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AN INVESTIGATION OF THE STABILITY OF FLOWS USING THE DIRECT METHOD OF LIAPUNOV

by

ERNEST LOSSON KISTLER

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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LIST OF SYMBOLS

\( a, \ell \) arbitrary constants in harmonic solution; (2.1)*

\( a_x, a_y, a_z \) components of absolute angular acceleration of a point along orthogonal Cartesian axes \( xyz \); (3.23)

\( a_r, a_t \) radial and tangential accelerations with respect to flight path; (B.1), (B.2)

\( A, B \) integration constants; (2.61), (2.62)

\( A \) reference area for aerodynamic coefficients; (3.2)

\( \hat{A} \) absolute acceleration of a point; (A.3)

\( \ell \) reference length for defining pitching moment; (B.10)

\( \bar{b}_{ijk}, \bar{b}_{ijk} \) parameters determined from properties of the material, usually taken constant; (3.6)

\( c \) wave velocity; (1.4)

\( \tilde{\varepsilon}_{(n)} \) stress couple; (A.27)

\( C_1, C_2, C_3, C_4 \) constants for a general harmonic solution; (2.2), (2.71)

\( C_D \) drag coefficient, \( D = \frac{1}{2} \rho AC_D \)

\( C_L \) lift coefficient, \( L = \frac{1}{2} \rho AC_L \)

\( C_{\alpha\alpha} \) slope of lift curve at equilibrium, \( \frac{\partial C_L}{\partial \alpha} \) (3.2)

\( C_{m_1}, C_{m_2} \) damping coefficients; (3.2)

---

*In List of Symbols, numbers in parenthesis refer to equations in the text.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$C_M$</td>
<td>Pitching moment coefficient, $M = g_i A L C_i$</td>
</tr>
<tr>
<td>$C_{Mx}$</td>
<td>slope of pitching moment curve at equilibrium, $\frac{dc_m}{d\omega}$; (3.2)</td>
</tr>
<tr>
<td>$C_{ij}, C_{ijk}$</td>
<td>parameters determined from properties of the material, usually taken as constant; (3.6)</td>
</tr>
<tr>
<td>$d$</td>
<td>characteristic dimension; (A.26)</td>
</tr>
<tr>
<td>$D$</td>
<td>drag force, component of resultant force, along velocity vector;</td>
</tr>
<tr>
<td>$\varepsilon_{kr}$</td>
<td>rate of deformation tensor; page 88</td>
</tr>
<tr>
<td>$\mathcal{E}$</td>
<td>set of all equilibrium sets; page 33</td>
</tr>
<tr>
<td>$f(s)$</td>
<td>known function given along boundary; page 17</td>
</tr>
<tr>
<td>$f_i(t)$</td>
<td>effective damping coefficient; (3.1), (3.2)</td>
</tr>
<tr>
<td>$f_2(t)$</td>
<td>effective spring constant; (3.1), (3.2)</td>
</tr>
<tr>
<td>$\hat{f}$</td>
<td>body force; (A.27)</td>
</tr>
<tr>
<td>$g$</td>
<td>a solution of the differential equation; (2.14)</td>
</tr>
<tr>
<td>$g_x, g_y, g_z$</td>
<td>components of body force $\hat{g}$, along $xyz$ axes; (2.33)</td>
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<td>$g_{ij}$</td>
<td>metric tensor; (3.13)</td>
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<td>$\hat{g}$</td>
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<td>$\hat{H}$</td>
<td>moment of momentum; (A.13)</td>
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<td>$\mathcal{K}(M)$</td>
<td>vector field; page 37</td>
</tr>
<tr>
<td>$\hat{\tau}, \hat{\sigma}, \hat{k}$</td>
<td>unit vectors along orthogonal Cartesian axes $\hat{x}, \hat{y}, \hat{z}$</td>
</tr>
</tbody>
</table>
$I$  second invariant of the deformation tensor; page 89
moment of inertia; (3.2)

$I_{ii}, I_{ij}$  moments and products of inertia;
(A.21)

$J_x, J_y, J_z$  components of momentum flux; (3.24)

$K_1, K_2, K_3, K_4$  alternate constants for harmonic solution; (2.71)

$K$  Hanks' coupling ratio; (3.17)

$\lambda$  alternate reference length for aero-
dynamic coefficients; (3.2)

$L$  lift force, component of resultant
force, normal to velocity vector;

$m$  element of manifold $\mathcal{M}$;
mass of moving element; (3.2)

$m_i$  sub-elemental mass within elemental
fluid particle; Fig. 20

$\mathcal{M}$  arbitrary manifold under consideration;

pitching moment, about $y$ axis;

$\mathcal{M}$  continuous mapping of input states
on $\mathcal{R}$ onto output states on $\mathcal{R}$;

$N$  range of real-valued functions in
metric space;
transport number associated with
second-order kinematics; (A.12)

$\rho$  perturbed solution; (2.16)

$-\rho$  hydrostatic pressure, mean pressure,
or average normal pressure at a point;

$$\rho = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$
$\mathbf{r}, \mathbf{q}, \mathbf{R}$ roll, pitch, yaw angular velocities, respectively about $xyz$ axes, usually used as perturbation rates; (3.2), (B.17)

$\mathbf{P}, \mathbf{Q}, \mathbf{R}$ total angular velocities about $xyz$ axes; (A.25)

$q$ dynamic pressure, $q = \frac{1}{2} \rho v^2$ ; (3.2)

$Q, Q_0$ two arbitrary sets; page 28

$Q$ arbitrary transportable fluid property; (A.7)

$\hat{r}$ position vector of point in fluid relative to fluid center of mass; (A.1)

$\hat{R}$ position vector of point within fluid element; (A.1)

$S_t$ state of the system; page 31, 32

$S$ surface area; (3.15)

$a$ a point on the contour; page 17

$\mathbf{a}$ alternate reference area for aerodynamic coefficients; (B.9)

$\mathbf{a}$ general isotropic stress tensor; (2.35)

$t$ time coordinate;

$\mathbf{t}_{(n)}$ force per unit area exerted by outside elements; (A.26)

$T_1$ time constant which defines effects due to rate of application of stress; (3.4)

$T_2$ time constant which defines effects due to gradual shearing in the fluid; (3.4)

$T$ time scale of the problem when second-order effects are considered; (3.26)
\( J \)  
invariant set of the dynamical system; page 33

\( u, v, w \)  
components of velocity vector along orthogonal cartesian coordinates; (1.1), (2.33)

\( U \)  
some neighborhood of the origin

\( \vec{u} \)  
resultant velocity at a point; (2.35)

\( U, V, W \)  
components of total velocity, or of mean velocity; (1.1), (1.4)

\( \nu(x,t) \)  
solution of the differential equation describing motion of the system;

\( \nu(t, x, t_0) \)  
well-defined solution beginning at \( x, t_0 \), where \( \nu(t_0, x_0, t_0) = x_0 \)

\( V(x) \)  
Liapunov function; (2.30)

\( \vec{V} \)  
absolute velocity; (A.1)

\( V, V_0 \)  
resultant velocity of the vehicle, along relative wind; (3.2), (B.1), (B.7), Fig. 21

\( x, y, z \)  
orthogonal Cartesian coordinates

\( X \)  
steady state solution; (2.16)

\( \vec{X} \)  
body force offset from center of mass; (A.27)

\( \alpha \)  
indicator of type of stability; (2.26 + 2.28)

angular displacement; (3.1)

angle of attack; (B.4), Fig. 21

\( \alpha, \beta \)  
transverse vibration of elastic solid; page 79

\( \alpha_\infty \)  
Couette flow parameter; (2.66)

\( \alpha_i \)  
wavelength of harmonic solution; (2.71)
\( \gamma \) flight path angle; Fig. 21
longitudinal vibration of elastic solid; page 79

\( \Gamma \) state space considered; page 25, 32

\( \delta \) selectable positive number; page 28

\( \delta_{ij} \) Kronecker delta;

\( \varepsilon_{ijk} \) permutation symbol;

\( \vec{\omega} \) vorticity vector; (2.38)
\[
\vec{\omega} = \vec{\nabla} \times \vec{u}
\]

\( \eta \) arbitrary positive number; page 25, 34

\( \eta_{ij} \) symmetric part of shear stress tensor; (3.19)

\( \Theta \) phase angle; (3.5)

\( \Theta, \Psi \) Euler angles describing pitch and yaw motion in \( \vec{x}z \) and \( \vec{xy} \) planes of the body fixed axes as the body rolls about the \( \vec{x} \) axis; (B.20)

\( \lambda \) second coefficient of viscosity; (2.36)
compressibility coefficient of elastic solid; page 81, 82

\( \mu \) viscosity, or first viscosity coefficient; (2.36)
rigidity coefficient for elastic solid; page 81, 82

\( \tilde{\mu} \) first viscosity coefficient for rotation; (3.20)

\( \mu_1, \mu_2, \mu_3 \) difference between given moments of inertia and principal moments of inertia; (B.16)

\( \nu \) kinematic viscosity, (2.38)

\( \nu_0 \) nominal kinematic viscosity; (3.4)
effective kinematic viscosity; (3.5)

\( \xi_{ij} \) skew-symmetric part of the shear stress tensor; (3.20)

\( \rho \) metric of the system; page 32

local density; (2.33)

\( \mathbf{\hat{r}} \) position vector of center of mass of fluid element; (A.1)

\( \mathbf{\hat{r}}_{xx}, \mathbf{\hat{r}}_{yy}, \mathbf{\hat{r}}_{zz} \) components of normal stress tensor; (2.34)

\( \mathbf{\hat{r}}_{xx}', \mathbf{\hat{r}}_{yy}', \mathbf{\hat{r}}_{zz}' \) portion of normal stress tensor components which depends on resistance to rate of deformation; (2.34)

\( \mathbf{\hat{r}}_{ij} \) rate of strain tensor; (3.3)

\( \mathbf{\hat{r}}_{\mathbf{x}}, \mathbf{\hat{r}}_{\mathbf{y}}, \mathbf{\hat{r}}_{\mathbf{z}} \) components of apparent normal stress due to second-order effects; (3.25)

\( \mathbf{\tau}_{ij} \) shear stress tensor; (3.3)

\( \mathbf{\tau}_{ij} \) components of apparent shear stress due to second-order effects; (3.25)

\( \mathbf{\hat{\tau}} \) torque (A.13)

\( \phi \) arbitrary harmonic solution; (2.1)

\( \phi \) solution to Orr-Sommerfeld equation; (1.4)

\( \mathbf{\hat{\phi}} \) vorticity parameter; (2.40)

\( \mathbf{\hat{\phi}} = \frac{\mathbf{\hat{\phi}}}{\rho} = \frac{\mathbf{\hat{\tau}}_{\mathbf{x}}}{\rho} + \frac{\mathbf{\hat{\tau}}_{\mathbf{y}}}{\rho} + \frac{\mathbf{\hat{\tau}}_{\mathbf{z}}}{\rho} \)

\( \phi \) trajectory in (\( \mathbf{x},\mathbf{y} \)) space; page 25

\( \psi \) stream function; (1.2)

\( \mathbf{\omega} \) angular velocity for rigid-body rotation; Fig. 18
domain of interest; page 40

subscripts

\( m \) maximum value

\( x, y \) partial differentiation of the parameter, holding \( x \) or \( y \) fixed; page 3

\( e_q \) equilibrium value;

\( n \) \( n \)-th component; (2.10)

superscripts and overlays

\( ^\wedge \) mean values; (2.49)

three-dimensional perturbation; (1.6)

\( ^\sim \) output state;

mean values;

two-dimensional perturbation; (1.6)

\( ', \) perturbation values; (2.49), (2.50)

differentiation with respect to spatial coordinate; (1.3), (1.4), (2.54)

\( (n) \) \( n \)-th derivative of \( x \); (2.6)

\( \cdot \) total derivative WRT time; (2.6), (2.14)

\( \star \) motion or solution;

special notation

\( \frac{d}{dt} \) total derivative; (A.1)

\( \frac{\delta}{\delta t} \) co-moving derivative; (A.1)
\vec{\nabla} \quad \text{del, or nabla operator;}
\quad \vec{\nabla} = \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}

\subseteq \quad \text{contained in;}
\in \quad \text{an element of;}
\exists \quad \text{there exists;}
\exists \quad \text{such that;}
\implies \quad \text{implies}
\equiv \quad \text{identical to}
\overset{\text{def}}{=} \quad \text{defined as}
I. INTRODUCTION

1. Statement of the Problem

The origin of turbulence is one of the classical unsolved problems of fluid mechanics. Historically the phenomena of flow transition is approached through a small-perturbation analysis of linearized equations describing the stability of motion of a viscous linear fluid element.

The problem is known to be non-linear. However, all early classical work treated the problem as linear to obtain the stability boundaries. Modern literature on hydrodynamic stability now emphasize that non-linearities must be accounted for to obtain improved stability boundaries. Customarily the fluid equations of motion are linearized. Then small perturbations are assumed, the equations of motion are modified accordingly, and again linearized. Only a few non-linear terms are ever retained in the stability analyses.

It is well known (1, 2)* that current stability boundaries are inadequate. The critical Reynolds number obtained by present techniques always predicts premature transition, and the result is excessively conservative.

*Numbers in parenthesis indicate References found on page 109.
This inadequacy of results using present techniques supplies the motivation for this thesis. This thesis is directed towards two parts of the problem. Because of the inadequacy of small-perturbation techniques for investigating the stability of a significantly non-linear system, a new mathematical technique is used; one which does not require linearization of the math model. The Liapunov Direct Method (3, 4) yields an evaluation of stability of the non-linear partial differential equations if;

(1) Suitable parameter can be found, whose trajectory in state space will truly indicate stability of the physical system, and

(2) If a Liapunov function can be constructed using the parameter*. The selection and use of such a parameter is presented herein.

A second basic inadequacy might be the math model itself (i.e. the Navier-Stokes equations). This question also is considered and inadequacies are described.

2. Discussion of the Problem

Review of Classical Stability of Parallel Flows

The stability analysis of classical fluid mechanics begins with the Navier-Stokes equations, which are

*Establishing the existence of a Liapunov function using the selected parameter is sufficient to guarantee stability of the parameter.
non-linear partial differential equations used to approximate a real fluid. Classically it is next postulated that the flow described by these equations consists of a mean flow combined linearly with small amplitude oscillatory flow using superposition concepts. Thus, for a two-dimensional flow, the fluctuations are separated from the mean flow by the assumption that total values may be given by

\begin{equation}
\begin{align*}
\mathbf{u} &= \mathbf{U}(y) + \tilde{u}(x,y,t) \\
\mathbf{v} &= \tilde{v}(x,y,t) \\
\rho &= \rho(x) + \tilde{\rho}(x,y,t)
\end{align*}
\end{equation}

(1.1)

where the tilde indicates a superposed two-dimensional fluctuation.

Substitution of these expressions into the two-dimensional Navier-Stokes equations produces additional non-linearities of the form \( \tilde{v} \tilde{u}_y \) where the subscript indicates differentiation with respect to the \( y \)-coordinate. The fluctuations are assumed to be small amplitude, and the entire set of equations is linearized. A stream function is introduced in the form

\begin{equation}
\psi(x,y,t) = \phi(y) e^{i(\alpha x - \beta t)}
\end{equation}

(1.2)

An arbitrary but small oscillation is assumed expanded in a Fourier series. The components of the disturbance

*Refer to List of Symbols, starting page v
are
\[
\tilde{u} = \frac{2\psi}{2y} = \phi'(y) e^{i(\alpha x - \beta t)}
\]
\[
\tilde{v} = -\frac{2\psi}{2x} = -i\alpha \phi(y) e^{i(\alpha x - \beta t)}
\]

(1.3)

These are substituted back into the linearized disturbance equation. A new equation is obtained which is then nondimensionalized. All velocities are divided by the maximum velocity \( U_m \) of the laminar flow, and all lengths by a selected reference length. The result is the fundamental stability equation. It is a differential equation for the disturbance postulated. Thus,
\[
(U - \zeta)(\phi'' - \alpha^2 \phi) - U'' \phi = -\frac{i}{\alpha R}(\phi''' - 2\alpha^2 \phi'' + \alpha^4 \phi)
\]

(1.4)

where \( \zeta = \frac{\beta}{\alpha} = \zeta_r + i \zeta_i \), and \( R = \frac{U_m S}{D} \)

denotes the Reynolds number characteristic of the laminar flow.

Equation (1.4) is the Orr-Sommerfeld equation, which rests at the center of past and current work on the problem of stability of parallel flows. It was arrived at by Orr in 1907 and independently by Sommerfeld in 1908. This equation remained unsolved until in 1929 Tollmien calculated the first neutral eigen-values and obtained a critical Reynolds number. The problem of stability thus was reduced to an eigen-value problem with various appropriate boundary conditions.
In order to formulate the boundary-value problem for the complete Orr-Sommerfeld equation together with the boundary conditions, it is necessary to write four particular solutions \( \phi_1, \phi_2, \phi_3, \phi_4 \) for this differential equation (1). Then, it is still difficult to deduce the general solution \( \phi \) from the four particular solutions in view of the two boundary conditions at \( y = \infty \). Tollmien circumvented the difficulty by assuming a velocity profile with a finite boundary thickness. Prior to the work of Tollmien, the viscous perturbation terms had been omitted completely and stability analyses had been conducted with the overly simplified frictionless disturbance equation, or Rayleigh Equation

\[
(U - \alpha)(\phi'' - \alpha^2 \phi) - U'' \phi = 0.
\]  

(1.5)

From this equation, an inviscid instability is found to occur whenever there is an inflection of the velocity profile. Furthermore, it is found that curvature of the profile is important. Viscosity effects on the mean flow are included in Eq. (1.5), but there are no effects of viscosity on the fluctuation. With Tollmien's work solving the complete stability equation, Eq. (1.4), including the effects of viscosity on the fluctuation, it was found that the results are drastically different, as is now well known. Thus, it was found that there are two fundamentally
different types of instability in laminar flow. Tollmien essentially proved that the effect of viscosity on disturbances must be taken into account not only in the immediate neighborhood of the wall but elsewhere in the flow and particularly in the neighborhood of the critical layer defined by that outward distance where \( U = c \). Betchov and Criminale (2)* show the great sensitivity of the stability boundaries to different assumed velocity profiles.

As for validation of Tollmien's analytical results by comparison with experiments, the existence of neutral or amplified oscillations of the type predicted could not even be detected. Hence the whole theory of small disturbances still met with expressions of doubt. It was not until 1943 that experimental results of Schubauer and Skramstad (5) finally confirmed that the process of amplification of small disturbances led to transition.

However, the entire process still was not clarified completely. As late as 1956, Clauser (6)** emphasized that the remaining knowledge denied is the turbulent mechanism itself, and the situation is only a little better to date. Advances were made in 1944 and 1945 when Lin reviewed and improved Tollmien's mathematical procedure and laid some

---

*R. Betchov and W. O. Criminale (2); p.64
**F. H. Clauser (6); p.22
foundation for expansion of the stability analysis. In 1944 and 1947, Landau (7) and Lees (8) initiated work on the stability of compressible flows, and this was continued by Dunn and Lin (9) in 1955. Only slowly did any work start on the theory of non-linear processes. Meksyn and Stuart (10) in 1951 were some of the originators of this work. Stuart (11) states that the stability problem in its general form must be considered non-linear, and later (12) he disproves a general conclusion (13) that the non-linear terms will stabilize. On the basis of linear theory the effect of Reynolds stresses on the mean motion is neglected.* In non-linear theory Stuart discusses why the interdependence of the mean and disturbance parts must be taken into account through non-linear terms.

Betchov and Criminale (2) also emphasize that when disturbances are amplified and become sufficiently large, certain non-linear effects suddenly open the door to random fluctuations, and the flow becomes turbulent. They present work on three-dimensional non-linear effects beginning with consideration of an oscillating layer of the form

\[ u_i = U_i(y, \bar{y}, t) + \hat{u}_i(x, y, \bar{y}, t) + \hat{u}_i(x, y, \bar{y}, t) \cdot \]  \hspace{1cm} (1.6)

*Several authors (2, 12, 14, 15) point out that the origin of turbulence is intimately tied to the Reynolds stresses, and that these effects must be included for any real understanding of the problem.
Here $U_i(y, \gamma, t)$ represents the mean flow, $\tilde{U}_i(x, y, t)$ represents an ordinary two-dimensional oscillation, and $\tilde{u}_i(x, y, \gamma, t)$ represents a three-dimensional oscillation with the same frequency and wave number as $\tilde{u}_i$. Their reasoning is that coupled oscillators often "lock-in" in non-linear mechanics.

Comments on the Classical Stability Analysis

Schlichting (1) and Pai (16) discuss theories resulting in either small perturbation techniques or total energy techniques. Struble (17), LaSalle and Lefshetz (3), Hahn (18), Krasovski (19), Zubov (20), Eckhaus (21), and others discuss a wide variety of ideas of what constitutes mathematical stability and practical stability. These basic concepts of what constitutes stability, and the theories for how one computes or establishes stability certainly will influence the analytical boundaries obtained. As Wang (4) points out, even the initial state space and transformation metrics in some of the more sophisticated stability techniques must be chosen very carefully so that the physical meaning of the problem is preserved.

