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Non-tangential Homotopy Equivalences

by

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Introduction. At the Seattle conference on Differential and Algebraic Topology in 1963, William Browder raised the following question (Problem 43, [5]): "If a 1-connected manifold M is the homotopy type of a topological group is it homeomorphic to a Lie group? Using the technique of [2] Hilton constructed a principal $S^3$ bundle over $S^7$, now known as Hilton's criminal, which answers Browder's question negatively, but it does not have the homotopy type of a Lie group. This leads us to the following modified question: If a 1-connected manifold M is the homotopy type of a lie group is it homeomorphic to a Lie group? One of the main purposes of this thesis is to answer this modified question negatively for a certain class of Lie groups.

At the same Seattle conference a related problem was posed (Problem 44): "It would be interesting to get some fresh examples of H-manifolds; i.e., manifolds which can be given the structure of an H-space. The known examples are the Lie groups, the 7-sphere, the real projective 7-space, and combinations of these. How can one construct H-manifolds which are different from these either in the topological sense or in the sense of homotopy type?" Although Hilton's criminal may be added to the list of known H-manifolds above, the number of examples which are not homeomorphic to Lie groups is still remarkably small. However any negative examples to either Browder's question or its modified form would give new H-manifolds in the topological sense. We shall give a
construction in this paper which for certain simply connected Lie groups yields countably many topologically distinct manifolds, each homotopy equivalent to the given Lie group, but none homeomorphic to any Lie group. The theorem from which this follows can actually be stated in a somewhat more general setting:

**Theorem.** Suppose \( N \) is a smooth closed \( \pi \)-manifold of dimension \( 4k \) \((k \geq 2)\) and \( L \) is a smooth, closed, simply connected \( \pi \)-manifold. Then there exists a countable sequence of smooth closed manifolds \( \{ M_i \} \) having the following properties:

1. no \( M_i \) is a \( \pi \)-manifold,
2. each \( M_i \) is homotopy equivalent but not homeomorphic to \( N \times L \),
3. \( M_i \) is not homeomorphic to \( M_j \) if \( i \neq j \).

The proof of the theorem is an application of Browder-Novikov surgery and a product formula for surgery obstructions obtained by Sullivan and Rourke. The procedure is as follows: for each positive integer \( i \) take the identity map \( l_N : N^{4k} \rightarrow N^{4k} \) and "sharp in" an index surgery obstruction to obtain \( l_N \# f_i : N \# \hat{W}_i \rightarrow N \), where \( \hat{W}_i \) has index \( 8 \cdot r \) (\( r \) is an integer depending on only \( k \)) and \( f_i \) is the collapsing map of \( W_i \) onto a disk. The product formula shows that crossing this problem with \( l_L : L \rightarrow L \) eliminates the surgery obstruction. The result of surgery on \( (l_N \# f_i) \times l_L : N \# \hat{W}_i \times L \rightarrow N \times L \) is a manifold homotopy equivalent but not homeomorphic to \( N \times L \), and using different \( \hat{W}_i \)'s yields topologically different results.
Since any manifold which is homotopy equivalent to an H-manifold is an H-manifold, this construction drastically augments the examples of H-manifolds within certain homotopy types. One direction further investigations along these lines might take is to seek a classification theorem for H-manifolds homotopy equivalent to $N^{4k} \times L$. There is no particular reason to suppose that we have constructed them all, and Novikov's classification results [9] do not apply since the manifolds we construct are not stably tangentially homotopy equivalent to $N \times L$.

Because a connected account of Browder-Novikov surgery is not easily accessible, sections 1-8 are devoted to a survey of definitions, theorems, and constructions which pertain to this and other topics which are used strongly in the proof of the theorem. In sections 9 and 10 the particular surgery problem is set up, and the main properties of the solution are established. The proof of the principal theorem is completed in section 11, and section 12 contains the applications to Lie groups and H-manifolds.

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contributed many valuable suggestions throughout the entire course of it. Finally, I want to express my thanks to Miss Kathy Vigil, who did the typing, frequently under trying circumstances.
We begin by giving a descriptive account of Browder-Novikov surgery, the essential tool used in this thesis. Most of this material can be found in S.P. Novikov's paper [9], but the form in which it is presented here is condensed from the lecture notes of a course on this topic given by Professor E.H. Connell at Rice University in the spring of 1967. We shall mainly state theorems, describe constructions, and summarize results without going into proofs in detail. Standard definitions and theorems from algebraic topology, differential topology, and bundle theory will be assumed. Throughout this paper $D^n$ denotes the closed $n$-disk, $S^{n-1}$ denotes its boundary, the $(n-1)$-sphere, and smooth means of class $C^\infty$.

1. **The definition of framed surgery.**

   **Def. 1.1.** Suppose $M^n$ and $L^{n+N}$ are smooth manifolds and $M^n \xrightarrow{1} L^{n+N}$ is a smooth embedding with trivial normal bundle $E_N$. Then a **framing** of $M^n$ in $L^{n+N}$ is a trivialization of $E_N$, that is a bundle map

   \[
   \begin{array}{ccc}
   M^n \times D^N & \xrightarrow{h} & E_N \\
   \downarrow P_1 & & \downarrow P \\
   M^n & \xrightarrow{id.} & M^n \\
   \end{array}
   \]

   The pair $(M^n, h)$ is called a **framed manifold** in $L^{n+N}$. 
If \( \partial M \neq \emptyset \), we require \( i(\partial M) \subset \partial L \) and \( i(\text{int } M) \subset \text{int } L \) so that \( (\partial M, h|_{\partial M \times D^N}) \) is a framed manifold in \( \partial L \).

**Def. 1.2.** If \( (M_1, h_1) \) and \( (M_2, h_2) \) are framed manifolds in \( S^{n+N} \), and \( (W^{n+1}, H) \) is a framed manifold in \( S^{n+N} \times I \) with \( \partial W = \text{disjoint union of } M_1 \) and \( M_2 \), and \( H(M_i) \subset S^{n+N} \times i \) \((i=0,1)\); then \( (M_1, h_1) \) and \( (M_2, h_2) \) are **framed cobordant** provided that \( h_i = H|_{M_i} \times D^N \) \((i=1,2)\). \( (W^{n+1}, H) \) is a **framed cobordism** between \( (M_1, h_1) \) and \( (M_2, h_2) \). By composing cobordisms and smoothing the result [7], it is easily seen that framed cobordism is an equivalence relation on framed manifolds in \( S^{n+N} \).

