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by

Kent LaVerne Goering

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Notation

A  cross sectional area
A_e effective area
b  flange width
b_e effective flange width
C_b bending coefficient
C_c \( \sqrt{\frac{2T^2E}{F_y}} \)
E  elastic modulus
F_a, f_a allowable and actual axial stresses
F_b, f_b allowable and actual bending stresses
F_{b_t} temporary allowable actual bending stress
F_e elastic column stress
F_v, f_v allowable and actual shearing stresses
F_y yield stress
G  gradient vector
g  amount of ineffective web height
h  web height
h_e effective web height
I  moment of inertia, I_x or I_y
I_x moment of inertia about x-axis
I_y moment of inertia about y-axis
KL/r maximum slenderness ratio
k  .522
L  length of member
L_b lateral unbraced length
L_x effective length for x-axis
$L_y$ effective length for y-axis
$M$ moment
$M_o$ dimensionless moment
$N$ number of elements
$P$ axial load
$P_q$ projection matrix
$Q_i$ transfer matrix
$R$ effective length parameter
$R_i$ intermediate support matrix
$R_y$ unbraced length parameter
$r$ radius of gyration
$S_{L_i}$ state vector at left end of element
$S_{R_i}$ state vector at right end of element
$t_f$ flange thickness
$t_w$ web thickness
$U$ dimensional web thickness
$U_q$ constraint gradient matrix
$U_i$ constraint gradient vector
$V$ shear
$V_o$ dimensionless shear
$V_l$ special dimensionless volume
$v$ volume
$W$ dimensionless flange width
$X$ dimensionless axial load
$x$ coordinate axis
$\bar{x}$ design variable vector
$Y$ dimensionless web depth
$Y_e$  effective dimensionless web depth 
$y$  coordinate axis 
$Z$  dimensionless area 
$Z_e$  dimensionless effective area 
$z$  longitudinal coordinate axis 
$z_1$  gradient projection vector 
$\alpha_i$  auxiliary or slack variable 
$\gamma$  material parameter 
$s$  constraint surface tolerance 
$\varepsilon$  convergence tolerance 
$\xi$  dimensionless longitudinal coordinate 
$\lambda$  LaGrange multiplier 
$\lambda$  dimensionless volume 
$\rho_x$  reciprocal slenderness ratio, x direction 
$\rho_y$  reciprocal slenderness ratio, y direction 
$\tau$  step size 
$\phi_a, \phi_a$  dimensionless actual and allowable axial stresses 
$\phi_b, \phi_b$  dimensionless actual and allowable bending stresses 
$\phi_E$  dimensionless elastic column stress 
$\phi_v, \phi_v$  dimensionless actual and allowable shearing stresses 
$\psi$  constraint
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I Introduction

Structural design is the synthesis of structures whose elements are simultaneously safe, economical, and aesthetically satisfactory. When aesthetics is not of primary importance, the design process is the search for an answer to the question, "Of all the structures which satisfy prescribed functional requirements, which one has the minimum cost, or, alternatively, the minimum weight?" This study is directed toward answering this question for single-span steel beam-columns of doubly symmetric I-shaped cross-section.

Few previous attempts have been made at optimizing a single member. Holt and Heithecker (5) found the optimum proportions for I-shaped laterally supported beams. Keller (7) found the theoretical optimum taper for elastic pin-ended columns of geometrically similar cross-section. Razani (12), Heithecker (4) and others have applied dynamic programming to the cost optimization of plate girders.

Much work in structural optimization has been in the design of frameworks. Most of the authors have assumed prismatic members, and some have assumed a continuous functional relationship between the radius of gyration of a member and its area, as was done for example by Brown and Ang(2). Since few studies have been made of optimum proportions for individual members, the shapes of such curves for optimum sections are unknown. The results of the present study
elucidate this aspect of the larger problem.

A. The Model

![Diagram of beam-column](image)

**Fig. 1**

The most general beam-column treated in this study is shown in Fig. 1. It has a specific length, L, and an I-shaped cross-section. Any statically admissible combination of end conditions is possible. The member is laterally supported at the ends and may be laterally supported either continuously or at a discrete number of points between the ends. The lateral bracing and end conditions determine a maximum unbraced length for the compression flange, $L_b$, and effective lengths for the x and y axis column buckling, $L_x$ and $L_y$.

The member is composed of a single grade of steel with yield stress, $F_y$, and elastic modulus, E. The web is proportioned so that web stiffeners are not required. Design details such as connections and bearing stiffeners are not considered.

The cross-sectional configuration of the member is assumed to be described by four design variables,

- $b =$ width of the flange
- $t_f =$ thickness of the flange
h = height of the web  
t_w = thickness of the web  
all of which are functions of z, the distance along the length of the member. The area of the cross-section is given by  
\[ A = 2bt_f + ht_w \]  
(1) 
The moment of inertia about the x-axis is given by  
\[ I_x = b t_f h^2 / 2 + t_w h^3 / 12 \]  
(2) 
which is a good approximation for thin flanges. The y-axis moment of inertia is  
\[ I_y = t_f b^3 / 6 \]  
(3) 
if the contribution of the web is neglected. The total volume, v, is given by  
\[ v = \int_0^L A \, dz \]  
(4) 
This is the objective function, the function to be minimized when a minimum weight solution is desired.

The beam-column is subjected to multiple loading systems which may consist of concentrated and distributed transverse loads and couples applied in the plane of the web and a compressive axial load, P, applied centrally to the ends of the member. The transverse loads and couples result in a maximum moment, M, and shear, V, which in general are functions of z. Distributed axial loads and axial loads applied between the ends of the member are not considered. The optimum beam-column must be proportioned to carry these loads and cause the total volume to be a minimum.
B. Functional Requirements

According to Schmit (15) a general problem in structural optimization has the following three basic properties:

1. a set of specifications and functional requirements which includes a.) the loading configuration, which in general may include multiple and thermal loads; b.) stress and deflection limitations; and c.) side constraints, which are frequently maximum and minimum values for the design variables;

2. a given technology or capability for analyzing the behavior of a trial structure;

3. a design criterion such as minimum cost or minimum weight.

The first of these properties, the set of functional requirements is considered to be a.) the loading system just described and b.) the stress and deflection limitations and side constraints imposed by the working stress provisions of the A.I.S.C. specifications (16). When a beam-column conforms to these specifications, the member is considered feasible. The lightest of all feasible designs is the minimum weight design.

The functional requirements are organized below as side constraints and restrictions on the bending stress, shearing stress, axial compressive stress, and combined axial load and bending. For equations which are dimensional, pounds and inches must be used.
Side Constraints

The web slenderness ratio, \( \frac{h}{t_w} \) must satisfy

\[
\frac{h}{t_w} \leq \min \left[ 260, \frac{14 (10)^6}{\sqrt{F_y} (F_y + 16,500)} \right]
\]  

(5)

If, however,

\[
\frac{h}{t_w} > \frac{8000}{\sqrt{F_y}}
\]  

(6)

then an effective web height is defined as

\[
h_e = \frac{8000 t_w}{\sqrt{F_y}}
\]  

(7)

Only web area included in this effective height is considered to be available to carry compressive stress due to axial load.

The flange slenderness, \( \frac{b}{t_f} \), must satisfy

\[
\frac{b}{t_f} \leq \frac{6000}{\sqrt{F_y}}
\]  

(8)

This value may be exceeded, however, if only the effective flange width given by

\[
b_e = \frac{6000 t_f}{\sqrt{F_y}}
\]  

(9)

is counted.

If \( KL/r = \max \left[ \frac{L_x A_x'^2}{I_x'^2}, \frac{L_y A_y'^2}{I_y'^2} \right] \), then \( KL/r \) must always be less than 200.

Stress Limitations

The actual stress due to bending, shear and axial compression, \( f_b \), \( f_v \), and \( f_a \), are calculated as follows:

\[
f_b = \frac{M h}{2 I_x}
\]  

(11)
\[ f_v = \frac{V}{ht_w} \]  
(12)
\[ f_a = \frac{P}{A_e} \]  
(13)

where the effective area, \( A_e \), is given by
\[ A_e = 2b_e t_f + h_e t_w \]  
(14)

In each case, these must be less than or equal to the corresponding allowable stresses, \( F_b \), \( F_v \), and \( F_a \).

**Allowable Axial Compressive Stress**

If \( KL/r \) defined by Eq. (10), is greater than \( C_c \) given by
\[ C_c = \sqrt{\frac{2\pi^2 E}{F_y}} \]  
(15)

then the allowable axial stress, \( F_a \), is given by
\[ F_a = \frac{149 (10)^6}{(KL/F)^2} \]  
(16)

which corresponds to elastic column buckling. If \( KL/r \) is less than \( C_c \), then
\[ F_a = \left[ 1 - \frac{(KL/F)^2}{2C_c^2} \right] \frac{F_y}{\eta} \]  
(17)

the factor of safety, \( \eta \), being given by
\[ \eta = \frac{5}{3} + \frac{3}{8} \frac{KL}{rc_c} - \frac{1}{8} \left( \frac{KL}{rc_c} \right)^3 \]  
(18)

which corresponds to inelastic column buckling.

**Allowable Bending Stress**

A section is considered compact if the following inequal-
ities are satisfied. Otherwise it is non-compact.

\[
\frac{b_e}{t_f} \leq \frac{1600}{\sqrt{F_y}} \quad \text{and} \quad \frac{h}{t_w} \leq \max \left[ \frac{13,300 \left(1 - 1.43 \frac{f_a}{F_a} \right)}{\sqrt{F_y}}, \frac{8000}{\sqrt{F_y}} \right]
\]  \quad (19)

Compact Sections

If the member is laterally supported, i.e., if

\[
L_b \leq \min \left( \frac{2400 b_e}{\sqrt{F_y}}, \frac{20(10)^6 b_e t_f}{h F_y} \right)
\]  \quad (20)

then \( F_b = 0.66F_y \). If not, the section is designed as a non-compact section.

Non-Compact Sections

If the member is laterally supported such that

\[
\frac{L_b}{r_t} \leq 4.0 \quad \text{where} \quad r_t^2 = \frac{t_f b_e^3}{12 (b_e t_f + h t_w)}
\]  \quad (21)

then \( F_b' = 0.6F_y \). Otherwise the lateral stability of the beam is considered by calculating

\[
F_b' = \max \left[ F_1, F_2 \right]
\]  \quad (22)

but non greater than \( 0.6F_y \), where

\[
F_1 = \left( 1 - \frac{L_b^2}{2 C_c C_b r_t^2} \right) 0.6F_y
\]  \quad (23)

and

\[
F_2 = \frac{12 (10)^6 b_e t_f}{L h}
\]  \quad (24)

\( C_c \) is defined by Eq. (15) and \( C_b \) is a bending coefficient dependent on the moment gradient and herein conservatively assumed equal to unity in accordance with A.I.S.C. 1.5.1.4.5.
The overall depth in Eq. (24) is taken as \( h \), a good approximation for thin flanges.

A reduction in flange stress may be required for thin webs. If

\[
\frac{h}{t_w} \leq \frac{24000}{\sqrt{F_b'}}
\]

then the allowable bending stress, \( F_b \), is given by \( F_b = F_b' \)
where \( F_b' \) is defined by Eq. (22). Otherwise,

\[
F_b = F_b' \left[ 1 - 0.0005 \frac{ht_w}{b_e t_f} \left( \frac{h}{t_w} - \frac{24000}{\sqrt{F_b'}} \right) \right]
\]

**Allowable Shearing Stress**

Elastic buckling controls the allowable shearing stress if

\[
\left( \frac{h}{t_w} \right)^2 \geq \frac{3.0038 (10)^3}{F_y}
\]

and the allowable stress is given by

\[
F_v = \frac{8.3149(10)^3}{\left( \frac{h}{t_w} \right)^2}
\]

Otherwise,

\[
F_v = \min \left[ F_3, F_4 \right]
\]

where

\[
F_3 = \frac{4.797.6 F_y^{3/2}}{h^{1/2} t_w}
\]

which corresponds to inelastic web buckling, and

\[
F_4 = 0.4 F_y
\]

which corresponds to yielding of the web.
**Combined Bending and Axial Compression**

The bending and axial compressive stresses must also satisfy the following restrictions:

\[
\frac{f_a}{F_a} + \frac{f_b}{F_b} \leq 1.0 \quad \text{for} \quad \frac{f_a}{F_a} \leq 0.15
\]

\[
\frac{f_a}{F_a} + \frac{C_m f_b}{(1 - \frac{f_a}{F_e})F_b} \leq 1.0 \quad \text{for} \quad \frac{f_a}{F_a} \geq 0.15
\]

where \(C_m\) is a coefficient dependent on the end moments and taken to be 1, and \(F_e = \frac{149(\rho_0)^6 I_x}{L_x^2 A}\).

It is noted that an increase in \(b/t_f\) for a given flange area increases \(I_y\) which increases the allowable axial stress when \(y\)-axis buckling controls. When it does not control, material may be taken from the flange while holding \(b\) fixed and placed in the web to increase \(I_x\), thereby increasing the allowable axial load. By extending this procedure, \(b/t_f\) for optimum columns attains the maximum value allowed by Eq. (8).

It is further noted that letting \(b/t_f\) reach its maximum value (Eqs. (19), compact and (8), non-compact) improves the allowable bending stress by improving the lateral-torsional properties through Eq. (23). If the ratio is exceeded, but the section is still acceptable through Eq. (9), savings in material can be made by reducing the dimensions to the maximum effective value. In no case is the allowable bending stress or axial load reduced. Hence constraint Eq. (8) may be tightened to an equality and \(t_f\) can be eliminated throughout the problem.
C. Dimensionless Variables

The problem may be recast in terms of the following dimensionless variables:

\[ W = \frac{E^{\frac{1}{2}} b}{F_y^{\frac{1}{2}} L_x} = \text{dimensionless flange width} \]

\[ Y = \frac{E^{\frac{1}{2}} h}{F_y^{\frac{1}{2}} L_x} = \text{dimensionless web height} \]

\[ U = \frac{E t_w}{F_y L} = \text{dimensionless web thickness} \]

Because \( t_f \) can be eliminated by Eq. (8), no dimensionless flange thickness needs to be defined.

Consistent with these definitions are the following dimensionless quantities:

\[ Z = \frac{E^{\frac{3}{2}} A}{F_y^{\frac{3}{2}} L_x^3} = \text{dimensionless area} \]

\[ J = \frac{E^{\frac{3}{2}} V}{F_y^{\frac{3}{2}} L_x^3 L} = \text{dimensionless volume} \]

\[ \gamma = \frac{Z}{L} = \text{dimensionless distance} \]

The dimensionless loading parameters of the problem are:

\[ X = \frac{E^{\frac{3}{2}} P}{F_y^{\frac{3}{2}} L_x^2} = \text{dimensionless axial load} \]

\[ M_0 = \frac{E^2 M}{F_y^3 L_x^3} = \text{dimensionless moment} \]

\[ V_0 = \frac{E^{\frac{3}{2}} V}{F_y^{\frac{3}{2}} L_x^3} = \text{dimensionless shear} \]
The lateral bracing and end conditions and grade of steel are included in the following dimensionless parameters:

\[ R = \frac{L_y}{L_x} = \text{effective length parameter} \]

\[ R_y = \frac{L_y}{L_b} = \text{unbraced length parameter} \]

\[ \gamma = \frac{E^{1/2}}{F_y^{1/2}} = \text{material parameter} \]

It is convenient to define another set of dimensionless variables, \( \rho_x \) and \( \rho_y \) such that

\[
\begin{align*}
\rho_x &= \frac{c_c I_x^{1/2}}{L_x A^{1/2}} \\
\rho_y &= \frac{c_c I_y^{1/2}}{L_y A^{1/2}}
\end{align*}
\]

reciprocal slenderness ratios

If dimensionless allowable stresses are defined as

\[
\begin{align*}
\phi_v &= \frac{F_v}{F_y} \\
\phi_b &= \frac{F_b}{F_y} \\
\phi_a &= \frac{F_a}{F_y}
\end{align*}
\]

then Eqs. (5) to (33) take the following form where dimensional constants have been non-dimensionalized by using a value of 29,000,000 psi for the elastic modulus.

