \approx \tilde{F} \times I \).

By proposition 4.4 and from the fact that \( M \) has

two different boundary components it follows that \( M \approx F \times I \).
§5 Existence of homeomorphisms

Theorem 5.1 Let $M$ and $N$ be manifolds which are irreducible, boundary irreducible and which do not contain 2-sided projective planes. (If $M$ is orientable and $\partial M = \emptyset$) suppose $M$ is sufficiently large. Let $\pi_1(N) \neq 1$. Let $f: (N, \partial N) \to (M, \partial M)$ be a map such that $\ker(f_*: \pi_1(N) \to \pi_1(M)) = 0$.

Then there exists a homotopy $f_t: (N, \partial N) \to (M, \partial M)$, $t \in I$ of $f = f_0$ such that either (a) or (b) holds:

(a) $f_1: N \to M$ is a covering map.

(b) $N$ is a line bundle over a closed surface and $f_1(N) \subset \partial M$.

If $f|_{\partial N}$ is locally homeomorphic, then the homotopy may be chosen constant on $\partial N$.

**Proof:**

By our hypothesis on $M$ there exists a hierarchy for $M = M_1$:

$$M_j, F_j \text{ 2-sided in } M_j, U(F_j) \subset M_j,$$

$$M_{j+1} = (M_j - U(F_j)) \quad , \quad j = 1, \ldots, n.$$

**Case 1:** $\partial M \neq \emptyset$ and $\partial F_j \neq \emptyset$ for each $j = 1, \ldots, n$.

Let $R$ be a boundary component of $N$ ($\partial N$ may be $\emptyset$). Since $N$ is irreducible and $\pi_1(N) \neq 1$ it follows that $R \neq S^2$. Furthermore $R \neq \mathbb{P}^2$. $f(R)$ is contained in some boundary component $S$ of $M$.

From the commutative diagram

$$
\begin{array}{ccc}
\pi_1(R) & \to & \pi_1(S) \\
(f|R)_* \downarrow & & \downarrow \\
\pi_1(N) & \to & \pi_1(M)
\end{array}
$$
and the fact that $N$ is boundary irreducible it follows from $\ker f^* = 0$ that $\ker (f|\partial R)^* = 0$. By theorem 2.1 $f|\partial R$ is homotopic to a covering map. Homotop $f: N \to R$ such that $f|\partial N$ is homotopic to a covering map. By a general position homotopy we achieve that $f^{-1}(\partial M) = \partial N$.

If $f|\partial N$ was locally homeomorphic, then the homotopies so far can be chosen constant on $\partial N$ and since all the following homotopies will be constant on $\partial N$ the last assertion of 5.1 will be proved.

For $r = 1$ we have proved the following

$\ast$ $f|f^{-1}(\partial M \sqcup \bigcup (F_r))$ is locally homeomorphic. Suppose $\ast$ is true for $r$. Let $N'$ be a component of $f^{-1}(M_r)$, let $f' = f|N'$. Then $f':(N', \partial N') \to (M_r, \partial M_r)$ and by induction hypothesis $f'|\partial N'$ is locally homeomorphic. By corollary 3.2 $f'$ is homotopic to a map $\hat{f}'$ which is transverse with respect to $F_r$, by a homotopy which is constant on $\partial N'$, and such that $\hat{f}'^{-1}(F_r)$ is a system of 2-sided incompressible surfaces in $N'$ which does not contain a $P^2$. From the commutative diagram

$$
\begin{array}{ccc}
N' & \xrightarrow{f'} & M_r \\
\downarrow i & & \downarrow \\
N & \xrightarrow{f} & M
\end{array}
$$

it follows from the fact that $\ker f^* = \ker i^* = 0$, that $\ker \hat{f}' = 0$. (Proof that $\ker i^* = 0$: $N'$ is a component of
\[ N = \bigcup_{j=1}^{r-1} f^{-1}U(F_j). \] Since \( f|f^{-1}U(F_j) \) is locally homeomorphic

\( f^{-1}F_j \) is a surface in \( N \), \( j < r \), which is incompressible by the
following diagram

\[
\begin{array}{ccc}
  f^{-1}F_j & \longrightarrow & N \\
  \downarrow & & \downarrow \\
  F_j & \longrightarrow & M
\end{array}
\]

Hence by lemma 1.6(b) \( \ker(i: N' \rightarrow N)_* = 0. \)

Now it follows from \( \ker f'_* = 0 \) and

from the commutative diagram

\[
\begin{array}{ccc}
  \hat{f}|G & \longrightarrow & F_r \\
  \downarrow & & \downarrow \\
  N' & \longrightarrow & M_r
\end{array}
\]

that \( \ker(\hat{f}'|G)_* = 0 \), for any component \( G \) of \( \hat{f}'^{-1}(F_r) \).