Betchov and Criminale (2) present one of the most up-to-date and comprehensive discussions of modern numerical techniques of analysis of slightly non-linear problems. The works discussed fall in the category of accepting the
Navier-Stokes model along with various postulated disturbances. Then small-perturbation techniques are used to determine asymptotic stability.

The Direct Method of Liapunov (3, 17, 18, 20) applied to the "full" non-linear partial differential equations via construction of a Liapunov function eliminates uncertainties associated with truncating a math model by assumption of small perturbations. This technique can be applied directly to the distributed mathematical model of the system, and it does not require knowledge of the perturbed motions of the complex original system of equations. However, it does require selection of a physically proper "indicator" and evaluation of that indicator.

Schlichting and Pai note that the sufficiency boundaries give too early a prediction of transition "since it takes a finite length of time for disturbances to grow." Necessary boundaries would result in a smaller region of instability and probably would shift the critical Reynolds number to a higher value more like experiment. However, they also would be unconservative since they might cause the prediction that a flow will be laminar instead of turbulent by having met conditions necessary for stability but possibly not sufficient for stability. A more plausible reason for the "delay" in growth of disturbances, as currently determined, will be related to the
mode or modes in which disturbance energy is stored. Con-
verses of some of the Theorems of Liapunov are available for establishing necessary conditions for stability (20), but (as noted) the results should be used with caution.

Comments on the Classical Math Model

The point never sufficiently made in the literature is that there are a whole host of types of non-linearities each yielding innumerable non-linear terms. One really is obliged to be objective about the possible effects of any of them. If one type of parameter (such as viscosity) or one type of non-linear term (such as the term \( \vec{u}\vec{u}_y \)) can now be claimed to be significant, then what about other types such as higher-order kinematic terms omitted from the basic Navier-Stokes equations themselves? What about the effects of linearizing the stress-strain constitutive equation?

Furthermore, why adopt a hypothesis (2)* that requires vorticity to be introduced fictitiously into a shear layer through a sheet of isolated, concentrated, point vortices whose vorticity vanishes except at certain points at which it becomes infinite? It is pure fiction to define mysterious vortices that reach out and induce velocity components

*R. Betchov and W. O. Criminale (2); p.159
in the mean flow, or bunch up to form rows of vortices, yet are still at separated points.

Such discussions and devices ignore Clauser's point that stability theory still does not include a physical mechanism for the production of turbulence out of a laminar flow. The only acknowledged mechanism is separation or insipient separation at a wall (or shear surface), where "as an oscillation develops, the fluid in contact with the wall acquires a vorticity just as the wheels of a landing aircraft gain angular momentum at touchdown." Even this explanation* is questionable in part, as will be shown.

No mathematical model will ever perfectly describe the physical situation. However, it is postulated in this thesis that transition begins with energy going into translational perturbations which increase in amplitude. As the translational perturbations continue to increase they show an unstable translational nature. However, because of three-dimensional, non-linear, and visco-elastic phase-shift effects these perturbations become unsymmetrical and begin to induce true rotational effects in the form of angular momentum of small but finite

*R. Betchov and W. O. Criminale (2); p. 50
fluid elements. Thus, rather than perturbation energy going into wildly divergent translational oscillations, viscoelasticity further causes it to couple with and increasingly excite rotational modes. These start as microscopic but continuum vortices and grow to finite vortices of completely random nature. Hence, a basic new idea is put forth that flow transition involves the creation of discrete vortices within a previously irrotational fluid. This is in addition to other currently accepted ideas about vortex creation at separation surfaces. Actually, when such discrete vortices start, they generate local conditions which probably cause insipient separations involved in other theories. The creation of such vortices requires a modified math model. The kinematical and constitutive effects involved in this process are described in Section III.

It is beyond the scope of this thesis to combine these several effects into one math model and evaluate their effects via the Liapunov Direct Method.
II. LIAPUNOV ANALYSIS OF CLASSICAL MATH MODEL

1. General Considerations in Stability Analysis

Prior to considering non-linear stability of fluid flows it is beneficial to review and collect some general concepts from non-linear mechanics, particularly stability of non-linear physical systems. Basically, all physical systems are non-linear if evaluated on a small enough scale. However, the complexities of mathematical analysis of non-linear systems require one to resort to approximations. The linearizations commonly practiced are approximation devices which may or may not be sufficiently valid for the purposes of the analysis.

In some problems a linearized treatment will yield gross answers which are quite acceptable in the sense that they might yield considerable information unavailable without making the approximation. In other cases, the linearization is completely worthless for producing useful results. Sometimes the inadequacy may be obvious, but in other cases an inadequacy may not be fully understood by the investigator. Frequently, previously unrecognized phenomena occur in non-linear systems and their effects are not appreciated when the investigator must rely on a linearized model. Sometimes phenomena occur in non-linear systems which absolutely cannot occur in a linear system. The purpose
of this section is to present information from the general field of non-linear mechanics which will be useful in evaluating non-linear fluid flows.

Analytical techniques used frequently to analyze non-linear oscillations might be categorized (17) as the perturbation method, the iteration method, the averaging method, and the method of harmonic balance. There are numerous texts on these methods. The perturbation method is an extension of the small perturbation technique applied to linear systems. It is applicable to non-linear equations in which a small parameter is associated with the non-linear term and an approximate solution is obtained as a power series where the terms involving the small parameter are raised to successively higher powers. Stuart (14), and Betchov and Criminale (2) describe current research work along these lines. All of the above analytical techniques must be restricted to equations with sufficiently small non-linearity.

The iteration method essentially involves solving the equation with certain terms neglected; then the resulting solution is inserted into the terms first neglected, and a second solution of improved accuracy is obtained.

The averaging method considers the trajectory of a representative point in phase space. The coordinates of the trajectory are expressed as a Fourier series, which is
then averaged over \(2\pi\) and truncated for a first approximation.

The method of harmonic balance has the widest utility for obtaining a periodic solution of a non-linear differential equation. The harmonic solution, to a first approximation, is written as

\[
\varphi(x) = a \cos x + b \sin x.
\]  

(2.1)

The periodic solution is then expanded into Fourier series with unknown coefficients, inserted into the original equation, and sine and cosine terms of the respective frequencies are set to zero separately. The unknown coefficients of the assumed solution are then obtained by solving the resulting simultaneous equations.

If \(\varphi_i(x)\) is a solution, then \((\varphi_1 + \varphi_2)\) and \(C \varphi_1\) are also solutions. If these solutions are known, a set of particular solutions can be written as

\[
\varphi(x) = \varphi_0 + \sum_{i=1}^{n} c_i \varphi_i,
\]

(2.2)

where one of the particular solutions is

\[
\varphi_i = C_i \cos \alpha_i x + C_2 \sin \alpha_i x.
\]

(2.3)
Hence, another suitable form for the solution is

\[ \varphi(x) = \varphi_0 + \sum_{i=1}^{m} C_i \cos \alpha_i x + \sum_{j=1}^{n} C_j \sin \alpha_j x. \] (2.4)

Finally, if a solution is harmonic, its derivatives also are harmonic.

The task of finding the general solution (i.e. the totality of all solutions of a partial differential equation) is a problem that hardly ever occurs. The task usually is to find a single solution. This is accomplished, when possible, by specifying further conditions to the differential equations. For the \( n+1 \) independent variables these additional restrictions usually refer to \( n \)-dimensional manifolds which sometimes appear as boundary conditions, sometimes as initial conditions, and sometimes as jump conditions at discontinuity surfaces of domains within which the solutions are to be found. Initial value problems are called Cauchy problems, and the initial data are called Cauchy data. For the Cauchy problem, usually values of \( \varphi(a) \) and \( \varphi'(a) \) are given which represent conditions at the beginning.

In other problems, conditions are given at two points \( a \) and \( b \) representing boundaries of the domain. That is, the conditions \( \varphi(a) \) and \( \varphi(b) \) or \( \varphi'(a) \) and \( \varphi'(b) \) are specified.
For example, consider Laplace's equation in two dimensions;

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (2.5)$$

and let a straight line \( y = c \) intersect the boundary of the region of interest at two points \( x = a \) and \( x = b \). Along this line \( \phi \) is a function of \( x \) only and satisfies a differential equation of the second order. It requires two conditions \( x = a \) and \( x = b \) for unique determination.

Varying \( c \), the unique determination of a function satisfying the second-order differential equation requires the assignment of one condition along the entire boundary. The following conditions are those most often encountered:

1. The value of the function \( \phi(s) = f(s) \) is given on the boundary, where \( f(s) \) is a known function of the coordinates of the point \( s \) of the contour. This is called the Dirichlet Problem, or first boundary-value problem;

2. The normal derivative of the unknown function, \( \frac{\partial \phi(s)}{\partial n} = f(s) \), is given on the boundary. This is the so-called Neumann Problem, or second boundary-value problem;
(3) The third or mixed boundary value is one where a linear combination

\[ f_1(s) \varphi(s) + f_2(s) \frac{d\varphi(s)}{dn} = f_3(s) \]

is given.

These boundary conditions uniquely define the solution of the equations of Laplace, Poisson, and Pockels.

2. **Classification of Singular Points**

The topological method of analysis is one of the important means of investigating non-linear oscillations. Furthermore, it yields insight into the physical problem by providing a visualization through trajectories.

Consider ordinary differential equations in the real domain. Two kinds of equations are encountered frequently. The first type is expressed as an equation of order \( n \):

\[ x^{(n)} = f(x, \dot{x}, \ldots, x^{(n-1)}, t) \]  \( (2.6) \)

where \( x^{(n)} \) is the \( n \)th time derivative of \( x \). The second type is expressed as a system of \( n \) equations of first order:

\[
\begin{align*}
\dot{x}_1 &= X_1(x_1, x_2, \ldots, x_n, t) \\
\dot{x}_2 &= X_2(x_1, x_2, \ldots, x_n, t) \\
&\vdots \\
\dot{x}_n &= X_n(x_1, x_2, \ldots, x_n, t) 
\end{align*}
\]  \( (2.7) \)
The single \( n^{th} \) order equation may be reduced to \( n \) first order equations, as can be seen by a single change of variable:

\[
\chi_1 = \chi, \quad \chi_2 = \dot{\chi}, \quad \cdots, \quad \chi_n = \chi^{(n+1)}.
\]  

(2.8)

Thus,

\[
\begin{align*}
\dot{\chi}_1 &= \chi_2 \\
\dot{\chi}_2 &= \chi_3 \\
&\quad \cdots \\
\dot{\chi}_{n-1} &= \chi_n \\
\dot{\chi}_n &= f(\chi_1, \chi_2, \cdots, \chi_n, t).
\end{align*}
\]  

(2.9)

Consider \( \chi_1, \cdots, \chi_n \) as the components of an \( n \)-vector \( \chi \), and \( X_1, \cdots, X_n \) as those of an \( n \)-vector \( X \). Then the system may be written in the simpler form

\[
\dot{\chi} = X(\chi, t) \quad \text{(heteronomous)}.
\]  

(2.10)

If \( X \) depends on \( \chi \) alone and not upon time, then

\[
\dot{\chi} = X(\chi) \quad \text{(autonomous)}.
\]  

(2.11)

The \( n \)-dimensional \( \chi \)-space is called phase space, and the \( (n+1) \)-dimensional space of the quantities

\( \chi_1, \chi_2, \cdots, \chi_n, t \)

is called motion space. Notation

\( \chi = \chi(t) \)

indicates that the components \( \chi_n \) of \( \chi \) are functions of \( t \). If these functions are continuous then
the point \((\mathbf{x}(t), t)\) of motion space moves along a segment of a curve as \(t\) runs from \(t_1\) to \(t_2\), \(t_1 \leq t \leq t_2\). This segment forms a motion of \(\mathbf{x}(t)\) in the motion space. The projection of a motion upon the phase space is called the integral curve, phase curve, or trajectory of the motion. Thus \((\mathbf{x}(t), t)\) for fixed \(\mathbf{x}\) is called a motion, and the set of all points \((\mathbf{x}(t), t)\) for \(t \in (-\infty, +\infty)\) is called the trajectory of the dynamical system.

For an autonomous system of two first-order differential equations

\[
\frac{dx}{dt} = X(x, y) \quad , \quad \frac{dy}{dt} = Y(x, y) \quad \quad (2.12)
\]

where \(X(x, y)\) and \(Y(x, y)\) are polynomials of \(x\) and \(y\), it is possible to eliminate \(dt\) and write

\[
\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} \quad \quad (2.13)
\]

The integral curves of this equation on the \(xy\)-plane are the trajectories in the state plane of variables \(x\) and \(y\), which themselves are called state variables. Once the integral curves of \(\frac{dy}{dx}\) are found on the state plane, solutions or motions \(x(t)\) and \(y(t)\) in motion space may be found. Thus, the oscillation of the system is viewed as
movement of a representative point \([x(t), y(t)]\) along integral curves which are trajectories of the representative point.

All points \((x, y)\) for which the two functions \(X(x, y)\) and \(Y(x, y)\) do not vanish simultaneously are called regular points or ordinary points. Any point \((x_o, y_o)\) for which \(X(x_o, y_o) = Y(x_o, y_o) = 0\) represents a trivial or singular solution and the point is called a singular point. A singular point represents an equilibrium condition since \(x(t)\) and \(y(t)\) are constant. This equilibrium condition may be stable or unstable.

Singular points can be classified according to the type of roots obtained from the characteristic equation of the system. It is interesting to note the state space representation of the integral curves. For the state plane, Hayashi (21) shows the following:

Fig. 1 Types of Singular Points in State Plane
For 3-space, five types of singular points attributed to Poincaré (21) are:

![Diagram of singular points](image)

**Fig. 2** Types of Singular Points in State Space

Generally speaking, a singularity is stable if the trajectory of the representative point moves into the singularity and unstable if it moves outward.
Indeed, there may be one or more regions of stability enclosed by a region of instability. The singular points in Fig. 1 and 2 are not to be considered all inclusive, but only representative of the complex motions which may occur in non-linear physical systems.

**Well-Posed Problems**

As noted by LaSalle and Lefschetz (3), even mathematicians for a long time were not concerned with the very existence of a solution of a system such as Eq. (2.10), the existence being taken for granted.

Furthermore, if a solution can be shown to exist, one may validly question the uniqueness of the solution.

Finally, and of main interest in this dissertation, is the stability of the solution. Hadamard, according to Ames (22), introduced these questions by asserting that a problem is well-posed if its solution exists, is unique, and stable. Hadamard further states that discontinuous dependence on the initial data precludes any physical meaning since physical data by their very nature are only approximate. That is, if the solution becomes unstable for small variations in the initial state space, then there is little hope of the problem having physical meaning because there is always an inaccuracy in the initial physical data. Now, this statement seems to be too strong, and experience bears this out. In general, a small change in any of the auxiliary data of
the problem should produce only a correspondingly small change in the solution. However, this certainly would depend on the "sensitivity" of the problem and criteria on how small is small. Thus some stability investigations should not be taken too seriously.

3. **Stability Concepts in Mechanics**

There are many different notions of stability of a vectorfield. However, basically, there are two or three broad categories of the stability concept \((17, 18)\), and then some important refinements within the categories \((23)\). Struble \((17)\) prefers three categories of the basic stability concept; Laplace, Liapunov, and Poincaré. The Liapunov concept has stringent requirements, which will be discussed in this section. In some cases the requirements are too stringent and flow is labeled unstable when for practical purposes it would be satisfactory to label it as stable. Poincaré established a less stringent concept often called orbital stability.

Definitions of the various stability concepts will vary somewhat with the author. Struble \((17)\) gives the following:

A system is **stable in the sense of Laplace** if it is ultimately bounded.

For Liapunov stability, he states that if solutions
or trajectories are once near together they must always remain near together. More precisely, a solution \( x(t) \) of

\[
\dot{x} = X(x^*, t)
\]  

(2.14)

is stable in the sense of Liapunov if, for each \( \eta > 0 \), there exists a \( \delta > 0 \) such that any solution \( y \) of the differential equation satisfying \( |x - y| \leq \delta \) for \( t = 0 \) also satisfies \( |x - y| \leq \eta \) for all \( t \geq 0 \).

There is no shearing of the trajectories.

The time-dependent comparison is frequently not of importance in stability of physical systems, so a more relaxed stability in the sense of Poincaré is appropriate:

A trajectory \( \dot{y} \) is stable in the sense of Poincaré, i.e. possesses orbital stability, if neighboring half paths which are once near \( \mathcal{R} \) remain near \( \mathcal{R} \).

A trajectory \( \dot{y} \) is asymptotically stable (orbitally or otherwise) if positive half paths which are once near \( \mathcal{R} \) actually approach \( \mathcal{R} \) in the limit as \( t \to \infty \).

In the absence of asymptotic stability the classical techniques typically fail because they center around comparison theorems which require that the linear part of the problem be dominant in determining stability. For example, the text by Eckhaus (21) presents an extensive
bibliography and extensive theory for certain classes of non-linear stability problems. The text is particularly pertinent as background for this dissertation because the theory is discussed in terms of the field of fluid mechanics. However, the theory is based essentially on two concepts: (a) asymptotic expansions with respect to suitably defined small parameters, and (b) series expansions in terms of eigen-functions. These require a system to be labeled unstable if it is not asymptotically stable. As already discussed, this really is inappropriate for the physical systems of fluid mechanics. The conventional methods suppose that at some initial instant the solution of the general problem consists of some known basic solution plus a perturbation, while boundary conditions are not perturbed. The initial values of the perturbations are considered given or can be chosen at will.

The question then arises as to how they should be chosen. Preferably, perturbations generally should be considered entirely arbitrary but suitably continuous functions of the phase-space coordinates. Furthermore, it should be considered that an arbitrary number of different perturbations can be introduced in an arbitrary order of succession an arbitrary number of times during the time interval of interest.
Only in certain problems may the stability be studied by use of a single perturbation as an initial value problem. This is valid only when the frequency of the forcing function and/or its phasing (directedness) will not materially enhance or retard the natural trajectory of the system. For example, in the oscillation of a simple spring-mass-dashpot system, if the forcing function is of not too large a magnitude and the frequency is much greater or much smaller than the natural frequency of the system then stability can be tested by a single perturbation as an initial value problem. But frequently the forcing function plays a continuing role other than that of initial perturber.

If the initial values of the perturbations are infinitesimally small then the classical stability theory assumes that the linearized system is approximately valid during some finite initial time interval. This leads to the construction of a solution to the non-linear problem by iteration procedures where the linearized solution is taken as the first approximation.

In problems of greater complexity, one may not know in advance where the singular point is, or even that it exists. Thus, when the linear part of the system does not play the dominant role in establishing stability of the system, and when other problems arise concerning knowledge of the singularity, then some different mathematical
techniques must be used.

4. **Practical Stability**

When one investigates a "practical" or real physical system, the mathematical model itself is an approximation. Then for the "less than perfect" model, one needs to know the size of the region of stability and compare this with estimates of the conditions to which the model will be subjected, with accuracy of these estimates, with requirements on system performance and finally with some practical criteria on stability. Many systems may be mathematically unstable about a singular point and yet may oscillate sufficiently close to the equilibrium state so that for all practical purposes they fit the intuitive concept of stable.

Again, consider the heteronomous system

\[ \dot{x} = X(x, t) , \quad t \geq 0 . \]  \hspace{1cm} (2.15)

The equilibrium state is the origin; i.e. \( X(0, t) = 0 \) for all \( t \geq 0 \). The perturbed system is

\[ \dot{x} = X(x, t) + \rho(x, t) , \quad t \geq 0 . \]  \hspace{1cm} (2.16)

Also let a number \( \delta \) and two sets \( Q \) and \( Q_0 \) be given such that \( Q \) is a closed bounded set containing the origin and \( Q_0 \) is a subset of \( Q \). Let \( x^*(t, x^*, t_0) \) be the solution of the perturbed system satisfying

\[ x^*(t_0, x^*, t_0) = x^* , \]  \hspace{1cm} (2.17)
and let \( P \) be the set of all perturbations \( p \) satisfying
\[ \|p(x,t)\| \leq \delta \]
for all \( t \geq 0 \) and all \( x \). If for all \( p \)
in \( P \), \( x^0 \) in \( Q_o \), and each \( t_o \geq 0 \), \( x^*(t,x^0,t_o) \) is in
\( Q \) for all \( t \geq 0 \), then the origin is said to be practically
stable (3). Solutions which start in \( Q_o \) remain thereafter
in \( Q \).

\[ \text{Fig. 3 Practical Stability} \]

Obviously the concept of practical stability is relative since the set \( Q_o \) is the set of initial states consid-
ered and the set \( Q \) is the set of acceptable states.

Thus, when one specifies practical stability, the set
\( Q \) represents a decision as to how close to the desired
state the system must operate to be called stable, the set
\( Q_o \) represents how well the initial states can be controlled,
and \( \delta \) represents the magnitude of the phase space pertur-
bation.
To summarize, a mathematical problem which is to correspond to physical reality should satisfy three requirements:

(1) The solution must exist - no mutually contradictory properties can be demanded;

(2) The solution should be unique - ambiguity should be excluded unless inherent in the physical system;

(3) The solution should be stable - the solution should depend continuously on the data.

Any problem satisfying these three requirements as given by Hadamard is called a well-posed or properly-posed problem. References (3) and (22) remark on improperly posed problems and give examples of meaningful but not properly posed problems. Various hydrodynamic problems involving the Navier-Stokes equations fall into this category of meaningful but not properly posed problems.