**Side Constraints**

The dimensionless web slenderness ratio, \( \gamma/U \), must satisfy

\[
\frac{\gamma}{U} \leq \min \left[ \frac{260}{\delta}, \frac{14 \gamma}{\sqrt{29(29+0.65\delta)}} \right]
\]  

(34)
If, however,
\[
\frac{V}{u} \geq \frac{8}{\sqrt{29}}
\]

then an effective web height is defined as
\[
\gamma_e = \frac{8u}{\sqrt{29}}
\]  

(35)

If
\[
\rho = \min \left[ \rho_x, \rho_y \right]
\]

(37)

where
\[
\rho_x^2 = \frac{3427}{2088} \frac{Y^2(Yu + \sqrt{29}w^2)}{Yu + \frac{\sqrt{29}w^2}{3}}
\]

(38)

\[
\rho_y^2 = \frac{3427}{6264} \frac{W^4}{R^2(Yu + \frac{\sqrt{29}w^2}{3})}
\]

(39)

then \( \rho \) must always be greater than \( \frac{c_e}{200} \).  

(40)

**Allowable Axial Compressive Stress**

If \( \rho \) is defined by Eq. (37), then the allowable axial stress is
\[
\phi_a = \frac{149}{5877} \rho^2 \quad \text{for} \quad \rho < 1
\]

(41)

which corresponds to elastic buckling, and
\[
\phi_a = \frac{12 \rho (2 \rho^3 - 1)}{40\rho^3 + 9\rho^2 - 3} \quad \text{for} \quad \rho \geq 1
\]

(42)

which corresponds to inelastic buckling.

**Allowable Bending Stress**

If the member is laterally supported such that
\[
\frac{12}{W^2} \left( \frac{R}{R_y} \right)^2 \left( 1 + \frac{U_Y}{\sqrt{29} W^2} \right) \leq \frac{1600}{Y^2}
\] (43)

then \( \phi'_b = 0.60 \). Otherwise the lateral stability of the beam is considered by calculating

\[
\phi'_b = \max \left[ \Phi_1, \Phi_2 \right]
\] (44)

but not greater than 0.60, where

\[
\Phi_1 = 0.6 \left[ 1 - \frac{3}{\pi^2 W^2} \left( \frac{R}{R_y} \right)^2 \left( 1 + \frac{U_Y}{\sqrt{29} W^2} \right) \right]
\] (45)

and

\[
\Phi_2 = \frac{2 W^2 R_y}{\sqrt{29} Y R}
\] (46)

A reduction in flange stress may be required for thin webs. If

\[
\frac{Y}{U} \leq 4.45669 \left( \phi'_b \right)^{\frac{1}{2}}
\] (47)

then \( \phi = \phi'_b \). Otherwise, the allowable bending stress is

\[
\phi = \phi'_b \left[ 1 - \frac{0.003 \sqrt{U_Y}}{\sqrt{29} W^2} \left( \frac{Y}{U} - 4.45669 \left( \phi'_b \right)^{\frac{1}{2}} \right) \right]
\] (48)

**Allowable Shearing Stress**

Elastic buckling controls the allowable shearing stress if

\[
\left( \frac{Y}{U} \right)^2 \geq 10.3578
\] (49)

and the dimensionless allowable stress is given by

\[
\phi'_v = \frac{2.8672}{\left( \frac{Y}{U} \right)^2}
\] (50)
Otherwise,
\[ \phi_v = \min \left[ \phi_3, \phi_4 \right] \]  \hspace{1cm} (51)

where
\[ \phi_3 = \frac{.89089}{(\frac{V}{u})} \]  \hspace{1cm} (52)

and
\[ \phi_4 = .4 \]  \hspace{1cm} (53)

The actual dimensionless stresses due to bending, shear, and axial compression, \( \phi_B, \phi_V, \phi_A \), are calculated as follows:
\[ \phi_B = \frac{6 M_o}{\gamma (\sqrt{29} w^2 + UY)} \]  \hspace{1cm} (54)
\[ \phi_V = \frac{V_o}{\gamma u} \]  \hspace{1cm} (55)
\[ \phi_A = \frac{X}{\sqrt{29} w^2 + Y_e U} \]  \hspace{1cm} (56)

In each case, these must be less than the corresponding allowable stresses. In addition, the bending and axial compressive stresses must satisfy the following restriction.

**Combined Bending and Axial Compression**

The interaction equations to be satisfied are
\[ \frac{\phi_A}{\phi_a} + \frac{\phi_B}{\phi_b} \leq 1.0 \quad \text{for} \quad \frac{\phi_A}{\phi_a} \leq .15 \]  \hspace{1cm} (57)
\[ \frac{\phi_A}{\phi_a} + \left(1 - \frac{\phi_A}{\phi_a}\right) \frac{\phi_b}{\phi_b} \leq 1.0 \quad \text{for} \quad \frac{\phi_A}{\phi_a} \geq .15 \]  \hspace{1cm} (58)
where
\[
\phi_e = \frac{149 (10)^6}{2 \pi ^3 \gamma^2} \rho_k^2
\]  
(59)

The set of equations (34) to (59) must be satisfied everywhere along the length of a feasible member. Since these requirements as presented strictly apply only to prismatic members, it is assumed that the changes in the design variables with \( \gamma \) are small enough that the expressions for the actual bending and shearing stresses, Eq. (54) and (55), are still valid. This is an assumption frequently made in plate girder design. A logical extension of the axial compression requirements to non-prismatic members may be made and is described later.
II Formulation of the General Problem

A. Calculus of Variations Approach

There is merit in first taking an overall view of the problem to see what is involved in solving it and to see what conclusions can be drawn from the general formulation without actually solving the problem. Minimum weight is considered the sole design criterion. Mathematically this is a calculus of variations problem in three independent variables, \( W, Y, \) and \( U, \) which are functions of the distance from the end of the member, \( \Psi. \) These three functions are to be determined which minimize the total volume, \( \mathcal{V}, \) where

\[
\mathcal{V} = \int_0^1 Z(\Psi) d\Psi = \int_0^1 \left( \frac{\sqrt{25}}{3} w^2 + YU \right) d\Psi
\]  

(60)

subject to \( \Psi_j(Y, U, W, \Psi) \geq 0 \quad j=1,...,3 \)  

(61)

The expression under the integral in Eq. (60), is equivalent to \( Z \) by Eqs. (1) and (8). The \( \Psi_j \) are constraints expressed by Eqs. (34) to (59). Each constraint must be satisfied everywhere along the length of the beam-column. The \( \Psi_j \) are dependent not only on the design variables, but also on the parameters of the problem: the external loads, the lateral bracing conditions, the fixity of the ends, and the type of steel.

The number of explicit constraints is taken as three. The first involves the combination of axial load and bending through Eqs. (57) and (58). The satisfaction of this very
complicated constraint implies the satisfaction of the axial load and bending restrictions independently. The second constraint is given by Eqs. (49) to (53) as a limitation on the shearing stress in the web. The third constraint is a side constraint which limits the $h/t_w$ ratio through Eq. (34). In addition to these, there are three implicit constraints which require each of the design variables to be non-negative.

Notice that the problem contains a simple objective function and quite complicated non-linear inequality constraints, particularly the constraint containing axial and bending stresses. The problem of guaranteeing a local optimum in this case involves certain derivatives of the constraining equations and both first and second order conditions are needed. Only first order necessary conditions will be discussed here.

Since the constraining equations are piecewise differentiable functions of the design variables at all but a small finite number of points, the necessary conditions which follow are valid everywhere except at these points. Here the corner conditions must also be considered (8).

A common way of handling inequality constraints in the calculus of variations originally due to Valentine (17) is used here. The inequality constraints are reduced to equality constraints by the use of auxiliary or slack variables $\alpha_i$, such that

$$\psi_j - \alpha_i^2 = 0 \quad j = 1, \ldots, 3$$

(62)
where the $\alpha_j$ are real functions of $\mathcal{S}$ to be determined.

If each constraint is multiplied by a Lagrange multiplier, $\lambda_j$, and integrated along the length, the problem is equivalent to minimizing

$$\mathcal{J} = \int_0^l \left[ Z(W, U, Y, \mathcal{S}) + \sum_{j=1}^3 \lambda_j (\psi_j - \alpha_j^2) \right] d\mathcal{S}$$

(63)

The first order necessary condition for the integral to be extremal is that the first variation be zero, or

$$\delta \mathcal{J} = \int_0^l \left[ \left( \frac{\partial Z}{\partial W} + \sum_{j=1}^3 \lambda_j \frac{\partial \psi_j}{\partial W} \right) \delta W + \left( \frac{\partial Z}{\partial Y} + \sum_{j=1}^3 \lambda_j \frac{\partial \psi_j}{\partial Y} \right) \delta Y \right.
\left. + \left( \frac{\partial Z}{\partial U} + \sum_{j=1}^3 \lambda_j \frac{\partial \psi_j}{\partial U} \right) \delta U + \sum_{j=1}^3 (\lambda_j - 2 \alpha_j) \delta \alpha_j + \sum_{j=1}^3 (\psi_j - \alpha_j^2) \delta \lambda_j \right] = 0$$

(64)

Since the coefficients of each part of the variation must vanish independently, the following equations form complete first order necessary conditions.

$$\begin{align*}
\frac{\partial Z}{\partial W} + \sum_{j=1}^3 \lambda_j \frac{\partial \psi_j}{\partial W} &= 0 \\
\frac{\partial Z}{\partial Y} + \sum_{j=1}^3 \lambda_j \frac{\partial \psi_j}{\partial Y} &= 0 \\
\frac{\partial Z}{\partial U} + \sum_{j=1}^3 \lambda_j \frac{\partial \psi_j}{\partial U} &= 0
\end{align*}$$

(65)

$$-2 \lambda_j \alpha_j = 0$$

(66)

$$\psi_j - \alpha_j^2 = 0$$

(67)

These are nine equations for the three unknown design variables, three auxiliary variables and three Lagrange multipliers.

From Eqs. (66) there are two cases for each constraint, either $\lambda_j = 0$ which indicates the constraint $\psi_j$ is inactive, or $\alpha_j = 0$ which indicates the constraint is active.
Suppose that none of the constraints are active at a particular point. Then the optimum member would have zero area at that point. If all constraints are active, then the three Eqs. (67) define the three design variables. In this case, the beam-column is constrained simultaneously by bending-axial load, shear, and the maximum $h/t_w$ ratio. It is fully-stressed in the sense that any increase in shear, bending moment, or axial load will cause the member to become unacceptable.

Since neither the shearing stress constraint nor the maximum $h/t_w$ constraint involves the flange configuration, Eq. (65a) becomes

$$\frac{2L}{3}W + \lambda_1 \frac{\partial \psi}{\partial W} = 0$$

(68)

If the bending-axial load constraint is not active, then $\lambda_1 = 0$ from Eq. (66a) and $W = 0$ from (68) whether the constraint derivative exists at that point, or not. Therefore, except for problems in which the optimum web can carry all the external loads without the flange (and this is not an I-shaped section) the optimum beam-column will always be fully-stressed with respect to bending and axial load.

Physically this means that if the flange is understressed, material may be removed from the flange without changing the shear carrying capacity or the maximum $h/t_w$ ratio, thereby resulting in a lighter member. This process may be continued until the beam-column is fully-stressed in bending-axial load.

Because the bending-axial load constraint is not independent of the web configuration, similar arguments cannot be
made to show that the web will be fully-stressed in shear. It has been shown (5) that for beams, the web will be either fully-stressed in shear or limited by the maximum $h/t_w$ ratio. For columns and beam-columns where Eq. (36) applies, it may be possible for the web to be limited by bending-axial load, not shear or $h/t_w$. For beam-columns, however, at least when shear controls the proportions of the web, the optimum weight design can be found among the fully-stressed designs.

These observations are useful but they do not show what the optimum values of the design variables are. Because the constraints are complicated, purely analytical solutions may never be possible in general. The only problems for which closed form solutions are known are those in which the removal of the dependence of the design variables on $\mathcal{S}$ is possible. The laterally-supported beam problem has been solved (5) analytically and the prismatic column solution is given in the present study.

B. Mathematical Programming Methods

For problems too difficult to handle analytically, several methods of mathematical programming are available for obtaining numerical solutions. The problem can be recast as a non-linear programming problem in which the objective function, $\mathcal{S}$, is a relatively simple function given by Eq. (60) and the constraints are a set of complicated non-linear functions given by Eqs. (34) to (59). The design variables are now the three independent dimensions of the cross-section at each of $N$ stations along the length of the
member, a total of 3N variables. Hence the number of variables is larger, but now they are just numbers, not functions. Symbolically the problem is to minimize the volume, \( \mathcal{J} \), or, equivalently, for equally spaced elements, the sum of the areas, \( Z_i \), given by

\[
\mathcal{J} = \sum_{i=1}^{N} Z_i = \sum_{i=1}^{N} \left( \frac{\sqrt{2g}}{6} w_i^2 + y_i u_i \right)
\]  

(69)

subject to \( m \) constraints at each point,

\[
\psi_j = \psi_j (w_i, y_i, u_i) \geq 0 \quad j = 1, \ldots, m
\]  

(70)

There are several methods for solving problems of this type, three of which are described here.

1. Gradient Projection Method

   The gradient projection method developed by Rosen (13) and others and applied to structural problems by Brown and Ang (2) may be applied to the beam-column problem. The objective function, \( \mathcal{J} \), is defined in 3N-dimensional space where \( N \) is the number of design variables. Each constraint, if satisfied as an equality constraint, describes a 3N - 1 dimensional hypersurface which separates the feasible and infeasible regions of the space with respect to that constraint. Taken together, these hypersurfaces form the boundary of the feasible region.

   The gradient, \( \mathbf{G} \), of the objective function is the vector of partial derivatives with respect to each design variable,

\[
\mathbf{G} = \nabla \mathcal{J} = \left[ \frac{\partial \mathcal{J}}{\partial w_j}, \frac{\partial \mathcal{J}}{\partial y_j}, \frac{\partial \mathcal{J}}{\partial u_j} \right] \quad j = 1, \ldots, N
\]  

(71)
In this case because the form of the objective function is simple, its gradient can be expressed analytically and evaluated exactly at any point. It is given by

$$ G = \left[ \frac{2\sqrt{29}}{3} \omega_j, U_j, Y_j \right] \quad j = 1, \ldots, N \quad (72) $$

The gradient vector represents the direction of steepest ascent or the direction of greatest rate of increase of the objective function. When no constraints are active, the optimum is sought by starting with a feasible design and moving by steps in the direction of $-G$ since a minimum rather than a maximum is being sought.

When constraints are active, it will not always be possible to take a step in the direction of $-G$ and still satisfy all the constraints. When this happens, it may still be possible to improve the design by moving tangentially along the boundary. The best tangential direction is the direction of the projection of $-G$ on the intersection of $q$ of the constraining hypersurfaces, where the $q$ constraints are the constraints for which $-G$ points away from the feasible region.