By theorem 2.1 it follows that \( \hat{f}'|G: G \rightarrow F_r \) is homotopic to

a covering map by a homotopy which is constant on \( \partial G \) except

possibly in the cases (1) (2) (3) below. Suppose we can

prove \( \hat{f}'|G: G \rightarrow F_r \) is homotopic to a covering for any component

\( N' \) and any \( G \), then, by induction (*) would be true for all

\( r = 1, \ldots, n+1 \). Now \( M_{n+1} \) is a ball. For any component \( N^* \)

of \( f^{-1}(M_{n+1}) \) we have \( N^* \neq N \), since \( \pi_1(N) \neq 0 \) and \( f_* \) is injective.

Since \( f|f^{-1}(\partial M_{n+1}) \) is a covering map, and since any covering

map onto a \( S^2 \) is a homeomorphism it follows that \( N^* \) is a ball,

since \( N \) is irreducible. Hence there is a homotopy of \( f|N^* \)

which is constant on \( \partial N^* \) such that afterwards \( f|N^* \) is a

homeomorphism. This proves case (a) of the theorem.
If there is a component $N'$ of $M_\tau$ and a $G$ such that $f|G: G \rightarrow F_\tau$ is not homotopic to a covering map, then by theorem 2.1 we have 3 possibilities (since $G \neq S^2$, $P^2$):

1. $F_\tau$ is a disc; $G$ is a disc, and the covering map $f|\partial G$ is not a homeomorphism.

2. $F_\tau$ is not a disc; Then $G$ is not a disc. By (2.1), $G$ is an annulus, and $f'|G: (G, \partial G) \rightarrow (F_\tau, \partial F_\tau)$ contracts to $(\partial F_\tau, \partial F_\tau)$.

3. $F_\tau$ is not a disc; $G$ is a moebiusband (2-sided), and $f'|G: (G, \partial G) \rightarrow (F_\tau, \partial F_\tau)$ contracts to $(\partial F_\tau, \partial F_\tau)$. In case (1) it is easily seen and in cases (2) and (3) it follows from lemma 2.3 that there exists a simple arc $\ell$ in $G$, $\ell \cap \partial G = \partial \ell$, such that $f' (\partial \ell)$ is one point and $f'|\ell : (\ell, \partial \ell) \rightarrow (F_\tau, f'(\partial \ell))$ contracts to a point. Now, since by hypothesis $\partial F_i \neq \emptyset$ for $i=1, \ldots, r$, we find an arc which joins $f'(\partial \ell)$ inside $$(\bigcup_{j=1}^r(F_j)$$ to $\partial M$. Lifting this arc to $N$ and composing it with $\ell$ we have a simple arc $k$ in $N$, such that $\partial k$ consists of 2 different points $p_1$ and $p_2$ in $\partial N$ and such that $f|k: (k, \partial k) \rightarrow (M, f(\partial k))$ contracts. Let $S$ be the boundary surface of $M$ which contains $f(p_1) = f(p_2)$. Let $R_1$ and $R_2$ be the boundary surfaces of $N$ which contain $p_1$ and $p_2$ respectively. Remembering that $R_1$ and $R_2$ are closed surfaces which cover $S$, we distinguish 3 cases:

(a) $R_1 \neq R_2$

Let $\alpha$ be any closed curve in $R_1$ with basepoint $p_1 \in R_1$. For some $n$, the lifting of $(f\alpha)^n$ is a closed curve $\alpha'$ in $R_2$. Then the curves $\kappa \alpha' \kappa^{-1}$ and $\alpha^n$ have the same image in $f_\ast n_1(N)$ since $f|k: (k, \partial k) \rightarrow (M, f(\partial k))$ contracted. Hence by lemma 4.5 and
since \( f_\ast \) is injective \( N \) is a product line bundle over a closed surface: \( N \approx R_1 \times I \).

