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APPLICATION OF FINITE-DIFFERENCE METHODS
TO THE STUDY OF NON-LINEAR PROBLEMS
IN HYDRODYNAMIC STABILITY

by

William David George

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN CHEMICAL ENGINEERING

Thesis Director's Signature:

J.D. Williams

Houston, Texas
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WITH APPRECIATION

TO MY

PARENTS
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>1</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>iv</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>vii</td>
</tr>
<tr>
<td><strong>CHAPTER I.</strong> INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td><strong>CHAPTER II.</strong> HISTORICAL DEVELOPMENT</td>
<td>7</td>
</tr>
<tr>
<td>1. Linear Theory</td>
<td>7</td>
</tr>
<tr>
<td>2. Non-Linear Theory</td>
<td>21</td>
</tr>
<tr>
<td><strong>CHAPTER III.</strong> MATHEMATICAL FORMULATION</td>
<td>29</td>
</tr>
<tr>
<td>1. Plane-Poiseuille Flow</td>
<td>29</td>
</tr>
<tr>
<td>2. Poiseuille Flow</td>
<td>34</td>
</tr>
<tr>
<td>3. Generalization</td>
<td>38</td>
</tr>
<tr>
<td><strong>CHAPTER IV.</strong> NUMERICAL APPROXIMATION AND SOLUTION</td>
<td>41</td>
</tr>
<tr>
<td><strong>CHAPTER V.</strong> COMPARISON OF BASIC METHODS</td>
<td>54</td>
</tr>
<tr>
<td><strong>CHAPTER VI.</strong> DISCUSSION OF RESULTS</td>
<td>66</td>
</tr>
<tr>
<td>1. Linear Plane-Poiseuille Flow</td>
<td>66</td>
</tr>
<tr>
<td>2. Linear Poiseuille Flow</td>
<td>72</td>
</tr>
<tr>
<td>3. Computational Aspects</td>
<td>76</td>
</tr>
<tr>
<td>4. Non-Linear Plane-Poiseuille Flow</td>
<td>78</td>
</tr>
<tr>
<td>5. Non-Linear Poiseuille Flow</td>
<td>84</td>
</tr>
<tr>
<td><strong>CHAPTER VII.</strong> CONCLUDING REMARKS</td>
<td>128</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>130</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (Cont.)

APPENDICES ................................................................. 135

I. Derivation of Expressions for Non-Linear Terms with Respect to the Harmonic Components ........................................... 136

II. Finite-Difference Approximations for Derivatives .................. 142

III. Derivation of Finite-Difference Approximations for the System of Partial Differential Equations .................. 143

IV. Derivation of Approximations for Certain Auxilliary Functions .......... 150

V. Derivation of Algorithm for Solution of Penta-Diagonal System of Linear Algebraic Equations .................. 153

VI. Derivation of Relationship Between Complex Eigenfunction \( \phi(y) \) and the Coefficients \( A(y,t) \) and \( B(y,t) \) ............ 162
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Stability Characteristics of Linear Plane-Poiseuille Motion</td>
<td>14</td>
</tr>
<tr>
<td>4.1</td>
<td>Flow Chart for Solution Procedure</td>
<td>54</td>
</tr>
<tr>
<td>6.1</td>
<td>Initial Stream Function and Velocity Fluctuation for Plane-Poiseuille Flow</td>
<td>91</td>
</tr>
<tr>
<td>6.2</td>
<td>Linear Oscillation for Plane-Poiseuille Flow Computed for $\alpha = 1$, $R = 10,000$</td>
<td>92</td>
</tr>
<tr>
<td>6.3</td>
<td>Effect of Channel Resolution on Accuracy of Linear Solutions in Plane-Poiseuille Flow (Growth Rate - 1)</td>
<td>93</td>
</tr>
<tr>
<td>6.4</td>
<td>Effect of Channel Resolution on Accuracy of Linear Solutions in Plane-Poiseuille Flow (Growth Rate - 2)</td>
<td>94</td>
</tr>
<tr>
<td>6.5</td>
<td>Effect of Channel Resolution on Accuracy of Linear Solutions in Plane-Poiseuille Flow (Period)</td>
<td>95</td>
</tr>
<tr>
<td>6.6</td>
<td>Effect of Channel Resolution on Linear Solution Behavior in Plane-Poiseuille Flow</td>
<td>96</td>
</tr>
<tr>
<td>6.7</td>
<td>Real and Imaginary Parts of Least Stable Plane-Poiseuille Flow Symmetric Eigenfunction ($\alpha = 1$, $R = 10,000$)</td>
<td>97</td>
</tr>
<tr>
<td>6.8</td>
<td>Combinations of $\alpha$ and $R$ Used for Verification of Linear Theory of Plane-Poiseuille Flow</td>
<td>98</td>
</tr>
<tr>
<td>6.9</td>
<td>Initial Stream Function Fluctuation for Poiseuille Flow</td>
<td>99</td>
</tr>
<tr>
<td>6.10</td>
<td>Initial Velocity Fluctuation for Poiseuille Flow</td>
<td>100</td>
</tr>
<tr>
<td>6.11</td>
<td>Curves of Constant $\alpha C_1$ and $C_1$ for the Least Stable Centerline Mode - Poiseuille Flow</td>
<td>101</td>
</tr>
<tr>
<td>6.12</td>
<td>Curves of Constant $\alpha C_1$ and $C_1$ for the Least Stable Wall Mode - Poiseuille Flow</td>
<td>101</td>
</tr>
<tr>
<td>Figure</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>6.13</td>
<td>Early Time Behavior of Linear Poiseuille Flow</td>
<td>102</td>
</tr>
<tr>
<td>6.14</td>
<td>Shape of $A(y,t)$ Corresponding to Successive Times for Which Centerline Peak is Maximum</td>
<td>103</td>
</tr>
<tr>
<td>6.15</td>
<td>Late Time Behavior of Linear Poiseuille Flow</td>
<td>104</td>
</tr>
<tr>
<td>6.16</td>
<td>Least Stable Poiseuille Flow Eigenfunction for $\alpha = 1, R = 10,000$</td>
<td>105</td>
</tr>
<tr>
<td>6.17</td>
<td>Effect of Non-Linear Terms on Stability in Plane-Poiseuille Flow</td>
<td>106</td>
</tr>
<tr>
<td>6.18</td>
<td>Effect of Reynolds Number on Non-Linear Solutions in Plane-Poiseuille Flow (Initial Amplitude 10%)</td>
<td>107</td>
</tr>
<tr>
<td>6.19</td>
<td>Effect of Reynolds Number on Non-Linear Solutions in Plane-Poiseuille Flow (Initial Amplitudes 2% and 5%)</td>
<td>108</td>
</tr>
<tr>
<td>6.20</td>
<td>Effect of Reynolds Number on Non-Linear Solutions in Plane-Poiseuille Flow (Initial Amplitudes 15%, 20% and 30%)</td>
<td>109</td>
</tr>
<tr>
<td>6.21</td>
<td>Magnitude of 2nd Harmonic at $y = 0.5$ for Non-Linear Plane-Poiseuille Flow</td>
<td>110</td>
</tr>
<tr>
<td>6.22</td>
<td>Variation of 2nd Harmonic Maximum Magnitude with Initial Amplitude for Non-Linear Plane-Poiseuille Flow</td>
<td>111</td>
</tr>
<tr>
<td>6.23</td>
<td>Critical Behavior for Finite-Amplitude Disturbances in Plane-Poiseuille Flow</td>
<td>112</td>
</tr>
<tr>
<td>6.24</td>
<td>Comparison of Plane-Poiseuille Flow Critical Curve with Earlier Work</td>
<td>113</td>
</tr>
<tr>
<td>6.25</td>
<td>Effect of Non-Linearity on Stability in Poiseuille Flow ($\alpha = 1, R = 10,000$)</td>
<td>114</td>
</tr>
<tr>
<td>6.26</td>
<td>Non-Linear Solution in Poiseuille Flow for $\alpha = 2.5, R = 3,000$ and Disturbance Amplitude $= 0.10$</td>
<td>115</td>
</tr>
<tr>
<td>Figure</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>6.27</td>
<td>Non-Linear Solution in Poiseuille Flow for ( \alpha = 0.35, R = 10,000 ) and Disturbance Amplitude = 0.10</td>
<td>116</td>
</tr>
<tr>
<td>6.28</td>
<td>Non-Linear Solution in Poiseuille Flow for ( \alpha = 0.50, R = 100,000 ) and Disturbance Amplitude = 0.19% in ( \psi ) or 3% in Velocity ( u )</td>
<td>117</td>
</tr>
<tr>
<td>6.29</td>
<td>Radial Distribution of Turbulence in Wake of Ring Airfoil of Radius 0.45 in</td>
<td>118</td>
</tr>
<tr>
<td>6.30</td>
<td>Radial Distribution of Mean Velocity Downstream of Ring Airfoil</td>
<td>118</td>
</tr>
<tr>
<td>6.31</td>
<td>Stream Function for Initial Mean Flow Distortion in Poiseuille Flow</td>
<td>119</td>
</tr>
<tr>
<td>6.32</td>
<td>Velocity Function for Initial Mean Flow Distortion in Poiseuille Flow</td>
<td>120</td>
</tr>
<tr>
<td>6.33</td>
<td>Related Vorticity Function for Initial Mean Flow Distortion in Poiseuille Flow</td>
<td>121</td>
</tr>
<tr>
<td>6.34</td>
<td>Effect of Reynolds Number on Viscous Decay of ( A_0 (0.7,t) ) - Max ( A_0 (r,0) = 5% ), Max ( A_1 (r,0) = 0.2% )</td>
<td>122</td>
</tr>
<tr>
<td>6.35</td>
<td>Effect of Reynolds Number on Behavior of ( A_1 (0.7,t) ) - Max ( A_0 (r,0) = 5% ), Max ( A_1 (r,0) = 0.2% )</td>
<td>123</td>
</tr>
<tr>
<td>6.36</td>
<td>Effect of Initial Amplitude of ( A_0 ) on Viscous Decay of ( A_0 (0.7,t) - \alpha = 1, R = 10,000 ), and Max ( A_1 (r,0) = 0.2% )</td>
<td>124</td>
</tr>
<tr>
<td>6.37</td>
<td>Effect of Initial Amplitude of ( A_0 ) on Behavior of ( A_1 (0.7,t) - \alpha = 1, R = 10,000 ), and Max ( A_1 (r,0) = 0.2% )</td>
<td>125</td>
</tr>
<tr>
<td>6.38</td>
<td>Effect of Initial Amplitude of ( A_1 ) on Viscous Decay of ( A_0 (0.7,t) - \alpha = 1, R = 10,000 ), and Max ( A_0 (r,0) = 5% )</td>
<td>126</td>
</tr>
<tr>
<td>6.39</td>
<td>Effect of Initial Amplitude of ( A_1 ) on Behavior of ( A_1 (0.7,t) - \alpha = 1, R = 10,000 ), and Max ( A_0 (r,0) = 5% )</td>
<td>127</td>
</tr>
</tbody>
</table>
## NOMENCLATURE

<table>
<thead>
<tr>
<th>Roman Letters</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Penta-diagonal system matrix.</td>
</tr>
<tr>
<td>A</td>
<td>Averaging operator.</td>
</tr>
<tr>
<td>A&lt;sub&gt;m&lt;/sub&gt;, B&lt;sub&gt;m&lt;/sub&gt;</td>
<td>Fourier expansion coefficients for mth harmonic of stream function disturbance, functions of y and t.</td>
</tr>
<tr>
<td>a&lt;sub&gt;j&lt;/sub&gt;, b&lt;sub&gt;j&lt;/sub&gt;, c&lt;sub&gt;j&lt;/sub&gt;, d&lt;sub&gt;j&lt;/sub&gt;, e&lt;sub&gt;j&lt;/sub&gt;</td>
<td>Diagonal elements of Matrix A taken left to right.</td>
</tr>
<tr>
<td>C&lt;sub&gt;m&lt;/sub&gt;, D&lt;sub&gt;m&lt;/sub&gt;</td>
<td>Vorticity variables associated with A&lt;sub&gt;m&lt;/sub&gt; and B&lt;sub&gt;m&lt;/sub&gt;, respectively.</td>
</tr>
<tr>
<td>c</td>
<td>Complex Orr-Sommerfeld eigenvalue.</td>
</tr>
<tr>
<td>E</td>
<td>Truncation error term.</td>
</tr>
<tr>
<td>F&lt;sup&gt;A&lt;/sup&gt;&lt;sub&gt;m&lt;/sub&gt;, F&lt;sup&gt;B&lt;/sup&gt;&lt;sub&gt;m&lt;/sub&gt;</td>
<td>Non-linear terms associated with differential equations for A&lt;sub&gt;m&lt;/sub&gt; and B&lt;sub&gt;m&lt;/sub&gt;, respectively.</td>
</tr>
<tr>
<td>G&lt;sup&gt;(n)&lt;/sup&gt;&lt;sub&gt;mn(j)&lt;/sub&gt;, H&lt;sup&gt;(n)&lt;/sup&gt;&lt;sub&gt;mn(j)&lt;/sub&gt;</td>
<td>Iteration invariant terms on right hand side of finite-difference equation for A&lt;sub&gt;m&lt;/sub&gt;(j) and B&lt;sub&gt;m&lt;/sub&gt;(j), respectively.</td>
</tr>
<tr>
<td>h</td>
<td>Dimensionless space increment, identical to Ay.</td>
</tr>
<tr>
<td>I&lt;sub&gt;n&lt;/sub&gt;</td>
<td>n&lt;sup&gt;th&lt;/sup&gt; order Bessel function of 1st kind.</td>
</tr>
<tr>
<td>J&lt;sub&gt;n&lt;/sub&gt;</td>
<td>n&lt;sup&gt;th&lt;/sup&gt; order modified Bessel function of 1st kind.</td>
</tr>
<tr>
<td>k</td>
<td>Iteration number.</td>
</tr>
<tr>
<td>k&lt;sub&gt;A&lt;/sub&gt;</td>
<td>Initial amplitude constant.</td>
</tr>
<tr>
<td>L</td>
<td>Differential operator (Eqn. 2.8), also lower triangular matrix.</td>
</tr>
<tr>
<td>L&lt;sub&gt;m&lt;/sub&gt;</td>
<td>Differential operator (Eqn. 3.60).</td>
</tr>
<tr>
<td>l</td>
<td>Reference length, taken as half the channel width for plane-Poiseuille flow and as radius of pipe for Poiseuille flow.</td>
</tr>
</tbody>
</table>
NOMENCLATURE (Cont.)

M  Truncation limit for number of harmonics.

m  Harmonic number.

N  Total number of intervals used for resolution of channel width.

$N_m^{\pm}(j), P_m^{\pm}(j), Q_m(j)$  Coefficients of unknown variables in finite-difference equations for $A_m(j)$ and $B_m(j)$.

n  Time step number.

p  Pressure.

R  Reynolds number $- U_0 \ell / \nu$.

$R_C$  Critical Reynolds number.

r  Dimensionless distance along radius of pipe.

$\mathfrak{D}_m$  Differential operator (Eqn. 3.61).

t  Dimensionless time.

U  Upper triangular matrix.

$U_0$  Reference velocity, taken as centerline velocity of undisturbed laminar flow.

u  Velocity component in direction of mean flow.

$\mathfrak{U}$  Velocity distribution for undisturbed laminar flow.

$V_m$  Identical notation for $\mathfrak{D}_m[B_m]$.

v  Velocity component in direction perpendicular to mean flow.

$W_m$  Identical notation for $\mathfrak{D}_m[A_m]$.

X  Vector of unknown quantities.

x  Longitudinal dimensionless space variable.
NOMENCLATURE (Cont.)

\[ y \]
Lateral Dimensionless space variable.

\[ y_{m(j)} \]
Terms that must be evaluated every iteration in finite-difference equations for \( A_m(j) \) and \( B_m(j) \).

Greek Letter

\[ \alpha \]
Wave number or number of complete cycles over a dimensionless length of \( 2\pi \).

\[ \alpha_j, \beta_j, \gamma_j \]
Elements of lower triangular matrix \( L \).

\[ \beta \]
Dimensionless frequency used in timewise periodic problems, \( \beta = \alpha c \).

\[ \nabla^2 \]
Laplacian operator.

\[ \nabla^4 \]
Biharmonic operator.

\[ \Delta t \]
Dimensionless time increment.

\[ \Delta y \]
Dimensionless space increment in \( y \)-direction.

\[ \delta^i_y \]
Finite-difference operator for \( i \)th derivative.

\[ \delta_j, \epsilon_j \]
Elements of upper triangular matrix \( U \).

\[ \gamma_i \]
Coefficient of \( A^i_m \) in partial differential equation for \( A_m \).

\[ \eta \]
Switch variable whose value is 0 for plane-Poiseuille flow and 1 for Poiseuille flow.

\[ \theta \]
Azimuthal direction, also quantity \( Rh^2/\Delta t \).

\[ \lambda_i \]
Coefficient of \( \frac{\partial A_m^i}{\partial t} \) in partial differential equation for \( A_m \).

\[ \nu \]
Kinematic viscosity.

\[ \xi \]
Vorticity variable.
NOMENCLATURE (Cont.)

\( \rho \)  
Density.

\( \sigma \)  
Coefficient of \( D_m \) in partial differential equation for \( A_m \).

\( \tau \)  
Coefficient of \( B_m \) in partial differential equation for \( A_m \).

\( \phi \)  
Eigenfunctions and other functions of \( y \).

\( \psi \)  
Total stream function.

\( \Psi \)  
Stream function for undisturbed laminar flow.

\( \hat{\psi} \)  
Disturbance stream function.

Subscripts

\( j \)  
Denotes grid point number.

\( m \)  
Denotes harmonic number.

Superscripts

\( k \)  
Denotes iteration number.

\( n \)  
Denotes time step number.

Overlines

\( \sim \)  
Normalized coefficients.

\( \sim \)  
Adjoint variable (Chapter V).

\( - \)  
Complex conjugate (Chapter V).

\( - \)  
Refers to steady laminar flow conditions.

\( ^\wedge \)  
Refers to deviation of variable from steady laminar flow.

NOTE: All of the symbols used are not defined here. Some of the symbols which have localized meaning are defined at appropriate places in the text.
CHAPTER I

INTRODUCTION

One of the most challenging problems in the field of fluid mechanics has been that of reaching an understanding of the basic physical mechanisms of the processes causing transition from laminar to turbulent flow. Although a fundamental problem, it still remains an enigma despite the continuous efforts of numerous stouthearted researchers over the past half century. It is now generally acknowledged that the earliest stages of transition to turbulence must be closely related to the problem of laminar instability. If a disturbance in a fluid motion is initially small, its growth or decay is determined by linear theory. However, if the disturbance grows, at some stage of development in the process the non-linear terms that were previously neglected in the analysis take control and subsequently govern the entire process. On the other hand if the initial disturbance is large enough, the process may be non-linear from the beginning. Thus, the question of infinitesimal versus finite disturbances is a significant one, and the effect of non-linear terms can be crucial. It would appear then that the capability to study the development of laminar instabilities far into the non-linear phases of the motion would provide a valuable tool toward gaining an understanding of incipient turbulent phenomena. It is with this fact in mind that the work reported here was undertaken.

More specifically, we will be concerned with stability in the general category of two-dimensional parallel shear flow and in
particular with plane-Poiseuille flow between parallel walls and Poiseuille flow in a straight circular pipe. Since both of these flows are characterized by parabolic velocity profiles, one might be tempted to conjecture that they would exhibit qualitatively similar stability characteristics. However, this is not the case, and in fact they are quite dissimilar in their basic behavior. Plane-Poiseuille flow has been shown to be unstable to infinitesimal disturbances for certain combinations of the basic parameters, Reynolds number and disturbance frequency, but stable to all frequencies for all Reynolds numbers below some critical value. On the other hand, it has been established that Poiseuille flow is universally stable with respect to small disturbances although experiments have repeatedly demonstrated that Poiseuille flow becomes turbulent for Reynolds numbers exceeding a critical value of about 2100. People have been trying to reconcile these conflicting facts for years but have been unable to establish any definite case concerning the origin of turbulence in Poiseuille flow. It is now generally felt that the most likely causes of instability are related to the behavior of finite-amplitude disturbances and/or non-axisymmetric disturbances, but these as yet have received very little theoretical or computational treatment. No one has ever predicted turbulence as a direct consequence of integrating the equations of motion, due to the horrendous nature of the computational problems inherent in the process. Nor do we set out to accomplish such a remarkable feat. We do, nevertheless, attempt to take a step in the right direction by treating the stability of spatially periodic flows for the aforementioned geometries, at least in an approximate sense.
It may ultimately turn out that direct integration of the
equations of motion in a three-dimensional region corresponding to
some physically realizable experiment is the only way in which turbu-
lence can be predicted theoretically. That is, it may be necessary for
finite-amplitude and three-dimensional effects to act simultaneously
and any simplification at all would destroy some vital feature(s) of
the actual physical behavior. Thus, the major obstacle to attacking
the general problem is that of dimensionality; and we must, at least
for the present, settle for something less than ultimate. Certainly,
we are restricted to two-dimensional studies for non-linear problems
and even these are somewhat recalcitrant. Consider for a moment some
of the problems associated with numerical integration of timewise
periodic disturbances propagating downstream in an infinite channel.
First of all, only a finite flow field can be used in the computation
to observe the disturbances as they are swept downstream. Therefore,
there is only a finite time for which observation is possible. This
flow field must in general be lengthy because instabilities of interest
are typically associated with large Reynolds numbers. Accordingly,
large distances downstream would be required for significant amplifica-
tion. There is also the question of what boundary conditions to use at
the open end of the channel. Ideally, a disturbance should always pro-
pagate into a region of undisturbed flow so that no reflections upstream
are created at the fictitious downstream boundary. Boundary conditions
corresponding to this situation would be extremely difficult to achieve.
Another difficulty arises from the fact that any numerical process from
which a quantitatively acceptable solution is to be achieved must be capable of accurate approximation of the eigenfunctions of the linear problem and their derivatives up to fourth order. These eigenfunctions vary rapidly over certain portions of the channel necessitating fine resolution for any computing grid in the direction perpendicular to the basic flow. Consequently, excessively large computer times would be involved.

Such considerations as these have convinced us of the necessity to restrict this work to investigation of disturbances that are periodic in space as opposed to those periodic in time. This may seem to limit the range of application to a degree in that one must consider disturbances as entirely arbitrary functions of the space variables if the realities of the physical world are to be represented. However, spatially periodic disturbances appear to be of considerable interest both from the point of view of the mathematical problem and as solutions to the general problem of stability. We admit that the simplifying assumptions should be regarded as being somewhat artificial; nevertheless, many qualitative features of the periodic solutions should apply to more general disturbances, particularly if growth or decay rates are not too large.

Preliminary work on the part of the author demonstrated the impracticality, at least for the present, of trying to obtain quantitative results by integrating the equations of motion in a complete two-dimensional periodically repeating region. The vast quantity of
computer time required for even a single problem coupled with the large number of parameters involved, prevented pursuit of the problem along these lines.

The course of action decided upon was that of harmonic analysis in which the disturbance stream function is expanded into a Fourier series containing unknown coefficients which depend upon position in the lateral channel direction and upon time. This series is inserted into the Navier-Stokes equations expressed in terms of disturbance stream function, whereupon the sine and cosine terms of the respective frequencies are separately set to zero. A system of coupled, non-linear one-dimensional partial differential equations for the unknown harmonic components is subsequently obtained. Integration of these equations using finite-difference techniques will yield the desired solutions whose growth or decay in time should provide some answers regarding stability of the basic flow to finite-amplitude disturbances.

We are now in a position to state the fundamental objectives of this investigation:

(1) To formulate, implement, and develop a practical and efficient numerical process for achieving solutions to the equations of motion which govern the behavior of disturbances in laminar parallel shear flow. Truncated Fourier series decomposition and finite-difference methods of approximation are to be employed.
(2) To demonstrate that the numerical process so developed can yield valid results in the limit to small disturbances by verifying linear theory for both plane-Poiseuille flow and Poiseuille flow through a circular pipe.