In order to find this projection, the gradient of each of the active constraints is needed, where

$$ U_j = \nabla \psi_j = \left[ \frac{\partial \psi_j}{\partial \omega_k}, \frac{\partial \psi_j}{\partial Y_k}, \frac{\partial \psi_j}{\partial U_k} \right] \quad j = 1, \ldots, m \quad k = 1, \ldots, N \quad (73) $$

is the gradient of the $j$th constraint. For the beam-column problem, these derivatives are most conveniently found by finite difference approximation.

Suppose there are $q$ active constraints at a point and
$U_q$ is the matrix of gradients of the $q$ hypersurfaces at that point,

$$U_q = [u_1, u_2, \ldots, u_q]$$  \hspace{1cm} (74)

Then the projection matrix, $P_q$, which gives the projection of any vector onto the $q$ supporting hyperplanes (hyperplanes tangent to the hypersurfaces at the point) is given by (13)

$$P_q = I - U_q \left[ U_q^T U_q \right]^{-1} U_q^T$$  \hspace{1cm} (75)

A unit vector $z_1$ which points in the direction of the projection of $G$ on the intersection of the $q$ supporting hyperplanes is

$$z_1 = \frac{P_q G}{|P_q G|}$$  \hspace{1cm} (76)

If $\overline{x}_j$ is a feasible set of design variables which corresponds to a point near the boundary of the feasible region, then a step along the intersection of the $q$ active constraint hypersurfaces is given by

$$\overline{x}_{j^{(m+1)}} = \overline{x}_j - \gamma z_1$$  \hspace{1cm} (77)

where $\gamma$ is a step size.

Because the hypersurfaces are non-linear, the new point, $\overline{x}_{j^{(m+1)}}$, may not lie in the feasible region. If this is true, an iteration back to the feasible region is necessary. It is not difficult to show (13) that

$$\overline{x}_{j^{(m+1)}} = \overline{x}_{j^{(m)}} - U_q \left[ U_q^T U_q \right]^{-1} w_q$$  \hspace{1cm} (78)
is an iteration perpendicular to \( z_1 \) in the direction of the feasible region. \( W_q \) is a vector of values of the constraints or a vector of the values, \( \alpha_i^2 \), and therefore is a measure of the distance from the feasible region. The step taken is shown in Fig. 2.

A local optimum is reached when the gradient becomes perpendicular to the active constraint set \( (|P_q G| = 0) \) and has no positive component along the unit normal of any supporting hyperplane, positive meaning into the feasible region.

The gradient projection method works well for problems with small numbers of constraints, particularly when the constraints are easily evaluated. These properties describe the prismatic beam-column problem which contains only three independent variables and two constraints. Application of the method may also be made to non-prismatic columns, but here the number of constraints is larger and therefore the size of the matrix to be inverted at each step is larger. In addition, it is computationally much more difficult to evaluate constraints and constraint derivatives, particularly those pertaining to axial load. Hence, for the non-prismatic case, the gradient projection method becomes complicated and time-consuming.

For other structural problems with large numbers of design variables, the complex method described by Mitchell and Kaplan(9) has been successful. It has the advantages of not requiring derivatives of constraint functions and not requiring manipulation and inversion of large matrices at each step. Perhaps this method might be useful in optimizing non-
prismatic beam-columns.

2. Grid Search Methods

Another method which is simpler than the gradient projection method is a direct grid search. Here a grid of points is placed around a feasible point which for the special case of two design variables might appear as shown in Fig. 3. The objective function and constraints are examined for each surrounding point on the grid.

![Fig. 3](image1)

![Fig. 4](image2)

If one of these points is more optimal, the central point of the grid is shifted to the most optimal of these points. When none of the surrounding points improve the design, the grid is compressed. The process is repeated until a sufficiently small grid does not contain points which have a lower objective function than the central point.

This method was tested for the prismatic beam-column problem which contains three independent variables and therefore a grid of 26 adjacent points. The results were entirely unsatisfactory when checked against the known analytical solutions for laterally supported beams, and were sensitive to both the initial feasible design and the relative grid dimensions.
The reason for the erroneous convergence is that only a few directions are investigated for each central point, not all directions as can be seen in Fig. 4. As drawn, only half of the surrounding grid points are feasible. Of the rest, suppose all give a higher value for the objective function. Then according to this scheme, the grid would be compressed. But it may happen that another point such as point P shown in Fig. 4 gives a lower value for the objective function. Under such circumstances the grid could be compressed repeatedly without finding the true optimum. Wilde (19) further discusses the inapplicability of grid search methods.

Goble and Razani (3) have applied the method to a two-dimensional plate girder design problem and have reported similar difficulties. They found better results using a 24 point grid as shown in Fig. 5, but the method is theoretically unsound and a larger set of grid points becomes very cumbersome in more than two dimensions. The refinements necessary to put the method on a sounder basis involve determination at each step of the best direction to move out of all directions, not just eight or twenty-four. But this is precisely equivalent to the gradient projection method.
3. Cost Optimization by Dynamic Programming

For certain structural problems which can be decomposed into sub-problems and these treated sequentially, dynamic programming can be a powerful method. Several authors have applied it to the minimum cost design of plate girders (4) (12).

When dynamic programming is applied to such problems, a table of available widths and thicknesses of steel plate is assumed known. Widths and thicknesses not listed in the tables are not examined. Corresponding to each plate size is a material cost. This cost information together with fabrication cost information is used to minimize the total cost of the structure.

The method can also be applied to the minimum cost design of beam-columns as follows. The beam-column is divided into a number of elements of equal length. Preliminary estimates are made for the maximum external moments and shears and for the column buckling load. The shears and moments are found exactly for determinate beams and for prismatic fixed-end beams. A set of web heights are assumed, e.g., ones which are optimal for laterally supported non-compact beams under the assumed set of moments and shears.

Using the initial moments and shears, the minimum allowable web thickness for each height is calculated and the minimum acceptable thickness from the web plate table is found. Then the smoothing procedure of dynamic programming (1) is used to balance welding and material costs.
so as to obtain a minimum cost design. The smoothing procedure is equivalent to treating the beam-column as a sequential n-stage decision-making process, each stage being an element, each state being a feasible plate size for that stage.

For the flange a two dimensional smoothing procedure is used. For each flange width listed in the table, the minimum feasible thickness is found. A smoothing over the feasible plate sizes then determines the minimum flange cost design.

Since the web heights were assumed, they must be adjusted and the process repeated. This can be accomplished by assuming the member to be composed of linearly tapered segments and making a grid search on the heights of the ends of the segments. When the best set of heights is found, the design is a local minimum cost design for the current set of design moments, shears, and axial load.

But in general the moments, shears, and column buckling load are now different from those assumed. A reanalysis must be made and a new optimum found using new shears, moments, and column buckling load. Since for zero fabrication costs, the procedure converges to a fully-stressed, not minimum weight, design, the corresponding design with fabrication costs included is not necessarily an optimum design. A procedure whereby the optimality of such designs can be checked is given by Goble and Razani (3).
This method has advantages over other methods for practical problems. Frequently, it more closely represents real problems in which the structures are cost oriented, an advantage that is partially offset by the inadequacy of available cost information. Also, the assumption of discrete plate sizes is more realistic.

The method has limitations in that an optimum cannot be guaranteed for the most general problems. Also, for variable depth beam-columns, the computing time may be high and computational difficulties can sometimes occur if the allowable axial load is greatly changed by small local changes in the cross sectional area.
III Optimum Prismatic Columns

The general methods discussed in the previous chapter are applied to specific problems here and in succeeding chapters. The optimum prismatic column problem consists of choosing the three design variables, W, Y, and U so as to minimize the area, Z, subject to the constraints given by Eqs. (34) to (59) where the load, X, the ratio of effective lengths, R, and the material parameter, $\gamma$, are known. An equivalent problem which is more easily solved is: Given Z, R, and $\gamma$, maximize X subject to the same constraints.

Since the design variables are numbers, not functions of position along the length, the calculus of variations problem discussed in Chapter II is reduced to a problem in the ordinary theory of maxima and minima. Hence, at most, differentiation of the objective function and constraints with respect to each design variable is required for solution.

The applicable functional requirements are Eqs. (41) and (42) which together give the allowable axial stress and Eqs. (36) and (8) which restrict the width/thickness ratios for the web and the flange. In order to make the two allowable stress formulas continuous at $\rho = 1$, Eq. (41) is taken to be $\phi_a = \frac{6}{23} \rho^2$ with an error of 0.2%.

It should be noted that $\phi_a$ increases monotonically with $\rho$ and therefore with the moment of inertia, I. Hence, for
a given area, \( Z \) is a maximum when \( I \) is. \( I \) reaches its maximum value when the width/thickness ratios attain their maximum values as given by Eq. (8,36). It should be noted that while the maximum web slenderness ratio of Eq. (36) may be exceeded, it is done only at the expense of effective area for carrying stress. Columns for which Eq. (35) is satisfied exactly are called columns with normal webs. Columns for which the maximum web slenderness of Eq. (35) is exceeded are called columns with slender webs and are treated separately.

A. Columns with Normal Webs

If the web takes on the maximum slenderness ratio of Eq. (35), the resulting equality can be used to eliminate one variable, say \( U \), giving

\[
U = \frac{129}{8} \gamma
\]

(79)

The dimensionless area is then

\[
Z = \frac{129}{3} \frac{W^2}{Z} + \frac{129}{8} \gamma^2
\]

(80)

The reciprocal slenderness ratios, \( \rho_x \) and \( \rho_y \), are

\[
\rho_x^2 = \frac{29}{12k^2} \frac{\gamma^2}{Z} \left[ \frac{Y^2}{Z} + W^2 \right]
\]

(81)

\[
\rho_y^2 = \frac{29}{36k^2} \frac{W^4}{Z R^2}
\]

(82)

where \( k^2 = \frac{174 \sqrt{29}}{3427} \).

The allowable stress constraint (Eqs. (41) and (42)) is a function of the minimum of \( \rho_x \) and \( \rho_y \). If for a given area, the smaller of \( \rho_x \) and \( \rho_y \) can be increased at the expense
of the other, then the largest $X$ results when $\rho_x = \rho_y = \rho$. When and if this is possible is found by eliminating $W$ and $Y$ among Eqs. (80), (81), and (82), giving $\rho_x$ as a function of $\rho_y$ and $Z$,

$$\rho_x^2 = \frac{2}{3k^2} \left( Z^{1/2} + 4kR\rho_y \right) \left( Z^{1/2} - 2kR\rho_y \right)$$  \quad (83)

Differentiation gives

$$2\rho_x \frac{d\rho_x}{d\rho_y} = \frac{2R}{3k^2} \left( Z^{1/2} - 8kR\rho_y \right)$$  \quad (84)

Eq. (84) shows that $\rho_x$ may be increased at the expense of $\rho_y$ when $\rho_y \geq \frac{Z^{1/2}}{8kR}$. The limiting case occurs when $\rho_y = \frac{Z^{1/2}}{8kR}$ and $\rho_x = \rho_y$. From Eq. (83) this leads to $R^2 = 1/48$. Hence, there are two cases. For values of $R^2 < 1/48$, $\rho_x$ will be smaller than $\rho_y$ and will govern the design. Otherwise they will be equal.

Case 1. ($R^2 \geq 1/48$)

It has been shown that in this case, $\rho_x = \rho_y = \rho$. This relation and Eqs. (80), (81), and (82), can be used to obtain an algebraic relation between $Z$, $R$, and $\rho$, for an optimum column. The result is

$$Z = \frac{87\sqrt{29}}{3427} \rho^2 \left[ 3 + 20R^2 - 2R\sqrt{6 + 36R^2} \right]$$  \quad (85)

The relation between $X$ and $Z$ can be found by solving Eq. (85) for $\rho$ and substituting into Eqs. (41) and (42). Because the resulting expression cannot be solved explicitly for $Z$ as a function of $X$ and $\rho$ and because the expression changes
form at \( \rho = 1 \), the implicit form of the solution is retained.

The other design variables may be found upon substitution of Eq. (85) into Eqs. (81) and (82). The result is:

\[
W = 3 \sqrt[3]{\frac{58}{3+27}} \rho \left[ R \sqrt{6+36R^2} - 2R^2 \right]^{\frac{1}{2}} \tag{86}
\]

\[
Y = 2 \sqrt[3]{\frac{174}{3427}} \rho \left[ 3 + 24R^2 - 4R \sqrt{6+36R^2} \right]^{\frac{1}{2}} \tag{87}
\]

Of interest also is the change in shape with \( R \), given by

\[
\frac{Y}{W} = \sqrt[3]{\frac{2}{3R}} \left[ \sqrt{6+36R^2} - 6R \right]^{\frac{1}{2}} \tag{88}
\]

Case 2. \( R^2 \leq 1/48 \)

Here there is one less relation between variables since \( \rho_y = \rho_x \) no longer holds. The solution is governed by \( \rho_x \) so that

\[
X = \phi_a (\rho_x) \cdot Z \tag{89}
\]

All but one design variable can be eliminated, say \( W \), through Eqs. (81) and (82) giving

\[
\rho_x^2 = \frac{2}{3 \sqrt{29}} \frac{Z - \sqrt{\frac{29}{3} W^2}}{k^2 Z} \left[ Z + \frac{2 \sqrt{29} W^2}{3} \right] \tag{90}
\]

For a fixed \( Z \), the optimum will occur when \( \frac{dX}{dW} = 0 \).

When the differentiation is carried out, the only solution is found to be

\[
Z = \frac{4 \sqrt{29}}{3} W^2 \tag{91}
\]

Back substitution of Eq. (91) into Eqs. (90) and (80) yields the solutions for case 2:
\[ Z = \frac{696 \sqrt{29}}{10,281} \rho_x^2 \]  
(92)

\[ W = \sqrt{\frac{174}{3427}} \rho_x \]  
(93)

\[ Y = 4\sqrt{\frac{87}{3427}} \rho_x \]  
(94)

\[ \frac{Y}{W} = 2\sqrt{2} \]  
(95)

The optimum dimensions in this case do not depend on \( R \) and the optimum \( Y/W \) ratio is constant. Also, the solutions for \( R^2 \geq 1/48 \) reduce to these solutions for \( R^2 = 1/48 \) as they must.

A curve of \( Z \) vs. \( X \) is presented in Fig. 6 for various values of \( R \). The dotted line marks the minimum allowable \( \rho \). It corresponds to an \( L/r \) of 200. The dashed line marks the change in allowable stress formulas from elastic to inelastic. For large \( X \), the optimum area curves approach a single curve for all \( R \). This is because the column is essentially yielding and the buckling stress for a given \( Z \) is nearly independent of \( \rho \).

The optimum depth is plotted in Fig. 7 for various values of \( X \) and \( R \). The optimum depth decreases with increases in \( R \) since more area is needed in the flange for lateral stiffness. Fig. 8 shows how the shape of an optimum column varies with \( R \).

When \( R \) is greater than 1, the case 1. solution applies and shows that the optimum column has a shallow web and a
Fig. 6. Optimum Area of Columns with Normal Webs vs. Axial Load for Various Lateral Bracing Conditions
Fig. 7. Optimum Web Height for Columns with Normal Webs vs. Axial Load
Fig. 8. Optimum Web Height/Flange Width for Columns with Normal Webs
wide flange. It is desired to see if a lighter column would result by rotating the column 90° about its longitudinal axis and redesigning it. This is done by assuming that two optimum columns have the same area, Z, but that $R = R_1 > 1$ in the first and $R_2 = \frac{1}{R_1} < 1$ for the second. This represents an interchange of the x and y axes. It is desired to show that $X_1$, the maximum load which can be carried by the first is less than $X_2$, the maximum load which can be carried by the second.

