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STABLE TANGENTIAL HOMOTOPY EQUIVALENCES

by

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Poincaré originally conjectured that the integral homology groups of a closed compact oriented manifold were enough to determine it. This conjecture was enhanced by the fact that it is true for 1 and 2-dimensional manifolds. Poincaré, however, later gave examples of homology 3-spheres, i.e., non-simply connected 3-manifolds with the integral homology of S^3. He then proposed the now famous, still unsolved conjecture:

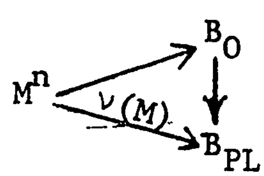
<u>The Poincaré Conjecture</u>: Every simply connected 3-manifold is homeomorphic to S^3.

If this conjecture is true a large class of 3-manifolds would be classified by their fundamental group. However, it is in this dimension that we run into the first examples of homotopy equivalent non-homeomorphic manifolds, the Lens spaces. So we see that even the homotopy type fails to determine the homeomorphism type of manifolds.

In more categorical language we are examining a relationship between two categories: first, the category of topological spaces which are the homotopy type of, say, CW complexes, and homotopy classes of maps (the Homotopy Category); and second, the category of topological manifolds and continuous maps (Top). We are able to prove every object of Top is an object of Hom. And we are asking if the map from Obj(Top) → Obj(Hom) is 1-1. There are many other questions that can be asked in this setting, and in order to examine these we will expand the range of categories at which we are looking. The PL category has as objects compact PL manifolds and as maps piecewise linear maps. The Differential Category (Diff) has as objects compact, C^∞ manifolds; and as maps, continuous functions. Of course, we have forgetful functors Diff → Top, Diff → Hom, PL → Top, and PL → Hom. Whitehead's triangulation theorem shows that there is a functor from Diff → PL. In this terminology we formulate many of the basic questions in geometric topology. Is the map Obj(PL) → Obj(Top) onto? (Are all topological manifolds triangulable?) Is the map 1-1? (The Hauptvermutung for PL manifolds). Is the map Obj(Diff) → Obj(PL) onto? (Are all PL-manifolds smoothable?) We could go on and on. Many of these questions have been answered, and progress is being made on the outstanding ones.

There are also several generalizations of the Poincaré conjecture to higher dimensions. For dimensions ≥ 5, it is known that all compact PL manifolds homotopy equivalent to S^n are PL homeomorphic to it. All topological manifolds homotopy equivalent to S^n are homeomorphic to it. However, there are numerous examples in dimensions ≥ 7 of smooth manifolds of the same homotopy type as S^n which are not diffeomorphic to it (of course, they are PL homeomorphic to it).

Two concepts have played a very important role in solving these problems as well as many others. The first is the use of bundles and classifying spaces. Each of the four categories has a natural category of bundles associated with it. (Vector bundles, PL microbundles or PL block bundles, topological microbundles and spherical fibrations [13]). The respective stable classifying spaces are B_O, B_{PL}, B_{TOP} and B_G. Of course, there are natural maps $B_O \to B_{PL} \to B_{TOP} \to B_G$. Each object in any of the four categories has a stable normal bundle (in the corresponding bundle theory) associated to it. It is possible to study objects in one category whose normal bundle has a higher structure. For instance, we can study PL manifolds whose PL normal bundle triangulates some vector bundle. This point of view was very fruitful in studying Diff $\to$ PL. As a matter of fact, Obj(Diff) $\to$ Obj(PL) has as image all such PL manifolds [10]. Thus we may reduce the problem of smoothing a given PL manifold, M^n, to that of lifting its classifying map $M^n \xrightarrow{\nu(M)} B_{PL}$ to B_O i.e.,



This problem in turn reduces to an Eilenberg-MacLane obstruction theory problem with obstructions in $H^i(M;\pi_i(PL/O))$.

In trying to solve the problem of the image of Diff $\to$ Hom the second technique emerged. First the image of Diff $\to$ Hom must be contained in P.D. $\subset$ Hom. P.D. is the subcategory of all Poincaré duality spaces. The appropriate notion of the normal bundle lifting is that the unique spherical fibration $\xi^N \to X^n$ with a Thom class $U_{n+N} \in H_{n+N}(T(\xi);Z)$ which is spherical, see [13], lifts to a vector bundle. Any element in $H^{-1}|U_{n+N}) \subset \pi_{n+N}(T(\xi))$ is called a normal invariant for X. Then if $\pi_1(X) = 0$ and the dimension of X is odd, or if the dimension of X is 4k and X satisfies the Hirzebruch Index formula, then X is realizable as a C^∞ closed manifold whose normal sphere bundle is ξ and which has the given normal invariant. X is always realizable as a PL manifold with given bundle off a point. The proof of this theorem requires Browder-Novikov smooth or PL surgery. (See e.g. [3] or [12]). The corresponding uniqueness theorem states that the bundle ξ over X and a normal invariant determines the resulting manifold up to sharping with a homotopy sphere. There are completely analogous statements and results in the PL category. Even though the statements in the two categories are identical, there is one significant difference. We have true uniqueness in the PL category since there is only one homotopy sphere in this category. This allows surgery techniques to bring more power to bear on problems in the PL category. Surgery on an n-manifold, M^n, consists in cutting out on $S^k \times D^{n-k}$ and attaching a copy of $D^{k+1} \times S^{n-k-1}$ along the boundary, $S^k \times S^{n-k-1}$. The resulting manifold is cobordant to the original one. The cobordism may be taken as

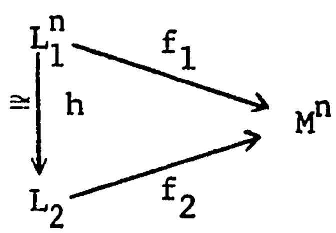
$M^n \times I \cup (k+1)$-handle, attached along $S^k \times D^{n-k-1} \times 1$.

Sullivan uses PL surgery along with the concept of fiber homotopically trivial PL bundles to study homotopy equivalences between PL n-manifolds for $n \geq 5$. He is able to give a collection of first order obstructions to such a map being homotopic to a PL homeomorphism. That is, to every $f : M^n \to N^n$, $\pi_1(M) = 0$, $n \geq 5$, and f a homotopy equivalence, there is a set of geometrically defined numbers whose vanishing is necessary and sufficient for f to be homotopic to a PL homeomorphism. Certainly, a necessary condition is that f be a tangential equivalence, i.e., $f^*\nu(N) \sim \nu(M)$, where $\nu(N)$ and $\nu(M)$ are the stable normal bundles of N and M respectively. This assumption, however, is not made in Sullivan's theory. One result of his research is a proof of the Hauptvermutung for simply connected n-manifolds ($n \geq 5$).