5. Persistent Disturbances

Another consideration is stability under persistent disturbances. A system might be stable in the neighborhood of a singularity if merely displaced from equilibrium by a sufficiently small amount. However, if a forcing function fluctuates and is persistent, it may add energy in such a way that the system cannot dissipate faster than the persistent input. Thus, for an oscillatory system, phasing is important in addition to the magnitude.
All real fluid flows are subject to persistent disturbance, but are dissipative, and do establish certain stable equilibriums. Thus, solutions could be unstable for small persistent disturbances but sufficiently damped so that for practical purposes they actually are stable. As disturbances increase in intensity it is reasonable to expect trajectories of a representative phase space point to digress farther from equilibrium. For sufficiently large disturbances (in the sense of a displacement in suitable state space) it also is reasonable to expect that disturbance, dissipation, and any pertinent transfer mechanisms might adjust so that the trajectory does diverge.

This would correspond to initial or later perturbations moving the "initial" state \( x^0 \in Q \) outside the allowable set \( Q_0 \) such that further \( x^*(t, x^0, t_0) \) eventually exceeds \( Q \). For dynamics of a fluid element, it is clear that the boundary of \( S_{t_0} \) will depend in a very complex way on the physical mechanisms in the system. For small disturbances the system may be essentially linear, but for finite disturbances various non-linear effects become important and considerable care is required in modeling the energy transfer mechanisms. Recall that failure to recognize and properly model the mechanism of energy transport to perturbations through viscosity caused an inadequacy in early classical hydrodynamic stability.
6. **Stability Concepts for General Metric Spaces**

Consider a dynamical system whose state at any fixed time $t$ is specified by $S_t$, an element of state space $T$ on which some metric $\rho$ is defined. The distance between two arbitrary states $S_t$ and $S_t'$ in $T$ at time $t$ is specified by $\rho(S_t, S_t')$, and they are to be regarded as identical when $\rho(S_t, S_t') = 0$. For a distributed-parameter dynamical system defined on a spatial domain $\Omega$, $S_t$ corresponds to a set of real-valued functions $\{X_i(t, x)\}$ $i = 1, 2, \ldots, N$ defined on $T$, or an element of a function space $T(\Omega)$, where $x$ is the spatial coordinate vector.

In more general situations where a distributed-parameter system is coupled to a lumped-parameter system having finite degrees of freedom (e.g., an elastic beam with lumped masses attached to it at various spatial points), $T$ may be taken to be the Cartesian product of a function space and a finite dimensional vector space $(3)$. The system motion originating from an initial state $S_{t_0} \in T$ at time $t_0$ is specified by $\Phi(t, t_0)S_{t_0}$, $t > t_0$, where $\Phi$ is a continuous operator on $T$, and for any fixed $\{t, t_0\}$ it maps $T$ into itself. The set of operators $\{\Phi(t, t_0)\}$, $0 \leq t_0$, $t < +\infty [-\infty < t_0, t < +\infty]$ has the properties of a semi-group [group]:

$$
\Phi(t, t_0) \Phi(t, t_0) = \Phi(t, t_0) \tag{2.18}
$$

$$
\Phi(t, t) = I
$$

where $I$ is the identity operator.
The semi-group property implies that the knowledge of the system states at any time completely determines its further behavior. In general, the problem of establishing precise conditions for which a given mathematical model has the above property is by no means trivial. This problem is closely related to the mathematical problem of "well-posedness" discussed in the previous Section.

Within the framework of dynamical systems outlined above, various definitions for stability in the sense of Liapunov can be established as follows:

**DEFINITION I:** An equilibrium state \( S_{eq} \) of a dynamical system is an element of the state space \( \mathcal{S} \) such that \( \rho\left[ \Phi(t, t_0)S_{eq}, S_{eq} \right] = 0 \) for all \( t \geq t_0 \).

The set of all equilibrium states will be called the equilibrium set \( \mathcal{E} \).

**DEFINITION II:** An invariant set \( \mathcal{J} \) of a dynamical system is a subset of \( \mathcal{S} \) such that for any initial state \( S_{i_0} \in \mathcal{J} \), its corresponding trajectory \( \Phi(t, t_0)S_{i_0} \), \( t \geq t_0 \) will remain in \( \mathcal{J} \).

For example, \( \mathcal{E} \) is an invariant set of the system. The distance of a particular state \( S \) from an invariant set \( \mathcal{J} \) is denoted by

\[
\rho(S, \mathcal{J}) = \inf \{ \rho(S, S') | S' \in \mathcal{J} \}, \tag{2.19}
\]
and the distance between a particular motion $\tilde{\Phi}(t, t_o) S_{t_o}$ and an invariant set $I$ is denoted by

$$\rho\left[\tilde{\Phi}(t, t_o) S_{t_o}, I\right] = \sup \left\{ \rho(S, I) \mid S \in \tilde{\Phi}(t, t_o) S_{t_o} \right\}. \quad (2.20)$$

Now precise definitions for various degrees of stability of an invariant set can be established (4), (20), (24)*:

**DEFINITION III:** An invariant set $I$ of a dynamical system is said to be stable (with respect to a metric $\rho$), if for every real number $\eta > 0$, there exists a real number $\delta(\eta, t_o) > 0$ such that

$$\rho(S_{t_o}, I) < \delta(\eta, t_o) \Rightarrow \rho[\tilde{\Phi}(t, t_o) S_{t_o}, I] < \eta \quad \text{for all } t \geq t_o. \quad (2.21)$$

If, in addition $\rho\left[\tilde{\Phi}(t, t_o) S_{t_o}, I\right] \to 0$ as $t \to +\infty$, then the invariant set $I$ is said to be asymptotically stable. If the foregoing conditions are satisfied for all initial states $S_{t_o} \in \Gamma$, then $I$ is said to be asymptotically stable in the large.

In the case where the stability of a particular equilibrium state $S_{eq}$ is of interest, it is convenient to formulate the system equations about $S_{eq}$ so that $S_{eq}$ corresponds to the null state. In applying various notions of stability in the sense of Liapunov to elastic, aeroelastic,

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*W. Hahn (24); p.76  
V. I. Zubov (20); p.22  
P. K. C. Wang (4); p.53
or fluid mechanic systems, one should confine the choice of the state space $\mathcal{F}$ to those whose elements are physically admissible state functions. Also, the metric defined on $\mathcal{F}$ should be selected so that the stability definitions are compatible with the physical situations.

In evaluating some $S_0 \in \mathcal{F}$ one may find that the equilibrium null state is not Liapunov stable with respect to $\rho_0$, i.e., there does not exist a $\delta$ for every $\eta$ such that

$$\rho_0(S_0, 0) < \delta \Rightarrow \rho_0[\phi(t, t_0)S_0, 0] < \eta \quad \text{for all } t > 0. \quad (2.22)$$

However, Liapunov stability with respect to $\rho_0$ might be achieved by restricting the initial states to a smaller class of functions $\mathcal{F}'$, such as a set whose magnitudes of the initial displacement tend to zero. Furthermore, one may be able to verify that for $S_0 \in \mathcal{F}'$, the null state is Liapunov stable with respect to some new metric $\rho_1$.

The above discussion emphasizes the fact that the choice of the initial state space and the metric is crucial in the formulation of elastic, aeroelastic, and fluid mechanic stability problems in the framework of Liapunov stability theory. In other words, one must define the class of admissible initial states or perturbations and a measure for their closeness in order to arrive at physically meaningful results.
Continuing along the lines of Wang, in some cases it may be meaningful to define Liapunov stability with respect to two metrics. Consider a dynamical system whose state transitions are defined by a set of operators \( \{ \Phi(t,0) \} \), \( t \geq 0 \), on \( \mathcal{S} \) into itself. Let there be a continuous mapping \( \mathcal{M} \) on \( \mathcal{S} \) onto its range \( \mathcal{T} \) (referred to as the output state space). Then if the output state vector is represented by \( \tilde{S}_t \), one possible form of \( \mathcal{M} \) may be

\[
\tilde{S}_t = \mathcal{M} S_t = \begin{pmatrix}
\int g_1(x)(\cdot) dx & 0 \\
0 & \int g_2(x)(\cdot) dx
\end{pmatrix}
\begin{pmatrix}
\varphi(x,t) \\
\frac{\partial \varphi(x,t)}{\partial t}
\end{pmatrix}.
\tag{2.23}
\]

Note that \( \mathcal{T} \) is a two-dimensional vector space. A measure of the closeness between two states in \( \mathcal{S} \), and between two output states in \( \mathcal{T} \) can be established by defining a metric \( \rho \) on \( \mathcal{S} \) and another metric \( \tilde{\rho} \) on \( \mathcal{T} \). Also, invariant sets and equilibrium sets in \( \mathcal{T} \) can be defined in the same manner given previously. Here it is assumed that \( \mathcal{M} \) maps an invariant set \( \mathcal{J} \) in \( \mathcal{S} \) onto a corresponding invariant set \( \tilde{\mathcal{J}} \) in \( \mathcal{T} \).

Then with respect to the above notations and assumptions, one may define Liapunov stability with respect to two metrics \( \rho \) and \( \tilde{\rho} \) as follows:

**DEFINITION IV:**  An output invariant set \( \tilde{\mathcal{J}} \) of a dynamical system with specified \( \mathcal{M} \) is said to be stable (with respect to two metrics \( \rho \) and \( \tilde{\rho} \)), if
for any real number \( \eta > 0 \) there exists a real number \( \delta > 0 \) such that
\[
\rho(s(\eta, t_0), J) < \delta \quad \Rightarrow \quad \tilde{\rho}[m \phi(t, t_0) s_{t_0}, \tilde{J}] < \eta
\]  
(2.24)
for all \( t \geq t_0 \). If in addition,
\[
\tilde{\rho}[m \phi(t, t_0) s_{t_0}, \tilde{J}] \to 0 \quad \text{as} \quad t \to +\infty,
\]  
(2.25)
then \( \tilde{J} \) is said to be asymptotically stable (with respect to two metrics \( \rho \) and \( \tilde{\rho} \)).

Abraham (23) gives the following unified definition of notions of stability in terms of continuity of set valued mappings. It should be noted that they vary somewhat from the definitions of other authors. The third definition has been modified here to highlight that it is a definition of asymptotic stability whereas generally speaking Lyapunov stability only requires an arbitrarily small distance between the solutions. When this distance approaches zero in the limit then the stability (orbitally, Lyapunov, or otherwise) is asymptotic.

**Proposition:** Let \( M \) be a manifold, \( X \in \mathcal{X}(m) \) and \( m \in M \) be \( \mathcal{R} \)-complete. Then \( m \) is \( \mathcal{R} \)-stable, with respect to \( X \) if for every \( \eta > 0 \) there is a \( \delta > 0 \) so that \( \rho(m', m) < \delta \) implies
\[
(1) \mathcal{R} = 0 \quad \tilde{\rho}(m', m') < \eta \quad (\tilde{\rho} \text{ is the Hausdorff metric})
\]  
(2.26)
- called orbital stability of Birkhoff
\( (2) \quad \alpha = \alpha \ ; \ \lim_{t \to \infty} \rho(m_{\sigma}, m_{\tau}') = 0 \quad (2.27) \)

- called asymptotic stability of Poisson

\( (3) \quad \alpha = \lambda \ ; \ \lim_{t \to \infty} \rho(m_{\lambda}, m_{\lambda}') = 0 \quad (2.28) \)

- called asymptotic stability of Liapunov

These are illustrated as follows:

\( \alpha = 0 \)

Fig. 4 Orbital Stability Concept

\( \alpha = \alpha \)

Fig. 5 Asymptotic Stability Concept
Fig. 6 Asymptotic Stability Concept in Sense of Liapunov

Thus for a trajectory $\tilde{\phi}$ in $n$-space surrounded by a tubing of radius $\delta$, if a trajectory once penetrates the inner tube $\delta$ it must thereafter remain within a slightly larger tube of radius $\eta$ for arbitrarily small tubes. Asymptotic stability further requires that trajectories once near each other approach each other in the limit as $t \to 0$, although one point may trace its trajectory faster than the follower. Liapunov asymptotic stability requires that shearing of the trajectories not occur; i.e., it requires that the points initially close remain close for all time.

7. Liapunov Direct Method

Consider the autonomous system of Eq. (2.11), with an equilibrium at $x = a$. To investigate the stability it is
convenient to take the fixed point under consideration as the origin. This is accomplished merely by a transformation to new coordinates $\mathbf{x}^* = \mathbf{x} - \mathbf{a}$, which are then relabeled so that the asterisk symbology is dropped. Thus the problem can always be shifted to investigate a basic system

$$
\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(\mathbf{0}) = \mathbf{0} \quad (2.29)
$$

and attention is on stability of the origin. Consider functions $\mathbf{V}(\mathbf{x})$ with the following properties:

1. $\mathbf{V}(\mathbf{x})$ is defined and continuous, with its first partial derivatives, in an open region $\mathcal{N}$ about the origin;

2. $\mathbf{V}(\mathbf{0}) = \mathbf{0} \quad (2.30)$

3. $\mathbf{V}(\mathbf{x})$ is positive at every other point in the neighborhood about the origin;

4. $\frac{d\mathbf{V}}{dt} \leq \mathbf{0}$ in $\mathcal{N}$

Such a function is called a Liapunov function.

Note that it must be positive definite, and the origin investigated is an isolated minimum of $\mathbf{V}$. For a particular system there may be many suitable functions meeting the requirements to be a Liapunov function. However, there may not be any. The task of finding or constructing a Liapunov function is not always easy. As a matter of fact,
this usually is one of the disadvantages of the method. Furthermore, the fact that a Liapunov function may not be unique leads to important questions as to which if any are physically meaningful in evaluating stability of a physical system. Once the initial hurdles of establishing a meaningful Liapunov function have been overcome, there finally remains the question of finding the optimal function.

If near an equilibrium state of a physical system the energy of the system is always decreasing then the equilibrium is stable. The Liapunov stability theorems are a generalization of this idea, and the Liapunov functions are simply extensions of the energy concept. There are a multitude of versions and extensions of the Liapunov stability theorems in the references. They are important to the host of non-linear systems requiring investigation. However, they fortunately are not necessary to investigate the functions selected in this thesis. For this thesis, the theorems of (3) are adequate:

STABILITY THEOREM: If there exists in some neighborhood $\Omega$ of the origin a Liapunov function $V(x)$, then the origin is stable;

ASYMPTOTIC STABILITY THEOREM: If $-\dot{V}$ is likewise positive definite in $\Omega$ then the stability is asymptotic.
These theorems have been proven rigorously (18, 20). They furnish sufficient conditions for stability but do not say whether the given conditions are necessary. There also has been developed a set of converse theorems (20, 24).

The Direct Method consists of finding a suitable function which meets criteria (1) through (3) of Eq. (2.30). The total derivative is then evaluated to determine those conditions for which \( \frac{dv}{dt} \leq 0 \). For these conditions, the function will become a Liapunov function and stability is assured.

In many cases a functional \( v(x,t) \) depending on the parameter \( t \) takes the place of the Liapunov function. Instead of \( V(x) \) which represents the Liapunov function along a motion, the theorem is constructed with

\[
v(t,x,t_0) = \sup \left\{ v(y,t) \mid y \in \mathbb{M} \subset \mathbb{P} \right\}.
\]  

(2.31)

Then, analogous to Eq. (2.30), to characterize the stability properties of an invariant set \( J \), consider a real functional with the following properties:

1. \( v(x,t) \) is defined for all \( t \geq 0 \) and for all \( x \) in a certain neighborhood of \( J \);

2. For each sufficiently small \( \eta > 0 \) \( \exists \eta_2 > 0 \) (2.32)

\[ \exists \, v(x,t) > \eta_2 \text{ for all } t \geq 0 \text{ provided } \rho(x,\delta) > \eta. \]
(3) \( \lim_{t \geq t_0} u(x,t) = \alpha \) uniformly with regard to \( t \geq t_0 \) if \( \rho(x,t) \to \alpha \).

(4) The function \( u(t,x,t_0) \) does not increase for \( t \geq t_0 \).

Conditions (1) through (4) are necessary and sufficient for stability (24)*.

8. **Liapunov Analysis of Classical Navier-Stokes Equations**

For three-dimensional flow, the classical equations of motion are:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = g_x + \frac{1}{\rho} \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = g_y + \frac{1}{\rho} \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \tag{2.33}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = g_z + \frac{1}{\rho} \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right)
\]

The assumption is made that

\[
\sigma_{xx} = -\rho + \sigma_{xx}'
\]

\[
\sigma_{yy} = -\rho + \sigma_{yy}'
\]

\[
\sigma_{zz} = -\rho + \sigma_{zz}'
\]

*W. Hahn (24); p.134
and in Gibbs notation, the equations of motion become:

\[
\dot{\mathbf{a}} = \frac{d\mathbf{u}}{dt} = \frac{2\mathbf{u}}{2t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{Q}{\rho} \frac{Q}{T} \mathbf{a} + \mathbf{q}.
\]  

(2.35)

Well known assumptions on stress symmetry and the stress-strain relationship for a linear viscous material give:

\[
\frac{d\mathbf{u}}{dt} = \frac{2\mathbf{u}}{2t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{\mathbf{q}}{\rho} - \frac{1}{\rho} \frac{Q}{T} \frac{\nabla}{\nabla} p + \frac{Q}{\rho} \nabla^2 \mathbf{u} + \frac{1}{\rho} \frac{Q}{T} \mathbf{a} + \frac{\mu}{\rho} \nabla (\nabla \cdot \mathbf{u}).
\]  

(2.36)

Then, for \( \lambda = -\frac{2}{3} \mu \), the equation of motion is:

\[
\frac{d\mathbf{u}}{dt} = \frac{2\mathbf{u}}{2t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{\mathbf{q}}{\rho} - \frac{1}{\rho} \frac{Q}{T} \frac{\nabla}{\nabla} p + \frac{Q}{\rho} \nabla^2 \mathbf{u} + \frac{1}{\rho} \frac{Q}{T} \mathbf{a} + \frac{\mu}{\rho} \nabla (\nabla \cdot \mathbf{u}).
\]  

(2.37)

If the flow is barotropic, and the body force is conservative and passes through the center of gravity of the fluid element, then the equation of motion in vorticity form (25) is:

\[
\frac{d}{dt} \left( \frac{\mathbf{v}}{\rho} \right) = \left( \frac{\mathbf{v}}{\rho} \cdot \nabla \right) \mathbf{u} + \frac{Q}{\rho} \nabla^2 \mathbf{v} - \frac{\mu}{\rho} \left[ \nabla (\frac{1}{\rho}) \times \left( \nabla \times \mathbf{v} \right) \right].
\]  

(2.38)

At this point the equation involves the direction as well as magnitude of the vorticity vector. The vorticity vector can be thought of as a vector originating from the origin of a three-dimensional axis system located instantaneously at the center of mass of the fluid element, and translating but not rotating or accelerating with the element. Such an axis system is an inertial system with
respect to which Newton's laws are valid. Classically, vorticity vector is just \( \mathbf{j} = 2 \mathbf{\omega} \), where \( \mathbf{\omega} \) is the absolute angular velocity for rigid-body rotation.

In a three-dimensional problem, the tip of the vorticity vector sweeps out a trajectory in 3-space which could be encased in a sphere about the origin. If the vorticity decreases, its rate of decrease could be measured by the rate of shrinkage of the bounding sphere. Similarly, an increase of vorticity could be measured.

If importance of orientation of the vorticity vector can be omitted in the criteria of stability, then the norm of the radius vector \( \mathbf{R} \) of the bounding sphere can be selected as the Liapunov function to be evaluated*. This would eliminate obvious complexities associated with criteria on randomness of vorticity for those instabilities developing into turbulence. Furthermore, this norm seems to be a parameter related to an energy of the vorticity or vorticity fluctuation and could be related further to some experimental parameter which already has been measured or could be measured. For example, for a fluctuation analysis the radius of the bounding sphere might be relatable to power

---

*Liberty is being taken with the terminology. Strictly speaking, the scalar function \( V(x,y,z) \) can only be called a Liapunov function after certain conditions are met, one of which is \( V \leq 0 \). Though \( V \) may exist, no Liapunov function exists unless all the criteria are met. Then, existence of a Liapunov function guarantees stability.
spectral density of turbulent fluctuations measured by a hot-wire anemometer. Actually, although the norm of the vorticity is a possible choice, the vorticity vector itself cannot be an appropriate choice for a Liapunov analysis. A Liapunov function must be a scalar which is always positive except at the origin. Squaring the vector is one way to assure the necessary positiveness to be a Liapunov function.

