BRITO, Dagobert Llanos, 1941-
ON THE LIMITS OF ECONOMIC CONTROL.

Rice University, Ph.D., 1970
Economics, theory

University Microfilms, A XEROX Company, Ann Arbor, Michigan
RICE UNIVERSITY

ON THE LIMITS OF ECONOMIC CONTROL

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

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May, 1970
ACKNOWLEDGMENTS

A dissertation is usually a student's first attempt to do original research and as a result his advisor is between Scylla and Charybdis. If he gives the student too much help, the whole exercise is without value; if on the other hand, he does not help the student enough, there is the danger that the project will take several years or more to complete. I was very fortunate. Professor David H. Nissen's guidance often saved me from pursuing sterile avenues of research; his judgment about what constitutes "good economics" gave me a benchmark against which I could judge my own work. However, he never tried to solve my problems for me. I am most grateful. I would also gratefully like to acknowledge the training and many helpful comments and suggestions I received from Professors Rui J. P. de Figueiredo and Richard D. Young.

The awarding of the doctorate is a rite of passage; it is thus fitting to acknowledge Professors Stanley M. Besen, Dwight S. Brothers and Gaston V. Rimlinger, who through their teaching and conversations have helped me develop as an economist. In particular, I would like to thank Professor Brothers, who first interested me in economics and was later responsible for my returning to Rice.

Patricia's own studies and research did not permit her to type, proofread and perform the usual uxorial chores; perhaps for that very same reason she understood what was involved in writing a dissertation. Her understanding and moral support are gratefully acknowledged. In addition, this dissertation would never have been completed in a year if she had not kept me out of Kay's.
INTRODUCTION

Whether the government should attempt to control the economy in the short run is an issue that has been debated for many years. Specifically, the issue is whether a policy based on observations of the economy is dominated in some sense by a policy for which the controls are constant in the face of short run fluctuations. Richard Thorn— with regard to monetary policy—considers this to be one of the most pressing issues yet to be resolved (10). Perhaps the best way of dating the start of the controversy is the 1938 Simons article in the Journal of Political Economy, "Rules vs Authorities in Monetary Policy" which gave the unfortunate name, Rules vs Authorities, to the controversy (9). Simons' thesis is that the uncertainty inherent in control by authorities generates noise in the system and is thus destabilizing. He states:

A democratic, free-enterprise system implies and requires for its effective functioning and survival a stable framework of definite rules laid down in legislation and subject to change only gradually and with careful regard for the vested interests of the participants in the economic game. It is particularly essential economically that there should be a minimum of uncertainty for enterprisers and investors as the monetary conditions in the future and, politically, that the plausible expedient of setting up authorities instead of rules with matters of such fundamental importance be avoided, or accepted only as a very temporary arrangement. The most important objective of a sound liberal policy apart from the establishment of highly competitive conditions in industry and narrow limitation of political control over relative prices should be that of securing a monetary system governed by definite rules.

Simons' position does not in itself preclude control based on observations of the state variables. However, in its original form, the problem is not clearly defined as has been pointed out by Viner (11) and
Whittlesey\textsuperscript{(12)}. First, it is not altogether clear what constitutes a rule, and second, formal written rules need not lead to less uncertainty in the system than informal, but well established procedures. As Viner points out: "...attempts to deal by simple rules with complex phenomena will lead to proliferation of rules or authorities, or of rules and authorities." An example would be to compare the uncertainty about government that exists in Great Britain, a country with an unwritten constitution, vis-à-vis a country with a very detailed constitution such as Mexico or Brazil.

Simons' ideas have been refined and further developed by Milton Friedman. Friedman has come, through a number of books, essays and papers, \textsuperscript{(2, 3, 4, 5)}, to be identified with the position that, given the lags and uncertainties in the American economy, attempts at control are more apt to be destabilizing than stabilizing. He has proposed that the government should follow a policy of increasing the money supply at a constant rate. In his view, this would have the virtue of avoiding major mistakes. These proposals have come to be identified with the rules side of the controversy.

In its new formulation the problem can be stated as follows: suppose that we are attempting to keep an economy on some given path. This path can be the result of some prior optimization or else it can represent some ad hoc goals such as full employment, price stability, a target rate of growth,...etc. Further let us assume that this path is feasible for the economy. Finally, suppose that we have a choice of the following classes of policies: first, policies based on predictors of the relevant state variables at some time in the future; second, policies based on observations of the state variables at the present time; third, policies that do not depend in the short run on
either estimates or observations of the state variable. Having first
defined a suitable metric on the notion of stability, our problem
is then to select the policy or policies that will result in the
stability for our system when: (1) there are errors in the measurement
of the state variable; (2) there are lags in the observation of the
state variable and/or in the action of the controller; (3) when the
control can be varied only in discrete amounts.

In this essay we will investigate this problem for the following
system: suppose that in the absences of disturbances the economy can
be described by the following equations:

\[ \dot{x} = f(x, u) \]  \hspace{1cm} (1)

where \( x \) is an \( n \)-vector whose components represent the stocks and prices
of the system, \( f(x, u) \) is a mapping from \( \mathbb{R}^{n+m} \) to \( \mathbb{R}^n \) satisfying the
Lipschitz condition everywhere on its domain of definition, \( u \) is a
control variable—for example, the rate of growth of the money supply.
Further, let us assume that the government—whose only interaction
with the system is through \( u \)—is trying to keep the system on some
path:

\[ x(t) = \hat{x}(t) \]  \hspace{1cm} (2)

Where the path (as mentioned before) is the result of some prior opti-
mization or some \textit{ad hoc} goal. The question we shall ask is: if (1)
is stable \*\footnote{The assumption that (1) is stable around (2) is the only assumption we make about (1). In the context of this essay, this is not restrictive. If the system were not stable—as has been argued by Minsky\footnote{Among others—then the whole controversy is moot in this context. In other words, a sufficient argument for control is that (1) is an unstable system.} among others—then the whole controversy is moot in this context. In other words, a sufficient argument for control is that (1) is an unstable system.} for some constant policy, \( u = u^*(t) \), is there some policy

\footnote{See the appendix for the formal definitions of the various kinds of stability discussed in this essay.}
based on observation of the state variable:

\[ u = \phi(x) \]  \hspace{1cm} (3)

that will improve the stability of the system when there are lags and/or errors.

The system described by (1) is very general and perhaps the following model will give some feeling for the results we seek.

