EADS, William Donovan, 1944-
FAULT ANALYSIS OF LINEAR TIME-INVARIANT NETWORKS.

Rice University, Ph.D., 1970
Engineering, electrical

University Microfilms, A XEROX Company, Ann Arbor, Michigan
RICE UNIVERSITY

FAULT ANALYSIS OF LINEAR
TIME-ININVARIANT NETWORKS

by

William Donovan Eads

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
ELECTRICAL ENGINEERING

Thesis Director's signature:

Houston, Texas

May 1970
FAULT ANALYSIS OF LINEAR TIME-INVARIANT NETWORKS

ABSTRACT
William D. Eads

Fault analysis is the study of the behavior of a system when its elements change value in some arbitrary and unknown manner from their nominal values. In this work results are derived for linear time-invariant networks containing lumped and distributed two-terminal elements. When input-output terminals have been specified, a given fault may or may not be distinguishable from some other fault. Single elements whose faults cause identical network response may be placed in similarity fault sets (SFS), or—for pairwise failures—in dual similarity fault sets (DSFS). Necessary and sufficient conditions for membership in a similarity fault set are presented; necessary conditions for elements of dual similarity fault sets are given. An efficient algorithm, requiring little more than the nominal frequency domain network solution, is given for the determination of the similarity fault set of a network.

Methods for isolating faults, using graphical or computational techniques, are given for both single and simultaneous dual faults. Each method requires a minimum number of frequency measurements of a minimal set of matrix transfer function elements in order to determine from which similarity fault set a fault has occurred. Consideration of measurement error is included in several of these methods. As an example of the methods of fault analysis introduced, a \( \mu A709 \) integrated
operational amplifier is modeled, and it is shown that a fault in any element of the model presented can be isolated.

Finally, the concept of local parameter identification is introduced, and conditions for local uniqueness of element values of a network, given input-output measurements, are found. This concept is useful if one is interested in isolating several simultaneous failures.
ACKNOWLEDGEMENTS

The author wishes to express his appreciation to his thesis director, Professor J. V. Leeds, for his guidance and many helpful suggestions. Dr. C.S. Burrus has been actively interested in the author's work on this thesis and has served on the thesis committee; for these the author is especially grateful. Professor R.D. Young also served on the thesis committee; his reading and review of this work are appreciated.

Thanks are due Dr. E.A. Feustel, whose conversations provided the author with many helpful suggestions on the thesis preparation.

The author is indebted to Mr. Charles Moore who aided in the derivation of several important results. Thanks are due Dr. Vidojko Ciric for his friendship and helpful confidence in the author's ability. Dr. M.L. Fontenot's encouragement and helpful advice are also greatly appreciated.

Thanks go to NSF for its support of 1966-1969 in the form of an NSF fellowship.

This thesis is dedicated to my wife, Anna Jo, whose patience and encouragement have made the effort worthwhile. Her careful editing and typing of the manuscript have been a special help to the author, and for these he is especially grateful.
## Table of Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td></td>
<td>Acknowledgements</td>
<td>iii</td>
</tr>
<tr>
<td>I</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>Statement of the Problem</td>
<td>7</td>
</tr>
<tr>
<td>III</td>
<td>Theoretical Derivations</td>
<td>14</td>
</tr>
<tr>
<td>IV</td>
<td>Fault Analysis of Networks with Distributed Elements</td>
<td>40</td>
</tr>
<tr>
<td>V</td>
<td>Computer Implementation of Theoretical Results</td>
<td>48</td>
</tr>
<tr>
<td>VI</td>
<td>An Example of Fault Isolation</td>
<td>60</td>
</tr>
<tr>
<td>VII</td>
<td>Component Parameter Identification</td>
<td>84</td>
</tr>
<tr>
<td>VIII</td>
<td>Conclusions</td>
<td>93</td>
</tr>
<tr>
<td></td>
<td>Appendix I</td>
<td>97</td>
</tr>
<tr>
<td></td>
<td>Appendix II</td>
<td>99</td>
</tr>
<tr>
<td></td>
<td>Appendix III</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>Appendix IV</td>
<td>102</td>
</tr>
<tr>
<td></td>
<td>References</td>
<td>105</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

The basic problem of interest and a discussion of the rationale for studying fault analysis are presented; historical development of this field, as related to the new material presented, is included.

1.1 Area of Interest and Its Justification

To detect and isolate faults in electrical networks, one would like to apply inputs to the network from a limited and well-defined set of points, to obtain measurement of only a specified set of output variables, and to determine from these measurements which network elements have failed. Why should one be limited to certain inputs and outputs? If voltages and currents could be measured arbitrarily within a given network, it would be a trivial task to determine the variables associated with each network element and to specify its value using only the knowledge of the relationship of voltage and current in that type element and performing simple arithmetic. Then the calculated value of each element is compared with the value it should have (the "nominal" value). Such a simple solution answers the question of element failure, if each network variable is measurable. However, in many physical networks of interest, the user is unable to arbitrarily excite points of interest and to measure arbitrary network variables.

Examples of these conditions are numerous. Any system physically isolated from persons who must measure its properties qualifies: orbiting satellites, remote or inaccessible missile systems, and under-
water telephone repeaters are examples of systems for which it is not feasible to remove and test individual components. Another group of practically inaccessible networks is that of integrated circuits, even before encapsulation (enclosure in a nonremovable container of all but the user-available terminals). Due to extremely small size of elements and difficulty of accurately placing probes in integrated circuits, only a few points within a circuit can be excited and/or measured. The growing number of increasingly complex integrated circuits is our prime motivation for isolating faults by external measurements. One may be interested not only in a method for isolating faults but also in the a priori determination of whether the cause of the network failure can be uniquely determined. Both these areas of interest are studied for the cases of single or multiple faults.

Although many circuits of interest are digital in nature, this study is restricted to linear networks. The value of a unified, well-defined approach to the isolation of faults in digital networks has resulted in extensive studies of their properties. However, since the development of linear integrated circuits has been somewhat slower than that of digital integrated circuits, and because these linear networks have until recently been relatively simple, theoretical research in isolating faults in linear networks has been meager.

1.2 Scope of Study of Fault Analysis

The networks under consideration have several properties of interest. First, and most important, it is assumed that the properties
of the nominal networks are completely known. That is, the network
graph, input-output terminals, and all element types, are known. In
addition, the value of each of these elements before a fault occurs is
known. Second, it is assumed that the system is linear and time-invar-
ant, and that no energy is stored in the network prior to the application
of excitations to the input terminals. The time-invariance assumption
specifically excludes intermittent failures, while linearity restricts
failures to those which do not bias active elements in a nonlinear
region. Thus, a network whose output is clipped or contains self-sus-
tained oscillations is not included in this study. The restrictions of
linearity and time-invariance allow a frequency domain solution of the
network. Two-terminal lumped elements—including resistors, capaci-
tors, inductors, mutual inductors, mutual capacitors, and each of the
four basic dependent sources—are allowed, so that a state-space
characterization of the network is possible.

Finally, faults are restricted to arbitrary changes in the values
of any elements of the network. Topology changes—except for open or
short-circuited elements causing the removal of a branch or the
coalescing of two nodes of the graph respectively—are specifically
excluded.

1.3 Previous Research in Fault Analysis

Little theoretical research has been done in the area of fault
analysis in linear networks. The first and most definitive research
in this field was that of Seshu and Waxman [1], who have considered only single faults for networks containing single inputs and single outputs. A suggested method of fault isolating includes the generation of an analytic expression of the transfer function in terms of each element and frequency, a formidable task unless the network under consideration is quite small. The analytic expression is evaluated using the nominal value of all but a single element, whose value changes by some pre-determined percentage. The magnitude of the transfer function is then calculated for each of several pre-selected frequencies. A mapping of the percentage change of the transfer function at a given frequency to the digits 0-9, and the concatenation of the digits corresponding to transfer function change at each of the chosen frequencies, provide a number which is paired with the changing element and its value. The above procedure is repeated for several changes of the element under consideration, and the entire process is repeated for each element of the network. The resulting set of numbers—each paired with its defining element and the value of that element—is then placed in a table in ascending order. To use this table to isolate element faults, one performs a frequency response of the failed network at the specified frequencies, transforms the change in magnitude of the response to one of the digits, and looks up this number in the table. The fault is then assumed to be the element paired with the number in the table.

One advantage of this method is the simplicity of the network measurements and analysis. Measurement accuracy may be poor,
since one is interested in determining only which of ten possible regions of transfer function change actually occurred. When measurements have been analyzed, those elements whose faults could have caused the network failure are usually known. Disadvantages include the fact that the transfer function is generated explicitly, a difficult and time-consuming process. To avoid immense tables of numbers, only a few changes in each element value can be allowed. Often many changes of an element value are necessary for a high probability of isolating a fault in that element. It is quite possible that no table entry exists for many single faults. The method of constructing the table from single digits severely limits one's ability to take advantage of accurate measuring instruments for which thousands (or, including phase measurements, millions) of different transfer function changes can be observed. Similarly, the table method is not easily extended to include several inputs and/or outputs. Finally, the proposed method is entirely heuristic; no general statements can be made about uniqueness of fault isolation, and interesting properties of changes in network functions due to changes in element values are ignored.

Following Seshu and Waxman, several similar works by Stahl, Maenpaa, and Stehman [2, 3] and others [4-8] have been primarily directed toward the development of efficient methods of forming the transfer function in terms of each element and toward the formation of the tables suggested by Seshu and Waxman. Reports that networks to be solved must be partitioned into single input, single output subnet-
works of about 20 elements or less—in order to obtain reasonable com-
putation time [3]—indicate that only partial success has been obtained
in constructing computer algorithms for fault analysis.

1.4 Presentation of New Material

In Chapter II, original definitions are introduced for a concise
statement of problems attacked. The question of uniqueness in the
detection of faults for the cases of single and simultaneous dual faults
is discussed in Chapter III, and conditions for uniqueness are derived.
Methods of determining which element(s) have failed are also intro-
duced. In Chapter IV the methods of determining uniqueness and iso-
lating faults are extended to networks containing two-terminal subnet-
works and distributed elements. The ease of determining uniqueness
of faults and of isolating these faults by the use of a computer, which
needs do little more than perform a circuit analysis, is discussed in
Chapter V. Chapter VI includes an example modeled from a commer-
cially available μA709 integrated circuit amplifier. Uniqueness of
isolation of single faults is demonstrated, and the properties of the
system, including the effect of measurement noise, is shown. Finally,
Chapter VII introduces a concept of parameter identification which is
useful for fault analysis; conditions for identifiability of all network
elements are derived.
II. STATEMENT OF THE PROBLEM

Important terms are defined. A precise statement of the problems to be solved is given.

2.1 Important Definitions

Let us consider a linear, time-invariant network \( N \), containing lumped elements only. Two terminal elements with the following relational operators are permitted:

\[
\begin{align*}
  f(t) &= Kg(t) \\
  f(t) &= K \int_{-\infty}^{t} g(u)du \\
  \text{or } f(t) &= K \frac{dg(t)}{dt}
\end{align*}
\]

where \( t \) is the independent variable,

\( f(t) \) is the voltage or current corresponding to the element under consideration,

\( g(t) \) is some other network voltage or current, and \( K \) is the element value under consideration.

It is noted that resistors, capacitors, and inductors result from equations (1a), (1b), and (1c) respectively when \( f \) represents the element voltage and \( g \) represents the current through the same element. The vector of network elements, ordered in an arbitrary but unchanging manner, is called the element vector \( \mathbf{P} \) and is of length \( \mathbf{N} \). The topology of \( N \), which includes the graph and element types, is assumed fixed, but the element vector may vary. The notation "\( N(P) \)" is
therefore used to denote the network with a particular element vector \( \mathbf{P} \).

The set of inputs available to the user of the network is denoted \( \mathcal{U} \); available outputs are the contents of the set \( \mathcal{Y} \). A useful description of the network and one which makes maximal use of the available terminals is the port description which pairs a common datum available terminal with each of the other available terminals. Each pair is then used for input and output so that \( \mathcal{Y} = \mathcal{U} \).

A network of the above form may be described by a set of linear ordinary time-invariant differential equations:

\[
\dot{x} = Ax + Bu \\
y = Cx + D_0u + \sum_{i=1}^{n} D_iu^{(i)}
\]

In the above equations, the following conventions are assumed:

- \( x \) — state vector
- \( u \) — input vector, length \( k \)
- \( u^{(i)} \) — \( i \)th derivative of input vector
- \( y \) — output vector, length \( h \).

For our network description, \( u \) is the ordered set \( \mathcal{U} \) of inputs; similarly, \( y \) is the set \( \mathcal{Y} \) ordered in some manner. Derivatives of inputs are necessary only when the system is improper. That is, the response to an impulse, \( \delta(t) \), at some input contains doublets \( (\delta^{(1)}) \) or higher order singularities. The upper limit, \( n \), of the sum in equation (2) corresponds to a highest order output of \( \delta^{(n)} \) in
the impulse response.

In order to precisely state the problems to be attacked, the following definitions are made.

Definition: The network solution \( S(N(P)) \) is the set of pairs of the form \( \{y(t), u(t): t \in (-\infty, \infty)\} \) where \( y \) is the response of \( N(P) \) due to the input \( u \), and where \( u \) is taken over all inputs for which outputs exist.

Definition: The nominal network is the network \( N \) whose element vector \( P \) is fixed at its nominal value \( P_o \), and is denoted \( N(P_o) \).

Definition: The nominal network solution is defined as the network solution \( S(N(P_o)) \).

The input-output behavior of \( N(P_o) \) is therefore completely specified by the nominal network solution. For networks of identical topology but different element values, the nominal network and its solution are incomplete, while the network solution for all sets of \( P \) is clumsy and large, motivating the following definitions:

Definition: A single fault is the unique element of \( P \) whose value differs from the corresponding element of \( P_o \).

If \( P \) and \( P_o \) do not differ in exactly one element, then there is no single fault associated with \( P \).
Definition: An **observable single fault** is a single fault of P for which
\[ S(N(P)) \neq S(N(P_0)). \]  
(3)

Multiple faults and observable multiple faults are similarly defined.

Definition: A **uniquely observable single fault** is an observable fault of P in which the network solution of N(P) differs from that of the network solution due to any other single fault.

Uniquely observable multiple faults are similarly defined.

In order to provide a method of handling the non-uniquely observable faults, collections of these faults can be considered as single entities if one is interested in the properties of N as faults occur:

Definition: A **similarity fault set** (SFS) is a set \( A = \{ a_1, \ldots, a_r \} \) of elements of N such that the following relationship holds:

Given a single fault \( a_i \in A \), and the element vector \( P_i \) of which its value is an element, then for each other element \( a_j \) of A, there is a corresponding element vector \( P_j \), for which \( a_j \) is the single fault, such that:

\[ S(N(P_i)) = S(N(P_j)). \]  
(4)

The above definition constitutes one basic topic of this thesis:

Which sets of elements are indiscernible as to the manner in which their values individually affect the network, as described from input-
output relationships? Those elements which are not discernible are grouped in a similarity fault set.

Remark: All faults which are not observable are members of the same SFS, \( F_{NO} \). This statement is a direct result of the definition of a SFS.

Definition: The collection of all similarity fault sets of \( N \) is called the \textbf{Fault Grouping} of \( N \).

Lemma: Each element of \( P \) belongs to exactly one SFS.

Proof: Case 1: If an element belongs to no SFS, then its effect upon the network solution of \( N(P) \) is unique; hence, it belongs to a SFS containing itself. This is a contradiction.

Case 2: If an element \( p \) belongs to two fault sets, say \( F_1 \) and \( F_2 \), \( F_1 \neq F_2 \), then at least one set must contain a second element. Assume without loss of generality that \( p^* \in F_1 \), \( p^* \neq p \); then by definition of SFS, \( p^* \in F_2 \). This must hold for all \( p^* \in F_1 \).

Thus \( F_1 \subseteq F_2 \). Similar statements must hold with the roles of \( F_1 \) and \( F_2 \) reversed. Thus \( F_2 \subseteq F_1 \); hence, \( F_1 = F_2 \), and a contradiction again results. Q.E.D.

The preceding lemma states that the similarity fault sets partition the set of all elements. This partition is useful in the respect that changes in the network solution caused by changes in any element of a particular SFS can be represented by the change in the solution.
caused by any one representative element of the SFS. The dimensionality of the set which must be considered when one attempts to locate failures in a linear time-invariant lumped network may be decreased without loss of information available from input-output relationships.

2.2 Statement of Problems

The first question to which this thesis is addressed is the determination of a systematic and analytic method for partitioning the set of elements of a given lumped linear time-invariant (LLTI) network into similarity fault sets. Once this has been done, a method of actually isolating the faulty network element must be found. At this point, the use of frequency domain analysis becomes helpful. It is known that [9] that the input-output relationships of linear time-invariant systems are completely characterized by the frequency response of the network. Because of the ease in analyzing frequency responses in LLTI networks, because of the ease in measuring frequency responses, and because of the completeness of the frequency domain solution in the above sense, the remainder of this thesis assumes frequency domain solutions of the network. How, then, can one make a minimum number of measurements at \( Y \), with sinusoids applied successively at \( U \) and be assured of identifying the proper SFS, when it is known that a single fault has occurred?