With the vorticity vector encased in a sphere, let the Liapunov function be the square of the radius \( R \); i.e.

\[
V(x, y, z) = \varphi_x^2 + \varphi_y^2 + \varphi_z^2 = R^2 ,
\]

where

\[
\hat{\varphi} = \left( \begin{array}{c} \varphi_x \\ \varphi_y \\ \varphi_z \end{array} \right) = \tau \varphi_x + \imath \varphi_y + \kappa \varphi_z .
\]

Thus the tip of the vorticity vector defines the coordinates of one point on the bounding sphere. The space considered is 3-space \((\varphi_x, \varphi_y, \varphi_z)\), the function \(V(x, y, z)\) exists, is positive definite, and the first partials exist and are continuous. Hence, for stability

\[
\frac{dV(x, y, z)}{dt} = \left( \frac{\partial V}{\partial \varphi_x} \right) \frac{d\varphi_x}{dt} + \left( \frac{\partial V}{\partial \varphi_y} \right) \frac{d\varphi_y}{dt} + \left( \frac{\partial V}{\partial \varphi_z} \right) \frac{d\varphi_z}{dt} \leq 0 ,
\]
or
\[
\frac{dV(xyz)}{dt} = 2 \phi_x \left\{ \frac{d\phi_x}{dt} \right\} + 2 \phi_y \left\{ \frac{d\phi_y}{dt} \right\} + 2 \phi_z \left\{ \frac{d\phi_z}{dt} \right\} \leq 0. \tag{2.42}
\]

Consider now Eq. (2.38) written in terms of the vorticity parameter \( \vec{\omega} = \frac{\vec{\omega}}{\rho} : 
\[
\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \nabla) \vec{u} + \frac{\mu}{\rho} \nabla^2 (\vec{\omega} \cdot \vec{u}) - \frac{\mu}{\rho^2} \left[ \nabla \left( \frac{1}{\rho} \right) \times (\nabla \times \vec{u} \rho) \right]. \tag{2.43}
\]

The three terms are expanded to give:
\[
\frac{d\vec{\omega}}{dt} = \vec{\tau} \left\{ \frac{d\phi_x}{dt} \right\} + \vec{j} \left\{ \frac{d\phi_y}{dt} \right\} + \vec{k} \left\{ \frac{d\phi_z}{dt} \right\}
\]
\[
= \vec{\tau} \left\{ \phi_x \frac{2u}{2x} + \phi_y \frac{2u}{2y} + \phi_z \frac{2u}{2z} - \frac{\mu}{\rho} \left( \frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_x}{\partial z^2} \right) \right\}
+ \frac{\mu}{\rho} \left( \frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_y}{\partial z^2} \right) + \frac{\mu}{\rho^2} \left[ \frac{\partial^2 \phi_y}{\partial x \partial z} \right]
+ \vec{j} \left\{ \phi_x \frac{2u}{2x} + \phi_y \frac{2u}{2y} + \phi_z \frac{2u}{2z} - \frac{\mu}{\rho} \left( \frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_y}{\partial z^2} \right) \right\}
+ \frac{\mu}{\rho} \left( \frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_y}{\partial z^2} \right) + \frac{\mu}{\rho^2} \left[ \frac{\partial^2 \phi_y}{\partial x \partial z} \right]
+ \vec{k} \left\{ \phi_x \frac{2u}{2x} + \phi_y \frac{2u}{2y} + \phi_z \frac{2u}{2z} - \frac{\mu}{\rho} \left( \frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} \right) \right\}
+ \frac{\mu}{\rho} \left( \frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} \right) + \frac{\mu}{\rho^2} \left[ \frac{\partial^2 \phi_z}{\partial x \partial y} \right]. \tag{2.44}
\]
Substituting into Eq. (2.42) gives:

\[
\frac{dV(x,y,z)}{dt} = 2\phi_x \left\{ \frac{d\phi_x}{dt} \right\} + 2\phi_y \left\{ \frac{d\phi_y}{dt} \right\} + 2\phi_z \left\{ \frac{d\phi_z}{dt} \right\} \leq 0
\]

\[
= \phi_x^2 \frac{\partial u}{\partial x} + \phi_y \phi_x \frac{\partial u}{\partial y} + \phi_x \phi_y \frac{\partial u}{\partial z} - \frac{\phi_x^3}{\rho} \left[ \frac{\partial^2 q_x}{\partial x^2} + \frac{\partial^2 q_x}{\partial y^2} - \frac{\partial^2 q_x}{\partial z^2} - \frac{1}{\rho} \frac{\partial^2 q_x}{\partial x^2} \right] \\
+ \phi_y \phi_x \frac{\partial v}{\partial x} + \phi_y^2 \frac{\partial v}{\partial y} + \phi_x \phi_y \frac{\partial v}{\partial z} - \frac{\phi_y^3}{\rho} \left[ \frac{\partial^2 q_y}{\partial x^2} + \frac{\partial^2 q_y}{\partial y^2} - \frac{\partial^2 q_y}{\partial z^2} - \frac{1}{\rho} \frac{\partial^2 q_y}{\partial y^2} \right] \\
+ \phi_x \phi_z \frac{\partial w}{\partial x} + \phi_y \phi_z \frac{\partial w}{\partial y} + \phi_z^2 \frac{\partial w}{\partial z} - \frac{\phi_z^3}{\rho} \left[ \frac{\partial^2 q_z}{\partial x^2} + \frac{\partial^2 q_z}{\partial y^2} - \frac{\partial^2 q_z}{\partial z^2} - \frac{1}{\rho} \frac{\partial^2 q_z}{\partial z^2} \right] \leq 0.
\] (2.45)

This expression can be set equal to zero for the case of neutral stability and can be solved for \( \frac{\mu}{\rho} \), giving:

\[
\frac{\mu}{\rho} = \frac{\left( \phi_x^2 \frac{\partial u}{\partial x} + \phi_y^2 \frac{\partial v}{\partial y} + \phi_z^2 \frac{\partial w}{\partial z} + \phi_x \phi_y \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \right)}{\left( \phi_x \left[ \frac{\partial^2 q_x}{\partial x^2} + \frac{\partial^2 q_x}{\partial y^2} - \frac{\partial^2 q_x}{\partial z^2} - \frac{1}{\rho} \frac{\partial^2 q_x}{\partial x^2} \right] \\
+ \phi_y \left[ \frac{\partial^2 q_y}{\partial x^2} + \frac{\partial^2 q_y}{\partial z^2} - \frac{\partial^2 q_y}{\partial y^2} - \frac{1}{\rho} \frac{\partial^2 q_y}{\partial y^2} \right] \\
+ \phi_z \left[ \frac{\partial^2 q_z}{\partial x^2} + \frac{\partial^2 q_z}{\partial z^2} - \frac{\partial^2 q_z}{\partial y^2} - \frac{1}{\rho} \frac{\partial^2 q_z}{\partial z^2} \right] \right)}{(2.46)}
\]
9. Results Simplified for Two-Dimensional Flows

The complexity of Eq. (2.46) reduces quickly when the problem is restricted to completely two-dimensional flow. This obviously eliminates the possibility that the vorticity vector may go unstable in orientation and leaves only the possibility that it may go unstable in magnitude. The Orr-Sommerfeld equation is an equation for steady two-dimensional incompressible flow, and exhibits an instability only in magnitude. The Squire Theorem says that a two-dimensional flow will go unstable for a two-dimensional disturbance before it will go unstable for a three-dimensional disturbance, and that it is conservative to analyze the flow as strictly two-dimensional. Unfortunately, this classical unstable two-dimensional problem usually predicts a critical Reynolds number that is much greater than experimentally determined transition Reynolds numbers. It is clear how three-dimensional effects could account for at least some of this discrepancy.

The neutral stability criteria for two-dimensional compressible flows becomes (from Eq. 2.46):

\[
\frac{\mu}{\rho} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = 0
\]  \hspace{1cm} (2.47)

where

\[
\tilde{\phi} = \kappa \tilde{\phi}_y
\]  \hspace{1cm} (2.48)
Actually the orientation of \( \mathbf{\omega} \) is not considered to be a good criterion of stability of the laminar flow as a predictor of turbulence because the flow might just change to another stable condition where the vorticity vector is oriented differently. Thus, the criterion needed seems to be one that involves fluctuations and randomness.

Let the vorticity be considered as the sum of a mean vorticity and a vorticity perturbation, and let growth of the perturbation be taken as the indicator of turbulence. That is, let

\[
\mathbf{\omega} = \mathbf{\omega}_{m} + \mathbf{\omega}'
\]

(2.49)

where \( \mathbf{\omega}_{m} \) may grow somewhat, and the perturbation \( \mathbf{\omega}' \) is not necessarily small.

Various choices could be made on which mean value of the flow should be taken: temporal, spatial, statistical, etc.

If the norm is a statistical mean, then fluctuations about it could be experimentally determined using hot wire anemometer techniques. Any rapid small-scale orientation fluctuation of the vorticity vector could be ignored, and attention could be directed towards: (a) energy level of the vorticity fluctuation, (b) energy level of the mean vorticity, and (c) approximate orientation of the mean vorticity. The envelope of the fluctuation energy becomes
the parameter of interest, and all the difficulties associated with criteria on randomness are eliminated so that in essence the problem is smoothed. Actually, this approach is taken by many experimentalists. It would be desirable to find a corresponding analytical parameter and a stability analysis technique using that parameter so that analytical and experimental work can be blended.

In addition to plotting an experimentally obtained vorticity fluctuation curve and observing the envelope, a criteria should be selected as to how large an envelope would be called "turbulent." Schematically, the plot of fluctuation data might be as indicated below, and a critical fluctuation could be determined.

![Graph](image)

**Fig. 7 Vorticity Fluctuation**
Hence, as an example, taking the criteria to apply to only the fluctuation, the completely two-dimensional classical case yields

\[
\frac{\mu}{\rho} \frac{\partial^2 \phi'}{\partial t^2} \left[ \frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} \right] = 0
\]  

(2.50)

for neutral stability. In terms of velocity gradients, it is

\[
\frac{\mu}{\rho^2} \left( \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} \right) \left\{ \frac{\partial^2}{\partial x^2} \left( \frac{1}{\rho} \left( \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} \right) \right) + \frac{\partial^2}{\partial y^2} \left( \frac{1}{\rho} \left( \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} \right) \right) \right\} = 0.
\]  

(2.51)

The Orr-Sommerfeld equation is for incompressible flow. Hence, for a comparison, if that assumption now is introduced, Eq. (2.51) becomes

\[
\frac{\mu}{\rho} \left( \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} \right) \left[ \frac{\partial^2}{\partial x^2} \left( \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} \right) \right] = 0
\]  

(2.52)

where \( \frac{1}{\rho^2} \) has been dropped with no loss of information.

It is at this corresponding point in the classical development that a stream function is introduced, and a stability equation obtained. A specific velocity profile is introduced, and the resulting equation is solved for disturbance wavelength versus Reynolds number to give neutral stability.
Equation (2.52) can be compared directly with a level of equation appearing the classical theory. Introducing the stream function

\[ \psi(x, y, t) = \phi(y) e^{i(\alpha x - \beta t)} \]  

(2.53)

where

\[ u' = \frac{\partial \psi}{\partial y} = \phi'(y) e^{i(\alpha x - \beta t)} \]

\[ v' = -\frac{\partial \psi}{\partial x} = -i \alpha \phi(y) e^{i(\alpha x - \beta t)} \]  

(2.54)

Equation (2.52) for neutral stability becomes

\[ \mathcal{V} \left[ \phi'' - \alpha^2 \phi \right] \left[ \phi''' - 2\alpha^2 \phi'' + \alpha^4 \phi \right] = 0 \]  

(2.55)

For real flows, it then is clear that

\[ \mathcal{V} = 0 \]  

(2.56)

removes the possibility of resistive instability. The first bracket gives particular solutions from

\[ \phi'' - \alpha^2 \phi = 0 \]  

(2.57)

which corresponds to most of the classical inviscid form called the Rayleigh equation. The second bracket gives particular solutions from

\[ \phi''' - 2\alpha^2 \phi'' + \alpha^4 \phi = 0 \]  

(2.58)

which is the viscous portion of the full Orr-Sommerfeld equation. The criteria given by Eq. (2.45) is considerably
more complete than the classical criteria developed from the Orr-Sommerfeld equation. It permits the consideration of general three-dimensional instability, yet reduces to the above two-dimensional form which yields essentially the same particular solutions as the classical criteria. The viscous solution is retained completely, although a portion of the inviscid solution (specifically that involving \( u'' \)) was lost. That part could be recovered by retaining the mean-flow terms in Eq. (2.47). Furthermore, a three-dimensional disturbance could be superimposed on a two-dimensional mean flow. Other possible combinations also could be formulated.

10. Flow Between Parallel Plates

Consider now the criteria of Eq. (2.47) applied to steady two-dimensional flow between parallel plates (26). Let

\[
\begin{align*}
    u &= u(y) \quad , \quad v = 0 \quad , \quad w = 0 \quad , \quad \frac{\partial}{\partial y} = 0 .
\end{align*}
\]  

(2.59)
The streamwise velocity component is independent of \( x \) thru continuity considerations, and \( \frac{\partial p}{\partial y} = 0 \) so that pressure depends on \( x \) only. Hence

\[
\frac{d\rho}{dx} = \mu \frac{d^2u}{dy^2} ,
\]

(2.60)

so that integration with respect to \( y \) gives

\[
u = \frac{1}{\mu} \frac{d\rho}{dx} \frac{y^2}{2} + Ay + B
\]

(2.61)
for the general solution of equilibrium velocity distribution. The constants $A$ and $B$ are arbitrary constants determined thru the boundary conditions for each specific type of flow.

**Couette Flow**

Consider steady flow in a channel where one of the plates is at rest and the other moving with a velocity $u$ parallel with the fixed plate. This flow is called Couette flow and is shown here in Fig. 8. The two arbitrary constants are determined from the boundary conditions:

$$
\begin{align*}
&@ y = 0 \quad , \quad u = 0 \\
&@ y = h \quad , \quad u = U.
\end{align*}
$$

That is, there is no slip at either wall. Hence

$$
A = \frac{U}{h} - \frac{1}{2\mu} \frac{dp}{dx} h \quad \text{(2.62)}
$$

$$
B = 0 \quad \text{(2.63)}
$$
The general solution for Couette flow is then
\[
\frac{u}{u} = \frac{y}{h} - \frac{h^2}{2\mu} \frac{d\sigma}{dx} \frac{y}{h} (1 - \frac{y}{h})
\] (2.64)

Non-dimensionalized by \( U \), the result is
\[
\frac{u}{U} = \frac{y}{h} + \alpha_c \frac{y}{h} (1 - \frac{y}{h})
\] (2.65)

where
\[
\alpha_c = \frac{h^2}{2\mu U} \left( - \frac{d\sigma}{dx} \right)
\] (2.66)

For pressure decreasing in the direction of flow, \( \alpha_c > 0 \).

For pressure increasing, \( \alpha_c < 0 \), and reverse flow begins to occur near the stationary wall as \( \alpha_c \) becomes less than -1.

Now for Couette flow,
\[
\frac{\phi_3}{\rho} = \frac{\sigma_x}{\rho} = - \frac{1}{\rho} \frac{d\sigma}{dy} \quad \text{since} \quad \nu = 0.
\] (2.67)

Hence, at most
\[
\frac{\phi_3}{\rho} = \frac{\sigma_x}{\rho} = - \frac{1}{\rho} \left( c_1 + c_2 y \right)
\] (2.68)

and
\[
\nabla^2 \phi_3 \equiv 0.
\] (2.69)

Thus, the equilibrium Couette flow is stable, via Eq. (2.47).

Any equilibrium flow (or any mean plus perturbation) that satisfies Eq. (2.47) is stable. Now Laplace's equation \( \nabla^2 \phi_3 = 0 \) satisfies (2.47), it is harmonic, and all its solutions are harmonic, as discussed earlier. Solutions
for certain higher-power velocity profiles will not have a Laplacian identically zero. They will then furnish conditions on the combination of flow parameters that do satisfy Eq. (2.47).

Written as a perturbation equation, Eq. (2.47) becomes

\[
\frac{\mu}{\rho} \left( \hat{\Phi}_y + \Phi'_y \right) \left[ \frac{\partial^2}{\partial x^2} \left( \hat{\Phi}_y + \Phi'_y \right) + \frac{\partial^2}{\partial y^2} \left( \hat{\Phi}_y + \Phi'_y \right) \right] = 0 .
\] (2.70)

A general harmonic solution \( \Phi_i \) for the perturbation could be written as

\[
\Phi_i = \left( C_1 \sinh \alpha_i x + C_2 \cosh \alpha_i x \right) \left( C_3 \sin \alpha_i y + C_4 \cos \alpha_i y \right)
= K_1 \sin \alpha_i y \sinh \alpha_i x + K_2 \cos \alpha_i y \sinh \alpha_i x
+ K_3 \sin \alpha_i y \cosh \alpha_i x + K_4 \cos \alpha_i y \cosh \alpha_i x .
\] (2.71)

Obtaining the second partials for the perturbation yields

\[
\frac{\partial^2 \Phi_i}{\partial x^2} + \frac{\partial^2 \Phi_i}{\partial y^2} \equiv 0 .
\] (2.72)

Since for Couette flow

\[
\frac{\partial^2 \hat{\Phi}_y}{\partial x^2} \equiv 0 , \quad \frac{\partial^2 \hat{\Phi}_y}{\partial y^2} \equiv 0 ,
\] (2.73)

the equilibrium and perturbed flow are stable for all finite amplitude disturbances. This agrees with general conclusions reached by several authors, as reported by Stuart (14), but never before proven.
Actually, this result is almost obvious and could have been obtained directly from the equation of motion in vorticity form for an incompressible flow. Without the use of Liapunov criteria, from Eq. (2.38)

$$\frac{d}{dt}\left(\frac{\vec{\omega}}{\rho}\right) = \frac{\mu}{\rho} \nabla^2 \vec{\omega}.$$  \hspace{1cm} (2.74)

Intuitively, if \(\frac{d}{dt}\left(\frac{\vec{\omega}}{\rho}\right) \leq 0\) the system is stable and does not diverge. Hence the same conclusions could have been reached about Couette flow. However, they would not hold in all cases. It is entirely possible for a measuring parameter to decrease for a while and then diverge. This is where the Liapunov criteria played an important role. It furnished the conditions which must be placed on the measuring parameter, and then furnished proven theorems for stability of those parameters meeting the conditions. One important condition was that there not be any local maximums in the neighborhood of the origin.

The entire problem becomes clear when it is recognized that for two-dimensional incompressible flow, Eq. (2.38) reduces to Eq. (2.74) which is just a diffusion equation. There is no source term and no convection term. Thus there cannot be any instability. For the two-dimensional compressible flow, the situation is no different when the velocity profile is at most quadratic. The same holds true for other two-dimensional flows.
Furthermore, since diffusion by itself is a stable process, as long as the process has reached an equilibrium vorticity distribution for the rate of input at the wall, the equilibrium distribution will be stable to all disturbances even with some convection. As long as the diffusion dominates the convection this will hold.

Finally, the two-dimensional problem may be concluded by the observation that if a flow can be represented by only a vorticity potential it is a stable rotational flow. The flow can only be unstable if it has source and/or convection terms in it.

*See M. J. Lighthill (34); p. 89
III. GENERALIZATIONS OF THE MATH MODEL

1. Comments on the Classical Math Model

The previous section presented a new technique of analyzing the stability of fluid systems. Classical approaches have centered around small-perturbation theory. Because of the complexity of the problem many approximations have been made, leading to the math model represented by the Navier-Stokes equations.

The introduction of the Liapunov Direct Method now offers a mathematical technique for establishing and solving more general criteria. It is beyond the scope of this thesis to actually develop specific solutions for a more general system, but possible inadequacies in the classical math model will be considered and some improvements will be suggested.

There are four categories thru which the math model could be inadequate. They are:

(1) Inadequacy of the kinematical description of the flow, due to omission of terms in a more general description of flow of a deformable media;

(2) Failure to couple equations for rotational motion with translational motion so that there can be an interchange of energy between these two modes;
Failure to include other possible modes of stress-strain interaction such as phase-shift to account for possible viscoelastic effects on a microscopic but continuum scale;

(4) Failure to allow the fluid to support a skew-symmetric stress tensor thru viscoelastic character of the fluid on a small scale.

It is felt that on a small scale there might occur an asymmetric fluid deformation somewhat like that for a flexible solid, due to motion interactions. Even though the fluid might initially be and remain isotropic in a gross sense, it is suggested that under continuous oscillatory disturbances (with resulting amplitude increases), there might develop a microscopic anisotropic character of the fluid. There might be coupling phenomena and nonlinear responses. As local perturbations increase, these normally negligible features of the real material could become important. They might actually result in the creation of initial vortices in a fluid previously completely irrotational. It is not suggested that elasticity of the fluid is such that a large "stretch" ever occurs. Instead, it is suggested that only a very small amount of elasticity is sufficient to provide an additional energy transport mechanism from mean flow to the perturbation, just like viscosity does.
Originally viscosity was omitted from fluid stability analyses because it was known to be small and thought to be completely negligible. However, incorporation of even a small viscosity provided a transport mechanism that greatly influenced stability boundaries.

Classical fluid mechanics is based on linear Stokesian fluids called Newtonian fluids. The ideas presented herein suggest that creation of vorticity within an irrotational fluid and the prediction of transition require a fluid math model which is not only non-Newtonian but non-Stokesian.

It has been established clearly (27, 28, 29) that Newtonian fluids are merely a small part of a vast host of fluids having distinguishable properties. Thus there are a host of constitutive equations of exceeding complexity and generality in the literature (27-34). These equations in their full generality are not suitable for engineering application, but they are important to provide a framework within which observed behavior under a wide range of conditions can be correlated. Furthermore, they serve as sources of simpler equations valid under more narrowly prescribed conditions of flow. Thus, in light of the general equations it becomes much clearer just how much has been discarded by hypothesizing the simpler equations.
The simple linear relation between the stress tensor and rate of deformation in a fluid is really a drastic hypothesis in terms of how much is discarded. Serrin (30) emphasizes the fact that this is only a hypothesis, is not derived from experiments, and cannot be proved by abstract reasoning. It is only a model, and the hypothesis rests on comparison (with experiment) of results obtained on the basis of the hypothesis. Of course, the better the comparison for many cases, then the better the hypothesis and one's faith in its validity over a wide range of conditions. One therefore must afford great respect to the Navier-Stokes equations as a worthy mathematical model for a predominance of fluid mechanics problems. Still, understanding the origin of turbulence and predicting the beginning of transition has not really yielded to analyses based on approximate solutions of Navier-Stokes equations (1,16).

