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PRESENTATIONS FOR THE ELEMENTARY GROUP,
AND THE FUNCTÖR $K_2$

by

Roger Keith Dennis

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TO MY WIFE

PERRY
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Introduction

This paper is a study of the abelian group $K_2(R)$ for certain rings $R$. The definition of $K_2(R)$ which appears below is due to J. Milnor [6].

For any ring $R$ with unit, $E(n,R)$ will denote the subgroup of $GL(n,R)$ generated by the elementary matrices. The Steinberg group $St(n,R)$ for $n \geq 3$ is the group defined by the generators $x_{ij}(r)$, where $i,j$ range over all pairs of distinct integers between 1 and $n$ and $r$ ranges over $R$, subject to the relations

1. $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$
2. $[x_{ij}(r), x_{jk}(s)] = x_{ik}(rs)$ for $i \neq k$
3. $[x_{ij}(r), x_{kl}(s)] = 1$ for $j \neq k$, $i \neq l$

where $r,s$ are any elements of $R$ and $[a,b] = aba^{-1}b^{-1}$.

We define $K_2(n,R)$ to be the kernel of the canonical epimorphism $\varphi_n : St(n,R) \longrightarrow E(n,R)$. By taking the direct limit we obtain $\varphi : St(R) \longrightarrow E(R)$. $K_2(R)$ is defined to be the kernel of $\varphi$. It is easily seen that $K_2(R)$ is the direct limit of the $K_2(n,R)$.

The problem of computing $K_2(R)$ is essentially that of determining a presentation for $E(R)$. In Section 2 we find a complete system of defining relations for $E(n,R)$
for rings which satisfy the conditions $M$ and $N_n$.

**Definition:** A ring $R$ satisfies condition $M$ if for every pair of elements $a_1, a_2$ of $R$, there exists a matrix $B \in E(2, R)$ such that $(a_1, a_2)B = (a_1', 0)$ for some $a_1' \in R$.

**Definition:** A ring $R$ satisfies the condition $N_n$ if whenever the matrix

$$
\begin{pmatrix}
I & 0 \\
0 & A
\end{pmatrix}
$$

is in $E(n, R)$, then $A$ is in $E(2, R)$. Here $I$ denotes an $(n-2) \times (n-2)$ identity matrix, $A$ a $2 \times 2$ matrix, and $0$ denotes a matrix of zeros of an appropriate size.

This presentation for $E(n, R)$ is obtained by a technique originally due to Magnus [5] in the case $R = \mathbb{Z}$. When the ring $R$ satisfies $M$ and $N_3$ we can write the elements of $E(3, R)$ in the "Magnus normal form" which allows us to make a reduction.

**Theorem 2.8:** Let $R$ be a ring for which every element of $E(3, R)$ can be written in the normal form. Let $\mathfrak{g}$ be a complete set of defining relations for $E(3, R)$. Then $\mathfrak{g}$ and the Steinberg relations (1), (2), and (3) are a complete set of defining relations for $E(n, R)$ for any $n \geq 3$. 
We also consider the possibility of "lifting" this normal form to the Steinberg group $\text{St}(n,R)$. When this can be done and when $R$ satisfies the condition $N'_n$, we obtain a stronger result.

**Definition:** A ring $R$ satisfies the condition $N'_n$ if when given a matrix

$$
\begin{pmatrix}
  d_1 & 0 \\
  \ddots & \ddots \\
  0 & \ddots & 0 \\
  * & \ddots & d_n
\end{pmatrix} \in E(n,R) \quad \text{and} \quad d_i = 1 \quad \text{for all} \quad i \neq k,
$$

then $d_k = 1$.

**Theorem 2.7:** Let $R$ be a ring which satisfies the condition $N'_n$ for every $n \geq 3$. Assume that every element of $\text{St}(3,R)$ can be written in the normal form. If $\mathcal{R}$ is a complete set of defining relations for $E(2,R)$, then $\mathcal{R}$ and the Steinberg relations form a complete set of defining relations for $E(n,R)$ for any $n \geq 3$.

As a result of Theorem 2.7, we study the relations for $E(2,R)$ in Section 3. First we define groups $\text{St}(2,R)$ and $K_2(2,R)$ and then apply a technique that was originally used by P. M. Cohn [4] to study $\text{GL}(2,R)$.

In the final section, we use our presentations for $E(n,R)$ to obtain information about $K_2(R)$. The following three theorems are the main results.
Theorem 4.1: Let $R$ be a ring which satisfies the condition $N'_n$ for every $n \geq 3$. Assume that every element of $\text{St}(3, R)$ can be written in the normal form. Then the groups $K_2(n, R)$ are abelian for every $n \geq 3$. The homomorphisms

$$K_2(n, R) \longrightarrow K_2(m, R)$$

are epic for every $m > n \geq 2$ and hence the homomorphisms

$$K_2(n, R) \longrightarrow K_2(R)$$

are epic for every $n \geq 2$.

Theorem 4.2: Let $R$ be a ring for which every element of $E(3, R)$ can be written in the normal form. Then the groups $K_2(n, R)$ are abelian for every $n \geq 4$. The homomorphisms

$$K_2(n, R) \longrightarrow K_2(m, R)$$

are epic for every $m > n \geq 3$ and hence the homomorphisms

$$K_2(n, R) \longrightarrow K_2(R)$$

are epic for every $n \geq 3$. 
Theorem 4.1 and the results of Section 3 yield:

**Theorem 4.5:** Let $R$ be a commutative discretely normed ring for which every element of $St(3, R)$ can be written in the normal form. Then the symbols generate $K_2(R)$.

Unfortunately the results are still incomplete. One would expect that the maps $K_2(n, R) \rightarrow K_2(n+1, R)$ in the cases under consideration are actually all isomorphisms for $n \geq 3$ in Theorem 4.1 and for $n \geq 4$ in Theorem 4.2.
1. **The Steinberg Group and \( K_2 \)**

In this section \( R \) will denote any ring with unit. Let \( E_{ij}(r) \in \text{GL}(n,R) \) denote the elementary matrix with entry \( r \) in the \((i,j)\)-th position. Here \( i \) and \( j \) can be any distinct integers between 1 and \( n \), and \( r \) can be any element of \( R \). Let \( E(n,R) \) denote the subgroup of \( \text{GL}(n,R) \) generated by all of the elementary matrices \( E_{ij}(r) \).

Observe that for any \( r, s \in R \) the following relations hold in \( E(n,R) \):

1. \( E_{ij}(r)E_{ij}(s) = E_{ij}(r+s) \)
2. \( [E_{ij}(r), E_{jk}(s)] = E_{ik}(rs) \) for \( i \neq k \)
3. \( [E_{ij}(r), E_{kl}(s)] = 1 \) for \( j \neq k, i \neq l \).

The following group was first introduced by Steinberg [9].

**Definition:** For \( n \geq 3 \) the **Steinberg group** \( \text{St}(n,R) \) is the group defined by the generators \( x_{ij}(r) \), where \( i, j \) range over all pairs of distinct integers between 1 and \( n \) and \( r \) ranges over \( R \), subject to the relations

1. \( x_{ij}(r)x_{ij}(s) = x_{ij}(r+s) \)
2. \( [x_{ij}(r), x_{jk}(rs)] = x_{ik}(rs) \) for \( i \neq k \)
3. \( [x_{ij}(r), x_{kl}(s)] = 1 \) for \( j \neq k, i \neq l \)
where \( r, s \) are any elements of \( R \).

The **canonical epimorphism**

\[
\varphi_n : \text{St}(n, R) \longrightarrow \text{E}(n, R)
\]

is defined by \( \varphi_n(\mathbf{x}_{ij}(r)) = \mathbf{E}_{ij}(r) \). \( \varphi_n \) is well-defined and a homomorphism as each defining relation between the generators \( \mathbf{x}_{ij}(r) \) is taken into a true relation in the \( \mathbf{E}_{ij}(r) \).

Passing to the direct limit as \( n \to \infty \), we obtain corresponding groups and a corresponding epimorphism

\[
\varphi : \text{St}(R) \longrightarrow \text{E}(R).
\]

**Definition:** The kernel of the homomorphism \( \varphi : \text{St}(R) \longrightarrow \text{E}(R) \) will be called \( K_2(R) \).

Milnor in [6] proves the following basic theorem about \( K_2(R) \).

**Theorem 1.1:** The group \( K_2(R) \) is precisely the center of the Steinberg group \( \text{St}(R) \).

The group \( K_2(R) \) may be thought of as all the nontrivial relations between the elementary matrices as any relation

\[
\mathbf{E}_{i_1 j_1}(r_1) \cdots \mathbf{E}_{i_k j_k}(r_k) = 1
\]

between the elementary matrices gives an element \( \mathbf{x}_{i_1 j_1}(r_1) \cdots \mathbf{x}_{i_k j_k}(r_k) \) of \( K_2(R) \).
Observe that $K_2$ is a covariant functor from the category of rings with unit to the category of abelian groups.

We now introduce some auxiliary groups in order to obtain information about $K_2$.

**Definition:** For $n \geq 3$, let $K_2(n,R)$ denote the kernel of the homomorphism $\varphi_n : St(n,R) \longrightarrow E(n,R)$.

Now there exist natural homomorphisms

$$\gamma_n : E(n,R) \longrightarrow E(n+1,R)$$

$$\beta_n : St(n,R) \longrightarrow St(n+1,R).$$

$\gamma_n$ simply takes the generator $E_{i j}(r)$ of $E(n,R)$ into the generator $E_{i j}(r)$ of $E(n+1,R)$. Similarly, $\beta_n$ takes the generator $x_{i j}(r)$ of $St(n,R)$ into the generator $x_{i j}(r)$ of $St(n+1,R)$. We obtain a commutative diagram with exact rows

$$
\begin{array}{cccccc}
1 & \longrightarrow & K_2(n,R) & \longrightarrow & St(n,R) & \longrightarrow & E(n,R) & \longrightarrow & 1 \\
& & \downarrow{\alpha_n} & & \downarrow{\beta_n} & & \downarrow{\gamma_n} & \\
1 & \longrightarrow & K_2(n+1,R) & \longrightarrow & St(n+1,R) & \longrightarrow & E(n+1,R) & \longrightarrow & 1
\end{array}
$$

where the homomorphism $\alpha_n$ is induced by $\beta_n$. The homomorphism $\gamma_n$ is always monic. However, it does not seem to be known in general whether or not $\alpha_n$ and $\beta_n$ are monic.
Theorem 1.2: $K_2(R)$ is canonically isomorphic to the direct limit $\lim_{\rightarrow} K_2(n,R)$.

Proof: Let $\bar{\alpha}_n : St(n,R) \rightarrow St(R)$ and $\bar{\gamma}_n : E(n,R) \rightarrow E(R)$ be the structure homomorphisms for the direct limit. Then we have the commutative diagram with exact rows

\[
\begin{array}{c}
1 \rightarrow K_2(n,R) \rightarrow St(n,R) \rightarrow E(n,R) \rightarrow 1 \\
\downarrow \bar{\alpha}_n \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \bar{\beta}_n \\
1 \rightarrow K_2(R) \rightarrow St(R) \rightarrow E(R) \rightarrow 1 \\
\end{array}
\]

where $\bar{\alpha}_n$ is the homomorphism induced by $\bar{\beta}_n$. By the universal property of direct limits, there exists a homomorphism $\theta : \lim_{\rightarrow} K_2(n,R) \rightarrow K_2(R)$ such that the following diagram commutes for every $n \geq 3$

\[
\begin{array}{c}
K_2(n,R) \\
\downarrow \bar{\alpha}_n \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \bar{\alpha}_n \\
\lim_{\rightarrow} K_2(n,R) \rightarrow \theta \rightarrow K_2(R) \\
\end{array}
\]

where $\bar{\alpha}_n$ denotes the structure homomorphism for the direct limit of the $K_2(n,R)$.

$\theta$ is epic: Let $x \in K_2(R)$. Then there exists an integer $k$ and an element $x_k \in St(k,R)$ such that $\bar{\alpha}_k(x_k) = x$. Hence we have
\[ 1 = \varphi(x) = \varphi \overline{\beta}_k(x_k) = \overline{\gamma}_k \varphi_k(x_k). \]

As \( \overline{\gamma}_k \) is monic, \( \varphi_k(x_k) = 1 \) and \( x_k \in K_2(k, R) \). Then we have

\[ x = \overline{\alpha}_k(x_k) = \theta \overline{\alpha}_n(x_k) \]

and hence \( \theta \) is epic.