**AN EXAMPLE**

Suppose that we have an economy with two sectors, public and private. The one good, which can be used either for investment or consumption, is produced in the private sector by a linearly homogeneous production function whose inputs are capital and labor. That is

\[ Y = F(K,L) \]  \hspace{1cm} (E-1)

where \( Y \) is the good, \( K \) is capital, \( L \) is labor and the usual neoclassical assumptions hold. Further we will assume that the labor force is growing at a rate \( u \)

\[ L = uL \]  \hspace{1cm} (E-2)

and capital depreciates at a rate \( \delta \),

\[ \dot{K} = I + G - \delta K \]  \hspace{1cm} (E-3)

where \( I \) is private investment and \( G \) is government investment. Further the government taxes an amount \( T \) which is used to provide for government consumption \( P \) and investment.

\[ T = G + P. \]  \hspace{1cm} (E-4)

Finally assume that the population saves a constant proportion \( \hat{S} \) of disposable income which is invested by the private sector

\[ I = \hat{S}(F(K,L) - T) \]  \hspace{1cm} (E-5)
Define \( k = K/L \), \( t = T/L \) and \( p = P/L \); we have
\[
\dot{k} = k\left(\dot{K}/K - \dot{L}/L\right)
\]
(E-6)
and recalling that because of linear homogeneity, \( f(k) = F(K,L)/L \)
we have combining (E-2) - (E-6).
\[
\dot{k} = \hat{S}f(k) + (1-\hat{S})t - p - (u+\delta)k
\]
(E-7)
Let us suppose that the Golden Rule capital/labor ratio is \( \hat{k} \). That
is to say that \( \hat{k} \) solves the equation
\[
f'(k) = (u + \delta)
\]
(E-8)
As can be seen in figure 1, \( \hat{k} \) maximizes feasible steady state per
capita consumption.

Figure 1

Let \( v = (1 + \hat{S})t - p \), (11) becomes
\[
\dot{k} = \hat{S}f(k) + v - (u + \delta)k
\]
(E-9)
and we can solve for the Loci of points in \( v-k \) space where \( k = 0 \).

![Graph showing the Loci of points in v-k space](image)

Figure 2-a

In figure 2-a we can determine the capital/labor ratio by picking the proper value of the control \( v \). In the phase diagram, figure 2-b, we see that not all capital/labor ratios result in stable equilibrium points; for example, the control \( v^* \) has the equilibrium points \( k^* \) and \( k^{**} \), but only \( k^{**} \) is stable; one can see by inspection of the
phase diagram that all equilibrium capital/labor ratios greater than \( \overline{k} \) will be stable. It is easy to show that the Golden Rule capital/labor ratio \( \hat{k} \) must be greater than \( \overline{k} \). So we know that given the appropriate \( v \), \( \hat{k} \) is a stable equilibrium; let \( \hat{v} \) be the appropriate value of the control.

![Figure 3](image)

Now suppose that \( S = \hat{S} + R \), where \( R \) is a random process with mean zero. This will cause the capital/labor ratio to fluctuate around \( \hat{k} \). (See fig. 3)

![Figure 4](image)

*Since \( \hat{k} \) is the Golden Rule capital/labor ratio, we know that \( f'(\hat{k}) = (u+\delta) \). At \( \hat{k} \), \( \hat{S}f'(\hat{k}) = (u + \delta) \). The neoclassical assumptions about the production function imply that \( f''(k) < 0 \); we assume \( \hat{S} < 1 \), therefore we know that \( \hat{k} > \overline{k} \).
We can construct a region $\mathcal{O}$ in $v-k$ space such that (E-9) is asymptotically stable in $\mathcal{O}$ for the control $v = \hat{v}$; that is to say that if (E-9) is perturbed but remains in $\mathcal{O}$, it will return to the equilibrium capital/labor ratio $\hat{k}$.

Now suppose that the government were to vary $v$ in response to the fluctuations in $k$; formally this is equivalent to saying that $v$ is a function of $k$, that is $v = \emptyset(k)$. In the context of this admittedly simple model we can now ask the following questions: Can the fluctuations of $k$ be reduced by some policy, $v = \emptyset(k)$ if:

1) there are lags in the observation of $k$ and/or in our ability to vary $v$;

2) there are errors in our measurement of the deviation of $k$ from $\hat{k}$;

It is these questions in a more general context that are the subject of this essay. However, we will return to this example to illustrate the results we develop.

**DEFINITIONS**

Since our object of interest is the deviation of our economic system from the target path, the problem can be considerably simplified by choosing an appropriate coordinate system. By making the following translation of coordinates

$$ y_i(t) = x_i(t) - \hat{x}_i(t) \quad (4) $$

and

$$ v = u - u^* \quad (5) $$

Equation (1) becomes

$$ \dot{y} = g(y,v) \quad (6) $$
and the objective of the government becomes to keep the system at the origin. Since we are interested in the behavior of the system in the neighborhood of the origin, we can expand (16) in a Taylor series around the origin and discard terms of second order or higher*. Equation (6) becomes

\[ \dot{y} = Ay + bv \]  

(7)

where \( A \) is a matrix whose typical element \( a_{ij} \) is

\[ a_{ij} = \frac{\partial g_i(0,0)}{\partial y_j} \]  

(8)

and \( b \) is a vector whose typical element \( b_i \) is

\[ b_i = \frac{\partial g_i(0,0)}{\partial v} < 0 \]  

(9)

Since (6) is stable the roots of \( A \) have negative real parts.

Let \( R \) be an \( n \)-vector random variable with mean zero and such that

\[ |R_i| \leq R_i = 1, n \]  

(10)

that perturbs (7) additively. That is to say that the economy is subjected to bounded random shocks whose distribution we will assume to be unspecified. It should not be assumed that \( R \) is an exhaustive list of all random elements in the economy; there will be random elements which for some reason will not affect the behavior of the system. For example, its frequency relative to rate of adjustment of that part of the economy which it effects may be so great that its effect may be self-canceling. We clearly do not have to worry about

*See Appendix

+We can define our coordinate system such that this is true.
such phenomena; however, it will be assumed that there exists random elements in the economy which can drive it from the target path.

The objective of policy, because of our new coordinate system, is to keep the economic system at the origin; because of the random shocks to which the system is subjected, this is not strictly possible. It can be shown* that for the control $v = 0$ there exists a closed set containing the origin which will contain all trajectories of the system. That is to say that for the control $v = 0$, once the system is in $M$, it will never leave $M$. In this world policy could take as its goal to keep the system as close to the origin as possible by minimizing the set width of $M$ in all directions, that is to minimize the maximum deviation of the system in all directions. In an $n$-th order system, this implies $n$ goals, i.e. $\min \max (y_i) \ i = 1, n$. In this system, as in systems that have been studied by Mundell and others (7) this requires $n$ instruments of policy. In a world with only $m$ instruments of policy, $m < n$, the government can only minimize the maximum deviation in $m$ directions. For mathematical convenience in this essay we will consider an economic system with only one instrument of policy. Further we will assume that the relevant political institutions have assigned a cost to deviations from the origin and that this cost is given by the quadratic form

$$y'Qy$$

and we can, without any loss in generality pick $Q$ to be the identity matrix. The existence of $Q$ is a heroic assumption. It would mean, for example that the system has resolved such problems as the social cost of unemployment vis-à-vis inflation.

* See below p.13
The goal of policy is then to minimize \( \max[y' I y] \); to express these ideas formally we make the following set of definitions.

**Definition.** Let \( y \in Y \) be an \( n \)-vector whose components represent the deviation of the system from the target path. Let \( z \in Z \) be an \( n \)-vector whose components represent the controlling authorities perception of the system. We assume that there exists some relationship \( \psi \) such that

\[
\psi: y \rightarrow Z
\]

Further we will define the error in perception \( e \) to be such that

\[
z = y + e,
\]

\( e \) will also be referred to as the error in observation of the state variable.