The dimensionless allowable stress, $\phi$, given by Eqs. (41) and (42), is a monotone increasing function of $\rho$. Hence for constant Z, if $\rho_1 < \rho_2$ then $X_1 = \phi(\rho_1)Z < X_2 = \phi(\rho_2)Z$.

For optimum columns, Z is given by Eq. (85). Therefore,

$$Z = \frac{87\sqrt{29}}{34+27} \rho_1^2 \left[3+20R^2-2R \sqrt{6+36R^2}\right] = \frac{87\sqrt{29}}{34+27} \rho_2^2 \left[3+\frac{20}{R^2} - \frac{2}{R} \sqrt{6+\frac{36}{R^2}}\right]$$

Since $\rho_1 > 0$ and $\rho_2 > 0$, $\rho_1$ is less than $\rho_2$ if and only if $\frac{\rho_1^2}{\rho_2^2} < 1$ which by Eq. (96) is true if and only if

$$\frac{3 + \frac{20}{R^2} - \frac{2}{R} \sqrt{6+\frac{36}{R^2}}}{3 + 20R^2 - 2R \sqrt{6+36R^2}} < 1$$

(97)

Rearranging, Eq. (97) is equivalent to

$$R^2 \left(10 - \sqrt{\frac{6}{R^2} + 36}\right) + \frac{\sqrt{6R^2+36}}{R^2} - 10 > 0$$

(98)

Eq. (98) is zero for $R = 1$. The slope at $R = 1$ is easily shown to be positive. The first term of Eq. (98) increases rapidly with R, the second increasing slowly until it becomes positive at $\frac{8}{\sqrt{6}}$, thereafter remaining positive as R in-
creases. Hence $1 < R^2 < 48$ implies that $X_1 < X_2$. For $R^2 < 48$ the expression for $Z$ is given by Eq. (92), but the same conclusion follows. Q.E.D. There are no optimum columns with normal webs for which $R > 1$.

B. Columns with Slender Webs

If the maximum web height/thickness ratio is exceeded, but if the cross-section would be acceptable with a portion of the height considered as removed, then by Eq. (36) the member is acceptable. There is the possible advantage of greatly increasing $I_x$ in exchange for a modest decrease in the effective area. The question arises as to whether or not there are optimum columns with a configuration as shown in Fig. 9. The portion of the web enclosed with dotted lines

![Fig. 9](image)

is present but is not included as effective area for the actual stress calculation. The width of the portion of the web not counted is denoted by $g$, an additional independent variable in the problem. The web height/thickness constraint can still be used to eliminate one variable, say $g$, but $U$ remains.

\[
U = \frac{\sqrt{2g}}{8} \frac{F_g \nu_2}{E \nu_2} (h - g)
\]  

(99)
The dimensionless area is
\[ Z = \frac{\sqrt{29}}{3} w^2 + YU \]  \tag{100}

The effective area is
\[ A_e = 2bt + (h-g)t_w \]  \tag{101}

and the dimensionless effective area is
\[ Z_e = \frac{\sqrt{29}}{3} w^2 + \frac{8}{\sqrt{29}} U^2 \]  \tag{102}

The maximum allowable load is given by
\[ X = \phi_a Z_e \]  \tag{103}

The expression for \( \rho_y \) is unchanged (Eq. (82) ) but \( \rho_x \) is given by
\[ \rho_x^2 = \frac{29Y^2}{12k^2L_x^2} [YU + \sqrt{29} W^2] \]  \tag{103}

It can be shown that there are no optimum columns for which \( \rho_y > \rho_x \). Hence there are two cases to consider.

Case 1. ( \( \rho_x = \rho_y \) )

The relation \( \rho_x = \rho_y \) taken together with Eqs.(103), (100) and (82) can now be used to obtain \( U, Y, \) and \( W \), in terms of \( Z \) and \( \rho \). Then by Eq. (102), \( Z_e \) can be obtained in terms of \( Z \) and \( \rho \). The result is
\[ Z_e = 2k\rho R Z^{\frac{\rho}{2}} + \frac{2Z^{\frac{\rho}{2}}}{3k^2\rho^2} [Z^{\frac{\rho}{2}} + 4k\rho R][Z^{\frac{\rho}{2}} - 2k\rho R]^2 \]  \tag{104}

For an optimum to exist, a necessary condition is
\[ \frac{dX}{d\rho} = 0 \]  \tag{105}

In addition, the solution must satisfy non-negativity requirements on \( g \) and \( h - g \). If \( g \) is to be non-negative,
then $Z_e \leq Z$. Using Eq. (104) this leads to
\[ 3k^2 \rho^2 \geq 2 [Z^{1/2} + 4k \rho R][Z^{1/2} - 2k \rho R] \]  
\[ (106) \]

From Eq. (99), $h - g$ is positive if and only if $U$ is positive. If $U$ is found in terms of $Z$ and $\rho$, then it can be shown that the following inequality insures that $U$ is positive.
\[ Z^{1/2} - 2k \rho R > 0 \]  
\[ (107) \]

In case elastic buckling controls (Eq. (41)), Eq. (105) has a solution, but inequalities (106) and (107) are violated. Hence there are no optimum solutions in the elastic range for columns with slender webs.

In case the inelastic buckling controls (Eq. (42)), Eqs. (42) and (104) are used to evaluate Eq. (105). The result is a high-order polynomial in $\rho$ and a cubic in $Z^{1/2}$ given by
\[ B Z^{3/2} + 12k^3 \rho^4 R^2 (2D - B) Z^{1/2} + 16k^3 \rho^3 R^3 (B - 3D - 3C/8R^4) = 0 \]  
\[ (108) \]

where
\[ B = 160 \rho^5 + 18 \rho^4 - 160 \rho^3 - 21 \rho^2 + 3 \]
\[ C = 40 \rho^5 + 18 \rho^4 + 20 \rho^3 - 12 \rho^2 + 3 \]
\[ D = 80 \rho^5 + 18 \rho^4 - 40 \rho^3 - 15 \rho^2 + 3 \]

The optimum solution must satisfy this equation, inequalities (106) and (107), and in addition be more optimum than the corresponding solution for columns with normal webs. A numerical investigation of equation (108), the inequalities, and the solution for columns with normal webs reveals that the region of applicability of this solution is that shown in Fig. 10. The inequalities (106) and (107) are violated to
the right of curve 1. The solution for columns with normal webs is more optimal to the left of curve 2.

Case 2. \((\rho_x \leq \rho_y)\)

For this case, since \(\rho_x = \rho_y\) no longer applies, one less design variable may be eliminated. Hence, there are two independent variables and two necessary conditions for an optimum,

\[
\frac{\partial X}{\partial \rho_x} = 0 \quad \frac{\partial X}{\partial \rho_y} = 0
\]  

(109)

It is easily shown that no solutions exist where elastic buckling controls.

If Eq. (109) is evaluated using (42) and (102), the following solution is found.

\[
Z = \frac{k^2 \rho_x^2}{48 \alpha \beta} \left[ 6 \alpha \beta + 3 \beta^2 + (6 \alpha + \beta) \sqrt{9 \beta^2 + 24 \alpha \beta} \right]
\]  

(110)

where

\[
\alpha = 18 \rho_x^4 + 80 \rho_x^3 - 9 \rho_x^2 + 3
\]  

(111)

\[
\beta = 160 \rho_x^5 + 18 \rho_x^4 - 160 \rho_x^3 - 21 \rho_x^2 + 3
\]  

(112)

The solution is independent of \(\rho_y\) and \(R\), though \(\rho_y R\) can be found from an intermediate result,

\[
(\rho_y R)^2 = \frac{\beta \rho_x^2}{64 \alpha} \left[ 1 + \sqrt{1 + \frac{8 \alpha}{3 \beta}} \right]
\]  

(113)

By letting \(\rho_x\) approach \(\rho_y\) in Eq. (113), the boundary of the region of applicability of this solution is found and is given by

\[
R^2 = \frac{\beta}{64 \alpha} \left[ 1 + \sqrt{1 + \frac{8 \alpha}{3 \beta}} \right]
\]  

(114)

This is the same kind of boundary as the boundary \(R^2 = 1/48\) for columns with normal webs.
The two solutions (Eqs. (108) and (110)) for \( g > 0 \) were computed numerically over the ranges of applicability and compared with optimum columns for which \( g = 0 \). The region over which each solution was found to be optimum is shown in Fig. 10.

It was expected that the advantage of increasing \( \rho \) would cause a wide applicability for columns with slender webs. However, as \( \rho \) increases in the inelastic range, the increases are not very beneficial since the column is essentially yielding, not buckling. Therefore since increases in \( g \) cause decreases in \( Z_e \), the loss in effective area can more than offset the gain in allowable stress. Hence for high values of \( \rho \), the columns with \( g = 0 \) are optimum. It is interesting to note that in regions where columns with slender webs are not optimal, they are only slightly heavier than optimum, even when the webs are very deep.

The percentage saving in column area for columns with slender webs is shown in Fig. 11, the maximum value being about 0.5 %. The percentage increase in \( Y/U \) ratio for these columns is shown in Fig. 12, the maximum being just over 30 %.

It can be concluded that over a wide range of loads and effective lengths, columns with normal webs can be taken as optimum, particularly since the slightly lighter solutions obtained with slender webs are found only at the expense of large increases in depth. However, since the optimum area is not highly sensitive to increases in depth, columns with much deeper webs can be designed without a large weight penalty.
Fig. 10. Region of Optimality of Columns with Slender Webs.
Fig. 11. Percentage Savings in Area of Columns with Slender Webs vs. Columns with Normal Webs
IV Optimum Non-Prismatic Columns

The problem considered here is the optimization of columns whose cross-section may vary along the length. Because of this variation, the problem is a calculus of variations problem as described in Chapter II, in which the optimum design variables are functions of $\mathcal{S}$, the position along the length. The variables are to be determined which minimize the volume, $V$, and carry a given load, $X$, where

$$V = \int_0^1 \left( \frac{1}{3} \frac{2g}{3} W^2 + YU \right) d\mathcal{S}$$

$$X = X(Y, U, W)$$

(115)

(116)

Consideration is restricted to columns with normal webs. For these problems, the web slenderness ratio, $Y/U$ is given by Eq. (35). If it is used to eliminate the thickness of the web, the problem is reduced to one of finding two functions, $Y$ and $W$, which carry the load and minimize $V$.

Because the constraining equation is complicated, no analytical solution is known except for pin-ended elastic columns whose cross-sectional shapes are geometrically similar along the length (7). Therefore a non-linear programming method such as the gradient projection method may be useful. Here the column is divided into a number of elements, $N$. The total number of design variables is $2N$. In order to apply the method, the partial derivatives of the constraint with respect to each of these variables must be
evaluated, if only approximately, for each step. Hence the practical application of the method depends on a capability for rapidly evaluating the constraint and on a reduction in the number of variables.

A. Critical Buckling Loads for Non-Prismatic Columns

A method for rapidly evaluating the axial load constraint is particularly troublesome. The pertinent functional requirements, Eqs. (41) to (42), are written in terms of an $L/r$ ratio and an allowable stress, neither of which is well-defined for a tapered column. It seems more reasonable to write the constraint in terms of an allowable load since any tapered column does have a critical buckling load, whether elastic or inelastic.

This critical load can be found quickly by using the method of transfer matrices described by Pestel and Leckie (10). If the column is divided into a number of elements, a transfer matrix gives the deflection, slope, moment, and shear at the right end of an element in terms of the same quantities at the left end of the element. The transfer matrix for an elastic prismatic axially-loaded element is given in Appendix I.

Extension of the method to include inelastic behavior is necessary in order to cover the full range of column problems. This is accomplished by deriving a tangent modulus, $E_t$, from the basic column strength curve established by
the Column Research Council (6). This curve is the basis for the inelastic column formula in the A.I.S.C. specifications (1.5.1.3.1). The tangent modulus is found by setting the critical stress equal to an equivalent elastic buckling stress. The result is 

\[ E_t = 4EP \left[ 1 - P/(AF_y) \right] / (AF_y). \]  

(117)

If the tangent modulus is assumed constant over each element, then the transfer matrix for an inelastic element will take exactly the same form as for the elastic element, but with \( E \) replaced by \( E_t \).

A theoretically exact transfer matrix for linearly tapered elements including exactly the change in \( E_t \) within an element due to the taper was also used. Because the components of the exact matrix are complicated infinite series, the calculation was prone to round-off error and required some additional time. The derivation of this more exact transfer matrix is given in Appendix I.

The transfer matrix is used as follows to calculate the critical buckling load. If \( S_{Li} \) is the vector of state variables at the left end of element \( i \),

\[
S_{Li} = \begin{bmatrix}
\text{displacement} \\
\text{slope} \\
\text{moment} \\
\text{shear}
\end{bmatrix}_{Li}
\]

then \( S_{Ri} = Q_i S_{Li} \), where \( Q_i \) is the transfer matrix for element \( i \), either exact or approximate,
\[ S_{Li} = \text{state vector at left end of element } i, \]
\[ S_{Ri} = \text{state vector at right end of element } i, \]
\[ i = 1,2,\ldots,n \text{ element index, left to right} \]
\[ S_{Li+1} = R_i S_{Ri} \]
\[ = R_i Q_i S_{Li} \]

where \( R_i \) is a matrix which takes care of intermediate boundary conditions such as intermediate supports (10). It is a unit matrix if there are no intermediate supports.

The variables at the right end of the column can then be calculated by \( S_{Rn} = Q_n R_{n-1} Q_{n-2} \cdots Q_1 S_{L1} \).

This gives four equations for the state vectors at each end or a total of eight unknowns. Four boundary conditions, two at each end, give the remaining equations for the complete solution of the problem. In the case of a pure column, the equations will be homogeneous and will possess a non-trivial solution if and only if the determinant of the coefficients of the unknowns is zero. For ordinary boundary conditions, this determinant is only second order since four of the variables are zero from the boundary conditions and two of the remaining four are uncoupled.

Since the transfer matrices are functions of the critical load, the quantity to be determined, the procedure is one of successive approximation. An estimate of the buckling load is made, the determinant is evaluated, a new load estimate is made, etc., until the load is determined within a small tolerance which causes the determinant to vanish.
Initial upper and lower bounds on the buckling load can be found easily. The minimum of the critical loads for the following two hypothetical columns will be a lower bound:

1. A column which is prismatic with the minimum radius of gyration of any element along the length of the real column.

2. A column which is prismatic with the minimum area of any element along the length of the real column.

The first of these will result in a lower bound for fully elastic columns; the second may govern for strongly inelastic columns, the area entering through $E_t$. The load which fully yields the smallest area of the column is an upper bound.

The lowest critical load is desired. In order to prevent calculation of a load corresponding to one of the higher modes, care must be taken in selecting improved estimates of the critical load. The value of the determinant can be visualized as a stiffness which decreases from a certain value to minus infinity, then starts again at plus infinity and drops to minus infinity, etc. The first crossing corresponds to the value desired, the others correspond to higher modes. The procedure described in Fig. 13 corresponds to a computer program subroutine SELEC which is listed in Appendix II. It successfully avoided the higher modes in all cases.