A framed surgery on a framed manifold \( (M, h) \) in \( S^{n+N} \) is, roughly speaking, the construction of a new manifold \( M' \), obtained by altering \( M \) in a prescribed way, together with a framed cobordism \( (W, F) \) between \( (M, h) \) and \( (M', F|_{M' \times D^N}) \). The construction of \( M' \) and \( W \) can be described in the following way. Let \( S^1 \subset M \) be smoothly embedded with trivial normal bundle \( E_{n-1} \). Then \( M' = (M - \text{int } E_{n-1}) \cup_{\partial D^{i+1} \times S^{n-1-l}} (D^{i+1} \times S^{n-1-l}) \)

and \( W = M \times I \cup_{\partial D^{i+1} \times D^{n-1}} D^{i+1} \times D^{n-1} \) where the identification is made along \( M \times 1 \). Thus \( W \) is obtained by adding an \((i+1)\)-handle to \( M \times I \). When \( N \) is sufficiently large there is no problem in embedding \( D^{i+1} \times D^{n-1} \) in \( S^{n+N} \times I \) with trivial normal bundle. (In addition one may assume that this is done in such a way that \( D^{i+1} \times D^{n-1} \cap S^{n+N} \times 1 = D^{i+1} \times \partial D^{n-1} \). This condition implies \( W \cap S^{n+N} \times 1 = M' \).) From the framing of \( M \) we have a product framing on \( M \times I \) in \( S^{n+N} \times I \). The question is: can this framing be extended to a framing on the
whole cobordism $W$? Novikov shows that it can be if $n > i$, $N \geq 1$, and the inclusion induced homomorphism $\pi_i(0_{n-i}) \to \pi_i(0_{N+n-i})$ is surjective, where $0_k$ denotes the group of orthogonal linear transformations of $\mathbb{R}^k$, euclidean $k$-space.

The relevant homotopy theoretic information is that $\pi_k(0_k) \to \pi_k(0_{k+N})$ is surjective if $N \geq 2$ and $k \neq 1, 3, 7$. If $k = 1, 3, 7$, the cokernel is $\mathbb{Z}_2$.

2. The Novikov problem.

Def. 2.1. If $E \overset{p}{\to} M$ is a closed disk bundle, denote by $T(E)$ the Thom space of $E$ obtained by collapsing the boundary sphere bundle to a point. This point will be the base point of $T(E)$ and will be denoted $\ast$. Note that if $E$ is an oriented bundle, $T(E)$ is a pseudomanifold with orientation class $[T(E)]$.

Let $M^N$ be a closed, oriented, simply connected smooth manifold, $M^N \subset S^{n+N}$ a smooth embedding with normal bundle $\nu(M)$, and $f:(S^{n+N}, \ast) \to (T(\nu(M)), \ast)$ a map of degree $+1$. Changing $f$ by a small homotopy, we can make it transverse regular with respect to $M \subset T(\nu(M))$. Then (1) $f^{-1}(M)$ is a closed, oriented, smooth manifold, (2) $f|f^{-1}(M)$ has degree $+1$, (3) $f: \nu(f^{-1}(M)) \to \nu(M)$ is a bundle map, and (4) we may further assume that $f(S^{n+N}-\nu(f^{-1}(M))) = \ast \in T(\nu(M))$. The Novikov problem is to determine whether it is possible to find a map $\tilde{f}:(S^{n+N}, \ast) \to (T(\nu(M)), \ast)$, homotopic to $f$ so that (1)-(4) above are satisfied and $\tilde{f}|f^{-1}(M)$ is a homotopy equivalence. The technique is to kill the kernel of $f|f^{-1}(M)$ in the fundamental group and in homology beginning in dimension
zero and working step-wise up to the middle dimension $[\frac{n}{2}]$ by doing surgery (framed, if $\nu(M)$ is trivial) and extending the map to $S^{n+N} \times I$ each time. Poincaré duality then implies that $\mathcal{F}|\mathcal{F}^{-1}(M)$ is an isomorphism on homology. Since $M$ is simply connected, the relative Hurewicz isomorphism theorem implies that $\mathcal{F}|\mathcal{F}^{-1}(M)$ is an isomorphism on homotopy. By Whitehead's theorem, we have that $\mathcal{F}|\mathcal{F}^{-1}(M)$ is a homotopy equivalence. Since $\mathcal{F}^{-1}(M)$ is obtained from $f^{-1}(M)$ by sequence of (framed) surgeries, $\mathcal{F}^{-1}(M)$ is (framed) cobordant to $\mathcal{F}^{-1}(M)$.

3. Surgery below the middle dimension. We now make more explicit what it means to kill the homology kernel of $f$ (henceforth we write $f$ instead of $f|f^{-1}(M)$) and under what conditions this can be done in dimensions less than $[\frac{n}{2}]$. Since $\pi_1(f^{-1}(M))$ is finitely generated, the procedure described below also applies to the kernel of

$$\pi_1(f) : \pi_1(f^{-1}(M)) \longrightarrow \pi_1(M) = 0.$$

Let us suppose that we have the Novikov problem with the following hypotheses:

(i) $n \geq 5$, $N$ large (say $N > n$), $1 < i < [\frac{n}{2}]$

(ii) $f : (S^{n+N}, *) \longrightarrow (T(\nu(M)), \infty)$ is transverse regular with respect to $M$, $M_1 = f^{-1}(M)$, $M_2 = M$, $\pi_1(M) = 0$.

(iii) $H_j(f) : H_j(M_1) \longrightarrow H_j(M_2)$ is an isomorphism for $0 \leq j < i$ and $x \in \ker H_1(f)$.