**Definition.** A **policy** is a mapping from observation space to control space. That is to say that our object of choice, a policy, is a function \( \emptyset \) such that if \( z \in Z \) and \( u \in U \) then:

\[
\emptyset: Z \rightarrow U
\]

For the policy \( v = 0 \) (i.e., \( u = u^+(t) \)) the system defined by equation (7) is

\[
\dot{y} = Ay + R
\]

Now consider the policy

\[
v = \emptyset(z) \tag{12}
\]

\[
z = \psi(y)
\]

our system becomes

\[
\dot{y} = Ay + bv + R
\]

\[
v = \emptyset(z) \tag{13}
\]

\[
z = \psi(y)
\]

**Definition.** A policy \( \emptyset(z) \) is said to be **permissible** in \( \Omega, \Omega \) in \( Y \), if the economic system given by equation (13) is asymptotically stable in \( \Omega \) for the policy.
Definition. A policy \( \phi(z) \) is said to be dominant in \( \Omega \) if the \( \min \max y'iy' \) associated with it is less than the \( \min \max y'iy' \) associated with the policy \( v = 0(u = u^*(t)) \) in \( \Omega \).

Definition. Let \( \Omega_1 \) be a subset of \( \Omega \) containing the origin and at least a segment, if the system represented by equations (7) is absolutely stable for \( v = 0 \) and there does not exist a dominant policy \( \Omega_1 \) then \( \Omega_1 \) is said to be a Friedman Region for (7).

Definition. Let \( \Omega_1 \) be a subset of \( \Omega \) containing the origin and at least a segment, if the system represented by equations (7) is absolutely stable at the origin for \( v = 0 \) and there does not exist a non-null policy that is permissible in \( \Omega_1 \), the \( \Omega_1 \) is said to be a Strong Friedman Region for (7).

THE EFFECT OF ERRORS IN THE MEASUREMENT OF THE STATE VARIABLE ON CONTROL

It is characteristic of economic systems that many of the state variables of the system can not be measured or can be measured only with great difficulty and/or large errors. Policy decisions are often based on proxies for the relevant variable; for example, "tightness" in the money market can not be measured and the money supply, the interest rate, and net free reserves are sometimes used as proxies in attempting to determine monetary policy; which one should be used has been a subject of some debate. The use of proxy variables, however, introduces errors into the act of control. It is thus not unreasonable to wonder whether attempts at control in a system with errors in the measurement of the state variable are apt to be destabilizing. That is to say whether a Friedman region (as defined previously) exists for a system with errors in the measurement of the state variable.

Let us consider the following variation of the system given by equation (13).
\[ \dot{y} = Ay + bv + R \]
\[ v = \emptyset(z) \]
\[ z = y + e \]

where as before, \( y \), \( b \), and \( c \) are \( n \)-vectors, \( A \) is a stable \( n \) by \( n \) matrix, \( v \) and \( z \) are scalers and \( R \) and \( e \) are \( n \) vector random processes with mean zero such that

\[
|R_i| \leq |\bar{R}_i| \\
|e_i| \leq |\bar{e}_i| 
\]

We are going to investigate the existence of a Friedman region around the origin for the economic system described by (13). It is convenient to assume a particular form for \( \emptyset(z) \), that is \( \emptyset(z) \) such that

\[
\frac{\partial \emptyset}{\partial \bar{z}_i} > 0
\]

for all \( i \) and prove the following proposition.

**Proposition 1.** Let \( \bar{y} \) be the solution to \( \min \max y'Iy \) for the policy \( v = 0(u = u^*(t)) \). Then if \( |\bar{y}_i| < |\bar{e}_i| \), there exists a Friedman region for the \( i \)-th variable. That is to say there does not exist a dominant policy based on observations of \( y_i \).

Before we attempt to prove Proposition 1, it is first necessary to prove a lemma.* Let \( M \) be an arbitrary closed and bounded set in \( n \)-space and let \( M' \) be its complement. Then for the system

\[ \dot{x} = f(x) \quad (F) \]

we have the following lemma:

**Lemma**

\( V(x) \) is a scaler function with continuous first partials for all \( x \) and

*This lemma is based on a lemma of Yoshizawa, see La Salle and Lefschetz, op. cit., p. 116.
M is a closed and bounded set in n-space. If \( \dot{V}(x) < 0 \) for all \( x \) in \( M' \) and if \( V(x_1) < V(x_2) \) for all \( x_1 \) in \( M \) and all \( x_2 \) in \( M' \), then each solution of (F) which for some time \( t_0 > 0 \) is in \( M \) can never leave \( M \).

Proof

By continuity, \( \dot{V}(x) \leq 0 \) for all \( x \) in Bndry \( M \); also \( V(x) \) is strictly increasing in Bndry \( M \), therefore no solution of (F) in \( M \) can cross Bndry \( M \).

Proof of Proposition

Pick a Liapunov function of the form

\[ V(y) = y'By \]

where \( B = \text{diag } (1/2, 1/2, \ldots, 1/2) \)

then

\[ (\text{grad}V)'y = \dot{y}'By + yB'y \]

By definition we know that \( \dot{V}(y) = (\text{grad}V)'y \)

so we have

\[ \dot{V}(y) = -y'Cy + \theta(z)b'Iy + R'Iy \]

(17)

where

\[ -C = A'B + B'A \]

and we can show that \( -C \) is a symmetric positive definite matrix.*

Let \( N = (x | \dot{V} \geq 0) \) and \( M = (y | y \leq |x| \) for all \( x \) in \( N \), we want to find the set \( M \) for a given \( \theta(z) \) and having determined \( M \) we want to maximize \( y'Iy \) subject to \( y \) in \( M \). The set \( M \) satisfies the conditions of the lemma for the function \( y'By \).

---

* \( C \) is in fact \( \frac{1}{2}(A + A') \) and the roots of \( A \) are negative.
We can simplify the problem considerably. Note, first, that \( \emptyset(z) \) and \( R'y \) are monotonic in \( e \) and \( R \) respectively; second, that \( \max y'y \) for \( y \) in \( M \) is equal to \( \max y'y \) for \( y \) in \( N \). We can, therefore, reduce this rather complex problem to a simpler problem;

![Diagram](image)

**Figure 5**

We can maximize \( y'y \) subject to

\[
\dot{V}(y) \geq 0 \\
R = \pm R \\
e = \pm e
\]

in the appropriate orthant where the signs of \( R \) and \( e \) are chosen such that \( \text{sign } e = -\text{sign } y \) and \( \text{sign } R = \text{sign } y \). Let us suppose that \( y \) is the solution to (18) for the policy \( \emptyset(z) = 0 \) for all \( z \). Let us now
consider the following maximization problem. Maximize
\[ c'y \]
subject to
\[ \dot{V}(y) \geq 0 \]
\[ R_i = \pm \bar{R}_i \]
\[ e_i = \pm \bar{e}_i \]
(18-a)
where \( c \) is chosen equal to \( \hat{y} \). It is easy to show that the set \( N \) is convex*, further, unless \( \hat{y} \) is the origin, it clearly has a non-null interior; we can then use the Kuhn-Tucker theorem to characterize the solution to problem (18-a). Note that (18) and (18-a) are related as follows, \( c'y > c'\hat{y} \) implies that \( y'Iy > \hat{y}'I\hat{y} \). The Lagrangian is
\[ L(y, \beta) = c'y + \beta \dot{V}(y) \]
(19)
and the saddle point conditions for a maximum are
\[ c' + \beta \text{grad} \dot{V}(y) \leq 0 \]
\[ [c' + \beta \text{grad} \dot{V}(y)]y = 0 \]
\[ \dot{V}(y) \geq 0 \]
\[ \beta \dot{V}(y) = 0 \]
(20)
Since \( c'y \) is linear we know that the extreme points must be boundary point of the constraint set. It then follows that \( \dot{V}(y) = 0 \). Also, because \( c'y \) is a linear functional we know that the solution to the optimization problem (18) is non-degenerate. In a small neighborhood of \( \hat{y} \), \( \theta(z) \) can be approximated by \( \theta(\hat{y} + \varepsilon) + \alpha'(y + \varepsilon) \) and to see what effect an increase in control would have on the solution we can differentiate

