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MICROWAVE CIRCUIT THEORY AND WAVE PROPAGATION
IN A FERRITE-LOADED E-PLANE
WAVEGUIDE JUNCTION

by

Fred M. Stuber

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I. INTRODUCTION

The waveguide junction studied in this work is of the type applied in the problem of E-plane circulators. Although the H-plane counterpart has been investigated to a considerable degree in recent years, no consistent theory is yet available for the intrinsic circulator mechanism in an E-plane device.

In an attempt to understand circulator action in a ferrite-loaded E-plane junction, the central portion is considered to be a cylindrical cavity, to which the waveguide arms are coupled through small slots. The cavity contains two flat ferrite discs positioned against the narrow walls. These discs are magnetized parallel to their axis and fill the cross-section of the cavity. The coupling through the waveguides is assumed to be very weak so that the unperturbed fundamental modes inside the cavity can be studied to see which can give rise to a circulation effect. The general field theory of the ferrite filled circular waveguide is discussed and the usual dependence upon the sign of the azimuthal variation is found, akin to Faraday Rotation. In an attempt to reduce the complexity of the problem, it was decided to study solutions with the longitudinal propagation constant equal to zero. This is an appropriate and successful
assumption in H-plane and stripline circulator theory but it is appreciated that this may not be entirely justified in the present problem. However, this approach has revealed an interesting set of inconsistencies when the waveguide is filled with ferrite.

Some network properties of the non-reciprocal waveguide junction are also studied. Attention is focused on three-port circulators. The matching theory of such devices is reviewed very briefly and a recently developed flow graph model for non-ideal circulators is analyzed in the light of this theory. For this purpose some decomposition theorems for two-port networks are derived. It is shown that a lossless, non-reciprocal two-port can be matched at both ends by the cascade connection of one two-port. The elements of this matching network are given in terms of the elements of the original non-reciprocal network.

An aspect of the ferrite-loaded cavity which is an interesting combination of microwave and network theory is the following. At the operating frequency, the central portion of the cavity is "non-propagating", i.e., the propagation constant is purely imaginary. The energy is coupled through this section into the ferrite portions by a mechanism very similar to the tunnel effect in quantum mechanics. For the study of
this cavity the ferrite discs are replaced by appropriate
dielectric discs, i.e., only the scalar components of the
ferrite are taken into account. The off-diagonal components
are regarded as perturbing quantities which do not alter
significantly the fundamental properties under considera-
tion. A lumped two-port model was recently given by Kelly\textsuperscript{3}
for a short evanescent window. It consists of two lengths
of propagating transmission line on either side of a lumped
shunt susceptance. Applying Kelly's model to the evanescent
cavity, some interesting properties can be anticipated. They
are investigated analytically and experimentally.

An error analysis is given for Kelly's model based on
the comparison of the transmission matrix elements of the
approximate and exact transmission line models. The error
analysis is verified in the application of the model to the
evanescent cavity.

The susceptance effect of the cutoff window may have a
variety of applications. For example, the lumped two-port
model may be used in filter design. In view of such appli-
cations, the input impedances into lumped and exact model
two-ports are calculated and compared for variable loads.
An attempt is made to establish ranges for parameter values
such as gap length, frequency, etc., for which the difference
between exact and approximate input impedance stays within certain bounds.
II. ON THE NETWORK THEORY OF CIRCULATORS

As mentioned in the introduction, it is possible to describe circulators in terms of network theory. Such investigations have been made for quite some time and the general network theory of circulators is well established.

One of the most useful results concerns the matching properties of circulators derived by Humphreys and Pennfield.4

In attempting to describe nonideal circulators, Hagelin6 recently suggested a flow graph model. It was used to derive some transmission properties of such a device. The match of these results with measurements was regarded as a justification of the model.

The purpose of this chapter is to derive some decomposition theorems on a general two-port network. These will then be used to show that Hagelin's model can be explained within the well-established circulator theories mentioned above. It will be shown that it is a direct consequence of Humphreys' model.
II.1 Circulator Models by Hagelin and Humphreys et al.

Both Humphreys' and Hagelin's model are related to the scattering matrix. Therefore, some of its properties pertinent to the problem are reviewed shortly.

The scattering matrix is one of the most important descriptions of linear electrical networks. It is particularly important in distributed systems such as microwave devices because its coefficients are directly connected to power flow and have simple physical interpretation as wave amplitudes. This leads to the possibility of visualizing scattering matrices with flow graph techniques, which is particularly advantageous when the problem is to determine the scattering matrix of a combination of networks with known matrices. The scattering matrix supplies all the circuit properties of a device at a given frequency, and terminal conditions may be changed without changing the magnitude of the scattering coefficients. Under certain conditions of physical symmetry, it is possible to derive $S$ from spacial considerations only.

The scattering matrix $S$ is defined by

$$b = S \ a \tag{2.1}$$

where $a$ is a column vector representing the incoming waves at the ports and $b$ the outgoing waves.
The above relation may be visualized by the following flow graph:

![Flow Graph Model of a Three-Port Scattering Matrix](image)

**Flow Graph Model of a Three-Port Scattering Matrix**

**Figure 1**

The correspondence of the above figure with Equation (2.1) may easily be established. For example,

\[ b_2 = s_{21} a_1 + s_{22} a_2 + s_{23} a_3 \]

If the junction is lossless, the scattering matrix is unitary and consequently the following relations hold:

\[ s_{11} s_{21}^* + s_{12} s_{22}^* + s_{13} s_{23}^* = 0 \]  \hspace{1cm} (2.2)

\[ s_{11}^2 + s_{12}^2 + s_{13}^2 = 1 \]  \hspace{1cm} (2.3)

\[ s_{11}^2 + s_{21}^2 + s_{31}^2 = 1 \]  \hspace{1cm} (2.4)
If \( s_{ij} \neq s_{ji} \) and all the ports are matched, the above scattering matrix transforms into

\[
S = \begin{bmatrix}
0 & 0 & s_{13} \\
0 & 0 & 0 \\
0 & s_{32} & 0
\end{bmatrix}
\]  

(2.5)

for a right hand system. This is the scattering matrix of a perfect circulator and can be represented by the following graph:

Flow Graph Model of an Ideal Circulator

Figure 2

Since it is generally not possible to construct three-ports that satisfy the conditions of an ideal circulator, it is necessary to connect external networks to achieve this goal. Attention was therefore focused on the problem of what class of three-ports could be transformed into an ideal circulator by placing reactive two-ports in series with each arm.
Humphreys and Davies\textsuperscript{4} studied the effect of placing a reactive two-port network in series with each terminal of an n-port lossless junction. In the case of a three-port junction, the scattering matrix is assumed to have the form of Equation (2.1). The symmetrical, lossless reciprocal two-ports that are supposed to be connected to each arm of the three-port have a scattering matrix of the form

\[
T = \begin{bmatrix}
\phi & t \\
t & \phi
\end{bmatrix}
\] (2.6)

From this, the scattering matrix \( R \) of the overall network is calculated. It can then be determined under what conditions \( R \) takes the form of the scattering matrix of an ideal circulator as defined in Equation (2.5). It turns out that for passive two-ports, \((\phi < 1)\) the conditions are as follows:

\[
S_{ij} < S_{ji}
\] (2.7)

The same conclusions were arrived at by Pennfield.\textsuperscript{5} A three-port network whose scattering coefficients satisfy Equation (2.7) is called a non-ideal circulator.

The model for the non-ideal circulator proposed by Hagelin consists of a matched lossless junction with a lossless two-port connected to each arm. In Figure 3 its flowgraph
is shown.

Flow Graph Model of a Non-Ideal Circulator

Figure 3

The scattering matrices of the two-ports are

\[
S = \begin{bmatrix}
  r_i & n_i \\
  m_i & t_i
\end{bmatrix}
\]  

(2.8)

For special reference planes, this can be written
\[ S_i = \begin{bmatrix} R_i & \sqrt{1 - R_i^2} e^{j\phi_i} \\ -\sqrt{1 - R_i^2} e^{-j\phi_i} & R_i \end{bmatrix} \] (2.9)

where \( |R_i| = r_i \) and \( \phi \) is arbitrary.

Using flow graph theory, the overall scattering matrix of the network in Figure 3 can be evaluated. Hagelin calculated the ratio between two off-diagonal elements of the scattering matrix. There is good agreement with measurements of the same ratio.

However, it will be shown that Hagelin's model may be derived directly from Humphreys' model. For this purpose some decomposition properties of two-port networks will be derived in the next section.

II.2 Decomposition Properties of Lossless Two-Port Networks

As in the case of three-ports, let us first summarize some common properties of two-ports in a convenient form.

Let the scattering matrix of a general two-port be

\[ S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} ae^{j\alpha} & be^{j\beta} \\ ce^{j\phi} & de^{j\delta} \end{bmatrix} \] (2.10)

If we assume losslessness, we can impose the unitary condition which results in the following conditions:
\[ a^2 + c^2 = 1 \quad (2.11) \]
\[ b^2 + d^2 = 1 \quad (2.12) \]
\[ s_{11}^* s_{12} + s_{21}^* s_{22} = 0 \quad (2.13) \]

Equation (2.13) can be decomposed into the following two equations:
\[ ab - cd = 0 \quad (2.14) \]
\[ \alpha + \delta = \beta + \gamma - \bar{n} \pm 2n \bar{n} \quad (2.15) \]

Equations (2.11), (2.12), (2.14), and (2.15) are independent and can be used to eliminate 4 of the 8 coefficients of the original scattering matrix of Equation (2.10). After some algebra we obtain
\[ a = d \quad (2.16) \]
\[ c = b = \sqrt{1 - a^2} \quad (2.17) \]

Equation (2.15) may be satisfied by fixing \( \alpha \), \( \beta \), and \( \gamma \) and letting
\[ \delta = \beta + \gamma - \alpha - \bar{n} \pm 2n \bar{n} \quad (2.18) \]

or by choosing \( \alpha \), \( \delta \), and an arbitrary angle \( \phi \) and letting
\[ \beta = \alpha + \phi \quad (2.19) \]

This gives
\[ s = \begin{bmatrix}
    ae^{j \alpha} & j \beta \\
    \sqrt{1 - a^2} e^{j \delta} & ae^{j (\beta + \gamma - \alpha - \bar{n} (1 \pm 2n))}
\end{bmatrix} \quad (2.20) \]
or
\[ S = \begin{bmatrix} ae^{j\alpha} & \sqrt{1 - a^2} e^{j(\alpha + \varphi)} \\ \sqrt{1 - a^2} e^{j(\varphi - \varphi + \Pi)} & ae^{j\varphi} \end{bmatrix} \] (2.21)

Equations (2.20) and (2.21) are the simplest possible description of a general lossless two-port. It should be noted that the only possible non-reciprocity is in the phase shift. If the two-port is reciprocal, we have
\[ \alpha + \varphi = \varphi - \varphi + \Pi \] (2.22)
and
\[ \varphi = \frac{1}{2} (\delta - \alpha + \Pi) \] (2.23)

The scattering matrix is now
\[ S = \begin{bmatrix} ae^{j\alpha} & \sqrt{1 - a^2} e^{j\frac{1}{2}(\alpha + \varphi + \Pi)} \\ \sqrt{1 - a^2} e^{j\frac{1}{2}(\alpha + \varphi + \Pi)} & ae^{j\varphi} \end{bmatrix} \] (2.24)

We are now going to derive the conditions under which a general, lossless two-port network \( S_g \) can be transformed into a matched two-port \( S_m \) by a cascade connected two-port \( S_c \). Let
\[ S_m = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} = \begin{bmatrix} 0 & |p| e^{j\alpha} \\ |q| e^{j\varepsilon} & 0 \end{bmatrix} \] (2.25)
be the scattering matrix of a matched two-port. It is assumed
to be lossless. Because of Equations (2.11) and (2.12) we
have
\[ |p| = |q| = 1 \]
(2.26)
\( \sigma \) and \( \xi \) may be arbitrary (Equation (2.13)).