Almost all continuing assaults on this problem (2) initiate with the point of view that stability analyses of closer and closer approximations to the Navier-Stokes equations eventually will yield acceptable comparisons of predicted with experimental transition Reynolds's numbers.

Slight deviations to alter the viscosity coefficient (2) are possibly a small step in the right direction since the stability boundaries obtained seem to be improved under certain conditions. However, it seems that some of
the lighter veined "Don'ts of Mathematical Modeling" suggested by Solomon Golomb (35) might also help in solving this long standing problem to the satisfaction of scientists and engineers. For example, "Don't eat the menu," and "avail oneself of a sort of legalized polygamy." That is, don't believe that the model is reality, and don't limit yourself to a single model.

Before moving on to a discussion of specific inadequacies in the classical math model it is very desirable to consider some analogous and non-linear physical systems to obtain insight into their motion, how they deform, and how modes couple. This is done next.

2. Stability of a Physical System

The concepts of stability discussed in Section II are abstract concepts. Much additional insight into the real mechanisms of fluid motion and into the physical significance of various types of stability or instability can be gained from analogies with other non-linear physical systems.

The analogy considered here for illustrative purposes is one between the motions of a vehicle in free flight and the motions of an element of fluid responding to external and internal forces. The basis for the analogy is postulation of a general element which at times acts more like an elastic solid element, and at other times more
like a fluid. This is accomplished thru combined properties of both viscosity and elasticity of the element. When acting like a fluid, the element has very weak but not zero elasticity. The small amount of elasticity alters the energy transfer mechanisms similar to the way viscosity changed the problem of classical hydrodynamic stability.

Appendix B presents briefly a simplified case of the motions of a body in flight. The restricted motions discussed are only three degrees of freedom: translations along X and Y axes, and rotation in the XY plane.

Consider an arbitrary body in motion:

![Diagram of fluid element with secondary inertial axes](image)

Fig. 9 Moving Fluid Element, with Secondary Inertial Axes

These three degrees of freedom correspond to the longitudinal modes for an airplane, although the axis symbols shown are those appropriate to fluid flow. Under more general conditions these modes couple with three additional modes known as the lateral modes. The lateral modes are translation along Z and two rotations, about X and Y axes.

If the body is not rigid then conditions arise where elastic modes also couple into the motion. Where elastic
frequencies are much higher than natural frequencies of the rigid body motion, then the body still may be analyzed by uncoupling the modes. Under restrictive but not invalid conditions, the three longitudinal equations may be separated into a "trajectory" motion and an "oscillatory" motion (36-41). This is the situation considered in Appendix B. The oscillatory motion is described by

\[ \frac{d^2 \alpha}{dt^2} + f_1(t) \frac{d\alpha}{dt} + f_2(t) = 0. \] (3.1)

This equation has a damping coefficient and spring constant which are heteronomous for motions of a missile in flight. In conventional flight dynamics symbology they are respectively

\[ f_1(t) = \frac{C_{\alpha x} \rho VA}{2m} - \frac{(C_{m y} + C_{m x}) \rho VA l^2}{2I} \]

\[ f_2(t) = \frac{d}{dt} \left( \frac{C_{\alpha x} \rho VA}{2m} \right) - \frac{C_{m y} C_{\alpha x} \rho^2 V_A^2 l^2}{4Im} - \frac{C_{m x} \rho V^2 A l}{2I}. \] (3.2)

The coefficient \( C_{\alpha x} \) represents a spring constant for translation and \( C_{M x} \) represents a spring constant for rotation. The coefficients \( C_{m y} \) and \( C_{m x} \) are damping coefficients for rotational motion, and are negative
when dissipative. When $C_{L\alpha}$ is positive and $C_{M\alpha}$ is negative, the system is said to be "statically" stable.

The coefficient $C_{M\alpha}$ determines the basic static stability of the rotational mode. If the system has no dissipation then $C_{m\theta}$ and $C_{m\phi}$ are zero. Still, with no dissipation, the system may exhibit an increase or decrease in amplitude of the oscillation, as seen by the fact that the effective damping coefficient $f_{i}(t)$ depends also on $\rho C_{L\alpha}V$, a spring stiffness term. With no dissipation the amplitude envelope could increase or decrease due to a spring constant changing with time. This system is analogous to the following torsional spring system, where a disc is displaced downward with rotation but no tilt. For a pure translational

![Diagram of a disc in translation and rotation](image)

**Fig. 10** Disc in Translation and Rotation

displacement the disc will oscillate vertically. However, through asymmetric stress in the spring, the vertical
oscillation will couple with the rotational degree of freedom and rotation will begin. The initial energy will transfer back and forth between the two modes, sometimes being all rotational, sometimes all translational, and sometimes mixed. Uncoupling of the two modes corresponds to conditions where frequencies, inertias, and spring stresses are such that the energy transfer mechanism is too slow and the motion remains predominantly in one mode or the other until dissipated.

Consider an idealized frictionless coil spring with no dissipation of the oscillation energy and no torsional stresses. The resulting motion can be modified if the spring is chilled or heated as the disc oscillates. Chilling will cause the amplitude to decrease because of a gradual increase in spring constant, meaning that the available displacement potential energy would result in less elongation. Sufficient heating softens the spring such that the amplitude will gradually increase.

Now consider the oscillatory rotational motion of a missile in atmospheric flight. Let \( \alpha \) represent the angular motion, and let \( C_M \) represent a moment coefficient defined by \( M = \frac{1}{2} \rho V^2 A \alpha C_M \). Then Fig. 11 illustrates static stability for a small neighborhood of \( \alpha = 0 \). For rigid bodies in free flight, which do not have a fixed constraint, the range of \( \alpha \) can be extended with repeated
ranges of the same stable or unstable characteristics. This will be used to illustrate dynamics of a general body. To what extent the same would hold for motion of a fluid element is not critical to the analogy.

If the body has a preferred orientation it will have a "statically stable" orientation. That is, if slightly rotated and then released, say from \( \alpha = 30^\circ \), it will try to return to the equilibrium orientation at \( \alpha = 0 \). The arrows in Fig. 11 show this return. However, other equilibrium orientations may exist, i.e. other singularities may exist in state space. Equilibrium B is such a singularity, and it is unstable. Any small perturbation from Position B would cause the motion to diverge, and some new stable equilibrium would be sought. A disturbance along positive \( \alpha \) would cause the motion to diverge from B towards Position C, and a disturbance in the other direction would cause the motion to diverge towards Position A.

These static stabilities and instabilities are associated with the spring constant \( f_2(t) \) in the absence of dissipation. If the flight were such that the elements of \( f_2 \) were constant, then the problem would be a Cauchy problem depending on rotational velocity and displacement at \( t = 0 \). If the Cauchy-Lipschitz Theorem is satisfied, then a solution other than the trivial solution exists and is unique.
Fig. 11 Rotational Static Stability

Fig. 12 Energy Levels
If the spring constant is heteronomous then additional considerations enter. The system could be statically stable about Equilibrium A and non-dissipative, yet the amplitude $\alpha$ still could begin to diverge due to auxiliary conditions on $f_j(t)$. The spring constant of a fluid dynamics problem is altered through boundary conditions, heat transfer, etc. and the problem is not an initial value problem dependant only on Cauchy data. The problem requires both initial values and boundary values, and is a complicated mixed-value problem.

The missile problem is easily envisioned in an energy-displacement state space (Fig. 12). When

$$\alpha = 0, \pm 360^\circ, \cdots,$$

the system has a stable equilibrium which may be taken as a reference condition of zero potential energy. When $\alpha = \pm 90^\circ, \pm 270^\circ, \cdots$, the system has maximum restoring torque but not maximum potential energy. In this very real problem, the position of

$$\alpha = \pm 90^\circ, \pm 270^\circ, \cdots,$$

represent conditions of steepest gradient on an energy phase plane. At positions of

$$\alpha = \pm 180^\circ, \pm 540^\circ, \cdots,$$

the system has an unstable equilibrium from which the motion will diverge to seek one of the neighboring stable equilibriums. Hence, if the singularity at $\alpha = \pm 180^\circ$ were selected as the origin of the phase place diagram, then the resulting motion would spiral outward and seek a stable condition located
at \( \text{Radius} = 180^\circ \). With the origin placed at the singularity \( \alpha = 0 \), the trivial solution is stable. Clearly the ball has a maximum potential energy above the reference level when \( \alpha = \pm 180^\circ \), and these points are unstable singularities. With the origin placed at a stable singularity, any displacement within the region of stability defined by the open set \( \alpha < 180^\circ \) will result in the ball moving back towards \( \alpha = 0 \). If the spring constant is autonomous the "hills" will not change size, and lacking dissipation the ball will oscillate forever within \( \pm \alpha_i \), where \( \alpha_i \) is the initial displacement.

Under this condition the motion can be plotted as \( \pm \alpha_i \) vs. \( t \) and the envelope of the oscillation will remain constant. Such an oscillation is called an undamped or neutral oscillation. For the classical eigen-value problem of hydrodynamic stability it is the condition of zero damping coefficient which produces the neutral stability boundary.

In the terminology of flight dynamics this is a neutrally damped oscillation, but does not represent neutral stability of the ball. It represents undamped oscillatory stability of the ball. Neutral stability of the ball would correspond to the ball rolling on a flat plane so that if displaced from the reference in any manner it would not change its potential energy, and would not
return to the origin unless guided back.

It is well recognized that hydrodynamic flows may have more than one stable equilibrium, and that one laminar flow may become unstable and change into another laminar flow without going turbulent. If the ball being discussed represented that fluid flow problem it is easy to see how finite perturbations of the ball greater than

$$\alpha = \pm 180^\circ$$

would kick the ball over into another valley; i.e. into motion about a different stable equilibrium.

For fluid flows, it is not known yet how the entire potential field should be represented. Outside the valley around \( \alpha = 0 \) there may or may not be other valleys. The topology may be flat or hilly. Only careful analysis of each and every individual fluid flow situation would provide the necessary topology (if the problem were simple enough). It is unlikely that there is any one or two universal topologies for fluid flow, but certainly many familiar patterns will reappear as new problems are analyzed. It is now easier to understand that turbulence is something more than just any unstable laminar flow. Indeed, it is a complex topology.

The velocity-displacement phase plane for the problem is the familiar stable center for \( \{ \alpha \mid |\alpha| < 180^\circ, 180^\circ < \alpha < 540^\circ \text{, etc.} \} \).

The trajectory spirals into a center when the system is dissipative. Both centers shown are stable
singularities for the regions within the dotted circles.

Fig. 13 Simple Velocity-Displacement Phase Plane
Outside the circles the system is stable for these two centers, and the trajectory will move to some new condition. Possibly it will be a different stable condition at greater displacement from the selected location for the origin for the problem. The three trajectories shown in Fig. 13 are dissipative.

The topology described for this problem could include a third dimension, as shown in Fig. 14. Then the problem would represent a ball moving across a corrugated sheet. Furthermore, if the system were oscillatory in two degrees of freedom, Fig. 14 would have to be altered in the third dimension. Suppose $\alpha$ and $\beta$ represent angle-of-attack
Fig. 14 Topology of a Simple Motion and angle-of-sideslip of the body, and suppose $C_M$ depends on both in the same way. Then $(\alpha = 0, \beta = 0)$ represents a center as before, but the topology for a single center would be illustrated as in Fig. 15.

Fig. 15 Two-Degree-of-Freedom Topology

The dashed line represents the extent of the stable $\alpha\beta$-range. The topology is representative of a ball
rolling around in a bowl whose curvature is such that the edges are parallel with the base. A vector from the origin to the ball would trace out an $\alpha\beta$-motion analogous to the motion of a missile entering the earth's atmosphere (described as $\theta\psi$-motion in Appendix B).

A second type of instability is already apparent. In flight dynamics it is commonly called "dynamic" instability since it is manifested through the effects of motion and is determined by magnitude and sign of the damping coefficient $f(t)$ of Eq. (3.2). As indicated, the coefficient may be heteronomous, depending on $C_{L\alpha}$, $C_{m_\gamma}$, $C_{m_\alpha}$, $\rho$, and $V$ which all may be functions of time directly or through their dependence on still other heteronomous phenomena.

Consider again the ball having only a single degree of freedom described by $\alpha$ (Fig. 11). The case of a coil spring with time-variant static instability corresponds to a case where the height of the mountains varies with time, either increasing or decreasing. Thus, the resulting motion has either an increasing or decreasing maximum amplitude even though the system is non-dissipative. This increasing or decreasing of maximum amplitude (i.e. of the envelope of $\alpha$ vs $t$) is called dynamic instability. Most authors associate dynamic instability only with non-dissipative systems, but this illustration shows the more
general case. The important point to note is that a system can be dissipative yet have the envelope of its maximum amplitude diverge due strictly to change in effective spring stiffness. This is the case for certain trajectories followed by a re-entry vehicle. Even though the system clearly is dissipative and can be made statically stable, an angle-of-attack divergence can occur as the vehicle passes the point of maximum dynamic pressure and begins to enter a portion of the trajectory where dynamic pressure decreases. Static stability is directly related to dynamic pressure. A decrease in dynamic pressure has the effect of a decrease in the spring constant of the problem, which for constant total energy of the system would cause the amplitude envelope to grow. For certain steep re-entries this effect can be predominant during a portion of the trajectory, and can result in the amplitude envelope growing even though true damping (energy dissipation) is occurring through the $C_{m_y}$ and $C_{m_x}$ terms. Thus a system can be statically stable but dynamically unstable in more than one way, and therefore divergent.

It is clear, therefore, that the same possibility exists for different types of stability or instability in physical systems such as viscoelastic fluids. It is quite easy for simplified models to fail to account for
such complex energy transfer mechanisms as mode coupling and changes in effective spring constant. It is particularly worth noting that coupling is strongly dependent on effective moments and products of inertia; refer to Appendix A and B for these effects.

Visualization of the motion becomes difficult or impossible for additional degrees of freedom. However, the complex motions are more understandable through visualization of simplified and analogous systems. One of the obstacles to progress in understanding and description of fluid flow problems has been this exceeding complexity due to generally multiple degrees of freedom. The analogies and simplified motions discussed herein give insight into the concepts to follow. The further application of such analogies and topological techniques seems to offer a rewarding path for future research. However, models are seldom if ever exact and they should be used with considerable caution to avoid grossly erroneous ideas and conclusions.

Vibrations of an Elastic Rectangular Parallelepiped

The analogy has just been drawn that dynamics of a fluid element is in many ways similar to a body in free
flight. A further analogy comes from Brillouin (42)*, and he suggests that the problem of the decomposition of the vibrations of a given solid material in normal coordinates may have significant implications in the theory of stability of liquids and gasses.

Consider cyclic conditions applied to a solid which may be anisotropic but homogenous. In writing an expression for the potential energy density, if all terms of second degree and higher in the derivatives \( \frac{\partial^2 u}{\partial x^2} \) of the displacement are neglected, then waves will be propagated without deformation. However, if higher order terms in the derivatives are used, the waves propagated by cyclic disturbances would be deformed in the course of time. This is analogous to the propagation of waves in both liquids and gases.

For a solid enclosed with smooth rigid walls, Brillouin shows that there are three proper vibrations of the same wavelength: one longitudinal vibration \( \gamma \) and two transverse vibrations \( \alpha, \beta \) . Furthermore, it is possible to decompose an arbitrary vibration into two opposite rotational motions instead of into two rectilinear vibrations \( \alpha, \beta \) .

* L. Brillouin (42); pages 335-370, 452-561.
Consider the longitudinal and transverse motions for the case of two dimensions \( a_3 = 0 \). The longitudinal motion is irrotational while that of the standing waves created by the transverse waves is decomposed into a series of neighboring vortices.

Fig. 16 Decomposition of an Arbitrary Vibration

These vortex motions in a solid are limited by the elastic reaction of the body and appear as vibrations. In a continuous medium where the rigidity would vanish \( (\mu = 0) \), that is, in a liquid, no elastic force would oppose these vortex motions and they would continue always in the same sense instead of showing an oscillating character.

It is interesting to consider that the classical linear perturbations of laminar hydrodynamic stability theory may produce such vortices since the gas or liquid never really has \( \mu = 0 \).
Fig. 17 Proper Vibrations of an Enclosed Solid

If an initially isotropic body is deformed by means of a shearing strain, it becomes anisotropic and the analysis of elastic waves must be modified. For wave propagation in an anisotropic medium there would be different wave velocities in the directions of the three axes, and the waves would no longer be exactly transverse or longitudinal.

If \( \mu \) vanishes the isotropic solid is transformed into a perfect liquid since the body no longer offers any resistance to shearing strains. It cannot propagate transverse waves (which were the ones decomposed into vorticity), although it retains a finite compressibility defined by \( \lambda \). Thus, when the rigidity coefficient \( \mu \) vanishes, the transverse motions become free and are not coupled with the longitudinal oscillations. When viscosity is taken into account, transverse waves can be
propagated with damping. With vanishing viscosity no elastic force opposes the transverse motions which are then free. These motions cause the regular vortex systems shown. The coefficients $\lambda$ and $\mu$ are the apparent Lame coefficients.

These suggestions by Brillouin support the postulate herein that turbulence can begin within an irrotational fluid thru viscoelasticity coupling the translational and rotational modes.

3. **Viscoelastic Effects**

It would be desirable to develop a physically better model of turbulence production. Towards this goal, it is informative to compare photographs (43) of the start of vortices with computer solutions of math models (2) which have built-in vortex sheets. The results look very much alike and so the model and computation seem to be good in part. But how do the infinite-strength point vortices originate in the first place? Generally it is reasoned that they were induced in the fluid by the upstream wall. Even if this were correct, the idea of point concentrations of infinite vorticity still isn't appealing.

As an alternative, this thesis suggests that translational perturbations can induce rotation through viscoelasticity and nonlinearities in the kinematics and constitutive equation.
For achievable rates of stress application, the elastic nature of a fluid such as air is negligible macroscopically. However, on the microscopic but continuum scale, a general elastic character is postulated to be an important part of the energy transport influencing dynamics of fluctuations, just as viscosity was found to be. Much current fluid mechanics research is now oriented this way. Betchov (2, 44) replaced the ordinary viscosity in the Orr-Sommerfeld equation with a complex viscosity representing the model of a Maxwell Fluid. Resulting analysis of a Blasius boundary layer showed that a delay of viscous stress was destabilizing. For a shear layer the delay of viscous stress was found to be stabilizing. Betchov first evaluated a Maxwell model of the form

\[(3.3)\]

Later, the partial derivative was replaced by a total derivative so that the equation would account for past deformations of the fluid particle instead of past local conditions. Frederickson (28) and Naghdi and Wainwright (45) would suggest that the derivative should be a Jaumann derivative to further account for effects of actual rotation of the fluid element. Betchov (44) considers the stability of a two-dimensional, incompressible, and parallel flow, with the assumption that the viscous
stresses are related to the Eulerian velocity by:

\[
\left[1 + T_1 \left(\frac{\partial}{\partial x}\right)\right] \tau_{ij} = \rho \nu_0 \left[1 + T_2 \left(\frac{\partial}{\partial x}\right)\right] \left\{ \left(\frac{\partial u_i}{\partial x_j}\right) + \left(\frac{\partial u_j}{\partial x_i}\right) \right\}.
\]

Density \( \rho \) and the nominal viscosity \( \nu_0 \) are taken to be constants. The time constant \( T_1 \) is included to define effects due to rate of application of the stress, and \( T_2 \) describes the rate of change of velocity gradients. Small perturbations proportional to \( e^{i(\alpha x - \beta t)} \) are assumed to obey an Orr-Sommerfeld equation where the ordinary viscosity is replaced by

\[
\nu(\alpha) = \nu_0 \left(\frac{1 - i \alpha \alpha T_2}{1 - i \alpha \alpha T_1}\right) = \nu_{oo} e^{-i\theta}.
\]

The constant \( \nu_{oo} \) is real positive and \( \theta \) indicates a phase angle.

Betchov gives stability boundaries of \( \alpha \delta \) vs. \( \frac{U \delta}{\nu_{oo}} \) for various values of \( \gamma = \frac{T_1 U_0}{\delta} \) and phase angles \( \theta \).

The larger the value of \( T_1 \), the larger the unstable region. For a given rate of strain, if the stress which produced it must now be replaced by \( \left[1 + T_1 \left(\frac{\partial}{\partial x}\right)\right] \tau_{ij} \) instead of \( \tau_{ij} \), this corresponds to a lag in the stress behind the rate of strain. Betchov's results show that such a lag in stress is destabilizing. Thus phasing as well as magnitude of the disturbances could be expected to influence stability.