In this paper, we will examine, for certain manifolds, the effect of the assumption that f is a normal bundle map. We are able to show that certain of the numbers are actually 0 in this case. Then for cross products of spheres, we are able to determine exactly which sets of numbers correspond to tangential maps. Then in the case of $M^n = S^k \times S^{n-k}$ we are able to show that there is at most one π-manifold homotopy equivalent to M^n but combinatorially distinct from it. We then show exactly in terms of n and k when this occurs. Then we make some progress in ascertaining how close the stable normal bundle and the homotopy type of a PL manifold come to determining that manifold. In fact we show a result, analogous to Novikov's smooth theorem, that for a simply connected manifold the stable normal bundle along with its homotopy type determine it up to a finite number of choices.

I would like to thank all the people who have spent time and effort in helping me both in this research and in general. Special thanks must go to Professor M. L. Curtis, my thesis advisor, for his kindly help and for making this research possible, to Professor E. H. Connell for the endless hours he has spent teaching me, and to Professor D. Sullivan for explaining his recent research to me. I also wish to thank Miss Janet Gordon and Mrs. Nancy Singleton for their patient help in preparing the manuscript.

To begin with, we review some of the Sullivan Theory that will be needed. The presentation is a brief summary of the material in [14] and [15]. Let M^n be a compact, PL n-manifold with possibly empty boundary. We define an equivalence relation on the set of all pairs (L^n, f), where L^n is a compact, PL n-manifold, and $f : (L^n, \partial L^n) \to (M^n, \partial M^n)$ is a homotopy equivalence of pairs. The equivalence relation is $(L_1, f_1) \sim (L_2, f_2)$ if and only if there is a PL homeomorphism $h : L_1 \to L_2$ so that the following diagram commutes up to homotopy.

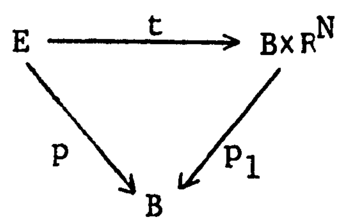


The homotopy is required to send ∂L_1 into ∂M at each stage. We denote the set of equivalence classes by $ht(M^n)$, read homotopy triangulations of M^n. Let G(n) be the space of homotopy equivalences of S^{n-1} onto itself. We will also use G(n) to denote its semisimplicial analogue, the singular complex of G(n). PL(n) is the semisimplicial complex of PL germ isomorphisms defined by Milnor [11]. G(n) has PL(n) as a subgroup complex, and we form G(n)/PL(n). Stabilizing gives us G/PL. This semisimplicial complex, as well as PL(n), PL, and B_{PL}, have finite π_1 and countable homotopy groups. Thus by [18] they are realizable as locally finite simplicial complexes.

<u>THEOREM</u>: There are natural maps $PL \xrightarrow{i} G$, $G \xrightarrow{\pi} G/PL$, $G/PL \xrightarrow{\eta_{PL}} B_{PL}$, and $B_{PL} \xrightarrow{j} B_G$, so that the following is an exact sequence of functors:

$$[\ ,PL] \xrightarrow{i_*} [\ ,G] \xrightarrow{\pi_*} [\ ,G/PL] \xrightarrow{\eta_*} [\ ,B_{PL}] \xrightarrow{j_*} [\ ,B_G].$$

Define a G/PL-bundle as a PL-bundle, $\{E \xrightarrow{p} B\}$, together with a fiber homotopy trivialization, $t : E \to B \times R^N$. $t|E-\{0\text{-section}\} : E-(0\text{-section}) \longrightarrow B\times R^N-\{0\}$ is of degree ± 1 on each fiber, and



is properly fiberwise homotopy commutative. The equivalence relation on this class of bundles is concordance. From the above exact sequence we see that G/PL classifies G/PL-bundles.

Define P_k, the surgery obstruction in dimension k, as the framed cobordism classes of framed, almost closed, PL submanifolds of D^{k+N} $(N>k)$. P_* is the mod four sequence $0, Z_2, 0, Z$ for $* \equiv 1,2,3,$ or $0 \bmod 4$. The element in Z corresponding to a framed 4K-manifold is its index divided by 8. The element in Z_2 corresponding to a framed (4K+2)-

manifold is its Kervaire Invariant [2]. To each G/PL-bundle, ξ, over a closed n-manifold, M^n, we assign an element of P_n called the surgery obstruction of ξ over M^n. Let ξ be

$$\begin{array}{ccc} E & \xrightarrow{t} & M\times R^N \\ \downarrow{\scriptstyle p} & & \\ M & & \end{array}$$

Then there is a proper homotopy of t to a map t′ which is transverse regular with respect to $M^n \times 0$, and $t'^{-1}(M^n \times 0) = M'^n$. The element in P_n is the difference of the indices divided by 8 or the Kervaire Invariant of $p : M'^n \to M^n$ depending on the modulus of n. If $n \geq 5$, and $\pi_1(M^n)=0$, t is properly homotopic to some t′ such that $t'^{-1}(M^n \times 0) \xrightarrow{p} M$ is an $([\frac{n}{2}]-1)$ connected map. The element of P_n is the only obstruction to finding a t′ properly homotopic to t with $t'^{-1}(M^n \times 0)$ homotopy equivalent to M^n. So to any map $f : M^n \to G/PL$ we have assigned an element in P_n. This element is 0 if (M^n, f) is a boundary in $\Omega_n(G/PL)$ [5], and thus the map factors to give a map $\mathcal{S} : \Omega_n(G/PL) \to P_n$, which is a homomorphism. $\mathcal{S}$ is called the surgery obstruction.

THEOREM: The composition $\pi_n(G/PL) \xrightarrow{\text{HUREWICZ}} \Omega_n(G/PL) \xrightarrow{\mathcal{S}} P_n$ is an isomorphism for all $n \neq 4$. It is monic and onto 2Z for n=4.

COROLLARY: $\pi_n(G/PL) \cong P_n$ for all n.

Now we define a natural transformation, ρ, from ht(M^n) to [M^n,G/PL]. Given $L^n \xrightarrow{f} M^n$, a homotopy equivalence, let $f' : M^n \to L^n$ be a homotopy inverse for f. f' can be approximated by an embedding, $i : M^n \subset L^n \times R^N$, for N>>n. The PL normal bundle of M^n in $L^n \times R^N$,E, (which exists by [11]) is PL homeomorphic to $L^n \times R^N$ by the PL half open h-cobordism theorem [8].

$$\begin{array}{ccccc} E & \xrightarrow[\cong]{h} & L^n \times R^N & \xrightarrow{f \times 1} & M^n \times R^N \\ \downarrow{\scriptstyle \pi} & & & & \\ M & & & & \end{array}$$

is the G/PL-bundle classified by $M \xrightarrow{\rho([L,f])} G/PL$.