Let
\[
S_g = \begin{bmatrix} r & m \\ n & t \end{bmatrix} = \begin{bmatrix} |r| e^{j\varphi} \sqrt{1 - |r|^2} e^{j\varphi_m} \\ \sqrt{1 - |r|^2} e^{j\varphi_n} |r| e^{j\tau} \end{bmatrix}
\]
(2.27)
represent a general, lossless two-port. Because of Equa-
tion (2.15) we have
\[ \tau = \varphi_n + \varphi_m - \tau - \varphi \]
(2.28)

Finally,
\[
S_c = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} |a| e^{j\alpha} & |b| e^{j\beta} \\ |c| e^{j\gamma} & |d| e^{j\delta} \end{bmatrix}
\]
(2.29)
stands for a general two-port.

When connecting networks in cascade the transmission
matrix of the overall network is obtained by multiplying
those of the individual networks. The transmission matrix
\( T \), that relates incident and reflected wave amplitudes on
the input side to those on the output side is commonly called
wave amplitude transmission matrix (to distinguish it from
the voltage-current transmission matrix). In terms of the parameters defined in Equation (2.1) we may write

\[
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} b_2 \\
a_2
\end{bmatrix}
\tag{2.30}
\]

The coefficients of $T$ may readily be solved for from Equations (2.1) and (2.30). $T$ becomes

\[
T = \begin{bmatrix}
1/s_{21} & -s_{22}/s_{21} \\
\frac{s_{11}}{s_{21}} & \frac{s_{12} s_{21} - s_{11} s_{22}}{s_{21}}
\end{bmatrix} \tag{2.31}
\]

Let us now connect $S_g$ and $S_c$ in cascade and determine under what circumstances the overall network results in a two-port $S_m$. Thus,

\[
T_g \times T_c = T_m
\]

or

\[
\begin{bmatrix}
1/n & -t/n \\
r/n & \frac{mn - rt}{n}
\end{bmatrix} \times \begin{bmatrix}
1/c & -d/c \\
a/c & \frac{cb - ad}{c}
\end{bmatrix} = \begin{bmatrix}
1/q & 0 \\
0 & p
\end{bmatrix}
\tag{2.32}
\]

This results in the following set of equations:

\[
\frac{1}{nc} (1 - at) = q \tag{2.34}
\]

\[
\frac{1}{nc} \left[ -d - t (cb - ad) \right] = 0 \tag{2.35}
\]

\[
\frac{1}{nc} \left[ r + a (mn - rt) \right] = 0 \tag{2.36}
\]
\[
\frac{1}{nc} \left[ -rd + \frac{(mn - rt)(cb - ad)}{nc} \right] = p \quad (2.37)
\]

The coefficients of \( S_c \) can now be solved in terms of the coefficients of \( S_g \).

\[
S_c = \frac{1}{rt - mn} \begin{bmatrix}
    r & -np \\
    -\frac{m}{q} & t\frac{p}{q}
\end{bmatrix}
\]

and, using Equations (2.25) and (2.29)

\[
S_c = e^{j\eta} \times \begin{bmatrix}
    |r| e^{j\varphi} & -\sqrt{1 - |r|^2} e^{j(\varphi_n + \sigma)} \\
    -\sqrt{1 - |r|^2} e^{j(\varphi_m - \epsilon)} & |r| e^{j(\tau + \sigma - \epsilon)}
\end{bmatrix}
\]

where \( \eta = -\left(\varphi_m + \varphi_n + \tau\right) \).

Note that the coefficients of \( S_c \) satisfy Equation (2.15), i.e., the two-port is lossless.

Now, if we impose reciprocity, the off-diagonal angles of \( S_c \) have to be equal. Thus,

\[
\varphi_m - \varphi_n = \sigma + \epsilon \quad (2.39)
\]

or

\[
\sigma = \varphi_m + \theta \quad (2.40)
\]

and

\[
\epsilon = -\varphi_n - \theta \quad (2.41)
\]

where \( \theta \) is an arbitrary angle. \( S_c \) becomes
\[ S_c = e^{j\eta} \begin{bmatrix} |r| e^{j\varphi} & -\sqrt{1-|r|^2} e^{j(\varphi_0 + \varphi_n + \theta)} \\ -\sqrt{1-|r|^2} e^{j(\varphi_0 + \varphi_n + \theta)} & |r| e^{j(2(\varphi_0 + \varphi_n + \theta) - \varphi - \varpi)} \end{bmatrix} \]

(2.42)

where Equation (2.28) was used.

According to Equation (2.39), \( \sigma \) and \( \varepsilon \) are not fixed uniquely yet. Therefore, we can impose another condition on \( S_c \). If we assume the network represented by \( S_c \) to be symmetrical, the angles of the diagonal elements of \( S_c \) have to be equal. Thus,

\[ \varphi = 2(\varphi_0 + \varphi_n + \theta) - \varphi - \varpi \]

(2.43)

from which we can calculate \( \theta \).

\[ \theta = \varphi/2 - \varphi_0 - \varphi_n + \varpi/2 \]

(2.44)

\( S_c \) now becomes

\[ S_c = e^{j\eta} \begin{bmatrix} |r| e^{j\varphi} & \sqrt{1-|r|^2} e^{j(\varphi + \varpi/2)} \\ \sqrt{1-|r|^2} e^{j(\varphi + \varpi/2)} & |r| e^{j\varphi} \end{bmatrix} \]

(2.45)

Moreover,

\[ \sigma = \varphi - \varphi_n + \varpi/2 \quad \text{and} \quad \varepsilon = \varphi_0 - \varphi - \varpi/2 \]

(2.46)

Alternatively, we could impose that \( S_m \) be reciprocal, i.e.,

\[ \sigma = \varepsilon \]. Then from Equation (2.39)
\[ \mathcal{S} = \mathcal{E} = \frac{\Phi_m - \Phi_n}{2} \]  

(2.47)

\[ S_c \text{ becomes (from Equation (2.38))} \]

\[
S_c = e^{j\eta} \begin{bmatrix}
|r| \ e^{j\Phi} & -\sqrt{1 - |r|^2} e^{j \frac{\Phi_m + \Phi_n}{2}} \\
-\sqrt{1 - |r|^2} e^{j \frac{\Phi_m - \Phi_n}{2}} & |r| e^{j(\Phi + \Phi_m - \Phi - \eta)}
\end{bmatrix}
\]

(2.48)

The foregoing results are summarized in the table that follows. The scattering matrices are represented in flow graph technique. For example, the flow graph for

\[ b = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \]

is

\[ a_1 \to S_{12} e^{j\Phi_{12}} \to S_{11} e^{j\Phi_{11}} \to S_{21} e^{j\Phi_{21}} \to S_{22} e^{j\Phi_{22}} \]

Flow graphs of two-ports are connected in cascade by a reciprocal, matched or two-port with no phase shift.

\[ S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \]

\[ \begin{array}{c}
- \quad 1 \\
\end{array} \]
### Table 1

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<td>$\emptyset + \sigma + \bar{\tau}$</td>
<td>$\emptyset_n - \varepsilon + \pi$</td>
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<tr>
<td></td>
<td>$\emptyset_m$</td>
<td>$\tau$</td>
<td>$\tau + \sigma - \varepsilon$</td>
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Non-Reciprocal, Non-Reciprocal, Non-Reciprocal

---

| 2 | | | | |
| | $\emptyset_n$ | $\emptyset + \emptyset + \theta + \bar{\tau}$ | $2(\emptyset_m + \emptyset_n + \theta)$ | $-\emptyset + \theta$ | 2.42, 2.40, 2.41 |
| | $\emptyset_m$ | $\emptyset_m + \emptyset_n + \theta + \bar{\tau}$ | $-\emptyset - \bar{\tau}$ | $-\emptyset_m - \theta$ | |

Non-Reciprocal, Reciprocal, Non-Reciprocal
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</tbody>
</table>
The magnitudes of the coefficients of the scattering matrices have been omitted in Table 1. Moreover, $S_c^*$ is represented instead of $S_c$ which has an additional phase factor $e^{j\eta}$ in every coefficient.

Alternatively, Table 1 may be regarded as indicating what kind of two-ports $S_c$ can transform a general, lossless, non-reciprocal two-port $S_q$ into a lossless, matched two-port. Of particular significance are #2 and #3. The "matching" two-ports consist of reactance networks. They may be either symmetrical according to #3 (such as an iris in waveguide), or asymmetric according to #2 (such as a step discontinuity).

II.3 Derivation of the Flow Graph Model from Humphreys et al. Model

Given the fact that a non-ideal circulator can be transformed into an ideal one by the addition of external networks, it will now be shown that Hagelin's model is a direct consequence of the previously derived properties of these two-port networks.

We recall that a non-ideal circulator (Figure 1) whose scattering coefficients satisfy the unitary condition can be transformed into an ideal circulator (Figure 2) by placing reciprocal two-port networks $R_j$ to its terminals. Now, let
us assume that a non-reciprocal two-port network \( NR_j \), (of
the form defined in Equation (2.20) or (2.21)) is connected
to each port of this combination, i.e., in cascade with the
reciprocal two-ports \( R_j \). The two-ports \( R_j \) and \( N_j \) can be
chosen such that their combination at each port results in
a matched two-port \( M_j \) (as defined in Equation (2.25)).
Clearly, the overall network \( \text{three-port} + R_j + N_j = \text{three-}
port + M_j \) is again a non-ideal circulator, since the addi-
tion of matched two-ports does not change any of the original
network's properties. It can now readily be concluded that
the combination of an ideal circulator \( = \text{non-ideal circu-
lator} + R_j \), according to Humphreys) and a non-reciprocal two-
port \( N_j \), as suggested by Hagelin, is indeed a model for the
non-ideal circulator.

II.4 Physical Analogies

The complete flow graph of one port and part of the in-
terior matched three-port of a non-ideal circulator, aug-
mented with external two-ports to give a matched three-port
is shown in Figure 4. All networks are assumed to be loss-
less.
Hagelin suggested that the interior matched three-port corresponds to the interior of the ferrite disk and that the non-reciprocal two-ports represent the discontinuity due to the ferrite-air interface. This postulate seems to require some further explanation.

First order field theory descriptions of circulators are usually based on a slight distortion (due to the applied dc magnetic field across the ferrite) of the degeneracy of the resonance frequencies of two counter-rotating TM modes in the center portion of the junction. If the magnetic field across the ferrite is adjusted properly, the standing wave pattern (sum of the two counter-rotating modes) is rotated such that port 3 is located at a field minimum while port 2 is at some value of the field equal to the field at the input port 1. There is complete transmission of the
energy between ports 1 and 2 while port 3 is isolated. It is therefore justified to regard the ferrite portion as the ideal circulator port in the equivalent circuit.

At the port the wave propagates from the ferrite into the air-filled waveguide. The permanent magnetic field across the ferrite is transverse to the direction of this propagation. If there is oblique incidence of the wave in the ferrite upon the ferrite-air interface, propagation across it is non-reciprocal. The discontinuity may therefore be represented by a non-reciprocal two-port. Finally, the reciprocal two-ports in the model correspond to external tuning devices, such as irises, stubs, or discontinuities in the waveguides used to cancel the reflection from the non-reciprocal inner two-ports.
III. FIELD THEORY OF FERRITE-LOADED E-PLANE JUNCTIONS

As an example of a ferrite-loaded E-plane junction, a three-port, E-plane circulator is shown in Figure 4a. It contains two flat ferrite discs positioned against the narrow walls of the waveguide junction. The ferrite discs are magnetized by a static magnetic field parallel to the ferrite axis. E-plane circulators, based on this geometry, have been built successfully.\textsuperscript{8,9} For analytical simplicity, the central portion of the junction is considered to be a cylindrical cavity to which the three waveguide arms are coupled through small slots.\textsuperscript{10} If the coupling through these slots is very weak, the unperturbed fundamental modes inside the cavity can be studied to see which can give rise to a circulation effect. The cross-section of the cavity is shown in Figure 4b.

Section 1 of this chapter is concerned with a review of the field theory for a circular waveguide completely filled with longitudinally magnetized ferrite. It is found that under the assumption of no field variation along the static magnetic field, i.e., the propagation constant, $\gamma = 0$, the general theory leads to TE modes only. This assumption
Figure 4a

3-Port E-Plane Junction Circulator

Figure 4b

Ferrite-Loaded E-Plane Junction Cavity
may be justified for ferrite discs whose thicknesses are small compared with the wave length.