An extension of the above objectives is the determination of
similarity fault sets for multiple failures of elements; specifically, double failures are of interest. Again, we would like to be able to make input-output measurements at only a minimum number of frequencies in order to be assured of uniquely identifying the two elements of the network which have failed.

Of practical interest is the question of the effect of measurement noise upon our ability to distinguish faults of elements of $\mathbf{N}$. Do additional frequency measurements help us isolate faults?

Finally, the question of component parameter identification—the ability to uniquely specify all the element values of a network given the frequency response characterization from available input-output terminals—must be discussed. This is equivalent to the problem of determining new element values when all elements change from their nominal value.
III. THEORETICAL DERIVATIONS

Mathematical procedures for determining single and dual similarity fault sets of $N$ are given in this chapter. The number of frequency measurements necessary for fault isolation, as well as algorithms for using these measurements are presented and discussed.

3.1 Dependence of Transfer Function Upon Element Value Change

Let us consider a linear time-invariant RLC active network $N$ with mutual inductance and capacitance. Its state-space representation may be written in general as:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + D_0 u + \sum_{i=1}^{n} D_i u^{(i)}
\end{align*}
\]

(1a) \hspace{1cm} (1b)

where $x$ is the vector of states, $u$ is the $k$-vector of inputs, and $y$ is the $h$-vector of outputs. Superscript "$(i)$" represents the $i^{th}$ derivative with respect to time. The nominal element vector $P_0$ is assumed.

Transforming the equations (1a, 1b) to the frequency domain, letting:

\[
D(s) = \sum_{i=0}^{n} D_i s^i ,
\]

(2)

we obtain the matrix input-output relationship:
\[ \mathbf{Y}(s) = \left[ \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}(s) \right] \mathbf{U}(s), \quad (3) \]

Equation (3) is written as:

\[ \mathbf{Y}(s) = \mathbf{T}_o(s) \mathbf{U}(s) \quad (4) \]

where \( Y \) is an \( h \times 1 \) vector, \( U \) is a \( k \times 1 \) vector and \( T_o \) is an \( h \times k \) matrix transfer function. Measurements on \( N \) can provide the value of an \( T_o \) at an arbitrary frequency \( s = j\omega \).

If \( P \) is allowed to vary from \( P_o \) in \( N \), equations (1), (2), (3), and (4) no longer represent the network. We may, however, augment equation (1) by introducing fictitious ports for each element of \( N \) which is allowed to vary. Rewriting (II.1), we have the general element relation expression:

\[ f_e(t) = K_o \cdot L(g(t)) \quad (5) \]

where \( K_o \) is the nominal value of element \( K \), \( L \) is the linear integral, differential or constant operator, and \( f_e \) and \( g \) are element voltages and/or currents. As \( K \) varies from \( K_o \), we write (5) as a branch equation (including more than one element):

\[ f_b(t) = (K_o + \Delta K) L(g(t)) \quad (6) \]

or using equation (5) :
\[ f_b(t) = f_e(t) + \frac{\Delta K}{K_0} f_e(t) \quad (7a) \]

Defining \( \Delta f_b \) as the fictitious port input and \( f_e(t) \) as the fictitious port output of \( N \):

\[ \Delta f_b(t) = \Delta \frac{K}{K_0} f_e(t) \quad (7b) \]

hence \( f_b = f_e + \Delta f_b \quad (7c) \)

we see that: 1) the port input is added to \( f_e \) to obtain the branch relationship required; 2) the element connected to the port represented by equation (7b) is a dependent source with no dynamics, i.e., it does not differentiate or integrate its input; 3) the value of the port element is the percentage change in the element value of the non-augmented network; and 4) the port output is the voltage or current, \( f_e \), of the element whose value changes. Equations (7a) and (7b) are shown schematically in Figure 1.

(a) \hspace{2cm} (b)

\[ \Delta v_b = \Delta \frac{K}{K_0} v_e \]

\[ \Delta i_b = \Delta \frac{K}{K_0} i_e \]

Figure 1
The network with changed element values may be written as:

\[ \dot{x} = Ax + Bu + B_a u_f \]  \hspace{1cm} (8a)

\[ y = Cx + \sum_{i=0}^{n} D_i u^{(i)} + D_a u_f \]  \hspace{1cm} (8b)

\[ y_f = C_f x + \sum_{i=0}^{n} D_{f_i} u^{(i)} + D_{af} u_f \]  \hspace{1cm} (8c)

where \( y_f \) and \( u_f \) are the vectors of augmented port output and input variables respectively. The length of each of these vectors is the number of element values allowed to change.

The relationship between port inputs and outputs is given for each element by equation (7b). In matrix form this relationship is:

\[ y_f = H u_f \]  \hspace{1cm} (9)

where the \( i^{th} \) diagonal term of the diagonal matrix \( H \) is the inverse of the percentage change of the \( i^{th} \) element value.

Transforming equations (8) and (9) and performing algebra, we obtain the overall transfer function:

\[ Y(s) = \left[ T_o(s) + R_1(s) Q^{-1}(s) R_2(s) \right] U(s) \]  \hspace{1cm} (10)

where:

\[ Q(s) = H - R_3(s) \]  \hspace{1cm} (11a)

\[ T_o(s) = C(sI - A)^{-1}B + D(s) \]  \hspace{1cm} (11b)

\[ R_1(s) = C(sI - A)^{-1}B_a + D_a \]  \hspace{1cm} (11c)
\[ R_2(s) = C_f(sI - A)^{-1}B + D_f(s) \]  \hspace{1cm} (11d)

and
\[ R_3(s) = C_f(sI - A)^{-1}B_a + D_{af} \]  \hspace{1cm} (11e)

If all elements are allowed to vary, then equation (10) is an exact statement of the dependence of the matrix transfer function upon frequency and all parameters. If a subset of the parameters change value, then only the proper columns of \( R_1 \), rows of \( R_2 \) and rows and columns of \( H \), \( R_3 \), and \( Q \) are retained to calculate the matrix transfer function:

\[ T(s) = T_0(s) + R_1(s)Q^{-1}(s) R_2(s) \]  \hspace{1cm} (12)

3.2 Conditions for Determination of Similarity Fault Sets

In order to find the similarity fault sets of \( N \), we need \( T(s) \) as a function of a single element only. Manipulation of equations (10) and (11) as the single element \( p_i \) changing from its nominal value gives:

\[ \Delta T(s) = \frac{R_1^i(s)R_2^i(s) \Delta p_i/p_i}{1 - R_3^i(s) \Delta p_i/p_i} \]  \hspace{1cm} (13)

where \( R_1^i \), \( R_2^i \), and \( R_3^i \) are respectively column, row, and scalar functions of frequency.

The above is a restatement of the bilinear function theorem [17] and provides the basis for the partitioning of the vector of elements \( \Delta P \).
into single fault sets. It is noted that $\frac{R_1^i(s) R_2^i(s)}{P_i}$ is a well-known function of the parameter changing. Indeed:

$$S_i^i(s) = \frac{\partial T(s)}{\partial p_i} \bigg|_{p_0} = \frac{R_1^i(s) R_2^i(s)}{p_i}$$  \hspace{1cm} (14)$$

where $S_i^i$ is called the sensitivity transfer function matrix with respect to element $p_i$. In a similar manner $R_3^i(s)$ is a scalar function of the element which changes, and we may write $r(i, s) = \frac{R_3^i(s)}{P_i}$ for $i^{th}$ element varying. Thus:

$$\Delta T(s) = \frac{S_i^i(s) \Delta p_i}{1 - r(i, s) \Delta p_i}$$  \hspace{1cm} (15)$$

From the above description of $N$, the set of non-observable faults is obvious.

Lemma: The SFS of non-observable faults is exactly that set:

$$F_{NO} = \{ p_2 : p_1 \in P, S_i^i(jw) = 0 \text{ for all } w \}.$$  

Proof: From (15), $S_i^i(jw) = 0$ for all $w$

$$\iff \Delta T(jw) = 0 \text{ for all } w \text{ and } \Delta p_i$$

$$\iff T(jw) = T_0(jw) \text{ for all } w \text{ and } \Delta p_i$$

$$\iff p_i \text{ is not observable.} \hspace{1cm} \text{Q.E.D.}$$
Equation (15) is a complete description of the network behavior as single faults occur. Also, since \( \Delta p_i \) must be arbitrary, according to our definition of observable faults and SFS, knowledge of the value of \( \Delta p_i \) does not directly affect the SFS classification of that element. The only remaining variables on the right hand side of equation (15) are \( r(i, s) \) and the matrix \( S^i(s) \) for \( i = 1, 2, \ldots, \# \).

Partitioning the elements into the proper SFS is now a matter of pairwise comparisons of certain properties of \( S^i(s) \) and \( r(i, s) \) with \( S^k \) and \( r(k, s) \), for each pair of elements \( p_i \) and \( p_k \).

Which properties of \( S^i(s) \) and \( r(i, s) \) are important? In Appendix I it is proved that the locus of points of each element of the matrix \( \Delta T(jw) \) in the complex plane, as \( \Delta p_i \) varies from \( -\infty \) to \( \infty \) and for \( w \) held constant, is a circle. For certain frequencies and network elements, this circle may be degenerate, and the locus of an element of \( \Delta T \) may become a straight line, or even the point \( (0, 0) \).

For the imaginary part of \( r(i, jw) \), written as \( \text{Im}(r(i, jw)) \), not zero, the circular locus of \( \Delta T_{m,n}(jw) \) is centered at:

\[
c_{m,n}(i, w) = \frac{-j\sigma_{m,n}(jw)}{2 \text{Im}(r(i, jw))},
\]

(16)

where \( \Delta T_{m,n} \) is the \( m, n \) element of the matrix \( \Delta T \), and \( \sigma_{m,n}^i \) is the \( m, n \) element of the matrix \( S^i \). The radius of the circle is given by:

\[
c_{m,n}(i, w) = \frac{1}{2} \left| \frac{\sigma_{m,n}^i(jw)}{\text{Im}(r(i, jw))} \right|
\]

(17)
Therefore, we see that at any given frequency the set of loci of an element of $\Delta T(j \omega)$ as each element of $N$ varies individually from its nominal value are circles or lines, each passing through the origin, since $\Delta p_i = 0$ implies $\Delta T = 0$. (See Figure 2.)

![Figure 2](image)

Let us consider the conditions that two elements are indistinguishable from input-output terminals. I.e., what conditions must be satisfied for two elements to belong to the same SFS? Let us define $C(i, \omega)$ as the matrix of centers $c_{m,n}(i, \omega)$.

Theorem III.1: A necessary condition for two elements of $N$, $p_i$ and $p_k$, to be members of the same SFS is that for all $\omega$:

(a) $C(i, \omega) = C(k, \omega)$

if $r(i, j \omega) \neq 0$ and $r(k, j \omega) \neq 0$

or

(b) $S^i(j \omega) = a(i, k)S^k(j \omega)$ otherwise.

Proof: Assuming $r(i, j \omega) \neq 0$ and $r(k, j \omega) \neq 0$, the loci of elements of $\Delta T(j \omega)$ are circles as $p_i$ or $p_k$ varies, with centers given by elements of $C(i, \omega)$ and $C(k, \omega)$ respectively. If $C(i, \omega_o) \neq$
C(k, \( w_o \)) for any \( w_o \), then for some \( m \) and \( n \),

\[
c_m, n(i, w_o) \neq c_m, n(k, w_o).
\]

But by elementary geometry it is known that circles with different centers can have no more than two points in common; hence, there must be a point on \( c_m, n(i, w_o) \) caused by a value of \( \Delta p_i \) which is not on the circular locus centered at \( c_m, n(k, w_o) \). Therefore, no value of \( \Delta p_k \) can cause the same response as that due to \( \Delta p_i \), and necessity of (a) is proved.

If either \( r(i, j w) = 0 \) or \( r(k, j w) = 0 \), the allowable loci of \( \Delta T \) due to that element are lines. Hence, the allowable loci for both elements must be lines. For incremental variations \( dp_i \) in \( p_i \) we may write:

\[
dT(j w) = S^i(j w) dp_i \tag{19}
\]

Similarly, incremental variations in \( p_k \) give:

\[
dT(j w) = S^k(j w) dp_k \tag{20}
\]

Equating \( dT(j w) \) caused by variations in each element, the result obtains:

\[
S^i(j w) dp_i = S^k(j w) dp_k \tag{21}
\]

or \( S^i(j w) = \alpha(i, k) S^k(j w) \tag{22} \)

where \( \alpha(i, k) = \frac{dp_k}{dp_i} \)
Thus necessity of (b) is proved. Q.E.D.

Usefulness of Theorem III.1 lies in the fact that if the loci of $\Delta T$ for variations in each pair of elements are unique, then each element lies in a different SFS.

Necessary but not sufficient conditions for membership in similarity fault sets have been presented. To obtain sufficient conditions, let us suppose that $p_i$ and $p_k$ are the elements whose distinguishability is under question. If they are indistinguishable, then for proper values of the change of each, the same matrix transfer function must result:

$$\Delta T(j\omega) = \frac{S^i(j\omega)\Delta p_i}{1 - r(i,j\omega)\Delta p_i} = \frac{S^k(j\omega)\Delta p_k}{1 - r(k,j\omega)\Delta p_k} \quad (23)$$

Using this equation, the following theorem can be proved:

**Theorem III.2**: Each SFS of a network $N$ is a collection of exactly those elements, $p_i$ and $p_k$, of $N$ for which the following relations hold:

(a) $S^i(j\omega) = \alpha(i,k)S^k(j\omega)$ \quad (24)

and

(b) $r(i,j\omega) = \alpha(i,k)r(k,j\omega) + g(i,k)$ \quad (25)

where $\alpha$ and $g$ are real scalars.

**Proof**: Necessity of (a) and (b):

From equation (23), observing the limit as $\Delta p_i$, $\Delta p_k \to 0$, it is seen that:
\[ S^i(jw) \frac{dp_i}{dp_k} = S^k(jw) \frac{dp_k}{dp_i} \quad (26) \]

or
\[ S^i(jw) = \frac{dp_k}{dp_i} S^k(jw) = \alpha(i,k) S^k(jw). \quad (27) \]

Substituting for \( S^i(jw) \) from equation (27) and solving equation (23) for \( \Delta p_k \) in terms of \( \Delta p_i \):

\[ \Delta p_k (1 - r(i,jw) \Delta p_i) = \alpha(i,k) \Delta p_i (1 - r(k,jw) \Delta p_k) \quad (28) \]

\[ \Delta p_k = \frac{\alpha(i,k) \Delta p_i}{1 - (r(i,jw) - r(k,jw) \cdot \alpha(i,k)) \Delta p_i} \quad (29) \]

Since \( \Delta p_k \) and \( \alpha(i,k) \) are independent of frequency, \( \delta(i,k) \not= \)
\( r(i,jw) - \alpha(i,k) r(k,jw) \) must also be frequency independent. Thus necessity is proved.

Sufficiency: Assuming conditions (a) and (b) hold, substitution of these conditions into the middle term of equation (23) gives:

\[ \Delta T(jw) = \frac{\alpha(i,k) S^k(jw) \Delta p_i}{1 - (\alpha(i,k) r(k,jw) + \delta(i,k)) \Delta p_i} \quad (30) \]

Substitution of

\[ \Delta p_i = \frac{\Delta p_k}{\alpha(i,k) + \delta(i,k) \Delta p_k} \quad (31) \]

in equation (30) demonstrates that there actually is a \( \Delta p_k \) for which \( \Delta T(jw) \) is the same as that caused by \( \Delta p_i \). Hence, \( p_i \) and \( p_k \)
are in the same SFS. Q.E.D.

Necessary and sufficient conditions have thus been derived for partitioning the elements into similarity fault sets.

The preceding theorem gives an indication of the simplicity in computing the fault grouping of similarity fault sets. More detailed computational aspects of determining the similarity fault sets will be found in Chapter V.

The ultimate simplicity in writing necessary and sufficient conditions for single failure indistinguishability leads one directly to the question of simultaneous multiple failures. Specifically, the problem of finding dual similarity fault sets (DSFS) for \( N \) is studied, and sufficient conditions for isolation of pairs of elements are given.

3.3 Conditions for Determination of Dual Similarity Fault Sets

Definition: A DSFS is a set of pairs of elements of \( N \) which satisfies the following relationship:

The pairs \((p_i, p_k)\) and \((p_m, p_n)\) are in the same DSFS if and only if an arbitrary change in the \( i^{th} \) and \( k^{th} \) elements of the nominal element vector \( P_o \) causes a change in the network solution which is indistinguishable from that caused by some changes in the \( m^{th} \) and \( n^{th} \) elements alone.