The fluid element should be taken initially undistorted. Then, as disturbances occur, in terms of effect lagging cause, the strain and rate-of-strain actually should lag the stress. The result is easily viewed in terms of strain relaxation. For a given value of the stress
tensor, if the stress is applied slowly in comparison to the relaxation time of the fluid, then no elastic effect would ever be noticed. That is, the parameter $T_i$ would be essentially zero. This properly models most of the fluid dynamic problems observed. However, there are fluid dynamic problems where steep gradients occur. Specifically, shock waves and concentrated high strength vortices are regions of strong gradients. Turbulent eddies could be very strong, fluctuate very rapidly, and be completely random. Hence it is suggested that such conditions involve a situation where the stresses fluctuate so rapidly that the fluid cannot respond in a normal manner. The rate of stress application then exceeds the rate of relaxation of the fluid. Classically, the assumption is made essentially that the relaxation rate is infinite, and this is far stronger than merely postulating that the relaxation rate is much quicker than rates of stress application. Specifically because of the stronger assumption, no strain is exhibited by the fluid. Strain is always relaxed faster than the forcing function can build it up.

The second effect in Eq. (3.4) involves a special case: time rate of change of rate of strain, i.e. the rate of change of the velocity gradients. The time constant $T_2$ was introduced to describe the possibility of sudden stresses resulting in a gradual shearing motion
in the fluid.

Betchov (44) discusses how in an oscillating turbulent flow, the massaging of the small eddies by the oscillation creates Reynolds stresses. He then notes that it is reasonable to question whether the Reynolds' stresses instantaneously follow the rate of strain, or respond with some lag. The lag would lead to the concept of an eddy viscosity varying with frequency and wavenumber of the wave. Of course, it is well accepted that the Reynolds' stresses are really a momentum transfer and feed energy from the mean flow to the perturbation.

Some of the principles that guide formulation of more general constitutive equations are as follows (27-32):

(1) coordinate invariance: Every constitutive equation must be stated in such a way that it is unaffected by choice of the coordinate system. If the equations are stated in tensorial form this will be satisfied.

(2) isotropy: The constitutive equation must reflect any symmetry or asymmetry of the material represented. If the material is isotropic (having no preferred directions of response) when it is in the undistorted state, then any rotation of the material coordinates must leave the constitutive equations unchanged when the reference configuration is taken to be an undistorted state. Truesdell's definition of a fluid requires the fluid to be isotropic.
and have the highest degree of symmetry. Furthermore, in his definition, any configuration can be taken as the reference configuration and treated as an undistorted state.

(3) **material indifference**: To be consistent with the notion that response of a material is independent of the observer, the constitutive equation must have a form such that it is not changed by an arbitrary rigid rotation of the frame of reference. That is, if the material and the system are both rotated at the same time the constitutive equation must remain the same. Even though the constitutive equation must obey the requirement of material indifference, the equations of motion do not.

**General Relation of Stress to Strain**

Frederick and Chang (33) suggest a more general equation of the form:

\[ \dddot{\mathbf{B}}_{i j k r} \dddot{e}_{k r} + \dddot{\mathbf{B}}_{i j k r} \dddot{e}_{k r} + \dddot{e}_{i j} = C_{i j} + C_{i j k r} e_{k r} + \dddot{C}_{i j k r} \dddot{e}_{k r} + \cdots \]  \hspace{1cm} (3.6)

Simple mechanical analogies are:

1. **generalized Hooke's law**: (elastic)
   \[ \sigma_{i j} = C_{i j k r} e_{k r} \]  \hspace{1cm} (3.7)

2. **linear viscous medium**: (viscous)
   \[ \sigma_{i j} = \dddot{C}_{i j k r} \dddot{e}_{k r} \]  \hspace{1cm} (3.8)
(3) Kelvin-Voigt body: (linear viscoelastic)
\[
\sigma_{ij} = C_{ijkr} e_{kr} + \tilde{C}_{ijkr} \dot{e}_{kr}
\]  
(3.9)

(4) Maxwell body: (linear viscoelastic)
\[
\tilde{B}_{ijkr} \dot{e}_{kr} + \sigma_{ij} = \tilde{C}_{ijkr} \dot{e}_{kr}
\]  
(3.10)

(5) others

It is interesting to note that, in effect, Stokesian fluids are Maxwell bodies at low rates of stress application, or Maxwell bodies with very low spring constants (i.e. very soft springs). Yet the Maxwell-body model should be modified so that a mechanism is available not only for springiness but also for transfer of energy from a translational displacement to a rotational displacement.

**Stokesian Fluid Postulates (31):**

(1) stress tensor \( \mathbf{T}_{ij} \) is a continuous function of the deformation tensor \( \mathbf{e}_{ij} \) and the local thermodynamic state, but independent of other kinematical quantities;

(2) the fluid is homogenous; i.e. \( \mathbf{T}_{ij} \) does not depend explicitly on \( \mathbf{x} \);

(3) the fluid is isotropic -, i.e. no preferred direction;
(4) when there is no deformation \( (\bar{\varepsilon}_{ij} = 0) \), the stress is hydrostatic \( (T_{ij} = -\rho \delta_{ij}) \).

The Stokesian fluid therefore is non-elastic and the relation between stress and strain is independent of the antisymmetric kinematical tensor.

For a compressible Newtonian fluid, the thermodynamic pressure is not equal to the mean pressure \( \rho_m \). The thermodynamic pressure is correctly given by

\[
\rho = \rho_m + \kappa I
\]

where \( \kappa \) = bulk viscosity

\[
\rho_m = -\frac{1}{3} T_{kk}
\]

All work done on a purely viscous fluid in shear is immediately dissipated as heat, whereas the work done on a perfectly elastic substance in shear is not dissipated but may be recovered by the elastic body regaining its original configuration.

It is suggested that the viscoelastic fluid exhibits both characteristics. The regain of elastic energy produces a phase shift enhancing instability, but a rotational retention of energy may account for delayed boundary layer transition to turbulence after laminar viscous instability theory shows instability to have developed. That is, according to classical theory the velocity perturbation may be going unstable and taking energy from the mean motion -- yet the perturbation may not be increasing without
bound if it can transfer energy on over into a rotational mode.

4. **Vorticity Orientation**

Actually, it is quite reasonable that the total vorticity vector might have preferred orientation. Possibly this preferred orientation will be distinctly one direction or another, rather than smoothly "liking" almost any orientation. Suppose meaningful stability of the flow includes more than just cases where the vorticity vector damps or diverges in magnitude while constantly directed in a preferred orientation. Meaningful stability could include motion where the vorticity vector begins to fluctuate rapidly in magnitude at a particular orientation. Then, as the energy of the fluctuation increases further, instead of the vector diverging in length it flips over to some new orientation stable for a different range of fluctuation energy. It is well known that the vorticity vector indeed is stably oriented differently for various flows. As a matter of fact there are distinct different stable orientations near a wall and further out in the stream (2).

This gives rise to one of the more frequent criticisms of the classical treatment as a prediction of turbulence in that it does not match with many experimental findings. Instability of a particular flow does not always result in
turbulence, but may merely result in a new stable laminar flow. The reason for this is now quite clear - the classical stability boundaries are not and never were based on stability of a generally adequate measuring parameter. It is clear that some sort of criteria on stability allowing deviations in orientation of the vorticity vector would be an improvement.

It has been emphasized in Section II that stability in a practical sense should be based on criteria regarding the acceptable level of motion around the origin. Classical hydrodynamic stability theory requires all flows not asymptotically stable in the translation parameters $u'$ and $v'$ to be classified unstable and proceeding to turbulence. This is not the case. The flow might be unstable in a small region about the origin, yet be oscillatory forever at some low level of fluctuation energy. Even if it did go divergent from that singularity it might proceed to stability for a different type of laminar flow instead of going to that random situation called turbulence.

Consider what the fluid element might look like while all these fluctuations are occurring. What could be occurring if the stably oriented but fluctuating vorticity vector flips to a new stable orientation, i.e. newly stable for a different range of fluctuation energy? Again, the larger-scale real world provides a clue. The dynamics
of all rigid bodies will act this way, as shown in Appendix B. There are different preferred modes of oscillation depending upon the type of excitation, but the motions also acquire a very strong dependence on instantaneous inertia characteristics of the body. For low excitation rates the body may respond smoothly with angular motion about the torque axis. However, if the torque is applied too rapidly, or if the angular rate gets too large, then the body may rather suddenly transfer some of the energy into rotation about a new axis. This is precisely the roll-coupling phenomena in flight of a slender high speed airplane. The required condition is merely a slight asymmetry of the body. Holstrom's work (41) on the nutations of a spinning slightly asymmetric body show the type of motions obtained (Appendix B). There should also be an analogy here to the concept of a kink in a vortex filament leading to formation of a large horseshoe vortex.

Thus, knowing that the vorticity vector in fluid flow may have several distinctly different preferred orientations at different places in the same flow, it seems reasonable that asymmetries of a rotating viscoelastic fluid element could provide the physical mechanism for conversion from one vorticity orientation to another. All that would be needed is for the fluid element to act instantaneously like an identifiable
element and have not zero but only very small elasticity so that it would hold together asymmetrically.*

This would also provide the needed mechanism for transfer of energy from a translational mode into a rotational mode. Hence, turbulence could originate from amplification of the classical translation modes to such a condition that strong density gradients develop, fluctuation frequency gets large, stresses are applied rapidly and changed rapidly, fluctuation energy begins to go divergent, and finally the asymmetries allow the energy to be transferred into a rotation mode.

5. **Hanks Coupling Criteria**

Hanks (46) put forth a theory of laminar flow stability very close to some of the concepts advanced by this thesis. He defines a stability parameter \( K \) representing the coupling ratio between the rate of change of angular momentum of a deforming fluid element and its rate of loss of momentum by frictional drag. The criteria is phenomenological in origin and the numerical value of

* Refer to Section III.2, and to Appendix B. There are actually two effects thru the postulate of some character of rigidity and elasticity of the element. There is the high frequency effect of wave propagation with all its possible wave superpositions and distortions. There also is the slower effect of non-classical kinematics of small continuum elements involving response times, asymmetries, mode coupling, memory, etc.
the constant is found only by experiment. It was found that when the coupling ratio $K$ reached a critical value $K = 404$, the element becomes unstable to rotational disturbances and turbulent eddies begin as a series of bursts which subsequently decay into fine scale eddies.

Hanks states that the turbulent eddy eruptions appear to be essentially non-linear propagations of unstable rotational motions and therefore surmises that the key mechanism in the development of a flow instability leading to transition to turbulent motion is one of the rotational momentum balance.

The development of his criteria is brief and lacks the necessary clarification of how such coupling occurs. He gives the time rate of change of linear momentum

$$\frac{\delta m^i}{\delta t} = \int \left\{ \frac{2}{\delta t} (\rho u^j) + \left( \rho u^i u^k \right)_i \right\} dV. \quad (3.11)$$

The integrand function is expanded as

$$\rho \frac{\partial u^i}{\partial t} + \frac{1}{2} \rho g^{jk} \left( u^j \right)_k + \rho g^{ij} \epsilon_{kpq} \omega^p u^q, \quad (3.12)$$

where

$$\omega^p = \epsilon^{pqr} g_{rs} u^s, = \text{vorticity of the flow}. \quad (3.13)$$

Thus the total rate of change of momentum is resolved into three parts:

(a) \( \rho \frac{\partial u^j}{\partial t} \) = instantaneous local rate of increase of momentum
(b) \[ \frac{1}{2} \rho g^i k (v^2),_k \] = rate of change of translational momentum (3.14)

(c) \[ \rho g^i k \epsilon_{kpq} \omega^p u^q \] = rate of change of angular momentum of flow.

He gives the resultant force on the material element as
\[ R^j = \int_V \rho f^j \, dV + \int_S T^{ji} n_i \, ds \] (3.15)

where
- \( f^j \) = external body force per unit mass
- \( T^{ji} \) = contravariant stress field tensor giving rise to a tractive force \( T^{ji} n_i \) on the surface \( ds \) having a covariant unit outward normal vector \( n_i \).

The surface force term is written as
\[ \int_S T^{jk} n_k \, ds = \int_V \left\{ - g^{ik} p_{,k} + \mathcal{P}^{ik} \right\} \, dV \] (3.16)

according to assumptions described earlier herein. The term \( \mathcal{P}^{ik} \) is the viscous stress tensor and \(-p\) is the mean normal stress. The negative of \( p^{ik} \),\(_{,k}\) represents the rate of loss of momentum per unit volume from the element due to the action of viscous traction, and Hanks gives it the symbol \( \tau^{ik} \)\(_{,k} \). He then forms the ratio
\[ K = \frac{\rho \left[ g^{jk} \epsilon_{jpq} \omega^p u^q \epsilon_{krs} \omega^r u^s \right]^{1/2}}{\left[ g_{jk} \tau^{jp} \tau^{kq} \right]^{1/2}} \] (3.17)
and states that when $K$ reaches $K = 404$ the rotational instability occurs. He lists additional references wherein further comparisons are made with experimental data. The important point is that a meaningful correlation involving a ratio between a translational parameter and a rotational parameter can be made.

The criteria advanced by Hanks indicates he is comparing total moment of momentum to total viscous traction. That is, he is comparing a contribution from the skew-symmetric part of the velocity gradient tensor to the total viscous stress tensor (after normal stress has been taken out). The viscous stress tensor also may be written as the sum of a symmetric and a skew-symmetric part. As well known, and reiterated herein, classically the conservation of moment of linear momentum is used as an argument for equating the two skew-symmetric tensors and removing them from the equation of motion. It is well known that this is done on the basis of the original assumption by Stokes that the fluid was purely viscous (i.e. no elasticity), could support no asymmetrical deformation, and therefore the shear tensor depends only on symmetric relative motion. Hanks never mentions that this is really a key point in the whole argument. The coupling he recognizes cannot occur without first refuting the applicability of the
viscous fluid hypothesis to modeling of the origin of turbulence. The viscous fluid hypothesis allows only symmetrical shear and "rigid body" rotation (a completely impossible physical situation for a fluid even if the model does give astoundingly useful results).

Further evidence that Hanks has not recognized this key mechanism is that he cites references to the fact that the source of turb in a turbulent stream is in the boundary layer near the wall. This is the classical and still current concept, and indeed this thesis does not disagree with it in part. However, this thesis does disagree that it is the only possible source of turbulence.

The previously discussed classical concept or model of a fluid element developing "rigid body" rotation at a wall similar to the rotation developed by airplane wheels as they touch a runway is somewhat realistic, but there must be an explanation then of why an element acts so rigid at one instant and yet at another instant out in the fluid it has no rigidity and cannot support asymmetric shear stress. Classically, the explanation is acknowledged to be lacking and is still lacking in Hanks papers.

It is really not too difficult to see that rate of application of stress must be involved. If a fluid element moving with its neighbors suddenly gets too close to the
wall and bumps the wall, the rate of application of asymmetrical shear stress on the surface of the element is exceedingly large. It is not surprising that the element displays a character of rigidity and spins. Now, having once acknowledged that under certain circumstances a fluid element acts like an identifiable continuum and displays some rigidity at the particular instant of contact, it is an invalid assumption to deny the further opportunity at some later time. It also is invalid to assume a rigid body model with these traits of a "fluid element" at one spatial position only (i.e. at the wall) and deny them elsewhere by assuming a purely viscous model everywhere else. The classical assumption of Stokes that a fluid is purely viscous is not correct. If it were correct a flow could never develop turbulence anywhere (wall or otherwise). The fallacy is that it is invalid to model a single continuous problem with two distinct, discontinuous models.

Since it is well known from experimental data that a laminar flow does develop turbulence at a wall, and since the purely viscous model does give good results for almost all flow phenomena not directly adjacent to the wall, the whole situation is rectified by properly structuring a single model so that the model fits both sets of known phenomena and results.
The clue as to how this should be done still lies in the analogy to airplane wheels on touchdown. In that situation the rate of stress application is extremely fast. Elsewhere in the flow the rate of stress application on a fluid element generally is not so fast because the "things" a fluid element touch are not rigid; they are soft and deformable, and stress is applied between fluid elements in a generally smooth continuous manner. Thus, rate of stress application should be recognized as important: maybe important only once in a while, but highly important in those certain situations.

Such a fluid with dependence on rate of application of stress exhibits one of several features called viscoelastic. There are many models for including such effects, some of which have already been noted. It is not within the scope of this thesis to say which if any of the current rheological models is best. This can only be done with further extensive analysis and experimental data. However, the following important characteristics of an improved model are postulated:

(1) it will exhibit non-homogeneity under adequately severe fluctuating rates of stress application;

(2) it will exhibit combined characteristics of a general Maxwell fluid and a Kelvin-Voigt solid.

(3) it will exhibit characteristics of fading memory;
(4) it will exhibit asymmetries and anisotropic features of the dynamics of a non-linear elastic solid experiencing finite deformations due to repeated, random, continuously distributed surface stresses;

(5) at high rates of stress application and/or high rates of rotation about an axis, the element will transfer energy between rotational modes, between translational and rotational modes, and possibly between microscopic and molecular modes.

Eventually experimental data will become available for evaluating the degree to which each of these features enters the problem. Then, there will still be differences depending on the fluid and the flow situation of interest.

Returning to the immediate task at hand, a three-dimensional viscoelastic model serves to remove the classical use of two separate "fluid" models (rigid, and purely viscous) in the same problem. Intermixed Maxwell-Kelvin-Voigt properties yield models which depend on a time constant multiplied by the first derivative of the stress tensor. For a time constant of zero the simple Maxwell fluid is identical to the Navier-Stokes fluid (depending on the temporal and spatial scale of the observation). However, if the rate of stress application becomes large enough (say approaching infinity, as is assumed to occur when a fluid element bumps the wall) then
the rate of stress application becomes important in the problem. The three-dimensional viscoelastic element will support asymmetric stress and will rotate (possibly with some asymmetric deformation) upon contact with the wall. Then as the element is cast away from the wall, some of the asymmetry may relax and result in partial restoration of initial shape. However, due to viscosity, it is certain that the original configuration will not be exactly returned. It would suffice to expect the element to have lost translational momentum and to have asymmetrically deformed to some new shape with increased angular momentum.

Now if this process can occur at the wall it can occur anywhere that the necessary conditions exist. If characteristics of rigidity or elasticity can occur when a fluid element bumps the wall, they can occur anywhere else that the rates of stress application becomes sufficiently large. Hence, if any viscoelasticity exists (and it must), then various flow situations can occur which will amplify disturbances to the condition wherein non-linearities (such as discussed herein, and by Hanks (46), Stuart (14) etc.) become important, gradients become severe, fluctuations become rapid, rates of stress application become important on a small enough temporal and spatial scale, and viscoelastic effects are commanded into action. Asymmetries will occur, rotations will
develop, and the random dynamics known as turbulence will begin.

Hanks correctly attacks the problem by phenomenologically determining a criteria relating growth of rotational angular momentum to a translational parameter such as rate of loss of momentum per unit volume. However, one should consider the various aspects of the change occurring.

6. Classical Equation with Skew-Symmetric Terms

The retention of the skew-symmetric terms in the equation of motion alters the dynamics and the stability criteria. Still, it does not provide a true source term.

Let the shear stresses $\tau_{ij}$ be decomposed into a symmetrical and skew-symmetrical parts. That is
\begin{equation}
\tau_{ij} = \frac{1}{2} (\tau_{ij} + \tau_{ji}) + \frac{1}{2} (\tau_{ij} - \tau_{ji})
= \eta_{ij} + \xi_{ij}
\end{equation}

Then assuming a stress-strain postulate:
\begin{equation}
\eta_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\end{equation}

A coefficient of proportionality for rotation is assumed to be of similar form:
\begin{equation}
\xi_{ij} = \tilde{\mu} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right).
\end{equation}

Equation (2.36) then becomes
\begin{equation}
\rho \frac{d\vec{u}}{dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) + \tilde{\mu} \nabla \times \vec{u}.
\end{equation}
Equation (3.21) is now put into vorticity form as before. Proceeding as before produces a modified equation for rate of change of vorticity, and then a modified stability equation. By inspection of Eq. (3.21) it is clear that the result will again omit any possibility of true vorticity production when there is none initially.

By analogy with Eq. (2.38) the result will be

$$\frac{d}{dt}(\frac{\Omega}{\rho}) = \left(\frac{\Omega}{\rho} \cdot \nabla\right) \vec{U} + \frac{\rho'}{\rho} \nabla^2 \vec{\Omega} - \frac{\rho}{\Omega} \left[\nabla \left(\frac{\rho}{\Omega}\right) \times (\nabla \times \vec{\Omega})\right] + \frac{\rho}{\Omega} \left[\nabla \left(\frac{\rho}{\Omega}\right) \times (\nabla \times \vec{\Omega})\right]$$

(3.22)

Obviously there is a distinct addition to the rate of change of vorticity. However there still is no increase in vorticity if vorticity initially is zero, hence this equation still contains no true source term.