Section 2 relates to the derivation of TE and TM wave equations directly from basic equations for $\delta = 0$. It is shown that only TE modes exist for the waveguide whose cross-section is filled with lossless ferrite. They are the same as those found in Section 1. However, they do not depend on the anisotropic properties of the ferrite and therefore are not suitable for a circulator cavity. However, TM modes depend on the ferrite properties and, if they existed, would give rise to circulator fields.

It is shown in Section 3 that TM modes might exist even in the case of a completely filled cross-section but only under certain assumptions that are different from those in Section 2. Thus, there appears to be discontinuous behavior between the cases where an air gap around the ferrite is equal to zero, i.e., $b/a = 1$ (Figure 4b), the case where it is made to approach zero, and the case for $b/a = 1$ where the ferrite is assumed to be lossy. This is compared with the "paradoxical" surface wave in ferrite-filled rectangular waveguides.
III.1 Review of the General Field Theory

A. Basic Equations

The displacement $D$ is related to the field strength $E$ by a scalar permittivity:

$$D = \varepsilon E \quad (3.1)$$

where the permittivity $\varepsilon$ is either real or complex.

Let a static magnetic field $H_o$ be produced in the medium by an external source. This produces in the medium a magnetic anistropy such that the induction $B$ and the field strength $H$ of an additional periodic magnetic field, superimposed upon $H_o$ are related by the equation

$$B = [\mu] H \quad (3.2)$$

where

$$[\mu] = \begin{bmatrix}
\mu & j\mu_a & 0 \\
-j\mu_a & \mu & 0 \\
0 & 0 & \mu_\parallel
\end{bmatrix} \quad (3.3)$$

The laws (3.2) and (3.3) hold only for periodic fields since $\varepsilon$ and $[\mu]$ are functions of the frequency $\omega$. Hence, the field vectors will be supposed to contain a time factor of the form $e^{j\omega t}$. $[\mu]$ is also a function of the permanent field $H_o$. It can be treated as spatially constant where the
static field \( H_0 \) is homogeneous within the ferrite.

Under the conditions specified above, Maxwell's equations can be written as follows for a source free region:

\[
\nabla \times E = jk_o B = jk_o [\mu] H \quad (3.4)
\]

\[
\nabla \cdot \mathbf{E} = 0 \quad (3.5)
\]

\[
\nabla \times H = jk_o \mathbf{E} \quad (3.6)
\]

\[
\nabla \cdot [\mu] H = 0 \quad (3.7)
\]

where \( k_o = \frac{\omega}{c} \) is the wave number.

We can eliminate either \( E \) or \( H \) from one of the above equations

\[
H = -\frac{j}{k_o} [\mu]^{-1} \nabla \times E \quad (3.8)
\]

where

\[
[\mu]^{-1} = 
\begin{bmatrix}
\bar{\mu} & j\bar{\mu}_a & 0 \\
-j\bar{\mu}_a & \bar{\mu} & 0 \\
0 & 0 & \bar{\mu}_{\parallel}
\end{bmatrix}
\quad (3.9)
\]

with

\[
\bar{\mu} = \frac{1}{\mu_\perp}, \quad \bar{\mu}_a = -\frac{\mu_a}{\mu_\perp}, \quad \bar{\mu}_{\parallel} = \frac{1}{\mu_{\parallel}}, \quad \mu_\perp = -\frac{\mu_a^2}{\mu} + \mu
\]

Substitution of (3.9) into (3.6) yields

\[
\nabla \left( \times \ [\mu]^{-1} \nabla \times E \right) = \mathbf{\varepsilon} k_o^2 E \quad (3.10)
\]

Equation (3.10) is the wave equation for an anisotropic medium.
It is possible to convert Maxwell's equations into a more convenient form. After considerable algebra, the following equations for the $z$ components of the fields can be derived:

\[
\left( \nabla_\perp^2 + a_e \frac{\delta^2}{\delta z^2} + b_e \right) E_z + c_e \frac{\delta}{\delta z} H_z = 0 \quad (3.11)
\]

\[
\left( \nabla_\perp^2 + a_m \frac{\delta^2}{\delta z^2} + b_m \right) H_z + c_m \frac{\delta}{\delta z} E_z = 0 \quad (3.12)
\]

where

\[\nabla_\perp = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\]

\[a_e = 1 \quad b_e = k_0^2 \varepsilon \mu_\perp \quad c_e = k_0 \mu_\parallel \frac{\mu_a}{\mu} \]

\[a_m = \frac{\mu_\parallel}{\mu} \quad b_m = k_0^2 \mu_\parallel \varepsilon \quad c_m = k_0 \varepsilon \frac{\mu_a}{\mu} \]

An equation can be obtained for $E_z$ separately by manipulating Equations (3.11) and (3.12). It can be written in the form

\[L(E_z) = 0 \quad (3.13)\]

where

\[L = \nabla^4 + \frac{\mu_\parallel}{\mu} \frac{\delta^4}{\delta z^4} + \frac{\mu_\parallel}{\mu} \nabla_\perp^2 \frac{\delta^2}{\delta z^2} + k_0 \varepsilon \left( \frac{\mu_\parallel}{\mu} \right) \nabla_\perp^2 + 2k_0^2 \varepsilon \mu_\parallel \frac{\delta^2}{\delta z^2} + k_0^4 \varepsilon^2 \mu_\parallel \mu_\perp \quad (3.14)\]
The same equation holds for the transverse field $E_\perp$ and the magnetic field $H$ is given by Equation (3.8).

It is possible to express the electric and magnetic fields in terms of a potential function $\nabla \psi$ as follows:

$$
E = \begin{bmatrix} u \end{bmatrix} \nabla \psi \quad (3.15)
$$

$$
H = \frac{\partial}{\partial z} \begin{bmatrix} v \end{bmatrix} \nabla \psi \quad (3.16)
$$

where

$$
\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} js & T & 0 \\ -T & js & 0 \\ 0 & 0 & jW \end{bmatrix}
$$

and

$$
\begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} jM & N & 0 \\ -N & jM & 0 \\ 0 & 0 & jR \end{bmatrix} \quad (3.17)
$$

with

$$
w = \frac{\mu_a}{\mu} \left( \frac{\mu_a}{\mu} + 1 \right) \nabla_\perp^2 \quad N = k_o \varepsilon \frac{\mu_a}{\mu}
$$

$$
M = \frac{1}{\mu k_o} \left( \varepsilon k_o^2 \mu + \partial^2 / \partial z^2 + \nabla_\perp^2 \right) \quad R = -\frac{\mu}{k_o \mu_\|} \nabla_\perp^2 \quad (3.18)
$$

B. Cylindrical Coordinate Systems

It is apparent from the nature of the gyromagnetic medium that the $z$ coordinate represents a preferred direction while the coordinates in the plane normal to $z$ can be chosen
arbitrarily. There is no loss of generality in considering cylindrical coordinate systems. Then the operator $\nabla_\perp$ in Equation (3.14) is independent of $z$. Consequently, the potential $\Psi$ can be built up in the form

$$\Psi(q_1, q_2, z) = Z(z) \phi(q_1, q_2)$$  \hspace{1cm} (3.19)

where $Z(z)$ is a harmonic function; e.g., $e^{\pm j \phi z}$. $q_1$ and $q_2$ are the coordinates in the planes $z = \text{constant}$. For any harmonic dependence on $z$, we can write

$$\frac{\partial}{\partial z} = j \phi, \hspace{1cm} \frac{\partial^2}{\partial z^2} = -\phi^2$$  \hspace{1cm} (3.20)

The operator $L$ can now be represented as follows:

$$L = \nabla_\perp^4 + p \nabla_\perp^2 + q$$  \hspace{1cm} (3.21)

where

$$p = k_o^2 (\varepsilon \mu_\perp + \mu_\parallel) - \phi^2 (1 + \frac{\mu_\parallel}{\mu})$$

$$q = \varepsilon \mu_\parallel \left( k_o^4 (\varepsilon \mu_\perp - 2k_o^2 \phi^2 + \frac{\phi^4}{\varepsilon \mu}) \right)$$

Moreover, it can be separated into the product of two second order operators

$$L = (\nabla_\perp^2 + \mathcal{H}_1^2)(\nabla_\perp^2 + \mathcal{H}_2^2)$$  \hspace{1cm} (3.22)

Multiplying this out and comparing it with Equation (3.21) we find that $\mathcal{H}_1$ and $\mathcal{H}_2$ are the solutions of the following quadratic equation:
\[ \Delta e^4 - p \Delta e^2 + q = 0 \]  

(3.23)

The roots of this equation are given by

\[ \epsilon^{1,2} = \frac{1}{2} \left[ \kappa_0^2 (\mu_1 + \mu_\parallel) - \left( 1 - \frac{\mu_\parallel}{\mu} \right) \delta^2 \right]^{\pm} \]

\[ + \left( \frac{1}{4} \left[ \kappa_0^2 (\mu_1 - \mu_\parallel) - \left( 1 - \frac{\mu_\parallel}{\mu} \right) \delta^2 \right]^2 + \delta^2 \kappa_0^2 \epsilon \mu_\parallel \frac{\mu_\parallel}{\mu} \right]^{1/2} \]

(3.24)

The potential now satisfies

\[ \nabla^2 \phi + \epsilon^{1,2} \phi = 0 \]  

(3.25)

There are two independent particular solutions \( \phi \) of Equation (3.24) corresponding to \( \epsilon_1 \) and \( \epsilon_2 \), respectively. The corresponding fields can be obtained from Equations (3.15) and (3.16).

C. Completely Filled Circular Waveguide with Longitudinal Magnetization

In a circular cylindrical system Equation (3.25) for the function \( \Phi \) takes the form

\[ \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + k^2 \Phi = 0 \]  

(3.26)

This equation is separable. The total potential function defined in Equation (3.19) now becomes

\[ \Psi = J_m (\epsilon^{1,2} \, r) e^{+j \delta z} e^{+jm \phi} \]  

(3.27)
We note that for a given \( m \) and a given \( r \) Equation (3.27) contains four partial waves: \( \mathcal{E}_1 \, e^{+jm \varphi} \); \( \mathcal{E}_2 \, e^{+jm \varphi} \). From Equations (3.15) and (3.16) the field components can be found. It results that the boundary conditions at \( r = a \), namely

\[
E_z = 0 \quad (3.28)
\]
\[
E \varphi = 0 \quad (3.29)
\]
cannot be satisfied by any partial wave individually. However, the sum of two partial waves with a different \( \mathcal{E} \) but the same \( m \) does satisfy them. Hence

\[
\psi = \left[ A_1 \, J_m(\mathcal{E}_1 \, r) + A_2 \, J_m(\mathcal{E}_2 \, r) \right] e^{jm(\varphi - \varphi z)} \quad (3.30)
\]
The field equation can now be obtained from Equations (3.15) and (3.16).

\[
E_r = j \left\{ \sum_i A_i \left[ S \, \mathcal{E}_i \, J'_m(\mathcal{E}_i \, r) + T_i \, \frac{m}{r} \, J_m(\mathcal{E}_i \, r) \right] \right\}
\]
\[
E \varphi = - \sum_i A_i \left[ T_i \, \mathcal{E}_i \, J'_m(\mathcal{E}_i \, r) + S \, \frac{m}{r} \, J_m(\mathcal{E}_i \, r) \right]
\]
\[
E_z = \gamma \sum_i A_i \, W_i \, J_m(\mathcal{E}_i \, r)
\]
\[
H_r = \left\{ \sum_i A_i \left[ M_i \, \mathcal{E}_i \, J'_m(\mathcal{E}_i \, r) + N \, \frac{m}{r} \, J_m(\mathcal{E}_i \, r) \right] \right\}
\]
\[
H \varphi = j \left\{ \sum_i A_i \left[ N \, \mathcal{E}_i \, J'_m(\mathcal{E}_i \, r) + M_i \, \frac{m}{r} \, J_m(\mathcal{E}_i \, r) \right] \right\}
\]
\[ H_z = -j \gamma^2 \left[ \sum_i A_i R_i J_m (\kappa_i r) \right] \]  

(3.31)

where \( i = 1, 2 \).