Let \( \tilde{M} \) be the covering of \( M \) which is associated to \( i_\ast(\pi_1(S)) \);
then there exists a copy \( \tilde{S} \) over \( S \), for which \( \pi_1(\tilde{S}) \to \pi_1(S) \) is an isomorphism. Now, since \( f_\ast \pi_1(N) = f_\ast \pi_1(R_1) = i_\ast \pi_1(S) = p_\ast \pi_1(\tilde{M}) \), there exists a lifting \( \tilde{f} : N \to \tilde{M} \) of \( f : N \to M \).

Let \( \tilde{f} \) be such that \( \tilde{f}(\partial N) \cap \tilde{S} \neq \emptyset \). Then since \( f|k : (k, \partial k) \to (M, f(\partial k)) \) contracts, it follows that \( \tilde{f}(k) \sim 0 \) and hence both of \( R_1 \) and \( R_2 \) are mapped into \( \tilde{S} \). Now since \( \tilde{M} \) (and \( \tilde{S} \)) are aspherical (lemma 1.7) and have the same fundamental group, it follows that \( \tilde{M} \) deformation retracts to \( \tilde{S} \).

(Proof:
Since \( \tilde{S} \) aspherical a given map \( 1 : \tilde{S} \to \tilde{S} \) extendable over \( \tilde{M} \) iff there exists \( h_\ast : \pi_1(\tilde{M}) \to \pi_1(\tilde{S}) \) such that

\[
\begin{array}{ccc}
\pi_1(\tilde{S}) & \xrightarrow{\tilde{f}_\ast} & \pi_1(\tilde{M}) \\
1 & \searrow & \swarrow h_\ast \\
& \pi_1(\tilde{S}) &
\end{array}
\]

commutes. Let \( h_\ast = 1 \). Then there exists an extension \( g : \tilde{M} \to \tilde{S} \) of \( 1 : \tilde{S} \to \tilde{S} \). Since \( g_\ast = 1_\ast(1 : \tilde{M} \to \tilde{M}) \), \( 1 \) and \( g \) are homotopic.)

If \( h_t : \tilde{M} \to \tilde{M} \) is the deformation retraction, then the homotopy

\( \phi_t : N \to M \) satisfies \( \phi_0 \tilde{f} = f \), \( \phi_1 \tilde{f}(N) \subset S \), which proves case (b) of the theorem.

(b) \( R_1 = R_2 \); \( (k, \partial k) \to (N, R_1) \) does not contract into \( (R_1, R_1) \):
Let \( \tilde{N} \) be the covering of \( N \) associated to \( i_\ast(\pi_1(R_1)) \). Let \( R' \) be a copy over \( R_1 \) for which \( \pi_1(R') \to \pi_1(R_1) \) is an
isomorphism. Let \( k' \) be a copy over \( k \), which originates at \( R' \) and has its other endpoint in a copy \( R'' \) over \( R_1 \). If \( R' = R'' \), then joining \( k' \) with an arc \( k^* \) in \( R' \) would give us a closed \( k' \cup k^* \) which lies over a closed curve \( k \cup k^* \), with \( k^* \in R_1 \). But by assumption, \( k \cup k^* \) is not homotopic to a curve in \( R_1 \), hence (by definition of \( \tilde{N} \)) \( k' \cup k^* \) is not closed in \( \tilde{N} \). Therefore \( R' \neq R'' \). As in (a) we prove that some non-null multiple of any closed curve in \( R' \) is homotopic to a curve in \( R'' \). (Let \( \tilde{\alpha} \in R' \) cover \( \alpha \in R_1 \). By (a), for some \( n \), \( \alpha^n \) is homotopic to \( k\alpha'k^{-1} \), with \( \alpha' \) a closed curve in \( R_1 \). Then the lifting \( k' \tilde{\alpha}'k'^{-1} \) with \( \tilde{\alpha}' \) in \( R'' \) is homotopic to \( \tilde{\alpha}^n \)). By lemma 4.5, \( \tilde{N} \cong R' \times I \), and hence \( \tilde{N} \to N \) is a 2-sheeted covering.