(3) Finally, to study the behavior of large or finite-amplitude disturbances for both of the basic geometrical situations in order to demonstrate the significant effect of non-linearity and in order to shed additional light onto the general nature of these flows. We would be particularly interested in demonstrating instability with respect to two-dimensional finite-amplitude disturbances in Poiseuille flow, if it exists.
CHAPTER II

HISTORICAL DEVELOPMENT

In order to establish some of the more salient points connected with the problem under consideration, it seems desirable to provide some background material and to review previous work on the subject. First, we will consider the results of investigations into what has been referred to as Linear Theory. This is to be followed by a discussion of more recent contributions to linear theory and finally of research into the vastly more complicated problem in which the effect of non-linearities are taken into account.

Linear Theory

Stability analysis in classical fluid mechanics begins with the Navier-Stokes equations which govern the behavior of a real incompressible fluid. It is assumed that the flows to be investigated consist of a mean flow combined linearly with a small amplitude oscillation. Linear theories of hydrodynamic stability are based on the perturbation equations obtainable by substitution of the mean and disturbance parts of the dependent variable into the Navier-Stokes equations neglecting all second and higher order terms. The fundamental assumption is that the disturbance is small so that the neglected terms do indeed fail to influence the overall behavior of the system.

The stability of a system with respect to an equilibrium state can be tested by investigating the response of the system to imposed disturbances, the response being obtained from the perturbation
equations. Since the source and nature of all spurious noises continuously acting on any real system of the physical world are unknown, the stability must be tested with respect to all possible disturbances. Such an all inclusive study is usually accomplished by expressing an arbitrary disturbance as the sum of a set of certain basic modes that constitute a complete set over the region of interest. The response of the system to any stimulus can be found by linear combination of the responses to the individual normal modes. If the system becomes non-linear due to the growth of small disturbances to significant size, then the superposition process is not applicable and one can no longer be assured of a stability test valid for all possible disturbances. Thus, non-linear analysis provides a mechanism for testing system stability with respect to a specified disturbance but provides no means of determining response to an arbitrary disturbance.

Let us become more specific by considering the stability of steady flow between parallel plates with respect to two-dimensional infinitesimal perturbations. The use of a two-dimensional disturbance is justified because of the work of Squire (1933) who showed that the problem of three-dimensional disturbances is exactly equivalent to a two-dimensional problem at a lower Reynolds number. Since a three-dimensional disturbance takes the form of a wave propagating in a direction oblique to that of the basic flow, the only component of the basic flow that will affect the disturbance is the one acting in that direction. Intuitively then, one can visualize the existence of a reduced effective Reynolds number; and as a consequence, sufficient
conditions for stability can be obtained directly from two-dimensional analysis.

The undisturbed flow is assumed to be parallel to one of the plane walls so that the velocity is dependent only on distance normal to the wall and so that there is no velocity component normal to the wall. This assumption is valid for flow far from the inlet region. Non-dimensionalization of the equations of motion is accomplished through introduction of a reference velocity $U_0$ and reference length $l$. Actual distances parallel and normal to the flow are denoted by $x \ell$ and $y \ell$. The corresponding velocity components are $uU_0$ and $vU_0$, and real time is $(l/U_0)t$. We always take $l$ to be half the channel width and $U_0$ to be the maximum channel velocity. It is convenient to express general two-dimensional motions in terms of stream function $\psi(x,y,t)$ defined such that

\begin{equation}
(2.1) \quad u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.
\end{equation}

The function $\psi(x,y,t)$ then satisfies the vorticity transport equation which, expressed in dimensionless form, is

\begin{equation}
(2.2) \quad \frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y} = R^{-1} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}
\end{equation}

where

\begin{equation}
(2.3) \quad \xi = -\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.
\end{equation}
To study a small disturbance from a stationary state $\bar{\psi}(y)$, we substitute

\[ (2.4) \quad \psi(x,y,t) = \bar{\psi}(y) + \hat{\psi}(x,y,t) \]

into (2.2) and retain only those terms linear in $\hat{\psi}(x,y,t)$. The resultant differential equation for $\hat{\psi}$ is linear and has coefficients that depend only on $y$. Consequently, we expect solutions cyclic in both $x$ and $t$.

\[ (2.5) \quad \hat{\psi}(x,y,t) = \phi(y)e^{i\alpha(x-ct)} \]

Substitution of this expression into the differential equation for $\hat{\psi}$ leads to an ordinary differential equation for $\phi(y)$, namely

\[ (2.6) \quad (\overline{u} - c)(\phi'' - \alpha^2 \phi) - \overline{u}'' \phi = -\frac{i}{\alpha R} (\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi) \]

with boundary conditions corresponding to vanishing wall velocities, i.e.,

\[ \phi = \phi' = 0 \text{ at } y = \pm 1. \]

This equation for $\phi(y)$ is the celebrated Orr-Sommerfeld equation arrived at independently by Orr in 1907 and Sommerfeld in 1908. It is a key equation forming the basis of all theoretical work on stability of two-dimensional parallel flows, although it takes on different forms and correspondingly different behavior depending on the geometry.

Earliest work on equation (2.6) was done by Lord Rayleigh (1880) who studied the inviscid form which bears his name,
(2.7) \[ (\bar{u} - c)(\phi'' - \alpha^2 \phi) - \bar{u}'' \phi = 0. \]

He was able to prove several important theorems dealing with laminar velocity profiles, the most important of which asserts that velocity profiles containing a point of inflection are unstable. The converse theorem, which provides a sufficiency condition, was not supplied until 1935 when Tollmien succeeded in doing so. According to these theorems plane-Poiseuille flow is stable in the absence of viscous forces. If one conjectures that viscous forces are stabilizing, one is forced into the fallacious conclusion that viscous plane-Poiseuille flow is universally stable. The dual role of viscosity was very confusing to the earliest workers, it being difficult to understand that viscosity has a damping effect on one hand but is actually the cause of instability on the other.

The full Orr-Sommerfeld equation is in reality an eigenvalue problem whose solution leads to a secular determinantal equation of form

\[ F(\alpha, R; c) = 0. \]

For each pair of values of the real parameters \( \alpha \) and \( R \), there exists a sequence of complex eigenvalues \( c \) whose imaginary part represents the amplification or decay of the disturbance (the corresponding eigenfunction) in time. Normally we are interested only in the eigenvalue with largest imaginary part - the least stable. If \( c_1 \) is
positive, the disturbance is amplified and is unstable; whereas if it is negative, the disturbance decays and is stable. The condition \( c_1 = 0 \) represents a curve in the \( \alpha - R \) plane, commonly referred to as the curve of neutral stability. It should be pointed out that an alternative interpretation can be given to the role of parameters \( \alpha \) and \( c \). The product \( \alpha c = \beta \) can be regarded as being real, whereas the individual factors \( \alpha \) and \( c \) are each complex. Real quantity \( \beta \) is then a dimensionless frequency parameter and \( \alpha \) is the corresponding complex eigenvalue. This interpretation is equivalent to consideration of timewise periodic disturbances that either grow or decay as they propagate downstream.

The Orr-Sommerfeld equation remained unsolved for a period of twenty-two years following its formulation until in 1929 Tollmien calculated the first neutral eigenvalues and obtained a critical Reynolds number beyond which instability begins. His success was facilitated by the work of Heisenberg (1924) who was the first to conclude instability for high Reynolds numbers but did not obtain a critical value. Much controversy developed during this period regarding some obscure points contained in Heisenberg's analysis. C. C. Lin (1944) later clarified many of these points and concluded in favor of instability.

Validation of Tollmien's analytical results by experiment was impossible to attain for a long period since the existence of small oscillations of the type predicted could not even be detected.
Hence, the theory was met with skepticism. However, the fundamental
diagnostic instrument, the hot-wire anemometer, became available and
in 1943 the classic experimental results of Schubauer and Skramstad
who used a vibrating ribbon to impose a controlled disturbance in a
boundary layer finally confirmed that the process of amplification of
small disturbances leads to transition. Theory and experiments agreed
with respect to eigenvalues and eigenfunctions but the transition pro-
cess was not well understood and still remains an enigma to this day.

With the appearance of the digital computer, the eigenvalue
problem began to be attached by direct numerical integration. Thomas
(1953), using special techniques to filter out spurious solutions
having rapid exponential growth that usually plague standard integra-
tion procedures, was able to tabulate the least stable eigenvalue for
a sequence of Reynolds numbers and wave numbers. His results validated
those obtained earlier by analytical methods and have become a present
day standard of accuracy. Shen (1954) made detailed calculations of
curves of $c_1$ using asymptotic power series methods. He found a minimum
critical Reynolds number 5360 at $\alpha = 1.05$ compared to Thomas' more
accurate figure of 5780 at $\alpha = 1.02$. These results are shown in
Figure 2.1.

Lin (1955) published a comprehensive monograph which gives a
systematic evaluation of the entire field of hydrodynamic stability
dealing primarily with asymptotic power series methods. This classic
volume settled many of the controversial points that had accumulated
FIGURE 2.1
STABILITY CHARACTERISTICS OF LINEAR PLANE - POISEUILLE MOTION (AFTER LIN 1955)
over the years and proved to be a milestone in the field.

More recent work connected with linear theory includes the calculations of Chen and Sparrow (1967) who investigated the entrance region of a parallel plate channel and found that the critical Reynolds number decreases monotonically with increasing distance from the channel entrance, approaching the fully developed value as a limit. That is, entrance region profiles were shown to be more stable than the fully developed profile.

Grosch and Salwen (1968) studied linear stability of plane-Poiseuille flow both for steady flow and for the case of a pressure gradient that is periodic in time. Their method was to expand the disturbance stream function in a complete set of functions satisfying the boundary conditions. The series was truncated and a Galerkin procedure was employed to yield a finite set of ordinary differential equations for the time-dependent coefficients. Calculations were carried out for both symmetric and anti-symmetric disturbances over a wide range of Reynolds numbers and wave numbers to obtain detailed information concerning the first ten or so eigenvalues. Their results agreed well with those on Thomas (1953) but are much more complete. Regarding the time-dependent flow situation, they found that small amplitudes of oscillation tended to stabilize while large amplitudes tended to destabilize the flow. The destabilization effect was particularly strong.
To backtrack in the chronology, we will now consider the more important and correspondingly more difficult case of Poiseuille flow in a straight circular pipe. Here the Orr-Sommerfeld equation takes the form

\[(L - a^2) - [\alpha c + i\alpha R]\left(\begin{array}{c} L - a^2 \end{array}\right) \phi = 0\]

where

\[L[\phi] = \phi_{rr} + \frac{1}{r} \phi_r - \frac{1}{r^2} \phi\]

with boundary conditions

\(\phi\) regular at \(r = 0\) and \(\phi = \phi_r = 0\) at \(r = 1\).

Sexl (1927) was the first to study the stability of axisymmetric disturbances in a pipe taking viscosity into account. He could not detect any instability but at the same time he could not prove stability for all Reynolds numbers. Pretsch (1941) formed an analogy between disturbances near the wall in Poiseuille flow and similar disturbances in plane-Couette flow which was believed to be stable. Pekeris (1948) found a family of eigenvalues corresponding to disturbances confined to a small region near the center of the pipe. Corcos and Sellars (1959) considered further both of these limiting cases but maintained that only a finite number of eigenvalues exist. All of these authors concluded in favor of stability. Schensted (1961) made an important contribution when she proved the existence of an infinite number of discrete eigenvalues and established their completeness by proving an expansion theorem for arbitrary functions satisfying the boundary conditions.
Experimental confirmation of theoretical findings came in 1959 when Leite studied stability of Poiseuille flow to small periodic oscillations superimposed on fully developed laminar flow. Disturbances were created by longitudinal oscillations of a thin sleeve adjacent to the pipe wall, response being measured downstream by means of hot-wire anemometer traverses. The measurements indicated complete and absolute stability to small disturbances (less than 0.1% of the maximum velocity in the mean flow) within the range of Reynolds numbers tested, up to 13,000. Large disturbances were created by means of a ring airfoil placed symmetrically in the flow field, reported by Kuethe (1956). It was found that transition to turbulent flow occurred whenever the disturbance amplitude exceeded a threshold value which decreases with increasing Reynolds number. The study further revealed that large disturbances propagate at a greater speed and decay more slowly than those generated by the sleeve. No quantitative measure of the threshold amplitude was possible due to departure from axial symmetry.

Gill (1965) considered stability of flow in a pipe from the same point of view as Leite, that is, spatial growth or decay of infinitesimal disturbances propagating downstream, presenting a most comprehensive account of the problem. He obtained uniformly valid asymptotic approximations for the eigenfunctions which agreed well with Leite. Wave speed and damping rate were calculated for each mode as a function of non-dimensional frequency. For each mode he found that damping rate is an increasing function of frequency for high frequencies approaching a limiting value as frequency is decreased. Notable also was Gill's success in finding an additional family of eigenfunctions overlooked by
previous workers (because they discarded some small but nevertheless vital terms) thus confirming the existence of an infinite set of eigenfunctions for given frequency and Reynolds number.

Extensive numerical calculations were made by Davey and Drazin (1969) for the eigenvalues of the problem under discussion. Results were obtained for both timewise and spacewise stability. Two numerical procedures were used (1) expansion in a power series in \( r \) near the centerline, continued from a small but finite value of \( r \) to the wall using Runga-Kutta integration, and (2) expansion of the solution into a complete set of orthogonal functions closely related to the problem. The results are most complete and are found to join accurately with known asymptotic results. Again, complete stability is indicated for axisymmetric disturbances.

As mentioned above, three-dimensional effects have been pointed to as a possible source of Poiseuille flow instability. Lessen et al. (1964) attempted to explore this possibility experimentally by oscillating a small vane or paddle, whose plane was parallel to the tube axis and whose motion was perpendicular to the tube axis, in the tube entrance region. The fluctuations gave rise to an azimuthally periodic disturbance of period one and also a periodic motion along the axis. The results, obtained over a range of frequencies and Reynolds numbers, indicated regions of stable and unstable flow similar in shape to a standard curve of neutral stability. No mention was made of the magnitude of the imposed disturbances. It should be mentioned
here that Squire's theorem for plane-Poiseuille flow does not hold for Poiseuille flow, so that two-dimensional disturbances in pipe flow are not more critical than those having a three-dimensional character. This experimental work was essentially repeated by Fox et al. (1968) since the earlier results were criticized on the basis that observed instabilities may have been caused by extraneous effects originating in the entrance region. The peak amplitude of the velocity disturbance, introduced by vibrating a thin flat spring perpendicular to the main flow, was maintained at less than 2% of the maximum steady state velocity. The findings again indicated that stability is both frequency and Reynolds number dependent to the first azimuthal mode with a critical Reynolds number of about 2150. It should be noted that the induced disturbances cannot be construed as being small; thus, non-linear effects very likely contributed to the observed transition.

A theoretical investigation, Lessen et al. (1968), made in an attempt to support the experimental findings discussed above, consisted of examination of the linear Orr-Sommerfeld equation in terms of the primitive variables. Tempewise stability was studied by letting \( \alpha \) be real and \( \zeta \) complex in spite of the fact that the experiments were performed on the basis of spacewise stability. No linear instability was found for any mode with an azimuthal periodicity of one.

A further attempt was made to investigate linear azimuthally periodic disturbances by Salwen and Grosch (1968). Their method
consisted of expansion of the perturbation velocity and pressure in a complete set of orthonormal functions satisfying the boundary conditions. Truncation of the expansion yielded a matrix differential equation for the time-dependent coefficients. Stability characteristics were determined from the eigenvalues of the matrix which were calculated numerically. For the symmetric case, their results were in good agreement with those of Pekeris (1948), indicating universal stability. Calculations carried out for the first five azimuthally varying modes indicated stability for wave numbers less than 10 and Reynolds numbers less than 10,000.

In a recent work Pedley (1969) considered stability of viscous flow in a rapidly rotating pipe and found the basic flow to be unstable for Reynolds numbers greater than 82.9. These results are surprising because of the widespread belief that rotation always has a stabilizing effect. Interesting also is the fact that the unstable modes take the form of growing spiral waves that are stationary relative to the rotating cylinder and twist in the opposite direction to the streamlines. The most rapidly growing disturbance has an azimuthal wave number which increases with R, the critical disturbance having an azimuthal wave number of one.

One last point should be made regarding linear theory. Betchov and Criminale (1967) have pointed out that no vorticity source term appears in the Orr-Sommerfeld equation for either axisymmetric pipe flow or plane-Couette flow and that this fact may be of fundamental
importance in trying to account for the variance in behavior between these flows and plane-Poiseuille flow. In this connection, Mott and Joseph (1968) investigated stability of flow between concentric cylinders with respect to the ratio $k$ of the outer to inner cylinder. The critical Reynolds number was found to be a monotone function of $k$, increasing without bound as $k \to \infty$ (Poiseuille flow) from the plane-Poiseuille flow limit at $k=1$. It should be noted that the vorticity source term does not vanish from the equations for $k>1$. Seeking to determine if the increase in stability was caused by skewing of the profile rather than inherent properties of the geometry, they generated a one-parameter family of increasingly skewed profiles and analyzed the Orr-Sommerfeld stability of these. Their analysis showed that the profiles which were deliberately skewed in annuli with fixed radius ratios exhibited an increased stability of the same general nature as that associated with reduction of the size of the inner cylinder. Similar results were found by Potter (1956) for a sequence of profiles continuously deformed from plane-Poiseuille motion to plane-Couette flow. These studies suggest that the absolute stability which linear theory associates with plane-Couette and pipe-Poiseuille flow is not a singular result but represents a limit result for profiles which increasingly depart from a least stable symmetric form.

Non-Linear Theory

Non-linear theory was set on its way by Meksyn and Stuart (1951) when they found periodic solutions for subcritical equilibrium flow between parallel plates. They considered a situation in which
the mean velocity field was assumed to be distorted by a fluctuation having the shape of the least stable Orr-Sommerfeld eigenfunction. Under these conditions the fluctuation interacts with itself to distort the mean flow which in turn interacts with the basic perturbation. The stability of the resultant modified mean equilibrium flow was examined by Meksyn and Stuart using asymptotic methods of analysis. They found a reduction in critical Reynolds number for amplitudes of fluctuation up to 0.04. For larger amplitudes their results indicate an increase in critical Reynolds number, but the validity of their assumptions beyond this point is subject to doubt. Notable also is the fact that an inflection point appears in the velocity profile for these finite-amplitude equilibrium conditions.

In the period between 1958 and 1962 there appeared a series of closely related papers by Stuart (1958, 1960) and Watson (1960, 1962) which laid the foundations for non-linear analysis. In these papers the concept of subcritical and supercritical regions is defined. It was postulated that within the neutral curve, the supercritical region, a disturbance of amplitude greater than some critical value would decay to this amplitude while disturbances of lesser magnitude would grow, approaching the critical value once more in the limit. Thus, supercritical equilibrium states might exist within the neutral curve that would be stable to small (or even large) disturbances. Similarly, the existence of unstable subcritical equilibrium states was hypothesized for the region external to the neutral curve. It
was presupposed that both subcritical and supercritical states could not exist simultaneously in the same basic flow although this was an open question.

Stuart (1958) presented an approximate integral or energy method treatment of the problem and arrived at a result conflicting with Meksyn and Stuart's (1951) earlier work. His results indicated the existence of supercritical equilibrium states but no subcritical instabilities. The basic method assumed that a disturbance retains its Orr-Sommerfeld shape as the amplitude grows. The amplitude to which a disturbance grows in this case would be determined by a balance of fluctuation energy production and viscous dissipation.

Stuart's later work (1960) in which the differential equations were attacked directly was motivated by a desire to resolve the conflicting results of the two aforementioned papers. Here Stuart formulated the problem in terms of the stability of the velocity profile of a two-dimensional traveling wave and expanded the stream function in terms of a complex Fourier series. All harmonics were assumed to travel at the same velocity. Separation of harmonic components led to a basic set of partial differential equations. Seeking some way to separate variables, he looked for solutions in the limit of \( c_1 \to 0 \) by expressing the harmonic components in terms of a single time-dependent amplitude function, the least stable eigenfunction, and certain unknown functions of \( y \). Then using approximations based on order of magnitude considerations he obtained a single ordinary
differential equation for the amplitude function. The solution of this equation depends heavily on the sign of a scalar quantity \( a^{(2)} \) which in turn depends on the shape of the disturbance. The differential equation for the amplitude function takes the form

\[
\frac{dA}{dt} = a^{(o)} A + a^{(2)} A^3
\]

where \( a^{(o)} = \omega c_1 \) vanishes on the neutral curve. Clearly, the sign of \( a^{(2)} \) governs whether or not equilibrium states are subcritical or supercritical in nature. Stuart enumerated the basic physical processes that contribute to \( a^{(2)} \). He showed that \( a^{(2)} \) can be decomposed into \( a^{(2)} = k_1 + k_2 + k_3 \) and that contributions arise from three basic physical processes (1) distortion of the mean flow, represented by \( k_1 \), (2) generation of the second harmonic \(-k_2\), and (3) distortion of the fundamental disturbance \(-k_3\). In addition, he showed that \( k_1 \) is negative and conjectured that \( k_2 \) is negative, but he did not determine \( k_3 \) and could not even speculate as to its sign although he was convinced that it played a dominant role.

Somewhat later, Reynolds and Potter (1967) made an important contribution which brought the previous work into sharper focus. The approach was that of formal expansion for the analysis of non-linear developments of an oblique wave in parallel flow and constituted an extension and modification of the methods of Stuart and Watson. The analysis centered about an equation for the amplitude of the velocity disturbance having the same form as equation (2.9). Choosing points on the neutral stability curve where \( a^{(o)} = 0 \), they found \( a^{(2)} \) to be
positive along the upper branch and negative along the lower branch. This behavior demonstrated that plane-Poiseuille flow does exhibit finite-amplitude subcritical instability with no evidence of existence of any supercritical equilibrium flow conditions. The implication is that catastrophic transition to turbulence would occur prior to reaching an equilibrium state. Further, their results (crude) indicate that relatively weak finite disturbances markedly reduce the critical Reynolds number. The main contribution supplied by this work, in addition to the development of an interesting mathematical technique, was that of demonstrating the fundamental importance of considering distortion of the fundamental fluctuation when examining finite-amplitude motions resulting from viscous instability.

Watson (1960) gave a more general and rigorous treatment than Stuart (1960) which represented modification and extension of his work. Stuart developed a limiting non-linear solution as $c_1 \rightarrow 0$, whereas, Watson developed a perturbation series. The same general conclusions were reached, i.e., a solution was found the behavior of which depended on the sign of an unknown parameter. Shortly thereafter, Watson (1962) investigated the more difficult problem of spatially growing disturbances using methods exactly analogous to his 1960 treatment of temporally growing disturbances. The same form of solution was again found to hold. He pointed out that for spatially growing disturbances the mean flow is not necessarily parallel and that in the timewise problem one has the freedom to specify constant
mean pressure gradient or constant mass flux with respect to the velocity profile, whereas in the spacewise problem no corresponding condition arises and one is forced into the condition of constant total mean flow.

Another notable work was that of Dixon (1966) who studied spatial stability of plane-Poiseuille flow and Poiseuille flow with respect to finite amplitude effects by integrating the Navier-Stokes equations in time-dependent form in a two-dimensional rectangular region. He demonstrated general feasibility of the finite-difference approach but fell short of desirable goals in that definitive quantitative results were, and indeed still are, impractical to obtain. This work will be discussed further in Chapter VI.

Dowell's (1969) investigation of non-linear plane-Poiseuille flow utilized Fourier decomposition of the streamwise spatial variable. A Galerkin expansion (see Chapter V) was made of the lateral variable which resulted in a complicated set of non-linear ordinary differential equations in time. The method is excellent for only one harmonic but computational requirements go out of sight rapidly with the addition of any more, due to the excessively large number of interaction terms involved. He solves a number of linear problems and is successful in getting series convergence when 40 or more channel modes are retained ($a=1, R=10,000$). One non-linear run is reported for which only the mean flow and fundamental fluctuation harmonics are retained. Further, only 16 channel modes are kept, yielding results that are only qualitative in nature. Dowell made a series of runs to see if a
large initial disturbance destabilizes a flow that is stable with respect to small disturbances, but was unsuccessful. It appears that he inadvertently used a Reynolds number smaller than the critical value below which flow is stable regardless of the size of a disturbance. (Chapter VI). The method of solution proposed by Dowell will be compared with ours in Chapter V.