The differential operators in these equations reduce, because of Equations (3.20) and (3.21), to numerical multipliers.

\[ T_{1,2} = \frac{k_o^2 \varepsilon \mu_\perp - \gamma^2 - \kappa_{1,2}}{\mu_\parallel k_o} \quad M_{1,2} = \frac{1}{\mu_\parallel k_o} \left( k_o ^2 \varepsilon \mu_\parallel - \gamma^2 - \kappa_{1,2} \right) \]

\[ S = \frac{\mu_a}{\mu_\parallel} \gamma^2 \quad N = k_o \varepsilon \frac{\mu_a}{\mu_\parallel} \gamma^2 \]

\[ W_{1,2} = -\frac{\mu_a}{\mu_\parallel} \gamma^2 \quad R = -\frac{T_{1,2} \gamma_{1,2}}{k_o \mu_\parallel \gamma^2} \]  

(3.32)

Substituting the resulting \( E_z \) and \( E_\varphi \) into Equations (3.28) and (3.29) we obtain a system of two homogeneous linear equations for the coefficients \( A_1 \) and \( A_2 \). Equating its determinant to zero we obtain a transcendental equation which, together with Equation (3.23) can be used to determine \( \gamma \), \( \kappa_1 \), and \( \kappa_2 \).

\[
\left( \frac{k_o^2 \varepsilon \mu_\perp - \gamma^2}{\kappa_1} \right) \frac{J_m'(\kappa_1 a)}{J_m(\kappa_1 a)} - \left( \frac{k_o^2 \varepsilon \mu_\parallel - \gamma^2}{\kappa_2} \right) \frac{J_m'(\kappa_2 a)}{J_m(\kappa_2 a)} - \frac{\mu_a}{\mu_\parallel} m \gamma^2 \left( \frac{1}{\kappa_1^2} - \frac{1}{\kappa_2^2} \right) = 0
\]

(3.34)
This equation reveals some important properties. The quantity $\gamma$ occurs in it only as $\gamma^2$; i.e., the direction of propagation of the wave does not affect the values of $\gamma$, $\mathcal{H}_1$, and $\mathcal{E}_2$, hence, the field configurations are not altered. On the other hand, $\mu_a$ and $m$ occur in their first power. The sign of $\mu_a$ determines the direction of the steady magnetization and the sign of $m$ determines the direction of rotation of the fields around the axis (see Equation (3.27)). $m$ and $\mu_a$ both affect the quantities $\gamma$, $\mathcal{H}_1$, and $\mathcal{E}_2$. Waves with right handed rotation with respect to the direction of the steady magnetization ($m$ pos.) have different propagation constants from waves with left handed rotation ($m$ neg.) This represents non-reciprocal behavior, affecting the rotation of the polarization vector (and not the propagation along the axis - since $\gamma$ occurs as $\gamma^2$).

The reflection of waves from plane boundaries normal to the waveguide axis and separating two media, one of which is gyromagnetic, cannot be dealt with by elementary methods. An exception is the case when the boundary is a short circuiting plate. Since the field components are the same for waves travelling in the positive and negative direction along the axis, they can be combined to a standing wave satisfying
the condition of zero transverse electric field at the shorting plate. For this reason it is also easy to write down the field component for a cylindrical cavity, choosing
\[ \psi = \frac{n \pi}{l}, \]
where \( l \) is the length of the cavity. However, for a ferrite dielectric interface it is necessary to represent fields in the form of series consisting of an infinite number of waves. The condition of continuity of the tangential field components at the plane boundary leads to an infinite system of linear equations even the approximate solution of which is complicated.

However, these problems may perhaps be avoided if one draws some simplifying assumptions. Under certain conditions the fields in the ferrite are independent of the coordinate along the steady magnetic field. In this case TE and TM modes are possible in a cylindrical guide, as will be shown in the next section.

III.2 Fields Independent of the Longitudinal Coordinate

As pointed out at the beginning of this chapter, the fields in the ferrite may be assumed independent of the coordinate along the static magnetic field under certain conditions. We then have from Equation (3.20).

\[ \frac{\partial}{\partial z} = j \psi = 0 \]

(3.35)
We are now going to apply this condition to the general solutions derived above and investigate its implications. Also, it will enable us to find a solution that is not contained in the general solution by applying the condition to Equations (3.11) and (3.12).

A. TE Modes From the General Solution

With Equation (3.35), Equation (3.24) becomes

$$\mathcal{E}_{1}^2 = k_0^2 \varepsilon \mu = \mathcal{E}_{1}^2$$

(3.36)

and

$$\mathcal{E}_{2}^2 = k_0^2 \varepsilon \mu = \mathcal{E}_{2}^2$$

(3.37)

For the characteristic equation, we obtain

$$
\left( \frac{k_0^2 \varepsilon \mu}{\mathcal{E}_{1}} - \mathcal{E}_{1} \right) \frac{J_m'(\mathcal{E}_{1} a)}{J_m(\mathcal{E}_{1} a)} - \left( \frac{k_0^2 \varepsilon \mu}{\mathcal{E}_{2}} - \mathcal{E}_{2} \right) \frac{J_m'(\mathcal{E}_{2} a)}{J_m(\mathcal{E}_{2} a)} = 0
$$

(3.38)

Using Equations (3.36) and (3.37), this reduces to

$$\frac{k_0^2 \varepsilon \mu - \mathcal{E}_{2}^2}{\mathcal{E}_{2}} \frac{J_m'(\mathcal{E}_{2} a)}{J_m(\mathcal{E}_{2} a)} = 0$$

or

$$J_m'(\mathcal{E}_{2} a) = 0$$

(3.39)

This equation determines \( \mathcal{E}_{2} \); i.e., the frequency for which the boundary conditions in Equations (3.28) and (3.29) are satisfied under the present conditions. Note that Equation
(3.38) is satisfied for any $\mathcal{E}_2$. For $\mathcal{Y} = 0$ the fields given in Equations (3.31) reduce to

$$\begin{align*}
H_z &= j A A \ T_2 k_o \mathcal{E} \ J_m (\mathcal{E}_2, r) \\
E_r &= j A \ T_2 \frac{m}{r} \ J_m (\mathcal{E}_2, r) \\
E_\phi &= -A_2 T_2 \mathcal{E}_2 J_m (\mathcal{E}_2, r)
\end{align*}$$

or

$$\begin{align*}
H_z &= B J_m (\mathcal{E}_2, r) \\
E_r &= \frac{B}{k_o \mathcal{E}} \frac{m}{r} J_m (\mathcal{E}_2, r) \\
E &= j \frac{\mathcal{E}_2}{k_o \mathcal{E}} J_m (\mathcal{E}_2, r)
\end{align*}$$

where

$$B = j A_2 T_2 k_o \mathcal{E}$$

We omitted the components with subscript 1 because the above fields satisfy the boundary conditions.

The following conclusions can be drawn with respect to the general solution based on the potential function of Equation (3.27) for $\mathcal{Y} = 0$:

1. The fields reduce to TE modes given by Equation (3.41) and are based on a single partial wave potential $\Psi (\mathcal{E}_2)$,

2. There are no TM modes,
3. The resonance frequencies for the TE modes are given by Equation (3.39), and

4. These modes satisfy the boundary condition of Equations (3.28) and (3.29).

B. Direct Derivation of TM and TE Modes

Applying Equation (3.35) to Equations (3.11) and (3.12) we find

\[ (\nabla_\perp^2 + k_o^2 \varepsilon \mu_\perp) E_z = 0 \]  \hspace{1cm} (3.42)

and

\[ (\nabla_\perp^2 + k_o^2 \varepsilon \mu_\parallel) H_z = 0 \]  \hspace{1cm} (3.43)

Thus, we have independent equations for \( H_z \) and \( E_z \). Equation (3.42) corresponds to a TM wave and Equation (3.43) to a TE wave. The solution of Equation (3.42) is

\[ E_z = A J_m(\kappa_\perp r) \]  \hspace{1cm} (3.44)

where

\[ \kappa_\perp^2 = k_o^2 \varepsilon \mu_\perp \]  \hspace{1cm} (3.45)

A factor \( e^{+jm\phi} \) is omitted in all equations for the field components. Using Equation (3.8) one obtains, for the other components

\[ H_r = \frac{A}{k_o} \left[ \pm \frac{\mu}{r} J_m(\kappa_\perp r) - \mu_\perp \mu_\parallel J_m'(\kappa_\perp r) \right] \]  \hspace{1cm} (3.46)

\[ H_\phi = \frac{A}{jk_o} \left[ \pm \frac{\mu_\parallel}{r} J_m(\kappa_\perp r) - \kappa_\perp \mu J_m'(\kappa_\perp r) \right] \]  \hspace{1cm} (3.47)
From Equation (3.43) we have

\[ H_z = B J_m (\omega \varepsilon_\parallel r) \quad (3.48) \]

where

\[ \varepsilon_\parallel = k_o^2 \varepsilon \mu_\parallel \quad (3.49) \]

From Equation (3.6) we find

\[ E_r = \frac{B m}{r k_o \varepsilon} J_m (\omega \varepsilon_\parallel r) \quad (3.50) \]

and

\[ E_\varphi = \frac{i}{k_o \varepsilon} B \omega \varepsilon_\parallel J_m' (\omega \varepsilon_\parallel r) \quad (3.51) \]

Recalling the results of the previous paragraph we observe that the TE modes derived here are identical with the TE modes derived from the general solution. However, the TM modes were not contained in the general solution for \( \varphi = 0 \). This is due to the fact that the general derivation was made for a completely filled guide (see Equations (3.28) and (3.29)). Thus, no boundary conditions were imposed on Equations (3.11) and (3.12) from which the TM modes were derived. In fact, the TM modes defined in Equations (3.48) through (3.51) disappear for \( r = a \), because, for this case we have

\[ H_z (r = a) = 0 \]

and

\[ H_r (r = a) = 0 \]
which results in
\[ A J_m (\alpha a) = 0 \] (3.52)

and
\[ \frac{A}{k_0} \left[ \pm \frac{m \mu_0}{r} J_m (\alpha a) - \alpha \mu_a J_m (\alpha a) \right] = 0 \] (3.53)

Equations (3.52) and (3.53) can only be satisfied for \( A = 0 \), i.e., for zero fields. Thus, for a completely filled guide there is no inconsistency between the modes resulting from the general potential and those derived in this paragraph. In either case we end up with TE modes only.

C. Transverse Modes in the Circulator Cavity

In view of the application of transverse modes to the circulator cavity it is clear that TE modes do not give rise to circulator action because they are independent of the static magnetic field and the anistropic properties of the ferrite. This is obvious from Equation (3.49) for the wave number which depends on \( \alpha_a \), a constant. Moreover, TE modes do not exist in the cavity in Figure 4b because the transverse electric field disappears at \( z = 0 \) and \( z = h \).

TM modes on the other hand, depend on \( \alpha_L \) (Equation (3.45)), which is a function of the static magnetic field. However, as previously mentioned, they do not exist if the cross-section of the cavity is completely filled because
Equations (3.52) and (3.53) can only be satisfied simultaneously with \( A = 0 \), i.e., zero fields.

In spite of these facts, however, there are indications that the TM modes may theoretically be possible even for the completely filled guide. These indications may be fortified by some results in the theory of wave propagation in a rectangular waveguide loaded with a transversely magnetized slab of ferrite. Let us briefly review the major points of these phenomena that relate to surface modes along the ferrite/air interface in such a structure.

In the theoretical study of rectangular waveguides, partially loaded with a transversely magnetized slab of lossless ferrite, there is a discontinuous behavior between the case where the air gap between the slab and the waveguide wall is made to approach zero, and the case where it is made identical zero. Lax and Button\(^1\) showed that if the slab of ferrite is against the wall of the waveguide, there is no surface mode propagating along this wall. Bresler, on the other hand, showed that if a gap exists there is always a surface mode, even if the gap is made to approach, but remain different, from zero. Gagné then measured surface modes in completely filled waveguides. He thus proved that Bresler's postulate is a purely mathematical concept that
has no physical meaning. Lax and Button's result did not hold because the ferrite was not free of losses and their theory was based on a loss-free ferrite. Therefore, to correctly represent the experimental behavior of a waveguide completely filled with a ferrite, a mathematical model is required which assumes losses. If, for reasons of simplicity, a lossless model is used, the only way to properly predict the existing surface modes is to assume, in accordance with Bresler, that the width of the air gap is not identical to zero.