By lemma 4.4, \( N \) is a line bundle over a closed surface.

If we join the endpoints of \( k \) by an arc \( k^* \) in \( R_1 \), the closed curve \( k^* \cup k \) does not belong to \( i_{k^*} \pi_1(R_1) \). Hence \( i_{k^*} \pi_1(R_1) \) and \( k^* \cup k \) generate \( \pi_1(N) \), and since \( f | k : (k, \partial k) \to (M, f(\partial k)) \) contracts, it follows that \( f_{k^*} \pi_1(N) \subset i_{k^*} \pi_1(S) \). Therefore we may construct as in case (a) a homotopy of \( f \) such that afterwards \( f(N) \subset S \), which again proves case (b) of the theorem.
(c) $R_1 = R_2$, $(k, \delta k) \rightarrow (N, R_1)$ contracts into $(R_1, R_1)$: $k$ is homotopic to an arc $\kappa$ in $R_1$, by a homotopy which keeps $\delta k$ fixed. $f(\kappa)$ is based loop in $S$, which is not contained in the subgroup $(f|_{R_1})_* \pi_1(R_1)$, because otherwise the lifting $\kappa$ of $f(\kappa)$ would be closed. But $f(\kappa) \sim f(k)$ in $M$, and $f(k) \sim 0$ in $M$ and hence $f(k) \sim 0$ in $S$ since $S$ is incompressible. In particular $f(\kappa)$ would belong to $(f|_{R_1})_* \pi_1(R_1)$. So this case can not occur.

Case II: $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and $\partial F_j = \emptyset$ for some $j$, $1 \leq j < n$.

We choose the hierarchy

$$M_j, F_j \subset M_j, U(F_j) \subset M_j, M_{j+1} = M_j - U(F_j), j=1, \ldots, n$$

such that the first $r-1$ surfaces are closed and $\partial F_1 \neq \emptyset$ if $1 \geq r$. Homotope $f$ such that $f$ is transverse with respect to $U F_j$ and $f^{-1}(U F_j)$ is a system of incompressible (closed) surfaces (by 3.2). Choose the homotopy such that in addition the following holds: If $F$ is one of the $F_j$'s ($j = 1, \ldots, r-1$) or if $F$ is a boundary component of $M$ and if $F'$ is parallel to $F$ in a regular neighborhood $U(F)$ in $M$, then $f^{-1}(F')$ is a system of (closed incompressible) surfaces and the number of components of $f^{-1}(F')$ is minimal.
By theorem 2.1 we may assume that there are regular neighborhoods $U(F_j)$ and $U(\partial M)$ such that $f|f^{-1}(\bigcup_{j=1}^{r-1} U(F_j)UU(\partial M))$ is a covering map on each component. (Because each component $G'$ of $f^{-1}(G)$ is different from $S^2$ and $\ker (f|G')_{\ast} = 0$).

Now, if $N'$ is a component of $f^{-1}(M_{\tau})$, we have $f' = f|N'$, $f': (N', \partial N') \to (M_{\tau}, \partial M_{\tau})$, $\ker f'_{\ast} = 0$ (this follows as in case I from lemma 1.6(b)). Since we are now in Case I theorem 5.1 shows that there is a homotopy of $f'$, constant on $\partial N'$ such that 5.1(a) or (b) holds. If for all components $N'$, $f'$ is homotopic to covering map, then 5.1(a) is proved.

Suppose this does not hold for a component $N'$ of $f^{-1}(M_{\tau})$.

Then, by 5.1(b), $N'$ is a line bundle over a closed surface, and $f_1'(N') \subset \partial M_{\tau}$.

