RICE UNIVERSITY

Long Period Leaking Modes

by

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<td>(\zeta)</td>
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\( \tilde{\phi} \) \hspace{1cm} \text{Laplace transformed variable}

\( k_{an} ; k_{\beta n} \) \hspace{1cm} \( s_{an} ; s_{\beta n} \)

\( \nu_\alpha ; \nu_\beta \) \hspace{1cm} \( (k^2 + k_{an}^2)^{1/2} ; (k^2 + k_{\beta n}^2)^{1/2} \)

\( f_{Br} \) \hspace{1cm} \text{Bromwich contour; } \int \hspace{1cm} \xi \rightarrow \infty \)

\( f_{Br^+} \) \hspace{1cm} \text{Upper Bromwich contour; } \int \hspace{1cm} \xi \rightarrow 0 \)

\( (++\text{)} \text{ sheet} \) \hspace{1cm} \text{The Riemann sheet for which } \text{Re}(\nu_\alpha) > 0 \text{ and } \text{Re}(\nu_\beta) > 0 \)

\( (+-) \text{ sheet} \) \hspace{1cm} \text{The Riemann sheet for which } \text{Re}(\nu_\alpha) > 0 \text{ and } \text{Re}(\nu_\beta) < 0 \)

\( (--) \text{ sheet} \) \hspace{1cm} \text{The Riemann sheet for which } \text{Re}(\nu_\alpha) < 0 \text{ and } \text{Re}(\nu_\beta) < 0 \)

\( (-+\text{)} \text{ sheet} \) \hspace{1cm} \text{The Riemann sheet for which } \text{Re}(\nu_\alpha) < 0 \text{ and } \text{Re}(\nu_\beta) > 0 \)

\( \phi^{++}, \pi^{--} \) \hspace{1cm} \text{Pi-modes on the (++) sheet; on the (--) sheet}

\( \Sigma^{++}, \Sigma^{--} \) \hspace{1cm} \text{Sigma-modes on the (++) sheet; on the (--) sheet}

\( \bar{\Pi}^{++}, \bar{\Pi}^{--} \) \hspace{1cm} \text{Lamb modes (Gilbert, 1964)}
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INTRODUCTION

In recent years increasing interest has been shown in the use of leaking modes in layered, elastic media as an adjunct to normal mode studies. Leaking or evanescent modes (Budden, 1961, pp. 131-132) are those modes which result from the complex roots of the secular function; they lie on lower Riemann sheets. They represent propagating modes of energy which are attenuated by leakage into the substratum as they travel in the wave-guide. Normal or trapped modes (Budden, 1961, pp. 134-135) result from the real roots of the secular function; these lie on the uppermost (++) Riemann sheet; they are unattenuated by leakage as they travel in the wave-guide. The normal modes (surface waves) give extensive information concerning the S-wave velocity distribution in the wave-guide, but practically no information concerning the P-wave velocity distribution. It has been suggested that by extending model analysis to leaking modes one may obtain information about the P-wave velocity distribution (Oliver and Major, 1960). Such an extension of modal analysis will allow a modal study of earlier parts of the seismogram.

Recent papers provide a firm theoretical foundation for the investigation of leaking modes in both acoustic and elastic wave-guides. Rosenbaum (1960) was the first to investigate leaking mode propagation in simple (single-layered) and complex (multi-layered) acoustic wave-guides; he showed how these leaking mode contributions may be uncovered by deforming the contours of integration. Phinney (1961)
extended these concepts to propagation in a simple, elastic wave-guide. Gilbert (1964) synthesized the mathematical theory and extended it to complex, elastic wave guides using Haskell's (1952) matrix formulation, and he briefly touched on the techniques for computing the excitation functions; he also presented numerical results on the dispersion and attenuation of a few leaking modes in simple wave-guides modeling the continental and the oceanic crusts. A more physical discussion of the importance and meaning of leaking modes is given by De Bremaecker (1967).

During this same period, other workers were attempting to explain the oscillatory motion following the arrivals of P and of S. Oliver and Major (1960) used leaking modes to explain the commonly observed PL phases. They used an approximate method, which is only valid for leaking modes with low attenuation, and computed dispersion curves for two models, a simple, acoustic wave-guide overlying an elastic substratum and a simple, elastic wave-guide overlying an elastic substratum. These curves were matched with data from recorded PL phases. Gilbert and Laster (1965), using this technique, were able to match theoretical results with data from model experiments of a simple, elastic wave-guide. This technique was successfully extended to more complicated wave-guides by Su and Dorman (1965).
Laster, et al (1965), using Harkrider's (1964) matrix formulation, computed excitation functions and synthetic seismograms for leaking modes in a simple, elastic wave-guide, and compared them to those obtained from model experiments. More recently, Abramovici (1968, a) computed the dispersion of leaking modes by directly solving the differential equations numerically. The spectral transfer functions (similar to excitation functions) associated with these modes were also computed for a random pressure source. He also computed (1968, b) the dispersion curves and transfer functions for a compressional source located in a liquid layer over an elastic substratum. A qualitative agreement between theoretical results and data indicate that leaking modes are important contributors to the total signal. De Bremaecker (1968) presented some dispersion curves for leaking modes on the (+-) sheet for the Canadian Shield model of Brune and Dorman (1963) with and without a low velocity P-channel. These results suggest that some of these modes are sensitive to the presence of this channel and may thus be used to determine the existence of this channel in the upper mantle. The excitation functions for these modes were not computed.

All of the above papers, with the exception of Laster, et al (1965), and Abramovici (1968, a and b) have been concerned either with the development of basic theory and techniques or with the application of leaking modes to a given problem. In only one
paper (De Bremaecker, 1968) is the effect of a low velocity P-channel on leaking modes discussed, and this paper is concerned only with the effect on the dispersive character of leaking modes on the (+-) sheet. Although both Laster, et al. (1965) and Abramovici (1968, a and b) use excitation functions extensively, neither analyzed the effect of changing the depth of the source.

It is, therefore, the purpose of this paper to analyze both the dispersive character and the excitation functions of all types of modes---π-, Σ--, Π+-, Σ+, P+-, P-+- in a complex, elastic wave guide containing a low velocity S- and P-channel. The effect of source depth and source type on these excitation functions will also be computed. In Section I, the theory of leaking mode propagation in complex, elastic wave-guides is extended to compute surface excitation functions for various source types and source depths in a manner similar to that for normal modes (Harkrider, 1964). The physical implications and meaning of leaking modes are discussed in Section II; in Section III the computational methods are explained. In Sections IV and V the dispersion curves and excitation functions for the given model are presented, and the effects of the low velocity P-channel and source depth are discussed. A brief comparison of the theoretical results with data is made in Section VI. Finally, the importance and limitations of leaking modes are discussed in Section VII.
SECTION I:
THEORY

Introduction. The problem is to determine the leaking mode
correlation to the modal response in a complex, elastic
wave-guide underlain by an isotropic, homogeneous, elastic
half-space (Figure 1). Each of the layers in the wave-guide
is assumed to be homogeneous and isotropic. The positive
z-axis points downward, and the boundaries of each layer
are plane-parallel and are perpendicular to the z-axis. The
source is located at an arbitrary point in the wave-guide
(x=D). The boundary conditions are:

1. The upper surface of the wave-guide (z=0) is
   stress-free.
2. All contacts are "welded" (i.e. the displacements
   and stresses are continuous across all boundaries).
3. Sommerfeld's radiation condition (Courant and
   Hilbert, 1962, p. 315-318) holds.

The equations applicable to this problem are the elastic
wave equations,

\[ \nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \]

(1)

\[ \nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \]

(2)
where $\phi$ scalar displacement potential,

$\psi$ = vector displacement potential,

$\alpha$ = P-wave velocity,

$\beta$ = S-wave velocity.

**The General Case.** The following derivation will be carried out in cylindrical coordinates; the derivation in cartesian coordinates is similar, and any significant deviations will be mentioned. The source will be considered to be axi-symmetric.

The cartesian analog is an infinite line source in the y-direction.