In our case of TM modes in an axially magnetized ferrite filling a circular waveguide, an analogous behavior may be expected for the following reasons. TM modes (with $\xi = 0$)

1. are not possible theoretically in a guide completely filled with a lossless ferrite,
2. do exist if an annular air gap is assumed between the ferrite and the guide, and
3. may be possible theoretically in the completely filled guide, if the ferrite is assumed to be lossy.

Point 1 was discussed in the second paragraph of this section. For point 2 let us investigate the case illustrated in Figure 4b where the ferrite only partially fills the waveguide. The fields in the ferrite are given by Equations (3.44) to (3.47)
and those in the air region can be obtained from

\[
(\nabla^2 + k_o^2 \varepsilon \mu_o) E_z = 0
\]

(3.54)

The solution can be written in the form of a linear combination of Bessel and Neumann functions whose coefficients are determined from the fact that \( E_z = 0 \) at \( r = a \). In the air region we have

\[
E_z = B C_m(\sqrt{\varepsilon_o} r)
\]

(3.55)

\[
H_r = \pm \frac{m}{r k_o \mu_o} B C_m(\sqrt{\varepsilon_o} r)
\]

(3.56)

\[
H_d = -\frac{j \varepsilon_o}{k_o \mu_o} B C_m'(\sqrt{\varepsilon_o} r)
\]

(3.57)

(where the common factor \( e^{\pm jm \varphi} \) has been omitted) and

\[
C_m(\sqrt{\varepsilon_o} r) = J_m(\sqrt{\varepsilon_o} r) - \frac{J_m(\sqrt{\varepsilon_o} b)}{N_m(\sqrt{\varepsilon_o} b)} N_m(\sqrt{\varepsilon_o} b)
\]

By using the condition of continuity of \( H_{\text{trans}} \) and \( E_{\text{trans}} \) at the ferrite/air interface, we obtain the characteristic equation for the system \(^{15}\)

\[
\frac{\mu \varepsilon_o a J_m'(\sqrt{\varepsilon_o} a)}{J_m(\sqrt{\varepsilon_o} a)} \pm m \mu_a = \frac{C_n'(\sqrt{\varepsilon_o} a)}{C_n(\sqrt{\varepsilon_o} a)} \mu_o a
\]

(3.58)

The solutions of this equation depend on \( m \), the direction of rotation of the fields, and therefore give rise to different
resonant frequencies of counterrotating modes, which is required for circulator action.

As suggested in point 3 a nontrivial solution of Equations (3.52) and (3.53) may be possible if the ferrite is assumed lossy, because in this case the wave number $\xi_1$ is complex.

Although considerable theoretical and experimental work will be required to clarify this point, these remarks indicate that the behavior of circulating TM modes may be similar to the surface modes in rectangular guides, studied by Lax and Button, Bresler and Gagné. It seems mandatory, for example, to prove experimentally the existence of TM modes for the completely filled round guide.
IV. CAVITY WITH EVANESCENT SECTION

In the last chapter we suggested consideration of the central region of a circulator as a cylindrical cavity. At the dimensions corresponding to an E-plane circulator, the central portion of the cavity would be "non-propagating" at the operating frequency, i.e., the propagation constant is purely imaginary. The situation is illustrated in Figure 5.

![Diagram of Rectangular Cavity with Non-Propagating Central Section]

Rectangular Cavity with a Non-Propagating Central Section

Figure 5

The energy is coupled through the non-propagating sections into the ferrite-loaded portions by a mechanism very similar to the tunnel effect in quantum mechanics.

Appendix A contains a literature survey that covers contributions to the problem of energy transfer through non-propagating sections of waveguides. Field theory and network theory concepts have been used to show under what conditions energy transfer through a non-propagating section of waveguide is possible. Several non-propagating sections
in cascade have filter properties. There has been some controversy about the explanation of these effects.

In addition, a lumped element equivalent circuit was recently given for a short non-propagating window.\(^3\) The lumped element model consists of two lengths of transmission line on either side of a shunt susceptance \(B_r\). The characteristic impedance of these transmission lines are equal to those in the propagating sections of the original case and the susceptance depends on the length of the cutoff section, the dielectric constants and the frequency. The susceptance is given by

\[
B_r = \frac{2\pi d \lambda}{\lambda_o^2} \frac{g_1}{(\varepsilon_2 - \varepsilon_1)} \tag{4.1}
\]

The non-propagating window in a uniformly loaded transmission line is illustrated in Figure 6. Below, in Figure 7, the lumped element representation of the window is shown, and in Figure 8, the exact transmission-line circuit. A detailed discussion of this model will be given later. For what immediately follows it is only important to note that \(B_r\) is negative, i.e., inductive for the case in which we are interested, and increasing for increasing length \(d\).
Figure 6
Cutoff Window in a Uniformly Loaded Waveguide

Figure 7
Lumped-Susceptance Model of the Window

Figure 8
Exact Transmission Line Model
We are now going to apply the lumped model to the case of the partially loaded cavity, illustrated in Figure 5. This will enable us to anticipate some novel properties of this cavity. Using Figure 7, we can obtain the resonance frequency by setting

\[ Y_{in1} = -Y_{in2} \]

where \( Y_{in2} \) is according to Figure 9 (which represents half of the cavity)

\[ Y_{in1} \quad Y_{in2} \]

and \( Y_{in1} = B_r/2 \) (half the equivalent susceptance given in Equation (4.1)). \( d/2 \) belongs to the model of the window as described in Figure 7 and \( t \) is the length of one of the propagating sections in the cavity. Recall that \( Y_{in1} \) is inductive and increasing with increasing length \( d/2 \). Now note that a short circuited length of transmission line may be inductive also, but that its inductance decreases with increasing length. The effect on the resonance frequency of
lengthening the non-propagating section of the cavity may therefore be simulated by

i) letting $Y_{in_1} = B_r$ with $d$ increasing, or

ii) letting $Y_{in_1}$ be equal to the input impedance into an appropriate length $\xi$ of a short circuited transmission line, with $\xi$ decreasing.

Due to the latter point, it is now clear that the resonance frequency $f_o$ of the cavity is increased if the length $d$ of the non-propagating section is increased. This behavior is very unusual and requires further attention. Normally, the resonance frequency of a cavity decreases upon increasing its dimensions.

Recalling that the susceptance $B_r$ is capacitive above cutoff it can be shown similarly that $f_o$ decreases for increasing $d$, if $f_o > f_c$ for $d = 0$, where $f_c$ is the cutoff frequency of the waveguide.

In the following section, the cavity of Figure 5 will be analyzed rigorously, and the range of variables over which the lumped model is a reasonable approximation will be determined.
IV.1 **Rectangular Cavity**

For experimental simplicity, this analysis and the measurements are made for a rectangular cavity. Figure 10 shows a rectangular cavity, coupled to a rectangular waveguide of the same transverse dimensions through a small hole and loaded with two pieces of dielectric material.

![Coupled Rectangular Cavity with a Non-Propagating Section](image)

**Figure 10**

The input impedance into a line with propagation constant $\dot{\gamma} = \alpha + j\beta$, characteristic impedance $Z_o$ and a termination $Z_L$ is

$$Z_{\text{in}} = Z_c \frac{Z_L + Z_c \tanh \dot{\gamma} l}{Z_c + Z_L \tanh \dot{\gamma} l} \quad (4.2)$$

For the input impedance into the first section, filled with
dielectric and terminated in a short circuit, we have, therefore

\[ Z_1 = Z_d \frac{0 + Z_d \tanh \frac{\phi_d}{2}}{Z_d + 0} = Z_d \tanh \frac{\phi_d}{2} \]

Using this impedance as the load for the adjacent section we can obtain \( Z_2 \) in the same manner. Similarly, for \( Z_3 \).

The coupling hole can be represented as a shunt susceptance

\[ Z_a = jX_L = j \frac{8 \beta r_o^3}{3ab} \frac{k_0 Z_o}{\beta} \]

Therefore, the input impedance through the coupling hole is

\[ Z_{in} = \frac{Z_a Z_3}{Z_a + Z_3} \]  \quad (4.3)

Clearly, resonance occurs for \( Z_a + Z_3 = 0 \).

The characteristic impedance of a TE mode is

\[ Z_c = -\frac{k}{j\phi} Z \]

where \( Z = \left( \frac{\mu}{\varepsilon} \right)^{\frac{1}{2}} = \frac{Z_o}{(\varepsilon_r)^{\frac{1}{2}}} \) with \( Z_o = \left( \frac{\mu_o}{\varepsilon_o} \right)^{\frac{1}{2}} \)

and \( k = (\varepsilon_r)^{\frac{1}{2}} k_o = (\varepsilon_r \omega^2 / \mu_o \varepsilon_o)^{\frac{1}{2}} \).

In the dielectric sections, the characteristic impedance is

\[ Z_d = -\frac{k_d}{j\beta} Z = \frac{k_0 Z_o}{\beta} \]
and in air

\[ Z_a = j \frac{k_o Z_o}{\alpha} \]

where

\[ \alpha = \left( k_c^2 - k^2 \right)^{\frac{1}{2}} \]

and

\[ \beta = \left( k^2 - k_c^2 \right)^{\frac{1}{2}} \]

\( k_c \) is the cutoff wave number. If the air filled section is propagating, \( \alpha \) has to be replaced by \( j \alpha \).

Using these expressions in the above resonance condition, Equation (4.3), we arrive, after some algebra, at

\[
8 \beta r_o^3 \left[ 1 - \tan^2(\beta t) - \tanh(\alpha d) \tan(\beta t) \frac{\beta^2 (\frac{\alpha}{\beta})^2}{\alpha \beta} \right] \\
+ \frac{3ab}{\alpha \beta} \left[ 2 \alpha \beta \tan(\beta t) + \beta^2 \tanh(\alpha d) \frac{\alpha^2}{\beta} \tanh(\alpha d) \cdot \tan^2(\beta t) \right] = 0 \quad (4.4)
\]

The sign in parenthesis holds if \( f_o > f_c \) or \( \gamma_a = j \alpha \), i.e., if the air portion is propagating.

For \( d = 0 \), or for \( \varepsilon_r = 1 \), \( d = t \), and \( f_o > f_c \), the resonance condition simplifies to an equation derived elsewhere. For \( d \to \infty \) it becomes independent of \( d \) and the resonance frequency approaches a constant value. It turns out that this value depends on \( t \) if \( \gamma_a = j \alpha \) and is equal to \( f_c \) if \( \gamma_a = \alpha \).
IV.2 Experiments

An experiment was set up to check the theoretical results obtained by solving Equation (4.4). A dominant propagating TE wave was launched from an air-filled X-band waveguide into a dielectric-filled Ku-band waveguide. This wave was then coupled through an iris into a cavity, partially filled with two pieces of dielectric ($\varepsilon_r = 2.77$) according to Figure 11. The short circuit to the right was a rectangular plunger which, together with the piece of dielectric next to it, could be moved along the guide. The resonance frequency was determined by monitoring the reflected power in the X-band waveguide. This was done for varying lengths $d$ of the air gap and for several different lengths $t$ of the dielectric in the cavity.

The cutoff frequency for the air-filled Ku-band waveguide is

$$f_{ca} = 9.48 \text{ GHz}$$

For the same guide filled with a dielectric with $\varepsilon_r = 2.77$ it is

$$f_{cd} = 5.70 \text{ GHz}$$

The above measurements and the solutions of Equation (4.4) are plotted in Figure 11. Curves #3 and #4 illustrate a case for which, at $d = 0$, the resonant frequency $f_0$ is
above the cutoff frequency of the air-filled guide. It is not surprising that $f_0$ decreases with increasing $d$ since all three regions in the cavity are propagating. For large $d$, $f_0$ approaches the cutoff frequency $f_c$.