Once the critical load is found, the allowable load is the critical load divided by a factor of safety. The
Definition of Symbols

- **P**: axial load
- **VS**: stiffness
- **NA**: iteration number (0 on output if converged)
- **AM**: min. cross-sectional area
- **FY**: yield stress
- **PA**: lower bound on axial load
- **VA**: stiffness corresp. to PA
- **PB**: upper bound on axial load
- **VB**: stiffness corresp. to PB
- **PI**: initial load
- **VI**: initial stiffness
- **NB**: 1 after VS has been negative or 0 otherwise
- **ND**: 1 if NA > 10 or 0 if NA < 10

```
START

No

NA = 1

Yes

PA = P
VA = VS
PI = P
VI = VS
PB = FY * AM

NB = 0

P = P + .1 * (PB - PA)

RETURN

No

vs : 0

Yes

vs : vi

No

PB = P
VB = VS

RETURN

No

NB : 0

Yes

VS : VB

No

GO TO BLOCK A

Yes

PA = P
VA = VS

PB = P
VB = VS

NB = 1

P = P + \frac{V(P - PA)}{(VA - V)}

No

NA = 10

Yes

ND = NA / 10

XND

P = P + 0.1 * (PB - PI)

P = P + 0.1 * (PB - PI)

(PB - PA) : P

100

Yes

NA = 0

RETURN
```

Fig. 13
factor of safety is 1.92 for fully elastic columns, but is a function of $KL/r$ for inelastic columns and varies from 1.67 to 1.92. Since $KL/r$ is not well-defined for a tapered column, an average factor of safety was used. It was found by substituting the average radius of gyration of all the elements and the effective length of the column into Eq. (18).

This procedure was applied to critical load problems for columns and beam-columns with fixed, pinned, or free ends, subject to the conditions of static stability. Lateral brace points for the beam were considered as pinned supports for the weak axis critical load calculation. Generalization to other end and intermediate support conditions is possible.

A computer program which uses this procedure is listed in Appendix II. The average time required to make a single critical load calculation with 24 elements is four seconds on the Burroughs B5500 computer. If critical loads about both strong and weak axes are required, twice this time is required.

B. Optimum Taper for Columns with Both Effective Lengths the Same

Because of the time involved in calculating the critical load, a set of more restricted problems with fewer variables was treated. The dimensionality can be cut in half by confining attention to those problems for which $L_x = L_y$. Then $\rho_x = \rho_y$ must hold at every point along the length of the column. This may be seen by considering a column in which
\( \rho_x = \rho_y \) everywhere except in a neighborhood of a point where, say, \( \rho_x < \rho_y \). Then if material is removed from the flange and \( 1/6 \) of it returned to the web, a savings in material has occurred without a change in the critical load. This process is worthwhile until \( \rho_x = \rho_y \). Similar arguments can be given for \( \rho_y < \rho_x \). Therefore, \( \rho_x = \rho_y \) and by Eqs. (81) and (82),

\[ Y^2 = 4c W^2 \]  

(118)

where \( c = -1 + \sqrt{2} \).

Substituting Eq. (118) into Eq. (115),

\[ V_l = \frac{\alpha (3c)}{2 + 3c} = \int_0^1 Y^2 d\mathcal{Y} \]  

(119)

The problem is to find \( Y(\mathcal{Y}) \) which satisfies the constraint and minimizes \( V_l \). Because all the design variables at a point can be found in terms of a single variable, \( Y \), the shapes of the cross sectional areas are geometrically similar along the length.

C. Optimum Linear Taper for a Pin-Ended Column

Consider now the optimum taper for a column in which \( Y(\mathcal{Y}) \) is given by

\[
Y = a_1 + a_2 \mathcal{Y} \quad \text{for} \quad \mathcal{Y} \leq \frac{1}{2} \\
= a_1 + a_2 (1 - \mathcal{Y}) \quad \text{for} \quad \frac{1}{2} \leq \mathcal{Y} \leq 1
\]  

(120)

where \( a_1 \) and \( a_2 \) are constants to be determined.

A column of this type is shown in Fig. 14. The dimensionless volume, \( V_1 \) is
\[ V_1 = a_1^2 + \frac{a_2 a_3}{2} + \frac{a_4^2}{12} \]  

(121)

The number of variables has now been reduced to only two, \( a_1 \) and \( a_2 \). The gradient projection method may be applied as mentioned or the inverse problem may be solved, giving, for a constant volume, \( V_1 \), the set of variables which maximizes \( P \) consistent with Eq. (121). If \( V_1 \) is the dimensionless volume associated with the optimum prismatic column \( (a_2 = 0) \) which will carry a given load \( X \), then the maximum load any linearly tapered column of the same volume can carry is determined by the procedure of Fig. 15.

D. Optimum Parabolic Taper for a Pin-Ended Column

The same procedure can be applied to a column with a parabolic taper by letting

\[ Y = a_3 + a_4 (s^2 - s^2) \]  

(122)

which gives

\[ V_1 = \int \frac{Y^2}{2} d y = a_3^2 + \frac{a_3 a_4}{3} + \frac{a_4^2}{30} \]  

(123)

Sample results for both linear and parabolic tapers are described as follows. An optimum prismatic beam is designed which has a length of 20 feet and carries a load of 10,000 pounds. If steel is used for which \( F_y = 36,000 \) psi, then the column is designed for a dimensionless load, \( X \), of .01085. The optimum proportions which result are
Fig. 15. Non-Prismatic Column Optimization
Y = 0.348 and W = 0.618. The volume, \( V \), for the column, is about 0.749.

Using this volume, but tapering the member linearly results in the following equation for the optimum depth:

\[
Y = 0.267 + 0.315 \gamma \quad \text{for } \gamma \leq \frac{1}{2}
\]

A column of these proportions will carry a load of \( X = 0.01295 \), 19% more load than the prismatic column. The slope of the web height of this column is about 1.1%.

If the optimum parabolic taper is determined, the equation for the web height is

\[
Y = 0.227 + 0.700(\gamma - \gamma^2)
\]

A column so tapered has a critical buckling load which is 24% greater than that of the optimum prismatic column of the same volume.

The optimum taper of arbitrary shape for the sample problems solved in this section was found analytically by Keller (7). The solution provides a theoretical upper bound on \( P \) of 1.33 times the critical load for a prismatic column of equal volume. The shape of the optimum arbitrary taper is described by a curve which decreases to zero at the ends. As it does, the area decreases to zero and the stress becomes infinite. Hence the solution, while theoretically sound, is unattainable practically.

An infinite stress cannot occur using the procedure described herein since a large reduction in area near the
ends causes a decrease in $E_t$ near the ends and corresponding decrease in the critical load. The 19% increase in the allowable load for an optimum linear taper and the 24% increase in the allowable load for the parabolic taper show that much of the advantage gained by tapering can be done with these simple kinds of tapers.

As the magnitude of the load increases, the optimum taper decreases to zero. This is because for highly inelastic columns, the column fails largely by yielding, not buckling, and decreases in area anywhere are harmful.
V Optimum Prismatic Beam-Columns

The problem to be considered here is the minimum weight design of a prismatic beam-column subjected to a given moment, $M_0$, and shear, $V_0$, in the plane of the web and a given axial load, $X$. Except for the restriction to prismatic members, it is the most general beam-column problem formulated in Chapter II. All of the functional requirements given by Eqs. (34) to (59) must be met and the full range of loadings from pure beam to pure column is considered.

Because the member is prismatic, there are only three design variables, $Y$, $U$, and $W$, none of which vary with $\theta$. The problem is to find values for these variables which satisfy the functional requirements and minimize the area parameter, $Z$.

A. Gradient Projection Method

Because of the algebraic complexity of the set of constraints, a special form of the gradient projection method was used to obtain a numerical solution. The first step in the application of the method is to generate an initial feasible design. Since the solution is known to lie on a boundary of the feasible region, the initial design is chosen so as to lie in a $\delta$-neighborhood of the bending-axial load constraint surface. This design is found by taking a height equal to the optimum height for a laterally-
supported beam (5). The web thickness corresponding to this height is taken to be the minimum which both carries the shear and satisfies the \( h/t_w \) limitations.

Using this web design, the minimum flange is determined by a one-dimensional search on the flange width variable, \( W \). The flange must satisfy the axial load and bending constraining and lie within a distance, \( \delta \), of the constraint boundary. This is equivalent to requiring that \( \alpha_i^2 \) be less than \( \delta \). For this study, \( \delta \) is taken to be \( 10^{-3} \).

Given this starting design, the step-by-step procedure used for finding the optimum design is as follows:

1. Evaluate the gradient, \( G \), of the objective function.
2. Calculate the partial derivatives of the active constraint surfaces with respect to each design variable using finite differences.
3. Assemble the projection matrix, using Eq. (75).
4. Take a step in the direction of the gradient projection on the intersection of the active constraint surfaces using Eq. (77) and a step size, \( \tau \).
5. If the new design is in the feasible region, go to step 7. If not, go to step 6.
6. Iterate back to the feasible region using Eq. (78).
7. If the current feasible design is not within a \( \delta \) -neighborhood of the bending-axial load constraint surface, reduce \( W \) until it is.
8. If the new design corresponds to a lower value of \( Z \), go to step 9. If not, go to step 10.
9. If the gradient vector, \( \mathbf{G} \), has a component toward the feasible region and perpendicular to the web thickness constraint surface, then the web thickness constraint becomes inactive. Go to step 11.

10. The local optimum is bracketed. Reduce the step size and return to the previous design. Go to step 11.

11. If the difference in the current and previous values of \( Z \) is less than \( \varepsilon \), output the optimum design. Otherwise, go to step 1.

More procedural detail can be found in the program listing in Appendix II.

The initial step size was taken as \( \tau = .01 \) maximum \((W,Y)\). \( \varepsilon \) was taken to be \( 10^{-7} \). The average computer time per problem was 6 seconds on the Burroughs B5500 computer.

The above procedure will converge to a local optimum. This local optimum is a global optimum if the region is convex and the objective function is strictly convex (18). Since neither of these conditions is satisfied, confidence that a global optimum has been found can only be produced by starting the procedure with a different initial design. This was done for two problems and identical results were obtained.

The method just described was applied to a wide range of problems for which \( R = 0 \) (fully supported laterally) or \( R = 1 \) (laterally braced only at the ends), \( M_0/V_0 = .25 \) (simply supported and uniformly loaded) or \( V_0 = 0 \) (end moments only).
Among the standard loading cases, these represent extremes since \( \frac{M_0}{V_0} \) is smallest for simply supported, uniformly loaded beams and largest for end moments, and the \( R \) values represent extreme lateral support conditions. Only members in which \( R_y = 1 \) and \( \gamma = 28.3823 \) (corresponding to steel with a yield stress of 36,000 psi) were investigated.

B. Optimum Simply Supported, Uniformly Loaded Beam-Columns

The beam-columns studied in this section are simply supported, uniformly loaded members. Curves of constant optimal area, \( Z \), for these problems are presented in Figs. 16 and 17 for a wide range of moments and axial loads. Fig. 16 represents a laterally supported beam-column (\( R = 0 \)) and Fig. 17 represents a beam-column supported laterally only at the ends (\( R = 1 \)). A comparison of the two sets of curves is made in Fig. 18. For the same set of loads, more area is required for \( R = 1 \) than for \( R = 0 \) as would be expected.

The increase in optimum \( Z \) as axial compression is added to pure beams is shown in Figs. 19 and 20 for \( R = 0 \) and \( R = 1 \). The curve for \( X = 0 \), the lowermost curve in Fig. 19, represents the optimum area variation with moment for a laterally supported beam. This curve checks identically with the analytical solution given by Holt and Heithecker (5) as do succeeding curves which describe this case. As axial load is added, more area is naturally required. As the moment becomes very small, the curves approach the optimum values for columns. The corresponding set of curves for laterally un-
supported beam-columns is given in Fig. 20.

Curves of optimum Z for columns and columns with small lateral loads are shown in Figs. 21 and 22 for $R = 0$ and $R = 1$. The two curves for these cases which correspond to zero moment are the same curves shown in Fig. 7 for pure columns.

Fig. 23 shows how the depth of an optimum laterally supported beam-column changes with moment. The curve corresponding to $X = 0$ changes slope when changes are made in the controlling set of constraint equations. For low moments, the web is very slender, so that near $M_o = .01$ the maximum allowable web height/thickness ratio is active (Eq. (34)). As the moment increases, elastic buckling of the web controls the web design, Eq. (50), and the flange stress reduction, Eq. (48), is active. There is a short transitional region near $M_o = .5$ and Eq. (48) becomes inactive as $M_o$ increases. Elastic web buckling continues to be active until another transitional region is reached near $M_o = 50$. Inelastic web buckling then becomes active (Eq. (52)), until for very high moments a final slope corresponds to yielding in the web, Eq. (53)

The higher curves of Fig. 23 correspond to beams with axial load. They show that a substantial amount of axial load can be added to a beam carrying substantial moment without appreciably changing the optimum depth. For small moments, however, the optimum depth eventually departs from
the optimum beam depth and seeks the optimum depth of the column.

A similar description of a laterally unsupported beam-column is seen in Fig. 24. Here the lateral-torsional buckling equations change as well as the equations for the shearing stress. The drastic decrease in the optimum depth of very lightly loaded beams is due to a change in the active lateral buckling equations from Eq. (45) to Eq. (46). For \( M_o \) greater than about .8, Eq. (45) is always active. The changes in slope that occur for higher moments correspond to the same changes in active web design equations as were seen in the laterally supported beam. Because the optimum depth drops so sharply near \( M_o = .8 \), the web again buckles inelastically below this value. It is interesting to note that the reduction in flange stress is never active for the shallower optimum sections which result when \( R = 1 \).

Figs. 25 and 26 give information about the depth from another point of view. They show that for columns to which have been added only light transverse loads, the depth tends to seek the optimum beam depth rather than the optimum column depth. This trend is more pronounced in the laterally supported members. All of the curves which correspond to members with combined loadings remain reasonably constant over much of the range of \( X \).

The optimum web depth/thickness ratios are shown in Figs. 27-30. The lowermost curves of Figs. 27 and 28 give
the optimum Y/U ratio for laterally supported and unsupported beams. The curves for beams with light axial loads are very close to those for no axial load. The curves for the substantial axial load, \( X = 10 \), show that the optimum Y/U ratio is not sensitive to the addition of large axial loads to beams. This is not unreasonable since moment and axial load interact directly through Eq. (57) and (58), while shear, which almost always determines the proportions of the web, does not interact with axial load directly.

Figs. 29 and 30 show that the optimum Y/U ratio increases rapidly with moment and then decreases as the moment and shear increase. The large increases in Y/U for low moments are possible since columns with slender webs are only slightly heavier than optimum columns.

Figs. 31 and 32 show how the optimum area is distributed between the web and the flange for various beam-columns. The influence of axial load on these curves is strong. This is expected since the size of the flange is greatly influenced by the magnitude of both \( M_0 \) and \( X \).

C. Optimum Columns with End Moments

The optimum design of columns to which have been added end moments but no shear was investigated. Figs. 33 and 34 show how the optimum area curves for these problems compare with the corresponding curves of Fig. 18. For a given set of loading conditions, less area is required since the beam-
column needs to carry no shear. For very light moments, the optimum design approaches the corresponding optimum design for simply supported uniformly loaded members, for which the shear is too low to restrict the web design.

The optimum depth variation is shown in Figs. 35 and 36. Again for columns with low moments the optimum depth quickly changes from the optimum column depth and seeks the optimum depth of the beam. The general appearance of this set of curves is remarkably similar to the corresponding curves simply supported, uniformly loaded beams, Figs. 25 and 26.

For columns with end moments, the web is always designed at its maximum allowable Y/U ratio, which for A36 steel is about 9.13.