Manipulating equations (10) and (11a,c,d) when two elements \( p_i \) and \( p_k \) are allowed to vary, it is seen that \( Q(s), H, R_1(s) \) and
R₂(s) are respectively 2 x 2, 2 x 2, h x 2, and 2 x k matrices. Appendix II contains the matrix manipulations of equation (12) resulting in:

\[
\Delta T(j\omega) = \frac{S^i(j\omega)\Delta p_i + S^k(j\omega)\Delta p_k + W^{i,k}(j\omega)\Delta p_i \Delta p_k}{1 - r(i,j\omega)\Delta p_i - r(k,j\omega)\Delta p_k + f(i,k,j\omega)\Delta p_i \Delta p_k}
\] (32)

To simplify the notation, we omit the functional dependence upon \( \omega \) of \( \Delta T \), \( S \), \( W \), \( r \), and \( f \). Also, \( \Delta p_i \) and \( \Delta p_k \) will be denoted respectively as \( \delta_i \) and \( \delta_k \). Hence:

\[
\Delta T = \frac{S^i_i\delta_i + S^k_k\delta_k + W^{i,k}_{i,k}\delta_i \delta_k}{1 - r(i)\delta_i - r(k)\delta_k + f(i,k)\delta_i \delta_k}
\] (33)

Changes in two other elements \( p_m \) and \( p_n \) give a similar expression:

\[
\Delta T = \frac{S^m_m\delta_m + S^n_n\delta_n + W^{m,n}_{m,n}\delta_m \delta_n}{1 - r(m)\delta_m - r(n)\delta_n + f(m,n)\delta_m \delta_n}
\] (34)

For \((p_i, p_k)\) in the same DSFS as \((p_m, p_n)\), we know there must be some \((\delta_m, \delta_n)\) which causes the same change in transfer function as a given \((\delta_i, \delta_k)\) pair. Thus, equating (33) and (34), we have:

\[
\frac{S^i_i\delta_i + S^k_k\delta_k + W^{i,k}_{i,k}\delta_i \delta_k}{1 - r(i)\delta_i - r(k)\delta_k + f(i,k)\delta_i \delta_k} = \frac{S^m_m\delta_m + S^n_n\delta_n + W^{m,n}_{m,n}\delta_m \delta_n}{1 - r(m)\delta_m - r(n)\delta_n + f(m,n)\delta_m \delta_n}
\] (35)
From Appendix II, the denominators of equation (35) are rational functions of frequency. The zeroes of these denominators specify the eigenvalues of the changed network as pairs of elements change, while the poles of these denominators are the poles of the functions \( r(\cdot, s) \) and \( f(\cdot, \cdot, s) \), i.e., the eigenvalues of the matrix \( A \). Since the zeroes of the denominator must be identical, as the poles obviously are, the denominator terms of both sides of (35) must be the same within some constant factor, \( K \). Hence:

\[
K \left( 1 - r(i)\delta_i - r(k)\delta_k + f(i,k)\delta_i\delta_k \right) = \\
1 - r(m)\delta_m - r(n)\delta_n + f(m,n)\delta_m\delta_n
\]

(36)

However, observing (36) for \( \delta_i = \delta_k = \delta_m = \delta_n = 0 \), we see that \( K = 1 \). Hence:

\[
r(i)\delta_i + r(k)\delta_k - f(i,k)\delta_i\delta_k = \\
r(m)\delta_m + r(n)\delta_n - f(m,n)\delta_m\delta_n
\]

(37)

Therefore, the numerator matrices must also be equal:

\[
S^i\delta_i + S^k\delta_k + W^i,k\delta_i\delta_k = S^m\delta_m + S^n\delta_n + W^m,n\delta_m\delta_n
\]

(38)

Lemma: For pair-wise faults \((p_i, p_k)\) and \((p_m, p_n)\) belonging to the same DSFS, it is necessary that:

\[
f(i,k)W^{m,n} = f(m,n)W^{i,k} \quad \text{for all } w.
\]

(39)

Proof: Performing simple algebra on equations (37) and (38) to remove the left hand terms \( f(i, k) \delta_i \delta_k \) and \( W^{i,k} \delta_i \delta_k \), the
results are obtained:

\[(r(i)w^{i,k} + f(i,k)s^{i})\delta_{i} + (r(k)w^{i,k} + f(i,k)s^{k})\delta_{k} = \] \hspace{1cm} (40)
\[(r(m)w^{i,k} + f(i,k)s^{m})\delta_{m} + (r(n)w^{i,k} + f(i,k)s^{n})\delta_{n} + \]
\[(f(i,k)w^{m,n} - f(m,n)w^{i,k})\delta_{m}\delta_{n} \]

Similarly, removing the right hand terms in \(\delta_{m}\delta_{n}\) from equations (37) and (38), the results obtain:

\[(r(i)w^{m,n} + f(m,n)s^{i})\omega_{i} + (r(k)w^{m,n} + f(m,n)s^{k})\delta_{k} + \] \hspace{1cm} (41)
\[(f(m,n)w^{i,k} - f(i,k)w^{m,n})\delta_{i}\delta_{k} = \]
\[(r(m)w^{m,n} + f(m,n)s^{m})\delta_{m} + (r(n)w^{m,n} + f(m,n)s^{n})\delta_{n} \]

Observing the effect of \(\delta_{m} = \delta_{n}\) large in (40) and \(\delta_{i} = \delta_{k}\) large in (41) we note two facts:

(a) \(\delta_{i}\) and \(\delta_{k}\) are of the order of \(\delta_{m}^{2}\), if
\[(f(i,k)w^{m,n} - f(m,n)w^{i,k}) \neq 0 \]

and (b) \(\delta_{m}\) and \(\delta_{n}\) are of the order of \(\delta_{i}^{2}\), if
\[(f(m,n)w^{i,k} - f(i,k)w^{m,n}) \neq 0 . \]

However, (a) and (b) are contradictory unless
\[f(i,k)w^{m,n} - f(m,n)w^{i,k} = 0 . \hspace{1cm} Q.E.D. \]

From the above lemma and equations (40) and (41), the linear equations:

\[(r(i)w^{i,k} + f(i,k)s^{i})\delta_{i} + (r(k)w^{i,k} + f(i,k)s^{k})\delta_{k} = \] \hspace{1cm} (42)
\[(r(m)w^{i,k} + f(i,k)s^{m})\delta_{m} + (r(n)w^{i,k} + f(i,k)s^{n})\delta_{n} \]
and
\[
(r(i)w^m,n + f(m,n)S^i)\delta_i + (r(k)w^m,n + f(m,n)S^k)\delta_k = \\
(r(m)w^m,n + f(m,n)S^m)\delta_m + (r(n)w^m,n + f(m,n)S^n)\delta_n
\]
are obtained. Equation (42) is a function of frequency. Hence, writing one element of (42) for two frequencies \( w \) and \( w_1 \), we have the matrix equation:
\[
\mathcal{I}_1(\mu, \nu, w, w_1) \begin{bmatrix} \delta_i \\ \delta_k \end{bmatrix} = \mathcal{I}_2(\mu, \nu, w, w_1) \begin{bmatrix} \delta_m \\ \delta_n \end{bmatrix} \tag{43}
\]
where
\[
\mathcal{I}_1(\mu, \nu, w, w_1) = \\
\begin{bmatrix}
(r(i)w^{i,k}_{\mu, \nu} + f(i,k)\sigma^{i}_{\mu, \nu})(w) & (r(k)w^{i,k}_{\mu, \nu} + f(i,k)\sigma^{k}_{\mu, \nu})(w) \\
(r(i)w^{i,k}_{\mu, \nu} + f(i,k)\sigma^{i}_{\mu, \nu})(w_1) & (r(k)w^{i,k}_{\mu, \nu} + f(i,k)\sigma^{k}_{\mu, \nu})(w_1)
\end{bmatrix}
\]
\( \mathcal{I}_2 \) is similarly defined.

One of two conditions must hold for equation (43):

(a) \( \mathcal{I}_1 \) is invertible for some \( w \neq w_1 \),

and for some \( \mu, \nu \) numerator matrix element;

(b) \( \mathcal{I}_1 \) is never invertible.

The consequences of the above statements are quite interesting. Let us first consider (a).

If the left hand side matrix is invertible for some \( w = w_2 \), then it must be invertible for almost all frequencies, and \( \delta_i \) and \( \delta_k \)
are solvable in terms of $\delta m$ and $\delta n$, i.e.:

$$
\begin{bmatrix}
\delta_i \\
\delta_k
\end{bmatrix} = G
\begin{bmatrix}
\delta m \\
\delta n
\end{bmatrix}
$$

(45)

where $G = \hat{\Phi}_1^{-1} \hat{\Phi}_2(\mu, \nu, w_2, w_1)$ is a $2 \times 2$ matrix.

Lemma: The matrix $G$ of equation (45) must be diagonal or anti-diagonal.

Proof: The existence of $G$ is dependent upon the fact that the matrix $\hat{\Phi}_1(\mu, \nu, w_2, w_1)$ is invertible. It is noted that each term of this matrix contains either $w_\mu, \nu$ or $f$; hence, $w_\mu, \nu(w_2)$ or $f(w_2)$ must be nonzero. Thus, the term $w \delta_i \delta_k$ or the term $f \delta_i \delta_k$ in equation (38) will be nonzero. Substituting for $\delta_i \delta_k$ from equation (45), we see that:

$$
\delta_i \delta_k = (g_{11}g_{21})(\delta_m)^2 + (g_{11}g_{22} + g_{12}g_{21})\delta_m \delta_n
$$

(46)

$$
+ (g_{12}g_{22})(\delta_n)^2
$$

However, from equation (32) terms in $(\delta_m)^2$ or $(\delta_n)^2$ must not exist. Therefore:

$$
g_{11}g_{21} = 0 \quad \text{and} \quad g_{12}g_{22} = 0
$$

(47)

Since $\delta_i$ and $\delta_k$ must be functions of both $\delta_m$ and $\delta_n$, it is not allowed that $g_{11} = g_{12} = 0$ or $g_{21} = g_{22} = 0$. The only remaining possibilities satisfying (47) are:
\[ g_{12} = g_{21} = 0 \quad (48) \]

or
\[ g_{11} = g_{22} = 0 \quad (49) \]

Equations (48) and (49) state respectively that \( G \) is either diagonal or anti-diagonal.

Q. E. D.

The above lemma leads directly to an important result.

Theorem III. 3. Condition (44a) corresponds to one of the following cases:

(a) \( p_i \) and \( p_m \) belong to the same SFS and \( p_k \) and \( p_n \) belong to the same SFS;

or

(b) \( p_i \) and \( p_n \) belong to the same SFS and \( p_k \) and \( p_m \) belong to the same SFS.

Proof: Condition (44a) results in equations (48) and (49) which state that either:

(a) \[ g_{12} = g_{21} = 0, \text{ implying that } \delta_i = g_{11} \delta_m \quad \text{and } \delta_k = g_{22} \delta_n. \]

These statements imply that single failures of \( \delta_i \) and \( \delta_m \) are not distinguishable. Similarly, \( \delta_k \) and \( \delta_n \) are indistinguishable in the way they change the input-output relationships;

or:

(b) \[ g_{11} = g_{22} = 0. \text{ In an exactly similar manner this implies that } \delta_i = g_{12} \delta_n \text{ and } \delta_k = g_{21} \delta_m. \]

Hence, for this case \( p_i \) and \( p_n \) belong to the same SFS while \( p_i \) and \( p_m \) belong to the same SFS.

Q. E. D.
Theorem III.3 states that elements of DSFS are no more than pairs of elements which singly belong to two similarity fault sets, except for the condition given by (44b). That is:

$$\det \hat{\delta}_1(\mu, \nu, w, w_1) = 0 \text{ for all } w, w_1, \mu, \nu . \quad (50)$$

This condition, although more difficult than (44a) to check in determining DSFS, is nevertheless manageable because it deals only with the parameters which occur in $\Delta T(jw)$ as a function of $\delta_i, \delta_k$, in equation (35). If equation (50) is not satisfied, and if single faults can be isolated, then $(p_i, p_k)$ are in no DSFS with any other pairs of elements.

3.4 Determination of Single Observable Faults from Input-Output Relationships

The completeness of $T(jw)$ for all $w$ as a description of the input-output relationships of $N$ was discussed in section 3.1. The number of frequency measurements actually necessary for the isolation of the single element (or the proper SFS), when it is known that no more than one element has failed, can be shown to be quite small for a network of arbitrarily large order.

Lemma: The matrices of circular loci caused by variations in two elements for which these matrices are not always identical, can agree at no more than $2M$ frequencies, where $M$ is the order of the network $N$. 
Proof: From equations (10), (11), and (16) it is seen that the center of any given circular locus is a rational function in $w$ of order no higher than $M$. Thus, two centers which agree at $2M + 1$ frequencies must be represented by the same rational function of $w$, and therefore must be identical. Q.E.D.

If the matrices of centers $C(i, w)$ and $C(k, w)$ do not agree at $2M + 1$ frequencies, then they must disagree almost everywhere.

This statement leads us to an important theorem:

Theorem III.4: If each pair of elements $p_i$ and $p_k$ of $N$ not in a common SFS violates condition (a) of Theorem III.2, then single faults can almost always be identified by measurement of $T(j, w)$ at one frequency.

Proof: Condition (a) of Theorem III.2 is necessary for the coincidence of all matrix circle centers, or in degenerate cases the proportionality of matrices of sensitivity functions, between two elements in the same SFS. Violation of this rule for all elements not in the same SFS implies from the above lemma that at almost any frequency any pair of elements $p_i$ and $p_k$ not in the same fault set have different circular loci, or their matrices of partial derivatives are not proportional to one another. Since the collection of all pairs of elements is finite, it is true that for almost any frequency all matrices of centers of circles or sensitivities are different. One may choose any one of these frequencies and be assured that the loci of reachable $\Delta T$ are
different for each pair of elements in different similarity fault sets. The "almost always" qualification in the theorem statement acknowledges the fact that any two different circular loci of elements of $\Delta T(j \omega)$ passing through the origin may intersect at one other point also. However, we note that this can occur only at a single pair of values for each pair of network elements. Q.E.D.

Even though most conceivable networks do, indeed, violate condition (a) of Theorem III.2 for all elements not in the same fault class, the following statement covers all cases.

Theorem III.5: Single failures of $N$ can almost always be localized to a single SFS with measurement of $\Delta T(j \omega)$ at two frequencies.

Proof: All pairs of elements in different SFS must, by Theorem III.2, violate either condition (a) or condition (b) of that theorem. For those pairs of elements which violate condition (a), Theorem III.4 holds, allowing any of an infinite collection of frequencies to be utilized for the first measurement, $\omega_1$. For remaining pairs $p_i$ and $p_k$ condition (b):

$$r(i, j \omega) = \alpha(i, k)r(k, j \omega) + \beta(i, k)$$

is violated. It is noted that $r(i, j \omega)$ and $r(k, j \omega)$ are rational functions of $\omega$. Therefore, condition (b) does not hold for almost all $\omega$. Since condition (b) is violated by only a finite collection of pairs of elements, all pairs of elements not violating condition (a) and in
different fault sets will violate (b) at almost every frequency. The second frequency, \( \omega_2 \neq \omega_1 \), is chosen so that condition (a) or (b) of Theorem III.2 is violated for each pair of elements in different similarity fault sets.

That two frequency measurements are sufficient is seen as follows: If measurement at two frequencies could not distinguish between \( p_i \) and \( p_k \) of different fault sets and if condition (a) or (b) is violated at one or both of \( \omega_1 \) or \( \omega_2 \), then Theorem III.2 would be contradicted. Thus \( \omega_1 \) and \( \omega_2 \) are sufficient frequencies of measurement to uniquely specify the SFS whose element has failed when a single failure occurs. Q.E.D.

It is important to note that the Theorem III.5 has greatly reduced the number of frequency measurements necessary for fault isolation, and that the choice of frequencies is almost arbitrary. The resulting freedom in choosing frequencies of measurement allows the user of an algorithm to minimize the sensitivity of that algorithm to ambiguities caused by noise. Also, more frequencies of measurement could be utilized if noise in measurements or the lack of sensitivity of particular parameters at certain frequencies becomes a problem. This aspect of failure detection is discussed via an example in Chapter VI.

As before, "almost always" refers to special values of failure for which circular loci intersect. These special cases could be placed in special sets, analogous to SFS, to indicate values of elements for which
isolation is not possible. It should be noted that these sets are a property of input-output variables defined, and are not due to any difficulty in the mathematical analysis.

An interesting problem is the determination of the single element of $N$ which has failed, given $\Delta T(j \ w_1)$, when the matrices of circles are different for each network element. I.e., we seek a mapping:

$$M : \{\Delta T(j \ w_1)\} \rightarrow \{p_i : i=1,2,\ldots,N\}$$ \hspace{1cm} (51)

where $\{\Delta T(j \ w_1)\}$ is the set of allowable $\Delta T(j \ w_1)$ for arbitrary nonzero changes of any single element and $p_i$ is a network element.

One inelegant but effective method for actually determining which element has failed would be the following:

(a) Place all network elements for which $\Delta T_{1,1}(j \ w_1)$ is an allowable change in the set $\Psi$.

(b) If $\Psi$ contains more than one element, select a new matrix element $\Delta T_{i,k}(j \ w_1)$, and remove from $\Psi$ those elements for which $\Delta T_{i,k}(j \ w_1)$ is not an allowable change.