We wish to examine the possibility of reducing $\ker H_1(f)$ by the subgroup generated by $x$. 
Step 1.  \( x \) is spherical (that is, in the image of the Hurewicz homomorphism). Some crucial facts which allow Novikov to do surgery in this context are that if \( f: (M, \partial M) \rightarrow (L, \partial L) \) is a map of degree +1, \( \ker H_1(f) \) is a direct summand of \( H_1(M, \partial M) \) and Poincaré duality and the universal coefficient theorem are valid for kernels. By this we mean that the usual Poincaré duality isomorphism restricts to give an isomorphism

\[
\begin{array}{c}
\text{PD: } \ker H^k(f) \xrightarrow{\approx} \ker H_{n-k}(f)
\end{array}
\]

and that the universal coefficient sequence restricts to the kernels to give, for example, \( \ker H^n(f) \approx \text{Hom}(\ker H_1(f), \mathbb{Z}) \oplus \text{Ext}(\ker H_{k-1}(f), \mathbb{Z}) \). Using these facts one can show that condition (iii) implies that the Hurewicz homomorphism maps \( \ker \pi_1(f) \) isomorphically onto \( \ker H_1(f) \).

Step 2. There is a smoothly embedded sphere \( S^i \subset M_1 \) representing \( x \). This follows by Whitney's embedding theorem [14] since \( 2i + 1 \leq n \).

Step 3. The closed disk normal bundle \( E_{n-i} \) of \( S^i \) in \( M_1 \) is trivial. Since \( \ker \pi_1(f) \approx \ker H_1(f) \), \( f|S^i \) is null homotopic. In fact \( f \) can be modified by a homotopy so that it sends all of \( E_{n-i} \) to a point. Since \( f \) is a bundle map of normal bundles, \( f \) pulls back tangent bundles stably, i.e., \( \tau(M_1) \approx f^* \tau(M_2) \). Restricting, we obtain \( \tau(M_1) \mid S^i \approx f^* \tau(M_2) \mid \text{point} \), which is trivial. Since \( \tau(S^i) \) is stably trivial, the standard equivalence \( \tau(M_1) \mid S^i \approx \tau(S^i) \oplus E_{n-i} \) implies that \( E_{n-i} \) is stably trivial. However the dimension of the fiber \( D^n \) is
larger than the dimension of the base $S^1$, so $E_{n-1}$ is actually trivial.

**Step 4.** $\pi_1(0_{n-1}) \to \pi_1(0_{n-1+N})$ is surjective since $i < n-1$ and $N \geq 2$.

In this case the framed surgery described in section 2 can be done and the map $f$ can be extended to the cobordism and to a map $F$ on $S^{n+N} \times I$ in such a way that $(F|S^{n+N}\times 1)^{-1}(M_2) = M_1$, the manifold $M_1$ after surgery on $x$. $F$ will be the required homotopy of $f$ and $F^{-1}(M)_{2}$ will be the cobordism between $M_1$ and $M_1$. The effect on the homology is to leave $H_j(M) (0 \leq j \leq i)$ undisturbed and to make $H_i(M) \cong H_i(M)/(x)$ where $(x)$ denotes the subgroup generated by $x$. Since ker $H_i(f)$ is finitely generated, a finite number of these surgeries makes $H_i(f)$ an isomorphism.

4. **Surgery in the middle dimension.**

We now survey the steps of surgery when $i = \lfloor \frac{n}{2} \rfloor$. Let $x \in \ker H_i(f)$ and suppose $H_j(f)$ is an isomorphism for $j < i$.

**Step 1.** $x$ is spherical by the same arguments as before. They did not depend on the fact the $i < \lfloor \frac{n}{2} \rfloor$.

**Step 2.** $x$ is represented by a differentiably embedded sphere. Since $n \geq 5$ and $\pi_1(M_1) = 0$, Whitney's embedding theorem is applicable even if $n = 2i$. 
Step 3. As before the normal bundle of $S^1 \subset M_1$ is stably trivial. Now if $n = 2i + 1$, the dimension of the fiber is greater than the dimension of the base, so it is trivial. If $n = 2i$ ($i = 3,7$), every stably trivial $D^i$ bundle over $S^1$ is trivial. If $n = 2i$ ($i$: even), the stably trivial $D^i$ bundles over $S^1$ are represented by the kernel of $\pi_{i-1}(0_i) \rightarrow \pi_{i-1}(0_{i+1})$, which is isomorphic to $\mathbb{Z}$. The Euler number of such a bundle is always even, and it is zero if and only if the bundle is trivial. It is a theorem that the Euler number is given by the intersection of $x$ with itself as a homology class so that the normal bundle is trivial if and only if $[x \cdot x] = 0$. The possibility of selecting $x$ to satisfy this condition will be discussed in section 5. If $n = 2i$ ($i$: odd, $i \neq 3,7$), it is not always possible to select $x$ so that the normal bundle will be trivial. This problem will be treated in section 6.

Step 4. If $n \neq 6,14$ and $N \geq 2$, $\pi_i(0_{n-1}) \rightarrow \pi_i(0_{n-1+N})$ is surjective so that the framing can be extended.

When $n$ is odd, we have seen that surgery can be done on any element of $\text{ker } H_1(f)$. By a rather complicated argument whose essential features may be found in the paper of Milnor and Kervaire [4], it can be shown that surgery may always be performed in such a way as to completely kill the kernel and make $f$ a homotopy equivalence. When $n$ is even, there is a problem with triviality of normal bundles, and this naturally divides into the two cases $n = 4k$ and $n = 4k+2$, 
which will be handled separately. There is also a framing problem when \( n = 6 \) or \( 14 \). The effect of middle dimensional surgery on the homology for even values of \( n \) is taken up in Theorem 8 of the next section.