*Suppose that \( x \) and \( y \) are in \( N \), then it can be shown that
\[ \dot{V}[ax + (1-a)y] = a\dot{V}(x) + (1-a)\dot{V}(y) - a(1-a)(x-y)'C(x-y) \]. Recall that \(-C\) is a positive definite matrix.
L(y, \hat{\beta}) with respect to \alpha. Further, we know from the Kuhn-Tucker theorem that \( \beta \geq 0 \). Differentiating and collecting terms, we have

\[
dL(y, \hat{\beta})/d\alpha = [c' - \hat{\beta}\text{grad}\hat{V}(y)]dy/d\alpha + d\beta/d\alpha \hat{V}(y) + \hat{\beta} \alpha \hat{V}(y)/\partial \alpha \tag{21}
\]

The saddlepoint conditions and the non-degeneracy of the solution reduces (21) to

\[
dL(y, \hat{\beta})/d\alpha = \hat{\beta} \partial \hat{V}(y)/\partial \alpha \tag{22}
\]

which is equal to

\[
dL(y, \hat{\beta})d\alpha_i = \hat{\beta} y_i b_i \hat{y}_i + \hat{e}_i \quad i = 1, n \tag{23}
\]

Since \( \hat{\beta} y_i b_i \hat{y}_i \) is negative and \( \hat{\beta} y_i b_i \hat{e}_i \) is positive, attempts to decrease \( \hat{y}_i \) given that \( \hat{y}_i < \hat{e}_i \) must in fact increase \( \hat{y} \). Since \( y'ly > c'y \) for all \( y > \hat{y} \), this proves the result we seek.

RETURN TO EXAMPLE

Let us apply our results to the example. The system equations are

\[
\dot{k} = \hat{S}f(k) - (u + \delta)k + v \tag{E-9}
\]

and in the neighborhood of \( \hat{k}, \hat{v} \) this can be expressed as

\[
\dot{k} = [\hat{S}f(\hat{k}) - (u + \delta)]k + v \tag{E-10}
\]
let
\[
x = k - \hat{k}
- u = v - \hat{v}
\]
\[
[\dot{Sf}(k) - (u + \delta)] = -a, \ a > 0
\]
The (E-10) becomes
\[
\dot{x} = -ax + u + R. \tag{E-11}
\]
Pick a Liapunov function of the form
\[
V(x) = \frac{x^2}{2} \tag{E-12}
\]
and note that this Liapunov function satisfies the conditions of the lemma.
\[
\dot{V}(x) = -ax^2 - xu + xR \tag{E-13}
\]
and let us consider a control or the form
\[
u = \alpha(x + e) \tag{E-14}
\]
so
\[
\dot{V}(x) = -(a + \alpha)x^2 + x(R - \alpha e) \tag{E-15}
\]
Now let us max x subject to
\[
\dot{V}(x) \geq 0 \tag{E-16}
\]
and R in \([-\bar{R}, +\bar{R}]; e \in [-\bar{e}, +\bar{e}]. \) This is equivalent to
\[
\max x
\]
subject to
\[
\dot{V}(x) \geq 0
\]
\[
R = +\bar{R}
\]
\[
e = -\bar{e}
\]
The Lagrangian is
\[
L(x|\beta) = x + \beta[-(a+\alpha)x^2 + x(\bar{R} + \alpha e)]. \tag{E-18}
\]
The solution to the maximization, \(\overline{x}\), is
\[
\overline{x} = \frac{\bar{R} + \alpha e}{a + \alpha} \tag{E-19}
\]
To find what effect a change in \( \alpha \) will have on our solution, we differentiate \( \bar{x} \) with respect to \( \alpha \)

\[
\frac{d\bar{x}}{d\alpha} = \frac{\bar{e}(a + \alpha) - (\bar{R} + \alpha \bar{e})}{(a + \alpha)^2}
\]

(E-20)

Combining terms we have

\[
\frac{d\bar{x}}{d\alpha} = \frac{\bar{e} - \bar{x}}{a + \alpha}
\]

(E-21)

This implies that the maximum deviation of the system can never be reduced below \( \bar{e} \) by control and if the maximum deviation of the system for the constant control is less than \( \bar{e} \) then control based on observations of the state variables is dominated and there exists a Friedman region.

**THE EFFECT OF LAGS ON PERMISSIBLE CONTROLS**

Milton Friedman has argued that the existence of lags in the reaction of the economy to policy which are long and variable precludes attempts to control the economy in the short run. In a statement to Congress he says\(^{(2)}\):

"Thus, you have a situation such that when the Federal Reserve System takes action today, the effect of that action may on some occasions be felt 5 months from now and on other occasions 2 years from now. That is the major reason why it is so difficult as a technical matter in the present state of our knowledge to know what measures one ought to take at a given time."

The argument is then that lags cause attempts at short-run control to be destabilizing because by the time corrective measures take effect, they are no longer appropriate. Thus we should not attempt to control the economy based on current observations. There has been much controversy about the existence, length and variability of the lags and the implications of lags.
However, little attention has been paid to the dynamic properties of economic systems characterized by lags in control. (Baumol\(^{(1)}\) and Phillips\(^{(8)}\) are two notable exceptions.) In this section we will investigate the effect of lags on the control of economic systems. We will show lags are not sufficient to cause the system to become unstable; that is to say that there does not exist a strong Friedman region for such systems. Of more economic interest, however, we will show that if the length of the lag exceeds a certain value \(T^+\), then there exists a region containing the origin such that attempts to control inside this region will increase the maximum deviation of the system, that is to say that a Friedman region exists for the system in question. Recall that our basic system is described by the equations

\[ \begin{align*}
\dot{y} &= Ay + bv + R \\
v &= \phi(z) \\
z &= \psi(y)
\end{align*} \]  \hspace{1cm} (13)

![Diagram](image)

**Figure 7**

Since \(b\) is a vector, control for (13) is restricted to the direction of \(b\) and control in the direction orthogonal to \(b\) is impossible. It is not unreasonable to assume that policy is some function of the distance
between the perceived state of the system and the \( b'y = 0 \) hyperplane.