(c) Repeat (b) until $\Psi$ contains only one element.

That the above process does, indeed, specify exactly one element is a direct consequence of Theorem III.4, whose necessary conditions are satisfied by the conditions of the above algorithm. For the special case of a single input, single output network, or one from which
all faults are distinguishable from a scalar input-output relationship,
the above algorithm collapses to the simple single step (a). A further
discussion of methods for locating single failures is found in Chapter V.

The isolation of double faults causing a \( \Delta T(jw) \) is much more
difficult than that of single faults for two basic reasons:

(a) There are \( \frac{N}{2} (N-1) \) pairs of elements causing double
faults as compared with \( N \) elements which may cause single
faults.

and (b) The equations for \( \Delta p_i \) and \( \Delta p_k \) in terms of \( \Delta T \) contain
a multiplicative nonlinearity in the two element values, while one
failure, \( \Delta p_i \), is linear in \( \Delta T \).

That is, for a single fault:

\[
\Delta T(jw) = \frac{S_i(jw) \Delta p_i}{1 - r(i,jw) \Delta p_i} , \tag{52}
\]

which rearranges to:

\[
(r(i,jw) \Delta T(jw) + S_i(jw)) \Delta p_i = \Delta T(jw) \quad ; \tag{53}
\]

while for two faults:

\[
\Delta T(jw) = \frac{S_i(jw) \Delta p_i + S_k(jw) \Delta p_k + W_i k(jw) \Delta p_i \Delta p_k}{1 - r(i,jw) \Delta p_i - r(k,jw) \Delta p_k + f(i,k) \Delta p_i \Delta p_k} \tag{54}
\]

rearranges to:
\[ (\tau(1)\Delta T + S^1)\delta_i + (\tau(k)\Delta T + S^k)\delta_k \]

\[ + (w^1, k - f(1, k)\Delta T)\delta_i \delta_k = \Delta T \]

or:

\[ \hat{\Delta}(\omega)\Delta p_i + \hat{B}(\omega)\Delta p_k + \hat{C}(\omega)\Delta p_1 \Delta p_2 = \Delta T(j\omega). \] (55b)

The solution of matrix equation (55b) for \( \Delta p_1 \) and \( \Delta p_k \) may be found using any element of the matrix equation for which \( \Delta T_{\mu, \nu} \neq 0 \) at frequencies \( \omega_1 \) and \( \omega_2 \):

\[ \hat{a}_{\mu, \nu}(\omega_1)\Delta p_i + \hat{b}_{\mu, \nu}(\omega_1)\Delta p_k + \hat{c}_{\mu, \nu}(\omega_1)\Delta p_1 \Delta p_2 = \Delta T_{\mu, \nu}(j\omega_1) \] (56)

\[ \hat{a}_{\mu, \nu}(\omega_2)\Delta p_i + \hat{b}_{\mu, \nu}(\omega_2)\Delta p_k + \hat{c}_{\mu, \nu}(\omega_2)\Delta p_1 \Delta p_2 = \Delta T_{\mu, \nu}(j\omega_2) \]

There are, in general, two distinct solutions to equation (56), so that even if it is known that \( p_i \) and \( p_k \) have failed, computation of two elements of the matrix equation or of one element at two frequencies will be necessary to uniquely identify the values of \( \Delta p_i \) and \( \Delta p_k \).

Conversely if \( p_i \) and \( p_k \) are not in the DSFS responsible for a given \( \Delta T(j\omega) \) due to changes in two element values, then any \( \Delta T(j\omega) \) caused by faults in \( p_i \) and \( p_k \) can agree with \( \Delta T(j\omega) \) only at a finite number of frequencies (since each is rational in \( \omega \)). Therefore, \( \Delta T \) and \( \Delta \hat{T} \) must disagree almost anywhere, and by proper choice of \( \omega_1, \omega_2 \) it is seen that:

(a) Solution of equation (56) gives at most \( (\Delta p_i^1, \Delta p_k^1) \) and \( (\Delta p_i^2, \Delta p_k^2) \)
(b) \( (\Delta p_1^1, \Delta p_k^1) \) results in \( \Delta T^1(j, w) \) while \( (\Delta p_1^2, \Delta p_k^2) \) results in \( \Delta T^2(j, w) \).

However, by proper choice of \( w_3 \):

(c) \( \Delta T(j, w_3) \neq \Delta T^1(j, w_3) \) caused by \( (\Delta p_1^1, \Delta p_k^1) \), and \( \Delta T(j, w_3) \neq \Delta T^2(j, w_3) \), caused by \( (\Delta p_1^2, \Delta p_k^2) \) since \( \Delta T \) differs from both \( \Delta T^1 \) matrices almost everywhere.

Therefore, by the above argument it is seen that three frequencies are sufficient for isolation of double faults. As in the case of single faults, the availability of almost all frequencies for measurements allows the user to minimize a sensitivity criterion of \( \Delta T \) with respect to elements of \( N \).

Simultaneous failure of more than two elements of \( N \) may not be dealt with effectively using the method of expression of \( \Delta T(j, w) \) as the ratio of a matrix polynomial to a scalar polynomial in the changes of several elements. This limitation is due to the fact that 1) the non-linear equations to be solved become unmanageable; 2) little can be said about uniqueness of values of any given set of failed elements; and 3) the number of sets of elements to be considered grows rapidly with the number of simultaneous failures. It is not unreasonable to expect that the best method for determination of multiple failures rests in the ability to identify all network elements. This method is discussed in Chapter VII.
IV. FAULT ANALYSIS OF NETWORKS

WITH DISTRIBUTED ELEMENTS

The method of fault analysis introduced in Chapter III for lumped elements is extended to include two-terminal distributed elements. The application of this extension to networks of subnetworks is discussed.

4.1 Definitions and Formulation of Network Equations

If the network \( N \) has \( l \) distributed two-terminal linear elements (i.e., having a voltage-current relationship \( L_i \) other than the integral, differential, or proportional operator), closed form time domain element and state equations do not exist in general. However, if the Laplace transform of each element relationship exists, the frequency domain network solution may be found. Assuming that each operator \( L_i \) is Laplace transformable, the distributed element relationships may be written:

\[
V_e^i(s) = Z_o^i(s) I^i(s) \quad i = 1, \ldots, \kappa .
\]  

(1)

where \( \kappa = \kappa + l \), \( V_e \) and \( I \) are element voltage and current variables respectively, and \( Z_o \) is the nominal impedance function.

In order to perform fault analysis on the distributed elements, we require that the manner in which the failed element changes value with frequency be known. The form of the failed element must be:
\begin{equation}
Z^i(s) = Z^i_0(s) \left[ 1 + \sum_{k=1}^{K(i)} z^i_k(s) \cdot d^i_k \right]
\end{equation}

where each $z^i_k(s)$ is a known function of frequency. For a fault in the $i^{th}$ distributed element, equation (1) is replaced by the branch equation:

\begin{equation}
V^i_b(s) = Z^i_0(s) \left[ 1 + \sum_{k=1}^{K(i)} z^i_k(s) \cdot d^i_k \right] I(s) ,
\end{equation}

\begin{equation}
\Delta V^i_b(s) = V^i_b(s) - V^i_e(s) = \sum_{k=1}^{K(i)} z^i_k(s) \cdot d^i_k \cdot V^i_e(s) ,
\end{equation}

\begin{equation}
i = \hat{i} + 1, \ldots, K .
\end{equation}

The failed element may therefore be considered as a branch containing the nominal element in series with a single voltage source dependent upon $V^i_e$ and whose gain is a function of frequency. Since lumped elements fail in the manner of equation (2) (with $K(i) = 1$, $z^i_1(s) = 1$, $d^i_1 = \Delta p_1$, and letting $\Delta V_b$ and $V_e$ represent arbitrary branch and element variables), equation (4) may be used to represent the element failure for each element of $N$. That is, $i = 1, \ldots, K$.

Definition: Any $k$ such that $d^i_k \neq 0$ is a mode of failure of the $i^{th}$ element.

Remark: The $i^{th}$ element has a single mode of failure if $d^i_k \neq 0 = d^i_n = 0$ for $n \neq k$. Lumped elements have only a single mode of failure.

Definition: $N$ is said to have a single fault if exactly one element fails,
and its failure has a single mode.

Multiple faults are similarly defined.

Analysis of N in the frequency domain may be accomplished by connecting the distributed elements to a second set of fictitious ports with inputs and outputs represented by:

\[
U_d(s) = \left[ I(s), \ldots, I_k(s) \right]^T
\]

and

\[
Y_d(s) = \left[ V_e, \ldots, V_k \right]^T
\]

respectively. The system equations may be written as in Chapter III, with the inclusion of an additional set of port relationships and transformed to the frequency domain:

\[
sIX = AX + BU + B_a U_f + B_b U_d
\]

\[
Y = CX + DU + D_a U_f + D_b U_d
\]

\[
Y_f = C_f X + D_f U_i + D_{af} U_f + D_{bf} U_d
\]

\[
Y_d = C_d X + D_d U + D_{ad} U_f + D_{bd} U_d
\]

where \( X, U, U_f, U_d, Y, Y_f, Y_d, D, D_f, \) and \( D_d \) are functions of frequency. The port constraint equations are written:

\[
Y_d = G(s) U_d
\]

\[
Y_f = H(s) U_f
\]

where \( G \) and \( H \) are diagonal matrices with diagonal 1st elements.
written respectively:

\[ g_{ii}(s) = Z_o^*(s)^{\frac{1}{i}} \quad i = 1, \ldots, \ell \]  \hspace{2cm} (9)

and

\[ h_{ii} = z_k^*(s)d_k^{i-1} \quad i = 1, \ldots, \kappa \]  \hspace{2cm} (10)

\( Y_f \) and \( U_f \) are now \( \kappa \)-vector fictitious port variables written to include the effect of faults in arbitrary elements:

\[
Y_f = \begin{bmatrix} V_e^{1} \cdots V_e^{\kappa} \end{bmatrix}^T,
\]

\[
U_f = \begin{bmatrix} \Delta V_b^{2} \cdots \Delta V_b^{\kappa} \end{bmatrix}^T.
\]

Substitution of (7) into (6d) gives:

\[
(G - D_{bd}) U_d = C_d X + D_d U + D_{ad} U_f \]

or

\[
U_d = E C_d X + E D_d U + E D_{ad} U_f \]

where \( E(s) = (G - D_{bd})^{-1} \)  \hspace{2cm} (14)

Thus (6a-6c) may be written:

\[
sIX = \hat{A} X + \hat{B} U + \hat{B}_a U_f \]

\[
Y = \hat{C} X + \hat{D} U + \hat{D}_a U_f \]

\[
Y_f = \hat{C}_f X + \hat{D}_f U + \hat{D}_{af} U_f \]

where \( \hat{A}(s) = A + B_b E C_d \),

and similar defining equations exist for each other matrix of equation (15). In general, each of these matrices is a function of frequency.

Equations (15) and (7) are at least formally identical to
equations (III.8) and (III.9) transformed to the frequency domain.

As in Chapter III, it follows that:

\[ Y(s) = \left[ \hat{T}_o(s) + \hat{R}_1(s) \hat{Q}^{-1}(s) \hat{R}_2(s) \right] U(s) \]  \hspace{1cm} (16)

where:

\[ \hat{Q}(s) = H(s) - \hat{R}_3(s) \]  \hspace{1cm} (17a)

\[ \hat{T}_o(s) = \hat{C}(s)(sI - \hat{A}(s))^{-1}\hat{B}(s) + \hat{D}(s) \]  \hspace{1cm} (17b)

\[ \hat{R}_1(s) = \hat{C}(s)(sI - \hat{A}(s))^{-1}\hat{B}_a(s) + \hat{D}_a(s) \]  \hspace{1cm} (17c)

\[ \hat{R}_2(s) = \hat{C}_f(s)(sI - \hat{A}(s))^{-1}\hat{B}(s) + \hat{D}_f(s) \]  \hspace{1cm} (17d)

\[ \hat{R}_3(s) = \hat{C}_f(s)(sI - \hat{A}(s))^{-1}\hat{B}_a(s) + \hat{D}_af(s) \]  \hspace{1cm} (17e)

If a subset of the elements of $N$ fails, then the proper columns of $\hat{R}_1$, rows of $\hat{R}_2$ and columns and rows of $\hat{H}$, $\hat{R}_3$ and $Q$ are included in equations (16) and (17).

4.2 Conditions for Similarity Fault Sets

For networks with distributed elements, the SFS is used to partition the modes of failure into equivalence classes, so that a fault of $d_{ik}$ cannot be distinguished from the single fault $d_{im}$ by input-output measurements if these faults are in a common SFS. How does one determine the SFS classification of modes of failure? If the single $i^{th}$ element fails, the change in the transfer function $T(s)$ is seen from equations (16) and (17) to be:
\[ \Delta \hat{\gamma}(s) = \frac{\hat{R}_1^i \hat{R}_2^i \left( \sum_{k=1}^{K(i)} z_k^i(s) d_k^i \right)}{1 - \hat{R}_3^i \left( \sum_{k=1}^{K(i)} z_k^i(s) d_k^i \right)} \] (18)

And if this failure is a single fault:

\[ \Delta \hat{\gamma}(s) = \frac{\hat{s}_k^i(s) d_k^i}{1 - \hat{r}_k^i(s) d_k^i} \] (19)

where

\[ \hat{s}_k^i(s) = \frac{\partial \hat{\gamma}(s)}{\partial d_k^i} = \frac{\hat{r}_1^i \hat{r}_2^i z_k^i(s)}{\hat{r}_3^i z_k^i(s)} \] (20)

and

\[ \hat{r}_k^i(s) = \hat{R}_3^i z_k^i(s) \] (21)

\( \hat{R}_1^i, \hat{R}_2^i, \hat{R}_3^i \) are the proper column, row, and scalar elements of \( \hat{R}_1^i, \hat{R}_2^i, \) and \( \hat{R}_3^i \) respectively.

The locus of each \( \Delta \hat{\gamma}_{m,n}(jw) \) as \( d_k^i \) varies (w held constant) is in general a circle, centered at:

\[ (C_k^i)_{m,n}(w) = -j \frac{(\sigma_k^i)_{m,n}(jw)}{2 \text{ Im} (\hat{r}_k^i(jw))} \] (22)

Equations (19) and (22) are the same as equations (III.15) and (III.16). Since Theorems III.1 and III.2 depend upon exactly these equations, they hold with only slight modifications. For example, the
extension of III.2 is given below:

Theorem IV: Each SFS of N containing distributed elements is a
collection of those modes of failure, \( d_k^i \) and \( d_n^m \), for which
the following relations hold:

\[
\begin{align*}
(a) \quad \hat{S}_k^i(jw) &= \alpha(i,k,m,n)\hat{S}_n^m(jw) \\
(b) \quad \hat{r}_k^i(jw) &= \alpha(i,k,m,n)\hat{r}_n^m(jw) + \beta(i,k,m,n)
\end{align*}
\]

where \( \alpha \) and \( \beta \) are real scalars.

The proof is identical to that of Theorem III.2 with obvious minor
changes in notation.

Testing the conditions of Theorem IV no longer has the simple
property of requiring only three frequency calculations for SFS deter-
mination. Since \( Z_i^i(s) \) is not necessarily rational in \( s \), no general
statement can be made about how conditions (a) and (b) can be veri-
fied computationally. Each network must be tested with the user's
knowledge of the behavior of each \( Z_i^i \) with frequency.

Isolation of single faults, or more properly the SFS of which
a mode of failure has occurred, also lacks the property of requiring no
more than three frequency measurements. Although the number of fre-
quency measurements required can be no more than the number of simi-
larity fault sets of \( N \), it would in fact be much less than this number
for most networks of interest. An a priori determination of the number
of frequency measurements necessary for isolation of faults in a net-
work containing distributed elements is, however, impossible.
4.3 Application of Method to Subnetworks

As the form of $Z^i(s)$ is unspecified, and as several modes of failure are allowed for each element, the extension of fault analysis of networks of non-lumped elements is directly applicable to networks containing interconnected two-terminal subnetworks. One value of this application is the ease of computing the impedance of each subnetwork and then combining them using equations (6-15), as compared with computing the solution of the entire network at once. This is particularly true when several subnetworks contain many elements. Similarly, the isolation of the subnetwork at fault is often sufficient for the user, and it requires less effort than isolation of the actual element at fault. Finally, determining the impedance and modes of failure of a subnetwork may be experimentally simpler than finding a lumped model for the subnetwork which displays both the proper impedance and mechanisms of failure. The major limitations of this extension to subnetworks lie in the requirement that the subnetworks be two-terminal and in the restriction (equation (2)) upon the mechanism of failure.
V. COMPUTER IMPLEMENTATION
OF THEORETICAL RESULTS

Network solutions and other computations necessary for determining SFS and DSFS are derived. Algorithms for fault isolation are discussed.