<u>THEOREM</u>: (1) If $\partial M \neq \phi$, n≥6, and $\pi_1(M) = \pi_1(\partial M) = 0$, this natural transformation, ρ, is a natural equivalence of functors. (2) If $\partial M = \phi$, n≥5, and $\pi_1(M)=0$, we have an exact sequence $0 \longrightarrow ht(M^n) \xrightarrow{\rho} [M^n,G/PL] \xrightarrow{s} P_n$. Notice that $\eta_{PL} \circ \rho([L,f])$ classifies the stable PL difference normal bundle of L^n and M^n.

It is possible to assign surgery obstructions over more general polyhedra than PL manifolds. Let M^n be a PL manifold with ∂M^n isomorphic to k disjoint copies of L^{n-1}. Look at the polyhedron, N, obtained from identifying the k copies of L^{n-1} together. Any polyhedran constructed in this manner is a Z_k-manifold. The Bockstein of N^n, δN, is by definition L^{n-1}. If $n=4\ell$, and ξ is a G/PL-bundle over N^n, we can define a surgery obstruction of ξ. It will be in Z_k. Again assuming $\pi_1(L) = \pi_1(M) = 0$ and

$n \geq 6$, it will be the only obstruction to finding t' properly homotopic to the trivializing map, t, which is transverse regular to $N \times 0$ with preimage homotopy equivalent to N^n. The proof of this is the same as the proof that the obstruction over a bounded manifold can be pushed off the boundary -- except for the fact that we must always change things by a multiple of k. The obstruction thus is only defined modulo k.

The characteristic variety theorem is a deep and central result in the study of G/PL-bundles and homotopy triangulations. It says a map $M^n \to G/PL$ is determined by surgery obstructions of the map restricted to certain submanifolds.

<u>THEOREM</u>: (Characteristic Variety). If $n \geq 6$, and $\pi_1(M) = \pi_1(\partial M) = 0$, then any G/PL-bundle, ξ, over M^n is determined by surgery obstructions of ξ restricted to a characteristic variety. A characteristic variety is a collection of singular submanifolds of M^n. They are broken up into 4 parts.

(i) A collection of $(4i+2)\mathbf{Z}_2$-manifolds $\{K_\alpha\}$ with $f : \cup K_\alpha \to M$ satisfying $\{f_*[K_\alpha]\}_\alpha$ is a basis of $A \oplus_{i>0} H_{4i+2}(X;Z_2)$. $A \subseteq H_2(X;Z_2)$ is the subgroup dual to $\text{kernel}(Sq^2 : H^2(M,Z_2) \to H^4(M,Z_2))$.

(ii) A collection of $4i$ $\mathbf{Z}_2r$-manifolds, $\{N_\alpha\}$ with $f : \underset{\alpha}{\cup} N_\alpha \to M^n$ where $\{f_*([\delta N_\alpha])\}_\alpha$ is a basis for [2 torsion of $\underset{i}{\oplus} H_{4i-1}(M;Z)$].

(iii) A collection $\{V_\alpha\}$ of Z_pr-manifolds for each odd prime p, of dimension 4N, so that $f : \underset{\alpha}{\cup} V_\alpha \to M$ and $\{f_\alpha[\delta V_\alpha]\}_\alpha$ is a basis of the odd torsion subgroup of $\Omega_{4*-1}(M) \underset{\Omega_*}{\otimes} \mathbf{Z}_{(odd)} \cong \underset{-1}{K}(M)$.

(iv) A collection $\{C_\alpha\}$ of singular closed oriented 4i-manifolds $g : \underset{\alpha}{\cup} C_\alpha \to M$ so that $\{g_*[C_\alpha]\}_\alpha$ is a basis for $\underset{i}{\oplus} H_{4i}(M,Z)/\text{Torsion}$.

There is a second half of the theorem which states that, except for dimension 4, all possible sets of surgery obstructions over the characteristic variety are realizable as obstructions of some homotopy equivalence. There is another way to view the surgery obstructions on submanifolds. Let $f : L^n \to M^n$ be a homotopy equivalence, and $N^k \subset M^n$ a submanifold. Then the surgery obstruction of f along N is defined by finding $\tilde{f}$ homotopic to f, transverse regular to N, and taking the difference of the indices divided by 8 or the Kervaire Invariant. These two points of view may be identified, i.e., the surgery obstruction of f on N^k is the surgery obstruction of $(\xi(L,f))|N$. This completes the résumé of Sullivan Theory.

We wish to study a subset of $ht(M^n)$. This is the subset of all $\{[L^n,f]\}$ so that f is a stable tangential homotopy equivalence. Clearly if $(L_1,f_1)\sim(L_2,f_2)$, f_1 is a stable tangential map if and only if f_2 is. We denote the subset by $stht(M^n)$. It corresponds to all $f : M^n \to G/PL$ which have 0 surgery obstruction, and which, when followed by η_{PL}, are trivial. This is because $\eta_{PL} \circ f$ classifies the stable PL difference normal bundle.

Proposition 1: If ξ is a G/PL-bundle over L^{4k} which is PL trivial, and L^{4k} is a closed C^∞ manifold, then the surgery obstruction of ξ is 0.

Proof:

$$\begin{array}{ccc} E & \longrightarrow & L \times R^N \\ \downarrow{\scriptstyle p} & & \\ L & & \end{array} \quad \text{is G/PL equivalent to} \quad \begin{array}{ccc} L \times R^N & \xrightarrow{t} & L \times R^N. \\ \downarrow{\scriptstyle p_1} & & \\ L & & \end{array}$$

We find t' properly homotopic to t which is transverse regular with respect to $L \times 0$. $t'^{-1}(L \times 0) = L'$ has a trivial PL normal bundle map in $L \times R^N$ by transversality. Thus $\tau(L') \oplus \theta_N \cong \tau(L \times R^N)|L'$. Let $p = p_1|L'$. Then we have $p^*\tau(L) \oplus \theta_N \cong \tau(L') \oplus \theta_N$. So that $p^*\tau(L) \underset{s}{\sim} \tau(L')$.

Thus $p^*(p_i(L)) = p_i(L')$ where p_i is the i^{th} Pontryagin class. Now let $L(p_i)$ be the Hirzebruch L classes. The Index formula tells us that $I(L') = \langle L(p_i(\tau(L')), [L'] \rangle$ and $I(L) = \langle L(p_i(\tau(L)), [L] \rangle$. But we have $L(p_i(L')) = p^*(L(p_i(L')))$. Thus, $\langle L(p_i(\tau(L')), [L'] \rangle = \langle p^*(L(p_i(\tau(L)))), [L'] \rangle$ $= \langle L(p_i(\tau(L))), p_*[L'] \rangle = \langle L(p_i(\tau(L))), [L] \rangle$. The result follows. Q.E.D.