An obvious metric is in fact \( d'z \) where \( d \) is a unit vector in the direction of \( b \). Let \( u = c'z \), our system becomes

\[
\begin{align*}
\dot{y} &= Ay + bv + R \\
v &= \emptyset(u) \\
u &= d'z \\
z &= \psi(y)
\end{align*}
\]  

(13a)

What we are assuming is that policy is insensitive to movements along directions over which it has no control. The assumption is made for mathematical convenience, but it does not seem unreasonable vis-à-vis economic behavior.

If we introduce a lag \( T \) in the action of the controller equations (13) become

\[
\begin{align*}
\dot{y} &= Ay + bv(t-T) \\
v &= \emptyset(u) \\
u &= d'z \\
z &= \psi(y)
\end{align*}
\]  

(24)

It is easy to show* that (24) is equivalent to the system

\[
\begin{align*}
\dot{y} &= Ay + bv(t-T_1) \\
v(t) &= \emptyset[u(t-T_2)] \\
u(t) &= d'z(t-T_3) \\
z &= \psi(y)
\end{align*}
\]  

(25)

*At time \( t \) the system responds to \( v(t) \) which is equal to \( v(t-T_1) \) which is equal to \( v(\emptyset[u(t-T_1-T_2)]) \) which in turn is equal to \( v(\emptyset[d'z(t-T_1-T_2-T_3)]) \) which is equal to \( v(t-T) \).
Where in the standard terminology $T_3$ is the recognition lag, $T_2$ is the reaction lag, and $T_1$ is the response lag. $T$ in the system (24) is equal to the sum of the three lags.

**Proposition 2:** For a system that is characterized by lags in the observation of the state variable and/or in the reaction of the controller, but no errors in the measurement of the state variables, there does not exist a Strong Friedman Region.

**Remarks**

1) We know from V. M. Popov's theorem that a sufficient condition for the existence of a permissible policy $\theta[u(t-T)]$ is that the Popov constant $h$ not be zero, thus the proof consists of demonstrating that

$$\lim_{T \to \infty} h \neq 0$$

2) Further suppose that the lag $T$ varies, then if we pick $h^+$ such that $h^+ = \min h$ for all $T$ the system will remain absolutely stable.

**Proof of Proposition 2**

Let $s$ be the operator $d/dt$; the linear part of (23) can be written as

$$(Is - A)y = bv$$

$$d'y - u = 0$$

Solving (26) for $u$ by Cramer's rule we have

---

* Aizerman and Gantmacher, Absolute Stability of Regulator System, p. 52.

+ Since we are assuming that there are no errors, $\psi(y) = y$. 

\[ -u = \frac{K(s)}{D(s)} \nu \]  

(27)

where \( D(s) \) is a polynomial in \( s \) of degree \( n \) and \( K(s) \) is a polynomial of degree \( m \) where \( n < m \). Let

\[ W(s) = -\frac{K(s)}{D(s)} \]  

(28)

We can replace \( s \) by \( iw \) and multiply the numerator and denominator of (28) by the complex conjugate of the denominator and separate the real and complex parts. We have

\[ W(iw) = R(w) + iI(w) \]  

(29)

Re-introducing a lag \( T \), we have again

\[ \dot{y} = Ay + bv(t-T) \]
\[ v = \phi(u) \]
\[ u = d'y \]  

(24)

Using a well known property of transfer functions* equation (29) becomes

\[ W(iw) = [R(w) + iI(w)] \exp(iwT) \]  

(30)

Recalling that \( \exp(iwT) \) equals \( \cos wT - i \sin wT \), equation (30) becomes

\[ W(iw) = R(w)\cos wT + I(w)\sin wT + i[I(w)\cos wT - R(w)\sin wT] \]  

(31)

and the Popov function that corresponds to the transfer function given by equation (31) is

\[ W^*(iw) = R(w)\cos wT + I(w)\sin wT + iw[I(w)\cos wT - R(w)\sin wT] \]  

(32)

---

To prove our proposition using the theorem of Popov, it is clearly sufficient to show that there exists a finite \( K^* \) such that

\[
\max_{w \to \infty} |\hat{\mathcal{W}}(iw)| \leq K^* \text{ for all } T
\]  

(33)

Since \( \sin wT \) and \( \cos wT \) are less than or equal to one for all \( T \) and \( w \) it is only necessary to show that

\[
\begin{align*}
wr(w) &\leq K' \\
wi(w) &\leq K''
\end{align*}
\]  

(34)

for all \( w \).

Since \( D(w) \) is a polynomial of degree \( n \) with negative roots the denominator of both \( R(w) \) and \( I(w) \) is a polynomial of degree \( 2n \) with negative roots; hence the denominator will not be zero for \( w \) non-negative. \( K(w) \) is a polynomial of degree \( m, m < n \); hence the numerators of \( wr(w) \) and \( wi(w) \) are polynomials of degree less than or equal to \( n + m + 1 \). Thus we see that both \( wr(w) \) and \( wi(w) \) are finite for positive finite \( w \) and since the degree of the numerator of both is less than or equal to the degree of the denominator, we have
\[
\lim_{w \to \infty} wR(w) = R^* \\
\lim_{w \to \infty} wI(w) = I^*
\]

\(R^*\) and \(I^*\) finite.

\[\frac{1}{K^*} u \]

**Figure 9**

Having shown that \(K^*\) is finite we know that the Popov sector is non-empty and this gives the result that we are trying to prove.

**Figure 10**

We have shown that the existence of lags in control is not sufficient to cause the system (23) to become unstable. However, for economic systems it is often the case that the nature of some controls is such that they can be applied only in discrete amounts; for example, it is hard to imagine Congress passing a 7.962 per cent surtax or the
Federal Reserve Open Market Committee making a two dollar transaction. Frequently the important problem is in choosing the proper control from the set of possible controls; thus the following proposition is of interest.

**Proposition 3:** Let $D$ be the set of policies which can be applied only in discrete amounts, then if $h$ is finite, there exists for the policies in $D$ a Friedman region.

**Proof:**

Let $v^*$ be the minimum possible policy action for all the policies in $D$, by the theorem of Popov, $v^*$ is permissible if and only if

$$v^* \leq hu$$

Now consider the region such that

$$|u| < |v^*/h|$$

It is clear that in this region the Popov condition is violated hence there exists a strong Friedman region.

**REMARKS**

Suppose that we have an economic system in which the central bank is attempting to control some weighted average of the rate of change of prices by varying the rate of growth of the money supply. Thus we have

$$z = d'y$$

where $y_i$ is the rate of change of the price of the $i$-th good and $d_i$ is the weighing factor. The control $v$ is the rate of growth of the money supply. Let the Popov constant for the system be $h$. If the central bank keeps the rate of growth of the money supply inside the Popov sector then the system will remain stable.
There are several problems; if the Popov sector is very small then the central bank is limited in the amount of control that it can apply and if the control is discrete, then the central bank must stop applying the control while still observing the disturbance that it is trying to control if it wants to avoid destabilizing the economy. There are strong political and psychological pressures on a central bank to decrease the rate of growth of the money supply when there is inflation and to increase the rate of growth of the money supply when there is deflation with the accompanying unemployment. There is a very real danger that if the Popov sector is small the central bank may destabilize the economy by over reacting. This thesis has been advanced by Professor Friedman (3) and others about the behavior of the Federal Reserve System.