The most recent calculations concerning finite-amplitude disturbances in plane-Poiseuille flow reported in the literature are those of Pekeris and Shkoller (1969a). Here the disturbance was expanded in terms of the eigenfunctions of the Orr-Sommerfeld equation with unknown coefficients that are functions of time, leading to a large finite set of ordinary differential equations. The computational requirements of the method, which seem staggering, will be discussed in Chapter V. They found instability for amplitudes exceeding certain critical values. A curve relating critical amplitude and Reynolds number is given but seems wrong and is definitely at variance with findings obtained using simpler methods reported in a companion paper. The companion paper (1969b) shows shifts in the position of the neutral curve as a function of an amplitude parameter. Solutions were obtained by means of a perturbation method in which the eigenfunctions and eigenvalues were expanded in a power series of the amplitude.

There have been no theoretical studies of non-linear disturbances in Poiseuille flow reported in the literature. The only computational treatment that we are aware of is that of Dixon (1966)
referred to earlier in connection with plane-Poiseuille motion. These results, though interesting from a qualitative point of view, contain no information of quantitative significance.

In the excellent book "Stability of Parallel Flows", Betchov and Criminale (1967) present a most comprehensive and up-to-date state of the art discussion concerning stability of parallel and almost-parallel flows of various kinds. Notable also is the earlier book by Eckhaus (1965) entitled "Studies in Non-Linear Stability Theory".
CHAPTER III

MATHEMATICAL FORMULATION

We have proposed to study the behavior in time of spatially periodic perturbations in an incompressible fluid flowing between parallel plates and through a circular pipe. In this chapter, appropriate equations for the solution of this problem will be obtained for each of the basic geometrical configurations. These equations will, in turn, be combined into a single generalized form for convenience in future manipulations.

Plane-Poiseuille Flow

We begin with the Navier-Stokes equations with non-dimensionalization based $\ell$, half the channel width and on $U_0$, the velocity of the undisturbed laminar flow at the center of the channel. The basic relationships between the non-dimensional and dimensional quantities are:

$$ x = \frac{x'}{\ell}, \quad y = \frac{y'}{\ell}, \quad u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0}, \quad p = \frac{p'}{U_0^2}, $$

$$ t = \left[ \frac{U_0}{\ell} \right] t', \quad \text{and} \quad R = \frac{U_0 \ell}{v} $$

The coordinate system is arranged such that the infinite parallel walls are situated at $y = \pm 1$. Basic flow occurs in $x$-direction with velocity component $u$. The fundamental equations of motion and continuity take the form
\[ u_t = uu_x + vu_y = -(p/\rho)_x + R^{-1}(u_{xx} + u_{yy}) \]

(3.2) \[ v_t + uv_x + vv_y = -(p/\rho)_x + R^{-1}(v_{xx} + v_{yy}) \]

(3.3) \[ u_x + v_y = 0. \]

Now it is more convenient computationally to work with the equations expressed in terms of vorticity and stream function variables. These can be obtained by elimination of pressure between equations (3.1) and (3.2).

Define new variables \( \xi \) & \( \psi \) such that

(3.4) \[ \xi = u_y - v_x \]

(3.5) \[ u = \psi_y \]

(3.6) \[ v = -\psi_x \]

Then, after some manipulation of equations (3.1) and (3.2), we arrive at the vorticity transport equation

(3.7) \[ \xi_t + u\xi_x + v\xi_y = R^{-1}(\xi_{xx} + \xi_{yy}) \]

where

(3.8) \[ \xi = \psi_{xx} + \psi_{yy} \]

Next, consider a stream function disturbance \( \hat{\psi}(x,y,t) \) to be superimposed on the steady laminar flow \( \bar{\psi}(y) \).
\[ (3.9) \quad \psi(x,y,t) = \tilde{\Psi}(y) + \hat{\psi}(x,y,t) \]

\[ (3.10) \quad \xi(x,y,t) = \tilde{\xi}(y) + \hat{\xi}(x,y,t) \]

\[ (3.11) \quad u(x,y,t) = \bar{u}(y) + \bar{u}(x,y,t) \]

\[ (3.12) \quad v(x,y,t) = \hat{v}(x,y,t) \]

Substitution of (3.9), (3.10), (3.11), and (3.12) into (3.7) and (3.8) yields the disturbance equation

\[ (3.13) \quad \hat{\xi}_t + u\hat{\xi}_x + v\hat{\xi}_y = R^{-1}(\hat{\xi}_{xx} + \hat{\xi}_{yy}) - \nu \nabla_y \]

where,

\[ (3.14) \quad \hat{\xi} = \hat{\psi}_{xx} + \hat{\psi}_{yy} \]

Finally, substitute (3.14) into (3.13) and drop the ^ notation to obtain the disturbance equation expressed in terms of stream function perturbation only.

\[ R^{-1}\nabla^4 \psi - \frac{\partial}{\partial t} [\nabla^4 \psi] - \bar{u} \frac{\partial}{\partial x} [\nabla^2 \psi] + \frac{d^2 \bar{u}}{dy^2} \frac{\partial \psi}{\partial x} \]

\[ (3.15) \quad = \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} [\nabla^2 \psi] - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} [\nabla^2 \psi] \]

where,

\[ (3.16) \quad \bar{u} = 1 - y^2 \]
We now make the finite Fourier expansion spoken of previously,

\begin{equation}
\psi(x,y,t) = \sum_{m=0}^{M} A_m(y,t) \cos \max + B_m(y,t) \sin \max
\end{equation}

Then,

\begin{equation}
\nabla^2 \psi = \sum_{m=0}^{M} \left[ A_m'' - (m\alpha)^2 A_m \right] \cos \max + \left[ B_m'' - (m\alpha)^2 B_m \right] \sin \max
\end{equation}

\begin{equation}
\nabla^4 \psi = \sum_{m=0}^{M} \left[ A_m^\text{iv} - 2(m\alpha)^2 A_m'' + (m\alpha)^4 A_m \right] \cos \max
\end{equation}

\begin{equation}
+ \left[ B_m^\text{iv} - 2(m\alpha)^2 B_m'' + (m\alpha)^4 B_m \right] \sin \max
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} [\nabla^2 \psi] = \sum_{m=0}^{M} \cos \max \frac{\partial}{\partial t} \left[ A_m'' - (m\alpha)^2 A_m \right] + \sin \max
\end{equation}

\begin{equation}
\times \frac{\partial}{\partial t} \left[ B_m'' - (m\alpha)^2 B_m \right]
\end{equation}

\begin{equation}
\frac{\partial}{\partial x} [\nabla^2 \psi] = \bar{u} \frac{\partial}{\partial x} \left[ \sum_{m=0}^{M} m \left[ B_m'' - (m\alpha)^2 B_m \right] \cos \max \right.
\end{equation}

\begin{equation}
- \left[ A_m'' - (m\alpha)^2 A_m \right] \sin \max \}
\end{equation}

\begin{equation}
\frac{d^2 \psi}{dy^2} \psi_x = \alpha \frac{d^2 \bar{u}}{dy^2} \sum_{m=0}^{M} m \left[ B_m \cos \max - A_m \sin \max \right]
\end{equation}

Substitute (3.19), (3.20), (3.21), and (3.22) into (3.15) to obtain a system of partial differential equations for the coefficients

\[ A_m(y,t) \& B_m(y,t) , m = 0, 1, \ldots, M. \]
Define

\begin{align*}
C_m &= A_m^{\prime\prime} - (\alpha \gamma) A_m \\
D_m &= B_m^{\prime\prime} - (\alpha \gamma) B_m
\end{align*}

The equations are then expressible as

\begin{align*}
R^{-1} \left[ (\alpha \gamma)^4 A_m - 2(\alpha \gamma)^2 A_m^{\prime\prime} + A_m^{\prime} \right] &- \frac{\partial A_m}{\partial t} + (\alpha \gamma)^2 \frac{\partial A_m}{\partial t} \\
&- \bar{u} \alpha D_m + \bar{u}' \alpha B_m = F_m^A
\end{align*}

\begin{align*}
R^{-1} \left[ (\alpha \gamma)^4 B_m - 2(\alpha \gamma)^2 B_m^{\prime\prime} + B_m^{\prime} \right] &- \frac{\partial B_m}{\partial t} + (\alpha \gamma)^2 \frac{\partial B_m}{\partial t} \\
&+ \bar{u} \alpha C_m - \bar{u}' \alpha A_m = F_m^B
\end{align*}

where $F_m^A$ and $F_m^B$ are the non-linear terms as derived in Appendix I.

The boundary conditions are $u(\pm 1) = v(\pm 1) = 0$, or expressed in terms of the harmonic components

\begin{align*}
A_m(\pm 1) &= \frac{\partial A_m}{\partial y} (\pm 1) = 0 \\
B_m(\pm 1) &= \frac{\partial B_m}{\partial y} (\pm 1) = 0
\end{align*}

We note that the only unstable eigenfunctions that exist according to linear theory are those that are even functions of $y$ with respect to stream function, the odd eigenfunctions all being strongly stable. For this reason we will concentrate our attention
on non-linear problems in which the fundamental fluctuation components ($A_1$ and $B_1$) are even functions. By looking at the equations (3.25) and (3.26), it is fairly easy to convince oneself that an initial even function for $A_1$ will remain even for all time, the non-linear interactions being completely consistent with this assertion. The equations also dictate that alternate harmonics are odd and even. Thus, we can cut the required computer time in half by solving the problem over only half of the channel using the following boundary conditions at the centerline.

$$A_m(0) = A_m''(0) = 0 \quad \text{for } m \text{ even}$$

$$A_m'(0) = A_m'''(0) = 0 \quad \text{for } m \text{ odd}.$$ 

Poiseuille Flow

Here again we begin with the Navier-Stokes equations non-dimensionalized with respect to the radius of the tube and the maximum velocity of the undisturbed laminar flow.

(3.29)  \[ u_t + uu_x + vu_r - \frac{p}{\rho} x - \frac{1}{r} \left( \frac{1}{r} (ru_r)_r + u_{xx} \right) \]

(3.30)  \[ v_t + uv_x + vv_r - \frac{p}{\rho} r + \frac{1}{r^2} \left( \frac{1}{r} (rv)_r \right)_r + v_{xx} \]

(3.31)  \[ u_x + \frac{1}{r} (rv)_r = 0 \]

Change to vorticity and stream function variables once more:
Define,

\begin{align*}
(3.32) \quad \xi &= r(u_r - v_x) \\
(3.33) \quad u &= \frac{1}{r} \psi_r \\
(3.34) \quad v &= -\frac{1}{r} \psi_x
\end{align*}

Whence, on elimination of pressure from (3.29) and (3.30) we obtain

\begin{align*}
(3.35) \quad \psi_t + u \xi_x + v \xi_r - 2v \xi/r &= R^{-1}(\xi_{xx} + \xi_{rr} - \xi/r) \\
(3.36) \quad \xi &= \psi_{xx} + \psi_{rr} - \psi_r/r
\end{align*}

Again, substitution of the disturbances (3.9) - (3.12) into (3.35) and (3.36) yields

\begin{align*}
(3.37) \quad \hat{\xi}_t + u \hat{\xi}_x + v \hat{\xi}_r - 2v \hat{\xi}/r &= R^{-1}(\hat{\xi}_{xx} + \hat{\xi}_{rr} - \hat{\xi}_r/r) \\
&\quad - v [\xi_r - 2\xi/r]
\end{align*}

where,

\begin{align*}
(3.38) \quad \hat{\xi} &= \hat{\psi}_{xx} + \hat{\psi}_{rr} - \hat{\psi}_r/r
\end{align*}

Now,

\begin{align*}
\bar{u} &= 1 - r^2 \\
\xi &= -2r^2
\end{align*}
Hence, the term in brackets vanishes, and on dropping the notation, the disturbance equation simplifies to

\begin{equation}
\xi_t + u\xi_x + v\xi_r - 2v\xi/r = R^{-1}(\xi_{xx} + \xi_{rr} - \xi_r/r)
\end{equation}

where,

\begin{equation}
\xi = \psi_{xx} + \psi_{rr} - \psi_r/r
\end{equation}

Equations (3.39) and (3.40) turn out to be identical to (3.35) and (3.36).

Finally, substitution of (3.33), (3.34), and (3.40) into (3.39) followed by some algebraic manipulation leads to the disturbance equation expressed in terms of stream function perturbation only.

\begin{align*}
R^{-1}\{&\psi_{xxxx} + \psi_{rrrr} + 2\psi_{xxrr} - \frac{2}{r}(\psi_{xxr} + \psi_{rrr}) + \frac{3}{r^2}(\psi_{rr} - \frac{1}{r}\psi_r) \\
&- \psi_{txx} - \psi_{ttr} + \frac{1}{r}\psi_{tr} - \check{u}(\psi_{xxx} + \psi_{xrr} - \frac{1}{r}\psi_{xr}) \}
\end{align*}

\begin{equation}
= \frac{1}{r}\psi_r[\psi_{xxx} + \psi_{xrr} - \frac{1}{r}\psi_{xr}] - \frac{1}{r}\psi_x[\psi_{rxx} + \psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r] + \frac{2}{r^2}\psi_x[\psi_{xx} + \psi_{rr} - \frac{1}{r}\psi_r]
\end{equation}

The Fourier expansion in x is made

\begin{equation}
\psi(x,r,t) = \sum_{m=0}^{M} A_m(r,t) \cos mx + B_m(r,t) \sin mx
\end{equation}
Substitution of (3.42) into (3.41) and subsequent simplification leads to equations for \( A_m(r,t) \) and \( B_m(r,t) \), namely

\[
(3.43) \quad R^{-1} \{ A_m^{iv} + \left[ \frac{-2}{r} \right] A_m'' + \left[ \frac{3}{r^2} - 2(\alpha^2) \right] A_m' + \left[ \frac{2(\alpha^2)}{r} - \frac{3}{r^3} \right] A_m + \left[ (\alpha^4) \right] A_m \} \\
+ \left( \alpha^2 \right) \frac{\partial A_m}{\partial t} - \frac{\partial A_m''}{\partial t} + \frac{1}{r} \frac{\partial A_m'}{\partial t} - \bar{u} \alpha D_m = f_m^A
\]

and

\[
(3.44) \quad R^{-1} \{ B_m^{iv} + \left[ \frac{-2}{r} \right] B_m'' + \left[ \frac{3}{r^2} - 2(\alpha^2) \right] B_m' + \left[ \frac{2(\alpha^2)}{r} - \frac{3}{r^3} \right] B_m + \left[ (\alpha^4) \right] B_m \} \\
+ \left( \alpha^2 \right) \frac{\partial B_m}{\partial t} - \frac{\partial B_m''}{\partial t} + \frac{1}{r} \frac{\partial B_m'}{\partial t} + \bar{u} \alpha C_m = f_m^B
\]

where,

\[
(3.45) \quad C_m = A_m'' - \frac{1}{r} A_m' - (\alpha^2) A_m
\]

\[
(3.46) \quad D_m = B_m'' - \frac{1}{r} B_m' - (\alpha^2) B_m
\]

The non-linear terms \( f_m^A \) and \( f_m^B \) are derived in Appendix I.

The boundary conditions are

\[
u(1) = \frac{1}{r} \psi_x(1) = 0 \,
\]

\[
v(1) = -\frac{1}{r} \psi_x(1) = 0 \,
\]

\[
v(0) = -\frac{1}{r} \psi_x(0) = 0 \,
\]

\[
u_r(0) = \frac{1}{r} \left[ \psi_{rr} - \frac{1}{r} \psi_r \right]_{r=0} = 0
\]
Expressed in terms of the harmonics these turn out to be equivalently

\begin{align}
(3.47) & \quad A_m(1) = B_m(1) = 0 \\
(3.48) & \quad A'_m(1) = B'_m(1) = 0 \\
(3.49) & \quad A_m(0) = B_m(0) = 0 \\
(3.50) & \quad A'_m(0) = B'_m(0) = 0
\end{align}

Generalization

The harmonic component disturbance equations for the two basic geometries can be combined into a single generalized formulation. Expressed in differential operator notation this generalization can be written as:

\begin{align}
(3.51) & \quad \frac{\partial}{\partial t} \mathcal{L}_m[A_m] = R^{-1} \mathcal{L}_m^2[A_m] - \bar{u} \bar{\alpha} \mathcal{L}_m[B_m] + (1-\eta) \bar{\alpha} B_{yy} B_m + F^A_m \\
& \quad \text{for } 0 \leq m \leq M \\
(3.52) & \quad \frac{\partial}{\partial t} \mathcal{L}_m[B_m] = R^{-1} \mathcal{L}_m^2[B_m] + \bar{u} \bar{\alpha} \mathcal{L}_m[A_m] - (1-\eta) \bar{\alpha} B_{yy} A_m + F^B_m \\
& \quad \text{for } 1 \leq m \leq M
\end{align}
(3.53) \[ \frac{\partial A}{\partial m} = -\frac{\alpha}{2} \frac{\partial A}{\partial m} ; \quad \frac{\partial B}{\partial m} = -\frac{\alpha}{2} \frac{\partial B}{\partial m} \]

(3.54) \[ \Theta = 1 + n(\frac{1}{y} - 1) \]

(3.55) \[ \hat{p}^A_m = \sum_{p=1}^{M} Q_{pp} \]

\[ \hat{p}^B_m = 2m\{A^0 \frac{\partial}{\partial y} L_0[A^0] - B_m \frac{\partial}{\partial y} L_0[A^0] \} \]

(3.56) \[ + \sum_{p=1}^{m-m} \{Q_{(m+p)p} + Q_{p(m+p)}\} + \sum_{p=1}^{m-1} R_{(m-p)p} \]

\[ \hat{p}^B_m = 2m\{A_m \frac{\partial}{\partial y} L_0[A^0] - A^0 \frac{\partial}{\partial y} L_0[A_m] \} \]

(3.57) \[ + \sum_{p=1}^{m-m} \{S_{(m+p)p} - S_{p(m+p)}\} + \sum_{p=1}^{m-1} T_{(m-p)p} \]

where,

\[ Q_{pp} = E_{pp} + H_{pp} \]

\[ R_{pp} = E_{pp} - H_{pp} \]

\[ S_{pp} = G_{pp} - F_{pp} \]

\[ T_{pp} = G_{pp} + F_{pp} \]
\[ E_{qp} = p A_q^* \mathcal{L}_q[B_q] - q \mathcal{I}_p[A_p]B_q \]

\[ F_{qp} = -p A_q^* \mathcal{L}_q[A_q] - q \mathcal{I}_p[B_p]B_q \]

\[ G_{qp} = p B_q^* \mathcal{L}_q[B_q] + q \mathcal{I}_p[A_p]A_q \]

\[ H_{qp} = -p B_q^* \mathcal{L}_q[A_q] + q \mathcal{I}_p[B_p]A_q \]

\[ \mathcal{L}_m[ ] = \frac{\partial^2[ ]}{\partial y^2} - \frac{n}{y} \frac{\partial[ ]}{\partial y} - (n\alpha)^2 [ ] \]

\[ \mathcal{I}_m[ ] = \frac{\partial}{\partial y} \mathcal{L}_m[ ] - \frac{2n}{y} \mathcal{L}_m[ ] \]

\[ \bar{u} = 1 - y^2 \]

\[ \eta = \begin{cases} 
0 & \text{for plane-Poiseuille flow} \\
1 & \text{for Poiseuille flow} 
\end{cases} \]
CHAPTER IV

NUMERICAL APPROXIMATION AND SOLUTION

In this chapter we will indicate the methods used to approximate the harmonic component disturbance equations as formulated in Chapter III in equations (3.51) through (3.63).

For computational purposes it is more convenient to express (3.51) and (3.52) directly in terms of the derivatives of $A_m$ and $B_m$. This is accomplished by substitution of the differential operator (3.60) into (3.51) and (3.52) which yields upon simplification

\[(4.1)\]
$\gamma_4 A_m^IV + \gamma_3 A_m''' + \gamma_2 A_m'' + \gamma_1 A_m' + \gamma_0 A_m + \lambda_0 \frac{\partial A_m}{\partial t} + \lambda_1 \frac{\partial A_m}{\partial t} + \lambda_2 \frac{\partial A_m}{\partial t}$

$+ \sigma D_m + \tau B_m = E_A^A_m$

\[(4.2)\]
$\gamma_4 B_m^IV + \gamma_3 B_m''' + \gamma_2 B_m'' + \gamma_1 B_m' + \gamma_0 B_m + \lambda_0 \frac{\partial B_m}{\partial t} + \lambda_1 \frac{\partial B_m}{\partial t} + \lambda_2 \frac{\partial B_m}{\partial t}$

$- \sigma C_m - \tau A_m = E_B^B_m$

where,

\[(4.3)\]
$C_m = A_m''' - \lambda_1 A_m' - (\alpha A_m)^2$

\[(4.4)\]
$D_m = B_m''' - \lambda_1 B_m' - (\alpha B_m)^2$

\[(4.5)\]
Defining $\gamma_1 = R^{-1} \gamma_1^*$
Then,

\[ \hat{\gamma}_4 = 1 \quad \hat{\gamma}_3 = -\eta(2/y) \quad \hat{\gamma}_2 = -2(\kappa x)^2 + \eta(3/y^2) \]

\[ \hat{\gamma}_1 = -\kappa y \hat{y}_2/y \quad \hat{\gamma}_0 = (\kappa x)^4 \]

\[ \lambda_0 = (\kappa x)^2 \quad \lambda_1 = \eta/y \quad \lambda_2 = -1 \]

\[ \sigma = -\kappa y \]

(4.6)

\[ \tau = (1-\eta) \bar{u}_{yy} \]

\[ \bar{u} = 1 - y^2 \]

\[ \eta = \begin{cases} 
0 & \text{for plane-Poiseuille flow} \\
1 & \text{for Poiseuille flow} 
\end{cases} \]

The equations (4.1) and (4.2) are to be discretized using Crank-Nicolson differencing in time, i.e.; knowing \( A_m \) and \( B_m \) at the \( n \)th time step, to advance to the \( (n+1) \)st step, we evaluate all terms in the equations at time \( (n+1/2)\Delta t \). The Crank-Nicolson scheme is characterized by small time truncation error and is well known for its superior stability properties. Fourth order space differencing will be employed, so that we expect an overall truncation error of order \( (\Delta y)^4 + (\Delta t)^2 \).

Define

\[ h \equiv \Delta y \]
\[ \delta_t Q_j = \frac{[Q_j^{(n+1)} - Q_j^{(n)}]}{\Delta t} \]

\[ \delta_y Q_j = \frac{[Q_{j-2} - 8Q_{j-1} + 8Q_{j+1} - Q_{j+2}]}{12h} \]

(4.7)

\[ \delta_y^2 Q_j = \frac{[-Q_{j-2} + 16Q_{j-1} - 30Q_j + 16Q_{j+1} - Q_{j+2}]}{12h^2} \]

\[ \delta_y^3 Q_j = \frac{[-Q_{j-2} + 2Q_{j-1} - 2Q_{j+1} + Q_{j+2}]}{2h^3} \]

\[ \delta_y^4 Q_j = \frac{[Q_{j-2} - 4Q_{j-1} + 6Q_j - 4Q_{j+1} + Q_{j+2}]}{h^4} \]

Now the difference operators \( \delta_y \) and \( \delta_y^2 \) have error terms proportional to \( h^4 \), whereas the error terms for \( \delta_y^3 \) and \( \delta_y^4 \), as defined, are of lower order, being proportional to \( h^2 \). Since fourth-order difference expressions for third and fourth derivatives involve three grid points to the left and to the right of the base point, it would be necessary to solve a set of linear algebraic equations having seven-diagonal matrix form. Hepta-diagonal systems require fifty percent more computing time to solve than do penta-diagonal systems. For this reason, we decided to use lower order implicit approximations for high derivatives. However, we are able to raise the overall accuracy up to \( h^6 \) universally by including error terms for the high derivatives that are explicitly evaluated and included in the right hand side of the resultant linear algebraic equations. These error terms involve fifth and sixth derivatives of the dependent variables, which remain small throughout the solution and do not vary appreciably with time. This technique is thought to be quite novel and has provided the needed improvement necessary to render the numerical process practical from a computational point of view, since far fewer
subdivisions are required to achieve any specified level of accuracy. Further discussion concerning the increase in accuracy as a result of this unique device will be given in Chapter VI.