The Kervaire Invariant of a framed manifold is defined as follows. First assume M^{4k+2} is 2k-connected. Then define the intersection pairing $H_{2k+1}(M) \otimes H_{2k+1}(M) \to Z$, by $a \otimes b \mapsto \langle PD^{-1}(a) \cdot PD^{-1}(b), [M] \rangle$. ($PD : H^*(M) \to H_{4k+2-*}(M)$ is the Poincaré duality isomorphism.) This is a bilinear, nonsingular, skew symmetric pairing. $H_{2k+1}(M)$ thus has a symplectic basis, i.e., $x_1, \ldots, x_r, y_1, \ldots, y_r$ so that

$x_i \cdot x_j = y_i \cdot y_j = 0$, $x_i \cdot y_j = \delta_{ij}$. We now define $\Phi : H_{2k+1}(M) \to Z_2$. Let $\alpha \in H_{2k+1}(M)$. α is spherical since M is 2k-connected. Let $f_\alpha : S^{2k+1} \to M$ be an embedding representing α. The normal bundle is a stably trivial 2k+1 bundle over S^{2k+1}. It is thus an element in Z_2. We take this as $\Phi(\alpha)$. The normal bundle is independent of the embedding so Φ is well defined. It can be shown that $\Phi(x+y) = \Phi(x) + \Phi(y) + [x \cdot y]_2$. Thus a proposition of Arf tells us $\Phi(M) = \sum_i \Phi(x_i) \cdot \Phi(y_i)$, for $x_1, \ldots, x_r, y_1, \ldots, y_r$ a sympletic basis, is independent of the basis. We have thus defined the Kervaire Invariant of a 2k connected, 4k+2, framed manifold. It can also be shown that if M^{4k+2} and N^{4k+2} are 2 such manifolds which are framed cobordant, $\Phi(M) = \Phi(N)$. To define the invariant of any 4k+2 framed manifold M, $\pi_1(M)=0$, recall that M is framed cobordant to a 2k connected framed manifold M′. Define $\Phi(M) = \Phi(M')$. By the above remark $\Phi(M)$ is well defined and is a framed cobordism invariant.

Let $\tau(S^{2k+1})$ denote the tangent D^{2k+1} bundle over S^{2k+1}. Embed

$$\begin{array}{ccc} D^{2k+1} \times D^{2k+1} & \subset & \tau(S^{2k+1}) \\ \downarrow p_1 & & \downarrow \pi \\ D^{2k+1} & \subset & S^{2k+1} \end{array}$$

as the bundle over a D^{2k+1} in S^{2k+1}. Take two copies of $\tau(S^{2k+1})$ and identify the $D^{2k+1} \times D^{2k+1}$ together by $(x,y) \mapsto (y,x)$. This is plumbing two spheres together. The boundary of the resulting manifold, X^{4k+2}, is a homotopy sphere. The Kervaire Invariant Conjecture is that it is not diffeomorphic to S^{4k+1}. Notice that X^{4k+2} is a π-manifold and if ∂X is diffeomorphic to S^{4k+1}, $X \underset{\partial X}{\cup} D^{4k+2}$ will be a π-manifold with Kervaire Invariant 1.

Proposition 2: If ξ is a G/PL-bundle over a π-manifold L^{4k+2}, and ξ is PL trivial, then the surgery obstruction for ξ is 0 provided that the Kervaire Invariant Conjecture [2] is true for (4k+2).

Proof: Again, we modify t by a homotopy until it is transverse regular with respect to $L^{4k+2} \times 0$. Now $t'^{-1}(L^{4k+2} \times 0)$ is trivial PL normal bundle in $L^{4k+2} \times R^N$, and thus is a π-manifold.

Claim: If the Kervaire Invariant Conjecture is true for (4k+2) if and only if all closed (4k+2) π-manifolds have 0 Kervaire Invariant.

Proof of Claim: Suppose M^{4k+2} is a π-manifold with Kervaire Invariant 1. Take a cell out of M. The resulting manifold has as boundary S^{4k+1}, and may be surgered modulo its boundary to a copy of X^{4k+2}. Thus ∂X would be diffeomorphic to S^{4k+1}. This is a contradiction.

But if M^{4k+2} and N^{4k+2} are π-manifolds and $f : M \to N$ is a degree 1 map, the Kervaire Invariant of f = the Kervaire Invariant of M — the Kervaire Invariant of N. This is true since $H_{2k+1}(M) \cong \text{Ker } f \oplus H_{2k+1}(N)$ and if $a \in \text{Ker } f$ and $b \in H_{2k+1}(N)$, $a \cdot b = 0$. Then we may take a symplectic basis for ker f and one for $H_{2k+1}(N)$ and together they give one for $H_{2k+1}(M)$. Thus the Kervaire Invariant of $p|t'^{-1}(L^{4k+2}\times 0)$ is 0 if the Kervaire Invariant Conjecture is true for 4k+2. Q.E.D. for the claim.

But if the Kervaire Invariant Conjecture is true for (4k+2), all (4k+2) dimensional π-manifolds have 0 Kervaire Invariant. Q.E.D.

Thus if $L^n \xrightarrow{f} M^n$ is a stable tangential homotopy equivalence, the numbers associated to the 4k-submanifolds by the G/PL-bundle, $\rho([L,f])$, must all be 0. This is so because they are surgery obstructions of the bundle restricted to 4k-submanifolds, and the bundle is PL trivial.

Corollary: $\mathrm{stht}(M^n)$ is finite if $n\geq 6$, and $\pi_1(M^n) = \pi_1(\partial M^n) = 0$.

<u>Proof</u>: There are only a finite number of "manifolds" in each of the four parts of the characteristic variety theorem. Each of the ones in the first three parts has a number in some $Z_p r$ assigned to it. Thus there are only a finite number of possibilities here. Those in the fourth category must have 0 assigned to them by the above remark. The corollary follows.

<u>Corollary</u>: There are only finitely many compact PL n-manifolds stably tangentially homotopy equivalent to M^n. (Assuming $n\geq 6$ and $\pi_1(M^n) = \pi_1(\partial M^n) = 0$).

<u>Proof</u>: Each such manifold represents a different element in $\mathrm{stht}(M^n)$. This implies a result of Novikov's [12].

Corollary: If $n \geq 6$, $\pi_1(\partial M) = \pi_1(M) = 0$, and M^n is a smooth compact manifold, then there are only finitely many smooth manifolds stably tangentially equivalent to M^n (up to diffeomorphism).

Proof: There are only finitely many such combinatorial manifolds and each of these has only finitely many smoothings [10]. Q.E.D.

For a certain class of manifolds we are able to completely specify those sets of numbers which, when assigned to the characteristic variety, correspond to stable tangential homotopy equivalences.

Proposition 3: $\pi_n(G/PL) \to \pi_n(B_{PL})$ has 0 kernel unless $n=4k+2$ and the Kervaire Invariant Conjecture is false for $(4k+2)$. In the latter case the kernel is Z_2.