RETURN TO EXAMPLE

Recall that the system can be described by the equation

\[ \dot{x} = -ax - u + R \tag{E-11} \]

and for the one dimensional case

\[ x = z \tag{E-23} \]

so we have

\[ (s + a)x = -u \tag{E-24} \]

\[ z - x = 0 \]
solving for $z$ in terms of $u$ we have

$$-z = \left[\frac{1}{a^2+a}\right]u \quad \text{(E-25)}$$

and writing the Popov function for a delay $T$ we have

$$P(iw) = \frac{(\cos wT - w\sin wT) - iw\sin wT + wcwT}{a^2 + w^2}$$

and for a given value of $T$, the values of $w$ for which the Popov function crosses the negative real axis are given by the odd solutions to the equation

$$\tan wT = -\frac{w}{a}.$$
RETURN TO THE PROBLEM OF LAGS

Let us once again consider the system

\[ \dot{y} = Ay + bv + r \]
\[ v = \emptyset(u) \]
\[ u = d'y(t - T) \]  

and let us assume that for the system (24), \( \hat{y} \) is the solution to

\[ \text{max } c'y \]
\[ \text{subject to } \]
\[ \dot{v}(y) \geq 0 \]

and let us ask the question: Under what circumstances will a change in \( \emptyset(z) \) improve our solution? It will be shown that a sufficient condition is that

\[ \text{sign } [d'y] \neq \text{sign } [u] \]

Figure 12

thus an obvious way to proceed is to investigate the loci of points from which the system can reach the point \( \hat{y} \) in an interval of length \( T \). If the point \( \hat{y} \) can, in this interval, be reached from points on the
other side of the d'y = 0 hyperplane, then it will be shown that a Friedman region exists.

**Proposition 4**: For the economic system described by equations (24) there exists a constant T⁺ such that if the lag in control T is greater than or equal to T⁺, then a Friedman sector exists for (24).

**Proof**

The system (24) can be written as

\[
\dot{y} = Ay + b \varnothing (d'r) + R
\]  
(36)

where

\[
r = y(t-T)
\]  
(37)

Now pick a Liapunov function of the form

\[
V(y) = 1/2 \ [y'Iy]
\]  
(38)

and note once again that for any set containing the origin, this satisfies the conditions of the lemma. Again, we want to maximize c'y subject to the condition that \(\dot{V}(y) \geq 0\) in the appropriate orthant with the sign of R picked in the appropriate manner. The Langrangian is

\[
L(y, \beta) = c'y + \beta[\dot{V}(y)]
\]  
(39)

and the saddle point conditions are

\[
\begin{align*}
c' + \beta \text{grad } \dot{V}(y) & \leq 0 \\
y[c' + \beta \text{grad } \dot{V}(y)] &= 0 \\
\dot{V}(y) & \geq 0 \\
\beta[\dot{V}(y)] &= 0
\end{align*}
\]  
(40)

Let \(\hat{y}\) and \(\hat{\beta}\) be the solutions to (40).

Replace \(\varnothing (d'r)\) by \(\varnothing (d'r) + ad'r\) and differentiate \(L(\hat{y}, \hat{\beta})\) with respect to \(a\). Differentiating, collecting terms, using the saddle-point conditions and the non-degeneracy of the solution of (40) (see
proof of Proposition 1) we have

\[
\frac{dL(y, \hat{\beta})}{d\alpha} = \hat{\beta} \frac{\partial V(y)}{\partial \alpha} = \hat{\beta} [y' I b] d'r
\]

(41)

The hyperplane \(d'y = 0\) portions the space \(Y\) into two half-spaces \(H\) and \(H'\) such that \(d'y > 0\) in \(H\) and \(d'y < 0\) in \(H'\). Assume that \(\hat{y}\) in \(H\) and that \(\hat{y}' I b < 0\). If \(\hat{y}' b > 0\) the proposition is true for all \(T\). Let \(Q(T)\) be the set of points such that \(y(t) = \hat{y}\) if and only if \(y(t-T)\) in \(Q\). That is to say that is the set of points from which (24) could reach \(\hat{y}\) in an interval of time of length \(T\).

Suppose that the intersection of \(Q(T)\) and \(H'\) is not null; let \(\overline{r}\) be contained in this intersection, then

\[
\frac{dL(y, \hat{\beta})}{d\alpha} = \hat{\beta}[\hat{y}' I b] d'r > 0.
\]

For \(T = 0\), \(Q(0) \neq \{\hat{y}\}\); further, it can be shown that there exists \(T = T^{++}\) such that \(Q(T^{++}) = M\) (recall that \(M\) is defined such that
the intersection of $Q(T)$ and $H'$ is null. Let $T^+$ be the largest such $T$. If the system lag exceeds $T^+$ then there exists a Friedman region for the economic system described by (24).

![Figure 14](image)

**REMARK**

If the lag exceeds $T^+$, it is not necessary to completely forego control of the economy; rather one can redefine $z$ as follows, if

\[
\begin{align*}
    d'y > q^+, z &= d'y \\
    d'y < -q^+, z &= d'y \\
    z &= 0 \text{ elsewhere,}
\end{align*}
\]

where the value of $q^+$ is picked large enough to exclude the intersection of $Q(T^+)$ and $H'$. (see fig. 13).

**RETURN TO EXAMPLE**

Recall that the behavior of our model is described by the equation

\[
\dot{x} = -ax + u + R 
\]  

(E-11)
and that without control the maximum deviation in the positive direction of our system is given by

$$x = \frac{R}{A}$$  \hspace{1cm} (E-26)

Let us introduce a lag $T^+$ into the control; to find the length of the lag is enough to cause a Friedman region, we can find the set of points from which the system can reach $\bar{x}$ in an interval of time of length $T^+$. If this set in the negative portion of the $x$ axis, then the lag can cause a Friedman region. An equivalent problem is to find the minimum time it takes the system to travel from the origin to $\bar{x}$. This can be done using Pontryagin's Maximum Principle; it is easier, however, to use the Maximum Principle to solve the dual to that problem, that is:

$$\max x(T)$$  \hspace{1cm} (E-27)

subject to

$$\dot{x} = -ax + u + R$$

$u$ in $U$

$R$ in $[-\bar{R}, \bar{R}]$

Where $U$ is the set of $\alpha x(t)$ for $-T^+ < t < 0$ such that $\alpha x(0) = 0$. Note that $u(t) = \alpha x(t-T^+)$. It is perhaps odd to consider the history of the system and system noise as control variables; the justification is that in a sense we are playing a game against a malevolent nature that is attempting to maximize our losses using the Maximum Principle.

\[\text{Figure 15}\]
The Hamiltonian is given by

$$H[x(t), u(t), R(t), p(t)] = p[-ax + u + R]$$  \hspace{1cm} (E-28)

and we have the conditions

$$\dot{p} = -\partial H / \partial x = + ap$$  \hspace{1cm} (E-29)

$$p(T) = \partial x(T) / \partial x \bigg| T^+ = 1$$

(E-29) implies that $p(t) > 0$ for $t$ in $[0, T]$, thus the Hamiltonian is maximized by picking the largest $u$ and $R$ in their respective constraint sets. For $R$ this is $\bar{R}$. Let us simplify the problem by assuming that the system was uncontrolled until $t = 0$; for $t$ in the interval $[-\infty, -T']$, $x(t) = -\bar{R}/a$ and $T'$ is picked to be the largest value of $t$ such that $x(T') = -\bar{R}/a$ and $x(0) = 0$.