\( \theta \text{ is monic:} \) Let \( y \in \text{Ker } \theta \). There exists an integer \( \ell \) and an element \( y_{\ell} \in K_2(\ell, R) \) such that \( \overline{\alpha}_\ell(y_{\ell}) = y \). We have

\[ 1 = \theta(y) = \theta \overline{\alpha}_\ell(y_{\ell}) = \overline{\alpha}_\ell(y_{\ell}) \]

and hence \( \overline{\beta}_\ell(y_{\ell}) = 1 \). Then there exists an \( m, m \geq \ell \), such that

\[ \beta_{m-1} \cdots \beta_{\ell}(y_{\ell}) = 1. \]

Then we have

\[ \alpha_{m-1} \cdots \alpha_{\ell}(y_{\ell}) = 1 \quad \text{and hence} \]

\[ y = \overline{\alpha}_\ell(y_{\ell}) = \overline{\alpha}_m(\alpha_{m-1} \cdots \alpha_{\ell}(y_{\ell})) = 1. \]

This completes the proof.

\textbf{Remark}: The first part of this section follows Milnor [6]. \( K_2 \) is also discussed in Swan [10].
2. The Magnus Normal Form

Let $S^n_i$ for $i=1, \ldots, n-1$ denote the subgroup of $\text{St}(n, R)$ generated by all elements of the form $x^i_{i+1}(r)$ and $x^i_{i+1}(r)$. $L S^n_i$ will denote the subgroup of $\text{St}(n, R)$ generated by all elements of the form $x^i_j(r)$ for $i > j$.

Now let $G_n$ be any homomorphic image of $\text{St}(n, R)$. $G^n_i$ will denote the image of $S^n_i$ in $G_n$. Similarly $L G^n_i$ will denote the image of $L S^n_i$ in $G_n$.

**Definition:** Let $G_n$ be a homomorphic image of $\text{St}(n, R)$. An element $g \in G_n$ can be written in the **Magnus normal form** if there exist elements $B^n_i \in G^n_i$, $j=1, \ldots, n-1$, $i=j, \ldots, n-1$ and an element $L \in L G^n_i$ such that

$$g = [B^n_{n-1} B^n_{n-2} \ldots B^n_1][B^n_{n-2} B^n_{n-3} \ldots B^n_2] \ldots [B^n_{n-1} B^n_{n-2}][B^n_{n-1}]L.$$  

This section will be devoted to deriving consequences of the existence of the Magnus normal form for every element of $\text{St}(n, R)$ and for $E(n, R)$. We also find conditions on the ring $R$ in order that every element of $E(n, R)$ for all $n \geq 3$ can be written in the Magnus normal form.

The consequences of the existence of a similar normal form for $\text{GL}(n, \mathbb{Z})$ were studied by Magnus in [5]. Magnus was able to give a presentation for $\text{GL}(n, \mathbb{Z})$ in terms of a presentation of $\text{GL}(3, \mathbb{Z})$. Our aim is to give a presentation for $E(n, R)$ in terms of a presentation for $E(3, R)$ for certain rings $R$. Under special conditions we will be able to get an even better result.
We now assume that for the ring R under consideration there is an integer n such that every element of E(n,R) can be written in the normal form. In particular, if \( a_1, a_2 \) are any elements of R, the matrix \( E_{1,n-1}(-a_1)E_{1,n}(-a_2) \) can be written in normal form. Hence there exist \( B_j^i \in E_n^i \), \( j=1, \ldots, n-1 \), \( i=j, \ldots, n-1 \) and an \( L \in LE_n \) such that

\[
E_{1,n-1}(-a_1)E_{1,n}(-a_2) = [B_{n-1}^1 \cdots B_{1}^1] \cdots [B_{n-1}^n]L.
\]

We then obtain

\[
E_{1,n-1}(a_1)E_{1,n}(a_2)B_{1}^{n-1} = L^{-1}[B_{n-1}^{n-1}] \cdots [B_{1}^{1} \cdots B_{1}^{n-2}]
\]

where \( B_{j}^i \in E_n^i \) denotes the inverse of \( B_{j}^i \). It is easy to see that for \( j \neq 1 \), the matrix \( B_{j}^1 \cdots B_{j}^{n-1} \) is of the form

\[
\begin{pmatrix}
I_{j-1} & 0_1 \\
0_2 & A_{n+1-j}
\end{pmatrix}
\]

where \( A_{n+1-j} \) is an \((n+1-j)\times(n+1-j)\) matrix which could possibly have all nonzero entries, \( I_{j-1} \) is the \((j-1)\times(j-1)\) identity matrix, and \( 0_1 \) and \( 0_2 \) are \((j-1)\times(n+1-j)\) and \((n+1-j)\times(j-1)\) zero matrices, respectively. Similarly, the matrix \( B_{1}^{1} \cdots B_{1}^{n-2} \) is of the form

\[
\begin{pmatrix}
C_{n-1} & 0_1 \\
0_2 & I_1
\end{pmatrix}
\]
where $C_{n-1}$ is an $(n-1) \times (n-1)$ matrix, $I_1 = (1)$, and $0_1$ and $0_2$ are $(n-1) \times 1$ and $1 \times (n-1)$ zero matrices, respectively. Thus

$$E_{1n-1}(a_1)E_{1n}(a_2)B_{1}^{n-1}$$

$$= \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I_{n-3} & 0 \\ 0 & A_3 \end{pmatrix} \cdots \begin{pmatrix} I_1 & 0 \\ 0 & A_{n-1} \end{pmatrix} \begin{pmatrix} C_{n-1} & 0 \\ 0 & I_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} I_1 & 0 \\ 0 & A_{n-1}^t \end{pmatrix} \begin{pmatrix} C_{n-1} & 0 \\ 0 & I_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

and hence we have the matrix equation

$$(a_1, a_2)B = (a_1', 0)$$

for some $a_1' \in \mathbb{R}$, where $B_{1}^{n-1} = \begin{pmatrix} I_{n-2} & 0 \\ 0 & B \end{pmatrix}$ with $B \in E(2, \mathbb{R})$.

We thus see that any ring $R$ for which every element of $E(n, \mathbb{R})$ can be written in the normal form for some $n$ satisfies the condition
M. For every pair of elements \( a_1, a_2 \) of \( R \), there exists a matrix \( B \in E(2, R) \) such that \( (a_1, a_2)B = (a'_1, 0) \) for some \( a'_1 \in R \).

If a ring \( R \) satisfies the condition \( M \), the above equation shows that the right ideal generated by \( a_1, a_2 \) contains the element \( a'_1 \). On the other hand, we have \( (a_1, a_2) = (a'_1, 0)B^{-1} \) and hence the right ideal generated by \( a_1, a_2 \) is generated by \( a'_1 \). Thus in any ring satisfying the condition \( M \), every finitely generated right ideal is principal.

We will now show that, if a ring \( R \) satisfies the condition \( M \) and the condition \( N_n \) defined below, then every element of \( E(n, R) \) can be written in the normal form.

\( N_n \). For every matrix \( \begin{pmatrix} I_{n-2} & 0 \\ 0 & A_2 \end{pmatrix} \in E(n, R) \), then \( A_2 \in E(2, R) \).

**Note:** If \( R \) is a commutative ring which satisfies \( M \), then \( R \) satisfies \( N_n \) for every \( n \geq 3 \).

For let \( A_2 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \), then as \( R \) satisfies \( M \), there exists a matrix \( B_2 \in E(2, R) \) such that \( (a_1, a_2)B_2 = (b_1, 0) \). Then we have

\[
\begin{pmatrix} I_{n-2} & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} I_{n-2} & 0 \\ 0 & A_2B_2 \end{pmatrix}
\]

where \( A_2B_2 = \begin{pmatrix} b_1 & 0 \\ b_3 & b_4 \end{pmatrix} \). By taking the determinant of
\[
\begin{pmatrix}
I_{n-2} & 0 \\
0 & A_2B_2
\end{pmatrix}
\]
we see that \( b_4 = b_1^{-1} \).

Now let \( C_2 = \begin{pmatrix} b_1^{-1} & 0 \\ 0 & b_1 \end{pmatrix} \)

\[
\begin{pmatrix}
1 & 1-b_1^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & b_1^{-1}
\end{pmatrix}
\in \text{E}(2, \mathbb{R}).
\]

Thus \( A_2B_2C_2 = D_2 \) where \( D_2 = \begin{pmatrix} 1 & 0 \\ b_3b_1^{-1} & 1 \end{pmatrix} \in \text{E}(2, \mathbb{R}) \) and hence
\( A_2 = D_2C_2^{-1}B_2^{-1} \in \text{E}(2, \mathbb{R}) \).

If \( R \) is a division ring, then clearly \( R \) satisfies \( M \).

By an argument similar to that given above, using the
Dieudonné determinant (see [1]) and the identity
\[
\begin{pmatrix}
(uv)^{-1} & 0 \\
0 & uv
\end{pmatrix}
\begin{pmatrix}
u & 0 \\
0 & v^{-1}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\
0 & uvu^{-1}v^{-1} \end{pmatrix} \in \text{E}(2, \mathbb{R})
\]
it can be shown that \( R \) satisfies \( N_n \) for all \( n \geq 3 \).

Now let \( R \) be a ring which satisfies \( M \) and \( N_n \) for
some \( n \geq 3 \). Let \( e \in \text{E}(n, R) \) and let \( e^{-1} = (e_{ij}) \).
As \( R \) satisfies \( M \), there exists a matrix \( A_1 \in \text{E}(2, R) \) such that
\( (e_{1n-1}, e_{1n})A_1 = (e'_{1n-1}, 0) \). Let \( B_1^{n-1} = \begin{pmatrix} I_{n-2} & 0 \\
0 & A_1 \end{pmatrix} \), then
we have \( e^{-1}B_1^{n-1} = \begin{pmatrix} e_{11} & \cdots & e_{1n-1} \\
* & \end{pmatrix} \). Now let \( A_2 \in \text{E}(2, R) \)
be such that \((e_{ln-2}', e_{ln-1}')A_2 = (e_{ln-2}', 0)\). Let
\[
B_{n-2} = \begin{pmatrix} I_{n-3} & A_2 \\ -A_2 & I_1 \end{pmatrix},
\]
then we have \(e^{-1}B_{n-1}B_{n-2} = \begin{pmatrix} e_{ll} \cdots e_{ln-2} & 0 \\ * & \end{pmatrix}\). Continue this process until we reach \(e^{-1}B_{n-1} \cdots B_{2}^{-1}B_{1}^{-1} = \begin{pmatrix} e & 0 \cdots 0 \\ * & \end{pmatrix}\). Since this is an invertible matrix, there is an \(f \in \mathbb{R}\) such that \(ef = 1\).

Let \(C_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f & -1 + fe \\ 1 & e \end{pmatrix} \in \mathbb{E}(2, \mathbb{R})\). Then \((e, 0)C_1 = (1, 0)\). We have just shown that whenever \(e \in \mathbb{R}\) has a right inverse, there is a matrix \(C \in \mathbb{E}(2, \mathbb{R})\) such that \((e, 0)C = (1, 0)\). We will use this statement several times.

Now let \(B_{1}^{-1} = B_{1}^{-1}C_{1}^{-1}\) where \(C_{1} = \begin{pmatrix} C_1 & 0 \\ 0 & I_{n-2} \end{pmatrix}\). This yields \(e^{-1}B_{n-1} \cdots B_{1}^{-1} = \begin{pmatrix} 1 & 0 \cdots 0 \\ * & \end{pmatrix}\).

We now apply the above process to clear the second row of the given matrix. We obtain \(B_{n-1}^{-1} \cdots B_{2}^{-1}\) such that
\[
e^{-1}[B_{n-1}^{-1} \cdots B_{1}^{-1}]B_{2}^{-1} \cdots B_{2}^{-1} = \begin{pmatrix} 1 & 0 \cdots 0 \\ a_0 & a_1 & 0 \cdots 0 \\ * & \end{pmatrix} = T.
\]

The matrix \(T\) is invertible. The equation \(TT^{-1} = I\) shows
that the first row of $T^{-1}$ must be $(1,0,\ldots,0)$. By computing the $(2,2)$ element in the equation $TT^{-1} = I$ we see that $a_1$ has a right inverse. Hence by our preceding remark, there exists a matrix $C_2 \in E(2,R)$ such that $(a_1,0)C_2 = (1,0)$. Now let $B_2 = B_2^2 C_2$, where $C_2 = \begin{pmatrix} I_1 & C_2 & I_{n-3} \end{pmatrix}$ and we obtain

$$e^{-1}[B_1^{n-1} \ldots B_1^1][B_2^{n-1} \ldots B_2^2] = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ & & & & \end{pmatrix}.$$ 

It is clear that we can continue this process, clearing out all of the elements to the right of the diagonal and leaving 1's on the diagonal until we reach the 2-nd row from the bottom. Assume we are now at this step. We have

$$e^{-1}[B_1^{n-1} \ldots B_1^1] \cdots [B_{n-2}^{n-1} B_{n-2}^{n-2}] = \begin{pmatrix} L_{n-2} & 0 \\ D & A_2 \end{pmatrix} = U$$

where $L_{n-2} \in LE_{n-2}$, $D$ is a $2 \times (n-2)$ matrix, and $A_2$ is a $2 \times 2$ matrix. By elementary column operations on $U$ (i.e., multiplying $U$ on the left by elementary matrices), we can change $U$ into $\begin{pmatrix} I_{n-2} & 0 \\ 0 & A_2 \end{pmatrix} \in E(n,R)$. By our hypothesis $N_n, A_2 \in E(2,R)$. Now let $B_{n-1}^{n-1} = \begin{pmatrix} I_{n-2} & 0 \\ 0 & A_{n-2}^{-1} \end{pmatrix}$. Then we have $e^{-1}[B_1^{n-1} \ldots B_1^1] \cdots [B_{n-1}^{n-1}] \in LE_n$. Thus there is an
\[
L \in \text{LE}_n \text{ such that }
\]
\[
e = [B_1^{n-1} \ldots B_1^1] \ldots [B_{n-1}^{n-1}] L.
\]

We have now completed the proof of the following.