If the dimensions of the dielectric were such that for $d = 0$, $f_0 = f_c$, the characteristic impedance of the air-filled guide and the input impedance into any length of it would be infinite. Consequently, increasing the separation $d$ has the effect of connecting an open circuit to the end of the propagating section of length $t$. The length $d$ has no influence on $f_0$. Curve #1 illustrates a situation very close to this case.

In curves #2 and #3, $f_0 < f_c$ for $d = 0$. For $d > 0$ and increasing, $f_0$ increases. Because of the near exponential pattern of the fields for increasing $d$, the coupling between the propagating sections is practically zero for $d$ larger than a few centimeters. $f_0$ becomes independent of $d$ and approaches the resonance frequency of a section of propagating guide of length $t$ shunted with the input impedance into an infinitely long cutoff section.
Resonance Frequency of a Rectangular Cavity with a Cutoff Section as a Function of the Length of the Cutoff Section

Figure 11
V. LUMPED ELEMENT EQUIVALENT CIRCUIT

FOR THE EVANESCENT WINDOW

In the previous chapter the lumped model for an evanescent window was successfully used to predict qualitatively the behavior of a cavity with a non-propagating section. The important question may now be raised as to how accurately and over what range of parameter values the model can be used to simulate a cutoff window. Very little is said in Kelly's work in this respect. Also, he does not discuss the approximations he used in the derivation of the model.

The purpose of the present chapter is to re-derive this model shortly, thereby investigating some points not previously mentioned. Then, an error analysis will be presented. The conclusions based on it will be verified by calculating the resonance frequency of the structure in Figure 5, using the lumped model.

V.1 Derivation of the Equivalent Susceptance

Figure 7 for the shunt susceptance equivalent circuit to be derived is repeated here for convenience.
Lumped-Susceptance Model of the Window

Figure 7

The wave amplitude transmission matrix was defined in Equation (2.30). For a length $d/2$ of transmission line it becomes

$$T_t = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \begin{bmatrix} \cos k_1 d/2 & j Z_1 \cos k_1 d/2 \\ j Y_1 \sin k_1 d/2 & \cos k_1 d/2 \end{bmatrix}$$

(5.1)

For a shunt element $j$

$$T_s = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ jB & 1 \end{bmatrix}$$

(5.2)

The transmission matrix of the network between ports $a$ and $b$ in Figure 7 is therefore given by

$$T = T_t \cdot T_s \cdot T_t = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

(5.3)
where
\[
\begin{align*}
A &= \cos^2 \eta - \sin^2 \eta - BZ_1 \cos \eta \sin \eta \\
B &= j(2Z_1 \sin \eta \cos \eta - BZ_1^2 \sin^2 \eta) \\
C &= j(2Y_1 \sin \eta \cos \eta + B \cos^2 \eta) \\
D &= \cos^2 \eta - \sin^2 \eta - BZ_1 \sin \eta \cos \eta
\end{align*}
\]

(5.4)

with
\[
\eta = \frac{k_1}{2}
\]

The normalized susceptance is obtained by setting the right hand side of Equation (5.3) equal to the transmission matrix of a length \(d\) of transmission line with propagation constant \(k_2\) and characteristic impedance \(Z_2\), i.e., equal to the transmission matrix of the portion \(a-b\) of the circuit in Figure 6. This results in the following set of equations:

\[
\begin{align*}
B/Y_1 &= - \frac{\cos k_2 d}{\frac{1}{2} \sin k_1 d} - \tan \frac{k_1}{2} d + \frac{1}{\tan \frac{k_1}{2} d} \\
(5.5) \\
B/Y_1 &= - \frac{Z_2 \sin k_2 d}{Z_1 \sin^2 \frac{k_1}{2} d} + \frac{2}{\tan \frac{k_1}{2} d} \\
(5.6) \\
B/Y_1 &= \frac{Y_2}{Y_1} \frac{\sin k_2 d}{\cos^2 \frac{k_1}{2} d} - 2 \tan \frac{k_1}{2} d \\
(5.7)
\end{align*}
\]

where
\[
Y_i = \frac{k_i}{2 \pi f \mu_o} = \frac{1}{\lambda_{gi} f \mu_o} + \frac{\lambda_i}{\lambda_{gi} \eta} = \frac{\lambda_o}{\lambda_{gi}} \eta_o
\]

and \(i = 1, 2\)
Also,
\[ \lambda_{gi} = \sqrt{\frac{\lambda_{i}}{1 - \left( \frac{\lambda_{i}}{\lambda_{c}} \right)^2}} = \frac{\lambda_{o}}{\sqrt{\varepsilon_{i} - \left( \frac{\lambda_{o}}{\lambda_{c}} \right)^2}} \]

and
\[ k_{i} = \frac{2\pi}{\lambda_{gi}} \]

Using these definitions, Equations (5.5), (5.6), and (5.7) can be written in terms of the following independent parameters:

\[ B, \lambda_{o}, \lambda_{c}, \varepsilon_{1}, \varepsilon_{2}, d \]

d and \( \lambda_{c} \) are given by the geometry of the device, \( \varepsilon_{1} \) and \( \varepsilon_{2} \) describe the media, and \( \lambda_{o} \) is given by the frequency.

Assuming, for example, that the size of the waveguide and the relative dielectric constants are fixed, we can consider

\[ \varepsilon_{1}, \varepsilon_{2}, \text{ and } \lambda_{c} \]

as fixed parameters. This leaves

\[ B, \lambda_{o}, \text{ and } d \]

free variables. The system of Equations (5.5), (5.6), and (5.7) could now be solved for these variables. Since these equations involve trigonometric functions the solutions are multivalued. Moreover, they are transcendental and can only be solved numerically.
It is possible to obtain an explicit approximate formula that relates the above quantities by ignoring Equations (5.5) and (5.6) and using some approximations on Equation (5.7) which relates the C elements or the inverse transfer impedance of the transfer matrices. The implications of ignoring Equations (5.5) and (5.6) are discussed later.

We have

\[ Y_1 = \frac{k_1}{\omega \mu_0}, \quad Y_2 = \frac{k_2}{\omega \mu_0} \]

Also, for small \( d \)

\[ \tan \frac{k_1}{2} d \approx \frac{k_1}{2} d, \quad \sin k_2 d \approx k_2 d \quad (5.8) \]

Equation (5.7) can now be rewritten as follows:

\[ \frac{B}{Y_1} = \frac{k_2}{k_1} \left( \frac{\sin k_2 d}{1 - \sin^2 \frac{k_1 d}{2}} \right) - 2 \tan \frac{k_1 d}{2} \]

\[ \approx \frac{k_2^2 d}{k_1} - k_1 d = \frac{d}{k_1^2} (k_2^2 - k_1^2) \quad (5.9) \]

After some algebra one obtains

\[ B_r = \frac{B}{Y_1} = \frac{2 \pi d \lambda g_1}{\lambda_0^2} (\varepsilon_2 - \varepsilon_1) \quad (5.10) \]

and

\[ B = 2 \pi f d (\varepsilon_2 - \varepsilon_1) \varepsilon_0 \quad (5.11) \]

This corresponds to Equation (4) in Reference 3.
Susceptances are commonly classified according to their sign. A negative term is called inductive, a positive term, capacitive. Thus, this classification depends only upon the sign and not upon the frequency variation of the susceptance. For $\varepsilon_2 > \varepsilon_1$, $B_r$ is positive and thus capacitive. This is in agreement with the findings in Reference 17. For $\varepsilon_1 < \varepsilon_2$, $B_r$ is negative and equivalent to an inductor. Conforming to the above classification, this could also be called a negative capacitor, a term used by Kelly. It makes sense to use this notation if one considers Equation (5.11) and conforms to the terminology of circuit theory, where a susceptance that is proportional to the frequency is called a capacitor.

V.2 Error Analysis

It is clear that the above derivation yields a susceptance in the lumped model such that its inverse transfer impedance (element C of transfer matrix) is approximately equal to the inverse impedance transfer matrix of the exact transmission line model. However, this is not the only approximation involved in the use of the lumped model as might be concluded from some remarks by Kelly.
The use of the model in a network usually involves several elements of its ABCD matrix, not just the element C. In the above derivation Equations (5.5) and (5.6) for the A and B matrix elements have not been considered. It therefore appears necessary to investigate the error introduced by using the susceptance defined in Equation (5.10). This can be done by comparing the matrix elements of the exact model with those of the lumped model and using the susceptance B as defined in Equation (5.10).

For the A element we have

$$\Delta A = \cos k_2 d - \left( \cos^2\frac{k_1 d}{2} - \sin^2\frac{k_1 d}{2} - B/Y_1 \cos\frac{k_1 d}{2} \sin\frac{k_1 d}{2} \right)$$

**exact model**

**lumped model**

Using $\sin x \approx x$, $\cos x \approx 1$, $\sin^2 x \approx 0$ and $\cos^2 x \approx 1$ for small $x$, we obtain for small $d$

$$\Delta A \approx 1 - \left[ 1 - 0 - \frac{d}{k_1} (k_2^2 - k_1^2) \frac{k_1 d}{2} \right]$$

Thus

$$\Delta A \approx \frac{(k_2^2 - k_1^2)}{2} d^2 \approx 0 \quad (5.12)$$

Similarly, for the B element

$$B = Z_2 \cos k_2 d - \left[ 2Z_1 \sin\frac{k_1 d}{2} \cos\frac{k_1 d}{2} - B/Y_1 Z_1 \sin^2\frac{k_1 d}{2} \right]$$

$$\approx Z_2 - k_1 Z_1 d \quad (5.13)$$
It can be seen from Equation (5.12) that \( A \) is well approximated for small window lengths \( d \). Equation (5.13) however, indicates the possibility of an appreciable error in \( B \).

At this point let us focus on the application of the window as an impedance transforming element. The input impedance into the central portion a-b of the network in Figure 7 loaded with an impedance \( Z_L \) is

\[
Z_{i_{n_k}} = Z_{11} - \frac{Z_{12}^2}{Z_{22} + Z_L} \quad (5.14)
\]

where

\[
Z_{11} = \frac{A}{C}
\]

\[
Z_{12} = \frac{1}{C}
\]

\[
Z_{22} = \frac{D}{C} = \frac{A}{C} = Z_{11}
\]

and \( A, C \) are given by Equation (5.4). Thus, \( Z_{i_{n_k}} \) is independent of \( B \) and the only approximation relevant to this problem is the linearization of the trigonometric functions for small \( d \).

We conclude that if this model is used as an impedance transforming element it gives a first order approximation to the transmission matrix, which is based on transmission line theory. However, if it is used such that the off-diagonal element \( B \) of the transmission matrix is involved
in the analysis, an appreciable error may be involved.

The calculations and measurements in Chapter IV on the cavity with an evanescent section provide a convenient opportunity to verify these conclusions. We shall compare the calculations of the resonance frequency of the cavity based on the lumped model with the results of Chapter IV.

The equivalent, lumped network for the cavity discussed in Chapter IV and illustrated in Figure 5 is shown in Figure 12.

![Lumped Model for Cavity with Evanescent Section](image)

**Figure 12**

The transmission matrix for the central portion a-a is given by Equation (5.4). The input impedance into this section, loaded with $Z_1 = jZ_d \tan \beta t$ can be written in terms of the
impedance matrix of section a-a as follows:

\[ Z_3 = Z_{11} - \frac{Z_{12}^2}{Z_{22} + Z_1} \]

where \( Z_{11}, Z_{22}, \) and \( Z_{12} \) can be determined from the transmission matrix. For the whole circuit in Figure 12, we have

\[ Z_1 + Z_3 = 0 \]

which is the resonance condition for the uncoupled cavity.

After some algebra this can be converted into

\[ 2 \cos \eta - BZ_d \sin \eta - 2 \tan \eta t \sin \eta - BZ_d \tan \eta t \cos \eta = 0 \]

(5.15)

\( B \) is defined in Equation (5.10) and all the other quantities are according to Chapter IV.