Translate the coordinates such that $-T' = 0$.

We solve the equation

$$\dot{x} = -ax + \bar{R}$$  \hspace{1cm} (E-30)

for $T'$ such that $x(T') = 0$, given that $x(0) = -\bar{R}/a$.

The solution to (E-30) is

$$x(t) = [-2\bar{R}/a] e^{-at} + \bar{R}/a$$  \hspace{1cm} (E-31)

and $x(T') = 0$ for $T' = \frac{\ln 2}{a}$

It is clear that $T^+ > \frac{\ln 2}{a}$, so for $t$ in $[0, T^+ - \frac{\ln 2}{a}]$

the motion of the system is described by

$$\dot{x} = -ax + aR/2 + R$$  \hspace{1cm} (E-32a)

and for $t$ in $[T^+ - \frac{\ln 2}{a}, T^+]$ the motion of the system is described by

$$\dot{x} = -ax + a[-2Ke^{-at} + K]$$  \hspace{1cm} (E-32b)
where $K = \overline{R}/a$ and the boundary conditions are chosen such that $x(0) = 0$ and $x(t)$ is continuous.

The solution of (E-32a) given that $x(0) = 0$ is

$$x(t) = -Ae^{-at} + A$$  \hspace{1cm} (E-33a)

where $A = \frac{R(a+2)}{2a}$

Introduce an integrating factor of the form $e^{at}$, equation (E-32b) becomes

$$\int d(xe^{at})/dt = -2akdt + \int aKe^{at}dt$$

The solution to (E-32b) is then

$$x(t) = -2akte^{-at} + \frac{aK}{a} + Ce^{-at}$$  \hspace{1cm} (E-33b)

where $C$ is a yet undetermined constant and we have the condition that $x(T^+) = \overline{R}/a$.

At this point the problem is best worked backwards; if the system is to be at $R/a$ at $t = T^+$, we need to find where the system has to be at $t = (T^+ - \frac{\ln 2}{a}) = T''$.

Translating our coordinate system such that $T'' = 0$, we can solve (E-33b) for $C$.

$$\frac{\overline{R}}{a} = -\frac{ak \ln 2}{a} + \frac{aK}{a} + \frac{C}{2}$$  \hspace{1cm} (E-34)

recalling that $K = \overline{R}/a$ we have

$$C = 2\overline{R}\left[\frac{a + a(\ln 2 - 1)}{a^2}\right]$$  \hspace{1cm} (E-35)

We now translate our system back to the origin and solve for $T''$ such that for (E-33a)

$$x(T'') = C$$  \hspace{1cm} (E-36)
and since $T^+ = T'' + \frac{\ln 2}{a}$ we now know the maximum lag that will not cause a Friedman region.

From (E-33a) we have

$$x(T'') = -Ae^{-aT''} + A$$

and the condition that $x(T'') = C$ gives us

$$T'' = \frac{1}{a} \ln \left[ \frac{A}{C - A} \right]$$  \hspace{1cm} (E-37)

which gives

$$T^+ = \frac{1}{a} \left\{ \ln \left[ \frac{\alpha + 2}{\alpha} \right] + \ln 2 \right\}$$  \hspace{1cm} (E-38)

If we plot $T^+$ as a function of $a$ we have

![Diagram](image)

Figure 16

and we see it is possible to compute $T^+$ from the parameters of the system. It is now possible to have meaningful theorems in the Samuelsonian sense about the length of lags.
CONCLUSIONS

We have shown that for very simple systems, lags in control and/or errors in the measurement of the state variable result in limitations to our ability to control the system. One would suspect that if there are limitations to our ability to control simple well behaved autonomous systems then it will be even more difficult to control the more complex systems that more accurately mirror reality. The assumptions we have made about the system are biased in favor of control.

There are many theoretical questions open; the most obvious and probably mathematically most difficult is whether the results can be extended to include a more general class of objective functions. Another extension of interest is to consider systems with multiple controls; this introduces the additional question of the choice of the instrument relative to the direction of desired control.

This is a theoretical paper so that it is necessary that all statements about policy be qualified. The key empirical questions are: 1) Whether or not the system is stable for a constant control; 2) Whether the lags are large enough to cause Friedman regions. The latter is not a tautological question. We saw in the example that we can compute the maximum lag that permits control as a function of such economic variable as savings ratios, capital/labor ratios...etc. Thus, for a given economic model, the statement: "Lags are destabilizing if they exceed a given magnitude $T^*$" is a meaningful theorem in the Samuelsonian sense.

The policy implications are almost too familiar to repeat and it is dangerous to make policy recommendations based on simple models. However, based on the results of this paper we can (with reservations)
conclude:

1) There is reason to suspect that constant controls (e.g. constant rate of growth of the money supply) may be preferable to controls based on observations of the state variables. This is not to be considered a universal rule; control based on observations of the state can be stabilizing when the system is driven far from the origin. This is consistent with Friedman's position as stated in Reference (4).

2) If there is to be control, then the variables upon which control is based should be picked such that errors in observation are minimized. Proxy variables should be avoided when the functional relationship between the variable observed and the variable controlled is unstable.
APPENDIX

Mathematics

Let us consider the following system of differential equations

\[ \dot{x} = f(x,t) \]  \((*)\)

where \(x\) is an \(n\)-vector and \(f(x,t)\) is a mapping from \(\mathbb{R}^{n+1}\) to \(\mathbb{R}^n\) satisfying the Lipschitz condition everywhere on its domain of definition.

Let \(\psi(t,t_o,x_o)\) be the solution to (*) for \(x(t_o) = x_o\).

Definition: A trajectory \(\Lambda(x_o,t_o)\) is defined as

\[ \Lambda(x_o,t_o) = \{ x | x = \psi(t,t_o,x_o), \text{ for all } t > t_o \}. \]

Definition: \(\Lambda(x_o,t_o)\) is stable for an initial time \(t_o\) and an initial condition \(x_o\) if for all \(\varepsilon > 0\) there exists \(\delta = \delta(x_o,t_o)\) such that if

\[ ||x_1 - x_o|| < \delta, \text{ then } ||\psi(t,t_o,x_1) - \psi(t,t_o,x_o)|| < \varepsilon \text{ for all } t > t_o. \]

Remarks

1) \(||x||\) is the Euclidean norm of \(x\).

2) This is equivalent to Samuelson's Stability of the Second Kind.\(^+\)

Definition: \(\Lambda(x_o,t_o)\) is asymptotically stable in \(\Omega\) if it is stable and

\[ \lim_{t \to \infty} ||\psi(t,t_o,x_1) - \psi(t,t_o,x_o)|| = 0 \]

for all \(x_1\) in \(\Omega\). An equivalent statement is given an \(\varepsilon > 0\) there exists \(T'\) such that

\[ ||\psi(t,t_o,x_1) - \psi(t,t_o,x_o)|| < \varepsilon \]

for all \(t > T'\) and all \(x_1\) in \(\Omega\).