**Theorem 2.1:** a) Let \( R \) be a ring such that for some \( n \) every element of \( E(n, R) \) can be written in the normal form. Then \( R \) satisfies the condition \( M \).

b) If \( R \) satisfies the condition \( M \) and the condition \( N_n \), then every element of \( E(n, R) \) can be written in the normal form.

**Corollary:** If \( R \) is a commutative ring and there exists an \( n_0 \) such that every element of \( E(n_0, R) \) can be written in the normal form, then every element of \( E(n, R) \) can be written in the normal form for all \( n \geq 3 \).

**Examples** of rings satisfying condition \( M \).

i) Any Euclidean ring. The proof that Euclidean rings satisfy \( M \) is a simple application of the Euclidean algorithm.

ii) Any valuation ring. A valuation ring is a commutative ring \( R \) such that given \( a, b \in R, a \neq 0, b \neq 0 \), then either \( a \) divides \( b \) or \( b \) divides \( a \).

iii) The skew polynomial rings \( F[x; S] \). \( F[x; S] \) consists of all polynomials in \( x \) with right coefficients in the field \( F \). Multiplication is given by the rule \( ax = xS(a) \) where \( a \in F \) and \( S \) is the given endo-
morphism of $F$. This ring satisfies $M$ as it has a right Euclidean algorithm. See Ore [8] or Cohn [2] and [3]. It can also be shown that this ring satisfies condition $N_n$ for all $n \geq 3$.

Before deriving any further consequences of the normal form, we will need certain results about the Steinberg group $St(n,R)$. The next four lemmas hold for any ring $R$.

**Lemma 2.2:** Every element of the group $LS_n$ can be written uniquely in the form

$$[x_{21}(r_{21})][x_{31}(r_{31})x_{32}(r_{32})\cdots[x_{n1}(r_{n1})\cdots x_{n,n-1}(r_{n,n-1})].$$

Hence $\varphi_n$ maps $LS_n$ isomorphically onto $LE_n$.

**Proof:** See Milnor [6, Lemma 9.15].

**Lemma 2.3:** Let $LS'_n$ denote the commutator subgroup of $LS_n$. Then $LS'_n$ is the subgroup of $LS_n$ generated by the elements $x_{ij}(r)$ where $i > j + 1$.

**Proof:** For $i > j + 1$ we have $[x_{i,j+1}(1),x_{j+1,j}(r)] = x_{ij}(r)$ and hence $x_{ij}(r) \in LS'_n$ for $i > j + 1$ and any $r \in R$. On the other hand, the relations (1), (2) and (3) in the Steinberg group show that any commutator of generators of $LS_n$ is either 1 or of the form $x_{ij}(r)$ with $i > j + 1$.

**Lemma 2.4:** Let $A \in S^i_n$ and let $L \in LS_n$. Then there exists an
$A' \in S^i_n$ and an $L' \in LS_n$ such that $LA = A'L'$.

**Proof:** By an induction on the number of factors in $A$, it suffices to consider the cases $A = x_{i+1,i}(r)$ and $A = x_{i,i+1}(r)$. In the first case let $A' = 1$ and $L' = Lx_{i+1,i}(r)$. This works as $x_{i+1,i}(r) \in LS_n$.

Now for the second case. By Lemma 2.2 we can write

$$L = [x_{21}(r_{21})] \cdots [x_{i+1,1}(r_{i+1,1}) \cdots x_{i+1,i}(r_{i+1,i})] \cdots$$

$$\cdots \cdots [x_{n1}(r_{n1}) \cdots x_{n,n-1}(r_{n,n-1})].$$

Modulo $LS'_n$, $LS_n$ is abelian; hence we can write

$$L = x_{i+1,i}(r_{i+1,i})[x_{21}(r_{21})] \cdots [x_{i+1,1}(r_{i+1,1}) \cdots$$

$$\cdots \cdots x_{i+1,i-1}(r_{i+1,i-1})] \cdots [x_{n1}(r_{n1}) \cdots x_{n,n-1}(r_{n,n-1})]C$$

where $C \in LS'_n$. Thus for any $i$, $1 \leq i \leq n-1$, and any $L \in LS_n$, we may write $L = x_{i+1,i}(s)L_0$, where (by the preceding statement and Lemma 2.3) $L_0$ is a product of elements of the form $x_{j,k}(t)$ where $j > k$ and $(j,k) \neq (i+1,i)$. To complete the proof, we will show that there is an $L_1 \in LS_n$ such that $L_0 x_{i,i+1}(r) = x_{i,i+1}(r)L_1$. For then

$$Lx_{i,i+1}(r) = x_{i,i+1}(s)L_0 x_{i,i+1}(r) = [x_{i+1,i}(s)x_{i,i+1}(r)]L_1$$

and we are done as $A' = x_{i+1,i}(s)x_{i,i+1}(r) \in S^i_n$. As $L_0$ is a product of elements of the form $x_{j,k}(t)$, where $j > k$
and \((j, k) \neq (i+1, i)\), it suffices to show that

\[
x_{jk}(t)x_{i, i+1}(r) = x_{i, i+1}(r)x_{jk}(t)L^*
\]

for some \(L^* \in \text{LS}_n\). But this follows immediately from the Steinberg relations:

1) \(x_{jk}(t)x_{i, i+1}(r) = x_{i, i+1}(r)x_{jk}(t)\) whenever \(j \neq i+1\) and \(k \neq i\).

ii) Now for \(j = i+1\), as \((i+1, k) \neq (i+1, i)\) and \(i+1 > k\), we have \(i > k\) and hence \(x_{i+1, k}(t)x_{i, i+1}(r) = x_{i, i+1}(r)x_{i+1, k}(t)x_{ik}(-rt)\) with \(x_{ik}(-rt) \in \text{LS}_n\).

iii) Now for \(k = i\), as \((j, i) \neq (i+1, i)\) and \(j > i\), we have \(j > i+1\) and hence \(x_{ji}(t)x_{i, i+1}(r) = x_{i, i+1}(r)x_{ji}(t)x_{j, i+1}(tr)\) with \(x_{j, i+1}(tr) \in \text{LS}_n\).

This completes the proof.

For any unit \(u \) or \(R\), let \(w_{ij}(u) \in \text{St}(n, R)\) be given by

\[
w_{ij}(u) = x_{ij}(u)x_{ij}(-u^{-1})x_{ij}(u).
\]

\(W_n\) will denote the subgroup of \(\text{St}(n, R)\) generated by all elements of the form \(w_{ij}(u)\).

**Lemma 2.5:** The following relations hold in \(\text{St}(n, R)\):
(i) \( w_{ij}(u)w_{ij}(-u) = 1 \)

(ii) \( w_{ij}(u) = w_{ji}(-u^{-1}) \)

(iii) \( w_{kl}(u)x_{ij}(r)w_{kl}(-u) = x_{ij}(r) \) for \( i, j, k, l \) distinct

(iv) \( w_{lt}(u)x_{ij}(r)w_{lt}(-u) = x_{lj}(-u^{-1}r) \) for \( t \neq j \)

(v) \( w_{ti}(u)x_{ij}(r)w_{ti}(-u) = x_{ji}(ur) \) for \( t \neq j \)

(vi) \( w_{lt}(u)x_{ji}(r)w_{lt}(-u) = x_{ji}(-ru) \) for \( t \neq j \)

(vii) \( w_{ti}(u)x_{ji}(r)w_{ti}(-u) = x_{ji}(ru^{-1}) \)

(viii) \( w_{ij}(u)x_{ij}(r)w_{ij}(-u) = x_{ji}(-u^{-1}ru^{-1}) \)

(ix) \( w_{ji}(u)x_{ij}(r)w_{ji}(-u) = x_{ji}(-uru) \).

Proof: See Milnor [6, §9].

Remark: Let \( n \geq 4 \) and let \( i \) and \( j \) be such that \( 2 \leq i, j \leq n-1 \) and \( i \neq j \). Then there is a \( w \in W_n \) such that

\[
wx_{i-1,i}(r)w^{-1} = x_{j-1,j}(r)
\]

\[
wx_{i,i-1}(r)w^{-1} = x_{j,j-1}(r)
\]

\[
wx_{i,i+1}(r)w^{-1} = x_{j,j+1}(r)
\]

\[
wx_{i+1,i}(r)w^{-1} = x_{j+1,j}(r).
\]

For if \( j \geq i+1 \), let \( w = w_{j-1,i-1}(1)w_{ji}(1)w_{j+1,i+1}(1) \), and if \( j \leq i-1 \), let \( w = w_{j+1,i+1}(1)w_{ji}(1)w_{j-1,i-1}(1) \). That this \( w \) works follows easily from parts (iii), (v), and (vii)
of the lemma.

**Lemma 2.6:** Let $G_3 = \text{St}(3,R)/K$ where $K$ is some normal subgroup of $\text{St}(3,R)$. If every element of $G_3$ can be written in the normal form, then every element of $G_n = \text{St}(n,R)/K(n)$ for $n \geq 3$ can be written in the normal form. $K(n)$ denotes any normal subgroup of $\text{St}(n,R)$ containing the image of $K$ under the natural inclusion $\text{St}(3,R) \rightarrow \text{St}(n,R)$.

**Proof:** As the elements in $S_n^i$, $i=1,\ldots,n-1$ clearly generate $\text{St}(n,R)$, the same holds for $G_n^i$ in $G_n^i$. Hence we need only show that the collection of elements which can be written in the normal form is a subgroup of $G_n^i$. It thus suffices to show that an expression of the form

\[(*) \quad g[B_1^{n-1} \cdots B_1^1] \cdots [B_{n-1}^{n-1}]L\]

can be written in the normal form where $B_j^i \in G_n^i$, $L \in LG_n$, and $g$ is one of the generators of $G_n^i$ for some $i$.

The following three facts will enable us to prove this lemma.

(A) $B_i^j B_j^i = B_j^i B_i^j$ for $B_i^i \in G_n^i$, $B_j^j \in G_n^j$ and $i \neq j$, $i \neq j+1$, $i+1 \neq j$.

(B) If $B_1^{i-1}, B_2^{i-1} \in G_n^i$, $B_1^i \in G_n^i$ (2≤$i$≤$n-1$) then there exist $B_3^{i-1} \in G_n^{i-1}$, $B_3^i$, $B_4^i \in G_n^i$ and $L \in LG_n$ such that $B_1^{i-1} B_1^i B_2^{i-1} = B_3^i B_3^{i-1} B_4^i L$. 

(C) If $A \in G_n^i$ and $L \in LG_n$, then there exist $A' \in G_n^i$ and $L' \in LG_n$ such that $LA = A'L'$.

As $G_n$ is a homomorphic image of $St(n,R)$, the Steinberg relations hold in $G_n$. (A) follows from the Steinberg relation (3). Again, as $G_n$ is a homomorphic image of $St(n,R)$, (C) holds, since it holds in $St(n,R)$ by Lemma 2.4.