The vector displacement, $\psi$, in any layer is given by

$$
(3) \quad \| \psi \| = (u, 0, w)
$$

and $\psi$, the vector potential field, by

$$
(4) \quad \| \psi \| = (0, \psi, 0).
$$

Writing (1) and (2) in component form yields

$$
(5) \quad \phi,_{rr} + \frac{1}{r} \phi, r + \phi,_{rz} = \frac{1}{\alpha^2} \phi, tt
$$

and

$$
(6) \quad \psi,_{rr} + \frac{1}{r} \psi, r + \psi, zz = \frac{1}{\beta^2} \psi, tt.
$$

The Laplace transforms of these equations are

$$
(7) \quad \tilde{\phi},_{rr} + \frac{1}{r} \tilde{\phi}, r + \tilde{\phi}, zz = \frac{\delta^2}{\alpha^2} \tilde{\phi}
$$
and

\( (8) \ \hat{\psi}, \ rr + \frac{1}{r} \ \hat{\psi}, \ r + \hat{\psi}, \ zz - \frac{1}{r^2} \ \hat{\psi} = \frac{\sigma^2}{p^2} \ \hat{\psi} \)

These equations are again transformed with a Hankel transform with respect to \( r \) (In cartesian coordinates, a Fourier transform with respect to \( x \) would be made.). Because of the different nature of equations (7) and (8), different transform kernels are used. For (7)

\[
(9) \ \hat{\phi}(k,z,s) = \int_{0}^{\infty} \hat{\phi}(r,z,s) \ r J_0(kr) \ dr,
\]

and for (8)

\[
(10) \ \hat{\psi}(k,z,s) = \int_{0}^{\infty} \hat{\psi}(r,z,s) \ r J_1(kr) \ dr.
\]

Equations (7) and (8) thus yield

\[
(11) \ \hat{\phi}, \ rr = (k^2 + s^2 /a^2) \ \hat{\phi}
\]

and

\[
(12) \ \hat{\psi}, \ rr = (k^2 + s^2 /b^2) \ \hat{\psi}.
\]

The solutions of these equations are

\[
(13) \ \hat{\phi} = R e^{\gamma a z} + R'e^{-\gamma a z}
\]

and

\[
(14) \ \hat{\psi} = S e^{\gamma b z} + S'e^{-\gamma b z}.
\]
The displacement field, $u$, in terms of the potential functions is

(15) \( u = \nabla \phi + \nabla \times \psi \);
therefore,

(16) \( u = \partial \phi / \partial r - \partial \psi / \partial z \)

and

(17) \( w = \partial \phi / \partial z + \frac{1}{r} \left( \frac{\partial}{\partial r} (r \psi) \right) \).

Substituting (13) and (14) into (16) and (17) and including the Bessel functions for clarity, the expressions for the doubly transformed displacements become

(18) \[ \ddot{u} = [-k(Re^\nu \alpha^Z + R'e^{-\nu} \alpha^Z) - \nabla \beta (Se^\nu \beta^Z - S'e^{-\nu} \beta^Z)] J_1 (kr) \]

(19) \[ \ddot{w} = \nu \alpha (Re^\nu \alpha^Z - R'e^{-\nu} \alpha^Z) + k(Se^\nu \beta^Z + S'e^{-\nu} \beta^Z)] J_0 (kr). \]

Stress is related to the displacement through the equations of linear momentum,

(20) \( \sigma = t^{zz} = \lambda (u, r + u/r + w, z) + 2\mu w, z \)

(21) \( \tau = t^{rz} = \mu (u, z + w, r). \)
Hence, the expressions for the transformed stresses may be written as

\[
(22) \quad \hat{\sigma} = [[\lambda s^2/\alpha^2 + 2\mu^2 \alpha^2][\text{Re}^\alpha \alpha^z + R'e^{-\nu} \alpha^z] + 2\nu \beta k[Se^\nu \beta^z - S'e^{-\nu} \beta^z]] J_0(kr)
\]

and

\[
(23) \quad \hat{\tau} = [2[ -\mu \nu \alpha[\text{Re}^\nu \beta^z - R'e^{-\nu} \beta^z] -
\mu[k^2 + \nu \beta^2][Se^\nu \beta^z + S'e^{-\nu} \beta^z]] J_1(kr).
\]

Writing (18, 19), (22), and (23) as a matrix equation yields

\[
\begin{bmatrix}
\hat{u} \\
\hat{w} \\
\hat{\sigma} \\
\hat{\tau}
\end{bmatrix} =
\begin{bmatrix}
-k & -k & -\nu \beta & \nu \beta \\
\nu \alpha & -\nu \alpha & k & k \\
\rho^2 s^2 + 2\mu k^2 & \rho^2 s^2 + 2\mu k & 2\mu \nu \beta k & -2\mu \nu \beta k \\
-2\mu \nu \alpha k & 2\mu \nu \alpha k & -\mu[k^2 + \nu \beta^2] & -\mu[k^2 + \nu \beta^2]
\end{bmatrix}
\begin{bmatrix}
e^{\nu \alpha z} \\
e^{-\nu \alpha z} \\
e^{\nu \beta z} \\
e^{-\nu \beta z}
\end{bmatrix} =
\begin{bmatrix}
R \\
R \\
S \\
S'
\end{bmatrix}
\]
where \( \hat{u} \equiv \hat{u}/J_1(kr) \); \( \hat{w} \equiv \hat{w}/J_0(kr) \); \( \hat{\sigma} \equiv \hat{\sigma}/J_0(kr) \);
\( \hat{\tau} \equiv \hat{\tau}/J_1(kr) \).

Equation (24) is equivalent to Harkrider's equation (14) (1964). This equation will be used in the problem at hand.

**Extension to Layered, Elastic Media.** The doubly transformed expressions for the surface displacements for a layered, elastic system will now be derived. The procedure followed in this derivation is similar to that presented by Haskell (1953).

The three boundary conditions mentioned previously may be written

\[
(25) \quad \left. \hat{\sigma}_1 \right|_{z=0} = \left. \hat{\tau}_1 \right|_{z=0} = 0;
\]

\[
(26) \quad \left. \hat{\sigma}_i \right|_{z=z_i} = \left. \hat{\sigma}_{i+1} \right|_{z=z_i}; \quad \left. \hat{\tau}_i \right|_{z=z_i} = \left. \hat{\tau}_{i+1} \right|_{z=z_i};
\]

\[
\left. \hat{u}_i \right|_{z=z_i} = \left. \hat{u}_{i+1} \right|_{z=z_i}; \quad \left. \hat{\varphi}_i \right|_{z=z_i} = \left. \hat{\varphi}_{i+1} \right|_{z=z_i}
\]

(27) \( \tilde{\sigma}_n, \tilde{\tau}_n, \tilde{u}_n, \) and \( \tilde{\varphi}_n \) approach 0 as \( z \) approaches \( \infty \).

The following notation (De Bremaecker, 1967) will be used:
\[ \begin{array}{cccc}
-k & -k & -\nu_1 & \nu_1 \\
\nu_1 & -\nu_1 & k & k \\
-\nu_1 \sigma_1^2 + 2\mu_1 \beta_1 & \beta_1^2 + 2\mu_1 \beta_1 & 2\mu_1 \nu_1 \beta_1 & -2\mu_1 \nu_1 \beta_1 \\
-2\mu_1 \nu_1 \sigma_1 & 2\mu_1 \nu_1 \beta_1 & -\nu_1 [k^2 + \nu_1 \beta_1^2] & -\nu_1 [k^2 + \nu_1 \beta_1^2] \\
\end{array} \]

\[ D_{\kappa_1} = \begin{array}{cccc}
\nu_1 \sigma_1 & 0 & 0 & 0 \\
0 & -\nu_1 \sigma_1 & 0 & 0 \\
0 & 0 & \nu_1 \sigma_1 & 0 \\
0 & 0 & 0 & -\nu_1 \sigma_1 \\
\end{array} \]

\[ \kappa_1 = (\sigma_1, \sigma'_1, \beta_1, \nu_1); \]

\[ \kappa_1 = (R_1, R'_1, S_1, S'_1); \]

\[ i \equiv \text{index of the layer.} \]
(28) \( \mathbf{t}_1 = \mathbf{k}_1 \cdot \mathbf{p}_1 \cdot \mathbf{T}_1 \)

for the \( i \)th layer. At the top of layer \( 1(z=0) \), (28)

becomes

(29) \( \mathbf{t}_1(0) = \frac{\mathbf{k}_1}{\mathbf{g}_1} \cdot \mathbf{T}_1 \) or \( \mathbf{t}_1 = \frac{\mathbf{k}_1}{\mathbf{g}_1} \cdot \mathbf{t}_1(0) \),

and, at the bottom this layer \( (z=z_1) \), (24) may be written as

(30) \( \mathbf{t}_1(z_1) = \frac{\mathbf{k}_1}{\mathbf{g}_1} \cdot \mathbf{p}_1(z_1) \cdot \mathbf{T}_1 \).