We are now in a position to write down the Crank-Nicolson difference equation corresponding to equation (4.1) for \( A_m(j) \).

\( E^iv \) and \( E^"" \) refer to the error terms in the fourth and third derivatives, respectively, which are required to raise the spatial accuracy up to fourth order.

Superscripts and subscripts in parentheses refer to time step and grid point number, respectively.

It is understood that the \( \gamma \)'s, \( \lambda \)'s, \( \sigma \) and \( \tau \) depend on spatial position \( y=jh \), although this dependence is not explicitly indicated in the equations.

Define the averaging operator

\[
A \left[ Q^{(n)} \right] = \frac{1}{2} \left[ Q^{(n+1)} + Q^{(n)} \right].
\]

The difference equations for the \( A_m(j) \) are then

\[
\begin{align*}
\gamma_4 A \left[ \delta_y^4 A_m^{(n)} \right] + \gamma_3 A \left[ \delta_y^3 A_m^{(n)} \right] + \gamma_2 A \left[ \delta_y^2 A_m^{(n)} \right] + \gamma_1 A \left[ \delta_y A_m^{(n)} \right] \\
+ \gamma_0 A \left[ A_m^{(n)} \right] + \lambda_1 \delta_t^\tau A_m(j) + \lambda_1 \delta_t^\delta_y A_m(j) + \lambda_2 \delta_t^2 \delta_y^2 A_m(j) \\
+ \sigma A \left[ E_{m(n,k)}^{(n)} \right] + \tau A \left[ E_{m(n)}^{(n)} \right] = A \left[ E_{m(n,k)}^{(n)} \right] - E_{m}^{(n)} \left( n \right) \right)
\end{align*}
\]
The actual substitutions and subsequent algebraic manipulations, which are given in detail in Appendix III, lead to a set of non-linear algebraic equations. These equations and an analogous set for the $B_{m(j)}$ are solved by iteration with respect to the non-linear coupling terms. This task is accomplished by considering the set of finite-difference equations in cyclic order. For each individual equation, the non-linear terms are explicitly evaluated utilizing the latest available updated values of the dependent variables. These terms are then added to the already known iteration invariant parts of the right hand side. Finally, the resultant set of linear algebraic equations are solved for new values of either $A_{m(j)}$ or $B_{m(j)}$. The process is repeated until certain convergence criteria, defined later, are satisfied. The iteration process is feasible only because changes in the non-linear terms are small to moderate when relatively large values are used for the time increment, so that the computation can proceed and a reasonable rapid pace with little penalty in the number of iterations required for convergence.

The resultant equations referred to are

\begin{equation}
[N_{m(j)}] A_{m(j-2)}^{(n+1)} + [P_{m(j)}^-] A_{m(j-1)}^{(n+1)} + [Q_{m(j)}] A_{m(j)}^{(n+1)} + [P_{m(j)}^+] A_{m(j+1)}^{(n+1)}

+ [N_{m(j)}^+] A_{m(j+2)}^{(n+1)} = H_{m(j)}^{(n)} + \gamma_{m(j)}^{(n+1)}
\end{equation}

and
\[(4.11)\]
\[
\left[ N^e_{m(j)} \right] B_{m(j-2)}^{(n+1)} + \left[ P^e_{m(j)} \right] B_{m(j-1)}^{(n+1)} + \left[ Q^e_{m(j)} \right] B_{m(j)}^{(n+1)}
\]
\[+ \left[ P^+_{m(j)} \right] B_{m(j+1)}^{(n+1)} + \left[ N^+_{m(j)} \right] B_{m(j+2)}^{(n+1)}
\]
\[= G_{m(j)}^{(n)} + Z_{m(j)}^{(n+1)}\]

The coefficients are defined in Appendix III and are not repeated here due to their exceptional length.

The N, P, and Q coefficients are time invariant and depend only on Reynolds number, wave number, \(y\), \(\Delta y\), and \(\Delta t\); accordingly, they need only be evaluated once, during the initialization phase. For plane-Poiseuille flow N, P and Q do not depend on \(y\) and thus reduce to constants. The G and H coefficients must be evaluated once each time step, while the coefficients containing the updated non-linear terms at the advanced time level must be re-evaluated each iteration.

Finally, the algorithm for solving penta-diagonal systems of linear algebraic equations is derived in detail in Appendix V. Briefly, the algorithm for solving the system \(AX = F\) is accomplished by factoring A into the product of an upper-triangular and a lower-triangular matrix.

\[A = LU\]

whence, the system becomes

\[LUX = F\]

Define \(UX = \lambda\)

Then \(LA = F\)
Thus, the solution consists of three basic steps

(a) Factorization of A into LU.

The elements of L and U can be determined uniquely from recursive relations involving only the elements of A. This phase can be executed once and for all at the beginning since the elements of A are time invariant.

(b) Calculation of $\lambda$ (forward substitution).

The elements of $\lambda$ can be computed successively from the elements of L obtained in (a) and the elements of the right hand side F.

(c) Calculation of X (back substitution).

The desired unknown quantities X require only knowledge of $\lambda$ obtained in (b) and of the elements of U obtained in (a).

For A having form

\[
A = \begin{bmatrix}
  c_1 & d_1 & e_1 & 0 \\
  b_2 & c_2 & d_2 & e_2 \\
  a_3 & b_3 & c_3 & d_3 & e_3 \\
  & & & & \\
  & & & & \\
  0 & a & b & c_N \\
\end{bmatrix}
\]
The complete solution algorithm is

\[ \gamma_1 = c_1 \]
\[ \delta_1 = d_1/\gamma_1 \]
\[ \varepsilon_1 = e_1/\gamma_1 \]
\[ \beta_2 = b_2 \]
\[ \gamma_2 = c_2 - \beta_2 \delta_1 \]
\[ \delta_j = (d_j - \beta_j \varepsilon_{j-1})/\gamma_j \quad 2 \leq j \leq N \]
\[ \varepsilon_j = e_j/\gamma_j \quad 2 \leq j \leq N - 2 \]
\[ \alpha_j = a_j \quad 3 \leq j \leq N \]
\[ \beta_j = b_j - \alpha_j \delta_{j-2} \quad 3 \leq j \leq N \]
\[ \gamma_j = c_j - \beta_j \delta_{j-1} - \alpha_j \varepsilon_{j-2} \quad 3 \leq j \leq N \]
\[ \lambda_1 = f_1/\gamma_1 \]
\[ \lambda_2 = (f_2 - \beta_2 \lambda_1)/\gamma_2 \]
\[ \lambda_j = (f_j - \beta_j \lambda_{j-1} - \alpha_j \lambda_{j-2})/\gamma_j \quad 3 \leq j \leq N \]
\[ x_N = \lambda_N \]

(4.15)  
\[ x_{N-1} = \lambda_{N-1} - \delta_{N-1} x_N \]

\[ x_j = \lambda_j - \delta_j x_{j+1} - \epsilon_j x_{j+2} \quad j = N-2, N-3, \ldots, 1 \]

A few computer operations can be saved by working with a normalized form of the difference equations obtained by division of both sides by \( N_m(j) \). Normalization will result in the following equation and will cause all of the \( \alpha_j \) to reduce to unity.

(4.16)  
\[ A_{m(j-2)}^{(n+1)} + [\tilde{P}_m(j)] A_{m(j-1)}^{(n+1)} + [\tilde{Q}_m(j)] A_{m(j)}^{(n+1)} + [\tilde{P}_m(j)] A_{m(j)}^{(n+1)} \]

\[ + [\tilde{N}_m(j)] A_{m(j)}^{(n+1)} = f_{m(j)}^{(n+1)} \]

For parallel plate flow \( \tilde{P}_m(j) = \tilde{P}_m = \tilde{P}_m \), which is independent of \( j \), and \( \tilde{N}_m = 1 \) since all of the odd derivatives vanish. The algorithm can then be executed with coefficients defined as follows

At wall:  \[ c_1 = \tilde{Q}_m + 1 \]  (see Appendix V of derivation of boundary conditions)

At centerline:

For odd \( m \): \[ a_N = 2, b_N = 2\tilde{P}_m, c_N = \tilde{Q}_m, c_{N-1} = \tilde{Q}_m + 1 \]

For even \( m \): \[ a_N = b_N = c_N = 0, c_{N-1} = \tilde{Q}_m - 1 \]

All other coefficients:

\[ a_j = e_j = 1 , \]

\[ b_j = d_j = \tilde{P}_m , \]

\[ c_j = \tilde{Q}_m . \]
For Poiseuille flow the algorithm cannot be simplified beyond normalization. Boundary conditions dictate that (Appendix V)

\[ c_1 = \tilde{\alpha}_m(1) + 1 \]

\[ c_{N-1} = \tilde{\alpha}_m(N-1) + \tilde{\alpha}_m^T(N-1) \]

\[ a_N = b_N = c_N = f_N = 0. \]

The algorithm for inversion of a penta-diagonal band matrix of dimension \( N \), which is normalized such that all of the \( a_j \) are unity, requires the following approximate computational effort:

- **Factorization** - \( 8N \) operations (addition or multiplication)
- **Forward substitution** - \( 4N \) operations
- **Back substitution** - \( 4N \) operations

**Summary of Solution Procedure**

The procedure utilized for the solution of the system of equations (3.51) through (3.63) is summarized as follows:

1. Read problem parameters, namely
   - No. harmonics, no. grid points, Reynolds number, wave number, initial amplitude of fundamental fluctuation, time increment, maximum no. of iterations per time step, no. of time steps to be executed.
(2a) Initialization Phase. (option one - initial start)

(a) Initialize all harmonic components \( A_m \) and \( B_m \).

In almost all cases all harmonic components are set to zero except \( A_1 \) which is specified to have the shape of the first eigenfunction of a closely related problem.

(b) Compute time invariant coefficients \( N_m^- (j) \), \( P_m^- (j) \), \( Q_m (j) \), \( P_m^+ (j) \), and \( N_m^+ (j) \).

(c) Execute factorization of \( A \) into \( LU \), i.e., compute quantities \( \delta_m (j) \), \( \epsilon_m (j) \), \( \alpha_m (j) \), \( \beta_m (j) \), and \( \gamma_m (j) \).

(d) Compute auxiliary functions \( C_m \) and \( D_m \) and various required derivatives and functions of \( A_m \) and \( B_m \) corresponding to the initial conditions.

(e) Compute initial values of the iteration invariant coefficients \( H_m (j) \) and \( G_m (j) \).

(f) Set time and time step to zero.

(2b) Initialization phase (option two - restart from tape)

(3) Set iteration number \( k \) to zero.

(4) Set harmonic number \( m \) to zero.

(5) If \( m = 0 \) then skip to (8) otherwise, continue.

(6) Re-evaluate non-linear terms \( F_m^A (j) \) and thus also \( y_m^{(n+1)} (j) \).

(7) Execute forward and back substitution to obtain \( A_m^{(n+1), k+1} (j) \).

(8) Compute auxiliary functions \( A_m' (j) \), \( C_m (j) \), and \( W_m (j) \) corresponding to \( A_m (j) \) obtained in (7).
(9) Re-evaluate non-linear terms $F_{m(j)}^B$ and thus also $Z_{m(j)}^{(n+1)}$.

(10) Execute forward and back substitution to obtain $P_{m(j)}^{(n+1,k+1)}$.

(11) Compute auxiliary functions $B_{m(j)}'$, $D_{m(j)}$, and $V_{m(j)}$ corresponding to $B_{m(j)}$ obtained in (10).

(12) If $m < M$ then increment harmonic number $m$ and return to (6) otherwise, continue.

(13) Execute iteration convergence test. If test fails then increment iteration number $k$ and return to (4) otherwise, continue.

(14) Execute output for current time step and write restart tape if five minutes have elapsed since last written.

(15) If step number has reached upper limit then stop, otherwise, continue.

(16) Re-evaluate iteration invariant terms $H_{m(j)}^{(n)}$ and $Q_{m(j)}^{(n)}$.

(17) Increment time and time step number.

(18) Return to (3).

The iteration criterion used in step (13) was

$$\begin{align*}
\max_m \left| \max_j \left[ A_{m(j)}^{(n+1,k+1)} - A_{m(j)}^{(n+1,k)} \right] \right| < 10^{-6}
\end{align*}$$
where \( k \) is the iteration number. A similar condition was simultaneously required of \( B_{m(j)}^{(n+1)} \).

This condition corresponds roughly to a requirement that all of the quantities \( A_{m(j)}^{(n+1)} \) and \( B_{m(j)}^{(n+1)} \) change by no more than one digit in the sixth significant figure during the preceding iteration.
FIGURE 4.1
FLOW CHART FOR SOLUTION PROCEDURE

READ PROBLEM PARAMETERS

INITIALIZE HARMONIC COMPONENTS $A_m$, $B_m$

COMPUTE TIME INVARIANT COEFF.

EXECUTE FACTORIZATION OF $A$ INTO LU

SET TIME & TIME STEP TO ZERO

COMPUTE INITIAL VALUE OF ITERATION INVARIANT COEFF.

COMPUTE AUXILIARY FUNCTIONS $A_m$, $B_m$

$C_m$, $D_m$, $V_m$, $W_m$

END INITIALIZATION

SET ITERATION NUMBER $k$ TO ZERO

SET HARMONIC NUMBER $m$ TO ZERO

$k + 1 \rightarrow k$

$m = 0$?

YES

EXECUTE FORWARD AND BACK SUBSTITUTION TO OBTAIN $A_m(j)$

COMPUTE $F^A_m(j)$

NO

$m + 1 \rightarrow m$

COMPUTE AUXILIARY FUNCTIONS $B_m$, $D_m$, $V_m$

EXECUTE FORWARD & BACK SUBSTITUTION TO OBTAIN $B_{m(j)}^{(n+1, k+1)}$

$m < M$?

YES

FAIL

ITERATION CONVERGENCE TEST

PASS

EXECUTE OUTPUT

LAST STEP?

NO

EVALUATE ITERATION INVARIANT TERMS

$t + \Delta t \rightarrow t$

STOP

YES
CHAPTER V

COMPARISON OF BASIC METHODS

We believe that the finite-difference method used in this investigation to be far superior to either of the two other methods reported in the literature for solving hydrodynamic stability problems with respect to spatially periodic perturbations. In this connection, we believe it worthwhile to describe the two other methods and then compare the computational effort required of all three.

Dowell (1969) proposed to discretize only the lateral space variable, leaving the time variable continuous, by means of a Galerkin process which can be described as follows. The solution is approximated by a linear combination of linearly-independent functions in an appropriate function space. These expansion functions are selected so that each member of the set satisfies the boundary conditions and, furthermore, the functions should be members of a larger class of functions which are complete in the sense that any arbitrary continuous function can be represented by a linear combination of this set in the region in question. Orthogonality is desirable but not required. The unknown expansion coefficients are determined so that the linear combination minimizes some measure of error in satisfying the equations. To be more specific, suppose that a solution of the equation $L[u] = 0$ is desired within domain $D$ where $u = u(x,y,t)$ and $L$ is a differential operator. Let the function $u$ be approximated by a function having the form
(5.1) \[ u = \sum_{j=1}^{N} c_j(t)v_j(x,y), \]

where the sequence of functions \( v_j \) possess the desired properties described above. The Galerkin method requires that the coefficients \( c_j(t) \) be chosen so that the following \( N \) ordinary differential equations be satisfied

(5.2) \[ \int_D L(u)v_j \, dx \, dy = 0, \quad j = 1, 2, \ldots, N \]

Thus, the measure of error to be minimized is the projection of the residual error with respect to satisfaction of the equation onto the subspace spanned by the expansion functions. Now any function orthogonal to all basis functions vanishes; therefore, in the limit as \( N \to \infty \) the function sequence \( u_N(x) \) should converge to a solution.

Dowell applied this procedure to the plane-Poiseuille flow problem by seeking a solution to (3.15) in the form of the expansion

(5.3) \[ \psi(x,y,t) = \sum_{m=1}^{N} \sum_{V=0}^{M} [A_{mV} \cos \nu \alpha x + B_{mV} \sin \nu \alpha x] \psi_m(y) \]

in which the expansion functions were chosen to be

(5.4) \[ \psi_m(y) = \cos(m-1)\pi y - \cos(m+1)\pi y. \quad (The \ walls \ are \ located \ at \ y = 0, 1) \]

The \( \psi_m(y) \) form a complete set but are non-orthogonal and also satisfy the boundary conditions. They are alternately odd and even functions of \( y \).
To apply the basic procedure, substitution of (5.3) and (5.4) into (3.15) and multiplication by \( \cos \alpha \psi_r(y) \) and \( \sin \alpha \psi_r(y) \) followed by integration over the channel-width in the \( y \) - direction and over the fundamental wavelength in the streamwise direction results in the following system of ordinary differential equations for \( A_{mv} \) and \( B_{mv} \).

\[
\sum \frac{Dm_{V}}{m} \frac{dA_{mv}}{dt} = \sum -A_{mv}D_{mV}^{2} - B_{mv}D_{mV}^{3} + \sum \sum \sum \sum \left[ C_{AB}^{vsu} A_{ms} B_{nu} + C_{BA}^{vsu} B_{ms} A_{nu} \right]
\]

(5.5)

\[
\sum \frac{Dm_{V}}{m} \frac{dB_{mv}}{dt} = \sum -B_{mv}D_{mV}^{2} + A_{mv}D_{mV}^{3} + \sum \sum \sum \sum \left[ C_{AB}^{vsu} A_{ms} B_{nu} + C_{BA}^{vsu} B_{ms} B_{nu} \right]
\]

(5.6)

(for \( r = 1, 2, \ldots, N \); \( v = 0, 1, 2, \ldots, M \))

Now consider the work requirement to evaluate the non-linear terms on the right hand sides of equations (5.5) and (5.6), which must be done at least twice per time step in order to integrate the equations forward in time. We take as a convenient and most reasonable basis, the case in which \( M = 3 \) and \( N = 40 \). Typically forty modes were required for convergence of series (5.3) for solution of the linear problem. The dummy variables \( m \) and \( n \) range from 1 to \( N \) while \( u \) and \( s \) range from 0 to \( M \); therefore, there are \( 4(4 \times 40)^3 = 16.4 \times 10^6 \) interaction
coefficients involved. These coefficients are integral functions of the expansion functions and in this case can be evaluated analytically. Storage locations, however, must be provided, which is clearly impossible for even this rather modest number of harmonics; consequently, time consuming tape or disk operations must be resorted to. Evaluation of the four-fold summation requires three operations per coefficient for each function evaluation. Thus, $6(16.4) \times 10^6 \approx 10^8$ operations are required each time step. An additional thirty percent or more operations would be required to evaluate the single summation and to solve the resulting linear system in order to obtain the quantities $\frac{dA_{mn}}{dt}$ and $\frac{dB_{mn}}{dt}$. A work load of this amount would tax even the most powerful hardware available today. One may as well solve the Navier-Stokes equations directly in a two-dimensional region. It is easy to see then why Dowell used only the mean flow and fundamental fluctuation for the single non-linear problem reported and even then only sixteen modes were carried, too few for series convergence. For this small problem there are about 10,800 interaction coefficients requiring some 65,000 operations per step for evaluation of the non-linear terms. This would be about as large a problem as could be handled without being forced into auxiliary storage.

Pekeris and Shkoller (1969) studied periodic disturbances of finite-amplitude by expanding the stream function into the series

$$
\psi = \psi_0 + \sum_{n=-\infty}^{\infty} f_n(y,t) e^{-i\omega nx}
$$

where

$$
f_{-n}(y,t) = \bar{f}_n(y,t)
$$

The bar denoting complex conjugate.
Substitution of (5.7) into the Navier-Stokes equation leads to perturbation equations which must be solved to obtain the components \( f_n(y,t) \). The expansion (5.7) is truncated at \( n = 3 \). The quantity \( \hat{f}_0(y,t) \) is represented by the series

\begin{equation}
\hat{f}_0(y,t) = \sum_{\sigma=1}^{S} A_{\sigma}(t) \cos [(\sigma - 1/2)\pi y].
\end{equation}

The other functions are expanded in terms of the even eigenfunctions \( \phi^\nu_n(y) \) of the Orr-Sommerfeld equation

\begin{equation}
\phi^\nu_n - 2\alpha^2 n^2 \phi^\nu_n + \alpha^4 n^4 \phi^\nu_n + 1i\alpha R[(1 - y^2 - c^\nu_n)(\phi^\nu_n - \alpha^2 n^2 \phi^\nu_n)
+ 2\phi^\nu_n] = 0 \quad \text{for } \nu = 1, 2, \ldots
\end{equation}

which satisfy the boundary conditions

\begin{equation}
\phi_n(\pm 1) = \phi_n'(\pm 1) = 0.
\end{equation}

The system adjoint to (5.8) is

\begin{equation}
\phi^\nu_n - 2\alpha^2 n^2 \phi^\nu_n + \alpha^4 n^4 \phi^\nu_n + 1i\alpha R[(1 - y^2 - c^\nu_n)(\phi^\nu_n - \alpha^2 n^2 \phi^\nu_n)
- 4y \phi^\nu_n] = 0.
\end{equation}

The adjoint eigenfunctions \( \phi^\nu_n \) and the eigenfunctions \( \phi^\nu_n \) share the same eigenvalues \( c^\nu_n \) and meet the orthogonality condition...
(5.12) \[ f_1 (\phi_n^\nu - \alpha_n^2 \phi_n^\nu) \phi_n^\nu = \delta_{\nu \sigma} \cdot \]

The \( \phi_n^\nu \) are normalized such that \( \phi_n^\nu(y = 0) = 1 \).

The three fluctuating harmonics are the expanded as

(5.13) \[ f_1(y,t) = \sum_{\nu=1}^{K} B_{\nu}(t) \phi_1^\nu(y) \]

(5.14) \[ f_2(y,t) = \sum_{\nu=1}^{K} D_{\nu}(t) \phi_2^\nu(y) \]

(5.15) \[ f_3(y,t) = \sum_{\nu=1}^{K} E_{\nu}(t) \phi_3^\nu(y). \]

Substitution of (5.8) into the perturbation equation for \( \hat{r}_0 \) and equating coefficients; and substitution of (5.13), (5.14), and (5.15) into the appropriate perturbations making use of the orthogonality condition (5.10) leads to the following system of ordinary differential equations for the unknown quantities \( A_0(t), B_\nu(t), D_\nu(t), \) and \( E_\nu(t) \).