5.1 Frequency Domain Network Analysis

It is assumed in the introduction that complete knowledge of the nominal network, called N, is known. For a complete description of the nominal network, the topology of the network, including a description of input-output variables, and the vector of nominal element values \( P_0 \) are required. Using this description of the network, any number of circuit analysis programs could produce the state equations

\[
\dot{x} = Ax + Bu \tag{1}
\]

\[
y = Cx + D_0u + \sum_{i=1}^{n} D_iu^{(i)}
\]

as well as the transfer function matrix

\[
T(s) = C(sI - A)^{-1}B + D(s) \tag{2}
\]

at any desired frequencies. It will now be shown that obtaining enough information about the system to determine each SFS, DSFS, and actual fault isolation requires a network solution of no higher order than the nominal network. The effect of element changes upon state and transfer functions can be represented by the inclusion of augmented
one-port subnetworks in series or parallel with each element, as may be seen by equation (III.7b):

\[ \Delta f_b(t) = \frac{\Delta K}{K_o} f_e(t) \]  

(3)

where \( \Delta f_b(t) \) is the port input variable and \( f_e(t) \) is the port output.

It may be noted that the form of (3) is simply that of a constant voltage-dependent voltage source or current-dependent current source for any given \( \Delta K \). The set of \( \{ \Delta f_b \} \) is labeled \( u_f \) and the set \( \{ f_e \} \) is labeled \( y_f \). Thus, the network with the augmented ports may be written:

\[ \dot{x} = Ax + Bu + B_a u_f \]  

(4a)

\[ y = Cx + \sum_{i=0}^{n} D_i u^{(i)} + D_a u_f \]  

(4b)

\[ y_f = C_f \dot{x} + \sum_{i=0}^{n} D_{fi} u^{(i)} + D_{af} u_f \]  

(4c)

and each of the above equations may be formed by available network programs. The transformed equations of interest for the computation of similarity fault sets and the isolation of single faults are:

\[ Y(s) = (T_o(s) + \Delta T(s)) U(s) \]  

(5)

where

\[ \Delta T(s) = \frac{R^i_1(s) R^i_2(s) \Delta \rho_i}{\rho_i^i - R^i_3(s) \Delta \rho_i} \]  

(6)
and \( R_1^i(s) = i^{th} \text{ column of } \{ C(sI - A)^{-1}B_a + D_a(s) \} \) \hspace{1cm} (7a)
\( R_2^i(s) = i^{th} \text{ row of } \{ C_f(sI - A)^{-1}B + D_f(s) \} \) \hspace{1cm} (7b)
\( R_3^i(s) = (i, i)^{th} \text{ element of } \{ C_f(sI - A)^{-1}B_a + D_{af} \} \) \hspace{1cm} (7c)

One notes immediately that the only involved computation is that of finding \( (j \omega I - A)^{-1} \) for each \( \omega \) of interest. Neither the computation of all the trees of \( N \) nor an explicit expression for \( T(s) \) in terms of all the elements of \( N \) has been necessary. Beyond the formation of the extra input-output matrices necessary for the fault isolation, only matrix multiplications and additions are needed to obtain:

\[
S^i(j \omega) = \frac{R_1^i(j \omega)R_2^i(j \omega)}{p_i}
\]

and \( r(i, \omega) = \frac{R_3^i(j \omega)}{p_i} \), \( i = 1, 2, \ldots, \hat{n} \).

The matrices \( S^i(j \omega) \) and polynomials \( r(i, \omega) \) are used to compute, when they exist, the matrices of centers of circular loci by the equation:

\[
C(i, \omega) = \frac{-j S^i(j \omega)}{2 \text{ Im}(r(i, \omega))}
\]

These centers may be used to plot the circular locus for each element at each transfer function and for each \( \omega \) of interest. Comparison of the matrices \( C(i, \omega) \), \( i = 1, 2, \ldots, \hat{n} \), quickly tell one the SFS grouping of elements of \( N \) if \( \hat{n} \) is sufficiently small. For large networks, a rather simple computer algorithm (see Appendix III) is used to compute the similarity fault sets of \( N \).
When the elements of \( N \) have been partitioned in similarity fault sets, a frequency (or frequencies) can be chosen to maximize the distance between the centers of the circles in order to cause the effect of changing any two elements to be as different as possible. After the network solution has been performed, including the computation of SFS grouping and:

(a) \( T_o(j\omega) \)

(b) elements of \( S^i(j\omega) \) and \( C(i,\omega), i = 1, 2, \ldots, k \),

and

(c) \( r(i,\omega) \) \( i = 1, 2, \ldots, k \),

for frequencies of interest on a larger high speed digital computer, the use of a small special purpose digital computer would be sufficient to isolate element faults.

5.2 Algorithms for Isolating Single Faults

It is assumed in this section that each SFS of \( N \) contains one element. No generality is lost in that any SFS containing more than one element is completely described by any representative. Thus, all but one element of each SFS are removed from the set of elements under consideration.

One of the most obvious, easily implemented, and easily used methods is a two-dimensional analogy of the catalog method of Seshu and Waxman, discussed in Chapter I. For a given frequency \( \omega_0 \) and a given element \( \Delta T_{m,n}(j\omega_0) \) of the \( \Delta T(j\omega_0) \) matrix, circles could be placed on the complex plane with centers at \( c^i_{m,n}(j\omega_0) \),
i = 1, 1, ..., h, and passing through the origin. For elements for which \( \text{Im}(\tau(i, w)) = 0 \), straight lines are drawn passing through the origin and forming an angle \( \theta_i = \frac{\sigma_{m,n}(jw_o)}{m,n} \) with the real axis.

It is known that if a single element fails, \( \Delta \tau_{m,n}(jw_o) \) must lie on one of the circular (or linear) loci. Circular (linear) loci are placed on separate planes for each matrix element \( \Delta \tau_{m,n}(jw_o) \). If each pair of elements of \( N \) violates condition (a) of Theorem III.2 at \( w_o \), then the intersection of the sets:

\[
F(m,n,w_o) = \{ p_i: \Delta \tau_{m,n}(jw_o) \text{ lies on circular (linear)} \}
\]

locus caused by varying \( p_i \) \hspace{1cm} (8)

must contain exactly one element for almost all \( \Delta T(jw_o) \)

\[
\{ P_{\text{fault}} \} = \bigcap_{m = 1, \ldots, h} \bigcap_{n = 1, \ldots, k} F(m,n,w_o)
\]

Therefore, one interested in locating a single fault need only use the \( h \cdot k \) plots of obtainable \( \Delta T(jw_o) \) by:

(1) measuring \( \Delta T(jw_o) \),

(2) noting the elements corresponding to loci of \( m,n \) element of \( \Delta T(jw_o) \) passing through the measured \( \Delta \tau_{m,n}(jw_o) \),

and

(3) choosing the one element in common for each \( \Delta \tau_{m,n}(jw_o) \).

It should be noted that perhaps not all the \( \Delta \tau_{m,n}(jw_o) \) add information about which element fails. For:
\[ h < h \approx k \]

it is clear that only \( h \) matrix element measurements are necessary for fault solution. In many cases, only a small percentage of the elements of the \( \Delta T(j \omega_o) \) matrix are needed for single fault isolation.

An example of the reduction of number of measurements of transfer function matrix elements is given in Chapter VI.

For large networks (more than about 50 elements) the graphical procedure above is of limited usefulness. The large number of loci on any given plane of \( \Delta T_{m,n}(j \omega_o) \), as well as the increasing number of planes necessary for isolation of failures would suggest that a computer algorithm would be necessary. This is especially true if the loci for various elements become similar enough that visual observation cannot distinguish the locus upon which \( \Delta T_{m,n}(j \omega_o) \) lies, even though the accuracy of measurement is sufficient for isolation of the fault. A computer program would be similar to the graphical technique except that:

the locus for each parameter would be compared with \( \Delta T_{m,n}(j \omega_o) \) initially, those parameters satisfying the equation:

\[
\text{Re}(\Delta T_{m,n}(j \omega_o) - C_{m,n}(i, \omega_o))^2 + \text{Im}(\Delta T_{m,n} - C_{m,n}(i, \omega_o))^2 = \| C_{m,n}(i, \omega_o) \|^2
\]

(10)

being retained for further consideration at succeeding \( m, n \) element comparisons.
If each pair of elements in different similarity fault sets do not violate condition (a) of Theorem III.2, the above procedure may not uniquely specify an element. To determine the element at fault, one need only determine a proper \( w_1 \), and compute

\[
\Delta p_i(m,n,w) = \frac{\Delta \tau_{m,n}(jw)}{\sigma^1_{m,n}(jw) + r(i,jw) \Delta \tau_{m,n}(jw)}
\]

for \( m = 1, \ldots, h \), \( n = 1, \ldots, k \), and \( w = w_0, w_1 \). If each of the \( 2hk \) values computed are identical, then \( p_i \) has failed. We are guaranteed that the above procedure isolates a unique fault by Theorem III.5, which states that single faults can be isolated by measurement of two frequencies. Figure 3 gives the basic steps of a

```
Set i = 1

Set u = 1
v = 1

i = i + 1

Is \( \Delta \tau_{u,v} \) on locus of \( p_i \)?

yes

Have all \( u,v \) pairs been used?

yes

Choose new \( u,v \) pair

no

\( p_i \) at fault

Figure 3
```
computer algorithm for finding single failures with no restrictive conditions.

A final interesting method for isolating faults when \( T(j \omega) \) is a scalar function and loci are unique is that of the use of some algebraic function \( F \) whose domain is the set of complex numbers and whose range is a set of integers representing the failed element:

\[
F : \Delta T(j \omega_o) \rightarrow I \quad i \in I, \ i = 1, 2, \ldots, n .
\] (12)

Such a function could be the sum of \( n \) other functions

\[
F_i : \Delta T(j \omega_o) \rightarrow \{0, i\} \quad i = 1, 2, \ldots, n .
\] (13)

such that \( F_i(\Delta T(j \omega_o)) = i \) if \( T(j \omega_o) \) lies on the locus

\[
\text{of } p_i
\]

\[
= 0 \ \text{otherwise}.
\] (14)

As measurement accuracy of any instrument is not perfect, equation (14) should actually be written:

\[
F_i(\Delta T(j \omega_o)) = i \ \text{if } T(j \omega_o) \text{ lies within measurement accuracy of the locus of } p_i.
\] (14')

An approximation of equation (14') is suggested:

\[
F_i(\Delta T(j \omega_o)) = i \exp(-\alpha_i \frac{2\pi}{i}(\Delta T(j \omega_o)))
\] (15)

where \( \alpha_i \) is suitably large, and :
\[ f_i(z) = R_e(z) \cdot R_e(z - 2C^i(j \omega_o)) + \text{Im}(z). \] (16a)
\[ \text{Im}(z - 2C^i(j \omega_o)) \]

when the locus of \( \Delta T(j \omega_o) \) is circular, and

\[ f_i(z) = R_e(z) \text{Im}(S^i(j \omega_o)) - \text{Im}(z) R_e(S^i(j \omega_o)) \] (16b)

when the \( i^{th} \) locus of \( \Delta T(j \omega_o) \) is linear.

It is seen that \( f_i(z) = 0 \) if \( z \) is on the \( i^{th} \) locus of \( \Delta T(j \omega_o) \),

and for small variations \( f_i \) varies linearly with the Euclidian distance \( d \) from \( z \) to the locus of \( \Delta T(j \omega_o) : f_i \approx k_i \cdot d \). Thus for small variations from the \( i^{th} \) locus:

\[ F_i(\Delta T(j \omega_o)) = i \exp(-\frac{2n}{\alpha_i k_i} \frac{2n}{a_m} d). \] (17)

\( \alpha_i \) is chosen so that at the measurement accuracy, \( a_m \),

\[ \frac{2n}{\alpha_i k_i} \frac{2n}{a_m} = 1 \] (18)

and \( n \) is an integer large enough that

\[ F_i(\Delta T(j \omega_o)) = i \exp(-\frac{d}{a_m} \frac{2n}{a_m}) \] (19)

has a nearly rectangular shape. A plot of \( F_1 \) versus \( d \) for several values of \( n \) may be found in Figure 4.

![Figure 4](image-url)
Thus an analytical function has been found which is nearly zero for
\( \Delta T(j \omega_j) \) not in the region of allowable loci, which is equal to the in-
teger \( i \) on the \( i^{th} \) locus, and is easily computed:

\[
F_i(\Delta T(j \omega_j)) = \sum_{i=1}^{2n} \exp \left( -\frac{f_i(\Delta T(j \omega_j))}{(k_i a m)^{2n}} \right) \tag{20}
\]

where each \( f_i \) is defined by equation (16a) or (16b).

5.3 Fault Isolation of Two Elements

It may be assumed that each pair of elements occupies a differ-
ent DSFS, since any DSFS may be represented by one of its pairs of
elements, the other pairs in the same DSFS being disregarded. Be-
cause of the nonlinear form of the equation

\[
\Delta T = \frac{S^i \Delta p_i + S^k \Delta p_k + W^i, k \Delta p_i \Delta p_k}{1 - r(i) \Delta p_i - r(k) \Delta p_k + e(i, k) \Delta p_i \Delta p_k} \tag{21}
\]

and because the region of reachable \( \Delta T(j \omega) \) for \( p_i \) and \( p_k \) varying
is no longer simple to express and has a nonzero measure in the plane,
a function from \( \Delta T \) measured at \( \omega_o \) and \( \omega_1 \) to a pair of integers
representing the failed elements in \( N \) is not feasible. For these same
reasons, the graphical method of 4.2 is not applicable for two faults.
The most straightforward method of isolating dual faults involves calcul-
lating the possible solutions \( \Delta p_i \) and \( \Delta p_k \) to equation (21) at the
frequencies \( \omega_o \) and \( \omega_1 \) for each pair of elements, then using these
choices at \( \omega_2 \) to calculate \( \Delta T(j \omega_2) \). That element pair for which
\[ \Delta T^C(j\ \omega_2) = \Delta T(j\ \omega_2) \]  

is then the only pair of elements which could have caused the network failure. Only one pair of elements results from solving equation (21) and checking with equation (22); this is a direct consequence of the fact that, from Chapter III, only three frequencies are necessary to isolate the DSFS responsible for a network failure. For an abbreviated diagram of the above method see Figure 5. (See following page.)

The number of computations necessary to isolate dual failures by the above method is:

(a) \( \frac{n}{3} (n - 1) \) solutions of quadratic equations parameter value pairs

(b) \((n - 1)\) forward solutions for comparison.

We see that the solutions of \( \frac{n}{3} (n - 1) \) quadratic equations are the most time-consuming part of the algorithm. Assuming a network of \( n = 50 \) elements, and a quadratic solution time of 100 milliseconds, about two minutes of computation time would be required to isolate dual failures. The isolation of more than two simultaneous faults, except for very small networks would be prohibitively expensive. For example, approximately \( \frac{n^3}{6} \) solutions of cubic equations would be required for the isolation of three simultaneous faults. The method of component parameter identification (Chapter VII) is more efficient for detection of more than two faults, whenever this method is applicable.
ALGORITHM FOR ISOLATION OF SIMULTANEOUS DUAL FAULTS

Read or calculate $S^i, W^{i,k}, r(i), f(1,k)$ for each $1, k \in \{0,1, \ldots, \#\}$ and $w = w_0, w_1, w_2$

Read or measure $T(jw)$

$w = w_0, w_1, w_2$

Set $i = 1, k = 2$

Do quadratic equations in $\Delta p_i, \Delta p_k$, for $T(jw_0)$ and $T(jw_1)$ have consistent solutions $(p^{1}_i, p^{1}_k)$ and $(p^{2}_i, p^{2}_k)$?

No

$i = \# - 1$?

Yes

$k = k + 1$

$i = 0$

No

Does $(\Delta p^{1}_i, \Delta p^{1}_k)$ or $(\Delta p^{2}_i, \Delta p^{2}_k)$ give $T(jw_2)$ measured?

Yes

$p_i$ and $p_k$ at fault

Figure 5
VI. AN EXAMPLE OF FAULT ISOLATION

The methods of fault analysis are applied to a model of any available network. Various properties of the network and of the methods of determining similarity fault sets and isolating faults are discussed.

6.1 Modelling the $\mu$A709 Integrated Circuit Operational Amplifier

One justification for isolating internal network failures by external observations is that linear integrated circuits are measurable only at certain points available to the user. Hence, it is proper that the example be a real integrated circuit. An important factor influencing the choice of circuits is that of network size. A readily available circuit analysis program, RAPID [11] allows up to 20 nodes and 39 elements, excluding sources. RAPID is extendable to include 72 total elements without requiring major modification. A final requirement is that of a reasonably accurate internal characterization of the integrated circuits to be modeled. For the above reasons the circuit to be modeled is a Fairchild $\mu$A709 integrated operational amplifier, which contains 13 transistors, 15 resistors, and 2 diodes. A schematic diagram of the circuit, along with approximate resistor values and typical gain vs. frequency curves, is available. The $\mu$A709 circuit description is given in Figure 6. The methods of fault analysis of Chapters III & V require a linear model of the amplifier. Thus transistors have been modeled using five elements; diodes have been modeled as resistors of the proper values; and entire biasing subnetworks have been replaced
μA709 AMPLIFIER

input frequency compensation

Figure 6
by an equivalent impedance.