Proof: Suppose $\alpha \in \pi_n(G/PL)$ is an element in the kernel of η_{PL}. Since S^n is a π-manifold, propositions 1 and 2 show that the surgery obstruction of $\alpha : S^n \to G/PL$ is 0, provided $n=4k$ or $n=4k+2$ and the Kervaire Invariant Conjecture is true for $(4k+2)$. But $\pi_n(G/PL) \longrightarrow \Omega_n(G/PL) \xrightarrow{s} P_n$ is monic, and $\alpha=0$ in these cases. Let X^{4k+2} be the Kervaire manifold with boundary a homotopy sphere, Σ^{4k+1}. If the Kervaire Invariant Conjecture is false, $X^{4k+2} \cup_{S^{4k+1}} D^{4k+2} = \hat{X}^{4k+2}$ is a smooth manifold.

Its tangent bundle is stably trivial off a point, since X is a π-manifold. The only obstruction to trivializing the tangent bundle lies in $H^{4k+2}(\hat{X}^{4k+2};\pi_{4k+1}(0)) \cong \pi_{4k+1}(0)$.

But it is in the kernel of the J-homomorphism which is monic in these dimensions. Thus $\hat{X}^{4k+2}$ is a π-manifold, (see [4]). Thus there is a map $f : X^{4k+2} \cup_{\Sigma} D^{4k+2} \longrightarrow S^{4k+2}$ with trivial normal bundle. The second statement follows. (From now on let $\hat{X}^{4k+2}$ denote $X^{4k+2} \cup_{\Sigma} D^{4k+2}$).

Proposition 4: If L^n is a closed π-manifold, and ξ is a G/PL-bundle over L^n which is PL trivial on the complement of a point and has 0 surgery obstruction, then ξ is PL trivial.

Proof: Let ξ be

$$\begin{array}{ccc} E & \xrightarrow{t} & L\times R^N \\ \downarrow{\scriptstyle \pi} & & \\ L & & \end{array}$$

As usual, we find a proper homotopy of t to t' which is transverse regular to $L\times 0$ with $t^{-1}(L\times 0)$ homotopy equivalent to L, $\pi : t^{-1}(L\times 0) \to L$ is a bundle map off a point. So $t^{-1}(L\times 0)$ is an almost π-manifold, i.e., $t^{-1}(L^n\times 0)$-int D^n is a π-manifold. Thus it has a smoothing and the smoothing induced on its boundary is completely determined by the index or Kervaire Invariant of $t'^{-1}(L\times 0)$-int D^n (see KM [9]). But this is the same as that of $L^n\times 0$-int D^n. Thus $\partial(t'^{-1}(L\times 0)\text{-int } D^n)$ is diffeomorphic to S^{n-1}. So $t'^{-1}(L\times 0)$ is a smoothable almost π-manifold. So, if $n\neq 0,1,2$ or $4 \bmod 8$, it is a π-manifold. If $n\equiv 0$ or $4 \bmod 8$ it is a π-manifold by an Index argument. If $n\equiv 1$ or $2 \bmod 8$, the obstruction to $t'^{-1}(L\times 0)$ being a π-manifold is in the kernel of the J homomorphism which is monic. So it vanishes, and

$t'^{-1}(L\times 0)$ is a π-manifold. Thus E is a π-manifold which means $\begin{matrix} E \\ \downarrow \\ L \end{matrix}$ is PL trivial. Q.E.D.

Now we consider $S^{k_1}\times\cdots\times S^{k_n}$, $1<k_1\leq\cdots\leq k_n$. The characteristic variety for it is the union of all $S^{k_{i_1}}\times\cdots\times S^{k_{ij_o}}$, such that $j_o<n$, $k_{i_1}<\cdots<k_{ij_o}$, and $(\sum_{1\leq j\leq j_o} k_{ij})$ is even. All of these are π-manifolds. So if $f : M\to S^{k_1}\times\cdots\times S^{k_n}$ is a stable tangential homotopy equivalence, the number assigned to $S^{k_{i_1}}\times\cdots\times S^{k_{ij}}$ is 0, unless $k_{i_1}+\cdots+k_{ij} = (4\ell+2)$, and the Kervaire Invariant Conjecture is false for $(4\ell+2)$. We call such a dimension a "bad" or "bad Kervaire" dimension. All other dimensions are "good".

<u>Proposition 5</u>: If $f : M\to S^{k_1}\times\cdots\times S^{k_n}$ is a homotopy equivalence which assigns 0 to all components of the characteristic variety which have "good" dimension, then f is a stable tangential homotopy equivalence.

<u>Proof</u>: <u>Claim</u>: If A and B are finite CW complexes $\Sigma(A\times B) \overset{h.e.}{\simeq} \Sigma(A\wedge B)\vee\Sigma A\vee\Sigma B$. For a proof of this see [7] p. 104. This enables us inductively to show that $\Sigma(S^{k_1}\times\cdots\times S^{k_n})$ is homotopy equivalent to a wedge of spheres. We will have one sphere, S^{ℓ}, for each $S^{k_{i1}}\times\cdots\times S^{k_{ij}}$, $i1<\cdots<ij$ where $\ell = (\Sigma k_{im}) + 1$. We have the classifying map $\varphi_f : S^{k_1}\times\cdots\times S^{k_n} \longrightarrow G/PL$. Suppose there are spaces

B_G and $B_{G/PL}$ and $\psi : B_G \longrightarrow B_{G/PL}$ so that

$$\begin{array}{ccc} \Omega B_G & \overset{\text{h.e.}}{\cong} & G \\ \downarrow \Omega\psi & & \downarrow \pi \\ \Omega B_{G/PL} & \cong & G/PL \end{array}$$

Now

$$\begin{array}{ccccc} & \varphi_f & \longrightarrow & & \tilde{f} \\ [S^{k_1}\times\cdots\times S^{k_n}, G/PL] & \cong [S^{k_1}\times\cdots\times S^{k_n}, \Omega B_{G/PL}] & \xrightarrow{\cong} & & [\Sigma(S^{k_1}\times\cdots\times S^{k_n}), B_{G/PL}] \\ \uparrow \pi_* & & & & \uparrow \psi_* \\ [S^{k_1}\times\cdots\times S^{k_n}, G] & \cong [S^{k_1}\times\cdots\times S^{k_n}, \Omega B_G] & \xrightarrow{\cong} & & [\Sigma(S^{k_1}\times\cdots\times S^{n_n}, B_G]. \end{array}$$

$\tilde{f}$ will map $\vee_{\ell} S^{\ell} \longrightarrow B_{G/PL}$; thus we may consider $\tilde{f} \in \oplus_{\ell} \pi_{\ell}(B_{G/PL})$.