(5.16) \[ \frac{dA_0}{dt} = -\frac{\pi^2(\sigma - 1/2)^2}{R} A_0 + 4\alpha Q \sum_{\nu=1}^{K} \sum_{\tau=1}^{K} [ \bar{E}_{\nu \tau} \Gamma_{\nu \tau}^{\nu \tau} + 2 \bar{E}_\nu \Gamma_{\nu \nu}^{\nu \nu} + 3 \bar{E}_\nu \Gamma_{\nu \nu}^{\nu \nu} ] \]
\[
\begin{align*}
\frac{dB^\nu}{dt} &= 1a \sum_{\sigma=1}^{S} A_{\sigma} B_{\sigma} Q^{\nu_{\sigma}} + 2a \sum_{\tau=1}^{K} A_{\tau} B_{\tau} Q^{\nu_{\tau}} + 2a \sum_{l=1}^{K} \sum_{j=1}^{K} E^l D_j V^{\nu_{l/j}} \\
&+ 2a \sum_{l=1}^{K} \sum_{j=1}^{K} \bar{E}^l D_j S^{\nu_{l/j}} \\
\frac{dD^\nu}{dt} &= 2a \sum_{\sigma=1}^{S} A_{\sigma} D_{\sigma} Q^{\nu_{\sigma}} + 4a \sum_{\tau=1}^{K} A_{\tau} E_{\tau} Q^{\nu_{\tau}} + 2a \sum_{l=1}^{K} \sum_{j=1}^{K} E^l D_j V^{\nu_{l/j}} \\
&+ 2a \sum_{l=1}^{K} \sum_{j=1}^{K} \bar{E}^l E_j U^{\nu_{l/j}} \\
\frac{dE^\nu}{dt} &= 3a \sum_{\sigma=1}^{S} A_{\sigma} E_{\sigma} Q^{\nu_{\sigma}} + 6a \sum_{\tau=1}^{K} A_{\tau} E_{\tau} Q^{\nu_{\tau}} + 2a \sum_{l=1}^{K} \sum_{j=1}^{K} E^l D_j T^{\nu_{l/j}} \\
\nu &= 1, 2, \ldots, K.
\end{align*}
\]

Where
\[
\begin{align*}
\Gamma^{\nu_{\sigma}}_{\sigma n} &= \int_0^1 \cos[(\sigma - 1/2)\pi y] \phi^\nu \phi_n dy \\
V^{\nu_{l/j}} &= \int_0^1 \phi_1 [\phi_1 \phi_2^j - \phi_1 \phi_2^j + 2\phi_1 \phi_2^j - 2\phi_2 \phi_2^j] dy \\
W^{\nu_{l/j}} &= \int_0^1 \phi_2 [\phi_1 \phi_2^j - \phi_1 \phi_2^j] dy \\
S^{\nu_{l/j}} &= \int_0^1 \phi_3 [2\phi_2 \phi_3^j + 3\phi_2 \phi_3^j - 2\phi_2 \phi_3^j] dy
\end{align*}
\]
\[
T_{\nu \lambda j} = \int_0^1 \tilde{\phi}_3^\nu \left[ 2\tilde{\gamma}_1^i \phi_2^j + \gamma_1^i \phi_3^j - \phi_1^i \gamma_2^j + 2\phi_1^i \gamma_3^j \right] \, dy
\]

\[
u_{\nu \lambda j} = \int_0^1 \tilde{\phi}_2^\nu \left[ 3\tilde{\phi}_3^i \gamma_3^j - \gamma_1^i \phi_3^j - 3\gamma_1^i \phi_3^j + \phi_1^i \gamma_3^j \right] \, dy
\]

\[
\gamma_n^\nu = \tilde{\phi}_n^\nu - \alpha^2 n^2 \phi_n^\nu
\]

\[
Q_{\nu \tau \phi} = \int_0^1 \tilde{\phi}_n^\nu \cos \left[ (\sigma - 1/2)\pi y \right] \left[ \gamma_n^\tau + \pi^2 (\sigma - 1/2)^2 \phi_n^\tau \right] \, dy
\]

The truncation limits adopted were \( \kappa = 21 \) and \( s = 60 \).

We have written out the Pekeris and Shkollr problem formulation in full in order to illustrate once more the almost staggering quantity of work involved in order to achieve a solution. First of all one is faced with the problem of computing accurately the first twenty-one eigenfunctions from equation (5.9) and analogously the first twenty-one adjoint eigenfunctions from equation (5.11) for \( n = 1, 2, \) and 3. This is in itself a task of major proportions which must be repeated for each geometrical situation. It is an especially difficult problem for flow in cylindrical geometry. Next, all of the interaction coefficients, complex integral functions of the eigenfunctions and adjoint eigenfunctions, must be obtained. This would constitute a second almost overwhelming computational task since over 200,000 complex coefficients are involved. Although far fewer coefficients are needed as compared to Dowell, use of some type of auxiliary storage is indicated. Finally, the 123 simultaneous ordinary differential
equations must be integrated forward in time. Again counting the
number of operations needed to evaluate the right hand sides of
equations (5.16) - (5.19), we find that in excess of $1.34 \times 10^6$ complex
operations are required each time step. Since two complex multiplica-
tions are carried out for each addition, this figure corresponds to
approximately $6.25 \times 10^6$ simple operations. While this number is not
too unreasonable, our method is significantly better with respect to
work load.

The finite difference method of solution proposed in this
thesis requires far less computational effort and far less storage
than the two methods just described. One reason is that no expansions
are made in the lateral spatial dimension resulting in excessively large
numbers of interactions.

For three harmonics, the non-linear terms for the finite-
difference formulation can be written as

\[
\begin{align*}
F_A^0 &= Q_{11} + Q_{22} + Q_{33} \\
F_A^1 &= 2[A_oD_1 - C_oB_1] + [Q_{21} + Q_{12}] + [Q_{23} + Q_{32}] \\
F_A^2 &= 2[C_oA_1 - A_oC_1] + [S_{21} - S_{12}] + [S_{32} - S_{23}] \\
F_A^3 &= 4[A_oD_2 - C_oB_2] + [Q_{31} + Q_{13}] + R_{11} \\
F_B^2 &= 4[C_oA_2 - A_oC_2] + [S_{31} - S_{13}] + T_{11} \\
F_A^3 &= 6[A_oD_3 - C_oB_3] + [R_{21} + R_{12}] \\
F_B^3 &= 6[C_oA_3 - A_oC_3] + [T_{21} - T_{12}]
\end{align*}
\]
It is a simple matter then to estimate the number of operations required of our process. Each $Q_{ij} + Q_{ji}$, $S_{ij} - S_{ji}$, $R_{ij} - R_{ji}$, or $T_{ij} - T_{ji}$ requires $18N$ operations; each $Q_{jj}$ requires $8N$ operations; and each $R_{jj}$ or $T_{jj}$ requires $10N$ operations. Thus, the non-linear terms require $18^4$ operations per grid point per iteration per time step. Calculation of $A'$, $B'$, $C$, $D$, $C'$, and $D'$ requires $58NM$ operations; and evaluation of the right hand side and execution of the forward and back substitution require $11NM$ operations. Since seven equations are involved, the subtotal becomes 657 operations per point per iteration per step. Using 51 grid points and 5 iterations per step, we estimate a grand total of about $16.4 \times 10^4$ operations per time step. This estimate does not include computation of subscript variables, time necessary for fetch and storage of elements of arrays in storage, and various other miscellaneous bookkeeping functions; however, the estimates made for the other methods do not include these times either, so that the basis of comparison is fair. The figures presented are for plane-Poiseuille flow. Poiseuille flow requires some fifty percent more computer time and twice as much storage since most coefficients are functions of position, whereas they are constants in the parallel plate problem. An overall comparison is given in the following table for a problem involving a mean flow and three harmonics.
<table>
<thead>
<tr>
<th>Method</th>
<th>Operations/Step x $10^{-6}$</th>
<th>Memory Requirement in words x $10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dowell</td>
<td>100.0</td>
<td>16.4</td>
</tr>
<tr>
<td>Pekeris &amp; Shkoller</td>
<td>6.23</td>
<td>0.4</td>
</tr>
<tr>
<td>George</td>
<td>0.164</td>
<td>0.005</td>
</tr>
</tbody>
</table>

In summary, the finite-difference method can be used quite efficiently on a computer of medium size and speed, and can utilize additional harmonics with correspondingly less penalty than the other methods. Also, it is more readily adaptable to geometrical configurations other than flow between parallel plates.
CHAPTER VI

DISCUSSION OF RESULTS

The numerical finite-difference techniques described earlier in this thesis have been applied extensively to the classical problems of hydrodynamic stability in plane-Poiseuille flow and Poiseuille flow. The results are presented in this chapter. First we will demonstrate the validity of the numerical methods employed in the overall solution scheme by showing that they are capable of accurate reproduction of arbitrarily selected results from linear theory. In this section account will be given of the effect of certain program parameters on stability and accuracy. This is to be followed by detailed discussions concerning the significant effect of non-linearity relative to the behavior of the two basic flows under investigation.

Linear Plane-Poiseuille Flow

First consideration due a new and unproven computational procedure is that relating to accuracy and numerical stability. We have attempted to achieve this goal by solving a number of problems involving small disturbances using a variety of combinations of Reynolds number, wave number pairs and comparing the computed results with analogous results derived from the linear theory of hydrodynamic stability.

Initial conditions in all cases (linear and non-linear) consisted of a fundamental perturbation having the shape of the first
eigenfunction of a closely related mathematical problem. We chose as a basis the family of symmetric eigenfunctions arising from the system

\[
\frac{d^4 \phi}{dy^4} = \alpha^4 \phi \quad \text{(Chandrasekhar - Appendix V)}
\]

\[\phi = \phi' = 0 \text{ at } y = \pm 1\]

namely,

\[
\phi_m(y) = \frac{\text{Cosh } \alpha_m y}{\text{Cosh } \alpha_m} - \frac{\text{Cos } \alpha_m y}{\text{Cos } \alpha_m}
\]

Where \( \alpha_m \) is the \( m^{th} \) root of the transcendental equation

\[
\text{Tanh } \alpha_m + \text{Tan } \alpha_m = 0.
\]

The eigenfunctions \( \phi_m(y) \) have been normalized such that

\[
\phi_m(0) = 1.
\]

The usual initial conditions used, then, were

\[
A_1(y,0) = k_A \phi_1(y).
\]

\[
A_m(y,0) \equiv 0 \text{ for } m \neq 1
\]

\[
B_m(y,0) \equiv 0 \text{ for } 0 \leq m \leq M.
\]

where \( k_A \) is the specified initial amplitude, a constant.

For problems that are solely linear, only \( A_1(y,t) \) and \( B_1(y,t) \) are involved, there being no interaction between the fundamental fluctuation and either the mean flow or other harmonics. The stream function
and velocity fluctuations corresponding to \( \phi_1(y) \) are shown in Figure 6.1.

As stated in Chapter II, solutions to the Orr-Sommerfeld equation are expressible in the complex exponential form

\[
(6.6) \quad \psi(x,y,t) = \phi(y)e^{i\alpha(x-ct)}
\]

where only the real part has physical significance.

We have written solutions having form

\[
(6.7) \quad \psi(x,y,t) = A(y,t) \cos \alpha x + B(y,t) \sin \alpha x.
\]

It is shown in Appendix VI that the correspondence between the two forms is such that

\[
(6.8) \quad A(y,t) = (\phi_r \cos \alpha c_r t - \phi_1 \sin \alpha c_r t)e^{\alpha c_1 t}
\]
\[
(6.9) \quad B(y,t) = - (\phi_1 \cos \alpha c_1 t + \phi_r \sin \alpha c_1 t)e^{\alpha c_1 t}
\]

Thus, computed solutions should have the form of a sine wave with period \( 2\pi/\alpha c_1 \) and growth or decay rate \( e^{\alpha c_1 t} \). The complex eigenvalue \( c \) has been obtained by various workers (Chapter II) as a function of \( \alpha \) and \( R \); therefore, we have a means of comparing computed solutions against a known standard. The eigenfunction \( \phi(y) \) can also be extracted from knowledge of \( A(y,t) \) and \( B(y,t) \) and the fact that \( \phi_r(0) = 1 \) and \( \phi_1(0) = 0 \).

Since linear solutions have the same form irrespective of lateral position in the channel, we will for convenience sake look at
solutions only at the channel centerline. In Figure 6.2 we show the oscillating solution computed for \( \alpha = 1, R = 10,000 \), which we designate as the model or base problem, using fifty spatial intervals and a time step of 0.30. Following the initial transient, a pure sine wave is realized which is characterized by period and growth rate parameters that are in exact agreement with linear theory. This is not always the case, however, as can be observed in Figure 6.3 where the effect of varying the number of grid points is shown. Clearly, the quality of the results deteriorates as the number of intervals decreases until finally the method completely falls apart. There appears to be a minimum resolution such that approximation of the high derivatives cannot be achieved in any sense, about fifteen intervals for fourth-order approximation. The enhancement of the quality of the computed linear solutions is dramatic when the technique discussed in Chapter IV is used to increase the order of overall spatial truncation error from second order to fourth order. It was this relatively simple device that provided the key which made the numerical method truly feasible from a computing time point of view. As can be seen from Figure 6.4, more than 200 intervals would be required to achieve the same accuracy using second-order methods as could be achieved with only 50 intervals using fourth-order methods. Most important is the fact that the more accurate solutions can be obtained with little or no penalty in computing time; that is, an \( N \)-interval problem would require essentially the same amount of time in either case.
The variation of period with number of intervals is shown in Figure 6.5 for the model case. We find here that the period is rather insensitive to changes in the space increment $\Delta y$ until such a large value is used that the solutions tend to degenerate. More striking is the effect of the error term on solution quality.

Graphic plotting of the complete oscillating solutions is a tiresome and time-consuming process. In order to gain some relief, we will often resort to plotting only the amplitude of the crests and troughs of the fluctuating wave against time on semi-log paper. Since linear solutions are exponential in character, we expect straight lines whose slopes are indicative of growth rate. Figure 6.6 illustrates several such solutions. We notice again how use of too few intervals can result in rapid decay in cases where moderate growth should occur. We see also how use of even fewer intervals causes the solution to degenerate to the point where a rather erratic wave form results, there existing periods of decay followed by abrupt amplification. The 50 interval solution is sufficiently accurate to be regarded as exact.

Dixon (1966) attempted to study the stability of Poiseuille and plane-Poiseuille flow by integrating the Navier-Stokes equation in a two-dimensional region. In his work Dixon used only 10 intervals in the lateral direction and also used second-order correct approximation formulae. Our findings strongly indicate that his results were more characteristic of the type of erratic behavior shown in Figure 6.6.
than of hydrodynamic solutions. This conjecture is supported by the fact that in his linear computations the solutions were not purely sinusoidal and were strongly dependent on position. In addition, he used a centerline boundary condition corresponding to anti-symmetric solutions in the stream function. As previously stated, anti-symmetric solutions are all extremely stable, which is not true of his results. Finally, we have found a case in which one of Dixon's low-amplitude solutions is almost identical to a corresponding high-amplitude solution. This fact suggests that the solution is being dominated by a type of numerical instability rather than true hydrodynamic solution behavior. The whole argument here is to emphasize the absolute necessity of fine resolution in order to achieve quantitative accuracy.

The complex eigenfunction \( \phi(y) \) generated by the model case is shown in Figure 6.7. We find that our computed function agrees well with the results of Thomas (1953) who tabulated \( \phi(y) \) at intervals of 0.01. The maximum deviation between our results and those of Thomas is about 2% and this occurs very near the wall where the eigenfunction itself is extremely small. At the centerline the error is something like 0.004%. The real part \( \phi_r \), more or less controls the period, whereas the imaginary part \( \phi_i \) controls the growth rate. Thus, it is easy to imagine how crude resolution and correspondingly crude approximation of this rapidly varying function could result in poor values for growth rate.
A number of linear model problems were solved using 50 intervals, varying the time step size. It was determined that period and growth rate were not affected by increasing the time step. The number of iterations required for convergence did, however, increase significantly, doubling and even trebling until finally no solution at all could be obtained. We found that a time step corresponding to about 20 intervals for each cycle in time of the solution was optimum. For non-linear problems this criterion was applied to the highest (most rapidly cycling) harmonic in choosing an appropriate time step.

Orr-Sommerfeld eigenvalues were computed for a variety of problems other than the model problem. We always obtained values that were within the accuracy to which we could determine known linear results from graphs found in the literature. Combinations of $\alpha$ and $R$ that were used are indicated in Figure 6.8. Fifty spatial intervals were found to be adequate even for Reynolds numbers up to 100,000. The large success that we had in verifying linear theory for a wide range of $\alpha - R$ combinations gave us great confidence in the numerical procedure.

**Linear Poiseuille Flow**

For problems in Poiseuille flow, the initial fluctuation was chosen to be the first eigenfunction of the system.
\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right] \phi = \alpha^2 \phi
\]

\[
\phi(1) = \phi'(1) = 0 \quad \text{(Chandrasekhar - Appendix V)}
\]

\[
\phi(0) = \phi'(0) = 0
\]

solutions of which are

\[
\phi_m(r) = r \left[ \frac{J_1(\alpha_m r)}{J_0(\alpha_m)} - \frac{I_1(\alpha_m r)}{I_0(\alpha_m)} \right] \quad 0 \leq r \leq 1
\]

where \( \alpha_m \) is the \( m \)th root of the equation

\[
J_1(\alpha_m) I_0(\alpha_m) = J_0(\alpha_m) I_1(\alpha_m).
\]

The \( \phi_m \) are normalized such that

\[
\max_{0 < r < 1} |\phi_m(r)| = 1
\]

As in plane-Poiseuille problems we take

\[
A_1(r,0) = k_A \phi_1(r), \quad \text{where } k_A = \text{initial amplitude}
\]

All other components are allowed to vanish initially. \( A_1(r,0) \) and the corresponding velocity fluctuation \( u_1 = \frac{1}{r} \frac{\partial A_1}{\partial r} \) are shown in Figures 6.9 and 6.10, respectively.

In plane-Poiseuille flow, there exist two families of eigenfunctions – those that are (1) symmetric and (2) anti-symmetric about the centerline. A similar situation obtains for Poiseuille flow, but
here the two families are represented by disturbances that are confined (1) to a small region near the tube centerline and (2) to a small region near the wall. These functions have been named center modes and wall modes, respectively, re Gill (1965A). The symmetric and anti-symmetric modes in plane-Poiseuille flow can be isolated merely by suitable choice of boundary conditions at the centerline. In contrast, there is apparently no convenient way to initially filter out either the center or wall modes of Poiseuille and indeed it isn't clear that one should even want to. The point which we are leading up to is that the initial fluctuation contains both types of modes and that these modes decay independently at different rates. Under non-linear flow conditions the modes, of course, interact with each other. Davey and Drazin (1969) have reported complete information regarding the eigenvalues corresponding to these modes. Data for the least stable modes are given in Figures 6.11 and 6.12. For \( \alpha = 1, R = 10,000 \) we have approximately

\[
\begin{array}{ccc}
\text{Centerline mode} & c_1 & 2\pi/\alpha c_1 \\
0.971 & 0.0285 & 6.47 \\
\text{Wall mode} & 0.287 & 0.076 & 21.9 \\
\end{array}
\]

We observe that the center mode is the least stable of the two by a rather large margin and that the wall mode has the longer period by a factor of 3.38. To check the results of our method against linear theory, we must compute a linear problem for a time sufficiently long
for the wall mode to decay to an insignificant level in comparison to the center mode. Some results for the linear model problem are shown in the accompanying figures. In Figure 6.13 we see the solution plotted at \( r = 0.5 \). For these early times, the two superimposed modes lead to a rather erratic wave form due to the difference in periods; however, the general trend is consistent quantitatively with center mode behavior. The shape of \( A(r,t) \) at several successive points in time for which the centerline peak reaches a maximum is shown in Figure 6.14 and illustrates the presence of both modes and the effect of the difference in period. Behavior at somewhat later times is given in Figure 6.15. Here no wall mode at all is evident. The error in decay rate is within the accuracy to which we can read Figure 6.11. The eigenfunction is difficult to extract exactly, but the shape to as near as we can determine is shown in Figure 6.16. We know that when \( A(r,t) \) reaches a maximum it becomes identical with either \( \phi_r \) or \( \phi_1 \) and that at the same instant, \( B(r,t) \) is \( \phi_1 \) or \( \phi_r \). If we do not happen to catch the results precisely at a maximum then there will be "some part" of each mode present in both \( A(r,t) \) and \( B(r,t) \). We recall that

\[
A(r,t) = (\phi_r \cos \alpha_r t - \phi_1 \sin \alpha_r t)e^{i\alpha c t}
\]

\[
B(r,t) = (\phi_1 \cos \alpha_r t + \phi_r \sin \alpha_r t)e^{i\alpha c t}
\]

If our purpose were only to determine \( \phi_r \) and \( \phi_1 \), we would use some less exotic method.
Computational Aspects

All calculations were carried out on a Burroughs B-5500 computer which uses a 48 bit word and has a 4 \( \mu \)s memory access time and 2 \( \mu \)s fixed point add time. Computer programming was carried out in Burroughs Extended Algol, an extremely flexible language whose compiler is tailored to the special characteristics peculiar to B-5500 hardware.

Solutions to the linear parallel plate problems discussed above were obtained at a rate of 0.7 seconds per time step or equivalently 85.7 steps per minute. Accordingly, a linear solution could be completed in less than five minutes at a cost of slightly over eight dollars. Linear problems in Poiseuille flow entail more computation requiring 0.9 sec/step.

Non-linear problems in plane-Poiseuille flow for which two harmonics are retained required 4 to 4 1/2 seconds per time step for solution depending on the number of iterations needed for convergence. Usually four or five iterations were sufficient but more were taken in a few situations in which the solution was rapidly changing relative to the size of the time increment. A dimensionless time increment of 0.3 was used for all parallel-plate problems except for very high amplitude cases in which the larger size of the second harmonic made it necessary to reduce the step size to a lower level. When the large time step could be used, integration was carried out to time
200 or thereabouts otherwise, the upper limit was nearer to time 100. Most non-linear runs required from forty-five minutes to one hour to complete. The use of more harmonics increases the computer time since each additional harmonic involves two additional partial differential equations and additional non-linear terms in all of the equations. Also the higher harmonics possess shorter periods necessitating use of smaller time increments. Use of four harmonics, for example, would require about five times as much computer time as a two harmonic problem. Here nine equations would be involved as opposed to five. Also, the time step must be halved since the period of the fourth harmonic is half that of the second harmonic. Recall that at least twenty steps per cycle are needed for numerical stability.

Poiseuille flow problems were executed at a computational rate of about 6.0 seconds per step. A time increment of 0.10 was used in all cases. The propagational velocity of a disturbance in circular geometry is somewhat faster than in parallel-plate geometry. Consequently, the same number of cycles of a disturbance can be computed in the same computer time for both geometries even though the time increments differ by quite a bit.

Rogers and Beard (1969) used a truncated Fourier expansion technique similar to ours for numerical approximation of wide gap Taylor vortices which appear as a secondary flow in viscous, incompressible fluid between rotating concentric circular cylinders. They
computed a succession of steady state flow patterns corresponding to variation of appropriate parameters, the integration not being carried out very far in time. In Rogers and Beard's method the non-linear terms were allowed to lag one time step; that is, the non-linear terms were evaluated entirely at time level nΔt. In accordance, a Taylor number dependent upper bound on Δt/(Δy)^2 was found to be required in order to maintain numerical stability. Necessary bounds ranged from 2 to 5. For Δy = 0.02 and an upper bound of 5 would leave Δt<0.002. Compare this bound with the two orders of magnitude larger values that we were able to use with ease. We carried out some experimentation to explore the possibility of letting the non-linear terms lag or even updating them only on the first few iterations but experienced inferior performance with respect to stability and accuracy. The motivation of course was to reduce the computational requirements since a large portion of time is spent evaluating the non-linear terms, particularly when a large number of harmonics are being retained.

Non-Linear Plane-Poiseuille Flow

Many solutions have been obtained for finite-amplitude disturbances in plane-Poiseuille flow. Except for a few special cases, computations were carried out using the mean flow and two fluctuating harmonics. As stated in equation (6.5), only the first harmonic is initialized to a non-vanishing function. This harmonic always interacts with itself to create an associated Reynolds stress which, if sufficiently large, excites both the zeroth and second
harmonics thus distorting the basic flow. Subsequently all harmonics interact with themselves and each other to induce further distortion. This distortion can strengthen the conversion of energy from the mean flow to the disturbance and consequently cause instability to occur at Reynolds numbers lower than the critical value given by linear theory. Our calculations have behaved in accordance with this mechanism.

The first non-linear problem solved corresponds to a point in the $\alpha - R$ plane near the apex of the neutral curve but just inside the stable or subcritical region. We chose to use $\alpha = 1$ and $R = 5,000$ in conjunction with a stream function disturbance amplitude of 0.10. The solution shown in Figure 6.17 represents the oscillating disturbance at a point on the channel centerline at one end of the region, i.e., at $y = 0$ and either $x = 0$ or $x = 2\pi/a$. The period and growth rate taken with respect to time remain invariant with position along the channel width, and the only change with position in the direction of flow is a shift in phase along the time axis. Also shown is the bounding envelope of maxima and minima of the oscillating disturbance function for the equivalent linear problem and a case corresponding to a linearly unstable problem. We observe that finite-amplitude instability does indeed exist for the situation where flow is stable according to linear theory. The case which is linearly unstable ($\alpha = 1$, $R = 7,000$) is made even more unstable with the addition of non-linearity. The growth rate exponent $\alpha_{1} = 7.93 \times 10^{-3}$
is almost quadrupled as compared to the linear value (2.0 x 10^{-3})
when the initial amplitude is 0.10 and is in fact larger than the
maximum possible linear value of 7.65 x 10^{-3} (see Figure 2.1).