1) Assume $N'$ is a twisted line bundle and $f'(\partial N') \subset F_j \times 0$ (where $U(F_j) = F_j \times I$ and $F_j = F_j \times \frac{1}{2}$). The component $\partial N'$ of $f^{-1}(F_j \times 0)$ bounds the line bundle $N'$. Hence a component of $f^{-1}F_j$ bounds a line bundle $U(N')$, which is a regular neighborhood of $N'$ in $N$. Take 2 copies $F'$, $F''$, of $F_j$ on different sides of $F_j$ in $\partial U(F_j)$. Then there are two copies $G' \in f^{-1}F'$, $G'' \in f^{-1}F''$ on different sides of $\partial U(N') \in f^{-1}F_j$, $G' \in U(N')$, say.
Homotope \( f: N \rightarrow M \), by a homotopy which is constant on \( N - U(N') \), such that afterwards \( f(U(N')) \subseteq F_j \). Then the inverse images of \( F' \) in \( N - U(N) \) are not altered, but \( G' \in U(N') \) has vanished, which contradicts our minimality condition.

ii) Assume \( N' \) is a twisted line bundle and \( f(\partial N') \subseteq \partial M \). Let \( G \) be that component of \( \partial M \) which contains \( f(\partial N') \). If \( \partial N' \) is contained in \( N \), then a component of \( f^{-1}(G \times \frac{1}{2}) \) bounds a line bundle \( U(N') \) in \( N \) (assume \( G = G \times 1 \)). Homotope \( f \), constant on \( N - U(N') \), such that afterwards \( f(U(N')) \subseteq G \times \frac{1}{2} \). Then the inverse image of \( G \) in \( U(N') \) has vanished, contradicting our minimality condition of \( f \). If \( \partial N' \) is contained in \( \partial N \), then, in fact \( \partial N' = \partial N \) and \( N \) is a line bundle, which gives 5.1(b).

The case that \( N' \) is a product line bundle may be treated similarly. Hence, theorem 5.1 is proved.

**Definition 5.2** Let \( M, N \) be manifolds, \( \psi: \pi_1(N) \rightarrow \pi_1(M) \) a homomorphism. \( \psi \) **respects the peripheral structure** iff the following holds. For each boundary surface \( G \) of \( N \), there exists a boundary surface \( F \) of \( M \), such that \( \psi(i_\# \pi_1(G)) \subseteq A \), where \( A \) is conjugate in \( \pi_1(M) \) to \( i_\# \pi_1(F) \).

The definition does not depend on the choice of the inclusion homomorphism \( i_\# \).

**Lemma 5.3:** Let \( N \) be a manifold. Let \( M \) be irreducible and boundary-irreducible. If \( M \) is orientable, let \( \pi_1(M) \) be infinite.
If $M$ is non-orientable, suppose there are no 2-sided projective planes in $M$. Let $\psi: \pi_1(N) \to \pi_1(M)$ be a homomorphism. Then there exists a map $f: (N, \partial N) \to (M, \partial M)$ with $f_\ast = \psi$ iff $\psi$ respects the peripheral structure.

**Proof:** By lemma 1.7, $M$ is aspherical. Therefore we may construct a map $f': N \to M$ which induces $\psi$. Suppose we have proved that if $G$ is any boundary component of $N$ and $g_0 = f'|G$, then there exists a homotopy $g_\tau: G \to M(\tau \in I)$, such that $g_1(G) \subset \partial M$. Then the lemma will be proved, since $M$ is aspherical.

($g_\tau: G \to M$ is exendable over $N$ iff there exists a homomorphism $h_\ast: \pi_1(N) \to \pi_1(M)$ such that $h_\ast i_\ast = (g_\tau)_\ast$, where $i: G \to N$ is the inclusion. Noting that $(g_\tau)_\ast = (f'|G)_\ast$, we may choose $h_\ast$ to be $f_\ast$).

So let $G$ be any boundary component of $N$. Then, since $f_\ast(i_\ast \pi_1(G)) \subset A$, a conjugate in $\pi_1(M)$ to $\pi_1(F)$ for a boundary component $F$ of $\partial M$, every curve of $g_0 G = f'(G)$ is homotopic to a curve on $F$. Hence the homotopy $g_\tau$ may be defined on the 1-skeleton $G^{(1)}$ of $G$. Every 2-cell $e^2$ of $G$ has $g_1(\partial e^2)$ in $F$, and since $\ker(\pi_1(F) \to \pi_1(M)) = 0$, we can map $e^2$ onto a 2-cell in $F$, i.e. we may extend $g_1: G^{(1)} \to M$ to a map $g_1: G \to M$.