7. Classical Equations with Second-Order Kinematics

Returning to Eq. (2.33) and retaining higher order and cross-product terms, the acceleration terms to second order are:

$$a_x = \frac{2u}{2t} + \frac{du}{dx} + \frac{u}{2y} + \frac{w}{2z} + \frac{u^2}{2} \frac{\partial^2 u}{\partial x^2} \Delta t + \frac{u^2}{2} \frac{\partial^2 u}{\partial y^2} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 u}{\partial z^2} \Delta t + \frac{\omega u}{2} \frac{\partial^2 u}{\partial x \partial y} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 u}{\partial x \partial z} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 u}{\partial y \partial z} \Delta t + \frac{\partial^2 u}{\partial z^2} \Delta t$$

$$a_y = \frac{3v}{3t} + \frac{dv}{dx} + \frac{v}{2y} + \frac{w}{2z} + \frac{u^2}{2} \frac{\partial^2 v}{\partial x^2} \Delta t + \frac{u^2}{2} \frac{\partial^2 v}{\partial y^2} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 v}{\partial z^2} \Delta t + \frac{\omega v}{2} \frac{\partial^2 v}{\partial x \partial y} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 v}{\partial x \partial z} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 v}{\partial y \partial z} \Delta t + \frac{\partial^2 v}{\partial z^2} \Delta t$$

$$a_z = \frac{3w}{3t} + \frac{dw}{dx} + \frac{w}{2y} + \frac{v}{2z} + \frac{u^2}{2} \frac{\partial^2 w}{\partial x^2} \Delta t + \frac{u^2}{2} \frac{\partial^2 w}{\partial y^2} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 w}{\partial z^2} \Delta t + \frac{\omega w}{2} \frac{\partial^2 w}{\partial x \partial y} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 w}{\partial x \partial z} \Delta t + \frac{\omega^2}{2} \frac{\partial^2 w}{\partial y \partial z} \Delta t + \frac{\partial^2 w}{\partial z^2} \Delta t$$

(3.23)
The originally retained classical terms are boxed. It is clear that the other terms involve rates of change of velocity gradients, a time scale, and when cast as momentum flux will involve a spatial scale. Following Schlichting (1), for constant density the fluxes of momentum at a surface of area $dA$ are

\[
\begin{align*}
    dJ_x &= dA \cdot \rho u^2 dt \\
    dJ_y &= dA \cdot \rho u v dt \\
    dJ_z &= dA \cdot \rho u w dt, \text{ etc.}
\end{align*}
\]

(3.24)

Thus, the momentum flux per unit area per unit time is an apparent stress

\[
\begin{align*}
    \sigma_x^x &= \rho u^2 \\
    \tau_{xy} &= \rho u v \\
    \tau_{xz} &= \rho u w \text{ etc.}
\end{align*}
\]

(3.25)

Thus, when Eq. (3.23) are utilized for a more accurate description of the acceleration of the fluid element and $\Delta t$ is taken as $T$ indicating the time scale, the equations of motion may be written as

\[
\begin{align*}
    \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial z} \right) &+ \frac{T}{2} \left\{ \sigma_x \frac{\partial^2 u}{\partial x^2} + \sigma_y \frac{\partial^2 u}{\partial y^2} + \sigma_z \frac{\partial^2 u}{\partial z^2} \\
    &+ 2 \tau_{xy} \frac{\partial^2 u}{\partial x \partial y} + 2 \tau_{xz} \frac{\partial^2 u}{\partial x \partial z} + 2 \tau_{yz} \frac{\partial^2 u}{\partial y \partial z} + \rho \frac{\partial^2 u}{\partial t^2} \right\} \\
    &= \rho g_x + \left( \frac{\partial^2 \eta_{xx}}{\partial x^2} + \frac{\partial^2 \eta_{yx}}{\partial x \partial y} + \frac{\partial^2 \eta_{yx}}{\partial y^2} \right) \\
    \text{etc.}
\end{align*}
\]

(3.26)
Compactly, the equation of motion is
\[
\rho \left[ \frac{3}{2} \frac{\partial \bar{u}}{\partial t} + \frac{1}{2} \nabla u^2 - \bar{u} \times (\nabla \times \bar{u}) \right] + \left[ \frac{\rho}{2} \frac{\partial \bar{\theta}}{\partial t} \right] + \frac{T}{2} \sum \tau_{i,j} \frac{\partial^2 u_k}{\partial x_i \partial x_j} = \nabla \rho + \bar{\theta} + \nabla \cdot \bar{\phi}'
\] (3.27)

where \( \bar{\phi}' \) now should retain skew-symmetric terms, and all the del operators are now second order. The curl of this equation produces a more general equation for time rate of change of vorticity, and it is found that true source terms do exist.
IV. CONCLUSION

The dynamics of a laminar flow have been investigated using an entirely different approach. The classical approach has been reviewed to establish state-of-the-art and point out areas of inadequacy. The classical approach has been found lacking both in the math model postulated and in the analytical techniques used to investigate stability of the model.

To establish a basis for the new work, general considerations in stability analysis of non-linear systems were described. Various concepts of stability were explained, and it was shown that classical concepts used in hydrodynamic stability theory are very limited in their ability to describe phenomena in multi-dimensional non-linear physical systems.

The Direct Method of Liapunov was introduced and well-established sufficiency theorems were selected for application to the Navier-Stokes equations of motion in vorticity form. Bounding of the vorticity vector was shown to provide several possible measuring parameters for a Liapunov analysis. The square of the radius of the bounding sphere was selected and used to establish a general three-dimensional criteria guaranteeing stability of the flow.
The criteria was then reduced to show that for a two-dimensional flow with a two-dimensional disturbance the solutions are the same as the inviscid and viscous parts of the Orr-Sommerfeld equation.

The criteria was applied to plane parallel flow and it was shown that Couette flow is stable for all disturbances; a result suspected but not previously proven. This result can be extended to draw further conclusions about stability of various velocity profiles.

The implications of the local vorticity vector changing orientation or going unstable in orientation as well as magnitude were related to viscoelasticity of the fluid. Hence, viscoelasticity was found to have a second previously unrecognized effect. In addition to its recognized role in providing a phase shift between stress and strain, it was shown to provide the key role in transfer of linear momentum to angular momentum. It holds an identifiable element together long enough for these two modes to couple and alter the dynamics of the system. An analogy of the dynamics of an airplane, missile, or spring-mass system was used to illustrate effects of inertial coupling, asymmetries, and time-variant damping and spring constant.

Second-order kinematics were shown to introduce terms which might be called bulk Reynolds stresses. These terms
arise very naturally when the derivative is recognized to be an approximation valid in the limit as the time scale approaches zero. If the time scale is not taken to zero but is only taken to that appropriate for turbulent flow with small vortices, the new terms provide a previously omitted momentum transport. It is a bulk transport analogous to the classical Reynolds stress which comes out of the first-order terms taken as a mean plus a perturbation.

The following topics are suggested for future research:

(1) Consideration of alternate measuring parameters;
(2) Investigation of suitable viscoelastic effects for providing both phase-shifts and coupling;
(3) Investigation of the effects of the identified second-order terms;
(4) Numerical and experimental determination of the conditions for the initial start of microscopic local rotations;
(5) Incorporation of the several identified effects into a unified math model. This model would probably combine momentum transport theory, vorticity transport theory, and theory for vorticity creation since it would seem to have the ability to match with key features of each in their respective regions of applicability.
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B. Stability of Motion


APPENDIX A
KINEMATICS OF AN ELEMENT

1. Acceleration of a Point

Consider an elemental cube of material under conditions such that it is a continuum. Let this cube translate, rotate, and distort in shape. Let a right-handed orthogonal axis system \((xyz)\) be located at the instantaneous center of mass (Point 0) of this cube. Then consider the velocity and acceleration of a neighbor \(P\) somewhere in the material. The absolute values are determined relative to a fixed or inertial axis system.

\[
\vec{\ddot{R}} = \vec{\ddot{\rho}} + \vec{\ddot{n}}
\]

\[
\vec{\ddot{\omega}} = \hat{\iota} \omega_x + \hat{j} \omega_y + \hat{k} \omega_z
\]

\[
\vec{V} = \frac{d\vec{\rho}}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{\rho}}{dt}
\]

\[
\vec{\ddot{v}} = \hat{\iota} x + \hat{j} y + \hat{k} z
\]

Fig. 18 Absolute Acceleration of a Point
The velocity of $P$ according to classical dynamics is:

$$
\vec{V}_p = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{x}}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{x}}{dt} + (\vec{\omega} \times \vec{r})
$$  \hspace{1cm} (A.1)

$\vec{V}_p$ = velocity of $P$

$\vec{\omega} \times \vec{r}$ = velocity due to rotation of the $(xyz)$ moving axes

$\frac{d\vec{r}}{dt}$ = velocity of $P$ relative to $(xyz)$ moving axes;

= $\dot{r} \hat{x} + \dot{r} \hat{y} + \dot{r} \hat{z}$

$V_o$ = velocity of moving origin;

= $\dot{r} \hat{x}_o + \dot{r} \hat{y}_o + \dot{r} \hat{z}_o$.

Hence

$$
\vec{V}_p = (\dot{r} \hat{x}_o + \dot{r} \hat{y}_o + \dot{r} \hat{z}_o) + \left[ \dot{r} \hat{x} + \dot{r} \hat{y} + \dot{r} \hat{z} \right] + \dot{r} (\vec{\omega}_y - \vec{\omega}_z) + \ddot{r} \left( \vec{\omega}_x - \vec{\omega}_y \right). 
$$  \hspace{1cm} (A.2)

The acceleration of $P$ according to classical dynamics is:

$$
\vec{A}_p = \frac{d\vec{V}_p}{dt} = \frac{d\vec{V}_o}{dt} + (\vec{\omega} \times \vec{V}_o)
$$

$$
= \left[ \frac{d\vec{V}_o}{dt} + \frac{\vec{V}_o^2}{2} + \frac{\vec{V}_o}{S} (\vec{\omega} \times \vec{r}) \right] + \vec{\omega} \times \left[ \vec{V}_o + \frac{\vec{V}_o}{S} + (\vec{\omega} \times \vec{r}) \right] 
$$

$$
= \vec{A}_o + \frac{\vec{V}_o^2}{S} + \left( \frac{\vec{V}_o}{S} \times \vec{r} \right) + \left( 2 \vec{\omega} \times \frac{\vec{V}_o}{S} \right) + \left[ \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right]
$$  \hspace{1cm} (A.3)

where

$$
\vec{A}_o = \frac{\vec{V}_o}{S} + (\vec{\omega} \times \vec{V}_o)
$$

$$
\frac{d\vec{r}}{dt} = \ddot{r} \hat{x} + \ddot{r} \hat{y} + \ddot{r} \hat{z}
$$

$$
\frac{d^2\vec{r}}{dt^2} = \dddot{r} \hat{x} + \dddot{r} \hat{y} + \dddot{r} \hat{z}
$$  \hspace{1cm} (A.4)
Performing the indicated operations yields
\[
\dddot{x}_p = \frac{1}{\rho} \left[ \dddot{x}_o + \omega_y V_{o_y} - \omega_x V_{o_x} + \dddot{z} + \dot{\omega}_y z - \dot{\omega}_x y + 2(\dot{\omega}_y \dot{z} - \dot{\omega}_z y) \\
+ y \omega_x \omega_y + z \omega_x \omega_z - x (\omega_y^2 + \omega_z^2) \right] \\
+ \frac{1}{\kappa} \left[ \dddot{y}_o - \omega_x V_{o_x} + \omega_z V_{o_z} + \dddot{x} + \dot{\omega}_z x - \dot{\omega}_x z + 2(\dot{\omega}_z \dot{x} - \dot{\omega}_x \dot{z}) \\
+ x \omega_x \omega_z + z \omega_z \omega_y - y (\omega_z^2 + \omega_x^2) \right] \\
+ \frac{\zeta}{\kappa} \left[ \dddot{z}_o + \omega_x V_{o_x} - \omega_y V_{o_y} + \dddot{y} + \dot{\omega}_x y - \dot{\omega}_y x + 2(\dot{\omega}_x \dot{y} - \dot{\omega}_y \dot{x}) \\
+ x \omega_x \omega_y + y \omega_y \omega_z - z (\omega_x^2 + \omega_y^2) \right].
\] (A.5)

The terms originating from \( \frac{\delta^2 \dddot{r}}{\delta t^2} \) and \( \frac{\delta^2 \dddot{r}}{\delta t^2} \) represent a distortion of the initial element under consideration. Eq. (A.5) gives a general description of the actual accelerations which would be experienced by a point somewhere in the fluid element. These terms only appear when the element has finite size and some ability to hold together rigidly or elastically.

Equation (A.3) is identical to the infrequently seen Jaumann derivative used by Oldroyd*. When applied to tensor quantity the operations of raising or lowering indices commute so that covariant and contravariant equations formulated in Jaumann derivatives are identical.

For clarification, note that the above derivation is indeed used in fluid mechanics, but for curved flows. In that case the angular velocity is due to the curved

*See Frederickson (11)
path of the mean flow. The \( \ddot{\omega} \) then does not account for rotation of the fluid element within the surrounding flow, or for rotation of the element relative to the moving axis system. The reason is that a classical Stokesian fluid element (in contrast to an element having some rigidity) is assumed to have no elasticity and can support no angular momentum.

Occasionally a flow is shown in vorticity form, where vorticity is defined as twice the rigid-body angular velocity, or

\[
\dot{\mathbf{\Omega}} = 2\mathbf{\omega} = \nabla \times \mathbf{v} .
\]  

Thus, the classical equation of motion in vorticity form is obtained by taking the curl of the classical linear equation of motion. When this is done, a portion of the correct general kinematical expression for angular moments will be missing because there is no term for intrinsic angular momenta or for elastic effects.

2. Acceleration of a Fluid Element with Higher Order Terms

Consider some fluid property \( \dot{Q} \) associated with \( \mathbf{P} \) at time \( t \). At some time later (\( t+\Delta t \)) the value of this property will change from \( \dot{Q}(\mathbf{r}, t) \) to \( \dot{Q}(\mathbf{r} + \mathbf{v} \Delta t, t + \Delta t) \).

Then \( \Delta \dot{Q} = \dot{Q}(\mathbf{r} + \mathbf{v} \Delta t, t + \Delta t) \). Using a Taylor's expansion,

\[
\dot{Q}(\mathbf{r} + \mathbf{v} \Delta t, t + \Delta t) = \dot{Q}(\mathbf{r}, t) + \left( \frac{\partial \dot{Q}}{\partial \mathbf{r}} \right)_{\mathbf{r}, t} \Delta t + \frac{1}{2} \left( \frac{\partial^2 \dot{Q}}{\partial \mathbf{r}^2} \right)_{\mathbf{r}, t} \Delta t^2 + \left( \frac{\partial \dot{Q}}{\partial t} \right)_{\mathbf{r}, t} \Delta t + \frac{1}{2} \left( \frac{\partial^2 \dot{Q}}{\partial t^2} \right)_{\mathbf{r}, t} (\Delta t)^2 + \cdots\]

(A.7)
where \( S \) denotes distance in the direction of velocity at point \( p \) at time \( t \).

\[
\frac{d\mathbf{Q}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{Q}}{\Delta t} = \lim_{\Delta t \to 0} \frac{Q(\mathbf{r} + \mathbf{v} \Delta t, t + \Delta t) - Q(\mathbf{r}, t)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \left[ \left( \frac{\partial \mathbf{Q}}{\partial t} \right)_{\mathbf{r}, t} + \frac{1}{2} \left( \frac{\partial^2 \mathbf{Q}}{\partial t^2} \right)_{\mathbf{r}, t} \Delta t + \cdots + \left( \frac{\partial \mathbf{Q}}{\partial S} \right)_{\mathbf{r}, t} \mathbf{v} + \frac{1}{2} \left( \frac{\partial^2 \mathbf{Q}}{\partial S^2} \right)_{\mathbf{r}, t} \mathbf{v}^2 \Delta t + \cdots \right].
\]

(A.8)

Thus, depending on how small \( \Delta t \) can be taken on an experimental or computational basis, additional terms might be important if the rates of change indicated by the higher derivatives ever become large enough.

In almost all cases the time scale can be taken small enough so that its product with the higher order derivatives is still small. This gives the customary value for the derivative at a point,

\[
\frac{d\mathbf{Q}}{dt} \to \frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial S} \mathbf{v},
\]

(A.9)
as $\Delta t \to 0$. However, the computation increment and the fluid element size are not shrunk to zero. Therefore, consider the next level of accuracy. For example, if the quantity $Q$ is taken to be the velocity of the flow, the components of acceleration correct to second order are

$$a_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 u_i}{\partial t^2} \right) \Delta t + u_i \left( \frac{\partial u_i}{\partial x_j} \right) + \frac{u_i u_j}{2} \left( \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right) \Delta t.$$  \hspace{1cm} (A.10)

Thus inflection of the velocity profile, cross gradients, and jerk terms appear. Furthermore, note that if $u_i$ and $u_j$ are fluctuating, the magnitude of certain terms depends on phasing of the velocity components. The increment $\Delta t$ is really only a time scale, or a characteristic time $T$. It is related to a spatial scale by

$$u_i \approx \frac{\Delta x_i}{\Delta t}$$  \hspace{1cm} (A.11)

where $\Delta x_i$ could be a characteristic length indicating the size of the elements being observed. Analogous to the mean Reynolds number, one might obtain an experimental correlation with a new transport number $N$, where

$$N = \frac{\rho u_i \Delta x_j}{\mu} = \frac{\rho u_i u_j \Delta t}{\mu}.$$  \hspace{1cm} (A.12)

This represents a momentum transport, and when this number gets large there are significant alterations in
the dynamics of the problem. It is clear that the classical acceleration terms are only an approximation which could be inadequate for various special flow regions.

In order to be consistent on the level of approximation being retained, it is necessary to note that a second-order nabla operator is being applied in Eq. (A.10) to obtain the third and fourth terms on the right-hand side. These two terms give \( \ddot{\mathbf{u}} \cdot \nabla \mathbf{u} \). Hence, in the equation of motion the second-order operation should also be used in obtaining divergence of the stress tensor. Finally, to obtain time rate of change of vorticity one should again consider the second-order nabla when taking the curl of acceleration.

3. Time Rate of Change of Moment of Momentum

Considering only surface forces, the torque on a small element of material is equal to the time rate of change of moment of momentum:

\[
\mathbf{\tau} = \frac{d}{dt} \left[ m (\mathbf{\hat{r}} \times \mathbf{\hat{v}}) \right] = \frac{\delta \mathbf{\hat{H}}}{\delta t} + \left( \mathbf{\hat{\omega}} \times \mathbf{\hat{H}} \right). \tag{A.13}
\]

Here \( \mathbf{\hat{H}} \) is the angular momentum of the element, not including any intrinsic internal angular momentum.