By (26),

\( \mathbf{t}_1(z_1) = \mathbf{t}_1(z_1) \)

or

(31) \( \mathbf{t}_1(z_1) = \frac{\mathbf{k}_1}{\mathbf{g}_1} \cdot \mathbf{p}_1(z_1) \cdot \mathbf{T}_1 \).

Taking (29) into account, this becomes

(32) \( \mathbf{t}_1(z_1) = \frac{\mathbf{k}_1}{\mathbf{g}_1} \cdot \mathbf{t}_1(0) \),

where \( \frac{\mathbf{A}_1}{\mathbf{g}_1} \equiv \frac{\mathbf{k}_1}{\mathbf{g}_1} \cdot \mathbf{p}_1(z_1) \cdot \mathbf{k}_1^{-1} \).

By continuing this procedure down through the \( n-1 \)

layers, one obtains

(33) \( \mathbf{t}_n(z_{n-1}) = \prod_{i=n-1}^{1} \frac{1}{\mathbf{g}_i} \cdot \mathbf{t}_i(0) \)

at the top of the substratum.
Noting that (29) at \( z = z_{n-1} \) in the substratum may be written as

\[
\xi_n(z_{n-1}) = \kappa_n^{-1} \cdot \xi_{n-1}(z_{n-1}),
\]

one may write (33) as

\[
(34) \quad T_n = \kappa_n^{-1} \cdot \prod_{i=n-1}^{i=1} \Delta_i \cdot \xi_1(0).
\]

Applying the boundary conditions, (25) and (27), (34) becomes

\[
\begin{bmatrix}
0 \\
\mathbf{R}_n' \\
\mathbf{R}_n \\
\mathbf{S}_n'
\end{bmatrix} = \mathbf{J}_n \cdot \begin{bmatrix}
\mathbf{g}_1(0) \\
\mathbf{g}_1(0) \\
0 \\
0
\end{bmatrix}
\]

where

\[
\mathbf{J}_n = \kappa_n^{-1} \cdot \prod_{i=n-1}^{i=1} \Delta_i.
\]

A Source Located in Layered Media. In this section, the previous derivation will be modified to allow for the presence of a source located at an arbitrary depth within the layered wave-guide. This derivation will be carried out in a manner similar to that of Harkrider (1964). A source is defined as a discontinuity in stress and/or displacement. Consider such a discontinuity to exist at \( D \) in layer \( s \). The doubly transformed source vector is:
\[ k_A = (\delta_\mu, \delta_\omega, \delta_\sigma, \delta_\tau) \]

Layer \( s \), containing the source, is broken up into two layers,

\( S^-, \) if \( z_{s-1} < z < D \),

and

\( S^+, \) if \( D < z < z_{s+1} \).

Now one may write

\[ k_{s+}(D) = k_{s-}(D) + \lambda. \]

Thus, the expression at the top of the \( s+1 \) layer becomes

\[ k_{s+1}(z_s) = \lambda_{s+} \cdot k_{s+} = \lambda_{s-} \cdot (k_{s-} + \lambda). \]

Since

\[ k_{s-} = \prod_{i=s-}^{1} \lambda_i \cdot k_1(0) \]

and

\[ \lambda_s = \lambda_{s+} \cdot \lambda_{s-}. \]

Equation (36) may be written as

\[ k_{s+1}(z_s) = \prod_{i=s}^{1} \lambda_i \cdot k_1(0) + \lambda_{s+} \cdot \lambda. \]
The expression at the top of the substratum thus becomes

\[
\hat{z}_n(z_{n-1}) = \frac{1}{\prod_{i=n-1} \mathbb{A}_1} \cdot \hat{z}_1(0) + \prod_{i=n-1} \mathbb{A}_1 \cdot \mathbb{S}.
\]

One thus obtains an expression analogous to (35):

\[
\begin{bmatrix}
0 \\
R'_n \\
0 \\
S'_n
\end{bmatrix} = \mathbb{J} \cdot \begin{bmatrix}
\hat{u}_1(0) \\
\hat{v}_1(0) \\
0 \\
0
\end{bmatrix} + \mathbb{J}' \cdot \mathbb{S},
\]

where \( \mathbb{J}' = \frac{1}{\mathbb{S}_{nn}} \cdot \prod_{i=n-1} \mathbb{A}_{1i} \).

Solving (39) for the surface displacements, one obtains

\[
\begin{bmatrix}
\hat{u}_1(0) \\
\hat{v}_1(0)
\end{bmatrix} = \frac{1}{S(s,k)} \cdot \begin{bmatrix}
\mathbb{J} \\
\mathbb{S} \\
\mathbb{S} \end{bmatrix},
\]

where \( S(s,k) = J_{41} J_{22} - J_{21} J_{42} \), the secular function;

\[
\mathbb{L} = \begin{bmatrix}
-J_{22} & J_{42} \\
J_{21} & -J_{41}
\end{bmatrix};
\]

\[
\mathbb{E} = \begin{bmatrix}
J'_{41} & J'_{42} & J'_{43} & J'_{44} \\
J'_{21} & J'_{22} & J'_{23} & J'_{24}
\end{bmatrix}.
\]
$S(s, k)$ and $L$ are independent of the source; $E$ and $\delta$ are independent respectively of source type and source location. This factorization will simplify later computations.

**Inversion.** Having obtained the doubly transformed expressions for the displacement field, one must now invert one of the transforms by the residue method. In this paper the Laplace transform is inverted first because, as Laster et al. (1965), suggested, the resulting dispersion curves are more meaningful. It is during the inversion of this transform by residues that both the normal mode contributions and the leaking mode contributions are encountered.

From (40), the expressions for the inverted surface displacement field in integral form are

\[
\begin{bmatrix}
u_{1}(0) \\ \psi_{1}(0)
\end{bmatrix} = \frac{1}{2\pi i} \int_{0}^{\infty} \int_{0}^{R} \left[ \frac{s \cdot \vec{E} \cdot \vec{Z}}{S(s, k)} \right] e^{st} \begin{bmatrix} J_{1}(kr) \\ J_{0}(kr) \end{bmatrix} \, ds \, dk.
\]

In order to invert the Laplace transform, one must carry out the integration along the Bromwich contour in the complex $s$-place, where $s = \xi + i\omega$ (Figure 2a). As a result of the reflection properties of $u$ and $\psi$, the Bromwich contour may be replaced by its upper half using the following relationship:
\[ A(t) = 2 \Re \int_{0+1\xi}^{0+1\xi} A(s) e^{st} ds. \]

The new Bromwich contour, \( Br_+ \), may be moved infinitely close to the imaginary axis of the \( s \)-plane (i.e. \( \xi \to 0 \)), since there are no poles in the first quadrant (The existence of such poles would cause the system to be unstable.). The presence of the two radicals, \( v_{\alpha_n} \) and \( v_{\beta_n} \), forces one to make four branch cuts in the \( s \)-plane in order to assure the analyticity of the function. The branch points are located at \( s = \pm i \alpha_n k \) and \( s = \pm i \beta_n k \). The branch cuts are chosen so that \( \Re(v_{\alpha_n}) \) and \( \Re(v_{\beta_n}) \) are greater than zero; this not only insures the analyticity of the function but also insures that the integrals will vanish on the infinite contours. The desired integral, \( \int_{\Gamma_1} \), may be broken down into the sum of four line integrals--- \( \int_{\Gamma_a}, \int_{\Gamma_b}, \int_{\Gamma_c}, \int_{\Gamma_d} \)--- as shown in Figure 2b. The integral along \( \Gamma_c \) vanishes due to the proscription placed on the branch cuts.