In Figure 6.18 we show curves representing solutions for a
whole sequence of Reynolds numbers ranging through the neutral curve,
all obtained using unity wave number and 0.10 initial amplitude. Thus
for disturbances having the shape shown in Figure 6.1 with an amplitude
of 10% of the undisturbed centerline laminar velocity, the critical
Reynolds number is reduced from 5815 to a value near 4125. A critical
Reynolds number reduction will be associated with each wave number and
amplitude for each disturbance shape. Another interpretation would be
to say that at the point (1,4125) of the $\alpha - R$ plane, the subcritical
equilibrium amplitude is 0.10; that is, initial disturbances slightly
larger than 0.10 will grow whereas smaller disturbances will decay to
zero. A critical or equilibrium amplitude is associated with each
point in the subcritical portion of the $\alpha - R$ plane.

Other solution curves are given in Figure 6.19 and 6.20 for
initial amplitudes varying from 2% to 30%. Each case represents some-
thing on the order of one hour of Burroughs B-5500 computer time. The
20% initial amplitude run with $R = 3500$ was carried out a second time
using four harmonics instead of only two. An interesting result was
that the two additional harmonics provided a slightly stabilizing
effect. The four harmonic centerline solution was almost identical to
the $R = 3000$ curve for the same initial condition. We suspected in
advance that this might be the case since Reynolds and Potter (1967)
observed similar stabilization when they studied the effect of neg-
lecting the second harmonic with respect to their work (Chapter II).
Seemingly more important than generation of higher harmonics are the
effects of modification of the mean flow by the Reynolds stress of
the oscillation coupled with a consequent modification of the funda-
mental shear wave oscillation itself.

At this point we will give an indication of the relative
sizes of the higher harmonics for several of the runs. The magnitude
of the second harmonic always increases from zero to an early maximum
and then decreases. The function may ultimately increase in the case
of unstable conditions but the decrease is continuous for stable
situations. Some indication of the behavior is given in Figure 6.21.
The absolute maximum spoken of is a fairly well defined function of
amplitude and does not seem to depend significantly on Reynolds number,
at least in the neighborhood of the neutral curve. At 2% initial
amplitude the maximum magnitude of the second harmonic is always two
orders of magnitude less than the first harmonic. The trend for larger
amplitudes is shown in Figure 6.22. Even for the largest amplitude
(0.3), the second harmonic is bounded by 0.029 or less than 10% of the
initial amplitude. Referring again to the four harmonic case, the
initial amplitude was 0.20 while in comparison the second, third, and
fourth harmonics were always less than 0.0172, 0.003, and 0.0015
respectively. It is our considered opinion that meaningful results can be obtained in most situations using only two harmonics.

Careful interpretation of the solution curves in Figures 6.18, 6.19, and 6.20 allowed construction of a curve relating Reynolds number to critical amplitude, but valid only for the single wave number used, i.e., for disturbances having a fundamental wave length of $2\pi$. Non-linear neutral Reynolds numbers were found by obtaining some measure of late-time slope for each curve as a function of $R$ and extrapolating to zero growth rate. Repetition of the process for a sequence of initial amplitudes led to Figure 6.23 which, due to the apparent stabilizing effect of higher harmonics, represents sufficient conditions for stability. Though not exact, we feel that the quality of the relation shown is quite good, especially for amplitudes less than 0.10 where critical amplitude increases with decreasing Reynolds number. The single most important characteristic of the curve is the existence of an absolute minimum Reynolds number below which the flow cannot be made unstable, no matter how large the initial amplitude. Also notable is the fact that non-linear growth is limited to amplitudes slightly in excess of 20%.

The experimental work of Whan and Rothfus (1959) and Sherlin (1960) dealt with transition in parallel plate flow. Whan and Rothfus made measurements of pressure drop and local fluid velocities in a smooth rectangular pipe having an aspect ratio of 20:1, while Rothfus' experiments involved observation of disturbances in colored dye injected
into flow in a rectangular pipe having 4:1 aspect ratio. Both investigations resulted in a predicted transition Reynolds number of 2700 based on average velocity and hydraulic radius or 1012 based on maximum velocity and half-channel width. Much later an experimental investigation of reverse transition from turbulent to laminar flow in a two-dimensional channel was carried out by Badrinarayanan (1968). Reverse transition occurred when Reynolds number of an initially turbulent flow was reduced below a certain value by widening the flow duct in the lateral direction. The critical Reynolds number for reverse transition was estimated on the assumption that the decay rates for all fluctuating quantities should tend to zero as this Reynolds number is approached. The value so determined was 2100±75 based on half the channel width and the maximum value of the mean velocity. Patel and Head (1968) analyzed previously obtained data and agreed with Badrinarayanan's figure of almost 2200 as an acceptable value for critical Reynolds number for reversion that is, the lowest Reynolds number for which fully developed turbulent flow can be maintained in a two-dimensional channel. They also state that the Reynolds number of the final laminar flow is unlikely to exceed that for which the laminar flow is formally stable. These experimental results, shown on Figure 6.23, are consistent with our computations. The ultimate computational achievement would be to construct a curve of critical Reynolds number vs. amplitude applicable to all possible disturbances. To accomplish this task would clearly be impossible due to the large number of parameters involved. A whole sequence of curves similar to
Figure 6.23 would have to be prepared each requiring large quantities of computer time. The lower locus of all point on all such curves would yield a critical Reynolds number – amplitude relationship for the particular disturbance used. Researchers seem to always study non-linear behavior with respect to the least stable eigenfunction of linear theory but there is no guarantee that this leads to absolute critical conditions because of inherent unpredictability of non-linear processes. A study of all eigenfunctions individually would also be insufficient since superposition principles are inapplicable. Presumably if one could generate an all inclusive critical curve, it would dip to an absolute minimum critical Reynolds number coincident with or slightly larger than the experimental value obtained by Badrinarayanan for reversion to turbulence.

Finally, Figure 6.24 shows a comparison of our critical curve with those obtained by earlier workers. All of the relations except ours and Pekeris (1969A) were obtained by methods that are valid only for small amplitudes. Pekeris' two relations were presented in companion papers and are inexplicably at variance with one another. The more drastically behaving curve (1969A), obtained by the method described in Chapter V, required many extremely complex calculations and as a result could possibly be in error.

Non-Linear Poiseuille Flow

The stability characteristics of Poiseuille flow with respect to non-linear disturbances is of great fundamental and practical
significance since it represents one of the oldest yet least well understood problems in fluid mechanics. In this connection, we have carried out numerous calculations for flow in this regime retaining always two fluctuating spatial harmonics. The initial conditions used were identical to those employed for linear studies (Figures 6.9 and 6.10). The wave number and Reynolds number parameters were chosen on the basis of the stability experiments of Fox et al, (1968) described in Chapter II. We selected pairs of values located well interior to the portion of the $\alpha - R$ plane shown to be unstable according to their work. For most of the runs an initial amplitude of 0.10 in the stream function was used. This 10% amplitude corresponds to a rather large disturbance in the velocity, particularly near the centerline, due to the inverse radius relation between stream function and velocity. Recall that $u = \psi_r/r$. In spite of the fact that large disturbances were involved, we were never able to detect any sign of absolute instability. The most extreme case computed is given in Figure 6.25. The response shown is the total stream function disturbance at a point midway between the tube centerline and wall and at one end of the spatially repetitious region, i.e., $\hat{\psi}(0, 0.5, t) = \sum_{j=0}^{2} A_j (0.5,t)$. We note that the function is rather distorted but that a definite trend in decay is present from the very start, although somewhat slowed in comparison to linear behavior. Some intermediate results were lost due to a hardware failure but were not recomputed as they did not appear to contain information of any great importance. The response does not oscillate about zero-mean
because of continuous monotonic growth of the zero wave number harmonic. Note the extremely rapid decay of the linear oscillation for this choice of parameters.

Solutions corresponding to other $\alpha - R$ combinations are shown in Figures 6.26, 6.27, and 6.28. In each case stability is clearly indicated. The disturbance represented in Figure 6.28 is much smaller in amplitude than the others but still is the same order of magnitude as those created and made unstable in the experimental work of Fox et al, and Leite. Rather striking is the rapid decay indicated for Reynolds number of 100,000.

The second harmonic becomes larger for circular geometry than for the parallel-plate situation, in some cases growing to the same order of magnitude as the fundamental. We did however recompute a portion of the solution shown in Figure 6.25 but using four harmonics instead of two. Although the response deviated a little from the two harmonic case, it was difficult to reach any definite conclusion concerning the overall effect of inclusion of the additional terms. Some peaks were slightly higher but correspondingly, troughs were also higher and the resultant wave height was about the same. For this run the second, third, and fourth harmonics were at most 56%, 12%, and 4% of the fundamental.

Seeking some legitimate way of creating an unstable Poiseuille flow situation, we began to consider an idea of Gill (1965B) who proposed a mechanism for instability. The motivation for Gill's work
was a desire to understand the experimental results of Leite reported by Kuethe (1956) which dealt with disturbances created by oscillation of a circular airfoil in an airstream flowing through a circular pipe. Gill's analysis of the mean velocity and velocity fluctuation profiles measured at various points downstream, shown in Figures 6.29 and 6.30, led him to the conclusion that slightly downstream of the airfoil, the deviation of the mean velocity from Poiseuille flow was 5 - 7%, whereas the amplitude of the periodic disturbance was only of order 0.5%. His thought was that the observed transition could have occurred as a result of an initial distortion of the mean flow rather than of the oscillation itself. Thus, the basic idea of Gill's paper was that instability might be achieved by starting with a small distortion of the laminar profile and superimposing an even smaller fluctuation on the flow field. It was shown that an initial mean flow profile which had a local maximum in the magnitude of \( \frac{1}{R} u_r \) is unstable provided its amplitude is greater than some threshold value. Now viscosity would tend to damp out the mean flow distortion so that if the instability is to be sustained, the fluctuation must have a growth rate large enough for non-linear effects to become important before the distortion is reduced to an ineffective level. This requirement leads to the existence of a critical Reynolds number which depends on the shape of the initial distortion and its amplitude, the shape of the periodic disturbance and its amplitude, and on the wave number. The proposed mechanism differs from classical approaches in that classically mean flow effects are secondary in the sense that they are produced by Reynolds stresses
resulting from a fluctuating disturbance interacting with itself. On the other hand, Gill suggests that the zero wave number component (the imposed change in the mean flow) plays a fundamental role.

With this idea in mind we constructed an initial mean flow stream function \( A_0 (r, \phi) \) (Figure 6.31) having an acceptable shape according to Gill's criteria. The mean velocity distortion is shown in Figure 6.32 and the related vorticity function, which must possess a local maximum is shown in Figure 6.33. Using an initial amplitude of 5\% for \( A_0 \) and 0.2\% for \( A_1 \), we made a series of stability calculations for \( \alpha = 1 \) and a sequence of Reynolds numbers. The responses shown in Figures 6.34 and 6.35 indicate that small fluctuations can be made to grow although the growth seems to be limited. The results are plotted at \( r = 0.7 \) since the fluctuation peak gravitates to that point. This is in qualitative agreement with experiment to the extent that unstable fluctuations move toward the wall while stable ones move in the opposite direction. The response for \( R = 10,000 \) grows by more than an order of magnitude but ultimately tends to die out. The fact of the matter is, however, that growth in this amount might possibly be enough to initiate transition if three-dimensional effects could be included in the calculation. We note the significant variation in the rate of decay of the mean flow harmonic with Reynolds number. Figure 6.37 indicates the variation in growth rate with initial amplitude of mean flow distortion, all other parameters being held fixed. Very little growth occurs for amplitudes less than 2\%. Decay of \( A_0 \) does not seem to be much influenced by its initial amplitude (Figure 6.36).
Finally, we observe the effect of initial amplitude of the fundamental fluctuation on its own growth in Figure 6.39. A striking feature of this graph is that for the two larger amplitude runs, growth of $A_1$ seems to be limited by an upper bound of 2.25%. The response corresponding to the extremely low amplitude run was not carried out far enough, but has grown by almost two orders of magnitude and appears headed toward an upper bound near the same level experienced by the other two cases.

The final conclusion to be reached concerning the set of solutions corresponding to Gill's idea is that growth can occur for a limited time but that on the basis of two-dimensional behavior we must rule in favor of ultimate stability with respect to finite amplitude disturbances.

In Chapter II we mentioned theoretical work of Lessen et al. (1968) and of Salwen and Grosch (1968) on studies of linear disturbances in three-dimensions both of whom found only stability. We must therefore conclude that both three-dimensional and finite-amplitude effects are needed in order to achieve sustained growth and transition. About the only other possibility that we can visualize is that linear three-dimensional disturbances could be unstable to helical modes that spiral in a direction opposite to the sense of the streamlines, i.e., to modes that vary azimuthally according to $e^{in\theta}$ where $n$ takes on negative values. This case seems to have been neglected by the previous workers, while use of negative $n$ does change the sign of several terms in the Orr-Sommerfeld equations.
Perhaps a three-dimensional study along the lines of Meksyn and Stuart (1951) would be in order in which a mean flow equation would be solved simultaneously with the Orr-Sommerfeld equation to determine equilibrium flow conditions. If computer solution is attempted, as opposed to asymptotic analysis, equilibrium equations for higher harmonics could also be included.
FIGURE 6.1
INITIAL STREAM FUNCTION AND VELOCITY FLUCTUATION FOR PLANE-POISEUILLE FLOW
A(0,t) = e^{C_i t} \cos C_r t
C_i = 0.003758
C_r = 0.2374
(\Delta y = 0.02, \Delta t = 0.30)

FIGURE 6.2
LINEAR OSCILLATION FOR PLANE-POISEUILLE FLOW COMPUTED FOR \alpha = 1, \ R = 10,000
EXACT 3.68

$E = O(h^4)$

$E = O(h^2)$

$\alpha = 1, \, R = 10,000$

FIGURE 6.3

EFFECT OF CHANNEL RESOLUTION ON ACCURACY
OF LINEAR SOLUTIONS
(GROWTH RATE -1)
\( \alpha = 1, \ R = 10.000 \)

**FIGURE 6.4**  
EFFECT OF CHANNEL RESOLUTION ON  
ACCURACY OF LINEAR SOLUTIONS IN PLANE-  
POISEUILLE FLOW  
(GROWTH RATE - 2 )
Figure 6.5
Effect of channel resolution on accuracy of linear solutions in plane-Poiseuille flow (period)

\( \alpha = 1, \ R = 10,000 \)

\( E = O(h^2) \)

\( E = O(h^4) \)

Exact value = 26.45
FIGURE 6.6
EFFECT OF CHANNEL RESOLUTION ON LINEAR SOLUTION BEHAVIOR IN PLANE-POISEUILLE FLOW

$\alpha=1, R=10,000$
$E = O(h^4)$
FIGURE 6.7
REAL AND IMAGINARY PARTS OF LEAST STABLE PLANE-POISEUILLE FLOW
SYMMETRIC EIGENFUNCTION
($\alpha = 1$, $R = 10,000$)
FIGURE 6.8
COMBINATIONS OF $\alpha$ AND $R$
USED FOR VERIFICATION OF
LINEAR THEORY OF PLANE-
POISEUILLE FLOW
FIGURE 6.9
INITIAL STREAM FUNCTION FOR FLUCTUATION FOR POISEUILLE FLOW
FIGURE 6.10
INITIAL VELOCITY FLUCTUATION FOR POISEUILLE FLOW
FIG. 6.11 CURVES OF CONSTANT \( a C_i \) \& \( \beta \) CR FOR THE LEAST STABLE CENTERLINE MODE - POISEUILLE FLOW

FIG. 6.12 CURVES OF CONSTANT \( a C_i \) \& \( \beta \) CR FOR THE LEAST STABLE WALL MODE - POISEUILLE FLOW
(AFTER DAVEY AND DRAZIN 1969)
Figure 6.13
EARLY TIME BEHAVIOR OF LINEAR POISEUILLE FLOW

FLUCTUATION AMPLITUDE AT $y=0.5$

- DECAY RATE $e^{-0.0285t}$
- MAXIMA
- MINIMA
FIGURE 6.14
SHAPE OF $A(y,t)$ CORRESPONDING TO SUCCESSIVE TIMES FOR WHICH CENTERLINE PEAK IS MAXIMUM
LATE TIME BEHAVIOR OF LINEAR POISEUILLE FLOW
LEAST STABLE POISEUILLE FLOW EIGENFUNCTION
FOR $\alpha = 1$, $R = 10,000$

FIGURE 6.16
FIGURE 6.18
EFFECT OF REYNOLDS NUMBER ON NON-LINEAR SOLUTIONS IN PLANAR POISEUILLE FLOW (INITIAL AMPLITUDE 10%)
FIGURE 6.19
EFFECT OF REYNOLDS NUMBER ON NON-LINEAR SOLUTIONS IN PLANE-POISEUILLE FLOW
(INITIAL AMPLITUDES 2% AND 5%)
FIGURE 6.20
EFFECT OF REYNOLDS NUMBER ON NON-LINEAR SOLUTIONS IN PLANE-POISEUILLE FLOW (INITIAL AMPLITUDES 15%, 20%, AND 30%)
Figure 6.21
Magnitude of second harmonic at $y = 0.50$ for non-linear plane Poiseuille flow
Figure 6.22 Variation of 2nd Harmonic Maximum Magnitude with Initial Amplitude for Non-Linear Plane-Poiseuille Flow
FIGURE 6.23
CRITICAL BEHAVIOR FOR FINITE AMPLITUDE DISTURBANCES IN PLANE POISEULLE FLOW
COMPARISON OF PLANE-POISEUILLE FLOW CRITICAL CURVE WITH EARLIER WORK
FIGURE 6.26

NON-LINEAR SOLUTION IN POISEUILLE FLOW FOR $\alpha = 2.5, R = 3000$
DISTURBANCE AMPLITUDE = 0.10
FIGURE 6.28

NON-LINEAR SOLUTION IN POISEUILLE FLOW FOR $a = 0.5, R = 100,000$
INITIAL DISTURBANCE AMPLITUDE $= 0.19\%$ IN $\psi$ OR $3\%$ IN VELOCITY $\mu$
FIG. 6.29 RADIAL DISTRIBUTION OF TURBULENCE IN WAKE OF RADIUS 0.45 IN.

FIG. 6.30 RADIAL DISTRIBUTION OF MEAN VELOCITY DOWNSTREAM OF RING AIRFOIL
(AFTER KUETHE 1956)
FIGURE 6.31
STREAM FUNCTION FOR INITIAL MEAN FLOW DISTORTION IN POISEUILLE FLOW
FIGURE 6.32

VELOCITY FUNCTION FOR INITIAL MEAN FLOW DISTORTION IN POISEUILLE FLOW
FIGURE 6.33
RELATED VORTICITY FUNCTION
FOR INITIAL MEAN FLOW
DISTORTION IN POISEUILLE FLOW
FIGURE 6.34

EFFECT OF REYNOLDS NUMBER ON VISCOSOUS DECAY OF $A_o(0.7, t)$

$A_o(r, o) = 5\%$, $A_1(r, o) = 0.2\%$, $\alpha = 1$
FIGURE 6.35

EFFECT OF REYNOLDS NUMBER ON BEHAVIOR OF $A_1(0.7, t)$

$\max A_0(r, o) = 5\%$, $\max A_1(r, o) = 0.2\%$

$\alpha = 1$
EFFECT OF INITIAL AMPLITUDE ON $A_0$ ON VISCOUS DECAY OF $A_0 (D.7, t )$, $a = 1$, $R = 10,000$, MAX $A_1 (r, 0 ) = 0.2$%
EFFECT OF INITIAL AMPLITUDE OF $A_0$ ON BEHAVIOR
OF $A_1(0.7,t)$  \( \alpha = 1 \), \( R = 10,000 \), MAX $A_1(r,o) = 0.2\%$
Figure 6.38
Effect of initial amplitude of $A_1$ on viscous decay of $A_0(0.7, t)$, $\alpha = 1$, $R = 10,000$, MAX $A_0(r, \theta) = 5\%$
Figure 6.39

Effect of initial amplitude of $A_1$ on behavior of $A_1(0.7,t)$. $e = 1$, $R = 10,000$, $\text{MAX } A_0(r,0) = 5\%$
CHAPTER VII

CONCLUDING REMARKS

1. We have developed a practical and efficient numerical method for obtaining approximate solutions to the equations of motion pertaining to two-dimensional disturbances superimposed on laminar parallel shear flow.

2. The numerical process was shown to be both stable and accurate when applied to problems in linear plane-Poiseuille flow and Poiseuille flow. Orr-Sommerfeld eigenvalues were computed to a high enough degree of accuracy to be considered exact by using only 50 channel subdivisions when fourth-order approximation in space was employed.

3. Non-linear results for problems in plane-Poiseuille flow confirmed that certain flows that are stable according to linear theory become unstable to finite-amplitude disturbances.

4. A curve relating Reynolds number to critical amplitude was constructed for a disturbance having fundamental wavelength $2\pi$. The curve demonstrated the existence of an absolute critical Reynolds number below which the disturbance cannot be made unstable, no matter how large its initial amplitude. Construction of similar curves for all wave numbers and disturbance shapes would presumably lead to an all inclusive critical locus whose minimum would touch the experimentally determined critical Reynolds number associated with reversion to turbulence, about 2200.
5. The computed plane-Poiseuille flow critical curve shows significantly less effect of amplitude than do those obtained by earlier workers using perturbation techniques and asymptotic analysis.

6. Computed results corresponding to Poiseuille flow through a circular pipe for which only the fundamental fluctuation was initialized indicated absolute stability. The $\alpha - R$ parameters chosen for these calculations were based on stability experiments carried out by Fox et al. (1968). Values chosen were located in the portion of the $\alpha - R$ plane shown to be unstable according to their work.

7. It was shown that growth of a Poiseuille flow disturbance can occur, at least for a limited time, by starting with a small distortion of the mean laminar velocity profile and superimposing a small fluctuation on the flow field. Ultimate stability does, however, prevail.

8. Since linear three-dimensional disturbances have been shown to be stable according to the work of Lessen et al (1968) and Salwen and Grosch (1968), it appears that both three-dimensional and finite-amplitude effects may be needed in order to realize sustained oscillation growth and resultant transition.
BIBLIOGRAPHY


Grosch, Chester E. and Salwen, Harold (1968) The stability of steady and time-dependent plane Poiseuille flow. J. Fluid Mech. 84, 177-205.


APPENDICES
APPENDIX I

In this appendix we derive expressions for the non-linear terms with respect to the harmonic components \( A_m(y,t) \) and \( B_m(y,t) \). These expressions for \( P^A_m(y,t) \) and \( P^B_m(y,t) \) must be evaluated once during each iteration of each time step as the integration of the partial differential equations proceeds forward in time.