From the path taken by the computer during computation, a general result may be obtained. It may be observed that beam-columns, whether optimum or not, are not highly sensitive to changes in depth. Frequently, moderate changes in depth cause only small changes in the cross-sectional area if the other variables are well-chosen.
Fig. 16. Optimum Area for Simply Supported, Uniformly Loaded Beam-Columns, $R = 0$
Fig. 17. Optimum Area for Simply Supported, Uniformly Loaded Beam-Columns, R = 1
Fig. 18. Comparison of Optimum Area for Simply Supported, Uniformly Loaded Beam-Columns, R 0 and R 1.
Fig. 19. Optimum Area for Simply Supported, Uniformly Loaded Beams and Beam-Columns, R = 0
Fig. 20. Optimum Area for Simply Supported, Uniformly Loaded Beams and Beam-Columns, R 1
Fig. 21. Optimum Area for Simply Supported, Uniformly Loaded Columns and Beam-Columns, $R_0$
Fig. 22. Optimum Area for Simply Supported, Uniformly Loaded Columns and Beam-Columns, R = 1
Fig. 23. Optimum Web Height for Simply Supported, Uniformly Loaded Beams and Beam-Columns, R = 0
Fig. 24. Optimum Web Height for Simply Supported, Uniformly Loaded Beams and Beam-Columns, $R = 1$
Fig. 25. Optimum Web Height for Simply Supported, Uniformly Loaded Columns and Beam-Columns, R 0
Fig. 26. Optimum Web Height for Simply Supported, Uniformly Loaded Columns and Beam Columns, $x/1$ and Beam Columns, $x/1$. 

Dimensionless Axial Load, $X$ 

Dimensionless Web Height, $Y$ 

$M_o = 0.5$ 

$M_o = 1$ 

$M_o = 2$
Fig. 27. Optimum Web Depth/Thickness for Beams and Beam-Columns, $R = 0$
Fig. 28. Optimum Web Depth/Thickness for Simply Supported, Uniformly Loaded Beams and Beam-Columns, R 1
Fig. 29. Optimum Web Depth/Thickness for Simply Supported, Uniformly Loaded Columns and Beam-Columns, $R = 0$
Fig. 30. Optimum Web Depth/Thickness for Simply Supported, Uniformly Loaded Columns and Beam-Columns, R 1
Fig. 32. Optimum Ratio of the Area of the Flange to the Area of the Web, R = 1
Fig. 33. Comparison of Optimum Areas for Beam-Columns with End Moments to Simply Supported, Uniformly Loaded Beam-Columns, $R = 1$
Fig. 34. Comparison of Optimum Areas for Beam-Columns with End Moments to Simply Supported, Uniformly Loaded Beam-Columns, R = 0
Fig. 35. Optimum Web Height for Columns and Beam-Columns with End Moments, R = 0
Fig. 36. Optimum Web Height for Columns and Beam-Columns with End Moments, $R' = 1$
VI Cost Optimization by Dynamic Programming

Frequently, the total cost of a structure is the most important design criterion, weight being important only as it affects the cost. A procedure is described herein which seeks to minimize the cost of non-prismatic beam-columns for which the design variables may only take on discrete values listed in a table. The procedure is similar to the one used by Heithecker (4) whose study of plate girder design was used as a basis, but his procedure has been extensively revised and generalized.

A. Procedure

The system considered is shown in Fig. 37. The beam-column is divided into a number of segments each of which has linearly tapered heights. The optimum set of heights is found by a grid search on the ends of the segments. The member is also divided into a number of elements, the number being variable but large enough to sufficiently approximate the continuous system. The web thickness and the flange width and thickness may change abruptly at element boundaries, but they are constant within any element.

Heithecker's basic algorithm was modified to include the effect of axial load. The critical load analysis for the beam-column is done in exactly the same way as described in
Fig. 37. Beam-Column Model
Chapter IV, but due to the form of the beam-column interaction equations, Eq. (57) and (58), three critical loads must be calculated. The smaller of the two elastic-inelastic critical loads, one taken in the plane of bending, the other taken perpendicular to the plane of bending is the basic quantity needed. But, for cases in which the actual load is greater than 15% of the allowable load, the fully elastic buckling load in the plane of bending is also needed. With axial load information available, the beam-column is designed as a beam in which the allowable bending stresses are reduced through the interaction formulas.

The beam analysis is done using the transfer matrices given in Appendix I. The analysis is theoretically exact for constant depth members and for the standard loading cases shown in Fig. 37. The design shears and moments are computed at element boundaries instead of at the midpoints of the elements.

The cost function and side constraints are identical to those used by Heithecker. The input costs are the material cost and six fabrication costs—the cost per segment, cost per flange width transition, cost per flange thickness transition, cost per web thickness transition, welding cost, and segment splicing cost. The side constraints are simple constraints which are minimum and maximum values for the design variables, coupled constraints which are limits on ratios of design variables, and fixed constraints which, for example, assure welding compatibility. The coupled and simple con-
straints may be violated in that order if they must be violated in order to find a feasible design.

The optimization procedure is carried out on several levels and is described schematically by Fig. 38. Dynamic programming is used in three ways: in the segment geometry procedure, and in the web smoothing and flange smoothing procedures, all of which fall within level C of Fig. 38. Discrete dynamic programming as described in Bellman and Dreyfus (1) is used. For the case of web smoothing the heights are known and the minimum feasible web thickness for each element has been determined. If there are n stages or elements and m states or possible thicknesses, then the optimal policy function is a matrix (mxn) which gives for each feasible web thickness and each element, the number of the cheapest web thickness at the element just to the left of that element. The optimal value function is a vector of order (m) which gives for each thickness the optimum cost associated with that thickness plus the optimal set of thicknesses of all the previous elements. The segment optimization and flange smoothing procedures are carried out in a similar fashion.

For beam-columns whose allowable column load is insensitive to local changes in area, this procedure converges rapidly to a fully-stressed design. Typical times on the Burroughs B5500 computer for beams with 24 elements and 20 available plate thicknesses are one minute for constant depth members and five minutes for 2 segment variable depth
Fig. 38. Computational Algorithm
members. These times are also approximately the times required per iteration in level A for beam-columns. The number of iterations required at level A can be very small if the initial estimate of the critical load is approximately correct. Hence some effort in making the first estimate is justified. The first estimate was taken to be the buckling load of a prismatic beam-column which will just carry the maximum moment in the span plus the axial load and which has the depth of an optimum non-compact laterally supported beam. This corresponds to the starting design that was used for the optimum prismatic beam-column study and is a good initial estimate.

Convergence on the axial load cycle (level A) can be a computational problem. If the loads and span are such that the column buckles in a highly inelastic way, then the member tends to be nearly prismatic. But the lateral loadings usually force the beam-column to be quite non-prismatic. When the buckling load is re-evaluated, it may be greatly different from that assumed. Improvement in convergence in such instances requires knowledge about how the variation in the dimensions at a point along the length affect the critical load. This knowledge is not readily available.

Dynamic programming can be applied to determinate beams because the optimal decisions can be made in stages. For beam-columns, however, the sequential nature of the problem has been destroyed. Changes in the cross-section at any point affect the critical buckling load and therefore the
optimal decisions at every other point. Because of this difficulty the procedure is presented here as a method for obtaining a minimum cost design corresponding to a fully-stressed condition for problems with combined loadings.

Schmit (14) and Razani (11) have shown that for indeterminate problems and problems with buckling constraints, the minimum weight design may not be fully-stressed. It is clear that for zero fabrication costs, the beam-column designed by this procedure will be a fully-stressed design. It is at present not clear whether or not it is an optimum design. Results of the prismatic beam-column study indicate that the web design is usually controlled by shear. If this is true at every point along the length, then the minimum weight design may be found among the fully-stressed designs. In these cases it is felt that the beam-columns designed by this procedure are optimal or nearly optimal.

B. Sample Problem

A sample problem is presented here. Extensive results have not been obtained, because of the large amount of computer time required.
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<td>0.142</td>
<td>10.000</td>
<td>17.499</td>
<td>0.000</td>
<td>120.000</td>
</tr>
</tbody>
</table>

---

MAX DEFLECTION: 2.04 IN  COLUMN BUCKLING LOAD: 105.5 KIPS  NO. OF COLUMNS: 900.1 KIPS

SMOOTHINGS: WEIGHT (LB) COST

198: 1024.2 112,422

PROCESSOR TIME = 6.6608 MINUTES
VII Summary and Conclusions

In this study, the techniques of optimization are applied to the optimum design of beam-columns. The general non-prismatic beam-column problem is formulated, and it is shown that the minimum weight beam-column must be fully-stressed with respect to combined axial load and moment at each point along the length.

A general analytical solution to minimum weight prismatic column design is presented. This solution shows that for columns with normal webs, similar bracing and end conditions result in optimum cross-sectional shapes which are geometrically similar. These shapes are such that both web and flange are as thin as possible consistent with the constraints. Columns with slender webs were found to be optimal for certain loads and lateral bracing conditions, but were only slightly heavier than optimum in other cases.

A method for evaluating the critical load of non-prismatic columns is given and applied to the optimum design of such members. Results of sample problems show that a 19% increase in allowable load is possible with a linear taper and 24% increase is possible with a parabolic taper.

An extensive set of prismatic beam-columns is optimized by the gradient projection method of non-linear programming.
The results show that the optimum depth increases with increase in both moment and axial load. The depth tends to seek the optimum beam depth rather than the optimum column depth. The optimum web depth/thickness ratio is usually determined by shear or a specified maximum allowable depth/thickness ratio even when a substantial axial load is present. The minimum area is not very sensitive to changes in web depth.

Finally, a method is presented for the minimum cost design of non-prismatic beam-columns using dynamic programming. The method is somewhat time-consuming and cannot guarantee an optimum when axial load is present. Despite these drawbacks, the method converges to a design which appears to be nearly optimal.

This study has resulted in a description of the optimum arrangement of materials for a large class of beam-columns. This information, while useful in itself, will be particularly helpful in further optimization studies of larger, more complex structures.
References


Appendix I

A transfer matrix (10) for a beam-column gives the four state variables, deflection, slope, moment, and shear, at the right end of an element of the beam-column in terms of the same variables at the left end. If \( S(z) \) is a vector of state variables and \( z \) the distance along the length, then the solution of the governing differential equation can be written as

\[
S(z) = B(z) \bar{c}
\]

where \( B(z) \) is a solution matrix which is 4x4 for columns and 5x5 for beam-columns and \( \bar{c} \) is a vector of constants of integration. Then,

\[
S(o) = B(o) \bar{c}
\]

or

\[
\bar{c} = B^{-1}(o) S(o)
\]

The transfer matrix is found by

\[
S(l) = \frac{B(l)B^{-1}(o)S(o)}{\text{transfer matrix}}
\]

where \( l \) is the length of the element.

Columns

For an elastic prismatic column, the transfer matrix is (10)

\[
\begin{bmatrix}
1 & \frac{l \sin \gamma}{\bar{y}} & \frac{l^2 (\cos \gamma - 1)}{EI \bar{y}} & \frac{l^3 (\sin \gamma - \gamma)}{EI \bar{y}^2} \\
0 & \cos \gamma & -\frac{l \sin \gamma}{EI \bar{y}} & \frac{l^2 (\cos \gamma - 1)}{EI \bar{y}^2} \\
0 & -\frac{p \sin \gamma}{\bar{y}} & \cos \gamma & \frac{l \sin \gamma}{\bar{y}} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

where \( \bar{y}^2 = \frac{p l^3}{EI} \).
For an inelastic linearly tapered column element, the governing differential equation is
\[(E_t I y'')'' + P y'' = 0)\]

where
\[E_t = \frac{4EP}{A(x) F_y} \left(1 - \frac{P}{A(x) F_y}\right)\]
\[I = \frac{b t_b h}{2} + \frac{t_w h^3}{12}\]

The equation can be rewritten as
\[\psi y'' + \phi y = \frac{\phi}{P} (q x + b)\]

where \(a_i\) and \(b_i\) are constants of integration, and
\[\psi = 4E (F_y t_w h + 2 b t_b) - P \left(\frac{1}{12} t_w h^3 + \frac{t_b h^2}{2}\right)\]
\[\phi = F_y (t_w h + 2 b t_b)^2\]

This equation can be solved in series form by repeated differentiation of the differential equation, as follows:
\[y'' = \frac{1}{\psi} (\varphi - \phi y) \quad \text{where} \quad \varphi = \frac{\phi}{P} (a_i z + b_i)\]
\[y''' = \frac{1}{\psi} (\varphi') - \frac{1}{\varphi} (\phi y' + \phi' y + \psi y'')\]
\[y'' = \frac{1}{\psi} (\varphi'') - \frac{1}{\psi} (\phi y'' + 2 \phi' y' + \phi'' y + \psi y''' + 2 \psi y''')\]
\[\vdots\]
\[y^{(n)} = -\frac{1}{\psi} \left[ \sum_{k=1}^{n} \binom{n-2}{k} \psi^{(k)} y^{(n-k)} + \sum_{k=0}^{2} \binom{n-2}{k} \phi^{(k)} y^{(n-k-2)} \right]\]

The displacement and slope at the right end of the element in terms of derivatives of the displacement at the left end are
\[y_r = y_o S_2 + y_o' S_{12} + a_S s_3 + b_S s_4\]
\[y_r' = y_o S_{21} + y_o' S_{22} + a_S s_{23} + b_S s_{24}\]

where
\[ S_{11} = 1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} A_n \quad S_{21} = \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} A_n \]

\[ S_{12} = \sum_{n=1}^{\infty} \frac{\rho^n}{n!} B_n \quad S_{22} = \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} B_n \]

\[ S_{13} = \sum_{n=1}^{\infty} \frac{\rho^n}{n!} C_n \quad S_{23} = \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} C_n \]

\[ S_{14} = \sum_{n=1}^{\infty} \frac{\rho^n}{n!} D_n \quad S_{24} = \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} D_n \]

and the \( A_n, B_n, C_n, \) and \( D_n \) are themselves series; e.g.,

\[ A_1 = 0 \]

\[ A_2 = -\frac{\phi'(\alpha)}{\psi'(\alpha)} \]

\[ A_3 = -\frac{1}{\psi'(\alpha)} \left[ \phi'(\alpha) + \psi'(\alpha) A_2 \right] \]

\[ A_4 = -\frac{1}{\psi'(\alpha)} \left[ \phi(\alpha) A_e + \phi''(\alpha) + \psi''(\alpha) A_e + 2 \psi'(\alpha) A_3 \right] \]

\[ \vdots \]

\[ A_n = -\frac{1}{\psi'(\alpha)} \left[ \sum_{k=1}^{n-1} \binom{n-2}{k-1} \psi^{(k)}(\alpha) A_{n-k} + \sum_{k=1}^{n-1} \frac{3(n-2)}{k-1} \phi^{(k-1)}(\alpha) A_{n-k-1} \right] \]

The transfer matrix is found by

\[ S(z) = B(z) \bar{c} \quad ; \quad S(\ell) = B(\ell) B(\alpha)^{-1} \bar{c} \]

where

\[ \bar{c} = \begin{bmatrix} y(\alpha) \\ y'(\alpha) \\ a_i \\ b_i \end{bmatrix} \]

The more exact transfer matrix is

\[
\begin{bmatrix}
S_{11} + PS_{14} & S_{12} & -S_{14} & -S_{13} \\
S_{21} + PS_{24} & S_{22} & -S_{24} & -S_{23} \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Beams

For beams, transfer matrix derivation is similar, but requires an extra column due to the non-homogeneous differential equation. The results for prismatic and linearly tapered I-shaped beams are summarized below.

\[
\begin{bmatrix}
y \\
y' \\
M \\
V \\
1
\end{bmatrix} = \begin{bmatrix}
1 & l & F_1 & F_2 & R_1 \\
0 & 1 & F_3 & F_4 & R_2 \\
0 & 0 & 1 & R_3 \\
0 & 0 & 0 & 1 & R_4 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
y \\
y' \\
M \\
V \\
1
\end{bmatrix}
\]

\text{Case 1}

\text{Case 2}

\text{Case 3}

The symbol A here denotes the area of the flange, $bt_f$.