The necessity of computing the two branch line integrals, \( \Gamma_a \) and \( \Gamma_b \), may be obviated by deforming the contour, \( \Gamma_1 \); such a deformation will uncover the leaking mode contribution. The contour, \( \Gamma_1 \), may be broken up into three segments --- \( \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) --- where \( \Gamma_2 \) extends from zero to \( i k \), \( \Gamma_3 \) extends from \( i \beta_n k \) to \( i \alpha_n k \), \( \Gamma_4 \) extends from \( i \alpha_n k \)
to \( i = (\text{Figure 3}) \). Integrals along each of these segments can be expressed as the sum of the pertinent residues and line integrals.

As the contours, \( \Gamma_{++0} (\Gamma_d) \) and \( \Gamma_{++\beta} \) cross no branch lines, they lie on the upper Riemann sheet ((++) sheet); therefore, noting that the integral along \( \Gamma_{++0} \) is purely real and vanishes on the infinite contour, one may write the following expression for the integral on \( \Gamma_2 \):

\[
(42) \quad \Re \left( \frac{1}{\pi} \int_{\Gamma_2} \right) = 2\Re (\Sigma \Re_{++}) - \frac{\Im}{\pi} \int_{++\beta} ;
\]

where \( \Sigma \Re_{++} \) is the sum of residues on the \((++)\) sheet lying in the region, \(-\infty < \Re(s) < 0 \) and \(0 < \Im(s) < k_{\beta_n} \).

As both \( \Gamma_{+-\beta} \) and \( \Gamma_{+-\alpha} \) cross one branch cut, they lie on a lower Riemann sheet, i.e. the (+-) sheet. As the integral vanishes on the infinite contour, the expression analogous to (42) for the integration along \( \Gamma_3 \) is

\[
(43) \quad \Re \left( \frac{1}{\pi} \int_{\Gamma_3} \right) = 2\Re (\Sigma \Re_{+-}) - \frac{\Im}{\pi} (\int_{\Gamma_{+-\beta}} - \int_{\Gamma_{+-\alpha}}) ;
\]

where \( \Sigma \Re_{+-} \) is the sum of residues on the (+-) sheet lying in the region, \(-\infty < \Re(s) < 0 \) and \(k_{\beta_n} < \Im(s) < k_{\alpha_n} \).
\( \Gamma_{-\alpha} \) has crossed both branch cuts and, hence, lies on the \((-\) \) sheet. The expression for the integration along \( \Gamma_4 \) is

\[
\text{Re} \int_{\Gamma_4} = 2\text{Re}(\Sigma R_{--}) + \frac{\text{Im}}{\pi} \int_{\Gamma_{-\alpha}},
\]

where \( \Sigma R_{--} \) is the sum of residues on the \((-\) \) sheet lying in the region, \(-\infty < \text{Re}(s) < 0 \)
and \( k_{\alpha n} < \text{Im}(s) < \infty \).

Using (42), (43), and (44), the expression for the desired integration along \( \Gamma_1 \) may be written

\[
\text{Re} \int_{\Gamma_1} = 2\text{Re}(\Sigma R_{++} + \Sigma R_{+-} + \Sigma R_{-+}) + \frac{\text{Im}}{\pi} \left( \int_{\Gamma_{+\beta}} + \int_{\Gamma_{-\alpha}} \right).
\]

One may combine the four line integrals into two line integrals which are integrated for \( \text{Re}(s) \) from zero to \( -\infty \) at \( \text{Im}(s) = \beta_{n k} \) and at \( \text{Im}(s) = \alpha_{n k} \) (Phinney, 1961). Their evaluation is not, however, germane to this study, as they do not give rise to modal contributions, and as their contribution dies off rapidly with time (Gilbert, 1964). These line integrals and their evaluation are further discussed by Rosenbaum (1960) and Phinney (1961).

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It is the residue contributions which result from poles on the various Riemann sheets that are of interest here. These poles occur at the zeroes of the secular function, $S(s,k)$, and give rise to the model contributions, both normal and leaking. The regions of interest are defined:

- on the $(++)$ sheet, as the region where $-\infty < \text{Re}(s) < 0$ and $0 < \text{Im}(s) < k_{\beta_n}$;
- on the $(+-)$ sheet, where $-\infty < \text{Re}(s) < 0$ and $k_{\beta_n} < \text{Im}(s) < k_{\alpha_n}$;
- on the $(--)$ sheet, where $-\infty < \text{Re}(s) < 0$ and $k_{\alpha_n} < \text{Im}(s) < \infty$.

Contributions from poles not lying in the regions of interest will not be considered in this paper.

The $R_{++}$ residues result from poles lying along the imaginary axis and, hence, their frequency is purely real. For this reason, there is no temporal attenuation associated with these modes, i.e. $\xi = 0$. They represent the normal (or trapped) modes which are discussed in detail by Pekeris (1948). The residues lying on the other Riemann sheets, $R_{+-}$ and $R_{-+}$, do not, in general, lie on the imaginary axis; hence, they are attenuated with time as they propagate $(\xi \neq 0)$. These are the leaking (or evanescent) modes.
Excitation Functions. Changing the first variable of integration from $s$ to $f$ where $s = if$, one may write the leaking mode contribution to the surface displacements as:

\[
\begin{bmatrix}
u_1(0) \\
\omega_1(0)
\end{bmatrix} = 2\text{Re} \int_0^\infty [\bar{E}_+(k_1t) + \bar{E}_-(kt)] dt.
\]

\[
\begin{bmatrix}
J_1(kr) \\
J_0(kr)
\end{bmatrix} = \frac{1}{ik} dk;
\]

where \( \bar{E}_+(k, t) = \sum_m \frac{[k \cdot \bar{E}_+ \cdot \bar{E}_-]}{\partial \tilde{S}(f, k) / \partial f} e^{ift} \) \( \big|_{f = f_{m+}} \)

\( \bar{E}_-(k, t) = \sum_m \frac{[k \cdot \bar{E}_+ \cdot \bar{E}_-]}{\partial \tilde{S}(f, k) / \partial f} e^{ift} \) \( \big|_{f = f_{m-}} \)
\( f_{m+} \) is the \( m \)th zero of \( S(f,k) \) on the \((+\)\) sheet;

\( f_{m-} \) is the \( m \)th zero on the \((-\)\) sheet.

The spatial excitation for the \( m \)th mode is defined as

\[
\Theta_{m+}(k) \equiv \text{Mod} \left( \tilde{E}_{m+}(k,t) \ e^{-if_{m+}t} \right)
\]

and

\[
\Theta_{m-}(k) \equiv \text{Mod} \left( \tilde{E}_{m-}(k,t) \ e^{-if_{m-}t} \right)
\]

The spatial excitation function is a function of only

the wave number, \( k \), which is, of course, related to the

frequency by the dispersion curve of the relevant mode.

To make the comparisons easier between the dispersion curves

and the excitation functions, \( \Theta \) has been plotted as a function

of \( \text{Re}(\nu) \) rather than \( k \) except for the \( \Pi_1 \) mode.

The excitation functions are of fundamental importance

in modal analysis since, for normal modes, they determine the

amplitude of any mode at any frequency. For leaking modes,

this amplitude is modified by the attenuation.

Dispersion. The leaking modes, as has been shown above, result

from complex roots in the \( s \)-plane of \( S(s,k) \). The location of

these roots varies with the wave number, \( k \), thus causing
dispersion. As \( k \) varies, the roots may:
(1) Move out of the region of interest;
(2) Move across branch lines and change sheets;
(3) Remain in the region of interest on the same sheet.

The most interesting behavior is that of the \( \Pi \) modes.

As will be seen below, these modes start on the physically meaningless \( \rightarrow \) sheet; they then cross both branch lines and enter the \( \leftarrow \) sheet in the region of interest.