Hence we have a map $g: (G \times 0) \cup (G^{(1)} \times I) \cup (G \times 1) \to M$. Taking a 3-cell $e^3$ of $G \times I$, $g(\partial e^3)$ is a singular 2-sphere in $M$ and since $\pi_2(M) = 0$ we can extend $g$ to $e^3$. Thus we have defined a homotopy $g: G \times I \to M$ as desired.
Corollary 5.4  Let M and N be manifolds which are irreducible and boundary irreducible. If M is orientable let M be sufficiently large. If M is non-orientable suppose there are no 2-sided projective planes in M. Suppose N is not a line bundle over a closed surface and \( \pi_1(N) \neq 1 \). If \( \psi: \pi_1(N) \to \pi_1(M) \) is an injection which respects the peripheral structure, then there exists a covering map \( f: N \to M \) which induces \( \psi \).

Proof:  By lemma 5.4 there is a map \( f'(N,\partial N) \to (M,\partial M) \) which induces \( \psi \). By 5.1 \( f' \) is homotopic to a covering map \( f: N \to M \).

Corollary 5.5:  Let M and N be manifolds which are irreducible, boundary irreducible and which do not contain any 2-sided \( \mathbb{P}^2 \)'s. If M is orientable, let M be sufficiently large.

Suppose \( \psi: \pi_1(N) \to \pi_1(M) \) is an isomorphism which respects the peripheral structure. Then there exists a map \( f: N \to M \), which induces \( \psi \) and such that either (a) or (b) holds:

(a) \( f: N \to M \) is a homeomorphism

(b) M is the product line bundle over a closed surface, \( N \) a twisted line bundle over the same surface and \( f: (N,\partial N) \to (M,\partial M) \) contracts into \( \partial M \).

Proof:  If N is not a line bundle, corollary 5.4 gives a 1-sheeted covering map. If N is a line bundle over a closed surface G we claim that M is a line bundle over a surface \( G' \) which is homeomorphic to G:

(1) Let N be not a product bundle, then \( p: \partial N \to G \) is a 2-sheeted covering of G and \( \pi_1(N) = \pi_1(G) \), if we choose the basepoint
in G. Now $p_\pi \pi_\pi_1(\partial N) \subset \pi_1(N)$ lies in the peripheral class determined by $\partial N$. Since $p_\pi \pi_\pi_1(\partial N)$ has index 2 in $\pi_1(N)$ it is a normal subgroup and the peripheral system of N is just the group $p_\pi \pi_\pi_1(\partial N)$. Hence we have an isomorphism $\psi: (\pi_1(N), p_\pi \pi_\pi_1(\partial N)) \rightarrow (\pi_1(M), 1_\pi \pi_\pi_1(F'))$ for a boundary component $F'$ of M. Let $\pi_1(M) \not\subset 1_\pi \pi_\pi_1(F')$, then $p_\pi \pi_\pi_1(\partial N) \nsim_{\pi_1} 1_\pi \pi_\pi_1(F')$. Let $\tilde{M} \rightarrow M$ be the covering associated to $1_\pi \pi_\pi_1(F')$. Let $\tilde{F}'$ be a copy over $F'$ for which $\pi_1(\tilde{F}') \rightarrow \pi_1(F')$ is an isomorphism. Now since $\pi_1(M)/1_\pi \pi_\pi_1(F') = Z_2$, $\tilde{M}$ is a 2-sheeted covering of M and there exists an other copy $\tilde{F}' \neq \tilde{F}'$ over $F'$. Since $\pi_1(\tilde{M}) = 1_\pi \pi_\pi_1(F')$, we can apply proposition 4.5. Therefore $\tilde{M} = \tilde{F}' \times I$ and from proposition 4.4 it follows that M is a line bundle over a closed surface $G'$. Since $\pi_1(M) \cong \pi_1(G')$ and since $\psi$ is an isomorphism, it follows that $G'$ is homeomorphic to G. If $\pi_1(M) = 1_\pi \pi_\pi_1(F')$ and if M contains more than one boundary component then again by 4.5 $M \cong F' \times I$. If M contains only one boundary component $F'$, M deformation retracts to $F'$ (since both are aspherical.) Hence there exists a deformation retraction of the double of M onto F. Composing this map with the inclusion $F \rightarrow M$ we would have a covering map from $\text{D}(M) \rightarrow M$ by Theorem 5.1, which is impossible.