The web height by $h+\alpha z$ where $\alpha$ is the slope of the taper.

$L$ is the element length. $B = \left(1+\frac{th}{12L}\right)/(1+\frac{\alpha h}{12L})$ and $B_{\alpha} = \left(1+\frac{th}{12L}\right)/(1+\frac{\alpha h}{12L})$

**Prismatic**

\[F_1 = -\frac{g^2}{2EI} \]

\[F_2 = -\frac{g^3}{6EI} \]

\[F_3 = -\frac{g}{E} \]

\[F_4 = -\frac{g^2}{2EI} \]

\text{Case 1. Concentrated Load}

**Prismatic**

\[R_1 = \frac{P(l-a)^3}{6EI} \]

\[R_2 = \frac{P(l-a)^2}{2EI} \]

\[R_3 = -P(l-a) \]

\[R_4 = -P \]

**Tapered**

\[R_1 = \frac{2P}{EA} \left\{ \left( \frac{LA}{th} + 1 + \alpha \frac{L}{h} \right) \left\{ \frac{-th}{EA} \left( a + \frac{h}{\alpha} \right) \log [B] - \frac{h}{\alpha} \log \left( 1 + \alpha \frac{th}{12L} \right) \right\} 
+ \frac{h}{\alpha} \left( 2 + \alpha \frac{L}{h} \right) \log \left( 1 + \alpha \frac{h}{12L} \right) \right\} \]

\[R_2 = \frac{2P}{EA} \left\{ \left( \frac{LA}{th} + 1 + \alpha \frac{L}{h} \right) \left\{ \frac{-th}{EA} \left( a + \frac{h}{\alpha} \right) \log [B] - \frac{h}{\alpha} \log \left( 1 + \alpha \frac{th}{12L} \right) \right\} 
+ \frac{h}{\alpha} \left( 2 + \alpha \frac{L}{h} \right) \log \left( 1 + \alpha \frac{h}{12L} \right) \right\} \]

\[R_3 = \frac{2P}{EA} \left\{ \left( \frac{L-a}{h} \right) \left[ \left( \frac{th}{EA} + \frac{h}{\alpha} \right) \log [B] + \frac{1}{\alpha} \right] \right\} \]

\[R_4 = \frac{2Pa}{EA} \left\{ \left( \frac{L-a}{h} \right) \left[ \left( \frac{th}{EA} + \frac{h}{\alpha} \right) \log [B] + \frac{1}{\alpha} \right] \right\} \]

**Tapered**

\[R_1 = \frac{2P}{EA} \left\{ \left( \frac{LA}{th} + 1 + \alpha \frac{L}{h} \right) \left\{ \frac{-th}{EA} \left( a + \frac{h}{\alpha} \right) \log [B] - \frac{h}{\alpha} \log \left( 1 + \alpha \frac{th}{12L} \right) \right\} 
+ \frac{h}{\alpha} \left( 2 + \alpha \frac{L}{h} \right) \log \left( 1 + \alpha \frac{h}{12L} \right) \right\} \]

\[R_2 = \frac{2P}{EA} \left\{ \left( \frac{L-a}{h} \right) \left[ \left( \frac{th}{EA} + \frac{h}{\alpha} \right) \log [B] + \frac{1}{\alpha} \right] \right\} \]

\[R_3 = \frac{2P}{EA} \left\{ \left( \frac{L-a}{h} \right) \left[ \left( \frac{th}{EA} + \frac{h}{\alpha} \right) \log [B] + \frac{1}{\alpha} \right] \right\} \]

\[R_4 = \frac{2Pa}{EA} \left\{ \left( \frac{L-a}{h} \right) \left[ \left( \frac{th}{EA} + \frac{h}{\alpha} \right) \log [B] + \frac{1}{\alpha} \right] \right\} \]
Case 2. Trapezoidal Load

Prismatic

\[ R_1 = \frac{(4q_1 + q_2)l^4}{120EI} \]
\[ R_2 = \frac{(3q_1 + 2q_2)l^3}{24EI} \]
\[ R_3 = -\frac{(2q_1 + q_2)l^2}{6} \]
\[ R_4 = -\frac{(q_1 + q_2)l}{2} \]

Case 3. Couple

Prismatic

\[ R_1 = -C \frac{(l-a)^2}{2EI} \]
\[ R_2 = -C \frac{(l-a)}{EI} \]
\[ R_3 = -C \]
\[ R_4 = 0 \]
Appendix II

Program Listings
Gradient Projection Method
YINC = 2 * VINC(1)
UINC = 2 * VINC(2)
RING = 2 * VINC(3)

UG(1,1) = ( PHI(1) = PHI(2) ) / YINC
UG(2,1) = ( PHI(3) = PHI(4) ) / UINC
UG(3,1) = ( PHI(5) = PHI(6) ) / UINC
UG(1,2) = ( PHI(1) = PHI(2) ) / YINC
UG(2,2) = ( PHI(3) = PHI(4) ) / UINC
UG(3,2) = ( PHI(5) = PHI(6) ) / UINC
G(1) = UB
G(2) = YU
G(3) = 2 * SIN * G7/3.
DO 10 I = 1, 2
DO 10 J = 1, 3
10 U(1, J) = UG( J, I )
DO 11 J = 1, 2
DO 11 I = 1, 3
11 UG( I, J ) = UG( I, J ) + UT( I, K ) * UG( K, J )
C
INVERT UG
D = UG(I, I) * UG(2, 2)
A1 = UG(1, 1) / D
UG(1, 1) = UG(2, 2) / D
UG(1, 2) = -UG(I, 2) / A1
UG(2, 1) = -UG(1, 2) / A1
UG(2, 2) = A1
DO 12 J = 1, 2
DO 12 I = 1, 3
UM(I+J) = 0
DO 12 K = 1, 2
UM(I+K) = UG(I+J) * UG(K+J)
C
CALCULATE PROJECTION MATRIX
DO 13 I = 1, 3
DO 13 J = 1, 3
IF ( I .LT. J ) 14, 15, 14
14 PG( I, J ) = 0,
DO 16 K = 1, 2
DO 15 J = 1, 2
PG( I, J ) = PG( I, J ) + UM(I+K) * UG(K+J)
16 DO 16 K = 1, 2
PG( I, J ) = PG( I, J )
C
CONTINUE
DO 41 I = 1, 3
RTEM(1) = 0,
DO 41 J = 1, 3
RTEM(J) = RTEM(1) + HT(I, J) * G(J)
DO 41 J = 1, 2
RM(I) = J,
DO 41 J = 1, 2
RM(J) = RM(I) + UG(I, J) * RTEM(J)
B1 = UM(I) / SQRT(2 * UG(1, 1))
B2 = RM(2) / SQRT(2 * UG(2, 2))
B1 = MAX(1, B2)
DO 40 I = 1, 3
ZS(I) = 0,
DO 40 J = 1, 3
ZS(I) = ZS(I) + Pu(I, J) * G(J)
D = SQRT(ZS(1) * ZS(1) + ZS(2) * ZS(2) + ZS(3) * ZS(3))
HETA = 4 * PI / D
IF ( ETA .GE. 49 ) 41, 47, 43
47 HETA = EPS / 22
43 IF ( D .GE. H ) 41, 49, 47
41 ETA = 0.001
49 ETA = ETA + 0.001
WRITE( 6, 2007 ) BETX, BI, D  
GO TO 100  
R 0185  
GO TO 100  
R 0188  
ZT(I) = ZT(I)  
W 0408  
18 K(I) = X(I) + TAU*Z(I)  
R 0408  
C THEN CORRECT BACK TO THE FEASIBLE REGION  
R 0411  
NIT = 0  
R 0412  
38 IND = 0  
R 0414  
CALL MCRHK( VC, X(1), X(2), YUNXX, WZ2T, IND )  
R 0414  
YX = X(1)  
R 0419  
UC = X(2)  
R 0420  
AC = X(3)  
R 0421  
CALL PHISM( XC, XG, YC, UC, R, PHA, PHA, PRE )  
R 0422  
CALL BEND(KME, X(1), X(2), X(3), R, RY, PHA, PHA, PRE, GAM,  
1 X(1), IND )  
R 0429  
IF( IND > 25, 27, 26  
R 0431  
nMS = SQRT( X(1)*X(1) + X(2)*X(2) )  
R 0434  
IF( nMS = DEL ) 29, 29, 20  
R 0437  
29 Z = SIN(X(3))*X(3)/3, *X(2)*X(1)  
R 0441  
IF( 7 > 28 ) 30, 30, 31  
R 0445  
30 IF( Z > 23 + EPS ) 32, 33, 33  
R 0448  
32 ZB = Z  
R 0452  
YB = X(1)  
R 0452  
UB = X(2)  
R 0453  
40 WRITE( 6, 2001 ) YB, UB, YB, UB, 28  
R 0455  
GO TO 34  
R 0471  
31 IF( ABS(Z-ZB) = EPS ) 33, 33, 34  
R 0472  
34 TAU = TAU/4.  
R 0476  
C PUNCH MESSAGE OPTIMUM IS BRACKETED  
WRITE( 6, 2002 ) X(1), X(2), X(3), Z  
R 0477  
GO TO 36  
R 0494  
C ITERATE BACK TO FEASIBLE REGION  
R 0496  
28 NIT = NIT + 1  
R 0497  
IF( NIT = 9 ) 55, 55, 55  
R 0498  
52 NIT = NIT/2  
R 0511  
PRINT NIT  
R 0502  
50 N1 = N1+1  
R 0513  
51 IF( N1 = 2, *41IF=1(1)  
R 0519  
41 IF( N1 = 2, *41IF=1(1)  
R 0519  
55 NQ = 1 + 1, 3  
R 0516  
57 NQ = 1 + 1, 3  
R 0521  
59 NQ = 1 + 1, 3  
R 0527  
19 CONTINUE  
R 0534  
WRITE( 6, 2003 ) X(1), X(2), X(3)  
R 0535  
C PRINT MESSAGE  
R 0545  
GO TO 36  
R 0551  
31 WRITE( 6, 2013 ) MMM  
R 0552  
WRITE( 6, 2013 ) YBB, XBB, XBB  
R 0561  
WRITE( 6, 2013 ) RY, RY, RFY  
R 0574  
WRITE( 6, 2013 ) YM, YM, YM, YM  
R 0587  
ARAI = SINH*B*MATH/6,  
R 0602  
ARAI = ARAT/C24,  
R 0604  
YM = YM/HR  
R 0606  
YO = YM/VR  
R 0607  
WRITE( 6, 2027 ) YM, YMM, YMM, YMM  
R 0607  
TF = TIME(T)/40,  
R 0623  
WRITE( 6, 2034 ) TF  
R 0623  
WRITE( 6, 2039 )  
R 0634  
109 CONTINUE
SURROUNGE WHCK( V, Y, U, YUMAX, A, IND )
C REVIRED
C SPECIAL VERSION FOR GRADIENT PROJECTION
IND = 0
CV = 45.324125/29.444
IF( CV - A ) 1, 2
1 PVALL = CV/2.89
GO TO 10
2 CV = 0.5*SQRT(0.34/29.444)/V
PVALL = AMIN( CV/2.49, 0.4 )
10 IF( Y/V = YUMAX ) 11, 12
12 IND = 1
13 AA = (Y/V) = YUMAX
AA = A*(1.0/AA) - PVALL
A = AMIN( -AA, -A )
RETURN
11 IF( Y/V = PVALL ) 1, 13, 12
END

SEGMENT 1 IS 685 LONG

START OF SEGMENT ********** 2
R 0000
R 0050
R 0000
R 0000
R 0056
R 0012
R 0014
R 0016
R 0091
R 0077
R 0031
R 0032
R 0033
R 0036
R 0038
R 0011
R 0064
R 0004
SEGMENT 2 IS 96 LONG
SUBROUTINE DEPTH(XH, YH, YB)
IF (XH < 1.3)
  RETURN
YB = 2.71833 XH**P
RETURN
16  (XH XH/(YH**1.5))
17  (XH XH = 1.16988 1+ 1.2)
1 YH = 2.33976
10  TO 20
2  (XH XH = 2.44372 ) 3+ 3.4
3  YH = 2.16392/SQR(XH)
10  TO 20
4  (XH XH = 3.25899 ) 5+ 5.6
5  YH = 1.350946
10  TO 20
6  (XH XH = 24.2816 ) 7+ 7.8
7  P = 1.7.
8  YB = 2.03827 XH**P
10  TO 20
8  (XH XH = 29.2246 ) 9+ 9.10
9  YH = 8.09156
10  TO 20
10  (XH XH = 192.459 ) 11+ 11.12
11  P = .39097
12  YH = 2.30709 XH**P
10  TO 20
13  (XH XH = 211.709 ) 13+ 13.14
14  YH = 1.73599
10  TO 20
15  P = 1./3.
16  YB = 2.71833 XH**P
RETURN
END

START OF SEGMENT *********** 3
R 0000
R 0003
R 0004
R 0008
R 0012
R 0014
R 0022
R 0024
R 0026
R 0031
R 0034
R 0036
R 0041
R 0043
R 0045
R 0050
R 0051
R 0055
R 0056
R 0061
R 0063
R 0065
R 0070
R 0072
R 0077
R 0079
R 0083
R 0085
R 0087
R 0088
R 0093
R 0094
R 0097
C
DESIGN FULLY STRESSED MEMBER WITH WEB DIMENSIONS EQUAL
C
TO THOSE OF A LATERALLY-SUPPORTED BEAM
C
START OF SEGMENT
************

CALL DEPTH(XM, V, XG, GAM, YUMAX, R, RYS, DEL, Y, U, W)
CALL WIDTH(XM, V, YMAX, GAM, W)
WRITE( 6, 1000 ) XM, V, XG, GAM, YUMAX, R
WRITE( 6, 1001 ) RYS, DEL, Y, U

IND = 0
WH = 1.25*U*Y
BSQ = W,WH/50.41(29.)
W = SURF( BSQ)

5 WINC = W+.05
CALL PHISM( XG, Y, U, W, PHA, PHA, PRE)
IF( PHA = 1., ) 7, 3
7 IF( PHA = 1., ) 10, 13

8 IND = 0
CALL RENDXY(XG, Y, U, W, RYS, PHA, PRE, GAM, RAT, IND)
WRITE( 6, 1002 ) Y, U, W, PHA, PRE, GAM, RAT, IND
IF( IND = 0., ) 8, 3
3 H = H + WINC
GO TO 5

4 WINC = W+.05
HF = W
H = H - WINC

21 CALL PRISM( XG, Y, U, W, PHA, PRE)
17 IF( PRE = 1., ) 10, 13
18 IND = 0
CALL RENDXY(XG, Y, U, W, RYS, PHA, PRE, GAM, RAT, IND)
WRITE( 6, 1003 ) Y, U, W, PHA, PRE, GAM, RAT, IND
IF( IND = 0., ) 19, 13
19 HF = H
H = H - WINC