5. The index of a 4k-manifold. Suppose \( V \) is an \( n \)-dimensional vector space over \( \mathbb{Q} \) and \( \psi : V \times V \rightarrow \mathbb{Q} \) is a

(1) symmetric, (2) non-singular, (3) bilinear form, i.e.,

(1) \( \psi(x,y) = \psi(y,x) \), (2) for any basis \( u_1, \ldots, u_n \) of \( V \),

\[ \det (\psi(u_i,u_j)) \neq 0, \]

(3) \( \psi(x+y,z) = \psi(x,z) + \psi(y,z) \) and

\[ \psi(x,y+z) = \psi(x,y) + \psi(x,z). \]

**Theorem 5.1.** There exists a basis \( u_1, \ldots, u_n \) for \( V \)

such that \( \psi(u_i,u_j) = t_i \delta_{ij} \) where \( t_i \in \mathbb{Q}, t_i > 0, i = 1, \ldots, r; t_i < 0, i = r+1, \ldots, n; \) and \( \delta_{ij} \) is the Kronecker delta. Moreover, \( r \) is independent of the choice of basis.

**Def. 5.1.** The index of the form \( \psi \), denoted \( I(\psi) \),

is \( r - (n-r) = 2r - n \).

**Theorem 5.2.** Suppose \( n = 2k, W \subseteq V \) is a vector subspace

of dimension \( k \), and \( \psi(x,y) = 0 \) for every \( x, y \in W \). Then \( I(\psi) = 0 \).

**Def. 5.2.** Let \( M^{4k} \) be a closed, oriented, topological manifold. Then \( \psi : H_{2k}(M; \mathbb{Q}) \times H_{2k}(M; \mathbb{Q}) \rightarrow \mathbb{Q} \) given by

\[ \psi(x,y) = [x \cdot y] \]

is a symmetric, non-singular, bilinear form. Define the index of \( M^{4k} \), \( I(M^{4k}) \), to be \( I(\psi) \).

(The intersection pairing \( [ \cdot, \cdot ] \) above is defined by

\[ [x \cdot y] = \langle \text{PD}^{-1}(y), x \rangle \]

where \( \text{PD} : H^{2k}(M) \rightarrow H_{2k}(M) \) is the Poincaré duality isomorphism.)
Theorem 5.3. If $M^{4k}_1$ and $M^{4k}_2$ are closed, oriented, topological manifolds, (1) $I(M_1 \cup M_2) = I(M_1) + I(M_2)$ where the union is disjoint, (2) $I(-M_1) = -I(M_1)$, (3) $I(M_1 \times M_2) = I(M_1) \cdot I(M_2)$, (4) if there is a compact, oriented manifold $N^{4k+1}$ such that $\partial N = M_1 - M_2$, then $I(M_1) = I(M_2)$.

Note that (4) says index is an oriented cobordism invariant. This is equivalent to the condition that manifolds which bound have zero index.

Theorem 5.4. (Hirzebruch [3]) For each positive integer $k$ there exists a polynomial $L_k$ of degree $k$ with rational coefficients such that if $M^{4k}$ is any closed, oriented, smooth manifold and $p_1 \in H^{4i}(M; \mathbb{Z})$ are the Pontrjagin classes of $M$, then $I(M^{4k}) = \langle L_k(p_1, \ldots, p_k), [M^{4k}] \rangle$.

Theorem 5.5. (Corollary of Theorem 5.4) Suppose $n = 4k$, $M^n_1$ and $M^n_2$ are closed, oriented, smooth manifolds, and $f: M_1 \rightarrow M_2$ is a degree 1 stable tangential equivalence. Then $I(M_1) = I(M_2)$ and thus the intersection pairing restricted to $\ker H_{2k}(f; \mathbb{Q})$ has index zero.

The proof of the second part of this theorem follows from the fact that under the hypotheses the index of the kernel of $H_{2k}(f; \mathbb{Q}) = I(M_1) - I(M_2)$.

This is the crucial fact which allows one to find homology classes represented by spheres whose normal bundles are trivial and such that doing surgery on them completely kills the kernel. We note that Theorem 5.5 requires that $f$ be a stable tangential map. Later we will want to consider cases where surgery is not possible in the middle dimension because $f$ is
not a stable tangential equivalence and the index of the kernel is not zero.

Using Poincaré duality and the universal coefficient theorem for kernels, it follows easily that \( \ker H_{2k}(f) \) is free if \( \ker H_i(f) = 0, \ i < \frac{n}{2} \). This makes the next sequence of definitions and theorems relevant to our discussion.

**Def. 5.3** Suppose \( G \) is a finitely generated free abelian group and \( \varphi: G \times G \to \mathbb{Z} \) is a symmetric, non-singular, bilinear form. (This time non-singular means there is a free abelian basis \( g_1, \ldots, g_n \) for \( G \) such that \( \det(\varphi(g_i, g_j)) = \pm 1 \).) Let \( V = G \otimes \mathbb{Q} \) so that \( V \) is a vector space over \( \mathbb{Q} \) and if \( g_1, \ldots, g_n \) is a free abelian basis for \( G \), \( g_1 \otimes 1, \ldots, g_n \otimes 1 \) is a basis for \( V \). Then we may define \( \psi: V \times V \to \mathbb{Q} \) by \( \psi(\varphi(a \otimes r_1, b \otimes r_2) = r_1 r_2 \cdot \varphi(a, b) \) so that \( \psi \) is a symmetric, nonsingular, bilinear form. Define the index of \( \varphi \), \( I(\varphi) \), to be \( I(\psi) \).

**Theorem 5.6** \([6]\) Suppose the rank of \( G \) is \( n \) and \( -n < I(\varphi) < n \). Then there is a non-zero element \( g \in G \) such that \( \varphi(g, g) = 0 \).

**Theorem 5.7** \([6]\) Suppose \( G \) has rank \( n \), \( I(\varphi) = 0 \), and \( \varphi(g, g) \) is even for every \( g \in G \). Then there exists a free abelian basis \( g_1, \ldots, g_n \) for \( G \) such that the matrix \( (\varphi(g_i, g_j)) \) has the form

\[
\begin{pmatrix}
0 & 1 & & & \\
1 & 0 & & & \\
& & & \ddots & \\
& & & & 0 & 1 \\
& & & & 1 & 0
\end{pmatrix}
\]
An immediate corollary is that if \( \varphi: G \times G \to \mathbb{Z} \) is a symmetric, non-singular, bilinear form with \( \varphi(g, g) \) even and \( I(\varphi) = 0 \), then \( G \) has even rank.