A side remark is due here with respect to the above equation for the resonance frequency. This same expression can be obtained directly by even mode analysis, i.e., by setting the input impedance into the right half of the cavity equal to infinity. Therefore, it can be concluded that this model only describes the even mode. This is understandable since in the odd mode the fields at the center of the cavity, i.e., at the location of the lumped susceptance in the equivalent circuit, are zero. The susceptance is short circuited.
The resonance frequency calculated on the basis of the lumped susceptance model is plotted in Figure 13 as a function of the unnormalized window length. The exact resonance frequency is also shown for comparison. As expected from the previous error analysis the two models are equivalent for \( d < 0.2 \) cms.
t = .79 cm

exact transmission line theory

lumped model theory

Gap Length $d$, cm

Calculated Resonance Frequency as a Function of Gap Length

Figure 13
VI. APPLICATION OF THE LUMPED MODEL

The susceptance effect of the cutoff window may have a variety of applications. For example, it may be used as a matching two-port in the arms of a circulator or, more generally, in the design of periodic structure type filters. Periodic structures are waveguides and transmission lines loaded at periodic intervals with identical obstacles. These may be reactive elements such as a diaphragm or a discontinuity periodically added into the equivalent circuit as shunt elements. Extensive design theories for such filters are available. Since it is possible to model the cutoff window in terms of a shunt susceptance, these theories may be directly applied in the design of periodic structures where the obstacles consist of cutoff sections. The circuit, or network analysis of periodic structures involves constructing an equivalent network for a single basic section or unit cell of the structure first. In this chapter we shall discuss in some detail the basic section for the case of a cutoff window.

For this purpose, let us calculate the input impedance into the central section of Figure 7, loaded with an impedance
$Z_L$. This impedance was given in Equation (5.14). Equation (5.14) can be written

$$Z_{\text{in}} = \frac{A(A + C Z_L) - 1}{(A + C Z_L) C}$$  \hspace{1cm} (6.1)

Letting $C = jC_r$ where $C_r$ is real and after some algebra we obtain

$$Z_{\text{in}} = \frac{R}{R^2 C_r^2 + (A - XC_r)^2} - j \frac{R^2 AC_r^2 + (A^2 + AXC_r - 1)(A - XC_r)}{R^2 C_r^3 + C_r (A - XC_r)^2}$$  \hspace{1cm} (6.2)

The input impedance into a cutoff transmission line of length $d$, loaded with $Z_L$ is

$$Z_{\text{in}} = \frac{Z_L + Z_c \tanh \alpha d}{Z_c + Z_L \tanh \alpha d}$$ \hspace{1cm} (6.3)

where $Z_c = j Z_a = \text{characteristic impedance}$

$\gamma = j \alpha = \text{propagation constant}$

From this we can obtain

$$Z_{\text{in}} = Z_a \frac{R Z_a \sech^2 \alpha d}{R^2 \tanh^2 \alpha d + (Z_a + X \tanh \alpha d)^2}$$

$$+ j Z_a \frac{R^2 \tanh \alpha d + (Z_a + X \tanh \alpha d) (X + Z_a \tanh \alpha d)}{R^2 \tanh^2 \alpha d + (Z_a + X \tanh \alpha d)^2}$$  \hspace{1cm} (6.4)
In order to compare the lumped susceptance model with the transmission line model, the difference between \( Z_{in_k} \) and \( Z_{in_e} \) as given in Equations (6.2) and (6.4), could now be determined. However, the resulting expressions are very cumbersome and unpractical. It was therefore decided to use a computer method based on Smith-Chart Techniques for this purpose (Appendix B).\(^{18}\)

VI.1 Normalization

In order to generalize these calculations it is desirable to introduce normalized parameters. Among many possibilities, the following seemed to be the most appropriate:

\[
\begin{align*}
 s_r &= \frac{d}{\lambda_c} \\
 \varepsilon_r &= \frac{\varepsilon_1}{\varepsilon_2} \\
 \lambda_r &= \frac{\lambda_2}{\lambda_c}
\end{align*}
\]  

where \( \lambda_c \) is the cutoff wave length and \( \lambda_2 \) is the free space wave length in medium 2 (central section, usually air).

The expression for the lumped susceptance \( B \)

\[
B/Y_1 = \frac{2\pi d}{\lambda_o^2} \times \lambda_g (\varepsilon_2 - \varepsilon_1)
\]  

(6.6)

can now be rewritten in terms of these parameters. We have
\[ \lambda_i = \frac{\lambda_o}{\sqrt{\varepsilon_i}} \]

and therefore,

\[ \lambda_1 = \lambda_2 \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^{\frac{1}{2}} = \frac{\lambda_2}{\sqrt{\varepsilon_r}} \]

Also,

\[ \lambda_{g1} = \frac{\lambda_1}{1 - (\lambda_1 / \lambda_2)^2} \]

\[ = \frac{\lambda_o}{\sqrt{\varepsilon_1}} \frac{\sqrt{\varepsilon_r}}{\sqrt{\varepsilon_r - \lambda_r^2}} \]

Equation (6.6) now becomes

\[ \frac{B}{Y_1} = \frac{2\pi}{\lambda_o} \frac{d}{\lambda_c} \lambda_c \frac{\lambda_o}{\sqrt{\varepsilon_1}} \frac{\sqrt{\varepsilon_r}}{\sqrt{\varepsilon_r - \lambda_r^2}} (\varepsilon_2 - \varepsilon_1) \]

\[ = \frac{2\pi}{\lambda_2 \sqrt{\varepsilon_2}} s_r \lambda_c \frac{\sqrt{\varepsilon_r \varepsilon_2}}{\sqrt{\varepsilon_2 \varepsilon_1}} \frac{1 - \varepsilon_r}{\sqrt{\varepsilon_r - \lambda_r^2}} \]

and finally

\[ \frac{B}{Y_1} = 2\pi (1 - \varepsilon_r) \frac{s_r}{\lambda_r \sqrt{\varepsilon_r - \lambda_r^2}} = B_r \quad (6.7) \]

In the computer programs mentioned above, the characteristic impedances of the various waveguide sections are used. In the case of rectangular waveguides, they can be written as follows:
\[ z_{ci} = \frac{\eta_o}{\varepsilon_i} \frac{1}{\sqrt{1 - \left( \frac{\lambda_i}{\lambda_c} \right)^2}} = \frac{\eta_o}{\sqrt{\varepsilon_2}} \frac{1}{\sqrt{\varepsilon_i/\varepsilon_2 - \lambda_r^2}} \]  

(6.8)

Due to the square root in Equation (6.7), the following relation has to be satisfied in order for \( B_r \) to be real:

\[ \varepsilon_r - \frac{\lambda_r^2}{\lambda_c^2} > 0 \]

(6.9)

If we have \( \varepsilon_2 = 1 \), for example

\[ \varepsilon_1 - \frac{\lambda_o^2}{\lambda_c^2} > 0 \]

and therefore \( \lambda_1 < \lambda_o \), i.e., medium 1 has to be propagating.

A relationship between \( B_r \) and \( \lambda_r \) can be found by computing \( \varepsilon_r \) from Equation (6.7). Solving for \( \varepsilon_r \), one has

\[ \varepsilon_r^2 - \left[ 2 + \left( \frac{B_r \lambda_r}{2 \pi s_r} \right)^2 \right] \varepsilon_r + \left[ \left( \frac{B_r \lambda_r}{2 \pi s_r} \right)^2 + 1 \right] = 0 \]

In order for \( \varepsilon_r \) to be real, the following must hold:

\[ 4 + 4 \left( \frac{B_r \lambda_r}{2 \pi s_r} \right)^2 + \left( \frac{B_r \lambda_r}{2 \pi s_r} \right)^4 - 4 \left[ \left( \frac{B_r \lambda_r}{2 \pi s_r} \right)^2 + 1 \right] > 0 \]

Thus,

\[ \left( \frac{B_r}{s_r} \right)^2 \left[ \frac{\lambda_r^2}{\pi^2} + \frac{B_r^2 \lambda_r^4}{16 \pi^4 s_r^2} - \frac{\lambda_r^4}{\pi^2} \right] > 0 \]
or 
\[ 1 + \left( \frac{B_r}{4 \pi s_r} \right)^2 - \lambda_r^2 > 0 \]

Finally,
\[ \frac{1}{\lambda_r^2} > 1 - \left( \frac{B_r}{4 \pi s_r} \right)^2 \]  \hspace{1cm} (6.10)

Because \( \frac{1}{\lambda_r^2} > 0 \) for any \( \lambda \), the above equation is always satisfied if
\[ \frac{B_r}{s_r} > 4 \pi \]

But if \( B/s_r \leq 4 \), \[ 1 - \left( \frac{B_r}{4 \pi s_r} \right)^2 > 0 \], therefore
\[ \lambda_r < \left[ \frac{1}{1 - \left( \frac{B_r}{4 \pi s_r} \right)^2} \right]^{\frac{1}{2}} \]  \hspace{1cm} (6.11)

In summary, we conclude that for all the parameters in Equation (6.7) to be real the following conditions have to be satisfied:

1. Medium 1 has to be propagating (from Equation (6.9)).
2. Values of \( B_r/s_r \leq 4 \pi \) can only be obtained within a limited wave length range given by Equation (6.11).

VI.2 Computed Comparison Between Lumped and Transmission-Line Model

We will now discuss the results of the proposed comparison
between the lumped and transmission line models. The network for which this will be done is illustrated in Figure 8. The difference between \( Z_{in} \) as calculated from the two models is determined using the computer methods discussed in Appendix B. The dependence of this difference on the load \( Z_L \), frequency, gap length, and dielectric constant is computed.

The mechanism of transformation across various boundaries and through various waveguide sections can be illustrated on the Smith Chart. Consider first the simplified case where \( Z_L \) is purely imaginary. The vector \( \Gamma_{el} \) in Figure 14 represents the reflection coefficient of such a case. Note that the load impedance is normalized with respect to the propagating section. Upon renormalization with respect to the cutoff section, the impedance becomes purely real \( \Gamma_{e2} \). Transformations through the cutoff section does not change the angle of the reflection coefficient but decreases its magnitude exponentially. We obtain \( \Gamma_{e3} \), which, upon renormalization with respect to the propagating section, becomes purely imaginary \( \Gamma_{e4} \). \( \Gamma_a \) represents the reflection coefficient as calculated on the basis of the lumped susceptance model. Moving through a propagating section of length \( d/2 \) rotates \( \Gamma_{a1} \) to \( \Gamma_{a2} \). Adding an inductive susceptance \( B_e \) results in \( \Gamma_{a3} \), which is rotated to \( \Gamma_{a4} \) upon
Transformation of an Imaginary Impedance
Through a Cutoff Section of Waveguide

Figure 14
transformation to plane 4 in Figure 8. Note that the susceptance added to $\Gamma_{a2}$ has such a magnitude that $\Gamma_{a4} = \Gamma_{e4}$, i.e., the two models are equivalent. This susceptance is now compared with the susceptance $B_r$, given by Equation (6.6). This can be done for varying loads (by varying the angle $\theta$ of $\Gamma_{e1} = \Gamma_{a1}$), frequency, etc.

Figure 15 shows the percentage difference

$$\Delta = (B_r - B_e)/B_r$$

as a function of gap length for various loads. As expected, the error is small for small $s_r$ (see Section V.2). Also it depends strongly on the load. Similar curves were obtained for constant loads but varying $B_r$ with various combinations of $\lambda_r$ and $\varepsilon_r$ ($\lambda_r$ according to Equation (6.11), $\varepsilon_r$ from Equation (6.7)). The dependences on all parameter changes were such that meaningful generalizations could not be made except for the following: for constant $B_r$ and load the error remained invariable for various combinations of $\lambda_r$ and $\varepsilon_r$. For this reason it was decided to analyze the same case but with a reflection coefficient of magnitude different from unity.

Figure 16 shows the reflection coefficient $\Gamma_e$ of a general admittance as it is transformed from plane 1 in
Calculated Percentage Error as a Function of Normalized Gap Length for Various Loads

Figure 15
Transformation of a Complex Impedance Through a Cutoff Section of Waveguide

Figure 16
Figure 8 to plane 4 according to transmission line theory. \( \Gamma_a \) is also indicated in this figure. The transformation from 2 to 3 corresponds to the addition of the parallel shunt susceptance \( B_r \), which in this case is determined from Equation (5.20). Note that \( \Gamma_e \) changes angle and magnitude when transformed from 1 to 4. It is obvious that both these changes cannot be simulated by adjusting the shunt susceptance, i.e., the arc \( \Gamma_{a_2} - \Gamma_{a_3} \). However, some \( B_r \) could be determined such as to balance the error between two components (e.g., angle and magnitude) and minimize it. Although this has been done on the computer the results are very impractical because their dependence on the load impedance, frequency, gap length, etc., is erratic.