To prove (C), choose $w \in \bar{W}_n$ given by the remark following Lemma 2.5 for $j = 2$ and $i = i$. $\bar{W}_n$ denotes the image of $W_n$ in $G_n$. Then we have $wB_1^{i-1}w^{-1}$, $wB_2^{i-1}w^{-1} \in G_n^1$, and $wB_1^{i-1}w^{-1} \in G_n^2$. Thus $w(B_1^{i-1}B_1^{i-1}B_2^{i-1})w^{-1}$ is contained in the image of $G_3$ in $G_n$. As every element of $G_3$ can be written in the normal form, the same holds for its homomorphic image in $G_n$. Hence there exist $\tilde{B}_3^2, \tilde{B}_4^2 \in G_n^2$, $\tilde{B}_3^1 \in G_n^1$, and $\bar{L}$ in the image of $LG_3$ contained in $LG_n$ such that

$$w(B_1^{i-1}B_1^{i-1}B_2^{i-1})w^{-1} = \tilde{B}_3^2 \tilde{B}_3^1 \tilde{B}_4^2 \bar{L}.$$ 

Let $B_3^i = w^{-1}B_3^i w \in G_n^i$, $B_4^i = w^{-1}B_4^i w \in G_n^i$, $B_3^{i-1} = w^{-1}B_3^i w \in G_n^{i-1}$ and $L = w^{-1}\bar{L}w \in LS_n$. Then we have

$$B_3^{i-1}B_3^{i-1}B_4^i = B_3^i B_3^{i-1}B_4^i \bar{L}.$$ 

By using (A), (B), and (C) we now complete the proof of the lemma. If $g \in G_n^{n-1}$, combine it directly with $B_1^{n-1}$. If $g \in G_n^{n-2}$ use (B) to write $gB_1^{n-1}B_1^{n-2}$ as $B_1^{n-1}B_1^{n-2}B_2^{n-1}L_1$. By applying (C) repeatedly, $L_1$ can be "moved" to the right, without changing the form of the
the expression. After the last application of (C) in this process, combine the element of \( L_G_n \) which occurs in this step with the element \( L \) at the right-hand end of the expression \((*)\). By (A), \( \overline{B}_2^{n-1} \) commutes with each of \( B_1^{n-3}, \ldots, B_1^1 \). Thus we can move it to the right and combine it with \( B_2^{n-1} \). This brings the expression \((*)\) into the normal form when \( g \in G_n^{n-2} \).

In a similar manner, by a finite number of applications of (A), (B), and (C) the expression \((*)\) can be brought into normal form for \( g \in G_n^i \), \( 1 \leq i \leq n-1 \). Done.

A ring \( R \) is said to satisfy the condition \( N_n' \) if when given a matrix

\[
\begin{pmatrix}
  d_1 & 0 \\
  \vdots & \ddots \\
  * & \cdots & * \\
  d_n
\end{pmatrix} \in E(n,R) \text{ and } d_i = 1 \text{ for all } i \neq k, \text{ then } d_k = 1.
\]

Note: (1) The conditions \( N_n' \) and \( M \) imply the condition \( N_n \). For Let

\[
\begin{pmatrix}
  I_{n-2} & 0 \\
  0 & A_2
\end{pmatrix} \in E(n,R)
\]

with \( A_2 = \begin{pmatrix}
  a_1 & a_2 \\
  a_3 & a_4
\end{pmatrix} \). Then there is a \( B_2 \in E(2,R) \) such that \( (a_1, a_2)B_2 = (b_1, 0) \). Thus
\[
\begin{pmatrix}
I_{n-2} & 0 \\
0 & A_2
\end{pmatrix}
\begin{pmatrix}
I_{n-2} & 0 \\
0 & B_2
\end{pmatrix} =
\begin{pmatrix}
I_{n-2} & 0 \\
0 & C_2
\end{pmatrix}
\]

where \( C_2 = \begin{pmatrix} b_1 & 0 \\ b_3 & b_4 \end{pmatrix} \). As \( b_1 \) is right invertible there exists a \( D_2 \in E(2, \mathbb{R}) \) such that \((b_1, 0)D_2 = (1, 0)\). Thus we have

\[
\begin{pmatrix}
I_{n-2} & 0 \\
0 & C_2
\end{pmatrix}
\begin{pmatrix}
I_{n-2} & 0 \\
0 & D_2
\end{pmatrix} =
\begin{pmatrix}
I_{n-2} & 0 \\
0 & E_2
\end{pmatrix}
\]

where \( E_2 = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \). By \( N'_n \), \( d = 1 \) and hence \( E_2 \in E(2, \mathbb{R}) \). Thus

\( A_2 = E_2 D_2^{-1} B_2^{-1} \in E(2, \mathbb{R}) \).

(2) By taking the determinant, we see that any commutative ring satisfies \( N'_n \) for all \( n \geq 2 \).

(3) It is also easy to show that the skew polynomial ring \( F[x; S] \) satisfies \( N'_n \) for all \( n \geq 3 \).

(4) Any ring \( R \) with two units \( u, v \) which do not commute, never satisfies \( N'_n \) for any \( n \geq 2 \). For the matrix

\[
\begin{pmatrix}
1 & 0 \\
\cdot & 1 \\
0 & c
\end{pmatrix}
\]

where \( c = uvu^{-1}v^{-1} \)

is in \( E(n, \mathbb{R}) \). This follows from the fact that
\[
\begin{pmatrix}
  u & 0 \\
  0 & u^{-1}
\end{pmatrix}
\]

is in \( E(2, R) \) for any unit \( u \) of \( R \) and from the identity

\[
\begin{pmatrix}
  (uv)^{-1} & 0 \\
  0 & uv
\end{pmatrix}
\begin{pmatrix}
  u & 0 \\
  0 & u^{-1}
\end{pmatrix}
\begin{pmatrix}
  v & 0 \\
  0 & v^{-1}
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 \\
  0 & uvu^{-1}v^{-1}
\end{pmatrix}.
\]

If \( R \) is a ring for which every element of \( St(n, R) \) can be written in the normal form and if \( R \) satisfies \( N_n' \), then there is a simple way of obtaining a presentation of \( E(n, R) \).

**Theorem 2.7:** Let \( R \) be a ring satisfying condition \( N_n' \) for every \( n \geq 3 \). Assume that every element of \( St(3, R) \) can be written in the normal form. If \( \mathcal{R} \) is a complete set of defining relations for \( E(2, R) \), then \( \mathcal{R} \) and the Steinberg relations form a complete set of defining relations for \( E(n, R) \) for every \( n \geq 3 \).

**Proof:** By Lemma 2.6 every element of \( St(n, R) \) can be written in the normal form for every \( n \geq 3 \). Let \( z \in K_2(n, R) \). Then there exist \( B_j^i \in S_n^i \), \( j=1, \ldots, n-1, i=j, \ldots, n-1 \), and \( L \in LS_n \) such that

\[
z = [B_{n-1}^n \ldots B_1^1] \ldots [B_{n-1}^1] L.
\]

We shall show that the homomorphism
\[ \psi : \text{St}(n, R)/N(\alpha) \longrightarrow \text{E}(n, R) \]

induced by \( \varphi_n \) is a monomorphism and hence an isomorphism. \( N(\alpha) \) denotes the normal subgroup of \( \text{St}(n, R) \) generated by the image of \( \alpha \) under the natural inclusion.

First observe that in the normal form there is exactly one element of \( S_n^1 \), namely \( B_1^1 \). Apply \( \varphi_n \) to the above equation and observe that \( \varphi_n \) of any of the terms except \( B_1^1 \) is a matrix whose first row is \((1, 0, \ldots, 0)\). Thus we obtain the matrix equation

\[
I = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & a & b & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & c \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & d \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \alpha_1 & \alpha_2 & \cdots & \cdots & \cdots & \alpha_n \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\]

where the first term is \( \varphi_n(B_1^{n-1} \cdots B_1^2) \), the second is \( \varphi_n(B_1^1) \), and the third is \( \varphi_n(B_2^{n-1} \cdots L) \). The first row of the above yields the equations \( 1 = a + b\alpha_1 \) and \( b\alpha_1 = 0 \), \( i=2, \ldots, n \).

Now if we have a \((k+j) \times (k+j)\) invertible matrix of the form

\[
T = \begin{pmatrix}
L_k & 0 \\
A & B_j
\end{pmatrix}
\]

where \( L_k \in LE_k \), then \( B_j \) must be an invertible matrix, for if \( T^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \) is the inverse of \( T \), then we obtain \( L_k Y = 0 \) and \( AY + B_j W = I_j \) from \( TT^{-1} = I \).

As \( L_k \) is invertible, \( Y = 0 \) and hence \( B_j W = I_j \). \( T^{-1}T = I \) yields \( WB_j = I_j \) and hence \( B_j \) is invertible.
If we let \( B = \begin{pmatrix} a_2 & \cdots & a_n \\ * \end{pmatrix} \), then our original equation (1) yields \((b, 0, \ldots, 0)B = (0, 0, \ldots, 0)\). As \( B \) is invertible, we must have \( b = 0 \). As \( l = a + ba_1 \), we then obtain \( a = 1 \).

Hence \( \varphi_n(B_1^1) = \begin{pmatrix} 1 & 0 \\ c & d \\ \hline 0 & \ddots & \ddots & 1 \end{pmatrix} \in E(n, R) \). By hypothesis \( N_n', d = 1 \). Thus \( \varphi_n(B_1^1) = E_{21}(c) \). As \( R \) is a complete set of defining relations for \( E(2, R) \), we have

\[
B_1^1 = x_{21}(c) \mod N(R)
\]

and thus

\[
z = [B_1^{n-1} \cdots B_1^2 x_{21}(c) \cdots [B_1^{n-1}] L] \mod N(R).
\]

Observe now that the hypothesis of Lemma 2.6 is satisfied here. Thus the facts (A), (B), and (C) which we derived in the proof of Lemma 2.6 are valid.

Now using (C), \( x_{21}(c) \) can be moved to the right. By using (A), (B), and (C) as in the proof of Lemma 2.6, we can move \( B_1^2, \ldots, B_1^{n-1} \) to the right and obtain a "smaller" normal form:

\[
z = [B_2^{n-1} \cdots B_2^2] \cdots [B_1^{n-1}] L \mod N(R).
\]

By repeating our above process, we can conclude that

\[
\varphi_n(B_2^2) = \begin{pmatrix} 1 & 0 \\ \hline 1 & c_2^{-1} \\ \hline 0 & \ddots & \ddots & 1 \end{pmatrix}.
\]
We now obtain \( \varphi_n(\mathbb{B}_2^2 x_{32}(-c_2)) = I \) and hence \( \mathbb{B}_2^2 x_{32}(-c_2) \in K_2(n, R) \). By the remark following Lemma 2.5, we can find a \( w \in \mathbb{W}_n \) such that \( w \mathbb{B}_2^2 w^{-1} \in S_n^1 \) and \( w x_{32}(-c_2) w^{-1} = x_{21}(-c_2) \). Then as \( \mathcal{A} \) is a complete set of defining relations for \( E(2, R) \), we have

\[
w \mathbb{B}_2^2 x_{32}(-c_2) w^{-1} = 1 \quad \text{mod } N(\mathcal{A})
\]

as \( \varphi_n(w \mathbb{B}_2^2 x_{32}(-c_2) w^{-1}) = 1 \). Thus we have

\[
\mathbb{B}_2^2 = x_{32}(c_2) \quad \text{mod } N(\mathcal{A}).
\]

Again, by using (A), (B), and (C) we obtain

\[
z = [\mathbb{B}_3^{n-1} \ldots \mathbb{B}_3^3] \ldots [\mathbb{B}_n^{n-1}] \tilde{L} \quad \text{mod } N(\mathcal{A}).
\]

Continue the process, and one finally obtains

\[
z = \mathbb{B}_n^{n-1} \tilde{L} \quad \text{mod } N(\mathcal{A}).
\]

By applying \( \varphi_n \) and looking at the equation on the matrix level, we see that \( \tilde{L} = x_{n, n-1}(r) \) for some \( r \) by using the unique representation of elements of \( L \mathbb{S}_n \). Hence \( \mathbb{B}_n^{n-1} \tilde{L} \in S_n^{n-1} \) and as \( N(\mathcal{A}) \) contains all of the \( 2 \times 2 \) relations, we have \( \mathbb{B}_n^{n-1} \tilde{L} \in N(\mathcal{A}) \). Thus \( z = 1 \mod N(\mathcal{A}) \) and hence \( \psi \) is monic.

Q.E.D.

**Theorem 2.8:** Let \( R \) be a ring for which every element of \( E(3, R) \) can be written in the normal form. Let \( \mathcal{A} \) be a complete set of defining relations for \( E(3, R) \). Then \( \mathcal{A} \) and the Steinberg relations are a complete set of defining relations for \( E(n, R) \) for every \( n \geq 3 \).
**Proof:** We have the isomorphism

\[ \text{St}(3, R)/K_2(3, R) \cong E(3, R). \]

Let \( \psi : \text{St}(n, R)/N(\mathcal{R}) \longrightarrow E(n, R) \) be the homomorphism induced by \( \omega_n \). \( N(\mathcal{R}) \) denotes the normal subgroup of \( \text{St}(n, R) \) generated by \( \mathcal{R} \) under the natural inclusion. Note that \( K_2(3, R) \) is generated by the image of \( \mathcal{R} \) in \( \text{St}(3, R) \).