Now we claim in good Kervaire dimensions +1, $\tilde{f}$ is zero on the corresponding spheres. If this is true then $\tilde{f}$ will factor through B_G, since in all bad Kervaire dimensions +1, $\psi_{\#}$ is onto. So that $\tilde{f} = \psi_*(g)$ for some g: $\Sigma(S^{k_1}\times\cdots\times S^{k_n}) \longrightarrow B_G$. Then $\pi_*(\Omega_g) = \varphi_f$. Thus $S^{k_1}\times\cdots\times S^{k_n} \xrightarrow{\varphi_f} G/PL \xrightarrow{\eta} B_{PL}$ is homotopic to 0 and corresponds to a tangential homotopy equivalence.

We show $\tilde{f}$ has the required form by induction on n. The result is trivial for n=1. Assume it is true for all $n \leq n_0 - 1$ and $\varphi_f : S^{k_1}\times\cdots\times S^{k_{n0}} \longrightarrow G/PL$. Let $\tilde{f} : \Sigma\, S^{k_1}\times\cdots\times S^{k_{n0}} \longrightarrow B(G/PL)$ be the "deloop" of

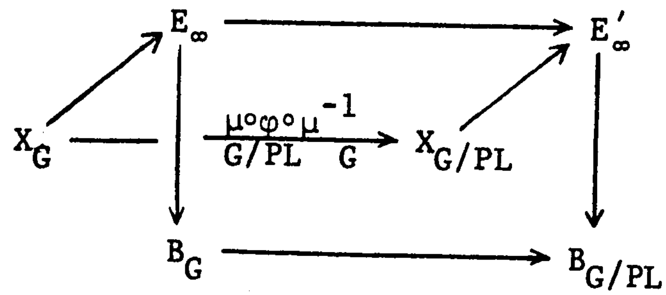
φ_f. $\tilde{f} \in \bigoplus_i \pi_i(B(G/PL))$. By induction $\tilde{f}|$(lower dimensional spheres) factors into B_G. Thus $\varphi_f|S^{k_1}\times\cdots\times S^{k_n}-\{pt\}$ factors through G. $(\Sigma(S^{k_1}\times\cdots\times S^{k_n}-\{pt\} = \vee$ spheres.)

If $\sum_{i=1}^{n_0} k_i$ is a "good" dimension, the surgery obstruction of φ_f will be 0 and Proposition 4 implies φ_f factors through G. Thus $\tilde{f}$ factors through B_G, but the only map $S^{\Sigma k_i+1} \longrightarrow B(G/PL)$ which factors through B_G is the trivial one in "good" dimensions +1. The claim is established. Actually a much stronger statement is true. The value of $\tilde{f}$ on any sphere in P_* is equal to the surgery obstruction of φ_f along the corresponding product of spheres. Since as H-spaces $G/PL = \Omega B(G/PL)$ this shows the surgery obstructions on the product of spheres is additive. This is not true for general manifolds.

It remains only to "deloop" $G \xrightarrow{\varphi} G/PL$. Boardman and Vogt, [1], prove G, G/PL are homotopy equivalent to infinite loop spaces X_G, $X_{G/PL}$. These, then, are associative on the nose with units

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G/PL \\ \cong\downarrow\mu_G & & \cong\downarrow\mu_{G/PL} \\ X_G & \longrightarrow & X_{G/PL} \end{array}.$$

They also show $\mu_{G/PL}\circ\varphi\circ\mu_G^{-1}$ may be taken as a map preserving all this structure. Under these circumstances the construction of Dold and Lashof, [6], gives a map between fibrations



where E_∞ and E'_∞ are contractible. The map on the base spaces is the required "deloop" for φ.

Q.E.D.

In the case of $S^k \times S^\ell$ we can give a direct geometric proof of this proposition using the previous one. We have a G/PL bundle, ξ, over $S^k \times S^\ell$ which has 0 surgery obstruction, and which is PL trivial over S^k and over S^ℓ. It is thus PL trivial over $S^k \vee S^\ell$ and then over $S^k \times S^\ell - \{pt\}$. Application of the previous proposition gives the result.

The characteristic variety theorem tells us all possible sets of numbers are realizable in this case. This implies that $\#(\text{st ht}(S^{k_1} \times \cdots \times S^{k_n})) = 2^\ell$, where ℓ is the number of "bad" dimensions of the form $k_{i_1} + \cdots + k_{i_j}$, $j<n$, $i_1 < \cdots < i_j$. Known "bad" dimensions are 2,6,14,30,62. The only possible "bad" dimensions are those of the form 2^k-2, [2].

Two things can account for non-zero elements in $\text{stht}(M^n)$. One is the existence of manifolds which are stably tangentially homotopy equivalent to M^n, but combinatorially distinct from it. The other is the existence of a self tangential homotopy equivalence of M^n which is not homotopic to a PL homeomorphism.

Now let $k_1+\cdots+k_n = n_0$, and suppose we can realize the element in $\mathrm{stht}(S^{k_1}\times\cdots\times S^{k_n})$ which is 1 on $S^{k_{i_1}}\times\cdots\times S^{k_{i_{j_0}}}$ and 0 on all other components of the characteristic variety by a self homotopy equivalence. (This implies that $k = \sum_{j=1}^{j_0} k_{i_j}$ is a "bad" dimension). Let $f : S^{k_1}\times\cdots\times S^{k_n} \longrightarrow S^{k_1}\times\cdots\times S^{k_n}$ be such a map. We may assume f is transverse regular to $S^{k_{i_1}}\times\cdots\times S^{k_{i_{j_0}}}$ with preimage M^k. Now M^k is a π-manifold with Kervaire Invariant 1. By the uniqueness theorem in smooth surgery, it is framed cobordant to a framing on $\hat{X}^k \# \tilde{S}^k$ for some $\tilde{S}^k \in \Gamma_k$. Now M^k smoothly embeds in S^{n_0+1} with trivial normal bundle (since $S^{k_1}\times\cdots\times S^{k_n}$ does). Thus using the Thom construction [5],[16], we may define $\eta : S^{n_0+1} \longrightarrow S^{n_0+1-k}$ so that $\eta \in \pi_{n_0+1}(S^{n_0+1-k})$ suspends to the coset of X^k in $\pi_k^s/F\Gamma_k$, where $F\Gamma_k$ is the subgroup of framings on homotopy k-spheres. We call such a pair (k, n_0+1) _a pair of type 1_. This gives a certain amount of negative information about which sets of numbers assigned to the characteristic variety are realizable by self homotopy equivalences. We have proven:

Theorem: If k is a "bad" dimension and (k,n+1) is not of type 1, and $M^n = S^{k_1}\times\cdots\times S^{k_{j_0}}\times S^{k_{j_0+1}}\times\cdots\times S^{k_r}$ and $\sum_{i=1}^{j_0} k_i = k$, then the manifold L^n realizing 1 on $S^{k_1}\times\cdots\times S^{k_{j_0}}$ and 0 on all other components of the characteristic variety is a π-manifold homotopy equivalent to M^n but combinatorially distinct from it.