22 IF( WINC = .01+DEL ) 20, 20, 21
20 H = HF
RETURN

13 WNF = W
W = .5( W + WNF )
WINC = ( W - WNF )*.*5
GO TO 29

1000 FORMAT( 4H C1, 6E14.8 )
1001 FORMAT( 4H C2, 6E14.8 )
1002 FORMAT( 4H C3, 6E14.8 )
1003 FORMAT( 4H C4, 6E14.8, 12 )
END
SUBROUTINE _BTHKF ( V, YB, YMAX, GAM, UB )
  IF ( V > 0 ) RETURN
  1 A = SQRT ( V )
  1 IF ( YB > 2.3398/4/A ) 1, 1, 2
  1 UB = 2.5*A*A/YB
  1 GO TO 10
  2 IF ( YB > 3.3809/A ) 3, 3, 4
  3 UB = 1.0844/A
  10 GO TO 10
  4 ALF = YMAX + 1.5
  1 IF ( YB > .60067*ALF/A ) 5, 5, 6
  5 P = 1/3.;
  5 UB = .7*(1.90*(YB+P)*((A+2.5+P))
  5 GO TO 10
  6 UB = YB/YMAX
  10 RETURN
END

START OF SEGMENT *********** 5

SUBROUTINE BENDCK ( Y, U, W, R, RY, PHA, PRA, PRE, GAM, A, IND )
  R 0000
  C REVISED NUM*COMPACT BENDING*AXIAL LOAD CHECKER
  R 0000
  K3 = ***
  R 0000
  STN = SRT(24.)
  R 0001
  RA = W/RY
  R 0002
  RA5 = RA+RA
  R 0003
  XLR = 12.*RA5+1. + U*Y/(STN+KS)/WS
  R 0005
  1 IF ( XLR = 1600.)/GAM+GAM ) 5, 5, 6
  5 PHB = *.6
  5 GO TO 7
  6 PISW = 92.*P(R/S)
  R 0010
  C CONSERVATIVELY TAKE ( U = 1 )
  R 0010
  PHB = AMAX ( .A* ( 1.+25.*XLR/PISW ) + 2.*KS/(STN+KS) )
  R 0020
  PHB = AMAX ( PHB + 0 )
  R 0031
  7 TEM = 1./SQR(2.*PHA)
  R 0035
  A = Y/U - 4.4*PHB*TEM
  R 0036
  1 IF ( A ) 5, 5, 0
  9 PHA = PHA+1. + .003*Y/U/GAM+GAM/(STN+KS)
  R 0041
  8 PHA = 6.*XY/((PHB+YA)/(STN+KS + U*Y))
  R 0051
  1 IF ( PRA = .15 ) 10, 10, 11
  10 AA = PHB + PRA - 1.
  R 0061
  AB = PHA - C + PHB - 1.
  R 0062
  1 IF ( AA ) 5, 5, 16
  R 0066
  C COEFF. CHosen in TAKEN to AE L
  R 0067
  11 AA = PHA + PRE*1.
  R 0069
  AB = PHA/C + PRE - 1.
  R 0071
  11 IF ( AA ) 15, 15, 16
  R 0075
  15 IF ( AA ) 14, 14, 14
  R 0078
  16 IND = 1
  R 0081
  14 A = AMAX ( AA, -AR )
  R 0082
  RETURN
R 0084
END

SEGMENT 5 IS 57 LONG

START OF SEGMENT *********** 6

SUBROUTINE BENDCK ( Y, U, W, R, RY, PHA, PRA, PRE, GAM, A, IND )
  R 0000
  C REVISED NUM*COMPACT BENDING*AXIAL LOAD CHECKER
  R 0000
  K3 = ***
  R 0000
  STN = SRT(24.)
  R 0001
  RA = W/RY
  R 0002
  RA5 = RA+RA
  R 0003
  XLR = 12.*RA5+1. + U*Y/(STN+KS)/WS
  R 0005
  1 IF ( XLR = 1600.)/GAM+GAM ) 5, 5, 6
  5 PHB = *.6
  5 GO TO 7
  6 PISW = 92.*P(R/S)
  R 0010
  C CONSERVATIVELY TAKE ( U = 1 )
  R 0010
  PHB = AMAX ( .A* ( 1.+25.*XLR/PISW ) + 2.*KS/(STN+KS) )
  R 0020
  PHB = AMAX ( PHB + 0 )
  R 0031
  7 TEM = 1./SQR(2.*PHA)
  R 0035
  A = Y/U - 4.4*PHB*TEM
  R 0036
  1 IF ( A ) 5, 5, 0
  9 PHA = PHA+1. + .003*Y/U/GAM+GAM/(STN+KS)
  R 0041
  8 PHA = 6.*XY/((PHB+YA)/(STN+KS + U*Y))
  R 0051
  1 IF ( PRA = .15 ) 10, 10, 11
  10 AA = PHB + PRA - 1.
  R 0061
  AB = PHA/C + PHB - 1.
  R 0062
  1 IF ( AA ) 5, 5, 16
  R 0066
  C COEFF. CHosen in TAKEN to AE L
  R 0067
  11 AA = PHA + PRE*1.
  R 0069
  AB = PHA/C + PRE - 1.
  R 0071
  11 IF ( AA ) 15, 15, 16
  R 0075
  15 IF ( AA ) 14, 14, 14
  R 0078
  16 IND = 1
  R 0081
  14 A = AMAX ( AA, -AR )
  R 0082
  RETURN
R 0084
END

SEGMENT 6 IS 108 LONG
SUBROUTINE PRISM ( XC, Y, U, V, PHA, PRA, PRE )

CONSTANTS XKA = SQRT(200) * XK = SQRT(174*XXA/38427 )

XKA = 5.385165
XK = 62299812
Z = XKA*U/3. + Y*U
RX = XKA*U*V + (Y*U + XKA*U*W)/( 12.*XXA*XK + Z )
R = SQRT(RX)

IF ( R ) 7, 7, 6

7 RH = RX
GO TO 4

6 RY = X**XKA/( 6.*XX*XK*SQRT(Z) )
RH = AMIN1( RH, RY )
9 IF ( RH = 1. ) 2, 2, 1
1 RS = RH*RH
PHAL = 12.*RH*(2.*RS+1.)/(40.*RS*RH + 9.*RS = 3. )
XKA = X*K*RS/XKA
GO TO 3
2 XXB = XX*RH/XKA
PHAL = 169.*XXB*XKB/XKA
3 IF ( Y/U = 8./XKA ) 4, 4, 5
4 ZE = Z
GO TO 6
5 ZE = XKA*XKB/3. + N.*U/U/XKA
PRA = XC/( ZE*PHAL )
PHAL = 169.*ZK*B*XKB/XKA
PRA = XC/( ZE*PHAL )
PHA = XC/ZE
RETURN
END

SEGMEN 7 15 82 LUNG
Critical Load Calculation
SUBROUTINE PCRTI(WF, TF, NW, TM, INDBP, XBP, NEAX,
XLB, NAX, NBX, ELMDX, SY, PE, PCR, FS )
DIMENSION WF(40), TF(40), HW(40), TM(40), INDBP(40), XBP(40),
AR(40), AI(40), AF(40), ALF(40), C(3,2), Q(4,2), U(3,3), CC(4)
C
C
COMMON/AI, PA, PB, VA, VB, PI, VI, N1
C
CALLS PINIT, EFFL, SELEC
C
IN = 0
XNEV = NEAX
DX = XLB/XNEV
DD 100 IP = 1, 3
IPASS = IP = 1
IFC IPASS = 1 ) 101, 101, 102
101 DD I = 1, NEAX
ALF(1) = ( HW[I+1] - HW[I]) /DX
H = HW[I]
IFC(ALF(1)) 104, 105, 105
104 H = HW[I+1]
105 ALF(I) = WF(I)*TF(I)
AR(I) = 2.*AF(I) + TM(I)*H
61 AI(I) = .5*(H*AF(I) + TM(I)*H)*H/12.
C
FIND MAX, MOM. OF INERTIA
XIN = AI(I)
DD 170 I = 1, NEAX
IFC( AI(I) = XIN ) 170, 170, 171
171 IX = I
XIN = AI(I)
170 CONTINUE
CALL EFFL( NA, NB, XK )
XIN = XIN/NA*NB*XK
GO TO 108
102 IF( INDBP(I) ) 100, 100, 107
107 DD IX = I, NEAX
H = HW(I)
IFC ALF(I) ) 106, 106, 103
106 H = HW[I+1]
103 AI(I) = TF(I)*WF(I)**3, /12. + H*( TM(I)**3. )
99 AR(I) = 2.*AF(I) + TM(I)*H
108 IPS = IP = 2
CALL PINIT( AR, AI, NA, NB, XL, H, MAXX, ELMDD, SY, P, ARMIN, IPS, FS )
1
NC = 1
62 DD IX = I, NEAX
IFC(IPASS = 1 ) 124, 124, 109
109 IF( INDBP(I) ) 124, 124, 116
116 IN = INDBP(I)
81 I = 1
DX = XBP(CN) = BI+DX
IF( DXA ) 200, 200, 117
117 NA = HW[I] + ALF(I)*DXA
ARA = 2.*AF(I) + TM(I)*NA
IFC(IPASS = 1 ) 119, 119, 120
119 AI(A) = .5*(H*NA*AF(I) + TM(I)*NA)*H/12.
GO TO 121
120 AI(A) = TF(I)*WF(I)**3, /12. + H*(TM(I)**3. )
121 IF( ALF(I) ) 122, 114, 114
122 AR(I) = AHA
AI(A) = AIA
DO 144 J = 1, 2
144 Q(K+J) = C(K+J)
IF( IN ) 110, 110, 201
200 IF( I = 1 ) 201, 201, 201
202 DAX = DX
IN = 0
GO TO 111
201 DD N = 1, 4
10 CC(N) = ( Q(1,2)*Q(N+1) - Q(1,1)*Q(N+2) )/Q(1,2)
CC(1) = 0.
DO 12 N = 1, 4
Q(N,1) = CC(N)
12 D(N,2) = 0,
Q(N,2) = 1,
GO TO 118
110 CONTINUE
IF( NB = 1 ) 145, 146, 147
145 V = Q(1,1)*Q(2,1) - Q(1,2)*Q(2,1)
GO TO 148
146 V = Q(1,1)*Q(3,1) - Q(1,2)*Q(3,1)
WRITE( 6, 1000 ) IPASS, NC, P, V
GO TO 148
147 V = Q(3,1)*Q(4,1) - Q(3,2)*Q(4,1)
148 CALL SELECT P, V, NC, ARMIN, XIM, SY, IPASS
1000 FORMAT(1H, 2I4, 2E15.8)
IF( NC ) 60, 161, 60
60 NC = NC + 1
IF( NC = 25 ) 62, 63, 63
63 WRITE( 6, 64 ) IPASS
64 FORMAT(24HNONCONVERGENCE= P IP = 13)
161 IF( IPASS = 1 ) 149, 150, 151
149 P1 = P
WRITE( 6, 1000 ) IPASS, NA, PE, P
GO TO 100
150 P1 = P
WRITE( 6, 1000 ) IPASS, NA, PC, FS
GO TO 100
151 PPA = P
IF( PPA = PC ) 152, 100, 100
152 PPA = PC
100 CONTINUE
WRITE( 6, 1000 ) NA, NB, PE, PC
RETURN
END

SEGMENT 3 IS 568 LONG
SUBROUTINE PMIT(AR, AI, KA, KB, XLN, N, E, SY, P, ARMIN, IND, FS)

C IF IND = 0, EULER FORMULA APPLIES
C = 1, PLASTIC FORMULA APPLIES
C = (-) AS INPUT, EULER WILL APPLY, IND WILL BE RETURNED
C WITH A VALUE OF ZERO IN THIS CASE
DIMENSION AR(40), AI(40)
CALL EFLK(KA, KB, XK)

IND=0
IND=-1

C FIND THE MINIMUM AREA
ARMIN=AR(1)
IND=1
DD 72 I=1+N
IF(AR(I)=ARMIN)71,72,72
71
IND=0
ARMIN=AR(I)
72 CONTINUE

C FIND THE MAX. KLR
IB=1
RSQ=AW(I)/AR(I)
RSQ=0.
DD 62 I=1+N
RSQ=AW(I)/AR(I)
RSQ=RSQ+RSQ
IF(RSQ=RSQ)81,62+62
61 IB=I
RSQ=RSQ
62 CONTINUE
XNEM = N
RSQ = RSQ/XNEM

C FIND P USING AN AISC BEAM OF AREA = ARMIN
RSQ=AW(I)/ARMIN
XLRS = XX*XX*XLN*XLN/RSQ
CCSQ=1.9792088*E/SY
XLRS = SQRT(XLRS)
CCSQ=SQRT(CCSQ)
IF(IND) 76, 102, 102
74 P=1.9000000, 1.92*ARMIN/XLRS
GO TO 75
73 P = ARMIN*(I, = 0.5*XLRS/CSCQ)*SY
IND=1
75 P=P

C FIND P USING AN AISC BEAM OF AREA = ARMIN
RSQ=RSQ
XLRS = XX*XX*XLN*XLN/RSQ
XLRS = SQRT(XLRS)
CCSQ=SQRT(CCSQ)
IF(IND) 84, 101, 101
101 IF(XLRS=CC8) 84, 101
84 P=1.9000000, 1.92*ARMIN/XLRS
GO TO 85
83 P = ARMIN*(I, = 0.5*XLRS/CSCQ)*SY
IND=1
85 IF(P=P) 94, 92
2 P=P
IND=IND
GO TO 5
4 IND=IND
5 RA = SQRT(RSQ)
XLRA = XRXLN/RA
IF (XLRA = CC) 55, 56, 56
55 RR = XLRA/CC
FS = 5./3. + 3.*HR/A, * RR+RR+RR/A,
GO TO 57
56 FS = 1.92
57 RETURN
END

SUBROUTINE SELEC (P, V, NA, ARMIN, XIM, SY, IPASS)
COMON/AI/ PA, PB, VA, VB, PI, VI, NB
IF (NA=1) x2+1
2 PA=P
VAA
PA=P
VA=N
NB=0
IF (IPASS) 10, 20, 19
20 PB = 149.X+8*1.92*XIM
GO TO 21
19 PB = 5*ARMIN
21 P = P + 1.*(PB*PA)
RETURN
1 IF(VI) x3.8
3 IF(NB) x5.8
6 IF(VB=V) x5.8
5 PB=P
VB=V
NB=1
10 P = .5*(PB+PA)
12 IF(ABS(PB - PA) = .01*P) 17, 17, 7
17 NA = 0
7 RETURN
9 PA=P
VAA
IF(NB) x11.10
11 IF(10*NA) x14.13
13 PPA, 1.* (PB + PI)
GO TO 12
14 NMD/NA/10
XND=1D
P = P + .1*(PB+PI)/(XND+10.
GO TO 12
8 PB=P
VB=V
NA=1
NB=0
PA=1
VA=VI
PPA=1.* (PB+PA)
RETURN
END

START OF SEGMENT ********** 5
1 R 0105
2 R 0107
3 R 0110
4 R 0111
5 R 0116
6 R 0117
7 R 0120
8 R 0123
SEGMENT 4 IS 148 LONG

SEGMENT 5 IS 106 LONG
SUBROUTINE EFFL(NA, NB, XX)
1 IF(NA = 1) 3, 2, 3
2 IF(NB = 1) 8, 5, 6
3 XX = 0, 65
4 GO TO 7
5 XX = 0, 8
6 GO TO 7
7 RETURN
END

START OF SEGMENT ********** 6
R 0000
R 0000
R 0000
R 0000
R 0012
R 0014
R 0016
R 0018
R 0020
R 0022
R 0024
R 0024
R 0029
R 0031
R 0033
R 0033
R 0035
R 0038
R 0041

SEGMENT 6 IS 40 LONG