Let \( m_i \) be an elemental sub-portion of the small volume under consideration, and let it be located such that it is considered to be a "point" mass at \( P \). Develop now the time rate of change of angular momentum of
this mass \( m_i \), keeping higher order terms, and then sum over the small volume. Evaluate first \( \frac{\delta \mathbf{H}}{\delta t} \) and then \( (\mathbf{\omega} \times \mathbf{H}) \).

\[
\mathbf{r} = \mathbf{r} + \mathbf{\tau} y + \mathbf{\tau} z
\]

\[
\mathbf{v}_R = \frac{\partial \mathbf{r}}{\partial t} = \frac{\delta \mathbf{r}}{\delta t} + (\mathbf{\omega} \times \mathbf{r})
\]

\[
\mathbf{\omega} = \mathbf{\omega}_x + \mathbf{\omega}_y + \mathbf{\omega}_z
\]

**Fig. 20** Incremental Mass at One Point of an Element

The term \( \frac{\delta \mathbf{H}}{\delta t} \) expands as:

\[
\frac{\delta \mathbf{H}}{\delta t} = \frac{\delta}{\delta t} \left[ m \left( \mathbf{r} \times \mathbf{v}_R \right) \right] = m \frac{\delta}{\delta t} \left( \mathbf{r} \times \mathbf{v}_R \right) + \left( \mathbf{r} \times \mathbf{v}_R \right) \frac{\delta m}{\delta t}
\]

\[
= m \left\{ \left( \frac{\delta \mathbf{r}}{\delta t} \times \frac{\delta \mathbf{r}}{\delta t} \right) + \left[ \frac{\delta \mathbf{r}}{\delta t} \times (\mathbf{\omega} \times \mathbf{r}) \right] + \left[ \mathbf{r} \times \frac{\delta^2 \mathbf{r}}{\delta t^2} \right] \right\} + \left[ \mathbf{r} \times \left( \mathbf{\omega} \times \frac{\delta \mathbf{r}}{\delta t} \right) \right] + \left[ \mathbf{\omega} \times \left( \mathbf{\omega} \times \mathbf{r} \right) \right] \}
\]

(A.14)

The term \( (\mathbf{\omega} \times \mathbf{H}) \) expands as:

\[
(\mathbf{\omega} \times \mathbf{H}) = m \left\{ \mathbf{\omega} \times \left( \mathbf{r} \times \frac{\delta \mathbf{r}}{\delta t} \right) + \mathbf{\omega} \times \left[ \mathbf{r} \times \left( \mathbf{\omega} \times \mathbf{r} \right) \right] \}
\]

(A.15)

Finally, combining all the terms in

\[
\frac{d\mathbf{H}}{dt} = \frac{\delta \mathbf{H}}{\delta t} + (\mathbf{\omega} \times \mathbf{H})
\]

(A.16)
the components along the three moving axes are:

_\mathbf{i} \text{ components:}

\begin{align*}
& m \left\{ y \dot{y} \omega_x - x \dot{x} \omega_y + \frac{y^2}{2} \omega_x - x^2 \omega_z + y \dot{y} - 3 z \dot{y} + y^2 \omega_x - x y \omega_y + z^2 \omega_x \right. \\
& \left. - x z \omega_z + y \dot{y} \omega_x - x \dot{x} \omega_y + z \omega_x - x z \omega_z + (y^2 - z^2) \omega_x \right. \\
& \left. - 3 y \dot{y} - x y \omega_x \omega_y + y z \omega_z^2 + x y \omega_x \omega_y + (x \dot{y} - y \dot{x}) \omega_y \\
& \left. - (3 \dot{x} - x \dot{z}) \omega_z \right\} + \frac{d m}{dt} \left\{ y \ddot{y} - x \ddot{x} + y^2 \omega_x - x y \omega_y + z^2 \omega_x - x z \omega_z \right\} \\
\end{align*}

(A.17)

_\mathbf{j} \text{ components:}

\begin{align*}
& m \left\{ - y \dot{x} \omega_x + x \dot{x} \omega_y + \frac{y^2}{2} \omega_x - x^2 \omega_z + z \dot{y} - x \dot{z} \omega_x + x^2 \omega_y \right. \\
& \left. + z \omega_y - y \dot{z} \omega_z - \dot{y} \dot{x} \omega_x + \dot{x} \dot{y} \omega_y + \frac{y^2}{2} \omega_x - y \dot{z} \omega_z + (3^2 \omega_x) \omega_x \omega_y \\
& \left. - x z \omega_z^2 - x y \omega_x \omega_y + x z \omega_x^2 + y z \omega_z^2 + x y \omega_x \omega_y + (y \dot{x} - z \dot{y}) \omega_z \\
& \left. - (x \dot{y} - y \dot{x}) \omega_x \right\} + \frac{d m}{dt} \left\{ 3 \dot{x} - x \dot{z} - x y \omega_x + x^2 \omega_y + z^2 \omega_y - y z \omega_z \right\} \\
\end{align*}

(A.18)

_\mathbf{k} \text{ components:}

\begin{align*}
& m \left\{ - z \dot{x} \omega_y + x \dot{x} \omega_z - z \dot{y} \omega_y + x \dot{y} \omega_z + x \dot{y} \omega_y - x z \omega_x + x^2 \omega_z \right. \\
& \left. + y^2 \omega_x^2 - z y \omega_y - \frac{z^2}{2} \omega_x - y \dot{y} \omega_z + y y \omega _2 + (x^2 y) \omega_x \omega y \\
& \left. - x y \omega_x^2 - y z \omega_x \omega_z + x y \omega_y^2 + x z \omega_y \omega_z + (z \dot{x} - x \dot{z}) \omega_x \\
& \left. - (y \dot{z} - z \dot{y}) \omega_y \right\} + \frac{d m}{dt} \left\{ x \dot{y} - y \dot{x} - x z \omega_x + x^2 \omega_z - z y \omega_y + y^2 \omega_z \right\} . \\
\end{align*}

(A.19)

Using the continuum assumption, consider the sum of point masses in the elemental volume, and introduce the notation of moment of inertia and product of inertia of
the elemental volume of fluid. By definition,

$$\int (y^2 + z^2) \, dm = m(y^2 + z^2) = I_{xx} \quad \text{moment of inertia}$$

$$\int xy \, dm = mxy = I_{xy} \quad \text{product of inertia.}$$

The derivatives of these are

$$\frac{d}{dt} [I_{xx}] = \frac{d}{dt} [m(y^2 + z^2)] = \frac{dm}{dt} (y^2 + z^2) + 2m(y\dot{y} + z\dot{z}) = \ddot{I}_{xx}$$

$$\frac{d}{dt} [I_{xy}] = \frac{d}{dt} [mxy] = \frac{dm}{dt} (xy) + m(x\dot{y} + y\dot{x}) = \ddot{I}_{xy}$$

(A.21)

These derivatives will account for terms containing $\dot{x}, \dot{y}, \dot{z}$, but second derivatives also appear.

Thus, consider

$$\frac{d^2}{dt^2} [I_{xx}] = \frac{d}{dt} \left[ \frac{d}{dt} (y^2 + z^2) + 2m(y\dot{y} + z\dot{z}) \right]$$

$$= \frac{d^2m}{dt^2} (y^2 + z^2) + \frac{dm}{dt} (2y\dot{y} + 2z\dot{z}) + 2 \frac{dm}{dt} (y\dot{y} + z\dot{z})$$

$$+ 2m \left[ y\ddot{y} + (\dot{y})^2 + z\ddot{z} + (\dot{z})^2 \right]$$

(A.22)

and

$$\frac{d^2}{dt^2} [I_{xy}] = \frac{d}{dt} \left[ xy \frac{dm}{dt} + m(x\dot{y} + y\dot{x}) \right]$$

$$= xy \frac{d^2m}{dt^2} + \frac{dm}{dt} (x\ddot{y} + y\ddot{x}) + \frac{dm}{dt} (y\dot{x} + x\dot{y})$$

$$+ m(y\dddot{x} + \dddot{y} x + \dddot{y} + \dddot{x} y)$$

(A.23)
Hence, asymmetric shear stresses at the surface of the element provide the following time rate of change of moment of momentum.

\[
\mathbf{\tau} = \dot{t} \left\{ I_{xx} \omega_x + I_{xx} \dot{\omega}_x - I_{xz} \omega_y - I_{xz} \omega_x \omega_y \\
+ I_{xy} \dot{\omega}_x \omega_z + 2 I_{yz} (\omega_z^2 - \omega_y^2) - I_{xy} \omega_y \\
- I_{xy} \dot{\omega}_y - I_{xz} \dot{\omega}_z + m \left[ (y^2 - z^2) \omega_y \omega_z \\
+ y \ddot{z} - z \ddot{y} \right] + \frac{dm}{dt} \left[ y \dot{z} - z \dot{y} \right] \right\}
\]

\[
+ \dot{j} \left\{ I_{yy} \omega_y + I_{yy} \dot{\omega}_y - I_{xy} \dot{\omega}_x - I_{yz} \omega_z - I_{yx} \omega_x \\
- I_{yz} \omega_z - I_{xz} (\omega_x^2 - \omega_z^2) - I_{xy} \omega_y \omega_z \\
+ I_{zy} \omega_y \omega_x + m \left[ (y \dot{z} - z \dot{y}) \omega_z - (x \dot{y} - y \dot{x}) \omega_x \\
+ z \ddot{x} - x \ddot{z} \right] + \frac{dm}{dt} \left[ z \dot{x} - x \dot{z} \right] \right\}
\]

\[
+ \dot{k} \left\{ I_{zz} \omega_z + I_{zz} \dot{\omega}_z - I_{xz} \dot{\omega}_x - I_{yz} \omega_y + (\omega_y^2 - \omega_x^2) I_{xy} \\
+ I_{xz} \omega_y \omega_z - I_{yz} \omega_x \omega_z - I_{yz} \omega_y - m \left[ z \dot{x} \omega_y \\
+ (z \ddot{x} - x \ddot{z} - z \ddot{y}) \omega_x - (x^2 - y^2) \omega_x \omega_y + x \ddot{y} \\
- y \ddot{x} \right] + \frac{dm}{dt} \left[ x \dot{y} - y \dot{x} \right] \right\}.
\]

(A.24)
If the contributions due to terms
\[ \vec{r} \times \frac{\delta \vec{r}}{\delta t} \quad \text{and} \quad \vec{r} \times \frac{\delta \vec{v}}{\delta t^2} \]
are dropped, then the resulting expression give equations classically seen in flight dynamics of a vehicle. They are

\[ \vec{\tau} = \hat{\tau} \left\{ \dot{\rho}I_{xx} + \rho \dot{I}_{xx} - \dot{Q}I_{xy} - Q \dot{I}_{xy} - \dot{R}I_{xz} - R \dot{I}_{xz} \right\} + \hat{\tau} \left\{ \dot{Q}I_{yy} + Q \dot{I}_{yy} - \dot{P}I_{xy} - P \dot{I}_{xy} - \dot{R}I_{yz} - R \dot{I}_{yz} \right\} + \hat{\kappa} \left\{ \dot{R}I_{zz} + R \dot{I}_{zz} - \dot{P}I_{xz} - P \dot{I}_{xz} - \dot{Q}I_{zy} - Q \dot{I}_{zy} \right\} \]  

(A.25)

If a fluid element were viscoelastic there would be an analogy of these equations or Eq. (A.3) to the dynamics of a fluid. The effects of element size, shape, and rate of change of mass with adjacent elements is indicated. If the fluid element can hold intrinsic internal angular momenta, then these equations would be modified further.

**Intrinsic Angular Momenta**

Dahler and Scriven (47), and Aris (31) show that the first moment of linear momentum does not describe all the angular momentum of a general fluid. Generally
it is assumed that the stresses are locally in equilibrium, hence the stress tensor is symmetric. The argument usually offered in support of this assumption is based on considering the rotational equilibrium of an element of material so small that its ratio of volume to surface area approaches zero. That is, if \( \mathbf{t}_{(n)} \) is the force per unit area exerted by elements outside, \( S \) is surface of the element, and if \( d \) is the characteristic dimension, Aris gives the classical argument that

\[
\lim_{d \to 0} \frac{1}{d^2} \iint_{S} \mathbf{t}_{(n)} \, ds = 0
\]  

(A.26)

and stresses are in local equilibrium. Fredrickson (28) and Aris (31) both point out that such a procedure is not permissible in general, since the continuum hypothesis cannot be applied to an element whose volume decreases indefinitely.

The continuum hypothesis can be applied to a very small but finite volume, and since most of the fluids of interest are polyatomic it seems there is the possibility of intrinsic angular momenta in fluids not normally considered polar. Polarization will obviously vary, and in many cases may be vanishingly weak. However, this possibility might be worth further investigation. In the case of polar fluids, Aris introduces a body torque per
unit mass, \( \rho \dot{\mathbf{g}} \), in addition to a body force \( \rho \ddot{\mathbf{f}} \), and a stress couple \( \mathbf{\tau}_{(n)} \) in addition to the normal stress \( \mathbf{t}_{(n)} \). He then equates the summation of torques on the element to the time rate of change of total angular momentum of a volume \( V \):

\[
\iiint (\rho \dot{\mathbf{g}} + \ddot{\mathbf{X}} \times \rho \ddot{\mathbf{f}}) \, dV + \iint (\mathbf{\tau}_{(n)} + \ddot{\mathbf{X}} \times \mathbf{t}_{(n)}) \, dS = \frac{d}{dt} \iiint \rho (\dot{\mathbf{\lambda}} + \dddot{\mathbf{X}} \times \dddot{\mathbf{v}}) \, dV. \tag{A.27}
\]

The total angular momentum is shown as the sum of moment of linear momentum \( \rho \dddot{\mathbf{X}} \times \dddot{\mathbf{v}} \) plus intrinsic angular momentum \( \rho \dot{\mathbf{\lambda}} \). This may be put in the form.

\[
\frac{d}{dt} \left[ \rho (\dot{\mathbf{\lambda}} + \dddot{\mathbf{X}} \times \dddot{\mathbf{v}}) \right] = \rho \dot{\mathbf{g}} + \dddot{\mathbf{X}} \times \rho \ddot{\mathbf{f}} + \dddot{\mathbf{v}} \cdot \mathbf{\dot{\mathbf{C}}} + \dddot{\mathbf{X}} \times (\dddot{\mathbf{v}} \cdot \mathbf{\tau}) + \dddot{\mathbf{T}}_x \tag{A.28}
\]

where \( T_x = \varepsilon_{ijk} T_{jk} \) is the antisymmetric part of the stress tensor.

Taking the vector product of \( \dddot{\mathbf{X}} \) with each element of Cauchy's Equation of Motion gives the equation for moment of momentum which, when subtracted from Eq. (A.28) gives

\[
\rho \frac{d}{dt} \dddot{\mathbf{X}} = \rho \dot{\mathbf{g}} + \dddot{\mathbf{v}} \cdot \mathbf{\dot{\mathbf{C}}} + \dddot{\mathbf{T}}_x \tag{A.29}
\]

for a constant density flow. Thus, \( \dddot{\mathbf{T}}_x \), the antisymmetric part of the stress tensor, can contribute to internal angular momentum.
APPENDIX B

AN INTERESTING ANALOGY

In missile flights, pitching motion can occur in any plane, but quite informative representative motion can be obtained by considering that the pitching motion occurs in the plane of the missile trajectory.

A simplified analysis by H. Unger* is based on the fact that in many cases the missile oscillatory motion about its center of gravity has negligible influence on the trajectory. This certainly is not always true, but in many cases the oscillatory equations can be investigated separately from the trajectory equations. This separation or uncoupling of equations produces distinct phugoid solutions, short period solutions, etc.

Consider the following force and moment diagram.

Fig. 21 Force and Moment Diagram

* See Friedrich, H. R., and F. J. Dore (38)
In Fig. 21 the forces included are gravitational force and aerodynamic forces due to lift and drag. Accelerations are radial and tangential to the flight path; $a_r$ and $a_t$ given by

$$a_r = - \left[ \frac{V^2 \cos \gamma}{(R+y)} \right] + V \dot{y} \quad (B.1)$$

$$a_t = \dot{V} \quad (B.2)$$

Consider aerodynamic moments due to static stability and pitch damping. The set of second-order equations describing the motions are then

$$-m \dot{V} - C_p g S - mg \sin \gamma = 0 \quad (B.3)$$

$$-mV \dot{\gamma} + \left( \frac{C_d}{\alpha} \right) \alpha g S + m \left\{ \frac{V^2}{R+y} - g \right\} \cos \gamma = 0 \quad (B.4)$$

$$-I \ddot{\theta} + C_{m \alpha} \left( \frac{\dot{\theta}^2}{2V} \right) g S \theta - \left( \frac{C_{m \alpha}}{\alpha} \right) \alpha g S \theta = 0 \quad (B.5)$$

$$\theta = \alpha + \gamma \quad (B.6)$$

The solution obtained by Unger depends on separating these equations into two sets of solutions, one describing the trajectory of the center of gravity and the other set describing the oscillatory motion about the center of gravity. Point-mass mechanics requires the assumption that the trajectory equations can be uncoupled from the oscillatory equations. Friedrich and Dore (38) present a study of such motions of a missile descending thru the atmosphere. Equations (B.1) thru (B.6) are the format
they used. Note the symbol $\gamma$ is used to represent both pitch rate (in $c_m$) and dynamic pressure. Also they neglect the pitch damping term $c_{m_\alpha}$, which is a justifiable assumption at high Mach numbers. Allen (37) follows the analysis of Friedrich and Dore, only he includes $c_{m_\alpha}$ primarily because the two damping derivatives are usually seen together as $(c_{m_\phi} + c_{m_\alpha})$. As a matter of fact, when obtaining the damping derivative by wind tunnel testing the whole derivative $(c_{m_\phi} + c_{m_\alpha})$ is obtained but the parts or contribution of each cannot be separated.

Reference (38) discusses the conditions for separating the equations into trajectory and oscillatory equations. Under proper conditions,

$$-m\dot{V}_o - C_{D_\alpha} g_o - mg \sin \gamma_o = 0 \tag{B.7}$$
$$-mV_o \dot{\gamma}_o + m\left(\frac{V_o}{R + y} - g\right) \cos \gamma_o = 0 \tag{B.8}$$

describe the trajectory of the missile, and

$$-mV_o \dot{\theta} + \left(\frac{C_{\dot{\alpha}}}{\alpha}\right) g_o S = 0 \tag{B.9}$$
$$-\dot{\theta} + \left[ C_{m_\phi}\left(\frac{\dot{\theta} + \ddot{\beta}}{2V_o}\right) + C_{m_\alpha}\left(\frac{\dot{\alpha} + \ddot{\beta}}{2V_o}\right)\right] g_o S \dot{\theta} + \left(\frac{C_{\dot{\alpha}}}{\alpha}\right) g_o S = 0 \tag{B.10}$$
$$\theta = \alpha + \gamma \tag{B.11}$$

describe the angular motion about the center of gravity.
The oscillatory equations can be put in the form
\[
\frac{d^2 \zeta}{dt^2} + f_1(t) \frac{d \zeta}{dt} + f_2(t) \zeta = 0
\] (B.12)

where
\[
f_1(t) = \frac{C_{\alpha} \rho V A}{2 m} - \frac{\left( C_{m_{\alpha}} + C_{m_{\beta}} \right) \rho V A l^2}{4 I m}
\] (B.13)
\[
f_2(t) = \frac{d}{dt} \left( \frac{C_{\alpha} \rho V A}{2 m} \right) - \frac{C_{m_{\alpha}} C_{\alpha} \rho V A l^2}{4 I m} - \frac{C_{\alpha} \rho V A l}{2 I}
\] (B.14)

This formulation of the oscillatory equation is equivalent to Equations 12 and 13 of Allen (37). The coefficients of \( f_1(t) \) and \( f_2(t) \) are time dependent because the missile is changing altitude. Allen assumes acceleration of gravity is constant with altitude, flight path is essentially a straight line, and variation of air density is the exponential function \( \rho = \rho_e e^{-\beta y} \).

Holstrom (41) presents an analysis of the effect that dynamic unbalance has on the motion of a spinning vehicle. The principal moments of inertia are given by the three values of \( I \) that satisfy the equation
\[
\begin{vmatrix}
I_x - I & I_{xy} & I_{xz} \\
I_{xy} & I_y - I & I_{yz} \\
I_{xz} & I_{yz} & I_z - I
\end{vmatrix} = 0
\] (B.15)

Instead of solving the expansion by the general formula for the cubic equation, an approximate solution is obtained by assuming that the given products of inertia
are small compared to the moments of inertia. This is equivalent to assuming that the given moments of inertia are almost equal to the principal moments of inertia, i.e.,

\[
\begin{align*}
I_1 &= I_x + \mu_1, \\
I_2 &= I_y + \mu_2, \\
I_3 &= I_z + \mu_3,
\end{align*}
\]  

(B.16)

where

\[
|\mu_1| \ll I_{xx}, \text{ etc.}
\]

Holstrom uses this approach to get Euler angles, equations of motion, and then approximate solutions under the assumption that \( \rho \gg q \) and \( \rho \gg r \), and \( I_z \approx I_y \).

The two equations of motion he solves are

\[
\begin{align*}
I_y \ddot{q} - \rho_0 \left( I_z + I_x \right) r - \rho_0^2 I_{xx} &= 0, \\
I_3 \dot{r} + \rho_0 \left( I_y - I_x \right) q + \rho_0^2 I_{xy} &= 0.
\end{align*}
\]  

(B.17)

The solutions are

\[
\begin{align*}
q &= N \cos (\omega t + \alpha) - \frac{I_{xy}}{I_y - I_x} \rho_0, \\
r &= -N \sin (\omega t + \alpha) - \frac{I_{xx}}{I_z - I_x} \rho_0,
\end{align*}
\]  

(B.18)

where

\[
\omega^2 = \frac{(I_y - I_x)(I_z - I_x)}{I_y I_z} \rho_0, \quad \lambda^2 = \frac{I_y (I_y - I_x)}{I_z (I_z - I_x)},
\]

(B.19)

and \( N \) and \( \alpha \) are constants of integration.
The expressions for the Euler angles are

\[
\theta = A \sin \left( (\varphi_0 - \omega) t + (\varphi_0 - \alpha) \right) \\
- B \sin \left( (\varphi_0 + \omega) t + (\varphi_0 + \alpha) \right) \\
+ C \sin \left( \varphi_0 t + \varphi_0 + \beta \right) + \gamma_1
\]

\[
\psi = -A \cos \left( (\varphi_0 - \omega) t + (\varphi_0 - \alpha) \right) \\
+ B \cos \left( (\varphi_0 + \omega) t + (\varphi_0 + \alpha) \right) \\
- C \cos \left( \varphi_0 t + \varphi_0 + \beta \right)
\]

where

\[
A = \left\{ 1 + \frac{1}{2} (\lambda - 1) \right\} \frac{N}{\varphi_0 - \omega}
\]

\[
B = \frac{1}{2} (\lambda - 1) \frac{N}{\varphi_0 + \omega}
\]

\[
C = \sqrt{\left( \frac{I_{xy}}{I_{y} - I_{x}} \right)^2 + \left( \frac{I_{xy}}{I_{z} - I_{x}} \right)^2}
\]

\[
\beta = \cos^{-1} \left[ -1 \frac{I_{xy}}{C(I_{y} - I_{x})} \right] = \sin^{-1} \left[ -\frac{I_{xy}}{C(I_{z} - I_{x})} \right]
\]

and \( \gamma_1 \) and \( \gamma_2 \) are constants of integration.

Finally, Holstrom shows the type of nutations experienced for a slightly asymmetric body (Fig. 21). The effects of inertia ratios and asymmetries clearly then is to induce couplings and random motions.
\[
\{1 + \frac{i}{2}(\lambda - 1)\} \frac{N}{\rho_0} < C < \{1 + \frac{i}{2}(\lambda - 1)\} \frac{N}{\rho_0 - \omega}
\]

\[
C = \{1 + \frac{i}{2}(\lambda - 1)\} \frac{N}{\rho_0} \quad C < \{1 + \frac{i}{2}(\lambda - 1)\} \frac{N}{\rho_0 - \omega}
\]

\[
C = \{- \frac{i}{2}(\lambda - 1)\} \frac{N}{\rho_0 - \omega} \quad C > \{1 + \frac{i}{2}(\lambda - 1)\} \frac{N_0}{\rho_0 - \omega}
\]

Fig. 21 Plots of Typical Motion for Spinning Asymmetric Body