For $S^k\times S^{n-k}$ we are able to prove a partial converse. First, recall we have shown $\mathrm{stht}(S^k\times S^{n-k})$ has

$$\begin{cases} 1 \text{ element, if } k \text{ and } n-k \text{ are "good" dimensions} \\ 2 \text{ elements, if exactly one of } k \text{ and } n-k \text{ is a "good" dimension} \\ 4 \text{ elements, if both } k \text{ and } n-k \text{ are "bad" dimensions} \end{cases}$$

We are assuming $1<k\leq n-k$ and $n\geq 5$. The characteristic variety is the even component(s) of $S^k \cup S^{n-k}$. We shall be talking about assigning certain combinations of 0's and 1's to S^k and S^{n-k}. Whenever we assign 1 to a manifold we are implicitly assuming that its dimension is "bad". However, if we assign 0 to a manifold we do not require that it be in the characteristic variety, i.e., have even dimension.

Embed $\hat{X}^k \subset S^n$ with trivial normal bundle. This is possible since $\hat{X}^k$ is a π-manifold and $2k\leq n$. Define $\eta : S^n \to S^{n-k}$ by the Thom construction applied to the framed submanifold $\hat{X}^k$. It sends $\hat{X}^k$ onto $x_0 \in S^{n-k}$ transversely. Now define $\eta' : S^n \to S^k\times S^{n-k}$ by $\eta'(x) = (p_0,\eta(x))$ and let

$Id\#\eta' : S^k \times S^{n-k} \# S^n \longrightarrow S^k \times S^{n-k}$. This is a tangential homotopy equivalence. It is transverse regular to $S^k \times \{x_0\}$ with preimage $S^k \cup \hat{X}^k$ (which has Kervaire Invariant 1). It is also transverse regular with respect to $p' \times S^{n-k}$ with preimage S^{n-k} (for $p' \in S^k$, $p' \neq p_0$). Thus $Id\#\eta'$ is a self homotopy equivalence representing 1 on S^k and 0 on S^{n-k}. This same argument shows that if (n-k,n) is of type 1, 0 on S^k and 1 on S^{n-k} is realizable by a self homotopy equivalence, $f : S^k \times S^{n-k} \longrightarrow S^k \times S^{n-k}$. It also shows that 1 on S^k and 1 on S^{n-k} is realizable by a map of the same combinatorial manifold into $S^k \times S^{n-k}$ as is 0 on S^k and 1 on S^{n-k}. Thus we have completely determined $\operatorname{stht}(S^k \times S^{n-k})$ unless n-k is "bad" and (n-k,n) is not of type 1, while (n-k,n+1) is of type 1. Note that if (a,b) is of type 1 then (a,b+1) is also. We have proved:

<u>Theorem</u>: $1 < k \leq n-k$, $n \geq 5$

(1) If k and n-k are "good" dimensions every $f : M \to S^k \times S^{n-k}$ which is a stable tangential homotopy equivalence is homotopic to a PL homeomorphism.

(2) If k is "bad" and n-k is "good" then all π-manifolds homotopy equivalent to $S^k \times S^{n-k}$ are PL homeomorphic to it. There is, however, a self homotopy equivalence which is not homotopic to a PL homeomorphism.

(3) If n-k is "bad" and (n-k,n) is of type 1 then all π-manifolds homotopy equivalent to $S^k \times S^{n-k}$ are PL homeomorphic to it; again there are self homotopy equivalences not homotopic to PL homeomorphisms.

(4) If n-k is "bad" and (n-k,n+1) is not of type 1, then there is exactly one PL π-manifold homotopy equivalent to $S^k \times S^{n-k}$ but combinatorially distinct from it.

As we said before, this leaves undetermined only the case when n-k is a "bad" dimension and n+1 is the smallest integer r so that (n-k,r) is of type 1. Toda's tables [17] tell us:

$$(6,n) \text{ is of type } 1 \Leftrightarrow n \geq 10$$

$$(14,n) \text{ is of type } 1 \Leftrightarrow n \geq 22$$

$$(30,n) \text{ is of type } 1 \Rightarrow n \geq 46.$$

Mahowald proved (30,53) is of type 1.

Novikov proved, using smooth techniques, that there is a π-manifold homotopy equivalent to $S^2 \times S^6$ but combinatorially distinct from it. This fits our results as (6,9) is not of type 1.

Lemma 1: Let k be a "bad" dimension. If (k,n) is of type 1, $n \geq \frac{3k}{2} + 1$.

Proof: Browder in [2] shows that any element of $\pi_{N+k}(S^N)$, N> k, corresponding to a manifold of Kervaire Invariant 1 is detected by a second order cohomology operation, namely the operation determined by h_r^2 in E^2 of the Adams spectral sequence for $2^{r+1}-2 = k$. Thus the homotopy element must be of filtration 2 in the spectral sequence. Since suspension in the unstable Adams spectral sequence preserves filtration and there is no element of filtration two in the unstable stem until 2^r, no homotopy element of Kervaire Invariant 1 can occur in $\pi_{s+k}(S^n)$ until $s = 2^r = \frac{k}{2} + 1$. So if (k,n) is of type 1, $n \geq k + \frac{k}{2} + 1$. Q.E.D.

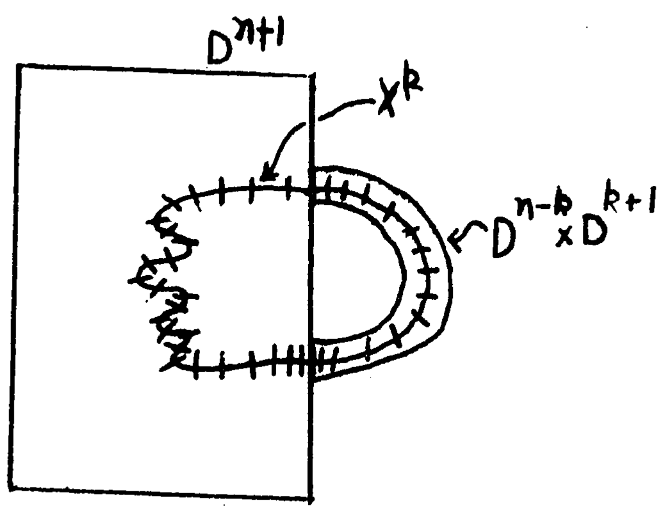
Lemma 2: If (k,n) is of type 1, the element α in $\pi_n(S^{n-k})$ of Kervaire Invariant 1 may be represented by f_α such that f_α is transverse regular to $p_0 \in S^{n-k}$ with preimage $\frac{k}{2} - 1$ connected.