Now we are ready to state the theorem which describes the effect on homology of doing surgery in the middle dimension when \( n \) is even.

**Theorem 5.8** Suppose \( M^{2l} \) is a compact, oriented, topological manifold, \( l \geq 2 \), \( \lambda_1 \) is a spherical class in \( H^1_t(M), (i=1, 2) \). Suppose \( [\lambda_1 \cdot \lambda_2] = \pm 1 \), \( [\lambda_1 \cdot \lambda_2] = \frac{1}{\lambda_2 \cdot \lambda_2] = 0} \). Also suppose there exists a subgroup \( G \) of \( H^1_t(M) \) with \( H^1_t(M) = [\lambda_1, \lambda_2] \oplus G \) and \( [\lambda_1 \cdot g] = 0 (i=1, 2), g \in G \). Let \( \overline{M} \) be \( M \) after surgery on \( \lambda_1 \). Then \( H^1_i(M) \approx H^1_i(M) \) for \( 0 \leq i < l \) and \( H^1_t(M) \approx G \).

So we see that under these conditions surgery on \( \lambda_1 \) kills the subgroup generated by \( \lambda_1 \) and \( \lambda_2 \). Now in the case \( n = 4k \) we can use Theorems 5.5 and 5.7 to apply Theorem 5.8.

**Theorem 5.9.** Suppose we have the Novikov problem and \( n = 4k \). Then there exists a map \( \tilde{f}: (S^{n+N}, \ast) \to (T^N(M), \ast) \), homotopic to \( f \), transverse regular with respect to \( M \), and such that \( \tilde{f}|\tilde{f}^{-1}(M): \tilde{f}^{-1}(M) \to M \) is a homotopy equivalence.

**Proof.** By doing surgery we can make \( H_i(f) \) an isomorphism for \( i < 2k \) and \( f^{-1}(M) \) simply connected. Then \( \ker H_{2k}(f) \) is free abelian, and the intersection pairing gives a symmetric, non-singular, bilinear form. Also if \( \lambda \in \ker H_{2k}(f) \), \( [\lambda \cdot \lambda] \) is even because, as we remarked in section 4, the Euler number of a stably trivial \( D^{2k} \) bundle over \( S^{2k} \) is even. By Theorem
5.5 the index of the intersection pairing restricted to the kernel is zero. By Theorem 5.7 there is a free abelian basis $g_1, \ldots, g_{2t}$ of $\ker H_{2k}(f)$ such that the matrix $[[g_i \cdot g_j]]$ has the form described in Theorem 5.7. Since $[g_i \cdot g_i] = 0$ for $i = 1, \ldots, 2t$, each $g_i$ is represented by an embedded sphere whose normal bundle has zero Euler number and hence is trivial, so surgery can be performed on the $g_i$'s. Since

$$\pi_k(0_k) \to \pi_k(0_{k+N})$$

is surjective for all even $k$, $N \geq 1$, we can extend the framing. By Theorem 5.8 these surgeries kill the kernel.

Q.E.D.

Now it is apparent what distinguishes the case $n = 4k$ from the case $n = 4k + 2$. In the latter case the intersection pairing is skew symmetric instead of symmetric, so index considerations cannot be applied. However Theorem 5.8 will still be applicable to the extent that the hypotheses can be obtained. It is this case which we now discuss briefly.

6. The Kervaire invariant of an $(4k+2)$-manifold.

Suppose $G$ is a finitely generated free abelian group and

$$[\cdot \cdot] : G \times G \to \mathbb{Z}$$

is a skew symmetric, non-singular, bilinear form.

**Def. 6.1.** A symplectic basis for $G$ is a free abelian basis $x_1, \ldots, x_r, y_1, \ldots, y_r$ such that $[x_i \cdot x_j] = [y_i \cdot y_j] = 0$ for all $i$ and $j$, $[x_i \cdot y_j] = \delta_{ij}$ for all $i$ and $j$.

This gives a matrix of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so we may reorder the basis elements and get $g_1, \ldots, g_{2r}$ such
that \((g^i_j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Theorem 6.1. If the form \([\cdot,\cdot]\) exists, then there is a symplectic basis for \(G\). Thus the rank of \(G\) is even.

Theorem 6.2. (C. Arf [1]) If \(\varphi: G \rightarrow \mathbb{Z}_2\) is a function satisfying \(\varphi(x+y) = \varphi(x) + \varphi(y) + [x \cdot y]_2\) (where \([x \cdot y]_2\) denotes reduction mod 2 of \([x \cdot y] \in \mathbb{Z}\)) and \(x_1, \ldots, x_s, y_1, \ldots, y_s\) and \(\overline{x}_1, \ldots, \overline{x}_s, \overline{y}_1, \ldots, \overline{y}_s\) are two symplectic bases for \(G\), then
\[
\sum_{i=1}^{s} \varphi(x_1) \varphi(y_1) = \sum_{i=1}^{s} \varphi(\overline{x}_1) \varphi(\overline{y}_1).
\]

Def. 6.2. Suppose \(G\) is a finitely generated free abelian group, \([\cdot,\cdot] : G \times G \rightarrow \mathbb{Z}\) is a skew symmetric, non-singular, bilinear form, \(x_1, \ldots, x_s, y_1, \ldots, y_s\) is a symplectic basis for \(G\), and \(\varphi: G \rightarrow \mathbb{Z}_2\) is a function satisfying \(\varphi(x+y) = \varphi(x) + \varphi(y) + [x \cdot y]_2\). Then the Arf invariant of \([\cdot,\cdot],\varphi\), denoted \(\hat{\varphi}\), is defined by

\[
\sum_{i=1}^{s} \varphi(x_1) \varphi(y_1).
\]

Theorem 6.3. [4] If \(\hat{\varphi} = 0\), then there is a symplectic basis \(x_1, \ldots, x_s, y_1, \ldots, y_s\) such that \(\varphi(x_1) = 0\ (i = 1, \ldots, s)\). If \(\hat{\varphi} = 1\), there is a symplectic basis \(x_1, \ldots, x_s, y_1, \ldots, y_s\) such that \(\varphi(x_1) = 0\ (i = 2, \ldots, s)\) and \(\varphi(x_1) = \varphi(y_1) = 1\).