a) Suppose M is a twisted line bundle. We have an isomorphism $\psi: (\pi_1(G), p_1 \pi_\pi_1(\partial N) \rightarrow (\pi_1(G'), p_2 \pi_\pi_1(\partial M))$ where $p_1: \partial N \rightarrow G$ and $p_2: \partial M \rightarrow G'$ are the 2-sheeted coverings. By theorem 2.1 there exists a homeomorphism $f: G \rightarrow G'$. Since $\psi p_1(\partial N) \subset p_2 \pi_\pi_1(\partial M)$ we may lift f to $\tilde{f}: \partial N \rightarrow \partial M$. Lifting $f^{-1}$ instead of f shows that $\tilde{f}$ is a homeomorphism. Now choose a canonical
system of simple closed curves $k_1, \ldots, k_n$ on $G$, $k_i \cap k_j = P$, the base point of $\pi_1(N)$, $(i \neq j)$, which cuts $G$ into a disc $D$. The line bundle over $k_i$ (induced from $N$) is either an annulus or a moebiusband and so is the line bundle over $fk_i$ (induced by $M$) respectively, since

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{f} & \mathcal{M} \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & G'
\end{array}
$$

commutes. So we may define a homeomorphism $h_n : i=1^n (\text{line bundle over } k_i) \cup_{i=1}^n (\text{line bundle over } fk_i)$ (such that $h_n = \psi$).

Cutting $N$ and $M$ along those annuli and moebiusbands we get two line bundles over discs $D$ and $fD$. Since these are product bundles, we may extend the homeomorphism defined on $(\partial D \times I) \cup (D \times 1/2)$ to a homeomorphism $D \times I \rightarrow fD \times I$.

This proves case (a) of 5.5.

(b) If $M$ is the product bundle, then by 5.1 case (b) of 5.5 will follow.

(2) Let $N$ be a product bundle: $N \approx F \times I$. We have an isomorphism $\psi$:

$$
\psi : (\pi_1(N), i_1*\pi_1(F_1), i_2*\pi_1(F_2)) \rightarrow (\pi_1(M), A, B)
$$

where $A$ and $B$ are conjugate to $i_1*\pi_1(G_1)$ and $\pi_1(G_2)$, for
some boundary components $G_1$ and $G_2$ of $M$, ($G_1 = G_2$ is possible).
If $\partial M$ contains more than one component, then since $i_1 \pi_1(F_1) = \pi_1(N)$ implies $i_1 \pi_1(G_1) = \pi_1(M)$, it follows from proposition 4.5 that $M \approx G_1 \times I$.

In this case theorem 2.1 tells us that case (a) of our corollary will follow. If $\partial M$ contains just one component, then $\psi^{-1}$ respects the peripheral structure. Clearly $N$ is sufficiently large. Then, by 5.1, either the map $g: M \to N$ (which induces $\psi^{-1}$) and exists by lemma 5.3) is homotopic to a one-sheeted covering which is impossible or $M$ is a line bundle over a closed surface $G'$ which is homeomorphic to $F$. In the latter case $\psi$ would map $\pi_1(N)$ onto $\pi_1(M)$ and $i_1 \pi_1(F_1) = \pi_1(N)$ onto the proper subgroup $i_1 \pi_1(\partial M)$ of $\pi_1(M)$, which is absurd. So this case cannot occur.

**Remark 1)** Case (b) of 5.5 does indeed exist. For, let $G$ be any closed surface, $N$ a twisted line bundle over $G$ and $M = G \times I$. Map $N \times G$ and $G \times 0 \subset \partial M$. Then if $f = ip$ we have an isomorphism $f_\pi: (\pi_1(N), i_1 \pi_1(\partial N)) \to (\pi_1(M), i_1 \pi_1(\partial M))$.

2) Given a group $G$ and subgroups $U_1, \ldots, U_n$ and a group $F$ and subgroups $V_1, \ldots, V_n$. If we define an isomorphism $\psi: (G, U_1, U_2, \ldots, U_n) \to (F, V_1, \ldots, V_n)$ to be an isomorphism not only from $G$ onto $F$ but also to map each $U_i$ onto $V_i$, then case (b) of 5.5 cannot occur and we would have no exception from case (a).