Plots of \( \nu \) vs. \( k \) are known as diagnostic diagrams (Gilbert, 1964). These diagrams may be somewhat hard to interpret; first, because they may include modes or parts thereof which do not lie in the regions of interest as defined here; secondly, because they tend to hide the alternation of plateaus and slopes which characterize dispersion curves on the \( \leftarrow \) sheet. Because of these facts and of the general use of dispersion curves in normal mode studies, I will use plots of \( \text{Re}(c) \) vs \( \text{Re}(\nu) \), \( \text{Im}(\nu) \) vs \( \text{Re}(\nu) \), and \( U \) vs \( \text{Re}(\nu) \), where \( c \) is the complex phase velocity \( (c = f/k) \), \( \nu \) is the complex frequency \( (\nu = f/2\pi) \), and \( U \) is the group velocity \( (U = dk/d\text{Re}(f)) \).
SECTION II:

PHYSICAL MEANING

Introduction. Having mathematically defined the leaking modes and shown how they arise from the solution of the problem, it is important to discuss their physical meaning. Such a discussion, nonrigorous as it may be, should bring new insight into the problem. For a more complete discussion of modal theory, the reader is referred to Budden (1961, pp. 69-253).

Normal modes represent paths of energy travel in the wave-guide at angles of total reflection for which the frequency and wavelength are such that constructive interference occurs. Similarly, leaking modes represent those paths for which the reflection is not total. In other words, the paths are such that reinforcement occurs, but part of the energy leaks out of the wave-guide as the result of incomplete reflection.

Leaking Modes. Two cases of incomplete reflection may occur in an elastic wave-guide. The energy may impinge on the upper boundary of the substratum at an angle such that it leaks into the substratum as both P-waves and S-waves. This occurs when the angle of incidence is less than both critical angles, i.e. when \( \theta > \alpha_n \). One may consider these paths to represent the paths of vertical and near vertical multiple reflections of exploration seismology. As expected, \( \Re(\xi_{\alpha_n}) \) and \( \Re(\nu_{\beta_n}) \) are negative for
these modes, indicating that both P-waves and S-waves leak out of the wave-guide for the modes on this sheet, (−−) sheet.

At vertical incidence (c = 0 and ν ≠ 0). P-waves and S-waves are uncoupled; as expected the secular function shows a similar behavior (Gilbert, 1964): it degenerates into two factors—-one dependent on the α 1's and the other β 1's. The first factor corresponds to multiply reflected P-waves, and these modes are designated as Π−− modes. The second factor corresponds to multiply reflected S-waves, and these modes are designated as Σ−− modes.

The other case of partial reflection occurs at angles lying between the two critical angles, i.e. β n < co n . For such paths, energy is lost as the result of leakage of S-waves into the substratum. For these modes Re(ω n ) is negative and Re(α n ) is positive; the roots lie on the (+−) sheet. The ray equivalent of these waves include channel waves, multiply reflected head waves, multiply reflected S-waves, and head waves (except the two head waves propagating along the top of the substratum). These modes are designated as Π+− and Σ+−.

In addition to the previously mentioned roots, there are at least two roots which begin at ω = 0, k ≠ 0, and, hence, cannot be classified as either Π or Σ modes. These are the complex Lamb roots (c−α n ). Only the roots originating on the (+−) and on the (−−) sheets are meaningful in the problem under discussion. These are the F−− and the F+− modes respectively (Gilbert, 1964). The physical interpretation of these modes is not clear.
Note that the present notation differs slightly from that of Gilbert (1964). The sheet designation of the \( \Pi \) and \( \Sigma \) modes refers to the sheet on which the mode contributes to the signal, not to the sheet on which the mode originates.

The magnitude of the attenuation factor, \( \xi \), is very dependent on the model used. As this factor represents the temporal attenuation resulting from leakage of energy into the substratum, its value depends on the number of reflections at the lower boundary of the wave-guide that occur in a given amount of time, the angle of incidence at this boundary, and the elastic impedance across the boundary. As a result of the first of these dependencies, \( \xi \), is inversely related to the thickness and directly related to the average velocity of the wave-guide. For this reason the relatively high attenuation found by authors who considered only simple, crustal wave-guides does not contradict the much lower attenuation computed for the present model. The relationship between the attenuation factor, \( \xi \), and of percent decrease /100 sec. is shown in Table I.

Finally, it should be noted that the spacing of the modes also depends on the model: it is directly related to the thickness and inversely related to the average velocity of the wave-guide. It is this relationship which makes it extremely difficult to compare computations made on models of greatly different thicknesses, e.g., models consisting one of the crust over a substratum and
<table>
<thead>
<tr>
<th>$\xi \times 10^3$</th>
<th>% Decrease /100 sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 0.1</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>0.1 ... 0.75</td>
<td>1 ... 7</td>
</tr>
<tr>
<td>0.75 ... 1.50</td>
<td>7 ... 14</td>
</tr>
<tr>
<td>1.50 ... 2.25</td>
<td>14 ... 20</td>
</tr>
<tr>
<td>2.25 ... 3.00</td>
<td>20 ... 26</td>
</tr>
</tbody>
</table>

**TABLE I.** Relationship between the attenuation factor and percent decrease in time (After De Bremaecker, 1968)
models consisting of the crust and upper mantle. Note, however, that this difficulty also exists for normal modes although it does not appear to have been mentioned in print.

**Empirical Rules.** The following empirical rules are useful in following the roots on the various Riemann sheets. The $\Pi$ modes start on both the ($\rightarrow$) sheet and the ($\rightarrow$) sheet at $c = \infty$ and $\nu \neq 0$. Those which begin on the ($\rightarrow$) sheet remain on this sheet at least within the region of interest. The roots which begin on the ($\rightarrow$) sheet generally move into the imaginary axis, cross both branch lines, and appear in the region of interest on the ($\leftarrow$) sheet. These modes are here called $\Pi\rightarrow$ modes. Since $\text{Re}(s) = 0$ or $\text{Im}(\nu) = 0$ as they change sheets, the origins of these modes on the ($\rightarrow$) sheet may be located by computing the secular function for $s = \xi + i\omega$ as $\xi \rightarrow 0$ and $k = \omega/\alpha_n$. $\Pi\rightarrow$ may be located by the same procedure.

The $\Sigma$ modes begin on both the ($\rightarrow$) and the ($\leftarrow$) sheets. Those starting on the ($\leftarrow$) sheet do not generally cross the branch lines in the region of interest, but migrate out of this region; these modes are designated $\Sigma\leftarrow$. Those that begin on the ($\rightarrow$) sheet remain on this sheet and move into the region of interest for $k = \omega/\alpha_n$. For this reason, one cannot compute the origins of these modes by the method given for $\Pi\rightarrow$ modes. Physically this means that the $\Sigma\leftarrow$ modes are attenuated for $c = \alpha_n$, while the $\Pi\rightarrow$ modes are not.
The \( E^\rightarrow \) modes then move toward the imaginary axis, cross one branch line at \( k = \omega / \beta_n \), and enter the region of normal modes on the (\( ++ \)) sheet. Since all roots for \( c < \beta_n \) are real, it would, however, appear that the (\( ++ \)) and the (\( +\leftarrow \)) sheet are physically identical for these roots. This behavior is shared by the \( n\rightarrow \), \( \bar{F}\rightarrow \), and \( \bar{F}\rightarrow \) modes, at least for the present model. One should thus be able to locate the end points of all modes on the (\( \leftarrow \)) sheet by computing the secular function at \( s = \xi + i \omega \), as \( \xi \to 0 \) and at \( k = \omega / \beta_n \). For an unknown reason, I was unable to use this procedure successfully.
SECTION III:
COMPUTATIONAL PROCEDURE

Initial Values. The roots of the secular function are first located on the various sheets for \( c^\infty \), \( w \neq 0 \). The approximate positions of the first few modes are found by replacing the complex wave-guide by a simple wave-guide having the average properties of the complex wave-guide. The roots for the \( \Pi \) modes on the (--) and (--+) sheets are given by (Gilbert, 1964).