Def. 6.3. Suppose \((M^n, h)\) is a framed manifold in \(S^{n+N}\) and \(n = 4k+2\). Then we define the Kervaire invariant of \((M, h)\) as follows. Perform framed surgery on \(M\) to make it 2k-connected,
to obtain a framed manifold \((M', h')\) which is framed cobordant to \((M, h)\) and such that \(\pi_1(M') = 0\) and \(H_1(M') = 0\) for \(i < 2k + 1\). Then \(H_{2k+1}(M')\) is a finitely generated, free abelian group and the intersection pairing

\[ [\cdot, \cdot] : H_{2k+1}(M') \times H_{2k+1}(M') \longrightarrow \mathbb{Z} \]

is skew symmetric, non-singular, and bilinear. For \(x \in H_{2k+1}(M')\) define \(\varphi(x) = 0\) if the normal bundle of the sphere representing \(x\) is trivial, define \(\varphi(x) = 1\) if its normal bundle is the non-trivial bundle over \(S^{2k+1}\). Then \(\varphi\) satisfies \(\varphi(x + y) = \varphi(x) + \varphi(y) + [x \cdot y]_2\) [9]. The Kervaire invariant of \(M\), \(KI(M)\), is the Arf invariant of \([\cdot, \cdot], \varphi\).

Pertinent facts about the Kervaire invariant are that it is a framed cobordism invariant and it is known to be zero unless \(4k + 2 = 2^t - 2\) \((k > 3)\). For \(k = 1, 3\) the Kervaire invariant depends on the framing. In dimensions 30 and 62 it is known that there are \(\pi\)-manifolds which have Kervaire invariant 1 for any framing, and it is conjectured that this is true for all dimensions of the form \(2^t - 2\) \((t > 4)\).

Now let us see how the Kervaire invariant fits into the Novikov problem in dimension \(4k+2\).

**Theorem 6.4.** Suppose we have the Novikov problem for \(n = 4k + 2, 2k + 1 \neq 3, 7\) \((\text{since } n \geq 5, 2k+1 \neq 1)\). Then there is a map \(\tilde{\gamma} : (S^{n+N}, \ast) \longrightarrow (T_N(M), \ast)\), homotopic to \(\gamma\), transverse regular with respect to \(M\), and such that

\begin{align*}
(1) & \quad \pi_1(\tilde{\gamma}^{-1}(M)) = 0 \\
(ii) & \quad H_1(\tilde{\gamma}) : H_1(\tilde{\gamma}^{-1}(M)) \xrightarrow{\approx} H_1(M), i \neq 2k + 1 \\
(iii) & \quad \ker H_k(\tilde{\gamma}) \approx \mathbb{Z} \oplus \mathbb{Z} \text{ or } 0 \text{ depending on whether } \tilde{\gamma} = 1 \text{ or } 0.
\end{align*}
Proof. Kill \( \pi_1(f^{-1}(M)) \) and the kernel of \( H_4(f) \) up to the middle dimension as usual. Then the intersection pairing on \( \ker H_{2k+1}(f) \) is skew symmetric, non-singular, and bilinear. For \( x \in \ker H_{2k+1}(f) \) define \( \varphi(x) = 0 \) or 1 depending on whether or not the normal bundle of the sphere representing \( x \) is trivial. (The situation here is quite similar to the index case; namely, the Arf invariant \( \delta \) of the kernel equals \( \text{KI}(f^{-1}(M)) - \text{KI}(M) \).) If \( \delta = 0 \), Theorem 6.3 gives a symplectic basis for \( \ker H_{2k+1}(f) \) \( x_1, \ldots, x_s, y_1, \ldots, y_s \) so that \( \varphi(x_i) = 0 \), \( i = 1, \ldots, s \). Then the normal bundle of the sphere representing \( x_i \) is trivial and surgery can be performed. Theorem 5.8 implies that the homology is killed. If \( \delta = 1 \), \( \varphi(x_i) = 0 \) \( i = 2, \ldots, s \), so everything but \( x_1 \) and \( y_1 \) can be killed by surgery leaving \( \ker H_{2k+1}(\mathcal{F}) \approx \mathbb{Z} \oplus \mathbb{Z} \). Since \( \pi_1(O_1) \to \pi_1(O_{1+N}) \) is surjective when \( i \neq 3,7, N > 1 \), the framing can be extended.

The cases \( n = 6, 14 \) are similar to the above using the Arf invariant of the skew symmetric intersection pairing. But in these cases the normal bundles are trivial, and the problem is the framing since \( \pi_1(O_1) \to \pi_1(O_{1+N}) \) has cokernel \( \mathbb{Z}_2 \) for \( i = 3,7, N > 1 \). Thus it is possible to have \( \mathbb{Z} \oplus \mathbb{Z} \) left in the kernel of \( H_{2i}(\mathcal{F}) \) for \( i = 3,7 \). If this happens, surgery can still be done to kill the kernel completely, but the framing will not extend.

We may now summarize the Novikov problem for closed manifolds. If \( n \) is odd or \( n = 4k \), there always exists a map \( f: (S^{n+N}, *) \longrightarrow (T_N(M), -) \), homotopic to \( f \) and
to obtain a framed manifold \((M', h')\) which is framed cobordant to \((M, h)\) and such that \(H_i(M') = 0\) and \(H_i(M') = 0\) for \(i < 2k + 1\). Then \(H_{2k+1}(M')\) is a finitely generated, free abelian group and the intersection pairing

\[ [\cdot, \cdot] : H_{2k+1}(M') \times H_{2k+1}(M') \rightarrow \mathbb{Z} \]

is skew symmetric, non-singular, and bilinear. For \(x \in H_{2k+1}(M')\) define \(\varphi(x) = 0\) if the normal bundle of the sphere representing \(x\) is trivial, define \(\varphi(x) = 1\) if its normal bundle is the non-trivial bundle over \(S^{2k+1}\). Then \(\varphi\) satisfies 

\[ \varphi(x+y) = \varphi(x) + \varphi(y) + [x \cdot y]_2 \quad (\forall x, y) \]

The Kervaire invariant of \(M\), \(KI(M)\), is the Arf invariant of \(\varphi\).