\(^+\)P. A. Samuelson, Foundations p. 261
Remarks

1) This means that from any initial condition the solution to $x$ will approach the origin as time becomes infinite.

2) This is equivalent to Samuelson's Stability of the First Kind.

3) We can choose our coordinate system such that the origin is the solution of $*$ whose stability is in question. The question is then to determine whether the origin is stable for $*$. In the rest of this appendix it will be assumed that such a translation has been made.

Definition: Let $V(x)$ be a continuous, differentiable scalar function. $V(x)$ is said to be a Liapunov function if and only if:

1) $V(x)$ is positive definite;

2) $\dot{V}(x) \equiv [\text{grad} V]'\dot{x}$ is negative semi-definite.

Theorem: (Liapunov) If there exists a Liapunov function in some neighborhood $\Omega$ of the origin, then the origin is stable.

Theorem: (Liapunov) If in addition, $\dot{V}(x)$ is negative definite then the origin is asymptotically stable.

Remark

The proof of these theorems can be found in La Salle and Lefschetz.

The following may give an intuitive feeling for the results.

Figure A-1
One can think of $V(x)$ as a bowl shaped surface whose bottom is the origin. $[\text{grad } V]' \dot{x}$ is a positive scalar multiple of the directional derivative of $V(x)$ in the direction of $\dot{x}$. If this derivative is decreasing, it means that the Liapunov function in the direction of $\dot{x}$ is decreasing, hence the solution to (*) is approaching the origin and the origin is asymptotically stable. If $V(x) = 0$, then the solution is moving along a level curve of $V(x)$ and the trajectory of (*) will be inside the smallest ball containing that level set of $V(x)$, hence the origin will be stable.

**Control Systems**

Consider a system of the form

$$\dot{x} = Ax + bv$$

$$v = \emptyset(z)$$

$$z = c'x$$

where $x$ is an n-vector, $A$ is a stable n by n matrix, $b$ and $c$ are n-vectors, $v$ and $z$ are scalars. $\emptyset(z)$ is assumed to be an arbitrary single-valued piecewise continuous real valued function defined for all real $z$ and satisfying the conditions $\emptyset(0) = 0$. Suppose also that $\emptyset(z)$ satisfies the condition

$$0 \leq \emptyset(z)/z \leq h.$$  

Figure A-2
Note that given an \( h \), the class of functions which satisfy condition (***), together with the system (**), define a class of systems. The feedback term \( \emptyset(z) \) is call a control, the object of control is to keep the system whose behavior is described by (**), at the origin.  

**Definition:** For any given \( h \), the class of systems (**), is absolutely stable if for any system in this class the origin is globally asymptotically stable.

Let \( G(iw) \) be the Fourier transform of the linear part of (**). The V. M. Popov absolute stability condition is:

**Theorem:** (Popov) The system (**), is absolutely stable for the condition (***) if there exists a real number \( q \) such that

\[
\text{Re}G(iw) + iq \text{Im}G(iw) + 1/h > 0.
\]

The proof of this theorem can be found in Aizerman and Gantmacher, *Absolute Stability of Regulator Systems*.

The Popov condition has an interesting geometrical interpretation. Let us define the Popov function \( G(iw)^* \) as

\[
G(iw)^* = \text{Re}G(iw) + w\text{Im}G(iw)
\]

and the Popov line as

\[
X - iqY + 1/h = 0.
\]

The Popov condition can be interpreted as follows: if the plot of the Popov function in the complex plane lies to the right of the Popov line (see figure A-3), then the Popov condition is satisfied.
Example

Let us compute $h$ for the system

\[ \dot{x} = -x + v(t-10) \]
\[ v = \theta(z) \]
\[ z = x \]

The Popov function is

\[ G(iw) = \frac{\cos 10w - w \sin 10w - iw(\sin 10w + w \cos 10w)}{1 + w^2} \]

and we can compute the following table
<table>
<thead>
<tr>
<th>( w )</th>
<th>( \text{Re}(\text{i}w)^+ )</th>
<th>( \text{Im}(\text{i}w)^+ )</th>
</tr>
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<tbody>
<tr>
<td>0.00</td>
<td>1.0000</td>
<td>0.0000</td>
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<tr>
<td>0.01</td>
<td>0.9939</td>
<td>-0.1097</td>
</tr>
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<td>-0.2181</td>
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<tr>
<td>0.03</td>
<td>0.9456</td>
<td>-0.3238</td>
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<td>0.9040</td>
<td>-0.4255</td>
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<tr>
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<td>0.1398</td>
<td>-0.9817</td>
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<tr>
<td>0.14</td>
<td>0.0313</td>
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<td>-0.0771</td>
<td>-0.9859</td>
</tr>
<tr>
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<td>-0.1844</td>
<td>-0.9700</td>
</tr>
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<td>0.17</td>
<td>-0.2890</td>
<td>-0.9425</td>
</tr>
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<tr>
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<td>-0.9501</td>
<td>-0.1708</td>
</tr>
<tr>
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<td>-0.9607</td>
<td>-0.0659</td>
</tr>
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<td>0.0390</td>
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<tr>
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<tr>
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<td>-0.9232</td>
<td>0.2446</td>
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<tr>
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<td>-0.3070</td>
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</tr>
<tr>
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<td>-0.6944</td>
</tr>
<tr>
<td>0.78</td>
<td>-0.4507</td>
<td>-0.6469</td>
</tr>
<tr>
<td>0.00</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Figure A-4
We can now determine \( h \) by plotting \( G(i\omega)^* \) in the complex plane. The procedure is not particularly elegant but, with a computer, it is not difficult to compute \( h \).

**Linear approximation of the system**

We were working with a differential equation system of the form

\[
\dot{x} = f(x,u)
\]

and we are interested in the stability properties of the path

\[
x = \hat{x}(t)
\]

and the corresponding control

\[
u = u^*(t)
\]

and making the appropriate transformation of coordinates

\[
y = x - \hat{x}(t)
\]

\[
v = u - u^*(t)
\]

we have

\[
\dot{y} = f[x(t) + y, u^*(t) + v] + \ddot{x}(t) = g(y,v,t)
\]

If we expand \( g(y,v,t) \) around the point \( \hat{x}, u^* \) and \( t_0 \) we have

\[
\dot{y} = Ay + bv + \beta(t-t_0)
\]

The \( A \) matrix and \( b \) vector are as defined previously. If we examine the \( \beta \) vector we have

\[
\beta = \left( \frac{\partial g}{\partial t} \right) = \left( \frac{\partial f}{\partial \hat{x}} \right) \ddot{x} + \left( \frac{\partial f}{\partial u^*} \right) \dot{u^*} + \ddot{x}
\]

In our approximation, we are making two assumptions; first, that the second order and higher terms in \( y \) and \( v \) are small enough to be neglected; second, that the target path has small enough variation for us to neglect the terms in \( \ddot{x}, \dddot{x} \) and \( \ddot{u^*} \) within the relevant planning
horizon. Without the second assumption, the mathematics of the problem is several orders of magnitude more difficult; further, the assumption makes sense from an economic standpoint. What we are assuming, for example, is that a central bank can ignore technical change and the rate of growth of the population when formulating short run monetary policy.