6.2 Transistor Models

The application of fault analysis is by definition limited to two terminal elements. Thus, transistors must be modeled using the elements listed in Chapter II. Is is at once clear that if the network is to have any dynamics, the transistor model must be more than a d.c. model since the basic circuit description of Figure 6 contains only resistors and transistors. One common transistor model is the hybrid-pi, a medium to high frequency version of which is shown in Figure 7.

![Figure 7](image)

Besides containing only five elements, including the feedback capacitor $C_{bc}$, this model contains no internal nodes, a constraint placed upon the model by the nodal restriction of RAPID. Element values for each transistor are assumed identical, as one would expect for integrated circuit elements, except for Darlington models of transistors $Q_3$, $Q_5$ and $Q_4$, $Q_6$ and the physically larger transistor $Q_{14}$. The choice of values for the elements of the hybrid-pi model, $C_{be}$, $C_{bc}$, $C_{ce}$, $g_{be}$ and $g_m$, although vital for the actual analysis of faults in a
\( \mu A709 \) on a production line, does not affect the SFS classification of the element and hence is not of overwhelming importance for the example. Therefore, numbers were chosen from Multistage Transistor Circuits [12] as typical of small signal, high frequency transistors:

\[
\begin{align*}
C_{be} & = 20 \text{ pf.} \\
C_{bc} & = 3.7 \text{ pf.} \\
C_{ce} & = 29 \text{ pf.} \\
g_{be} & = 0.32 \text{ mmho} \\
g_{m} & = 32 \text{ mmho}.
\end{align*}
\]  

RAPID circuit analysis program was used to model the Darlington transistor pairs \((Q_3, Q_5)\) and \((Q_4, Q_6)\) as single transistors consistent with their models as interconnected pairs. In Appendix IV the element values of the single transistor are shown to be:

\[
\begin{align*}
C_{be} & = 10.3 \text{ pf.} \\
C_{bc} & = 12 \text{ pf.} \\
C_{ce} & = 28.5 \text{ pf.} \\
g_{be} & = 0.00314 \text{ mmho} \\
g_{m} & = 32 \text{ mmho}.
\end{align*}
\]  

With the above element values, the behavior of the single transistor is similar to that of a pair, each of which has element values as given in equation (1).

The only remaining transistor is \( Q_{14} \), a power transistor with
a large diffusion area, which is modeled with relatively large capacitance and admittance values:

\[
\begin{align*}
C_{be} & = 40 \text{ pf.} \\
C_{bc} & = 7.4 \text{ pf.} \\
C_{ce} & = 58 \text{ pf.} \\
g_{be} & = 0.64 \text{ mmho} \\
g_{m} & = 64 \text{ mmho}.
\end{align*}
\]

6.3 Simplification of Internal Structure of Model

The structure of the \(\mu A709\) is unnecessarily complicated for a.c. analysis of the circuit and for fault analysis. Several elements are used as bias elements only. \(R_8 - R_{11}, Q_{10},\) and \(Q_{11}\) form a constant current source for the emitter circuitry of \(Q_1\) and \(Q_2\) and provide a resistor path between \(V^-\) and the emitter of \(Q_{16}\). A single 25 kohm resistor replaces this entire subnetwork. \(Q_7\) and \(R_5\), used as an impedance transformation, are replaced by a 200 ohm resistance. \(R_3, R_4,\) and \(Q_5\) provide bias only and are removed; \(Q_9\) only transforms d.c. levels and is replaced by a 1 kohm resistance. Output transistor \(Q_{13}\) is, according to the manufacturer \([13]\), essentially a forward biased diode and is removed. The reduced and simplified model of the \(\mu A709\) is shown in Figure 8. Original element placement has been followed as much as possible to aid in clarity; however, transistors and resistors have been relabeled. Transistors are modeled using the hybrid-pi configuration discussed previously with element values as
μ A709 AMPLIFIER — LINEAR MODEL

Figure 8-A
### FIGURE 8B

<table>
<thead>
<tr>
<th>Number of Element</th>
<th>Location</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>R&lt;sub&gt;1&lt;/sub&gt;</td>
<td>25 kohm</td>
</tr>
<tr>
<td>2</td>
<td>R&lt;sub&gt;2&lt;/sub&gt;</td>
<td>25 kohm</td>
</tr>
<tr>
<td>3</td>
<td>R&lt;sub&gt;3&lt;/sub&gt;</td>
<td>.2 kohm</td>
</tr>
<tr>
<td>4</td>
<td>R&lt;sub&gt;4&lt;/sub&gt;</td>
<td>10 kohm</td>
</tr>
<tr>
<td>5</td>
<td>R&lt;sub&gt;5&lt;/sub&gt;</td>
<td>10 kohm</td>
</tr>
<tr>
<td>6</td>
<td>R&lt;sub&gt;6&lt;/sub&gt;</td>
<td>1 kohm</td>
</tr>
<tr>
<td>7</td>
<td>R&lt;sub&gt;7&lt;/sub&gt;</td>
<td>25 kohm</td>
</tr>
<tr>
<td>8</td>
<td>R&lt;sub&gt;8&lt;/sub&gt;</td>
<td>.075 kohm</td>
</tr>
<tr>
<td>9</td>
<td>R&lt;sub&gt;9&lt;/sub&gt;</td>
<td>20 kohm</td>
</tr>
<tr>
<td>10</td>
<td>R&lt;sub&gt;10&lt;/sub&gt;</td>
<td>30 kohm</td>
</tr>
<tr>
<td>11</td>
<td>R&lt;sub&gt;11&lt;/sub&gt;</td>
<td>1 kohm</td>
</tr>
<tr>
<td>12</td>
<td>Q&lt;sub&gt;1&lt;/sub&gt;, C&lt;sub&gt;be&lt;/sub&gt;</td>
<td>20 pf.</td>
</tr>
<tr>
<td>13</td>
<td>Q&lt;sub&gt;1&lt;/sub&gt;, C&lt;sub&gt;bc&lt;/sub&gt;</td>
<td>3.7 pf.</td>
</tr>
<tr>
<td>14</td>
<td>Q&lt;sub&gt;1&lt;/sub&gt;, C&lt;sub&gt;ce&lt;/sub&gt;</td>
<td>29 pf.</td>
</tr>
<tr>
<td>15</td>
<td>Q&lt;sub&gt;1&lt;/sub&gt;, g&lt;sub&gt;be&lt;/sub&gt;</td>
<td>.32 mmho</td>
</tr>
<tr>
<td>16</td>
<td>Q&lt;sub&gt;1&lt;/sub&gt;, g&lt;sub&gt;m&lt;/sub&gt;</td>
<td>32 mmho</td>
</tr>
<tr>
<td>17</td>
<td>Q&lt;sub&gt;2&lt;/sub&gt;, C&lt;sub&gt;be&lt;/sub&gt;</td>
<td>20 pf.</td>
</tr>
<tr>
<td>18</td>
<td>Q&lt;sub&gt;2&lt;/sub&gt;, C&lt;sub&gt;bc&lt;/sub&gt;</td>
<td>3.7 pf.</td>
</tr>
<tr>
<td>19</td>
<td>Q&lt;sub&gt;2&lt;/sub&gt;, C&lt;sub&gt;ce&lt;/sub&gt;</td>
<td>29 pf.</td>
</tr>
<tr>
<td>20</td>
<td>Q&lt;sub&gt;2&lt;/sub&gt;, g&lt;sub&gt;be&lt;/sub&gt;</td>
<td>.32 mmho</td>
</tr>
<tr>
<td>21</td>
<td>Q&lt;sub&gt;2&lt;/sub&gt;, g&lt;sub&gt;m&lt;/sub&gt;</td>
<td>32 mmho</td>
</tr>
<tr>
<td>22</td>
<td>Q&lt;sub&gt;3&lt;/sub&gt;, C&lt;sub&gt;be&lt;/sub&gt;</td>
<td>10.3 pf.</td>
</tr>
<tr>
<td>23</td>
<td>Q&lt;sub&gt;3&lt;/sub&gt;, C&lt;sub&gt;bc&lt;/sub&gt;</td>
<td>12 pf.</td>
</tr>
</tbody>
</table>
24  $Q_3 \ C_{ce}$  28.5 pf.
25  $Q_3 \ g_{be}$  .00314 mmho
26  $Q_3 \ g_{m}$  32 mmho
27  $Q_4 \ C_{be}$  10.3 pf.
28  $Q_4 \ C_{bc}$  12 pf.
29  $Q_4 \ C_{ce}$  28.5 pf.
30  $Q_4 \ g_{be}$  .00314 mmho
31  $Q_4 \ g_{m}$  32 mmho
32  $Q_5 \ C_{be}$  20 pf.
33  $Q_5 \ C_{bc}$  3.7 pf.
34  $Q_5 \ C_{ce}$  29 pf.
35  $Q_5 \ g_{be}$  .32 mmho
36  $Q_5 \ g_{m}$  32 mmho
37  $Q_6 \ C_{be}$  20 pf.
38  $Q_6 \ C_{bc}$  3.7 pf.
39  $Q_6 \ C_{ce}$  29 pf.
40  $Q_6 \ g_{be}$  .42 mmho
41  $Q_6 \ g_{m}$  32 mmho
42  $Q_7 \ C_{be}$  40 pf.
43  $Q_7 \ C_{bc}$  7.4 pf.
44  $Q_7 \ C_{ce}$  58 pf.
45  $Q_7 \ g_{be}$  1.14 mmho
46  $Q_7 \ g_{m}$  64 mmho
given below:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Transistors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Q₁, Q₂, Q₅, Q₆</td>
</tr>
<tr>
<td>2</td>
<td>Q₃, Q₄</td>
</tr>
<tr>
<td>3</td>
<td>Q₇</td>
</tr>
</tbody>
</table>

6.4 **Input-Output Description**

The rationale for choosing an integrated circuit as an example is the availability of only a specified number of terminals in the network. Therefore, the voltages and currents which can be applied and measured at these terminals form the inputs and outputs, respectively, of the model:

\[
\begin{align*}
\mathbf{u} &= \begin{bmatrix} e_1, e_2, e_3, e_4, e_5, e_6 \end{bmatrix}^T \\
\mathbf{y} &= \begin{bmatrix} i_1, i_2, i_3, i_4, i_5, i_6 \end{bmatrix}^T 
\end{align*}
\]  

(4)

The power supply connections, \( V^+ \) and \( V^- \), are assumed at the same potential (a common node) for a.c. analysis. These assumptions correspond to the actual manner in which one would bias, apply inputs to, and measure outputs from the real \( \mu \)A709 amplifier. The output equation is of the form:

\[
Y(s) = T(s) U(s)
\]  

(5)

where \( T(s) \) is the \( 6 \times 6 \) port admittance matrix of the amplifier.
6.5 Programming the Fault Analysis Algorithm

The model of Figure 5 with input-output relationships given by equations (4) and (5) was solved by the program RAPID. The total number of elements of the modeled system was 52, including:

6 independent voltage sources,
18 resistors,
21 capacitors,
and 7 dependent current sources. As the algorithm for fault analysis requires a fictitious port in series or parallel with each element under consideration, a current source was placed in parallel with each element of the model except the independent voltage sources.

RAPID and a matrix manipulation program EQNS were used to form the system of equations:

\[
\begin{align*}
\dot{x} &= Ax + Bu + B_1\ddot{u} + B_a u_f \\
y &= Cx + Du + D_1\ddot{u} + D_a u_f \\
y_f &= C_i x + D_i u + D_{1f}\ddot{u} + D_{af} u_f
\end{align*}
\]

where the state vector of the system was found to be a seven-vector:

\[
x = \begin{bmatrix}
v_{be} \text{ of } Q_1 \\
v_{be} \text{ of } Q_3 \\
v_{ce} \text{ of } Q_3 \\
v_{ce} \text{ of } Q_4 \\
v_{ce} \text{ of } Q_5 \\
v_{be} \text{ of } Q_6 \\
v_{ce} \text{ of } Q_6
\end{bmatrix}
\]
y and u are six-vectors, and \( y_f \) and \( u_f \) are 46-vectors.

The necessity of derivatives of inputs is seen by the several occurrences of loops of capacitors and voltage sources. It is interesting to note here that even though 46 fictitious sources were added to the system, while RAPID can solve systems with no more than 20 sources, the linearity of the system in \( u_f \) allowed the repeated solution of the model with subsets of \( u_f \) included. The effects of all the sources were found with three solutions. The program EQNS was then used to partition the matrices:

\[
\hat{B}_i = \begin{bmatrix} B_i & B_{ai} \end{bmatrix}
\]

and

\[
\hat{D}_i = \begin{bmatrix} D_i & D_{ai} \\ \_ & \_ \\ D_i & D_{af_i} \end{bmatrix} \quad i = 1, 2, 3
\]

where "i" represents the ith subset of \( u_f \) included in a given repeated solution. Finally, EQNS was used to join the submatrices \( B_{ai} \), \( D_{ai} \), \( D_{af_i} \) to form:

\[
B_a = \begin{bmatrix} B_{a1} & B_{a2} & B_{a3} \end{bmatrix},
\]
\[
D_a = \begin{bmatrix} D_{a1} & D_{a2} & D_{a3} \end{bmatrix},
\]

and

\[
D_{af} = \begin{bmatrix} D_{af1} & D_{af2} & D_{af3} \end{bmatrix}.
\]

The program FAULT was given the state and output equations (6) and was used to determine various properties of interest for fault analysis, including the nominal solution, the sensitivity matrices, the functions \( r(i, j w ) \), \( i = 1, \ldots, 46 \), and the centers of circles for
frequencies. This program also was used to determine the similarity fault sets, dual similarity fault sets, and effects of measurement noise upon fault identification.

6.6 Determining Single Faults of the \( \mu A709 \)

The frequencies of interest in the circuit of Figure 8 were determined to be in the range:

\[
10^4 < \omega < 2 \times 10^8 \text{ rad/sec} \tag{10}
\]

by finding the spectrum of the matrix "A" whose eigenvalues were found to be negative-real and within the range of inequality (10). The network solution \( T(j \omega) \) was initially performed for the set \( \omega_1 = 10^4 \), \( \omega_2 = 10^6 \), and \( \omega_3 = 10^8 \). The matrices:

\[
C_i(j \omega) = \frac{-jS_i(j \omega)}{2 \text{Im} r(i,j \omega)} \tag{11}
\]

calculated by the program FAULT, were quite similar for several sets of elements \( (p_4, p_{22}, \text{ and } p_{26}; p_7, p_{11}, p_{30}, \text{ and } p_{31}) \) at \( \omega = 10^4 \), implying that measurement at this frequency could not yield fault isolation among elements of these sets. However, for \( \omega = 10^6 \) and \( \omega = 10^8 \), a rather simple ordering of the \( C_i(j \omega) \) matrices demonstrated rather quickly the fact that:

Remark: Each element of the \( \mu A709 \) operational amplifier,

Figure 8, is a member of a different SFS. Hence, the arbitrary failure of any element of this circuit can
almost always be isolated.

It is important to note that the truth of the statement above is not dependent upon the fact that certain element values were chosen for the model; rather it holds because of the structure of the model and the location of available terminals. Therefore, if it is known that Figure 3 gives an adequate model for the \( \mu A709 \) amplifier, then with arbitrarily accurate measurements, any single failure can be isolated.

The fact that the matrices of circular loci are unique for each element at a single frequency means that the measurement of \( T(j \omega) \) need be made at only one frequency in order to isolate single faults. That any failure can be isolated by measurements at one frequency is seen.

However, measurement of a \( 6 \times 6 \) matrix of transfer functions is apparently necessary. It was stated in Chapter V that sometimes not all matrix transfer function elements are necessary. The example chosen has been found to be one such network. Assuming that the easiest functions to measure are driving-point rather than transfer admittances, the matrices of centers (or sensitivities), evaluated at \( \omega = 10^8 \text{ rad/sec} \), may be ordered in the following manner:

(a) Degenerate cases \( p_5, p_{10}, p_{18}, p_{28}, p_{33}, p_{44} \) and \( p_{46} \), in which the loci of \( \Delta T \) are lines, are ordered by \( S^i(j \omega) \), while other elements, whose loci of \( \Delta T \) are circles, are ordered using circles \( C^i(j \omega) \);

(b) Set \( i = 1 \);

(c) If \( \text{Re} \, c^r_{ii}(j \omega) < \text{Re} \, c^s_{ii}(j \omega) \), then \( p_T \) pre-orders \( p_s \);
\( \text{Re} \sigma_{11}^{R}(jw) < \text{Re} \sigma_{11}^{S}(jw) \)

(d) If (c) does not hold, then increment \( i \) and repeat (c) until either (c) holds or \( i > 6 \).

If each element is related to each other by some pre-order, then the elements are ordered. Hence, the matrices of centers of circles are different for each pair of elements, and single faults can always be isolated. That is, each SFS contains one circuit element.

For the example solved, it is indeed true that the elements are ordered by the driving-point admittances. Hence, any failure can be isolated by measurement of driving-point admittances only. It may be further noted that if \( c_{22}(jw) \) is omitted from step (b) above, an ordering of the elements still exists.