Pertinent facts about the Kervaire invariant are that it is a framed cobordism invariant and it is known to be zero unless \(4k+2 = 2^t - 2 + 1\). For \(k = 1, 3\) the Kervaire invariant depends on the framing. In dimensions 30 and 62 it is known that there are \(\mathbb{R}\)-manifolds which have Kervaire invariant 1 for any framing, and it is conjectured that this is true for all dimensions of the form \(2^t - 2 + 1\) (\(t > 4\)).

Now let us see how the Kervaire invariant fits into the Novikov problem in dimension \(4k+2\).

**Theorem.** Suppose we have the Novikov problem for 

\[ n = 4k + 2, i = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \] (since \(n > 5, 2k+1 \neq 1\). Then there is a map

\[ f : (S^{n+N}, *) \rightarrow (T_N(M), *) \]

homotopic to \(f\), transverse regular with respect to \(M\), and such that

\[ (i) \quad H_i(f^{-1}(M)) = 0 \]

\[ (ii) \quad H_i(f^{-1}(M)) \rightarrow H_i(M), \quad i \neq 2k+1 \]

\[ (iii) \quad \text{for } H_1(?), \quad \mathbb{Z} \div \mathbb{Z} \text{ or } 0 \text{ depending on whether } \]

\[ n = 1 \text{ or } 0 \]
Proof. Kill \( \pi_1(f^{-1}(M)) \) and the kernel of \( H_4(f) \) up to the middle dimension as usual. Then the intersection pairing on \( \ker H_{2k+1}(f) \) is skew symmetric, non-singular, and bilinear. For \( x \in \ker H_{2k+1}(f) \) define \( \varphi(x) = 0 \) or 1 depending on whether or not the normal bundle of the sphere representing \( x \) is trivial. (The situation here is quite similar to the index case; namely, the Arf invariant \( \check{\varphi} \) of the kernel equals \( KI(f^{-1}(M)) - KI(M) \).) If \( \varphi = 0 \), Theorem 6.3 gives a symplectic basis for \( \ker H_{2k+1}(f) x_1, \ldots, x_s, y_1, \ldots, y_s \) so that \( \varphi(x_i) = 0 \), \( i = 1, \ldots, s \). Then the normal bundle of the sphere representing \( x_i \) is trivial and surgery can be performed. Theorem 5.8 implies that the homology is killed. If \( \varphi = 1 \), \( \varphi(x_1) = 0 \) \( i = 2, \ldots, s \), so everything but \( x_1 \) and \( y_1 \) can be killed by surgery leaving \( \ker H_{2k+1}(f) \cong \mathbb{Z} \oplus \mathbb{Z} \). Since \( \pi_1(O_i) \to \pi_1(O_{i+N}) \) is surjective when \( i \neq 3, 7, N > 1 \), the framing can be extended.

The cases \( n = 6, 14 \) are similar to the above using the Arf invariant of the skew symmetric intersection pairing. But in these cases the normal bundles are trivial, and the problem is the framing since \( \pi_1(O_i) \to \pi_1(O_{i+N}) \) has cokernel \( \mathbb{Z}_2 \) \( i = 3, 7, N > 1 \). Thus it is possible to have \( \mathbb{Z} \oplus \mathbb{Z} \) left in the kernel of \( H_{2i}(f) \) for \( i = 3, 7 \). If this happens, surgery can still be done to kill the kernel completely, but the framing will not extend.

We may now summarize the Novikov problem for closed manifolds. If \( n \) is odd or \( n = 4k \), there always exists a map \( f: (S^{n+N}, *) \to (T_N(M), *) \), homotopic to \( f \) and
transverse regular with respect to $M$ such that $f|f^{-1}(M)$: $f^{-1}(M) \rightarrow M$ is a homotopy equivalence. If $n = 4k + 2$, there is an $f'$ homotopic to $f$ such that $H_1(f|f^{-1}(M))$ is an isomorphism for $i \neq 2k + 1$ and $\ker H_{2k+1}(f|f^{-1}(M)) = 0$ or $\mathbb{Z} \oplus \mathbb{Z}$ depending on whether or not the Arf invariant of the intersection pairing on $\ker H_{2k+1}(f|f^{-1}(M))$ vanishes.

7. **Plumbing.** Let $E^k$ be the total space of the tangent disk bundle of $S^k$, $D^k_1$ a small disk in $S^k$, and let

$$
\begin{align*}
D^k_1 \times D^k & \xrightarrow{f} E^k \\
D^k_1 & \xrightarrow{\text{incl.}} S^k
\end{align*}
$$

be a local trivialization of $E^k$, i.e., $f$ is a bundle map. Denote by $E_1$ and $E_2$ distinct copies of $E^k$ and by $f_i$ the map $f$ into $E_i$, $i = 1, 2$.

**Def. 7.1.**

$$
X^{2k} = \frac{E_1 \cup E_2}{f_1(x, y) \sim f_2(y, x)}
$$

be obtained by plumbing together two copies of the tangent disk bundle of $S^k$.

**Theorem 7.1.** (i) $X^{2k}$ is a smooth $\pi$-manifold of dimension $2k$.

(ii) $\partial X^{2k}$ is a homotopy sphere (a smooth manifold which has the homotopy type of a sphere).

**Theorem 7.2.** The following matrix $A$ is symmetric non-singular, takes on even values, and has index 8.
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

By this we mean that if \( G \) is a free abelian group of rank 8, \( g_1, \ldots, g_8 \) is a free basis for \( G \), and a bilinear function \( \varphi: G \times G \rightarrow \mathbb{Z} \) is defined by \( \varphi(g_i, g_j) = A_{ij} \); then \( \varphi \) is symmetric and non-singular, \( \varphi(x, x) \) is even for every \( x \in G \), and \( I(\varphi) = 8 \).