\[
\phi = \frac{\alpha}{4\pi} (2n-1) \pm i \frac{\alpha}{4\pi} \log \frac{\rho \alpha_{n-\alpha}}{\rho \alpha_n + \alpha_{n+\alpha}}
\]

where \( \bar{\alpha} \equiv \text{average } \alpha \text{ in the complex wave-guide}; \)
\( \bar{\rho} \equiv \text{average density in the complex wave-guide}; \)
\( H \equiv \text{thickness of the complex wave-guide}; \)
\( \alpha_n \equiv \alpha \text{ in the substratum}; \)
\( \rho_n \equiv \text{density in the substratum}. \)

The roots for the \( \Sigma \) modes on the (--) and (++) sheets are given by (49) where \( \bar{\alpha} \) is replaced by \( \bar{\beta} \), the average \( \beta \) in the complex wave-guide. These roots can be computed by hand. Such computations give the approximate location and spacing of these modes for \( c^\infty \). Such information will help in the computation of the secular function at \( k=0, \nu \neq 0 \).
As has been previously discussed, the secular function
degenerates into two factors at \( k=0, \nu \neq 0 \). As a result of
this degeneracy and of further algebraic simplifications,
these two factors are readily computed. In order to locate
the origin of these modes, the moduli of each of the factors
are computed for a given range of complex \( \nu \). The range and
spacing of \( \nu \) is chosen from information obtained in the
aforementioned approximation procedure. The inverses of these
moduli are contoured, and the trial values of the roots are
taken at the peaks. The approximate origins of the \( \Pi^{--}, E^{+-}, \)
and \( E^{--} \) modes are thus readily found. The \( E^{+-} \) are then
followed as \( c \) decreases to \( a_n \). The values of \( E^{+-} \) at \( c=a_n \)
are then used as the initial points of the \( E^{+-} \) modes in the
region of interest. The procedure for obtaining the initial
points of the \( \Pi^{+-} \) and \( \bar{\Pi}^{--} \) modes has been discussed in a previous
section. The origin of the \( \bar{\Pi}^{+-} \) mode is at \( \nu = 0, c=a_n \).

**Dispersion Curves.** The computational method for the calculation
of the dispersion curves is similar to that of De Bremaecker (1968).
The secular function is written as a function of complex phase
velocity, \( c \), and complex frequency, \( \nu \), instead of real \( k \) and
complex \( f \), where \( \nu = 2\pi f \) and \( c = 2\pi \nu / k \). As \( k \) is real, the imaginary
parts of \( \nu \) and \( c \) are related by

\[
\text{Im}(c) / \text{Im}(\nu) = \text{Re}(c) / \text{Re}(\nu).
\]
The value of the secular function is then computed using Dunkin's modification of the Haskell-Thomson matrix formulation (Dunkin, 1965; Haskell, 1953). The roots of this function are found by Newton's iteration scheme. The method of iteration is dependent on the nature of the expected dispersion curve. On the (→) sheet where the curves are, in general, extremely steep in the Re(c)-Re(ν) plane, Re(c) is taken as the independent variable, and ν is taken as the dependent variable. On the other hand, there are regions on the (←) sheet where the curves are fairly flat. In these regions, Re(ν) is considered the independent variable, and c is considered the dependent variable.

After the initial points are found, the curves are followed by extrapolating forward from the three previous points to obtain a new initial guess for the next point. In order to preclude the "jumping" of modes, limits are placed on these extrapolated values. In the first case, Re(ν) is extrapolated by 5% at most, and Im(ν) by 10%. Similar limits are placed on the extrapolation of c in the second case.

The criteria for "declaring" a root are two-fold; two successive root approximations must lie within a given tolerance, ε; the real part and the imaginary part of the secular function must change signs between c + ε/2 and c - ε/2 or ν + ε/2 and ν = ε/2, as the case may be. It is probable that the latter
restriction is too strong. Points where the secular function changes signs between \( c + \bar{\epsilon}/2 \) and \( c - \bar{\epsilon}/2 \) or \( v + \bar{\epsilon}/2 \) and \( v - \bar{\epsilon}/2 \), where \( \bar{\epsilon} = \text{conj}(\epsilon) \), should also be accepted. The addition of this feature would probably little affect the results, but would increase the speed of computation. This and some other desirable features are now in standard use (Woebner, 1969).

**Group Velocity.** Before the computation of the group velocity curves, \( U \) vs. \( \text{Re}(\nu) \), the dispersion curves were decimated, as small errors in \( \text{Re}(c) \) or \( \text{Re}(\nu) \) could cause large errors in \( U \). The decimation procedure used was that of De Bremaecker (1968); the minimum spacing between adjacent points was limited to \( \text{Re}(\epsilon) \).

The group velocity was computed from a fifth-order polynomial. These results were spot-checked with analytically determined group velocities (Harkrider, 1964; Woebner, 1969).

In general, the error is less than 2%, but errors of up to 10% were observed at the endpoints of the dispersion curves. For this reason, these points were discarded in the group velocity computations.

**Excitation Functions.** The excitation functions, (47) and (48), consist of four factors --- \( J_1 \), \((\partial S(f,k)/\partial f)^{-1}\), \( P \), \( \mathcal{L} \). The first two are independent of source and need be computed only once for each mode. \( J_1 \) was computed by directly evaluating \( J_1 \);
Dunkin's method could be employed, but this was not necessary in the present case. The partial derivative, \( \frac{\partial S(f,k)}{\partial f} \), may be rewritten

\[
(50) \quad \frac{\partial S(f,k)}{\partial f} = \frac{1}{k} \frac{\partial S(c,k)}{\partial c}
\]

The right-hand expression in (50) was computed analytically using a method developed by Woeber (1969). \( \frac{\partial S}{\partial f} \) was computed in the same manner as \( \frac{\partial f}{\partial f} \), i.e. \( \frac{\partial f}{\partial f} \) was computed directly. \( \frac{\partial f}{\partial f} \) is considered to be an input parameter.
SECTION IV:

DISPERSION AND GROUP VELOCITY CURVES

Introduction. The phase and group velocity curves have been computed for a slightly simplified version of the CIT 11 GB model (Figure 4) used by Julian and Anderson (1968). This model contains a 279 km low-velocity S-channel and a 54 km low-velocity P-channel. It consists of 6 layers and a substratum, the latter at a depth of 653 km.

The (--) Sheet. Figure 5 shows the dispersion curves, c vs. Re(\omega), for the fifteen \Pi-- modes, and Figure 6 shows these curves for the first twenty-six \Sigma-- modes. These curves have very steep initial slopes and tend to flatten as the mode number increases and the phase velocity decreases. In general, the \Pi-- curves are steeper and, in fact, the \Pi1-- curve slopes slightly in the opposite direction. The attenuation factor, \xi, for the \Pi-- modes is shown in Figure 7. The lower order \Pi-- modes show an overall decrease in \xi as the frequency increases; the higher modes show relative maxima and minima.

The attenuation of the \Sigma-- modes (Figure 8) is, in general, less than that of the \Pi-- modes, and the variation of the attenuation is quite different. For the higher modes (n>3), the attenuation curves are marked by relatively sharp maxima. They also tend to show a slight increase as the mode number increases; this tendency is also seen for the \Pi-- modes.
The group velocity curves for both the $\Pi$- and the $\Sigma$- modes (Figures 9 and 10) are characterized by sharp maxima over a large range of group velocities ($2.0 \text{ km/sec} < U < 7.0 \text{ km/sec}$). In many cases the group velocities for the $\Sigma$- and $\Pi$- modes go through minima following the maxima, then rise to a velocity of approximately 4.0 km/sec., and oscillate around this velocity for large ranges of frequency. These modes are, therefore, not necessarily restricted to the later parts of the seismogram; some of them (or parts thereof) may contribute energy in the interval between the arrival of $P$ and that of $S$.

The modes on the (---) sheet seem to be little affected by the presence of the low-velocity channels since their curves are qualitatively similar to those reported by Gilbert (1964) and Abramovici (1968, a and b), neither of whom analyzed models containing such channels.