These calculations were an attempt to establish ranges of values for the various parameters such that the error stays within certain limits. However, due to the non-linear character of the Smith Chart, this is difficult. A small deviation in the magnitudes of the reflection coefficients, for example, may result in a large percentage error in the left half of the Smith Chart and in practically zero error at some location in the right half (where the reflection coefficient vector is tangential to a constant resistance circle). It can be concluded from numerous calculations that it is
impossible to establish meaningful criteria for the parameters such that the errors stay within certain bounds. An evaluation of the lumped model in special applications such as the cavity in the previous section is therefore necessary. The only general remark that can be made is that the error goes to zero for $d$ approaching zero. Thus, the lumped model is a good approximation for the evanescent window for relative gap lengths of a few percent. Under these conditions it may therefore lend itself successfully to other applications, such as filter design.
VII. CONCLUSIONS

It has been shown that when the longitudinal propagation constant $\zeta \neq 0$, circulating modes are possible in a circular waveguide, completely filled with ferrite, since $\zeta$ depends on the sign of the azimuthal variation of the fields. But previous H-plane circulator analysis was based on separable modes, i.e., TE and TM that can exist when $\zeta = 0$. However, it has been shown that TM modes cannot exist in a ferrite-filled circular guide if $\zeta = 0$. TE modes have been shown to exist, but these solutions are trivial in the present problem since they cannot cause circulation. It is suggested that the TM modes can only exist if an air space is present either next to the conducting wall or in the center. An alternative solution for the ferrite-filled guide may be possible if losses are assumed. The requirement that either losses or air-gaps exist is reminiscent of the thermodynamic paradox discussed by Lax and Button, Bresler, and Gagné. It is suggested that this may be the equivalent problem in cylindrical coordinates and this investigation may lay out the theory needed to tackle the problem.
The work on the cavity with a non-propagating section has shown that there are two distinct types of behavior. They depend on whether the resonance frequency $f_{ro}$ for zero length of the evanescent section is above or below the cut-off frequency $f_a$ of this section. If $f_{ro} < f_a$, the resonance frequency $f_r$ decreases for increasing gap length and approaches $f_a$. If $f_{ro} > f_a$, $f_r$ increases and approaches a constant value that depends on the lengths of the propagating sections of the cavity. This structure has been analyzed using transmission line theory and computer methods. The results show good agreement with the measurements.

The error analysis of Kelly's two-port model for an evanescent window has shown that the A, C, and D elements of its transmission matrix are first order approximations of the coefficients of the transmission matrix of the exact transmission line two-port model. The B element may deviate appreciably from the exact element. It is concluded that the model may be successfully used in applications where the B element is not involved. Such a case is the calculation of the resonance frequency of the evanescent cavity, where the A, C, and D elements only are involved. The behavior in this case is shown to be accurately predicted for short gap lengths. Generally, the B element is not involved if the
two-port is used in cascade connections.

In order to analyze such a case in more generality, the input impedances into lumped and exact model two-ports have been calculated and compared for varying loads and gap lengths. It has been found that the behavior of the error upon parameter variations cannot be described in a concise manner. However, the mechanism of transforming an impedance through a non-propagating section of transmission line has been clarified. This made it possible to account for the complex behavior of the analyzed error. Also, it has been confirmed that the lumped model is a good approximation of the evanescent window at short gap lengths. Under these conditions it may be successfully employed in other applications, such as filter design.

In the study of the network theory for a three-port junction, Hagelin's flow graph model for a non-ideal circulator has been explained within the well known matching theory of such devices. For this purpose it has been shown that a matched two-port network can be decomposed into the cascade connection of two two-ports, one of which is non-reciprocal. Among other possibilities, the second one may be reciprocal and non-symmetrical or reciprocal and symmetrical.
APPENDIX A

Review of Evanescent Mode Waveguides

Cullen et al.\(^1\) investigated the phenomenon of power transmission through a finite section of cutoff waveguide and explained it in terms of an incident and reflected field. The electric and magnetic field of an evanescent TE mode exhibit no variation of phase with position and the electric field is in quadrature with the magnetic field at all points. The time average Poynting vector is therefore zero and there is no power flow. If, however, we have a second evanescent mode decaying exponentially in the opposite direction along the guide, the two together carry an amount of power depending on their amplitudes and the phase angle between them. This is due to the fact that the interference between the two fields results in a field whose E and H components are not in quadrature.

Another useful way of considering waveguides in their evanescent mode is with the impedance concept. For a dominant evanescent guide the input impedance is essentially a pure inductive reactance. Also, a Z matrix can be derived whose elements are imaginary. In lumped network theory it
is well known that total energy transfer can be obtained through such a complex reactive network by a suitable termination. In the case of a short length of cutoff waveguide, for example, it results in the voltage transfer coefficient being infinity for a termination such that its magnitude is equal to the magnitude of the characteristic impedance of the guide. The circuit behaves as a series resonant circuit.

Hupert and Ott\textsuperscript{20} used the filter effect of a cutoff section of waveguide to give an electromagnetic analog of the tunnel effect. The quantum mechanical particle energy \( E \) corresponds to the operating frequency in the analogous electromagnetic representation. The only numerical difference is that in quantum mechanics the phase constant \( K \) of the propagation sector and the corresponding attenuation constant \( \alpha \) of the evanescent sector are somewhat different functions of \( E \) and \( \omega \), respectively. The tunneling in the microwave experiment occurs at any input frequency between the cutoff frequencies of the dielectric-loaded and the air-filled waveguide.

In a later publication Hupert\textsuperscript{21} investigated the electromagnetic analog of a quantum mechanical potential function representing two finite barriers and two infinite walls. The analog consists of a cavity containing two "non-propagating"
sections. The calculation of series and parallel resonances of this system were outlined. A graphical method was given for the solution of two transcendental equations for the resonant frequencies.

In two almost identical papers Hupert\textsuperscript{22,23} considered a slightly different situation, modeling two finite barriers. At the center frequency the air sectors were described by an approximate $Z$ matrix and the transmission coefficient is calculated. Its poles are investigated and the dependence on the length $l$ is discussed. Losses were accounted for by parallel resistances in the equivalent lumped element circuit. The analogy with the quantum mechanical tunneling effect was established and it was shown that selective transmission or tunneling through long barriers is possible.

In another publication Hupert and Virgil\textsuperscript{24} calculated and measured the $Q$ factors of the same resonators assuming near exponential voltage distribution along the air-filled guide. The measured dependence of the $Q$ factors on the length $l$ showed good agreement with the calculations. In summary, it was shown that

i) an evanescent mode resonator relying on a pure TE mode is feasible,

ii) such a resonator behaves in agreement with the
approximate equivalent circuit theory applicable to the corresponding types of waveguides.

iii) the lumped capacitance effect obtained elsewhere in the form of studs and diaphragms\textsuperscript{25} can be realized by a short sector of waveguide in a propagating mode located between two evanescent sectors.

The tuning screws at the junctions between the propagating and cutoff waveguides and the screws used in the filter of Craven\textsuperscript{25} led Nicholson and Powell\textsuperscript{26} to believe that the band pass characteristics were, in fact, due to coupled TEM modes, supported by the screws in the direction normal to the direction of propagation. It was inferred that these filters are waveguide-coupled forms of combline filters. Such filters consist of a number of coupled cylindrical resonators between common and ground planes. Each resonator is somewhat less than $\lambda/4$ long, short circuited at one end and capacitively loaded at the other. The broad dimension of the cutoff waveguide is considered to be the ground plane separation of the combline structure. The resonant modes are launched by the screws at the junctions, TEM in nature, and not subject to cutoff effects. Two broad band filters designed on the basis of their theory show good agreement with the calculated response characteristics.
Although it seems reasonable to explain filters of the structure just described, on the basis of coupled TEM mode theory, this concept becomes invalid either in the case of waveguides periodically filled with dielectric\textsuperscript{24} or if the capacitance in the middle of the cutoff section can be introduced by some means other than screws. This was pointed out quantitatively by Craven.\textsuperscript{26} He showed that the screws at the end of the cutoff sections (used for tuning out the junction effect) which would be necessary for launching the TEM waves can be eliminated.

Although Mok\textsuperscript{27} showed that in the lossless case a conventionally capacitive iris becomes inductive below cutoff, it can be shown that in the presence of finite loss the capacitive iris remains capacitive over a small range of frequency below cutoff. Thus, using such an iris instead of a screw in the middle of the cutoff section seems to eliminate the similarity with a combline structure. A simple device, based on these arguments was constructed by Craven.\textsuperscript{26}

It is therefore quite clear that some evanescent waveguide filters can be explained on a TEM mode basis while some others can only be described by the evanescent mode concept. Also, referring to the work by Cullen\textsuperscript{19} the application of transmission line theory to non-propagating modes
does not introduce conceptual difficulties as inferred by Nicholson and Powell.
APPENDIX B

Computer Programs

All the programs were written in FORTRAN IV for the IBM 7040 and the B5500.

The cord method was chosen for the solution of the transcendental equations because the derivatives of these equations were not available.

The transmission line calculations were performed with a set of programs written for Smith Chart operations, the accuracy of which was checked against the solutions of Equation (4.4). A selection of these is briefly described below.

ZTOY is a program inverting the unnormalized impedance into its unnormalized admittance. The relevant equation for this transformation, written in a form suitable for our purpose is

\[
\tilde{Y} = \frac{1}{Z_o^2} \left[ \frac{1 + (-\tilde{\Gamma})}{1 - (-\tilde{\Gamma})} \frac{1}{Z_o} \right]
\]

where \( \tilde{Z} \) = unnormalized impedance

\( \tilde{\Gamma} \) = complex reflection coefficient

\( Z_o \) = characteristic impedance
The given impedance \( \tilde{Z} \) is first converted into the reflection coefficient \( \tilde{\rho} \) with subroutine ZTOS.\(^+\) Then \( \tilde{\rho} \) is rotated through 180° using ROTATE.\(^+\) Then the expression in parenthesis can be executed using STOZ\(^+\) with the rotated \( \tilde{\rho} \) as its input. Finally, the real and imaginary parts of the result are multiplied by \( 1/Z_o^2 \) in order to obtain the \( \tilde{Y} \) components.

\text{YTOZ} is the inverse of \text{ZTOY} and governed by the equation

\[
\tilde{\rho} = \left[ \frac{(\tilde{Y} Z_o^2) - \tilde{Z}_o}{(\tilde{Y} Z_o^2) + \tilde{Z}_o} \right] e^{j \tilde{\Pi}}
\]

The expression in the brackets can be obtained from \text{ZTOS}\(^+\) with \((\tilde{Y} Z_o^2)\) as its input. Then the multiplication by \( e^{j \tilde{\Pi}} \) is performed by rotating the result by \( \tilde{\Pi} \)° using ROTATE.\(^+\) Finally, the impedance \( \tilde{Z} \) can be obtained from \text{STOZ}.

The problem of impedance transformation through a cut-off section is illustrated in Figure 8. Continuity of the tangential \( E \) and \( H \) at the interfaces \( a \) and \( b \) requires the wave impedance to be continuous. Thus, across the boundary we have, for the impedance

\[
Z_L' = Z_L
\]

\(^+\)For these subroutines see Reference 18.
and for the reflection coefficient

\[ \vec{r} (1 = d) = \frac{\hat{Z}_L - \hat{Z}_a}{\hat{Z}_L + \hat{Z}_a} = \vec{r}_L \]

PSTEPF is the subroutine written for the last equation and is similar to STEP.+

In the cutoff section we have

\[ \vec{r} (1) = \vec{r}_L e^{-2\alpha L} \]

i.e., moving towards the boundary a leaves the angle of the reflection coefficient unchanged and modifies the magnitude exponentially.

Subroutine CSTEPP converts the reflection coefficient to the right of boundary a to the one on the left.
REFERENCES