Proof: We know by lemma 1 that $n \geq \frac{3k}{2} + 1$. We find f_α transverse regular to p_0 with preimage M^k representing α. We now do the standard type surgery argument in S^n. Suppose inductively that $f_\alpha^{(i-1)}$ may be chosen homotopic to f_α with preimage M^k_{i-1} which is i-1 connected for $i \leq \frac{k}{2} - 1$. Now $M^k_{i-1} \subset S^n$ and $M^k_{i-1} \times I \subset S^n \times I$. Take some $g : S^i \to M^k_{i-1}$ a generator for $H_i(M)$. Since $i < \frac{k}{2}$, g may be taken as an embedding of S^i into $M^k_{i-1} \times 1$ with trivial normal bundle. Now $g(S^i)$ bounds a disk in $S^n \times 1$, and since $k + i + 1 \leq \frac{3k}{2} < n$ we may assume D^{i+1} intersects M^k_{i-1} only in its boundary and orthogonally. We give its normal bundle in $S^n \times [0,2]$ trivialization, $D^{i+1} \times D^{n-i} \longrightarrow \nu(D^{i+1})$. This induces an (n-1) frame on $S^i \times 1$ in $S^n \times [0,2]$, τ^{n-i}. Let τ_0^{n-k} be the framing of $M^k \times 1 \subset S^n \times 1$ restricted to S^i. Now $\pi_i(O(k-i)) \longrightarrow \pi_i(O(n-i))$ is onto since $i < \frac{k}{2}$. Thus we may pick out a k-i frame τ^{k-i} on S^i in M^k so that $(\tau_0^{n-k}, \tau^{k-i}) = \tau^{n-i}$. Now embed $S^i \times D^{k-i}$ by τ^{k-i} into M^k_{i-1} and attach $D^{i+1} \times D^{k-i}$ by it. The framing τ_0^{n-k} then extends over $D^{i+1} \times 0$ and thus over $D^{i+1} \times D^{k-i}$. Then scoop out $D^{i+1} \times D^{k-i}$ so that $D^{i+1} \times D^{k-i} \cap S^n \times 1 = D^{i+1} \times \partial D^{k-i}$. This gives a framed cobordism of M^k_{i-1} in $S^n \times I$ to a manifold with one less generator in H_i. The Thom construction on this cobordism provides the homotopy from $f_\alpha^{(i-1)}$ to $f_\alpha'^{i}$. Proceeding in this way we may kill all of H_i. The result follows by induction on i. Q.E.D.

Proposition: If n-k is bad and (n-k,n+1) is of type 1, then all manifolds homotopy equivalent to $S^k \times S^{n-k}$ are PL homeomorphic to it.

If (n-k,n) is of type one, this proposition follows from theorem 1. It also extends our results to all cases.

Proof: We may assume that M^{n-k} is a $\frac{n-k}{2} - 1$ connected, framed manifold of Kervaire Invariant 1 which embeds in S^{n+1} with trivial normal bundle. Let $M_0^{n-k} = M^{n-k}\text{-int } D^{n-k}$. Then $(M_0^{n-k}, \partial M_0^{n-k}) \subset (D^{n+1}, S^n)$ with trivial normal bundle. Use the framing of $\partial M_0^{n-k} \subset S^n$ to attach $D^{n-k} \times D^{k+1}$ to S^n. $M_0^{n-k} \cup D^{n-k} \times \{0\}$ is then a framed submanifold of $S^{n-k} \times D^{k+1}$, with $M_0 \cup D^{k+1} \times 0 = \hat{M}_0$ homologous to $S^{n-k} \times 0$.



We define a map from this manifold to $D^{n-k}\times D^{k+1} \cup D^{n-1}\times D^{k+1}$ attached along $S^{n-k-1}\times D^{k+1}$. Send the right hand $D^{n-k}\times D^{k+1}$ to $D^{n-k}\times D^{k+1}$ by the identity. Send M_0^{n-k} onto $D^{n-k}\times 0$ and use the framing to extend to a regular neighborhood. We now wish to extend this map over all of D^{n+1}. Furthermore we wish the rest of D^{n+1} to miss $S^{n-k}\times 0$. The obstructions to doing this lie in $H^*(D^{n+k},M^{n-k}\times D^{k+1};\pi_{*-1}(S^{n-k}\times S^k))$. However, $H^*(D^n,M^{n-k}\times D^{k+1};\pi_{*-1}(S^{n-k}\times S^k)) = H^{*+1}(M^{n-k};\pi_{*-1}(S^{n-k}\times S^k))$. The only possible non zero group is

$H^{\frac{n-k}{2}}(M^{n-k};\pi_{\frac{n-k}{2}-2}(S^{n-k}\times S^k))$. But $n \geq \frac{3(n-k)}{2} \Rightarrow k \geq \frac{n-k}{2}$

and this group also is 0. The resulting f maps $(D^{k+1}\times S, S^k\times S^{n-k})$ to $(D^{k+1}\times S^{n-k}, S^k\times S^{n-k})$. It is a homotopy equivalence of $D^{k+1}\times S^{n-k} \longrightarrow D^{k+1}\times S^{n-k}$ since $\hat{M}_0$ is homologous to S^{n-k}. It sends $p\times S^k \longrightarrow p\times S^k$ for $p \in D^{k+1}$ and thus is an $\cong$ of $\pi_k(S^{n-k}\pi S^k) \longrightarrow \pi_k(S^{n-k}\times S^k)$. Thus it is also a homotopy equivalence where restricted to the boundary. It is transverse regular to $0\times S^{n-k}$ with preimage a manifold of Kervaire Invariant 1. Now embed $S^{n-k}\times I$ in $D^{k+1}\times S^{n-k}$ so that $S^{n-k}\times\{0\} = \{0\}\times S^{n-k}$ and $S^{n-k}\times 1 = p_0\times S^{n-k}$ where $p_0 \in \partial D^{k+1}$. Homotope f until it is transverse regular to $S^{n-k}\times I$ leaving it fixed near $\hat{M}_0\times 0$ and so that it still sends $S^k\times S^{n-k} \longrightarrow S^k\times S^{n-k}$. Let $g = f|S^{n-k}\times S^k$. g is transverse regular to $p_0\times S^{n-k}$ with a preimage which is framed cobordant to $\hat{M}_0$, and thus has Kervaire Invariant 1. g then is a self homotopy equivalence $S^k \times S^{n-k} \longrightarrow S^k \times S^{n-k}$ which realizes 1 on S^{n-k}.

Q.E.D.

This allows us to replace parts 3 and 4 of theorem 1 by:

(3′) If n-k is "bad" there is exactly 1 π-manifold homotopy equivalent to $S^k \times S^{n-k}$ but combinatorially distinct from it if and only if (n-k,n+1) is not of type 1. Otherwise all π-manifolds homotopy equivalent to $S^k \times S^{n-k}$ are PL homeomorphic to it. In this case there will be a self homotopy equivalence not homotopic to a PL homeomorphism.

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