Therefore, only the measurement of driving-point admittances at ports 1, 3, 4, 5, and 6 are necessary for single fault isolation of the \( \mu A709 \) amplifier model discussed. The above represents a significant reduction in measurement as compared with the most general case of 36 admittances measured at two frequencies, and is most gratifying.

In order to isolate faults in the model under discussion, a final program would require the following vectors, calculated at \( w = 10^8 \) rad/sec:

(a) \( \hat{T}(jw) = [\tau_{11}, \tau_{33}, \tau_{44}, \tau_{55}, \tau_{66}]^T(jw) \) \hspace{1cm} (12)

(b) \( \hat{C}^i(jw) = [c_{11}^i, c_{33}^i, c_{44}^i, c_{55}^i, c_{66}^i]^T(jw) \) \hspace{1cm} (13)

for \( i \neq 5, 10, 18, 28, 33, 44, 46 \).

and \( \hat{S}^i(jw) = [\sigma_{11}^i, \sigma_{33}^i, \sigma_{44}^i, \sigma_{55}^i, \sigma_{66}^i]^T(jw) \) \hspace{1cm} (14)
for \( i = 5, 10, 18, 28, 33, 44, 46 \).

Also of value would be a table relating each element to the port(s) whose measurement is necessary to uniquely identify a failure of that element. Then the measurement at port 1 would be compared with only those elements for which measurements at port 1 are necessary for identification. If no agreement were found, the admittance of port 3 would be compared to the list of elements identifiable from port 3 above. The process would continue until, at some port, agreement is found, and the failed element isolated. Such a table for the example may be found in Table 1. The existence of this table reduces the number of comparisons necessary to uniquely identify a failure. An example of the loci of elements observable from port 3 is shown in Figure 9. This corresponds to one of the five planes necessary for fault isolation when the graphical method of Chapter IV is used.

The computer time necessary to perform the single fault analysis of the 46 element model is perhaps interesting to note. Time usage on a Burroughs B-5500 computer (excluding compilation time) is given below:

<table>
<thead>
<tr>
<th></th>
<th>Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RAPID 3 sets of sources</td>
<td>2.80</td>
</tr>
<tr>
<td>EQNS</td>
<td>.50</td>
</tr>
<tr>
<td>FAULT 3 frequencies</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td>Total</td>
</tr>
<tr>
<td></td>
<td>4.50</td>
</tr>
</tbody>
</table>

It should be noted that approximately two minutes of computation time would be necessary simply to produce the state and output equations,
<table>
<thead>
<tr>
<th>Number of Element</th>
<th>Port Comparisons Necessary for Isolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 3</td>
</tr>
<tr>
<td>2</td>
<td>1, 3</td>
</tr>
<tr>
<td>3</td>
<td>1, 3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1, 3, 5</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>5, 6</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>1, 3, 5</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>1, 3</td>
</tr>
<tr>
<td>18</td>
<td>3, 4</td>
</tr>
<tr>
<td>19</td>
<td>1, 3</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
</tr>
</tbody>
</table>
LOCI OF $\Delta \tau_{33}$ AS ELEMENT VALUES CHANGE

$\omega = 10^8 \text{ rad/sec}$
and solve these at 3 frequencies. Viewed from that perspective, only about two and one-half minutes are actually required to determine the similarity fault sets of the model and produce the matrices of circles to be used in an isolation algorithm. Computation time is quite reasonable for single fault analysis.

6.7 Noise Sensitivity of the \( uA709 \) Model

An initial question of interest to the user of a failure isolation algorithm is whether one can determine that an element has changed by at least some minimum percentage \( \frac{\Delta p_{\text{min}}}{p} \). Since we can write the change in any matrix transfer function element when a single element \( p_i \) varies as:

\[
\Delta \tau_{m,n}(jw) = \frac{\sigma_{m,n}^i(jw) \Delta p_i}{1 - r(i,jw) \Delta p_i}
\]  \hspace{1cm} (15)

If \( \varepsilon = \frac{\Delta p_{\text{min}}}{p} \), where \( p \) is any element in \( N \), is small enough then the approximation:

\[
\Delta \tau_{m,n}(jw) = \sigma_{m,n}^i(jw) \Delta p_i
\]  \hspace{1cm} (16)

is valid, and a measurement accuracy of:

\[
\left| \frac{\Delta \tau_{m,n}(jw)}{\tau_{m,n}(jw)} \right| = \left| \frac{p_i}{\tau_{m,n}(jw)} \sigma_{m,n}^i(jw) \right| \cdot \varepsilon \hspace{1cm} (17)
\]

is necessary to detect from measurement of \( \tau_{m,n} \) that the \( i^{th} \) element has changed. To detect a change in the \( i^{th} \) element, it is neces-
sary only to detect it from one of the \( \tau_{m,n} \). Thus, the measurement accuracy required is no more than:

\[
\alpha_m^i(jw) = \max_{m,n} \left( \frac{\Delta T_{m,n}(jw)}{\tau_{m,n}(jw)} \right) = \max_{m,n} \left( \frac{P_i \sigma_{m,n}^i(jw)}{\tau_{m,n}(jw)} \right) \epsilon \quad (18)
\]

And, since detection of a failure at any frequency is sufficient, we define:

\[
A_m^i = \max_{w} \left( \frac{a_m^i(jw)}{P_i \sigma_{m,n}^i(jw)} \right) = \max_{w} \left( \frac{P_i \sigma_{m,n}^i(jw)}{\tau_{m,n}(jw)} \right) \epsilon \quad (19)
\]

If one wishes to detect the change of any element in the network, the overall measurement accuracy, \( A_m^i \), required is:

\[
A_m^i = \min_{i} \left( \frac{A_m^i}{A_m^i} \right) = \min_{i} \left( \frac{P_i \sigma_{m,n}^i(jw)}{\tau_{m,n}(jw)} \right) \epsilon \quad (20)
\]

For the example network, the most difficulty is encountered in detecting changes in element \( p_7 \), for which \( A_m^i = 0.002 \epsilon \); thus, letting \( \epsilon = 0.1 \), a measurement accuracy of approximately 0.02 percent is necessary to detect a ten percent change in any element.\(^1\) If the measurement accuracy available is specified, as \( A_m^i \), then it is valid to ask the minimum percentage change of each element which may be detected by external measurements. The mathematics are similar, and the result follows

---

\(^1\) That such accurate measurements are necessary to detect changes in \( p_7 \) means simply that these transfer functions are insensi
tive with respect to this element. This is not a bad condition unless one is primarily interested in detecting changes in the element value.
immediately:

\[
\varepsilon_i = \min_{\omega} \min_{m,n} \left| \frac{\tau_{m,n}(j\omega)}{\sigma_{m,n}(j\omega)} \right| A_m
\]

(21)

A plot of the number of elements for which \( \varepsilon_i \geq \varepsilon \) versus \( \varepsilon \) for \( A_m = .001 \) is given in Figure 10. In this plot the number of elements for which \( \Delta T \) is undetectable versus the percentage change in element value is presented for measurement accuracy of .1 percent. It may be noted that, except for three elements \( (p_7, p_{20}, \text{ and } p_{25}) \), 10 percent element changes are detectable from some port at some frequency.

The question of ambiguity in failure isolation due to measurement error is a much more broad question than that of simply observing that an element has failed. Little can be said generally about measurement error difficulties in isolating failures. However, it is possible, using the algorithms of Chapter V, to know when the cause of a network failure could be one of several sources. For the graphical method, the circles may become widened into rings, having thickness of twice the unnormalized measurement accuracy. A failure \( \Delta T_{u,v}(j\omega) \) could be caused then by any element whose ring of loci pass through that point for each matrix element. A method of comparison suitable for computer analysis need simply allow for measurement error in comparisons of measured \( \Delta T \) with the locus for each parameter. (The graphical and comparison methods are seen to be equivalent except in their manner of implementing an algorithm.) The functional approach:
\[ F : \Delta T(j\omega) \rightarrow \mathbf{I} , \] (22)

may include measurement error in the form of \( F \); such a form was discussed in Section 5.2. Any ambiguity in isolating the failed elements in the \( \mu A709 \) model under discussion would be observed by the methods discussed above, and is dependent upon the actual value of the element which failed.

6.8 Determining Dual Similarity Fault Sets in the \( \mu A709 \) Model

As a direct result of Theorem III.3, failure of two elements may be isolated for any pair of elements if:

(a) each SFS contains only one element, (23)

and

(b) \( \det \xi_1(i,k,w,w_1) \neq 0 \) for some \( w_1, w \)

and for all element pairs \( i,k \).

From the above discussion of single faults, it was determined that each element was in a different SFS. Hence, condition (a) is automatically satisfied. The determination of the validity of condition (b) would require the computation of \( \frac{3}{2} (46)(45) = 1035 \) 2x2 determinants for each of 36 matrix elements, clearly an impractical task. However, the fact that single faults are analyzable would cause one to suspect that each dual fault might also produce unique network solutions.

6.9 Conclusions

A nontrivial model of a real network has been developed, in which input-output terminals correspond exactly to those available to the user, and whose internal structure is a reasonably simplified model of the
structure specified by the manufacturer. Solution of the model for an assumed set of element values indicated that single failures can be observed and isolated if there is no measurement error, and that the accuracy of measurement of real instruments is sufficient for the detection of small changes of almost all elements.

Especially interesting is the unexpected result that only driving-point admittances need be measured, that measurement at terminal 2 is not necessary, and that measurements of one frequency are sufficient for single fault isolation.
VII. COMPONENT PARAMETER IDENTIFICATION

When several parameters \(^1\) of a network \(N\) change in value, it becomes impractical to search through the entire set of combinations of parameters which could have failed. A solution of the entire parameter vector is then a feasible approach to the isolation of failures. Conditions on uniqueness of the solution of the element vector of \(N\) are derived.

7.1 Definition of Component Parameter Identification

Definition: A network \(N\) is said to be component parameter identifiable (CPI) if a given network solution (input-output) corresponds to a unique parameter vector, \(P_0\).

A weaker definition, suitable for the study of fault analysis, will prove useful.

Definition: A network \(N(P_0)\) is said to be locally parameter identifiable (LPI) if there exists an \(\varepsilon < 0\) such that the network solution of \(N(P)\) is not equal to the network solution of \(N(P_0)\) for any

\[
P \neq P_0 \quad \text{such that} \quad \|P - P_0\| < \varepsilon \quad (1)
\]

\[
\|P - P_0\| < \varepsilon \quad (2)
\]

---

\(^1\) The term "parameters" of this section corresponds to the term "element" used throughout the thesis and is used in order to be consistent with the terminology of the literature.
The LPI definition states that there is no continuum of \( P \) vectors, passing through \( P_0 \), which have the same network solution. Because \( P_0 \) is known for fault analysis, and because the parameter vector is not expected to change drastically in parameter space when a fault occurs, the question of whether many nearby parameter vector solutions exist for the failed network is meaningful. If \( N(P_0) \) is not LPI, then one has no hope of determining which parameters fail by finding a new solution for all the network parameters.

7.2 **Local Parameter Identification - A State Formulation**

From Chapter III, the matrix transfer function for parameter vector \( P_0 \) can be written as that due to an arbitrary parameter vector \( P_1 \) with a correction term. Using a convenient notational form, with the understanding that matrices \( T, A, B, C \) and \( D \) are functions of frequency as well as \( P_0 \) and \( P_1 \), we may write:

\[
T(P_0) + T(P_1) + A(P_1)B(P_0, P_1)C(P_1), \quad (3)
\]

where:

\[
B(P_0, P_1) = D(P_0) + \begin{bmatrix}
\frac{1}{p_1} & 0 \\
\frac{\Delta p_1}{\Delta p_1} & \ddots \\
0 & \ddots & \frac{1}{p_i} \\
\end{bmatrix}
\]

\[
P_1 = [p_1^1, p_2^1, \ldots, p_i^1]^T, \quad (5)
\]
and \( \Delta P \overset{D}{=} P_0 - P_1 = [\Delta p_1, \Delta p_2, \ldots, \Delta p_\#]^T \).

To avoid singularities in equation (4), \( P_1 \) is chosen initially such that no \( \Delta p_i \) is zero. This is always possible, since \( P_1 \) is arbitrary.

In order to determine whether \( N \) is LPI, it is sufficient to prove that, at least locally, the only parameter vector with matrix transfer function \( T_o \) is \( P_0 \). To verify (or disprove) this assumption, we may write a matrix equation in \( P \), letting \( \Delta T = T(P_0) - T(P_1) \):

\[
E(P) \overset{D}{=} \Delta T - A(P_1)B(P, P_1)C(P_1)
\]

(6)

By observation of equation (3), \( E(P_0) = [0] \). Are there any other vectors in a neighborhood of \( P_0 \) for which \( E = [0] \)?

It is helpful to define a scalar function which is zero only when \( E \) is the zero matrix:

\[
f(P) = \| E(P) \|.
\]

(7)

where \( \| E \| = \sum_{i,j} e_{ij}^* e_{ij} \)

(8)

(The superscript asterisk denotes complex conjugation.)

Remark: In this chapter the limits on sums will be understood to include each element of the matrix or vector which the corresponding index variable subscripts. Thus, if \( E \) is a \( k \times \# \) matrix,

\[
\sum_{i,j} e_{ij} \overset{D}{=} \sum_{i=1}^{k} \sum_{j=1}^{\#} e_{ij}
\]

(9)
Therefore, from equations (6), (7), and (8):

\[
f(P) = \sum_{i,j} (\Delta \tau_{ij} - \sum_{m,n} a_{im} b_{mn} c_{nj}) (\Delta \tau^* - \sum_{m,n} a^*_{im} b^*_{mn} c^*_{nj})
\]

(10)

where subscripted \( \Delta \tau, a, b, c \) variables are elements of \( \Delta \tau, A, B, \) and \( C \) matrices respectively.

Defining:

\[
\nabla f(P) = \left[ \frac{\partial f}{\partial P_1}, \frac{\partial f}{\partial P_2}, \ldots, \frac{\partial f}{\partial P_\hat{n}} \right]^T
\]

\( P = \hat{P} \) \hspace{1cm} (11)

it is seen that \( f(P_0) = 0 \) \hspace{1cm} (12)

and \( \nabla f(P_0) = 0 \), \hspace{1cm} (13)

since:

\[
\frac{\partial f}{\partial P_k} = \sum_{i,j} (\Delta \tau_{ij} - \sum_{m,n} a_{im} b_{mn} c_{nj}) (\Delta \tau^* - \sum_{m,n} a^*_{im} b^*_{mn} c^*_{nj})
\]

(14)

\[
+ (\Delta \tau^*_{ij} - \sum_{m,n} a^*_{im} b^*_{mn} c^*_{nj}) (\Delta \tau - \sum_{m,n} a_{im} b_{mn} c_{nj})
\]

and the first term of each product is zero.

Theorem VII: A sufficient condition for \( P_0 \) to be a unique local solution to equation (6), is that the matrix:

\[
J_f(\hat{P}) = \begin{bmatrix}
\frac{\partial^2 f}{\partial P_1^2} & \ldots & \frac{\partial^2 f}{\partial P_1 \partial P_\hat{n}} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial P_\hat{n} \partial P_1} & \ldots & \frac{\partial^2 f}{\partial P_\hat{n}^2}
\end{bmatrix}
\]

\( P = \hat{P} \) \hspace{1cm} (15)

be positive definite at \( P_0 \).

Proof: This result obtains because \( E(P) = [0] \) exactly when \( f(P) = 0 \).
As \( f \) is continuous and contains continuous first and second derivatives in \( P \), we may expand \( f(P) \) in a Taylor's series about the point \( P_o \):

\[
f(P) = f(P_o) + \nabla f(P_o)(P-P_o) + \frac{1}{2}(P-P_o)^T J_f(P_o)(P-P_o) + O(P-P_o)^3
\]

Substituting from equations (10) and (11), and assuming \( P - P_o \) is small enough that terms of \( O(P - P_o)^3 \) are negligible with respect to terms of \( O(P - P_o)^2 \), we write:

\[
f(P) = \frac{1}{2}(P - P_o)^T J_f(P_o)(P - P_o) . \tag{16}
\]

\( J_f(P_o) \) positive definite implies that for any \( P \neq P_o \),

\[
(P - P_o)^T J_f(P_o)(P - P_o) > 0 . \tag{17}
\]

Hence, \( f(P) > 0 \) for \( P \neq P_o \) and \( P \) in a neighborhood of \( P_o \), and \( P \) is not an allowed parameter vector. \( \Box \).