The non-linear terms can be written as

\[
\frac{\Theta(x,y,t)}{\kappa} = \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \mathcal{L}[\psi] - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \mathcal{L}[\psi] - \frac{2n}{y} \mathcal{L}[\psi]
\]

where,

\[
\mathcal{L}[\psi] = \psi_{xx} + \psi_{yy} - \frac{n}{y} \psi_y
\]

and

\[
\kappa = 1 + \eta \left( \frac{1}{y} - 1 \right)
\]

The expansion of \( \psi \) into its harmonic components is

\[
\psi(x,y,t) = \sum_{m=0}^{M} A_m(y,t) \cos \max + B_m(y,t) \sin \max
\]

Substitution of this series into the expression for \( \Theta \) yields

\[
\frac{\Theta(x,y,t)}{\kappa} = \left[ \sum_{q=0}^{M} A'_q \cos q\alpha x + B'_q \sin q\alpha x \right] \left[ \sum_{p=0}^{M} \rho \Delta D_p \cos p\alpha x - \rho \Delta C_p \sin p\alpha x \right] + \left[ \sum_{q=0}^{M} -q\alpha B'_q \cos q\alpha x + q\alpha \sin q\alpha x \right] \left[ \sum_{p=0}^{M} \nu \Delta D_p \sin p\alpha x + \nu \Delta C_p \cos p\alpha x \right]
\]
where,

\[ C_p = A''_p - \frac{n}{y} A'_p - (pa)^2 A_p, \quad D_p = B''_p - \frac{n}{y} B'_p - (pa)^2 B_p, \]

\[ V_p = D'_p - \frac{2n}{y} D_p, \quad W_p = C'_p - \frac{2n}{y} C_p \]

\[ \eta = \begin{cases} 
0 & \text{for plane-Poiseuille flow} \\
1 & \text{for Poiseuille flow} 
\end{cases} \]

Multiplying the series out and collecting like terms we obtain

\[ \frac{\Theta}{\kappa} = \alpha \sum_{q=0}^{M} \sum_{p=0}^{M} \left[ pA'_q D_q - qB'_q W_p \right] \cos qax \cos pax \]

\[ + \left[ -pA'_q C_p - qB'_q V_p \right] \cos qax \sin pax \]

\[ + \left[ pB'_q D_p + qA'_q W_p \right] \sin qax \cos pax \]

\[ + \left[ -pB'_q C_p + qA'_q V_p \right] \sin qax \sin pax \]

Define:

\[ E_{qp} = pA'_q D_p - qW_p B_q \]

\[ F_{qp} = -pA'_q C_p - qV_p B_q \]

\[ G_{qp} = pB'_q D_p + qW_p A_q \]

\[ H_{qp} = -pB'_q C_p + qV_p A_q \]
Then,

\[
\Theta(x,y,t) = \sum_{q=0}^{M} \sum_{p=0}^{M} E_{qp} \cos qx \cos px + F_{qp} \cos qx \sin px \\
+ G_{qp} \sin qx \cos px + H_{qp} \sin qx \sin px \\
= 1/2 \sum_{q=0}^{M} \sum_{p=0}^{M} E_{qp} [\cos (q-p)x + \cos (q+p)x] \\
+ F_{qp} [\sin (q+p)x + \sin (p-q)x] \\
+ G_{qp} [\sin (q+p)x + \sin (q-p)x] \\
+ H_{qp} [\cos (q-p)x - \cos (q+p)x]
\]

Define:

\[
Q_{qp} = E_{qp} + H_{qp} \\
R_{qp} = E_{qp} - H_{qp} \\
S_{qp} = G_{qp} - F_{qp} \\
T_{qp} = G_{qp} + F_{qp}
\]

Then,

\[
\frac{2 \Theta}{\alpha x} = \sum_{q=0}^{M} \sum_{p=0}^{M} Q_{qp} \cos (q-p)x + R_{qp} \cos (q+p)x \\
+ S_{qp} \sin (q-p)x + T_{qp} \sin (q+p)x
\]
Thus,

\[ \sum_{m=0}^{M} F_m^A \cos \alpha x = \frac{\alpha k}{2} \sum_{q=0}^{M} \sum_{p=0}^{M} Q_{qp} \cos(q-p)\alpha x \]

\[ + R_{qp} \cos(q+p)\alpha x \]

or on combining like harmonics we have:

\[ \left[ F_0^A - \frac{\alpha k}{2} \left( R_{oo} + \sum_{p=0}^{M} Q_{pp} \right) \right] + \sum_{m=1}^{M} \left[ F_m^A - \frac{\alpha k}{2} \sum_{p=0}^{M-m} \left[ Q_{(m+p)p} + Q_{p(m+p)} \right] \right. \]

\[ - \frac{\alpha k}{2} \sum_{p=0}^{M} R_{(m-p)p} \right] \cos \alpha x = 0 \]

Thus,

\[ F_0^A = \frac{\alpha k}{2} \sum_{p=0}^{M} Q_{pp} \]

\[ F_m^A = \frac{\alpha k}{2} \left\{ \sum_{p=0}^{M-m} \left[ Q_{(m+p)p} + Q_{p(m+p)} \right] + \sum_{p=0}^{m} R_{(m-p)p} \right\} \]

For \( 1 \leq m \leq M \)

Similarly,

\[ \sum_{m=0}^{M} F_m^B \sin \alpha x = \frac{\alpha k}{2} \sum_{q=0}^{M} \sum_{p=0}^{M} S_{qp} \sin(q-p)\alpha x + T_{qp} \sin(q+p)\alpha x \]

or

\[ \left[ F_0^B - \frac{\alpha k}{2} \left( T_{oo} + \sum_{p=0}^{M} S_{pp} \right) \right] \sin 0 \]

\[ + \sum_{m=1}^{M} \left[ F_m^B - \frac{\alpha k}{2} \sum_{p=0}^{M-m} \left[ S_{(m+p)p} - S_{p(m+p)} \right] - \frac{\alpha k}{2} \sum_{p=0}^{m} T_{(m-p)p} \right] \sin \alpha x \]
Thus,

\[ P_m^B = \frac{\alpha \kappa}{2} \left[ \sum_{p=0}^{M-m} \left( S_{(m+p)p} - S_{p(m+p)} \right) + \sum_{p=0}^{M} T_{(m-p)p} \right] \quad \text{for } 1 \leq m \leq M \]

Define

\[ P_m^A = \frac{\alpha \kappa}{2} \hat{P}_m^A \]

\[ P_m^B = \frac{\alpha \kappa}{2} \hat{P}_m^B \]

We then simplify by stripping off the zero wave number interactions, i.e., \( p = 0 \), to obtain finally

\[ \hat{P}_0^A = \sum_{p=1}^{M} Q_{pp} \]

\[ \hat{P}_m^A = 2m \left[ A' D_m - W'_B m \right] + \sum_{p=1}^{M-m} \left[ Q_{(m+p)p} + Q_{p(m+p)} \right] \]

\[ + \sum_{p=1}^{m-1} R_{(m-p)p} \quad \text{for } 1 \leq m \leq M \]

\[ \hat{P}_m^B = 2m \left[ W A_m - A'_C m \right] + \sum_{p=1}^{M-m} \left[ S_{(m+p)p} - S_{p(m+p)} \right] \]

\[ + \sum_{p=1}^{m-1} T_{(m-p)p} \quad \text{for } 1 \leq m \leq M \]

It is understood that the \( Q \) and \( S \) summations are omitted when \( m = M \) and that the \( R \) and \( T \) summations are omitted when \( m = 1 \).
The individual terms in the series written out in full are:

\[ Q_{(m+p)p} + Q_{p(m+p)} = p \left[ A_{m+p}' D_p - B_{m+p}' C_p + V_{m+p} A_p - W_{m+p} B_p \right] \]

\[ + (m + p) \left[ V_p A_{m+p} - W_p B_{m+p} + A_p D_{m+p} - B_p C_{m+p} \right] \]

\[ R_{(m-p)p} = p \left[ A_{m-p}' D_p + B_{m-p}' C_p \right] - (m-p) \left[ W_p B_{m-p} + V_p A_{m-p} \right] \]

\[ T_{(m-p)p} = p \left[ B_{m-p}' D_p - A_{m-p}' C_p \right] + (m-p) \left[ W_p A_{m-p} - V_p B_{m-p} \right] \]

\[ S_{(m+p)p} - S_{p(m+p)} = p \left[ B_{m+p}' D_p + A_{m+p}' C_p - W_{m+p} A_p - V_{m+p} B_p \right] \]

\[ + (m + p) \left[ W_p A_{m+p} + V_p B_{m+p} - B_p D_{m+p} - A_p C_{m+p} \right] \]

\[ Q_{pp} = p \left[ A_p' D_p - W_p B_p - B_p' C_p + V_p A_p \right] \]
APPENDIX II

In this appendix the finite difference expressions used to approximate various derivatives in the partial differential equations are tabulated. These expressions were obtained from Collatz (1960) Appendix Table III.

<table>
<thead>
<tr>
<th>Formulae</th>
<th>The next non-vanishing term of the Taylor expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q'<em>0 = \frac{1}{12h} (Q</em>{-2} - 8Q_{-1} + 8Q_1 - Q_2)$ +</td>
<td>$+ \frac{1}{30} h^4 Q_0^v$</td>
</tr>
<tr>
<td>$Q''<em>0 = \frac{1}{12h^2} (-Q</em>{-2} + 16Q_{-1} - 30Q_0 + 16Q_1 - Q_2)$ +</td>
<td>$+ \frac{1}{90} h^4 Q_0^{vi}$</td>
</tr>
<tr>
<td>$Q'''<em>0 = \frac{1}{2h^3} (-Q</em>{-2} + 2Q_{-1} - 2Q_1 + Q_2)$ +</td>
<td>$- \frac{1}{4} h^2 Q_0^v$</td>
</tr>
<tr>
<td>$Q'''<em>0 = \frac{1}{h^4} (Q</em>{-2} - 4Q_{-1} + 6Q_0 - 4Q_1 + Q_2)$ +</td>
<td>$- \frac{1}{6} h^2 Q_0^{vi}$</td>
</tr>
<tr>
<td>$Q''''<em>0 = \frac{1}{2h^5} (-Q</em>{-3} + 4Q_{-2} - 5Q_{-1} + 5Q_1 - 4Q_2 + Q_3)$ +</td>
<td>$O(h^2)$</td>
</tr>
<tr>
<td>$Q'''''<em>0 = \frac{1}{h^6} (Q</em>{-3} - 6Q_{-2} + 15Q_{-1} - 20Q_0 + 15Q_1 - 6Q_2 + Q_3)$ +</td>
<td>$O(h^2)$</td>
</tr>
</tbody>
</table>
APPENDIX III

In this appendix we derive a finite difference approximation to the system of fourth-order, one-dimensional partial differential equations for the Fourier harmonic components. Substitution of appropriate finite difference expressions for the derivatives, as tabulated in Appendix II, into the partial differential equations and subsequent simplification leads ultimately to a penta-diagonal system of linear algebraic equations which must be solved each iteration of each time step as the integration progresses. The approximation so obtained is fourth-order correct in space. Strict fourth-order correct differencing for the third and fourth derivatives would lead to a hepta-diagonal system of equations; however, we employ second-order correct formulae in the implicit scheme and add to the equations an approximation of the error term to bring the overall accuracy up to fourth-order. This technique, which reduces the computer time by fifty percent, will become clear in the derivation. Crank-Nicolson type differencing in time is used throughout, that is, in advancing the solution from time level \(n\Delta t\) to time level \((n+1)\Delta t\), the equations are evaluated at ficticious time level \((n+\frac{1}{2})\Delta t\) through an averaging process.

The system of partial differential equations to be solved are:

\[
\begin{align*}
\gamma_4 \frac{A_m^{IV}}{A_m} + \gamma_3 \frac{A_m^{III}}{A_m} + \gamma_2 \frac{A_m^{II}}{A_m} + \gamma_1 \frac{A_m^{I}}{A_m} + \gamma_0 A_m + \lambda_0 \frac{\partial A_m}{\partial t} + \lambda_1 \frac{\partial A_m^I}{\partial t} + \lambda_2 \frac{\partial A_m^II}{\partial t} + \sigma D_m + \tau B_m &= R_m^A, \quad 0 < m < M
\end{align*}
\]
\[ \gamma_4 B_m^{1v} + \gamma_3 B_m^{m} + \gamma_2 B_m'' + \gamma_1 B_m' + \gamma_0 B_m + \lambda_0 \frac{\partial B_m}{\partial t} + \lambda_1 \frac{\partial B_m'}{\partial t} + \lambda_2 \frac{\partial B_m''}{\partial t} \]

\[ - \sigma C_m - \tau A_m = F_m^B, \quad 1 < m < M \]

where,

\[ C_m = A_m'' - \lambda_1 A_m' - (\max)^2 A_m \]

\[ D_m = B_m'' - \lambda_1 B_m' - (\max)^2 B_m \]

Define \( \gamma_i = R^{-1} \hat{\gamma}_i \) for \( 0 \leq i \leq 4 \)

Then,

\[ \hat{\gamma}_4 = 1 \quad \hat{\gamma}_3 = -\eta(2/y) \quad \hat{\gamma}_2 = -2(\max)^2 + \eta(3/y^2) \]

\[ \hat{\gamma}_1 = -\eta \hat{\gamma}_2/y \quad \hat{\gamma}_0 = (\max)^4 \]

\[ \lambda_0 = (\max)^2 \quad \lambda_1 = \eta/y \quad \lambda_2 = -1 \]

\[ \sigma = -\bar{\Omega} \max \]

\[ \tau = (1-\eta) \bar{\Omega}_{yy} \max \]

\[ \bar{u} = 1 - y^2 \]

\[ \eta = \begin{cases} 0 \text{ for plane-Poiseuille flow} \\ 1 \text{ for Poiseuille flow in a circular pipe} \end{cases} \]

Consider the equations for \( A_m \)

\[ \gamma_4 A_m^{1v} + \gamma_3 A_m^{m} + \gamma_2 A_m'' + \gamma_1 A_m' + \gamma_0 A_m + \lambda_0 \frac{\partial A_m}{\partial t} + \lambda_1 \frac{\partial A_m'}{\partial t} + \lambda_2 \frac{\partial A_m''}{\partial t} \]

\[ + \sigma D_m + \tau B_m = F_m^A \]
We discretize by utilizing the expressions in Appendix II along with Crank-Nicolson differencing in time:

Notation: Superscripts and subscripts in parentheses refer to time step and grid point number, respectively.

$E^v$ and $E^m$ refer to the error terms in the fourth and third derivatives.

$h \equiv \Delta y$

\[
\frac{\gamma_4}{2h} \left[ A_m^{n+1} - 4A_m^{n+1} + 6A_m^{n+1} - 4A_m^{n+1} + A_m^{n+1} + A_m^{n+1} \right] + \gamma_4 E_m^{n+1}
\]

\[
\left[ -4A_m^{n+1} + 3A_m^{n+1} - 4A_m^{n+1} + A_m^{n+1} + A_m^{n+1} \right] + \gamma_3 E_m^{n+1}
\]

\[
\frac{\gamma_3}{4h^3} \left[ -A_m^{n+1} + 2A_m^{n+1} - 3A_m^{n+1} + 2A_m^{n+1} - A_m^{n+1} + A_m^{n+1} \right] + \gamma_3 E_m^{n+1}
\]

\[
\frac{\gamma_2}{2h^2} \left[ 16A_m^{n+1} - 30A_m^{n+1} + 16A_m^{n+1} - A_m^{n+1} + A_m^{n+1} \right] + \gamma_2 E_m^{n+1}
\]

\[
\frac{\gamma_1}{2h} \left[ A_m^{n+1} - 8A_m^{n+1} + 8A_m^{n+1} - A_m^{n+1} + A_m^{n+1} \right] + \gamma_1 E_m^{n+1}
\]

\[
\lambda_t \left[ A_m^{n+1} - A_m^{n+1} + A_m^{n+1} \right] + \frac{\lambda_1}{2h} \left[ A_m^{n+1} - 8A_m^{n+1} + 8A_m^{n+1} \right]
\]

\[
- A_m^{n+1} - A_m^{n+1} + 8A_m^{n+1} - 8A_m^{n+1} + A_m^{n+1} \]

\[
+ \frac{\lambda_2}{12h} \left[ -A_m^{n+1} + 16A_m^{n+1} - 30A_m^{n+1} + 16A_m^{n+1} \right]
\]

\[
- A_m^{n+1} + A_m^{n+1} - 16A_m^{n+1} - 30A_m^{n+1} + 16A_m^{n+1} + A_m^{n+1} \]

\[
- A_m^{n+1} + A_m^{n+1} - 16A_m^{n+1} - 30A_m^{n+1} + 16A_m^{n+1} + A_m^{n+1}
\]
Define $\theta = Rh^2/\Delta t$

Multiply through by $2Rh^4$ and rearrange to obtain

$$
\begin{align*}
&= \frac{1}{2} \left[ \frac{pA(n+1)}{m} + \frac{pA(n)}{m} \right] - \frac{\sigma}{2} \left[ D_m(n+1) + D_m(n) \right] - \frac{\tau}{2} \left[ B_m(n+1) + B_m(n) \right] \\
&= \left[ 1 - \frac{1}{2} \hat{\gamma}^3 h - \frac{1}{12} \hat{\gamma}^2 h^2 + \frac{1}{12} \hat{\gamma} h^3 + \frac{1}{6} \lambda_1 h \theta - \frac{1}{6} \lambda_2 \theta \right] A_m(n+1) \\
&+ \left[ -4 + \hat{\gamma}^3 h + \frac{4}{3} \hat{\gamma}^2 h^2 - \frac{2}{3} \hat{\gamma} h^3 + \frac{1}{3} \lambda_1 h \theta + \frac{8}{3} \lambda_2 \theta \right] A_m(n+1) \\
&+ \left[ 6 - \frac{5}{2} \hat{\gamma}^2 h^2 + \hat{\gamma}^2 h^4 + 2\lambda_0 h^2 \theta - 5\lambda_2 \theta \right] A_m(n+1) \\
&+ \left[ -4 - \hat{\gamma}^3 h + \frac{4}{3} \hat{\gamma}^2 h^2 + \frac{2}{3} \hat{\gamma} h^3 + \frac{4}{3} \lambda_1 h \theta + \frac{8}{3} \lambda_2 \theta \right] A_m(n+1) \\
&+ \left[ 1 + \frac{1}{2} \hat{\gamma}^3 h - \frac{1}{12} \hat{\gamma}^2 h^2 - \frac{1}{12} \hat{\gamma} h^3 - \frac{1}{6} \lambda_1 h \theta - \frac{1}{6} \lambda_2 \theta \right] A_m(n+1) \\
&= - \left[ 1 - \frac{1}{2} \hat{\gamma}^3 h - \frac{1}{12} \hat{\gamma}^2 h^2 + \frac{1}{12} \hat{\gamma} h^3 - \frac{1}{6} \lambda_1 h \theta + \frac{1}{6} \lambda_2 \theta \right] A_m(n+1) \\
&- \left[ -4 + \hat{\gamma}^3 h + \frac{4}{3} \hat{\gamma}^2 h^2 - \frac{2}{3} \hat{\gamma} h^3 + \frac{4}{3} \lambda_1 h \theta - \frac{8}{3} \lambda_2 \theta \right] A_m(n+1) \\
&- \left[ 6 - \frac{5}{2} \hat{\gamma}^2 h^2 + \hat{\gamma}^2 h^4 - 2\lambda_0 h^2 \theta + 5\lambda_2 \theta \right] A_m(n+1) \\
&- \left[ -4 - \hat{\gamma}^3 h + \frac{4}{3} \hat{\gamma}^2 h^2 + \frac{2}{3} \hat{\gamma} h^3 + \frac{4}{3} \lambda_1 h \theta - \frac{8}{3} \lambda_2 \theta \right] A_m(n+1) \\
&- \left[ 1 + \frac{1}{2} \hat{\gamma}^3 h - \frac{1}{12} \hat{\gamma}^2 h^2 - \frac{1}{12} \hat{\gamma} h^3 - \frac{1}{6} \lambda_1 h \theta + \frac{1}{6} \lambda_2 \theta \right] A_m(n+1) \\
&+ \left[ \frac{pA(n)}{m} - \sigma D_m(n) - \tau B_m(n) \right] + \left[ \frac{pA(n+1)}{m} - \sigma D_m(n+1) - \tau B_m(n+1) \right] \\
&- 2Rh^4 \left[ \gamma_4 E^{1V}_m + \gamma_3 E^m_m \right](n)
\end{align*}
$$
Next consider the error term:

From Appendix B

\[ E_{m}^{1V} = -\frac{1}{6} h^2 A_{m}^{V1} \quad \text{and} \quad E_{m}^{\prime\prime} = -\frac{1}{4} h^2 A_{m}^{V} \]

Substitution of expressions for \( A_{m}^{V1} \) and \( A_{m}^{V} \) leads to

\[ -2Rh^4 \left[ \gamma_4 E_{m}^{1V} + \gamma_3 E_{m}^{\prime\prime} \right] (n) = 2Rh^4 \left[ \frac{1}{6} \gamma_4 h^2 A_{m}^{V1} + \frac{1}{4} \gamma_3 h^2 A_{m}^{V} \right] (n) \]

\[ = h^6 \left[ \frac{1}{3} A_{m}^{V1} + \frac{1}{2} \gamma_3 A_{m}^{V} \right] (n) \]

\[ = \frac{1}{3} \left[ A_{m(j-3)}^{(n)} - 6A_{m(j-2)}^{(n)} + 15A_{m(j-1)}^{(n)} - 20A_{m(j)}^{(n)} + 15A_{m(j+1)}^{(n)} - 6A_{m(j+2)}^{(n)} \right] \]

\[ + A_{m(j+3)}^{(n)} \]

\[ + \frac{1}{4} \chi_h \left[ -A_{m(j-3)}^{(n)} + 4A_{m(j-2)}^{(n)} - 5A_{m(j-1)}^{(n)} + 5A_{m(j+1)}^{(n)} \right] \]

\[ - 4A_{m(j+2)}^{(n)} + A_{m(j+3)}^{(n)} \]

or

\[ -2Rh^4 \left[ \gamma_4 E_{m}^{1V} + \gamma_3 E_{m}^{\prime\prime} \right] (n) = \left[ \frac{1}{3} - \frac{1}{4} \chi_h \right] A_{m(j-3)}^{(n)} \]

\[ + \left[ \chi_h - 2 \right] A_{m(j-2)}^{(n)} + \left[ 5 - \frac{4}{5} \chi_h \right] A_{m(j-1)}^{(n)} \]

\[ + \left[ - \frac{20}{3} \right] A_{m(j)}^{(n)} + \left[ 5 + \frac{4}{5} \chi_h \right] A_{m(j+1)}^{(n)} \]

\[ - \left[ \chi_h + 2 \right] A_{m(j-2)}^{(n)} + \left[ \frac{1}{3} + \frac{1}{4} \chi_h \right] A_{m(j+3)}^{(n)} \]