As every element of \( \text{St}(3, R)/K_2(3, R) \cong E(3, R) \) can be written in the normal form, then every element of \( \text{St}(n, R)/N(\mathcal{R}) \) can be written in the normal form by Lemma 2.6.

Let \( z \in \text{St}(n, R) \), then there exist \( B_j^i \in S_n^i, j=1, \ldots, n-1, \ i=j, \ldots, n-1 \) and \( L \in L S_n \) such that

\[ z = [B_{n-1}^1 \ldots B_1^1] \ldots [B_{n-1}^n \ldots B_1^n]L \mod N(\mathcal{R}). \]

If \( z N(\mathcal{R}) \in \text{Ker} \ \psi \), then by applying \( \psi \) to the preceding equation we obtain

\[ I = \omega_n(B_{n-1}^1 \ldots B_1^1) \omega_n(B_{n-1}^2 \ldots B_1^n) \omega_n(B_{n-1}^{n-1} \ldots B_1^{n-1} L) \]

\[ I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & \cdots & * \end{pmatrix} \begin{pmatrix} a & b & \cdots & 0 \\ c & d & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}. \]

By the argument given in the proof of Theorem 2.7, \( a = 1 \) and \( b = 0 \). It is then easy to see that \( d \) must be a unit
as \( \varphi_n(B_1^1) \) is an invertible matrix. The identity

\[
\begin{pmatrix}
  d^{-1} & 0 \\
  0 & d
\end{pmatrix} =
\begin{pmatrix}
  1 & 1-d^{-1} \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  -1 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 1-d \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  d^{-1} & 1
\end{pmatrix}
\]

shows that \( \begin{pmatrix}
  d^{-1} & 0 \\
  0 & d
\end{pmatrix} \in \mathbb{E}(2, \mathbb{R}) \). Let

\[
A_o = x_{23}(1-d^{-1}) x_{32}(-1) x_{23}(1-d) x_{32}(d^{-1}) \in S_n^2.
\]

Then

\[
\varphi_n(a_o) =
\begin{pmatrix}
  1 & d^{-1} & 0 \\
  d^{-1} & d & 0 \\
  0 & 1 & \ddots & \ddots
\end{pmatrix}.
\]

Now let \( A_1 = [w_{21}(1) \ w_{32}(1)B_1^1 \ w_{32}(-1) \ w_{21}(-1)]^{-1} \in S^2_n \). We obtain

\[
\varphi_n(A_1A_o) =
\begin{pmatrix}
  1 & d^{-1} & 0 \\
  d^{-1} & c' & 1 & \ddots & \ddots \\
  0 & 1 & \ddots & \ddots
\end{pmatrix}
\]

and hence

\[
\varphi_n(A_1A_oB_1^1) =
\begin{pmatrix}
  1 & d^{-1} & 0 \\
  d^{-1} & c' & 1 & \ddots & \ddots \\
  0 & 1 & \ddots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
  1 & c & d & 0 \\
  c & 1 & \ddots & \ddots & \ddots \\
  0 & 1 & \ddots & \ddots & \ddots
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 0 \\
  d^{-1}c & 1 \\
  c'c & c'd & 1 & \ddots \\
  0 & \ddots & \ddots & \ddots
\end{pmatrix}
\]
Thus \( \varphi_n(A_1A_0B_1^1) = \varphi_n(x_{21}(d^{-1}c)x_{31}(c'c)x_{32}(c'd)) \). But both \( A_1A_0B_1^1 \) and \( x_{21}(d^{-1}c)x_{31}(c'c)x_{32}(c'd) \) lie in the subgroup of \( \text{St}(n,R) \) generated by \( x_{i,j}(r), i\leq j, j\leq 3 \) and as \( N(\mathfrak{R}) \) is the normal subgroup generated by \( K_2(3,\mathfrak{R}) \) we must have

\[
A_1A_0B_1^1 = x_{21}(d^{-1}c)x_{31}(c'c)x_{32}(c'd) \mod N(\mathfrak{R}).
\]

Let \( L_0 = x_{21}(d^{-1}c)x_{31}(c'c)x_{32}(c'd) \) and let

\[
A = A_1A_0 \in S^2_n,
\]

then

\[
z = [B_1^{n-1}\ldots B_1^2 A^{-1} AB_1^1]\ldots [B_{n-1}^{n-1}]L \mod N(\mathfrak{R})
\]

and

\[
z = [B_1^{n-1}\ldots (B_1^2 A^{-1})L_0]\ldots [B_{n-1}^{n-1}]L \mod N(\mathfrak{R}).
\]

As in the proof of Theorem 2.7, (A), (B), and (C) hold. Thus the same technique used there works here.

We obtain then

\[
z = [B_2^{n-1}\ldots B_2^2]\ldots [B_{n-1}^{n-1}]L \mod N(\mathfrak{R}).
\]

We now repeat the above process, until we reach

\[
z = [\overline{B}_{n-2}^{n-1} \overline{B}_{n-2}^{n-2}] [\overline{B}_{n-1}^{n-1}]\overline{L} \mod N(\mathfrak{R}).
\]

Applying \( \varphi_n \) and looking at this equation on the matrix level, we see that \( \overline{L} = x_{n-1,n-2}(r_1)x_{n,n-2}(r_2)x_{n,n-1}(r_3) \)

for some \( r_1, r_2, r_3 \in R \). Thus \( \overline{B}_{n-2}^{n-1}, \overline{B}_{n-2}^{n-2}, \overline{B}_{n-1}^{n-1}, \) and \( \overline{L} \)

lie in the subgroup of \( \text{St}(n,R) \) generated by the elements
of the form $x_{ij}(r)$ for $n-2 \leq i, j \leq n$, and as

$$\varphi_n(B^n_{n-1} B^n_{n-2} B^n_{n-1} L) = 1,$$

we have that $B^n_{n-1} B^n_{n-2} B^n_{n-1} L \in N(\mathfrak{n})$ as $N(\mathfrak{n})$ is the normal subgroup of $\text{St}(n, R)$ generated by $K_2(3, R)$ and hence contains all of the 3x3 relations. Thus $z \equiv 1 \mod N(\mathfrak{n})$ and hence $\psi$ is an isomorphism. Q.E.D.

To show that every element of $\text{St}(n, R)$ can be written in the normal form, by Lemma 2.6 it suffices to show this for $n = 3$. The following lemma gives a further reduction.

**Lemma 2.9:** Every element of $\text{St}(3, R)$ can be written in the normal form if and only if every element of the form $x_{13}(r)B$ can be written in the normal form, where $r \in R$ and $B \in S_3^1$.

**Proof:** The necessity of this condition is obvious. To prove sufficiency, we need only show that the subset of elements of $\text{St}(3, R)$ which can be written in the normal form is a subgroup of $\text{St}(3, R)$, as the elements of $S_3^1$ and $S_3^2$ generate $\text{St}(3, R)$. It thus suffices to show that

$$g[B_1^2 B_1^1][B_2^2]L$$

can be written in the normal form, where $g$ is any generator of $S_3^1$ or $S_3^2$. Thus $g$ may be any one of the elements $x_{21}(r), x_{31}(r), x_{32}(r), x_{23}(r), x_{12}(r)$. All of the first three elements are in $\text{LS}_3$ and hence we are done by applying
Lemma 2.4 three times. If \( g = x_{23}(r) \), we can simply combine \( g \) with \( B_1^2 \). The only remaining case is when \( g = x_{12}(r) \).

Note the following identities:

\[
[x_{13}(u)x_{12}(v)]x_{23}(w) = x_{23}(w)[x_{13}(u+vw)x_{12}(v)]
\]

\[
[x_{13}(u)x_{12}(v)]x_{32}(w) = x_{32}(w)[x_{13}(u)x_{12}(v+uw)].
\]

Hence \( x_{12}(r)B_1^2 = B_1^2 x_{13}(u) x_{12}(v) \) for some \( u, v \in \mathbb{R} \). We now obtain

\[
x_{12}(r)B_1^2 B_1^1 B_2^2 L = B_1^2 x_{13}(u) x_{12}(v)B_1^1 B_2^2 L.
\]

By hypothesis, there exist \( A_1^2, A_2^2 \in S_3^2 \), \( A_1^1 \in S_3^1 \), and \( L_1 \in LS_3 \) such that

\[
x_{13}(u) x_{12}(v)B_1^1 = A_1^2 A_1^1 A_2^2 L_1.
\]

Then we have

\[
x_{12}(r)B_1^2 B_1^1 B_2^2 L = B_1^2 A_1^2 A_1^1 A_2^2 L_1 B_2^2 L.
\]

By Lemma 2.4 we can write \( L_1 B_2^2 = \overline{B}_2^2 L_2 \) for some \( B_2^2 \in S_3^2 \) and some \( L_2 \in LS_3 \). Now let \( C_1^2 = B_1^2 A_1^2 \in S_3^2 \),

\( C_2^2 = A_2^2 B_2^2 \in S_3^2 \), \( C_1^1 = A_1^1 \in S_3^1 \), and \( L_3 = L_2 L \in LS_3 \), then we have

\[
x_{12}(r)B_1^2 B_1^1 B_2^2 L = (B_1^2 A_1^2)A_1^1 (A_2^2 B_2^2)(L_2 L)
\]

\[
= C_1^2 C_1^1 C_2^2 L_3
\]
which is in the normal form. This completes the proof of the lemma.

We are now in a position to give examples of rings for which every element of $\text{St}(n,R)$ for $n \geq 3$ can be written in the normal form.

**Theorem 2.10:** Let $R$ be a local ring (not necessarily commutative) that satisfies the property:

If $a, b \in R$, $a \neq 0$, $b \neq 0$, then either $a = bc$ for some $c \in R$, or $b = ad$ for some $d \in R$.

Then every element of $\text{St}(n,R)$ for $n \geq 3$ can be written in the normal form.

**Proof:** By Lemmas 2.6 and 2.9, it suffices to prove that $x_{13}(r)B$ can be written in the normal form in $\text{St}(3,R)$ for any $r \in R$ and any $B \in \text{S}_3$. First of all, every element $B \in \text{S}_3$ can be written as

$$B = x_{12}(r_m)x_{21}(s_m)\ldots x_{12}(r_1)x_{21}(s_1)$$

for some $s_i, r_i \in R$.

We have

$$x_{13}(r)B = x_{13}(r)Bx_{23}(a)x_{23}(-a)$$

$$= x_{23}(g_m a)x_{13}(r + f_m a)Bx_{23}(-a)$$

where $f_m$ and $g_m$ are defined recursively by
\[ f_0 = 0, \quad g_0 = 1 \]
\[ g_{i+1} = s_{i+1} f_i + g_i \]
\[ f_{i+1} = r_{i+1} g_{i+1} + f_i. \]

This follows from the identity:

\[ x_{12}(r_{i+1}) x_{21}(s_{i+1}) x_{23}(g_i) x_{13}(f_i) = x_{23}((g_i + s_{i+1} f_i) x_{13}((r_{i+1} (g_i + s_{i+1} f_i) + f_i)) x_{12}(r_{i+1}) x_{21}(s_{i+1}). \]

Now if \( f_m \neq 0 \) and \( r = f_m c \) for some \( c \in \mathbb{R} \), let \( \alpha = -c \), then

\[ x_{13}(r)B = x_{23}(-g_n c)Bx_{23}(-c) \]

which is clearly in the normal form.

Thus we need only take care of the two cases:

- \( f_m = 0 \) or \( r \neq f_m c \) for any \( c \) in \( \mathbb{R} \). In either case, observe that \( f_m \) is not a unit.

Note the identities:

\[ [x_{32}(u)x_{31}(v)]x_{12}(w) = x_{12}(w)[x_{32}(u + vw)x_{31}(v)] \]
\[ [x_{32}(u)x_{31}(v)]x_{21}(w) = x_{21}(w)[x_{32}(u)x_{31}(v + uw)]. \]

Then we obtain

\[ x_{13}(r)B = x_{32}(1)x_{32}(-1)x_{13}(r)x_{12}(r_m)B'. \]
\[ = x_{32}(r)x_{13}(r)x_{12}(r_m+r)x_{32}(-1)B' \]
\[ = x_{32}(l)x_{13}(r)x_{12}(r_m+r)B'x_{32}(u)x_{31}(v) \]

where \( B = x_{12}(r_m)B' \) and \( u \) and \( v \) are some elements of \( R \).

Now consider the problem of writing \( x_{13}(r)x_{12}(r_m+r)B' \) in the normal form. By our first computation

\[ x_{13}(r)x_{12}(r_m+r)B' = x_{23}(g'_m\gamma)x_{13}(r+f'_m\gamma)x_{12}(r_m+r)x_{23}(-\gamma) \]

where \( g'_m \) and \( f'_m \) are defined in the same manner as \( g_m \) and \( f_m \). Note that the only change is that \( r_m \) is replaced by \( r_m+r \) and hence \( g'_i = g_i, i=0,1,\ldots,m \), and \( f'_i = f_i, i=0,\ldots,m-1 \).