The (---) Sheet. The dispersion curves of the $\Pi+$- and the $\Sigma+$- modes for frequencies less than 0.076 Hz ($\tau > 13 \text{ sec}$) are shown in Figure 11. The $\Sigma+$- modes, with the exception of the first two modes, are marked by plateaus which break up the otherwise steeply sloping curves. A similar effect is seen in the curves for the $\Pi+$- modes; in this case, however, the relative importance of the plateaus and the slopes is reversed. The plateau is best developed for $\Pi2+$- mode which seems to asymptotically approach the $P$-wave velocity of the low-velocity channel, 7.75 km/sec. It is likely that the slope of the curve will steepen at
a higher frequency and that the phase velocity will decrease to $\beta_n$.

The dispersion curves for $\bar{E}^\rightarrow$ and $\bar{F}^\rightarrow$ are also shown in Figure 11. It should be noted that $\bar{E}^\rightarrow$ is the extension below the cut-off frequency of the second shear mode rather than of the first one. The $\bar{F}^\rightarrow$ mode is the extension of the fourth shear mode. The behavior of the $\bar{F}^\rightarrow$ mode differs from the one shown by Gilbert (1964). His curves indicate that this mode changes from the $(\rightarrow)$ sheet to the $(\leftarrow)$ sheet and back again many times; the behavior shown here is similar to that shown by Abramovici (1968, a).

Figure 12 shows the attenuation factor for the first ten $E^\rightarrow$ modes. These curves are marked by broad minima and maxima superimposed on an overall decrease when the frequency increases. The minima correspond to the previously mentioned plateaus of the dispersion curves. The group velocity curves for these modes (Figure 13) are marked by broad maxima at velocities between 7.0 km/sec and 9.0 km/sec; these regions also correspond to the plateaus of the dispersion curves. There are also broad, flat regions between 3.5 km/sec and 4.5 km/sec; these regions correspond to the flattening of the dispersion curves (as $c$ approaches $\beta_n$), but they begin at frequencies substantially lower than cut-off.

The attenuation curves for the $\Pi^\rightarrow$, the $\bar{E}^\rightarrow$, and the $\bar{F}^\rightarrow$ modes (Figure 14) are also marked by broad minima and maxima.
The maxima are associated with the steeper portions of the
dispersion curves, and the minima with the plateaus. Group
velocities (Figure 15) also shows relative minima and maxima.
The maxima lie between 7.0 km/sec and 9.0 km/sec, much as
they did for the \( \Sigma \) modes; they are also associated with
the slightly attenuated, plateau regions. The minima lie
between 2.0 km/sec and 4.0 km/sec and correspond to the steeper,
more highly attenuated portions of these modes.

**Pseudo-modes** From what precedes, it is clear that the \( \Pi \) and
the \( \Sigma \) modes combine to form two sets of almost continuous
curves (see Figure 11). The first set, here called \( \Pi \) pseudo-modes
(Figure 16), is formed by the plateaus of the curves; it is
characterized by low attenuation and high group velocity; its
phase velocity depends on \( \alpha_1 \) but not on \( \beta_1 \) (Woeber, 1969). The
other set, here called \( \Sigma \) pseudo-modes (Figure 17), is formed by
the slopes and exhibits the reverse characteristics, i.e. high
attenuation, low group velocity, dependence on \( \beta_1 \) but not on \( \alpha_1 \).

In order to simplify the notation the pseudo-modes will be designated
as \( P_1, P_2, \ldots \) and as \( S_1, S_2, \ldots \). Extreme care is necessary during
the computations in order to follow the modes rather than the
pseudo-modes (see Woeber, 1969).

It should be noted that a situation similar to, but less
complex than, the one just described was noted by Andrianova
et al. (1965, pp. 39-45) and by Chang (1968) for normal Love modes.
It can also be detected in Figure 2 of Abramovici (1968, b).

Because of their low attenuation, the pseudo-modes are almost entirely trapped in the wave-guide. Their ray analog are thus channel waves and P head waves.

The development of pseudo-modes is not the result of the presence of the low-velocity P-channel because they also exist in models not containing such a channel (De Bremaecker, 1968) and in leaking plate modes (Tamm and Weiss, 1959). The curves labelled Π and Σ by De Bremaecker (1968) were incorrectly identified: these curves are generally composed of almost interconnecting segments and are really pseudo-modes.
SECTION V:
EXCITATION FUNCTIONS

Introduction. Using the method discussed in Section III, spatial excitation functions were computed for typical examples of leaking modes on both the (--) sheet and the (+-) sheet. These functions were computed for a source consisting of a unit discontinuity in "w" at one of three different locations---above, in, and below the low-velocity P-channel, respectively point A, B, and C in Figure 4. As a result of the reciprocity principle (Courant and Hilbert, 1962, p. 354), one may consider these excitation functions to represent the vertical and horizontal displacements at each of the depths for a surface source.

Note that, in general, this is not an advantageous procedure: all things being-equal it is much more laborious to compute the excitation functions than to compute the displacements (or stresses) at depth for a given displacement (or stress) at the surface. It is thus generally preferable to do the latter and use the reciprocity principle to find the excitation functions. It is only because the partial derivatives were needed for other purposes (see Woeber, 1969) that Harkrider's method was advantageous in the present case.

In order to also examine the effect of the nature of the source, excitation functions were computed for a source consisting of each of four discontinuities: in w, in u, in σ, or τ---at point A for two modes on the (--) sheet.
The (⊥) Sheet. The excitation functions for the fundamental modes on the (⊥) sheet, $\Pi_{l-}$ and $\Sigma_{l-}$, are shown in Figures 18 and 19 on a semi-log scale. The notation $w>LVL$ means that the source is a discontinuity in $w$ below the $P$ low-velocity layer. The curves for $\Pi_{l-}$ are plotted versus wave-number because of the mode's anomalous behavior with respect to frequency. The amplitude of these functions increases as the source moves downward in the wave-guide, but the form of the curves is basically unchanged. $\Pi_{l-}$, even though its dispersion curve is anomalous, is more highly excited than $\Sigma_{l-}$. The vertical displacement ($w$) curve of $\Sigma_{l-}$ is marked by rapid oscillations while the rest of the curves are remarkably smooth. The rapid oscillations which occur at low amplitudes may be the result of rounding errors in the numerical calculations.

The curves for higher modes on this sheet (Figures 21 and 21), $\Pi_{10-}$ and $\Sigma_{11-}$, are marked to a lesser extent by oscillatory behavior. In the case of $\Sigma_{11-}$, however, the maxima can be correlated with minima of group velocity and vice-versa. The amplitude generally increases as the source depth increases, but this is less marked than for the fundamental modes.

The (⊥) Sheet. The excitation functions for the fundamental modes on the (⊥) sheet—$P\perp$, $\Pi_{l\perp}$, and $\Sigma_{l\perp}$—are shown in Figures 22, 23, and 24. The peaks of these functions may be
correlated with minima of the group velocity curves. The
curves of Π4− are most interesting: Their peaks correspond
to those parts of the mode which have been included on the
Σ pseudo-modes; those included in the Π pseudo-modes are
less excited. Furthermore, the surface displacements are
generally smaller for a source located in the low-velocity
P-channel, especially on the Π pseudo-mode parts of the curve.
Similar results are shown for Σ7− (Figure 25).

The curves for Π2− (Figure 26) are slightly different.
In this case, the excitation functions for a source at A have
a lower amplitude than those for the other two source depths.
For the part of the curve on the Π pseudo-mode, the amplitudes
for a source at B (in the channel) are often higher than those
for the source at C.

The excitation functions for Π− (Figure 27) are similar
to those for Π− but are of much higher amplitude. In fact,
portions of the Π− curves show the highest amplitudes of those
studied, a fact of great potential importance in seismogram
interpretation.