Substitute this correction term back into the main difference equation to yield finally

\[ \left[ N_{m(j)} - \right] A_{m(j-2)}^{(n+1)} + \left[ P_{m(j)} - \right] A_{m(j-1)}^{(n+1)} + \left[ Q_{m(j)} \right] A_{m(j)}^{(n+1)} + \left[ P_{m(j)}^{+} \right] \]

\[ A_{m(j+1)}^{(n+1)} + \left[ N_{m(j)}^{+} \right] A_{m(j+2)}^{(n+1)} = H_{m(j)}^{(n)} + Y_{m(j)}^{(n+1)} \]
Where

\[ N_{m(j)} = 1 - \frac{1}{2} \hat{\gamma}_3 h + \frac{1}{12} h^2 \left[ \hat{\gamma}_1 h - \hat{\gamma}_2 \right] + \frac{1}{6} \theta \left[ \lambda_1 h - \lambda_2 \right] \]

\[ P_{m(j)} = -4 + \hat{\gamma}_3 h + \frac{2}{3} h^2 \left[ 2\hat{\gamma}_2 - \hat{\gamma}_1 h \right] + \frac{4}{3} \theta \left[ 2\lambda_2 - \lambda_1 h \right] \]

\[ Q_{m(j)} = 6 - \frac{5}{2} \hat{\gamma}_2 h^2 + \hat{\gamma}_0 h^4 + \theta \left[ 2\lambda_0 h^2 - 5\lambda_2 \right] \]

\[ P_{m(j)}^+ = -4 - \hat{\gamma}_3 h + \frac{2}{3} h^2 \left[ 2\hat{\gamma}_2 + \hat{\gamma}_1 h \right] + \frac{4}{3} \theta \left[ 2\lambda_2 + \lambda_1 h \right] \]

\[ N_{m(j)}^+ = 1 + \frac{1}{2} \hat{\lambda}_3 - \frac{1}{12} h^2 \left[ \hat{\gamma}_1 h + \hat{\gamma}_2 \right] - \frac{1}{6} \theta \left[ \lambda_1 h + \lambda_2 \right] \]

\[ H_{m(j)}^{(n)} = -\hat{N}_{m(j)} A_{m(j-2)}^{(n)} - \hat{N}_{m(j)} A_{m(j+2)}^{(n)} - \hat{P}_{m(j)} A_{m(j-1)}^{(n)} - \hat{P}_{m(j)} A_{m(j+1)}^{(n)} \]

\[ \hat{N}_{m(j)}^{(n)} = -\hat{N}_{m(j)}^A A_{m(j-2)}^{(n)} - \hat{N}_{m(j)}^A A_{m(j+2)}^{(n)} - \hat{P}_{m(j)}^A A_{m(j-1)}^{(n)} - \hat{P}_{m(j)}^A A_{m(j+1)}^{(n)} \]

\[ \hat{P}_{m(j)}^{(n)} = -4 + \hat{\gamma}_3 h + \frac{2}{3} h^2 \left[ 2\hat{\gamma}_2 + \hat{\gamma}_1 h \right] - \frac{4}{3} \theta \left[ 2\lambda_2 - \lambda_1 h \right] + \left[ 5 - \frac{5}{4} \hat{\gamma}_3 h \right] \]

\[ \hat{Q}_{m(j)} = 6 - \frac{5}{2} \hat{\gamma}_2 h^2 + \hat{\gamma}_0 h^4 - \theta \left[ 2\lambda_0 h^2 - 5\lambda_2 \right] - \left[ \frac{20}{3} \right] \]

\[ \hat{P}_{m(j)}^+ = -4 + \hat{\gamma}_3 h + \frac{2}{3} h^2 \left[ 2\hat{\gamma}_2 + \hat{\gamma}_1 h \right] - \frac{4}{3} \theta \left[ 2\lambda_2 + \lambda_1 h \right] + \left[ 5 + \frac{5}{4} \hat{\gamma}_3 h \right] \]

\[ \hat{N}_{m(j)}^+ = 1 + \frac{1}{2} \hat{\gamma}_3 h - \frac{1}{12} h^2 \left[ \hat{\gamma}_1 h + \hat{\gamma}_2 \right] + \frac{1}{6} \theta \left[ \lambda_1 h + \lambda_2 \right] - \left[ 2 + \hat{\gamma}_3 h \right] \]

\[ \hat{R}_{m(j)} = \frac{1}{3} - \frac{1}{4} \hat{\gamma}_3 h \]

\[ \hat{R}_{m(j)}^+ = \frac{1}{3} + \frac{1}{4} \hat{\gamma}_3 h \]
Similarly, the difference equation for $B_{m(j)}^{(n+1)}$ may be written as

$$\begin{bmatrix} N_{m(j)}^{-1} & B_{m(j-2)}^{(n+1)} & B_{m(j-1)}^{(n+1)} & B_{m(j)}^{(n+1)} & P_{m(j)}^{-1} \\
 & B_{m(j+1)}^{(n+1)} & B_{m(j)}^{(n+1)} & B_{m(j+1)}^{(n+1)} & P_{m(j)}^{+1} \end{bmatrix}$$

$$= G_{m(j)}^{(n)} + Z_{m(j)}^{(n+1)}$$

where

$$G_{m(j)}^{(n)} = -N_{m(j)} B_{m(j-2)}^{(n)} - N_{m(j)}^{+} B_{m(j+2)}^{(n)} - P_{m(j)}^{-} B_{m(j-1)}^{(n)} - P_{m(j)}^{+} B_{m(j+1)}^{(n)}$$

$$- \hat{G}_{m(j)} B_{m(j)}^{(n)} + \hat{P}_{m(j)}^{-} B_{m(j-3)}^{(n)} + \hat{P}_{m(j)}^{+} A_{m(j+3)}$$

$$Z_{m(j)} = R_n^4 \left[ P_{m(j)}^B + \sigma C_{m(j)} + \tau A_{m(j)} \right]$$
APPENDIX IV

In this section we derive approximations for several auxiliary quantities necessary for solution of the system of partial differential equations, namely, $A_m', B_m', C_m', D_m', W_m$, and $V_m$.

(1)\[ A_m'(j) = \frac{1}{12h} \left[ A_m(j-2) - 8A_m(j-1) + 8A_m(j+1) - A_m(j+2) \right] \]
\[ = \frac{2}{3h} \left[ A_m(j+1) - A_m(j-1) \right] - \frac{1}{12h} \left[ A_m(j+2) - A_m(j-2) \right] \]

(2)\[ C_m = A_m'' - \frac{n}{y} A_m' - (\alpha x)^2 A_m \]
\[ C_m(j) = \frac{1}{12h^2} \left[ A_m(j-2) + 16A_m(j-1) - 30A_m(j) \right. \]
\[ + 16A_m(j+1) - A_m(j+2) \left] - \frac{n}{12hy} \left[ A_m(j-2) + 8A_m(j-1) \right. \]
\[ \left. + 8A_m(j+1) - A_m(j+2) \right] - (\alpha x)^2 A_m(j) \]

or
\[ C_m(j) = \frac{1}{12h} \left[ \frac{1}{h} + \frac{n}{y} \right] A_m(j-2) + \frac{2}{3h} \left[ \frac{2}{h} + \frac{n}{y} \right] A_m(j-1) \]
\[ - \left[ \frac{5}{2h^2} + (\alpha x)^2 \right] A_m(j) + \frac{2}{3h} \left[ \frac{2}{h} - \frac{n}{y} \right] A_m(j+1) \]
\[ - \frac{1}{12h} \left[ \frac{1}{h} - \frac{n}{y} \right] A_m(j+2) \]

(3)\[ W_m = C_m' - \frac{2n}{y} C_m \]

Now \[ C_m = A_m'' - \frac{n}{y} A_m' - (\alpha x)^2 A_m \]

Therefore,
\[ W_m = A_m^{\prime\prime} - \frac{n}{y} A_m^{\prime\prime} + \left[ \frac{n}{y^2} - (\alpha t)^2 \right] A_m' \]

\[ - \frac{2n}{y} \left[ A_m^{\prime\prime} - \frac{n}{y} A_m' - (\alpha t)^2 A_m \right] \]

Rearrange to obtain

\[ W_m = A_m^{\prime\prime\prime} + K^{(1)} A_m^{\prime\prime} + K^{(2)} A_m' + K^{(3)} A_m \]

Where,

\[ K^{(1)} = - \frac{3n}{y} \]

\[ K^{(2)} = \frac{3n}{y^2} - (\alpha t)^2 \]

\[ K^{(3)} = \frac{2(\alpha t)^2 n}{y} \]

Substitution of suitable derivative approximations yield

\[ W_m(j) = \frac{1}{8h^3} \left[ A_m(j-3) - 8A_m(j-2) + 13A_m(j-1) - 13A_m(j+1) \right. \]

\[ + 8A_m(j+2) - A_m(j+3) \left. \right] + \frac{K^{(1)}}{12h^2} \left[ - A_m(j-2) + 16A_m(j-1) \right. \]

\[ - 30A_m(j) + 16A_m(j+1) - A_m(j+2) \left. \right] + \frac{K^{(2)}}{12h} \left[ A_m(j-2) \right. \]

\[ - 8A_m(j-1) + 8A_m(j+1) - A_m(j+2) \left. \right] + \frac{K^{(3)}}{12h} A_m(j) \]

Finally upon simplification we have

\[ W_m(j) = \left[ \frac{1}{8h^3} \right] A_m(j-3) + \left[ - \frac{1}{8h^3} - K^{(1)}_i + \frac{K^{(2)}_i}{12h} \right] A_m(j-2) \]

\[ + \left[ \frac{13}{8h^3} + \frac{4K^{(1)}_i}{3h^2} - \frac{2K^{(2)}_i}{3h} \right] A_m(j-1) + \left[ \frac{K^{(3)}_i}{3h} \right] A_m(j) \]

\[ - \frac{5K^{(1)}_i}{2h^2} \right] A_m(j) + \left[ - \frac{13}{8h^3} + \frac{4K^{(1)}_i}{3h^2} + \frac{2K^{(2)}_i}{3h} \right] A_m(j+1) \]

\[ + \left[ \frac{1}{h^3} - \frac{K^{(1)}_i}{12h^2} - \frac{K^{(2)}_i}{12h} \right] A_m(j+2) - \left[ \frac{1}{8h^3} \right] A_m(j+3) \]
Analogous expressions for $B_m'$, $D_m$ and $V_m$ are obtained by substituting $B_m(j)$ for $A_m(j)$ in the above expressions for $A_m'$, $C_m'$ and $W_m$, respectively.
APPENDIX V

The fundamental difference equations derived in Appendix III have the special form

\[(1)\quad AX = f\]

where \(A\) is a penta-diagonal band matrix.

A simple algorithm for solving the equations which takes advantage of the special form is derived as follows:

\(A\) is factored into the product of an upper-triangular matrix and a lower-triangular matrix

\[A = LU\]

whence (1) becomes

\[LUX = f\]

Define \(UX = \lambda\)

Then \(LA = f\)

Thus the three basic steps in the solution are

(a) Factorization of \(A\) into \(LU\)
(b) Calculation of \(\lambda\) (forward substitution)
(c) Calculation of \(X\) (back substitution)
Now write $A = LU$ out in full and perform the matrix multiplication.

\[
\begin{bmatrix}
  c_1 & d_1 & e_1 \\
  b_2 & c_2 & d_2 & e_2 \\
  a_3 & b_3 & c_3 & d_3 & e_3 \\
  & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & d_{N-1} \\
  & & & & & & & & & & & & a_N \\
  & & & & & & & & & & & & b_N \\
  & & & & & & & & & & & & c_N
\end{bmatrix}
\begin{bmatrix}
  \gamma_1 \\
  \beta_2 \\
  \alpha_3 \\
  \vdots \\
  \alpha_N \\
  \beta_N \\
  \gamma_N
\end{bmatrix}
= 
\begin{bmatrix}
  1 & \delta_1 & \epsilon_1 \\
  1 & \delta_2 & \epsilon_2 \\
  1 & \delta_3 & \epsilon_3 \\
  & & \cdots \\
  1 & & \cdots \\
  1 & & \cdots \\
  1 & & \cdots \\
  1 & & \cdots \\
  & & \cdots \\
  & & \cdots \\
  1
\end{bmatrix}
\begin{bmatrix}
  \gamma_1 \delta_1 & \gamma_1 \epsilon_1 \\
  \beta_2 \gamma_2 & \beta_2 \delta_1 + \gamma_2 \\
  \alpha_3 \delta_2 + \beta_3 & \alpha_3 \epsilon_2 + \beta_3 \gamma_2 \\
  \alpha_N \delta_N + \beta_N & \alpha_N \epsilon_N + \beta_N \gamma_N \\
  \alpha_4 & \alpha_4 \delta_2 + \beta_4 \\
  \alpha_5 & \alpha_5 \delta_3 + \beta_5 \\
  \alpha_6 & \alpha_6 \delta_4 + \beta_6 \\
  \alpha_7 & \alpha_7
\end{bmatrix}
\]
Hence we find the following relationships existing among the elements of A, L and U

\[
\begin{align*}
  a_3 &= \alpha_3 \\
  a_4 &= \alpha_4 \\
  \vdots \\
  a_N &= \alpha_N \\
  b_2 &= \beta_2 \\
  b_3 &= \alpha_3 \delta + \beta_3 \\
  b_4 &= \alpha_4 \delta + \beta_4 \\
  b_5 &= \alpha_5 \delta + \beta_5 \\
  c_1 &= \gamma_1 \\
  c_2 &= \beta_2 \delta + \gamma_2 \\
  c_3 &= \alpha_3 \epsilon + \beta_3 \delta + \gamma_3 \\
  c_4 &= \alpha_4 \epsilon + \beta_4 \delta + \gamma_4 \\
  d_1 &= \gamma_1 \delta \\
  d_2 &= \beta_2 \epsilon + \gamma_2 \delta \\
  d_3 &= \beta_3 \epsilon + \gamma_3 \delta \\
  e_1 &= \gamma_1 \epsilon \\
  e_2 &= \gamma_2 \epsilon \\
  e_3 &= \gamma_3 \epsilon \\
  \text{etc.} & \quad \text{etc.}
\end{align*}
\]

Inverting these expressions, we find the following recursive formulae for the elements of L and U in terms of the elements of A.

\[
\begin{align*}
  \gamma_1 &= c_1 \\
  \delta_1 &= d_1 / \gamma_1 \\
  \epsilon_2 &= e_1 / \gamma_1 \\
  \beta_2 &= b_2 \\
  \gamma_2 &= c_2 - \beta_2 \delta_1 \\
  \delta_j &= (d_j - \beta_j \epsilon_{j-1}) / \gamma_j \\
  \epsilon_j &= e_j / \gamma_j \\
  \alpha_j &= a_j \\
  \beta_j &= b_j - \alpha_j \delta_{j-2} \\
  \gamma_j &= c_j - \beta_j \delta_{j-1} - \alpha_j \epsilon_{j-2}
\end{align*}
\]

\( \text{for } 2 \leq j \leq N-1 \)

\( \text{for } 2 \leq j \leq N-2 \)

\( \text{for } 3 \leq j \leq N \)

\( \text{for } 3 \leq j \leq N \)

\( \text{for } 3 \leq j \leq N \)
If the elements of $A$ are invariant with respect to time, the quantities $\alpha_j$, $\beta_j$, $\gamma_j$, $\delta_j$ and $\epsilon_j$ need only be calculated once. They must be recalculated whenever Reynolds number, grid spacing $h$, or time step $\Delta t$ is changed.

Next, consider the forward substitution phase $L\lambda = f$

\[
\begin{bmatrix}
\gamma_1 \\
\beta_2 & \gamma_2 \\
\alpha_3 & \beta_3 & \gamma_3 \\
\vdots & \vdots & \vdots \\
\alpha_N & \beta_N & \gamma_N \\
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\vdots \\
\lambda \\
\end{bmatrix}
=
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_N \\
\end{bmatrix}
\]

Direct multiplication yields time relations:

\[
\lambda_1 = \frac{f_1}{\gamma_1}
\]
\[
\lambda_2 = \frac{(f_2 - \beta_2\lambda_1)}{\gamma_2}
\]
\[
\lambda_j = \frac{(f_j - \beta_j\lambda_{j-1} - \alpha_j\lambda_{j-2})}{\gamma_j}, \quad 3 \leq j \leq N
\]

Finally, consider the back substitution phase $UX = \phi$

\[
\begin{bmatrix}
1 & \delta_1 & \epsilon_1 \\
1 & \delta_2 & \epsilon_2 \\
1 & \delta_3 & \epsilon_3 \\
\vdots & \vdots & \vdots \\
1 & \delta_{N-1} & \epsilon_{N-2} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_N \\
\end{bmatrix}
=
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\vdots \\
\lambda_N \\
\end{bmatrix}
\]
\[ x_N = \lambda_N \]
\[ x_{N-1} = \lambda_{N-1} - \delta_{N-1} x_N \]
\[ x_j = \lambda_j - \delta_j x_{j+1} - \epsilon_j x_{j+2} \quad j = N-2, N-3, \ldots, 1 \]

Now consider a normalized form of the difference equations for some \( m \).

\[
A_m(j-2) + \begin{bmatrix} \tilde{p}_m^{-} \\ \tilde{\omega}_m^{-} \end{bmatrix} A_m(j-1) + \begin{bmatrix} \tilde{q}_m^{-} \\ \tilde{p}_m^{+} \end{bmatrix} A_m(j) + \begin{bmatrix} \tilde{q}_m^{+} \\ \tilde{\omega}_m^{+} \end{bmatrix} A_m(j+1) \\
+ \begin{bmatrix} \tilde{n}_m^{+} \end{bmatrix} A_m(j+2) = f_m(j)
\]

Specializing to the parallel plate problem we find

\[ \tilde{p}_m^{-} = \tilde{p}_m^{+} \quad \text{and} \quad \tilde{n}_m^{+} = 1 \]

since all odd derivatives fail to appear in the equations for this particular geometry.

Also, the coefficients are independent of position in this case. The boundary conditions at the wall are

\[
A_m(j=0) = \frac{\partial A_m}{\partial y}(j=0) = 0
\]

Writing the difference equation for \( j = 1 \) we find

\[
A_m(-1) + \begin{bmatrix} p_m \end{bmatrix} A_m(0) + \begin{bmatrix} \omega_m \end{bmatrix} A_m(1) + \begin{bmatrix} p_m \end{bmatrix} A_m(2) + A_m(3) = f_m(1)
\]
But \( A_m(0) = 0 \) and \( A_m(-1) = A_m(+1) \)

\[
\begin{bmatrix}
\tilde{Q}_m + 1 \\
\tilde{P}_m
\end{bmatrix}
A_m(1) + 
\begin{bmatrix}
\tilde{P}_m
\end{bmatrix}
A_m(2) + A_m(3) = f_m(1)
\]

Hence \( c_1 = \tilde{Q}_m + 1 \)

The boundary conditions at the centerline are

\[
\begin{bmatrix}
\frac{\partial A_m}{\partial y}
\end{bmatrix}_{j=N} = \begin{bmatrix}
\frac{\partial^3 A_m}{\partial y^3}
\end{bmatrix}_{j=N} = 0 \text{ if } m \text{ is odd}
\]

or

\[
A_m(j=N) = \begin{bmatrix}
\frac{\partial^2 A_m}{\partial y^2}
\end{bmatrix}_{j=N} = 0 \text{ if } m \text{ is even}
\]

Write out the difference equation for \( j = N \) we have

\[
A_m(N-2) + \begin{bmatrix}
\tilde{P}_m
\end{bmatrix} A_m(N-1) + \begin{bmatrix}
\tilde{Q}_m
\end{bmatrix} A_m(N) + \begin{bmatrix}
\tilde{P}_m
\end{bmatrix} A_m(N+1) + A_m(N+2) = f_m(N)
\]

For \( m \) odd: \( A_m(N-2) = A_m(N+2) \) \& \( A_m(N-1) = A_m(N+1) \)

\[
\begin{bmatrix}
2
\end{bmatrix} A_m(N-2) + \begin{bmatrix}
2\tilde{P}_m
\end{bmatrix} A_m(N-1) + \begin{bmatrix}
\tilde{Q}_m
\end{bmatrix} A_m(N) = f_m(N)
\]

For \( m \) even: \( A_m(N) = 0, \) \( A_m(N-1) = -A_m(N+1), \) and \( A_m(N-2) = -A_m(N+2) \)

Thus the equation vanishes.
Next write the difference equations for $j = N-1$,

$$A_{m(N-3)} + \left[ \tilde{p}_m \right] A_{m(N-2)} + \left[ \tilde{q}_m \right] A_{m(N-1)} + \left[ \tilde{p}_m \right] A_{m(N)} + A_{m(N+1)} = f_{m(N-1)}$$

Then,

for $m$ odd:

$$A_{m(N-3)} + \left[ \tilde{p}_m \right] A_{m(N-2)} + \left[ \tilde{q}_{m+1} \right] A_{m(N-1)} + \left[ \tilde{p}_m \right] A_{m(N)} = f_{m(N-1)}$$

Thus, $c_{N-1} = \tilde{q}_m + 1$

for $m$ even:

$$A_{m(N-3)} + \left[ \tilde{p}_m \right] A_{m(N-2)} + \left[ \tilde{q}_{m-1} \right] A_{m(N-1)} = f_{m(N-1)}$$

Thus, $b_{N-1} = \tilde{q}_m - 1$

Summarizing the effect of the boundary conditions on the solution algorithm we have:

At wall:

$$c_1 = \tilde{q}_m + 1$$

At centerline:

$m$ odd:

$$a_N = 2, b_N = 2\tilde{p}_m, c_N = \tilde{q}_m, c_{N-1} = \tilde{q}_m + 1$$

$m$ even:

$$a_N = b_N = c_N = 0, c_{N-1} = \tilde{q}_m - 1$$
All other coefficients are:

\[ a_j = e_j = 1 \]
\[ b_j = d_j = \tilde{p}_m \]
\[ c_j = \tilde{q}_m \]

\( \tilde{p}_m \) and \( \tilde{q}_m \) are independent of position

The forward and back substitution calculations require 8N computer operations (multiplication or addition). Notice that by normalizing the equations such that \( a_j = e_j = 1 \), we save 3N operations in the factorization phase and N operations during the forward substitution phase.

For the Poiseuille flow case, the algorithm cannot be simplified except for normalization to make \( \tilde{N}_m = 1 \). This will reduce the number of computer operations slightly since all of the \( \alpha_j \)'s become unity. The coefficients all depend on position. The wall boundary condition is the same as the parallel plate case, i.e.,

\[ c_1 = \tilde{q}_{(j=1)} + 1. \]

To obtain the boundary condition near the pipe centerline we write for \( j = N-1 \)

\[ A_m(N-3) + \left[ \tilde{p}_m(N-1) \right] A_m(N-2) + \left[ \tilde{q}_m(N-1) \right] A_m(N-1) + \left[ \tilde{p}_m(N-1) \right] A_m(N) \]
\[ + \left[ \tilde{N}_m(N-1) \right] A_m(N+1) = f_m(N-1) \]

We require, \( A_m(N) = \left[ \frac{\partial A_m}{\partial y} \right]_{j=N} = 0 \)

or \( A_m(N) = 0, \ A_m(N+1) = A_m(N-1) \)
which leads to

\[ A_{m(N-3)} + \left[ \tilde{f}_{m(N-1)} \right] A_{m(N-2)} + \left[ \tilde{q}_{m(N-1)} + \tilde{n}_{m(N-1)}^+ \right] A_{m(N-1)} = f_{m(N-1)} \]

Thus

\[ c_{N-1} = \tilde{q}_{m(N-1)} + \tilde{n}_{m(N-1)}^+ \]
APPENDIX VI

In this appendix, we obtain the relationship between the complex eigenfunctions \( \phi(y) \) associated with traditional linear theory of hydrodynamic stability and the coefficients, \( A(y,t) \) and \( B(y,t) \), of the harmonic components used in this work.

Normally, the disturbance stream function is written as

\[
\psi(x,y,t) = \phi(y)e^{i\alpha(x-ct)}
\]

where \( \alpha \) is real and \( c \) is complex.

We wish to change this expression around such that the real part has form

\[
\psi(x,y,t) = A(y,t) \cos \alpha x + B(y,t) \sin \alpha x
\]

Therefore, we write

\[
\psi = \phi e^{i\alpha x} e^{-i\alpha r t} e^{\alpha c t}
\]

\[
\psi = e^{\alpha c t} \left[ \begin{array}{c} \phi_r + i\phi_i \\ \phi_r - i\phi_i \end{array} \right] \left[ \begin{array}{c} \cos \alpha x + i \sin \alpha x \\ \cos \alpha r t + i \sin \alpha r t \end{array} \right]
\]

Let \( \xi_1 = \phi_r \cos \alpha x - \phi_i \sin \alpha x \)

\( \xi_2 = \phi_i \cos \alpha x + \phi_r \sin \alpha x \)

Then,

\[
\psi = e^{\alpha c t} \left[ \xi_1 + i\xi_2 \right] \left[ \cos \alpha r t + i \sin \alpha r t \right]
\]
\[ \psi = e^{\alpha c_1 t} \left[ (\xi_1 \cos \alpha c_r t - \xi_2 \sin \alpha c_r t) \\
+ i(\xi_2 \cos \alpha c_r t + \xi_1 \sin \alpha c_r t) \right] \]

Thus,
\[ \psi = Q_1 + iQ_2 \]

where,
\[ Q_1 = e^{\alpha c_1 t} \left[ (\phi_r \cos \alpha x - \phi_1 \sin \alpha x) \cos \alpha c_r t \\
- (\phi_1 \cos \alpha x + \phi_r \sin \alpha x) \sin \alpha c_r t \right] \]

or on rearrangement,
\[ Q_1 = e^{\alpha c_1 t} \left[ (\phi_r \cos \alpha c_r t - \phi_1 \sin \alpha c_r t) \cos \alpha \\
- (\phi_1 \cos \alpha c_r t + \phi_r \sin \alpha c_r t) \sin \alpha \right] \]

The desired expressions are therefore found to be
\[ A(y,t) = (\phi_r \cos \alpha c_r t - \phi_1 \sin \alpha c_r t) e^{\alpha c_1 t} \]
\[ B(y,t) = - (\phi_1 \cos \alpha c_r t + \phi_r \sin \alpha c_r t) e^{\alpha c_1 t} . \]

This correspondence holds for linear theory only.