We shall show that by an appropriate choice of \( \gamma \) that we have \( r+f'_m\gamma = 0 \). Now we have

\[ f'_m = (r_m+r)g_m+f_{m-1} \]
\[ = rg_m+(r_mg_m+f_{m-1}) \]
\[ = rg_m+f_m. \]

If \( f'_m \) were a unit, then by choosing \( \gamma = -(f'_m)^{-1}r \), we would obtain our objective of making \( r+f'_m\gamma = 0 \). So assume \( f'_m \) is not a unit. By hypothesis \( f_m \) is not a unit. As \( R \) is local, \( rg_m \) is not a unit as \( f'_m = rg_m+f_m \).

Now observe that for every \( i \), at least one of \( g_i \) and \( f_i \) is a unit. This is certainly true if \( i = 0 \), as \( g_0 = 1 \). We complete the proof by induction. Assume the
result holds for i-1. If $g_i$ and $f_i$ are both nonunits, then as $f_i = r_i g_i + f_{i-1}$ and R is local, $f_{i-1}$ must be a nonunit. Now $g_i = s_i f_{i-1} + g_{i-1}$, and again as R is local, $g_{i-1}$ must be a nonunit. This contradiction completes the proof.

Hence as $f_m$ is not a unit, $g_m$ must be a unit. Now if $f_m = 0$, let $\gamma = -g_m^{-1}$, then $r + f_m \gamma = r + (rg_m)(-g_m^{-1}) = 0$. If $f_m \neq 0$, we have that $f_m = rd$ for some $d \in R$. d is not a unit, for if it were $r = r_m d^{-1}$, a contradiction. As R is local, $g_m + d$ is a unit. Let $\gamma = -(g_m + d)^{-1}$ in this case. Then $r + f_m \gamma = r + r(g_m + d)(-(g_m + d))^{-1} = 0$.

Thus in any case, we have chosen $\gamma$ so that $r + f_m \gamma = 0$. Hence we obtain

$$x_{13}(r)B = x_{32}(1)x_{13}(r)x_{12}(r + r)B'x_{32}(u)x_{31}(v)$$

$$= x_{32}(1)x_{23}(g_m \gamma)x_{13}(0)x_{12}(r + r)B'x_{23}(-\gamma)x_{32}(u)x_{31}(v)$$

$$= [x_{32}(1)x_{23}(g_m \gamma)][x_{12}(r + r)B'][x_{23}(-\gamma)x_{32}(u)]x_{31}(v)$$

which is in the normal form. Q.E.D.

Examples: Fields, division rings, and valuation rings all satisfy the hypotheses of this theorem.

The next result [Proposition 2.12] is found in Milnor [6], but as it is not stated exactly the same way, a proof will be given here.
Definition: In $\text{St}(n,R)$ we will say that an element $z$ involves no more than $k$ indices, if there is an expression for $z$ in terms of the generators $x_{ij}(r)$ such that the set consisting of all of the indices $i,j$ that appear in this expression has no more than $k$ elements.

Lemma 2.11: Let $R$ be any ring. Let $P_{n_o,n}$ by the subgroup of $\text{St}(n,R)$ generated by all of the elements of the form $x_{in_o}(r)$ for $i=1,\ldots,n_o-1,n_o+1,\ldots,n$. Then every element of $P_{n_o,n}$ can be written uniquely in the form

$$x_{ln_o}(r_1)\cdots x_{n_o-1,n_o}(r_{n_o-1})x_{n_o+1,n_o}(r_{n_o+1})\cdots x_{nn_o}(r_n).$$

Hence the canonical homomorphism $\varphi_n$ is monic when restricted to $P_{n_o,n}$.

Proof: Obvious by the first and third Steinberg relations.

Proposition 2.12: If $z \in K_2(n,R)$ and $z$ involves no more than $n-1$ indices, then $z$ is in the center of $\text{St}(n,R)$.

Proof: If $z$ involves no more than $n-1$ indices, then there is some expression for $z$ in which some index does not appear, say $n_o$. Now observe that

$$x_{ij}(r)P_{n_o,n}x_{ij}(-r) \leq P_{n_o,n} \text{ if } i \neq n_o \text{ and } j \neq n_o \text{ as }$$

$$x_{ij}(r)x_{kn_o}(s)x_{ij}(-r) \text{ is equal to either } x_{kn_o}(s) \text{ if } j \neq k \text{ or to } x_{in_o}(rs)x_{kn_o}(s) \text{ if } j = k.$$ 

Now as $z$ is a product of $x_{ij}(r)$ with $i \neq n_o$ and
j \neq n_o$, we have that $z p_{n-o, n}^{-1} \subseteq p_{n-o, n}$. As $\varphi_n(z) = 1$, we have for each $p \in p_{n-o, n}$

$$\varphi_n(z p_{o}^{-1}) = \varphi_n(z) \varphi_n(p) \varphi_n(z)^{-1} = \varphi_n(p).$$

Since $\varphi_n$ is monic, $z p_{o}^{-1} = p$. Thus $z$ commutes with every element $x_{i n-o}(r)$ of $St(n, R)$ when $i \neq n_o$. An analogous argument shows that $z$ commutes with every element of the form $x_{n-o i}(r)$ for $i \neq n_o$. Hence $z$ commutes with $[x_{k n-o}(r), x_{n-o l}(1)] = x_{k l}(r)$ for all $k \neq n_o$ and all $l \neq n_o$. Thus as $z$ commutes with every generator of $St(n, R)$, $z$ is in the center of $St(n, R)$.

The following theorem seems to indicate the significance of the fact that every element of $St(n, R)$ can be written in the normal form.

**Theorem 2.13:** Let $R$ be a ring satisfying $N'_n$ for all $n \geq 3$ and assume every element of $E(3, R)$ can be written in the normal form. Let $\mathcal{A}$ be a complete set of defining relations for $E(2, R)$. Then $\mathcal{A}$ and the Steinberg relations form a complete set of defining relations for $E(3, R)$ if and only if every element of $St(3, R)$ can be written in the normal form.

**Proof:** Theorem 2.7 gives the sufficiency of the condition.

First observe that the hypothesis that $\mathcal{A}$ and the Steinberg relations are a complete set of defining relations for $E(3, R)$ implies that every element of $K_2(3, R)$
is a product of conjugates of the images of the elements of \( \mathfrak{g} \) in \( \text{St}(3,\mathbb{R}) \).

By the preceding lemma, every element in the image of \( \mathfrak{g} \) in \( \text{St}(3,\mathbb{R}) \) is in the center of \( \text{St}(3,\mathbb{R}) \) and hence every element of \( K_2(3,\mathbb{R}) \) is a product of images of elements of \( \mathfrak{g} \). That is, we may assume every element in \( K_2(3,\mathbb{R}) \) is a product of the form

\[
x_{12}(r_m)x_{21}(s_m)\cdots x_{12}(r_1)x_{21}(s_1)
\]

for some \( r_i, s_i \in \mathbb{R} \). By the remark following Lemma 2.5, there is a \( w \in W_3 \) such that

\[
w(x_{12}(r_m)x_{21}(s_m)\cdots x_{12}(r_1)x_{21}(s_1))w^{-1}
\]

\[
= x_{23}(r_m)x_{32}(s_m)\cdots x_{23}(r_1)x_{32}(s_1)
\]

\[
= x_{12}(r_m)x_{21}(s_m)\cdots x_{12}(r_1)x_{21}(s_1).
\]

The last equality holds as the element \( x_{12}(r_m)\cdots x_{21}(s_1) \) is in the center of \( \text{St}(3,\mathbb{R}) \).

Now as \( \text{St}(3,\mathbb{R})/K_2(3,\mathbb{R}) \) is isomorphic to \( E(3,\mathbb{R}) \) and every element of \( E(3,\mathbb{R}) \) can be written in the normal form, we can lift this to \( \text{St}(3,\mathbb{R}) \). If \( g \in \text{St}(3,\mathbb{R}) \), then there exist \( B_j^1 \in S_j^1, L \in LS_3 \), and a \( z \in L_2(3,\mathbb{R}) \) such that

\[
g = z[B_1^2 B_1^1][B_2^2]L.
\]

By our preceding observation, \( z \in S_3^2 \). Hence
\[ g = [(z B^2_1) B^1_1][B^2_2] L \]

is the normal form.

**Corollary:** Let \( R \) be a ring satisfying \( N_n' \) for all \( n \geq 3 \) and assume every element of \( E(3, R) \) can be written in the normal form. Let \( \mathcal{R} \) be a complete set of defining relations for \( E(2, R) \). Then \( \mathcal{R} \) and the Steinberg relations form a complete set of defining relations for \( E(n, R) \) for all \( n \geq 3 \) if and only if every element of \( St(n, R) \) can be written in the normal form.

**Proof:** Theorem 2.7 gives the sufficiency of the condition. Necessity follows from the preceding theorem and Lemma 2.6.

**Example:** Let \( R = \mathbb{Z} \), the ring of integers. Then by Milnor [6], \( E(n, \mathbb{Z}) \) has the Steinberg relations and the single other relation

\[ (w_{12}(1))^4 = 1 \]

as a complete set of defining relations. Hence every element of \( St(n, \mathbb{Z}) \) for all \( n \geq 3 \) can be written in the normal form.

**Remark:** The computation of Milnor depends on a presentation of \( GL(n, \mathbb{Z}) \) given by Magnus in [5]. Magnus' presentation for \( GL(n, \mathbb{Z}) \) in turn depends on the presentation for \( GL(3, \mathbb{Z}) \) given by Nielsen in [7]. If one could show
directly that every element of \( \text{St}(3, \mathbb{Z}) \) could be written in the normal form, the laborious computations used by Nielsen in obtaining a presentation for \( \text{GL}(3, \mathbb{Z}) \) could be avoided. For Nielsen (very easily) shows that a presentation for \( \text{SL}(2, \mathbb{Z}) = \text{E}(2, \mathbb{Z}) \) is given by

\[
    w_{21}(1)w_{12}(1) = 1
\]

\[
    (w_{12}(1))^4 = 1 .
\]

The first relation becomes a consequence of the Steinberg relations when \( n \geq 3 \).
3. The Relations in $E(2, R)$

The essential ideas of this section are due to Cohn [4]. His results do not immediately apply to our situation as he considers the groups $GL(2, R)$. However, his methods will work to give presentations of $E(2, R)$ in certain cases.

First of all, we will define a group which will play the role of the Steinberg groups of the previous sections.

**Definition:** The Steinberg group $St(2, R)$ is the group defined by the generators $x_{12}(r)$ and $x_{21}(r)$, where $r$ ranges over $R$, subject to the relations

1) \( x_{ij}(r)x_{ij}(s) = x_{ij}(r+s) \)

2) \( w_{ij}(u)x_{ij}(r)w_{ij}(-u) = x_{ji}(-u^{-1}ru^{-1}) \)

3) \( w_{ji}(u)x_{ij}(r)w_{ji}(-u) = x_{ji}(-uru) \)

for every $r, s \in R$ and for every unit $u \in R$ where

\( w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u) \).

Note that by conjugating $w_{ij}(u)$ by itself we obtain

\( w_{ij}(u) = w_{ji}(-u^{-1}) \).

As before we define the canonical epimorphism

\[ \phi_2 : St(2, R) \longrightarrow E(2, R) \]

by \( \phi_2(x_{ij}(r)) = E_{ij}(r) \).
We will let $K_2(2, R)$ denote the kernel of $\varphi_2$. As in the general case, we have the commutative diagram with exact rows

$$
\begin{array}{cccc}
1 & \longrightarrow & K_2(2, R) & \longrightarrow & St(2, R) & \longrightarrow & E(2, R) & \longrightarrow & 1 \\
& & \downarrow \alpha_2 & & \downarrow \beta_2 & & \downarrow \gamma_2 & & \\
1 & \longrightarrow & K_2(3, R) & \longrightarrow & St(3, R) & \longrightarrow & E(3, R) & \longrightarrow & 1
\end{array}
$$

where $\gamma_2$ and $\beta_2$ are the obvious homomorphisms and $\alpha_2$ is the homomorphism induced by $\beta_2$.

Following Cohn [4] we define

$$X(r) = x_{12}(-r)w_{12}(1)$$

in $St(2, R)$. Observe that the elements of the form $X(r)$ generate $St(2, R)$ as

$$x_{12}(r) = X(-r)X(0)^{-1}$$

and $$x_{21}(r) = X(0)^{-1}X(r).$$

Let $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ for any unit $u$ of $R$. $H_2$ will denote the subgroup of $St(2, R)$ generated by all elements of the form $h_{ij}(u)$, $u$ a unit of $R$.