The Effect of Source Type. The excitation functions, using the
four sources previously mentioned, for the Σ4− and the Π3− modes
are shown in Figures 28 and 29. All of the sources are located at
A. By suitably combining these four sources, one may represent
any physically meaningful source.
While varying the source type does not greatly change the form of the excitation functions, there are interesting differences. The curves for sources of the $\delta u$ and $\delta r$ type are very similar, as are those for the $\delta w$ and $\delta \sigma$ type. A similar relationship was seen earlier (equations 18, 19, 22, and 23); the integral expressions for $u_1(0)$ and $r_1(0)$ contain $J_1(kr)$, and those for $w_1(0)$ and $\sigma_1(0)$ contain $J_0(kr)$. A fundamental difference between $\Pi$-- and $\Sigma$-- modes is also shown in these figures. For the $\Sigma$-- modes, the radial displacement excitation is of higher amplitude than that for the vertical displacement for all sources; the opposite effect is seen for $\Pi$-- modes. This effect is also shown by the other modes studied on the ($\frac{\pi}{2}$) sheet. It is also interesting to note that the $\Sigma$-- modes are best excited by sources containing $\delta u$ and $\delta r$, whereas, $\Pi$-- modes are best excited by sources containing $\delta w$ and $\delta \sigma$. This is to be expected, as the $\Sigma$-- modes are predominantly shear modes and the $\Pi$-- modes are predominantly dilatational modes.

**Generalizations.** Some generalizations concerning excitation functions result from this analysis. The form of the excitation functions seem to be little affected by the change of source depth; the amplitude, however, may be profoundly affected. While the relationship between amplitude and source depth is simple for the fundamental modes on both sheets, it may become quite complicated for the higher modes.
The form of the excitation functions appear to be roughly the inverse of the group velocity curve for the given mode. A relationship also noted for Love waves (Chang, 1968). This indicates that $\partial S/\partial f$ which appears in the denominator of the excitation function (46) --- is numerically related to the group velocity. The reason for this is not entirely clear.

The excitation of the $P$ pseudo-modes is of lower amplitude than that of the $S$ pseudo-modes. This may be the result of the low-velocity $P$-channel, as similar results have been reported for normal Love modes in the presence of a low-velocity $S$-channel (Andrianova, et al., pp. 39-45, 1965; Chang, 1968); on the other hand, it may be indirectly related to the fact that the amplitudes of $P$ is generally smaller than that of $S$. The tendency for the pseudo-modes to be best excited by sources in the low-velocity $P$-channel suggest that these pseudo-modes do, in effect, represent channel waves at the frequencies here considered.
SECTION VI:
COMPARISON WITH DATA

Introduction. In order to determine whether these leaking modes appear on a seismogram in the form theoretically computed, dispersion curves were computed from the record of a recent earthquake. The earthquake, located at 18.3N, 100.8W (Guerrero, Mexico) and at a depth of 60 km., which occurred September 25, 1966, at 6:02:26.4 GMT was recorded at the Rice seismographic station. The great circle distance from the epicenter to the recording station was 1381.1 km. or 12.43°. As the purpose was only to compare the form of the theoretical dispersion curves with those obtained from these data, only the vertical component of displacement was analyzed. The data, shown in Figure 30, have been doubly summed (first by 13, then by 15) and decimated by 5 to attenuate high frequencies (Blackman and Tukey, pp. 129-135, 1958).

Procedure. Dispersion curves were computed from these data using Sato's single-station method as modified by Chang (1968). The curves were computed from the phase spectra obtained in various time-frequency windows using the following formula (Chang, 1968):

\[ c = f / (2\pi n + 3\pi / 4 - \phi_f) \]

where \( r \) = the radial distance from the source, 
\( \phi_f \) = the phase at frequency, \( f \), and
\( n \) = an arbitrary, positive integer.
By varying n, one obtains a series of possible phase velocity curves for a given window. This time-frequency window should be chosen so that a single mode is isolated within it (Chang, 1968); this is obviously impossible in this case (see Figures 9 to 17). It is, however, possible to choose a window in which part of the pseudo-modes may be isolated, and this was done.

It should be noted here that the dispersion curves computed from the data are only approximate since the phase angle at the source is unknown and since the phase response of the recording instrument was not taken into account. Such factors, which are approximately constant over the frequency ranges studied, should not significantly alter the shape of the curves.

Results. The comparison of these dispersion curves with those computed for the model in question are shown in Figure 31. The shapes of the curves are very similar; only the comparison for $\tilde{\Phi}^–$ shows marked differences. In fact, the close fit, even with all the aforementioned possibilities for error, obtained for $S_1$ and $S_2$ appears to indicate that the $S$-wave velocity distribution in the model is approximately correct for the paths of these modes. The fits for $\tilde{\Phi}^+$, and for $P_1$, $P_2$, and $P_3$ are less good; this may indicate that the $P$-wave velocity distribution is less valid. The fit is, however, good enough to indicate that a low-velocity $P$-channel may be present.
This preliminary comparison shows that it is possible to extract these pseudo-modes from the record and that a comparison with the theory can be made with a view towards solving the "inverse problem." The partial derivatives computed by Woeber (1969) should be very useful for this purpose.
SECTION VII: CONCLUSIONS

Some important implications may be drawn from this study:

1. It is obvious that the complexity of the dispersion curves will make the use of leaking modes difficult, but this difficulty may be substantially reduced by using the concept of pseudo-modes. The pseudo-modes carry information on the P-wave velocity distribution not contained in normal modes; hence, they will be a useful adjunct to normal mode studies. They will also be useful in regions where the Herglotz-Wiechert transformation is inapplicable.

2. The presence of a low-velocity P-channel is most marked on the (++) sheet. In the frequency range analyzed, modes or pseudo-modes corresponding to head waves are not observed, but they are observed at higher frequencies (Woeber, 1969). The actual physical effect of the low-velocity P-channel is not entirely clear; computations now in progress may shed more light on this problem.

3. The dependence of the excitation functions on source depth is simple only for the fundamental modes. The form of these curves is generally inversely related to the group velocity curves and is little affected by source depth. As expected, the vertical displacement is greater for the ++-modes than is the radial displacement; further the ++- modes are optimally excited by a
source containing discontinuities in \( \omega \) or \( \sigma \). A similar relationship is seen for the \( \Sigma \) modes: in this case, the radial component is greater than the vertical component, and the modes are optimally excited by a source containing discontinuities in \( u \) or \( \tau \). It is expected that similar results would be noted on the \( (+-) \) sheet if the pseudo-modes were studied in this manner. Such results would probably not be apparent for the \( \Pi \) modes because these designations have lost their physical meaning.

4. The preliminary comparison of the theoretical dispersion curves with the approximate curves obtained from a seismogram indicate that leaking modes may be extracted, and comparisons made without extreme difficulty. The good fit for the \( \Sigma \) pseudo-modes indicates that the shear velocity distribution in the model is similar to that along the path of the seismic waves. The worse fit for the \( \Pi \) pseudo-modes indicates that the assumed P-wave velocity distribution is less accurate. The fit may, however, indicate that a low-velocity P-channel is present.

5. The results of this paper indicate that model fitting will be possible much as it is for normal modes, although it will be appreciably more difficult.
BIBLIOGRAPHY


REGION OF INTEREST ON THE 
(---) SHEET

REGION OF INTEREST ON THE 
(+--) SHEET

REGION OF INTEREST ON THE 
(+++) SHEET

3
0
35 km $a_1 = 6.58 \text{ km/sec}$ $\beta_1 = 3.55 \text{ km/sec}$ $\rho_1 = 2.9$
34 km $a_2 = 8.05 \text{ km/sec}$ $\beta_2 = 4.60 \text{ km/sec}$ $\rho_2 = 3.5$
69 km $a_3 = 7.75 \text{ km/sec}$ $\beta_3 = 4.31 \text{ km/sec}$ $\rho_3 = 3.41$
54 km
125 km

50 km

C $a_4 = 8.19 \text{ km/sec}$
225 km $\beta_4 = 4.55 \text{ km/sec}$ $\rho_4 = 3.6$
348 km $a_5 = 8.84 \text{ km/sec}$
102 km $\beta_5 = 4.92 \text{ km/sec}$ $\rho_5 = 3.8$
450 km $a_6 = 9.82 \text{ km/sec}$
203 km $\beta_6 = 5.4 \text{ km/sec}$ $\rho_6 = 3.95$
653 km $a_7 = 10.6 \text{ km/sec}$ $\beta_7 = 5.8 \text{ km/sec}$ $\rho_7 = 4.15$

4