7.3 **Computation of the Matrix \( J_f(P_o) \)**

Computation of \( J_f(P_o) \) follows, with some algebra, from equations (4), (6), (7), (14), and (15). From equation (14), it is necessary to compute \( \frac{\partial b_{m,n}}{\partial p_k} \). The method of matrix modification \[14\] is applicable since only one diagonal element of \( B^{-1}(P, P_1) \) changes as the limit in \( p_k \) is taken:

\[
b_{m,n}(p_k \neq p_k^o, p_1) = b_{m,n}(P_o, P_1) + \gamma_k b_{m,k} b_{k,n} \tag{18}
\]
where:

\[ \gamma_k = \frac{h_k}{h_k b_{kk} + 1} \]  

(19)

and:

\[ h_k = \frac{p_k - p_k^0}{(p_k - \frac{1}{2})(p_k^0 - p_k^1)} \]  

(20)

Thus:

\[ \frac{\partial b_{m,n}}{\partial p_k} \bigg|_{p_o} = \frac{b_{m,k} b_{k,n} \frac{\partial \gamma_k}{\partial p_k} \bigg|_{p_o}}{b_{m,k} b_{k,n} \frac{\partial h_k}{\partial p_k} \bigg|_{p_o}} = \frac{b_{m,k} b_{k,n}}{(p_k^0 - \frac{1}{2})^2} \]  

(21)

Thus, from equation (14):

\[ \frac{\partial f}{\partial p_k} = \frac{1}{(p_k^0 - \frac{1}{2})^2} \left( \sum (\Delta \tau_{ij} - \sum a_{im} b_{mn} c_{nj})(- \sum a_{im}^* b_{mn}^* c_{nj}^*) \right) \]

\[ + (\Delta \tau_{ij}^* - \sum a_{im}^* b_{mn}^* c_{nj}^*) (- \sum a_{im} b_{mn}^* c_{nj}) \]  

(22)

Differentiating equation (22) with respect to \( p_k \) and evaluating the result at \( P = P_o \), the second partial derivative is:
\[
\frac{\partial^2 f}{\partial p_k \partial p_h} = \frac{1}{(p_k - p_h)^2} \sum_{i,j,m,n} \left[ (-\sum a_{im} b_{mn} c_{nj}) + (-\sum a_{im} b_{mknj})^* \right] \\
+ \left[ (-\sum a_{im} b_{mknj}) + (-\sum a_{im} b_{mknj})^* \right] \\
= \frac{2}{(p_k - p_h)^2} \sum_{i,j,m,n} R \left\{ \sum a_{im} b_{mknj} b_{mknj} \sum a_{im} b_{mknj} \right\} (23)
\]

The matrix \( J_f(p_o) \) represented element-wise by equation (23) is positive definite if and only if the matrix \( G \) whose elements are:

\[
\delta_{kh} = R \left\{ \sum a_{im} b_{mknj} \sum a_{im} b_{mknj} \right\} (24)
\]

is also positive definite. The above simplification occurs because:

\[
J_f(p_o) = L^T G L (25)
\]

where \( L \) is nonsingular diagonal and:

\[
\ell_{ii} = \frac{\sqrt{2}}{(p_i - p_i)^2} (26)
\]

In order to facilitate the computation of \( G \), it is seen from equation (24) that:

\[
\delta_{kh} = R \left\{ \sum a_{im} b_{mknj} \sum a_{im} b_{mknj} \right\} (27)
\]

The matrix products necessary for the computation of equation (27) are
AB and BC. Each element of G is simply a sum of the real part of the products of two elements of each of these matrices.

7.4 Remarks and Conclusions

\( J_f(P_0) \) positive definite is sufficient for LPI of \( N(P_0) \). This statement is true for any frequency for which one wishes to evaluate \( J_f \) or \( G \). It should be noted that even if \( J_f \) is not positive definite for any frequency, other sufficient conditions exist for LPI:

Lemma: A sufficient condition for LPI of \( N(P_0) \) is that:

\[
K(w_1, \ldots, w_m) = \sum_{i=1}^{m} J_f(P_0, w_i)
\]

be positive definite for any choice of \( m, w_1, \ldots, w_m \).

Proof: Equation (6) must hold for all frequencies if \( P \) in a neighborhood of \( P_0 \) is a solution to that equation. Thus, if \( K \) is positive definite for any \( w_1, \ldots, w_m \), then in a neighborhood of \( P_0 \),

\[
\sum_{i=1}^{m} f(P,w) \approx \frac{1}{2} \sum_{i=1}^{m} (P - P_0)^T K(w_1, \ldots, w_m)(P - P_0) > 0
\]

(29)

Since \( f \geq 0 \) for all \( P, w \) there must exist some \( w_i \) such that:

\[
f(P, w_i) > 0.
\]

This implies that \( P \) does not satisfy equation (6) at \( w_i \). Q.E.D.

Necessary conditions for LPI of \( N(P_0) \) are well defined in
terms of higher even-order tensors of partial derivatives of $f$ at several frequencies. However, the difficulty of computation of these higher order partial derivatives and their properties of interest causes these conditions to be practically useless. Networks for which the sufficient conditions are not also necessary conditions are referred to as singular in the calculus of variations and depend on special values of $P_0$.

A final note concerning local parameter identification as compared with global component parameter identification is in order. Calahan [15] poses the question, in terms of continuously equivalent networks, of whether all global parameter solutions which produce the same input-output relationships in $N$ can be reached by a continuous path of solution. That such is not possible is proved by R.W. Newcomb [16], by use of an example of a six-element network whose driving-point impedance may be caused by exactly two sets of parameters.
VIII. CONCLUSIONS

8.1 Results Derived

The problem of identifying and isolating failures in lumped and
distributed linear, time-invariant networks from input-output relation-
ships has been studied. Necessary and sufficient conditions for the
ability to isolate single faults have been found, along with conditions
concerning the minimum grouping of elements which can be distinguished
when a single fault occurs. It has been established that one or, under
certain well defined conditions, two frequencies is sufficient to deter-
mine from which minimum grouping a single fault has occurred; methods
for isolating this group have been proposed and discussed.

The case of several simultaneous faults has been considered,
and, for the case of two faults, sufficient conditions for the theoretical
ability to isolate these faults have been presented. An algorithm for the
isolation of dual faults was proposed; however, it was noted that, except
for small networks, the difficulty in isolating dual faults is considerable.
Although the manner in which fault analysis can be extended to include
three or more faults is seen to be similar to that of single and dual
faults, the algebraic difficulty and the excessive computations required
have precluded the study of these cases.

The proposed methods for obtaining minimum groups of single
faults and for isolating these faults have been discussed in terms of
computational difficulty, as related to the computation necessary to
solve the nominal system. Computer algorithms for fault isolation have been discussed. To demonstrate the practicality of the single fault results, a model of a \( \mu A709 \) integrated circuit operational amplifier has been analyzed using a previously available circuit analysis program along with programs to compute the minimum single fault grouping. It was determined that a fault in any element of the model could, indeed, be isolated from the terminals available to the user; furthermore, the number of transfer function measurements required was shown to be significantly less than the maximum possible number of measurements. That only one frequency of measurement is necessary to isolate a failure was shown for this example. The effect of measurement noise typical of that in available instruments was computed and discussed; the results were generally favorable for detecting the existence of small changes in element values.

Finally, an alternate method of isolating several failures was discussed. Beginning with the nominal network, several iterative methods obtain, in the limit, element values for which the network's input-output relationships are equivalent to that of the failed network. Sufficient conditions have been presented which, except for a possible finite number of isolated set of element values, allow one to state that the obtained elements are unique. This result allows one to use the identification method with some assurance that the predicted element faults actually are the same as the elements which have failed.
8.2 Open Questions for Future Work

The assumptions upon the network under consideration suggest several avenues of further study. That the nominal network is precisely known before its failure is often not true. Linear integrated circuits, in which element tolerances are quite large, provide an excellent counter-example. It is quite possible that some definite statements could be made concerning identification and isolation of faults if it is known, say, that the nominal network elements lie in some region in element space. Also, the method of local parameter identification could be used in some networks to determine each element value. Also, a probabilistic model for the element values could perhaps be used to determine most likely element faults. The problem of incomplete models would also be included in the general field of isolation of failures in partially undefined networks.

A second area of extension which is of practical importance is the inclusion of faults resulting in nonlinear networks. Several possible means of including these failures are possible. The frequency response could be broadened to include harmonic response. Or a table look-up procedure, such as that proposed by Seshu and Waxman, might be quite useful for faults resulting in nonlinear responses; a catalog of most likely nonlinear failures could make a table of responses small enough for usefulness.

Finally, the restriction of two-terminal networks might be removed so that faults in transistors as transistors could be isolated.
This would also allow the user to isolate failures of subnetworks within a larger network without the necessity of considering each element in each subnetwork. The faulty subnetwork, once isolated, could then be replaced without the knowledge of which element had failed. Work in the area of faults in interconnected subnetworks would require some result concerning the manner in which an allowed failure in one subnetwork would affect the solution of the interconnected network and, in particular, the input-output relationships of the entire network.

The ability to determine quickly and easily how failures of two terminal elements affect the network solution, and the resulting ability to isolate the failed element permit the application of fault analysis to almost any network for which the frequency domain solution can be computed.
APPENDIX I

The change in transfer function due to a change $\Delta p$ in a single element may be written:

$$
\Delta T(j\omega) = \frac{s^p \Delta p}{1 - r_p(\omega) \Delta p},
$$

(1)

Rewriting, in simplified form, the expression for any matrix element of equation (1):

$$
x = \frac{a u}{1 - r u},
$$

(2)

where $x = \Delta r_{i,k}(j\omega)$, $a = \sigma_{i,k}(j\omega)$, $u = \Delta p$, and $r = r_p(j\omega)$.

Thus:

$$(x_r + jx_i)(1 - r_r u - j r_i u) = (a_r + j a_i) u
$$

(3)

where $R_e\{x\} = x_r$, $I_m\{x\} = x_i$, and similarly for variables $r$ and $a$.

Equating real and imaginary parts of (3):

$$
x_r(1 - r_r u) + x_i r_i u = a_r u
$$

(4a)

$$
x_i (1 - r_r u) - x_r r_i u = a_i u
$$

(4b)

Removing $u$ from (4), the result obtains:

$$
\frac{x_r}{a_r + x_r r_r - x_i r_i} = \frac{x_i}{a_i + x_i r_r + x_r r_i}
$$

(5)
Hence
\[ r_i x_r^2 + a_i x_r + r_i x_i^2 - a_r x_i = 0 \]  \hspace{1cm} (6)

or for \( r_i \neq 0 \),
\[ x_r^2 + \left(\frac{a_i}{r_i}\right) x_r + \left(\frac{a_i}{2r_i}\right)^2 + x_i^2 - \left(\frac{a_i}{r_i}\right) x_i + \left(\frac{a_i}{2r_i}\right)^2 = \frac{a_r^2 + a_i^2}{\left(2r_i\right)^2} \]  \hspace{1cm} (7)

Hence
\[ \left(\frac{x_r + a_i}{2r_i}\right)^2 + \left(\frac{x_i - a_r}{2r_i}\right)^2 = \frac{a_r^2 + a_i^2}{4r_i^2} \]

and the locus of \( x \) as \( u \) varies is a circle centered at \( \frac{-ja_i}{2r_i} \) and

with radius \( \frac{a_i}{2r_i} \) whenever \( r_i \neq 0 \). \hspace{1cm} Q.E.D.
APPENDIX II

$\Delta T(j\omega)$ AS TWO ELEMENTS VARY

From equations (III.12) and (III.11a) the change in $T(j\omega)$ as two elements $p$ and $q$ vary is given by:

$$\Delta T = A Q^{-1} C,$$  \hspace{1cm} (1)

and

$$Q = \begin{bmatrix} \frac{p}{\Delta p} & 0 \\ \Delta p & q \\ 0 & \Delta q \end{bmatrix}^{-D},$$  \hspace{1cm} (2)

where $A$ contains the columns of $R_1$ corresponding to elements $p$, $q$

$C$ contains the rows of $R_2$ corresponding to elements $p$ and $q$

and $D$ contains the intersections of rows and columns corresponding to elements $p$ and $q$ of $R_3$.

The dependence of $A$, $C$, $D$, $Q$, and $\Delta T$ upon frequency is not shown explicitly.

Equation (2) may be written:

$$Q = \begin{bmatrix} \frac{p}{\Delta p} - d_{11} & -d_{12} \\ -d_{21} & q - \Delta q - d_{22} \end{bmatrix}$$

Hence:

$$Q^{-1} = \frac{1}{(p-d_{11}q)(q-d_{22}p) - d_{12}d_{21}q\Delta q} \begin{bmatrix} q\Delta p - d_{22}q\Delta p - d_{12}q\Delta q \\ \frac{d_{12}q\Delta p}{q\Delta p - d_{22}q\Delta p} - \frac{d_{21}q\Delta p}{q\Delta p - d_{22}q\Delta p} \end{bmatrix}$$  \hspace{1cm} (3)
Each element of $\Delta T$ is thus seen to be a ratio of polynomials in $\Delta p$ and $\Delta q$ with a denominator term $(p - d_{11}\Delta q)(q - d_{22}\Delta p) - d_{12}d_{21}\Delta p\Delta q$, and a numerator term in $\Delta p$, $\Delta q$, and $\Delta p \Delta q$ only. Thus:

$$
\Delta T = \frac{J\Delta p + K\Delta q + L\Delta p\Delta q}{1 - \frac{d_{11}\Delta p}{p} - \frac{d_{22}\Delta q}{q} + \frac{(d_{11}d_{22} - d_{12}d_{21})}{pq}\Delta p\Delta q}
$$

(4)

Setting $\Delta p$ and $\Delta q$ to zero in turn, we see that:

$$
\Delta T(j\omega) = \frac{S^p(j\omega)\Delta p + S^q(j\omega)\Delta q + W^{p,q}(j\omega)\Delta p\Delta q}{1 - r(p,j\omega)\Delta p - r(q,j\omega)\Delta q + f(p,q,j\omega)\Delta p\Delta q}
$$

where $W^{p,q}(j\omega) = L$ and $f(p,q,j\omega) = \frac{1}{pq} \det D$. 
APPENDIX III

DETERMINATION OF SIMILARITY FAULT SETS

Read network information
Compute matrices
A, B, B_a, C, C_f,
D, D_a, D_f, D_af

Compute
S^i(jω), r(i,ω)
i = 1, 2, ..., ℎ
ω = ω_0, ω_1

Set K = 1, L = 2

Does
S^K(jω_0) = (K,L)S^L(jω_0),
r(K,ω_0) = α(K,L)r(L,ω_0) + β(K,L)
α, β real?

Yes

Does
S^K(jω_1) = α(K,L)S^L(jω_1),
r(K,ω_1) = α(K,L)r(L,ω_1) + β(K,L)?

Yes
Place p_L in a new SFS

No

Does K + 1 = L?

No

Yes

Place p_L in SFS containing p_K

Does L = ℎ?

No

L = L + 1
K = 0

End

Yes
APPENDIX IV

The hybrid-pi model of a common emitter transistor

\[ \begin{align*}
&+ & \bullet & 3.7 & + \\
&32 & 32v_{be} & 29 & + \\
& v_{be} & \bullet & 20 & - \\
& - & \bullet & \text{Values given in pf. and mmho.} \\
& - & \bullet & v_{ce} \\
\end{align*} \]

Figure A-1

may be used in the Darlington connection of two identical transistors:

\[ \begin{align*}
&+ & \bullet & Q_1 & Q_2 & + \\
& v_i & \bullet & v_1 & v_2 & - \\
& - & \bullet & \end{align*} \]

Figure A-2

The resulting model contains ten elements, reducible to nine by combining parallel capacitors. To obtain a five element model of this transistor pair, the admittance matrix of the circuit of Figure 2 was computed by RAPID circuit analysis program.

\[
Y(s) = \frac{1}{s + .4} \begin{bmatrix}
.5 + 2.05 + 22.4s^2 & 8.1s + 12.5s^2 \\
12.6 + 29.2s + 19.9s^2 & 19s + 40.5s^2
\end{bmatrix}
\] (1)

where units of \( Y \) are mmho and units of \( s \) are \( 10^9 \) rad/sec.

It may be noted that the asymptotic behavior of \( y_{12}(s) \) as \( s \to \infty \) represents the admittance of \( C_{be} \) and is found to be
\[ C_{bc} = \lim_{s \to \infty} \frac{y_{12}(s)}{s} = 12.0 \text{ pf}. \] (2)

Similarly, observing asymptotic behavior of \( y_{22}(s) \):

\[ C_{bc} + C_{ce} = \lim_{s \to \infty} \frac{y_{22}(s)}{s} \] (3)

Thus \( C_{ce} = 28.5 \text{ pf}. \)

The effect of \( C_{be} \) is found from \( y_{11}(s) \):

\[ C_{be} + C_{bc} = \lim_{s \to \infty} \frac{y_{11}(s)}{s} \] (4)

Thus \( C_{be} = 10.3 \text{ pf}. \)

The value of \( g_{be} \) may be found by observing the behavior of the input admittance at low frequency:

\[ g_{be} = \lim_{s \to \infty} y_{11}(s) = 0.00314 \text{ mmho}. \] (5)

Finally, the gain of the active source \( g_m \) may be found by use of \( y_{21} \) at low frequencies:

\[ g_m = \lim_{s \to \infty} y_{21}(s) = 32.0 \text{ mmho}. \]

Therefore, the hybrid-pi model of a Darlington pair of transistors of Figure 1 is given below:
The reduction of the model from nine to five elements actually involves the removal of a pole-zero pair from each element of the admittance matrix in such a manner that the asymptotic behavior of the transistor is preserved.
REFERENCES