**Theorem 3.1**: Let $R$ be any ring. Then in $St(2, R)$ the following relations are valid:

(i) $X(r)X(0)X(s) = h_{21}(-1)X(r+s)$
(ii) \( X(r)h_{12}(u) = h_{12}(u^{-1})X(uru) \)

(iii) \( X(0)^2 = h_{21}(-1) \)

(iv) \( X(r)^{-1} = h_{12}(-1)^2X(0)X(-r)X(0) \)

(v) \( X(r)X(u)X(s) = X(r-u^{-1})h_{12}(-u^{-1})h_{21}(-1)X(s-u^{-1}) \)

where \( r, s \) are any elements of \( R \) and \( u \) is any unit of \( R \).

Moreover, these relations imply that every element \( g \in \text{St}(2, R) \) can be written in the standard form

\[
g = h \cdot X(r_1) \cdots X(r_k)
\]

where \( h \in H_2 \), for \( 1 \leq k \leq k \), \( r_1 \) is neither zero nor a unit, and in case \( r = 2 \), \( r_1 \) and \( r_2 \) are not both zero.

**Proof:** First observe that

\[
(w_{12}(1))^2 = (w_{21}(-1))^2 = h_{21}(-1).
\]

(i) \( X(r)X(0)X(s) = x_{12}(-r)w_{12}(1)x_{12}(0)w_{12}(1)x_{12}(-s)w_{12}(1) \)

\[
= x_{12}(-r)(w_{12}(1))^2 x_{12}(-s)w_{12}(1)
\]

\[
= (w_{12}(1))^2 x_{12}(-r)x_{12}(-s)w_{12}(1)
\]

\[
= h_{21}(-1)x_{12}(-r-s)w_{12}(1)
\]

\[= h_{21}(-1)X(r+s). \]

(ii) First we observe that
\[ w_{12}(u)w_{12}(v) = w_{12}(v)w_{12}(-v)[x_{12}(u)x_{21}(-u^{-1})x_{12}(u)]w_{12}(v) \]
\[ = w_{12}(v)x_{21}(-v^{-1}uv^{-1})x_{12}(vu^{-1}v)x_{21}(-v^{-1}uv^{-1}) \]
\[ = w_{12}(v)w_{21}(-v^{-1}uv^{-1}) \]
\[ = w_{12}(v)w_{12}(vu^{-1}v) \]

and similarly \( w_{12}(u)w_{12}(v) = w_{12}(uv^{-1}u)w_{12}(u) \).

Hence

\[ X(r)h_{12}(u) = x_{12}(-r)w_{12}(1)w_{12}(u)w_{12}(-1) \]
\[ = x_{12}(-r)w_{12}(1)w_{12}(-1)w_{12}(u^{-1}) \]
\[ = x_{12}(-r)w_{12}(u^{-1}) \]
\[ = w_{12}(u^{-1})x_{21}(uru)w_{12}(-1)w_{12}(1) \]
\[ = w_{12}(u^{-1})w_{12}(-1)x_{21}(-uru)w_{12}(1) \]
\[ = h_{12}(u^{-1})X(uru). \]

(iii) Letting \( r = s = 0 \) in (i) we obtain

\[ X(0)^3 = h_{21}(-1)X(0) \]

and hence

\[ X(0)^2 = h_{21}(-1). \]

(iv) Letting \( s = -r \) in (i) we obtain
\[ X(r)^{-1} = X(0)X(-r)X(0)^{-1}h_{21}(-1)^{-1} \]
\[ = X(0)X(-r)X(0)h_{21}(-1)^{-2} \]
\[ = h_{21}(-1)^{-2}X(0)X(-r)X(0) \]
\[ = h_{12}(-1)^2X(0)X(-r)X(0). \]

We have used the fact that \( h_{21}(-1) = w_{12}(1)^2 \) is in the center of \( \text{St}(2, \mathbb{R}) \) and the identity \( h_{21}(1)^{-1} = h_{12}(-1) \).

(v) We will need two preliminary results before proving (v). First if \( u \) is a unit

\[ X(u)X(u^{-1})X(u) = x_{12}(-u)w_{12}(1)x_{12}(-u^{-1})w_{12}(1)x_{12}(-u)w_{12}(1) \]
\[ = w_{12}(1)x_{21}(u)w_{12}(1)x_{21}(u^{-1})x_{12}(-u)w_{12}(1) \]
\[ = w_{12}(1)^2x_{12}(-u)x_{21}(u^{-1})x_{12}(-u)w_{12}(-1)w_{12}(1)^2 \]
\[ = h_{21}(-1)h_{12}(-u)h_{21}(-1) \]
\[ = h_{21}(-1)^2h_{12}(-u). \]

Next we have

\[ X(r)X(s)^{-1} = X(r)h_{12}(-1)^2X(0)X(-s)X(0) \]
\[ = h_{12}(-1)^2X(r)X(0)X(-s)X(0) \]
\[ = h_{12}(-1)^2h_{21}(-1)X(r-s)X(0) \]
\[ = h_{12}(-1)X(r-s)X(0) \]
and similarly \( X(r)^{-1}X(s) = h_{12}(-1)X(0)X(s-r). \)

Now let \( u \) be a unit of \( R \), then

\[
X(r)X(u)X(s) = X(r)X(u^{-1})^{-1}h_{21}(-1)^2h_{12}(-u)X(u^{-1})^{-1}X(s)
\]

\[
= h_{12}(-1)X(r-u^{-1})X(0)h_{21}(-1)^2h_{12}(-u)h_{12}(-1)X(0)X(s-u^{-1})
\]

\[
= X(r-u^{-1})X(0)h_{12}(-u)X(0)X(s-u^{-1})
\]

\[
= X(r-u^{-1})h_{12}(-u^{-1})X(0)^2X(s-u^{-1})
\]

\[
= X(r-u^{-1})h_{12}(-u^{-1})h_{21}(-1)X(s-u^{-1}).
\]

It remains only to show that every element of \( \text{St}(2,R) \) can be put into the standard form. We already know that \( \text{St}(2,R) \) is generated by the elements \( X(r) \). Now by (iv) we can replace the inverse of any \( X(r) \) by a product of an element of \( H \) and other \( X \)'s which all have positive exponent. By (ii) all of the elements of \( H \) can be moved to the left without introducing any new \( X \)'s. Thus every element of \( \text{St}(2,R) \) can be written in the form

\[
h X(r_1) \cdots X(r_k).
\]

Now if for some \( i, 1<i<k \), we have \( r_i = 0 \), then the expression can be shortened by using (i) and then (ii). If for some \( i, 1<i<k \), \( r_i \) is a unit, then the expression can be shortened by using (v) and then (ii). If \( k=2 \) and \( r_1 = r_2 = 0 \), then (iii) can be used to eliminate the \( X \)'s. This completes the proof of the theorem.
We will now investigate the nature of the relations in $E(2, R)$ for a specific class of rings.

**Definition:** A ring $R$ is discretely normed if there is a mapping $|\cdot|$ from $R$ to the nonnegative real numbers such that

1. $|r| = 0$ if and only if $r = 0$
2. $|r+s| \leq |r| + |s|$
3. $|rs| = |r||s|$
4. $|r| \geq 1$ for all $r \neq 0$ with equality only if $r$ is a unit of $R$
5. There exists no $r \in R$ such that $1 < |r| < 2$.

Now let $\mathbb{Z}[t_1, t_2, \ldots]$ denote the ring of noncommuting polynomials on countably many generators over the integers. Define a sequence of polynomials by

$$e_{-1} = 0, \quad e_0 = 1$$

$$e_k(t_1, \ldots, t_k) = e_{k-1}(t_1, \ldots, t_{k-1})t_k - e_{k-2}(t_1, \ldots, t_{k-2}).$$

We then have that

$$\varphi_2(x(r_1)\cdots x(r_k)) = \begin{pmatrix} r_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} r_k & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e_k(r_1, \ldots, r_k) & e_{k-1}(r_1, \ldots, r_{k-1}) \\ -e_{k-1}(r_2, \ldots, r_k) & -e_{k-2}(r_2, \ldots, r_{k-1}) \end{pmatrix}$$
which follows by a simple induction argument.

**Lemma 3.2:** Let \( r_1, \ldots, r_k \) (\( k \geq 1 \)) be any elements of a discretely normed ring \( R \) such that the \( r_i \) for \( i > 1 \) are neither zero nor units of \( R \). Then

\[
|e_k(r_1, \ldots, r_k)| \geq |e_{k-1}(r_1, \ldots, r_{k-1})|
\]

and

\[
e_k(r_1, \ldots, r_k) \neq 0.
\]

**Proof:** See Cohn [4, Lemma 5.1].

**Theorem 3.3:** Let \( R \) be a discretely normed ring. Then \( K_2(2,R) \subseteq H_2 \).

**Proof:** Let \( z = h X(r_1) \cdots X(r_k) \) be an element of \( K_2(2,R) \) written in standard form. Then we have

\[
1 = \varphi_2(z)
\]

\[
= \varphi_2(h)\varphi_2(X(r_1) \cdots X(r_k))
\]

which is just

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}
\begin{pmatrix}
e_k(r_1, \ldots, r_k) & e_{k-1}(r_1, \ldots, r_{k-1}) \\
-e_{k-1}(r_2, \ldots, r_k) & -e_{k-2}(r_2, \ldots, r_{k-1})
\end{pmatrix}
\]

The \((1,2)\) position yields

\[
e_{k-1}(r_1, \ldots, r_{k-1}) = 0
\]

which contradicts Lemma 3.2 if \( k > 2 \). Hence \( k \leq 2 \). If \( k = 1 \), we have
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}\begin{pmatrix}
r_1 & 1 \\
-1 & 0
\end{pmatrix}
\]

and hence \( \alpha = 0 \) which is a contradiction since \( \alpha \) is a unit. If \( k=2 \), we have

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}\begin{pmatrix}
r_1 & 1 \\
-1 & 0
\end{pmatrix}
= \begin{pmatrix}
\alpha r_1 & r_2 \\
r_1 & \alpha r_1
\end{pmatrix}
= \begin{pmatrix}
\alpha r_1 & r_2 \\
r_1 & \alpha r_1
\end{pmatrix}
\]

Thus \( \alpha r_1 = \beta r_2 = 0 \) and hence \( r_1 = r_2 = 0 \) which yields a contradiction as \( z \) was assumed to be written in standard form.

In all cases we have that \( z = h \) and hence

\( K_2(2, R) \subseteq H_2 \).

Done.

Remark: This theorem enables us to give a presentation for any discretely normed ring. First observe that

\[
h_{12}(u)h_{21}(u) = w_{12}(u)w_{12}(-1)w_{21}(u)w_{21}(-1)
\]

\[
= w_{12}(u)w_{12}(-1)w_{12}(-u^{-1})w_{12}(1)
\]

\[
= w_{12}(u)w_{12}(-1)w_{12}(1)w_{12}(-u)
\]

\[
= 1.
\]
Hence any element \( h \in H_2 \) may be written as

\[
h = h_{12}(u_1)^{\epsilon_1} \ldots h_{12}(u_k)^{\epsilon_k}, \quad \text{where } \epsilon_i = \pm 1.
\]

Thus if \( h \in K_2(2,R) \) and \( h \) is in the above form we have

\[
1 = \varphi_2(h) = \begin{pmatrix}
\epsilon_1 & 0 \\
0 & -\epsilon_1
\end{pmatrix} \ldots \begin{pmatrix}
\epsilon_k & 0 \\
0 & -\epsilon_k
\end{pmatrix}
= \begin{pmatrix}
\epsilon_1 \ldots \epsilon_k & 0 \\
0 & -\epsilon_1 \ldots -\epsilon_k
\end{pmatrix}
\]

and hence \( u_1^{\epsilon_1} \ldots u_k^{\epsilon_k} = 1 \) and \( u_1^{-\epsilon_1} \ldots u_k^{-\epsilon_k} = 1 \). Thus a complete set of defining relations for \( E(2,R) \) are the relations \( h_{12}(u_1)^{\epsilon_1} \ldots h_{12}(u_k)^{\epsilon_k} \), where \( \epsilon_i = \pm 1 \) and \( u_1, \ldots, u_k \) are units of \( R \) such that \( u_1^{\epsilon_1} \ldots u_k^{\epsilon_k} = 1 \) and \( u_1^{-\epsilon_1} \ldots u_k^{-\epsilon_k} = 1 \), and the relations used in defining \( St(2,R) \).

In the case that \( R \) is a commutative ring, it is easy to see that all of the relations in \( H_2 \) are a consequence of the relations \( h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1} \).

**Note:** \( K_2(2,R) \subseteq H_2 \) implies that \( K_2(2,R) \) is an abelian group. For by relations (2) and (3) for \( St(2,R) \) any
element of $H_2$, conjugates a generator $x_{ij}(r)$ of $\text{St}(2, \mathbb{R})$ into some other generator $x_{km}(s)$ of $\text{St}(2, \mathbb{R})$. Hence if $z \in K_2(2, \mathbb{R})$

$$zx_{ij}(r)z^{-1} = x_{km}(s).$$

Now apply $\varphi_2$ and obtain

$$\varphi_2(zx_{ij}(r)z^{-1}) = IE_{ij}(r)I^{-1}$$

$$= \varphi_2(x_{km}(s))$$

$$= E_{km}(s).$$

Thus $k=i$, $m=j$, and $r=s$. Hence

$$zx_{ij}(r) = x_{ij}(r)z$$

for every generator $x_{ij}(r)$ of $\text{St}(2, \mathbb{R})$. Thus $K_2(2, \mathbb{R})$ is in the center of $\text{St}(2, \mathbb{R})$.

Examples of discretely normed rings:

i) Any division ring. Define $|\cdot|$ by $|0| = 0$ and $|r| = 1$ if $r \neq 0$.

ii) $\mathbb{Z}$, the ring of integers.

iii) The ring of algebraic integers in any imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ where $d$ is positive and square free, $d \neq 1, 2, 3, 7, 11$. The norm is the absolute value function induced by the one on the complex numbers.
iv) The polynomial ring in any number of indeterminants over a field. The norm is given by \(|0| = 0\) and \(|r| = 2^{d(r)}\) for \(r \neq 0\), where \(d(r)\) is the usual degree function.

v) The free associative algebras (polynomial rings in non-commuting indeterminants) with the norm function defined as in iv).

For a discussion of these examples and others see Cohn [4, §5].
4. **The Normal Form and $K_2$**

For rings which admit the normal form in $St(n,R)$ or $E(n,R)$ for every $n \geq 3$ we can derive certain partial results about $K_2(R)$.

**Theorem 4.1:** Let $R$ be a ring which satisfies the condition $N'_n$ for every $n \geq 3$. Assume that every element of $St(3,R)$ can be written in the normal form. Then the groups $K_2(n,R)$ are abelian for every $n \geq 3$. The homomorphisms

$$K_2(n,R) \longrightarrow K_2(m,R)$$

are epic for every $m > n \geq 2$ and hence the homomorphisms

$$K_2(n,R) \longrightarrow K_2(R)$$

are epic for every $n \geq 2$.

**Proof:** By Theorem 2.7 a complete set of defining relations for $E(n,R)$ is given by $\mathcal{R}$ and the Steinberg relations, where $\mathcal{R}$ is a complete set of defining relations for $E(2,R)$. Thus $K_2(n,R)$ is generated as a normal subgroup of $St(n,R)$ by the image of $\mathcal{R}$. By Proposition 2.12 each element in the image of $\mathcal{R}$ is in the center of $St(n,R)$ if $n > 2$. Hence if $n > 2$, $K_2(n,R)$ is generated by the images of the elements of $\mathcal{R}$. For $n = 2$, $K_2(2,R)$ is generated as a normal subgroup of $St(2,R)$ by the image of $\mathcal{R}$. These last two statements clearly imply that the homomorphisms
\[ K_2(n, R) \longrightarrow K_2(m, R) \]

are epic for \( m > n \geq 2 \).

As \( K_2(n, R) \) for \( n \geq 3 \) is generated by a collection of elements in the center of \( \text{St}(n, R) \), \( K_2(n, R) \) is contained in the center of \( \text{St}(n, R) \) and hence is abelian.

That the homomorphisms

\[ K_2(n, R) \longrightarrow K_2(R) \]

are epic for every \( n \geq 2 \) follows from the fact that the homomorphisms

\[ K_2(n, R) \longrightarrow K_2(m, R) \]

are epic for \( m > n \geq 2 \) as \( K_2(R) \) is the direct limit of the \( K_2(n, R) \).

**Theorem 4.2:** Let \( R \) be a ring for which every element of \( E(3, R) \) can be written in the normal form. Then the groups \( K_2(n, R) \) are abelian for every \( n \geq 4 \). The homomorphisms

\[ K_2(n, R) \longrightarrow K_2(m, R) \]

are epic for every \( m > n \geq 3 \) and hence the homomorphisms

\[ K_2(n, R) \longrightarrow K_2(R) \]

are epic for every \( n \geq 3 \).

**Proof:** By Theorem 2.8 a complete set of defining relations for \( E(n, R) \) for \( n \geq 3 \) is given by \( \rho \) and the Steinberg
relations, where \( \mathfrak{R} \) is a complete set of defining relations for \( E(3,R) \). Thus \( K_2(n,R) \) for \( n \geq 3 \) is generated as a normal subgroup of \( \text{St}(n,R) \) by the image of \( \mathfrak{R} \). By Proposition 2.12 each element in the image of \( \mathfrak{R} \) is in the center of \( \text{St}(n,R) \) for every \( n \geq 4 \). Hence for \( n \geq 4 \), \( K_2(n,R) \) is a central subgroup of \( \text{St}(n,R) \) (and hence is abelian) which is generated by the image of \( \mathfrak{R} \). This clearly implies that the homomorphisms

\[
K_2(n,R) \rightarrow K_2(m,R)
\]

are epic for every \( m > n \geq 3 \).

The homomorphisms

\[
K_2(n,R) \rightarrow K_2(R)
\]

are epic for \( n \geq 3 \) as the homomorphisms

\[
K_2(n,R) \rightarrow K_2(m,R)
\]

are epic for \( m > n \geq 3 \) and \( K_2(R) \) is the direct limit of the \( K_2(n,R) \).

Now following Milnor [6] we will exhibit certain elements of \( K_2(n,R) \) for \( n \geq 3 \). Let \( u,v \) be units of the commutative ring \( R \).

**Definition:** The element \( \{u,v\} = [h_{ij}(u), h_{ik}(v)] \) (\( j \neq k \)) of \( K_2(n,R) \) for \( n \geq 3 \) is called a symbol.
$h_{ij}(u)$ denotes the element $w_{ij}(u)w_{ij}(-1)$ in $St(n, R)$. It is easy to check that $[u, v]$ is an element of $K_2(n, R)$.

**Lemma 4.3:** Let $n \geq 3$. The symbol $[u, v]$ does not depend on the choice of the indices $i, j, k$ used to define it. The following identities hold:

(i) $[u, v] = h_{ik}(uv)h_{ik}(u)^{-1}h_{ik}(v)^{-1}$

(ii) $[u, v][v, u] = 1$

(iii) $[uv, w] = [u, w][v, w]$

(iv) $[u, vw] = [u, v][u, w]$

(v) $[u, 1-u] = 1$ if both $u$ and $1-u$ are units

(vi) $[u, -u] = 1$ if both $u$ and $1-u$ are units

where $u, v, w$ are any units of the commutative ring $R$.

**Proof:** See Milnor [6, §9].

**Note:** As all of the maps are compatible, we can also consider $[u, v]$ to be an element of $K_2(R)$.

**Lemma 4.4:** If $R$ is commutative and $K_2(n, R) \subseteq W_n$ for some $n \geq 3$, then the symbols $[u, v]$ generate $K_2(n, R)$.

**Proof:** See Milnor [6, §9].
**Theorem 4.5:** Let $R$ be a commutative discretely normed ring for which every element of $\text{St}(3, R)$ can be written in the normal form. Then the symbols generate $K_2(R)$.

**Proof:** As $R$ is commutative, $R$ satisfies the conditions $N_n'$ for every $n \geq 3$. Hence by Theorem 4.1 the homomorphism

$$K_2(3, R) \longrightarrow K_2(R)$$

is epic. If we show that $K_2(3, R) \subseteq W_3$. Then we will be done by Lemma 4.4. By Theorem 3.3 we have $K_2(2, R) \subseteq H_2$. As the homomorphism $K_2(2, R) \longrightarrow K_2(3, R)$ is epic by Theorem 4.1 and as $H_2$ is carried into $H_3$, we have that $K_2(3, R) \subseteq H_3 \subseteq W_3$. This completes the proof of the theorem.

**Remark:** If one could show that every element of $\text{St}(3, F[x])$ ($F$ a field) can be written in the normal form, then the above theorem would yield the conjecture that

$$K_2(F[x]) \cong K_2(F).$$

For by applying the functor $K_2$ to the homomorphisms

$$F \longrightarrow F[x]$$

and

$$F[x] \longrightarrow F$$

where the first is the inclusion map and the second is given by $x \mapsto 0$, we see that $K_2(F)$ is a direct summand of $K_2(F[x])$. As $F[x]$ is a discretely normed ring, the
extra hypothesis that every element of \( \text{St}(3,F[x]) \) could be written in the normal form would (by Theorem 4.5) yield the fact that \( K_2(F[x]) \) is generated by the symbols. But the only units of \( F[x] \) lie in \( F \). Thus the homomorphism \( K_2(F) \rightarrow K_2(F[x]) \) would be epic and hence an isomorphism.

It is conceivable that the homomorphisms
\[ K_2(n,R) \rightarrow K_2(R) \]
could be epic for every \( n \geq 2 \) and yet the homomorphism \( K_2(n,R) \rightarrow K_2(m,R), \ m > n, \) might never be epic. However, even in the general case there is a relation between the normal form and the condition that \( K_2(2,R) \rightarrow K_2(R) \) be epic.

**Theorem 4.6:** Let \( R \) be a ring which satisfies the condition \( M \) and the condition \( N_3 \). Assume the homomorphism \( K_2(2,R) \rightarrow K_2(R) \) is epic. Then if \( x \in \text{St}(3,R) \), there exists an \( n \) (depending on \( x \)) such that \( \overline{x} = \overline{B}_1^2 \overline{B}_1^1 \overline{B}_2^2 \overline{L} \) in \( \text{St}(n,R) \) for some \( \overline{B}_1^2, \overline{B}_2^2 \in S_n^2, \overline{B}_1^1 \in S_n^1 \) and \( \overline{L} \) in the image of \( LS_3 \) in \( LS_n \). (\( \overline{x} \) denotes the image of \( x \) in \( \text{St}(n,R) \)).

**Proof:** As \( R \) satisfies \( M \) and \( N_3 \), the normal form exists in \( E(3,R) \). Hence there exist \( B_1^2, B_2^2 \in S_3^2, B_1^1 \in S_3^1 \), \( L \in LS_3 \) and \( z \in K_2(3,R) \) such that
\[
x = z B_1^2 B_1^1 B_2^2 L.
\]
Now let $z' \in K_2(2, R)$ be such that $z'$ and $z$ have the same image in $K_2(R)$. As $K_2(R)$ is the direct limit of the $K_2(n, R)$, there is an $n$ such that $\overline{z} = \overline{z}'$, where the bar denotes the images of the elements $z$ and $z'$ in $\text{St}(n, R)$. Thus in $\text{St}(n, R)$

$$\overline{x} = \overline{z} \bar{B}_1^2 \bar{B}_1^1 \bar{B}_2^2 \overline{L}$$

$$= \overline{z} \bar{B}_1^2 \bar{B}_1^1 \bar{B}_2^2 \overline{L}.$$ 

Now $\overline{z}'$ involves only the indices 1 and 2 and $\overline{z}'$ is in $K_2(n, R)$, hence by Proposition 2.12, $\overline{z}'$ lies in the center of $\text{St}(n, R)$. Thus

$$\overline{x} = \bar{B}_1^2 (\overline{z} \bar{B}_1^1) \bar{B}_2^2 \overline{L}$$

which is in the normal form.

**Remark:** A partial converse of this theorem exists: Let $R$ be a ring which satisfies the conditions $M$ and $N'_n$ for all $n \geq 3$. Then if for every element $x \in \text{St}(3, R)$ there exists an $n$ and elements $\bar{B}_1^2, \bar{B}_2^2 \in S_n^2, \bar{B}_1^1 \in S_n^1$, and $\overline{L}$ in the image of $LS_3$ in $LS_n$ such that

$$\overline{x} = \bar{B}_1^2 \bar{B}_1^1 \bar{B}_2^2 \overline{L}$$

in $\text{St}(n, R)$, then the homomorphism

$$K_2(2, R) \longrightarrow K_2(R)$$

is epic. This can be proven in a manner similar to that used in Theorem 2.7. First prove that any element of $\text{St}(n, R)$ can be written in the normal form provided it is taken sufficiently far out in the sequence. Then proceed as in Theorem 2.7. At each step
in the process used one must put a certain product into the normal form. This can be done by looking at the element farther out in the sequence. As the process involves only a finite number of steps there is an $n_0$ sufficiently large so that all steps can be carried out in $\text{St}(n_0, R)$ exactly as they were in the proof of Theorem 2.7.
REFERENCES