**Def. 7.2.** Construct the smooth manifold \( Y^{4k} \) by plumbing together 8 copies of the tangent disk bundle of \( S^{2k} \) according to the following scheme.

\[
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc
\end{array}
\]

**Theorem 7.3.** (i) \( Y \) has the homotopy type of the one point union of 8 \( 2k \) - spheres. Hence it is \((2k-1)\) - connected and \( H_{2k}(Y) = \bigoplus_8 \mathbb{Z} \).

(ii) \( Y \) is a \( \pi \)-manifold

(iii) \( \partial Y \) is a homotopy sphere

(iv) If the 0-sections of the disk bundles used to construct \( Y \) are taken as the free abelian basis \( g_1, \ldots, g_8 \) for \( H_{2k}(Y) \), the matrix of intersections \( [g_i \cdot g_j] \) is the matrix \( A \). Hence \( I(Y) = 8 \).

**Def. 7.3.** If \( M^1 \) and \( M^2 \) are closed, oriented smooth manifolds, denote by \( M_1 \# M_2 \) the connected sum of \( M_1 \) and \( M_2 \).
If $M_1^n$ and $M_2^n$ are oriented, smooth manifolds with non-empty connected boundary, denote by $M_1 \# M_2$ the connected sum along the boundary of $M_1$ and $M_2$. Note that $\partial(M_1 \# M_2) = \partial M_1 \# \partial M_2$ in the bounded case.

**Theorem 7.4.** For any positive integer $r$, $Y_4^{k \# \ldots \# Y_4^{k}}_r$ is a $\pi$-manifold with index $8r$ and boundary $\partial Y_4^{k \# \ldots \# \partial Y_4^{k}}_r$.

8. **Relative surgery for bounded manifolds.**

We shall also need to refer to the relative Novikov problem for bounded manifolds, so we give a brief summary of the results.

The hypotheses for the relative Novikov problem are the following:

(i) $n \geq 5$, $N > n$

(ii) $M^n$ is a simply connected, compact, oriented smooth manifold

(iii) $(M^n, \partial M^n) \subset (D^{n+N}, \partial D^{n+N})$ is a proper, smooth embedding with trivial normal bundle $\nu(M)$. (Proper means $\partial M^n \subset \partial D^{n+N}$ and $\text{int } M^n \subset \text{int } D^{n+N}$.)

(iv) $f: (D^{n+N}, \partial D^{n+N}) \rightarrow (T(\nu(M)), T(\nu(\partial M)))$ is a proper map of degree $+1$.

(v) $f|_{\partial D^{n+N}}$ is transverse regular with respect to $\partial M$

(vi) $H_1(f|_{f^{-1}(\partial M)}): H_1(f^{-1}(\partial M)) \rightarrow H_1(\partial M)$ is an isomorphism for $i = \lfloor \frac{n}{2} \rfloor$ and $i = \lceil \frac{n}{2} \rceil - 1$.

**Theorem 8.1.** Suppose we have the relative Novikov problem. Then $f$ is homotopic relative to $\partial D^{n+N}(\text{leaving } \partial D^{n+N} \text{ fixed})$ to a smooth map $\tilde{f}$ which is transverse regular with respect to
M such that \( \pi_1(\tilde{f}^{-1}(M)) = 0 \) and \( H_j(\tilde{f}^{-1}(M)) \cong H_j(M) \) for \( 0 \leq j < \left[ \frac{n}{2} \right] \). Moreover \( \tilde{f} \) satisfies

1. For \( n = 2k + 1 \), \( H_k(\tilde{f}^{-1}(M)) \cong H_k(M) \)

2. For \( n = 2k \), \( k \) even, if \( I(f^{-1}(M)) = I(M) \), \( H_k(\tilde{f}^{-1}(M)) \cong H_k(M) \).

3. For \( n = 2k \), \( k \) odd, \( k \neq 3, 7 \), \( \text{ker } H_k(\tilde{f}|\tilde{f}^{-1}(M)) \approx \mathbb{Z} \oplus \mathbb{Z} \) or 0 depending on whether the Arf invariant defined by triviality of normal bundles is 1 or 0.

4. For \( n = 6, 14 \), \( \text{ker } H_k(\tilde{f}|\tilde{f}^{-1}(M)) = \mathbb{Z} \oplus \mathbb{Z} \) or 0 depending on whether the Arf invariant defined by the possibility of extending framing is 1 or 0. (As in the closed case, surgery can always be performed, but the framing will not extend if the Arf invariant is 1.)

**Theorem 8.2.** Suppose we have the relative Novikov problem and \( f|f^{-1}(\partial M) \) is a homotopy equivalence. If we have a case where \( \text{ker } H_k(\tilde{f}|\tilde{f}^{-1}(M)) = 0 \), then \( \tilde{f}|\tilde{f}^{-1}(M) \) is also a homotopy equivalence.

Note that the outstanding difference between the closed case and the bounded case is the necessity to assume \( I(f^{-1}(M)) = I(M) \) in order to complete the surgery. This happens because Theorem 5.4 does not hold for bounded manifolds. Thus in the bounded case, the index may be a non-vanishing obstruction to surgery in dimension 4k. This concludes the survey of background material.
9. **Construction of the surgery problem.** We now proceed to the construction of the specific surgery problem whose solution gives the result we are working toward.

Consider the bounded manifold $Y^{4k}$ described in Theorem 7.3, and let $r$ be the order of $\partial Y$ in the group of homotopy spheres $bP_{4k}$ [4]. If $W$ is the connected sum $Y \# \ldots \# Y$, $\partial W$ is diffeomorphic to $S^{4k-1}$ by the choice of $r$. Attaching $r$ a $4k$-disk to $W$ by a diffeomorphism on the boundary, we obtain a closed, smooth manifold $\hat{W}$.

**Proposition 9.1.** $\hat{W}$ is a closed smooth manifold satisfying

1. $\text{I}(\hat{W}) = 8r$,
2. $\hat{W}$ is $(2k-1)$-connected
3. $\hat{W}$ is not a $\pi$-manifold.

**Proof.** (1) and (2) follow from the corresponding facts about $W$. Since all the Pontrjagin classes of a $\pi$-manifold are zero, Hirzebruch's index theorem (Theorem 5.4) implies that all smooth, closed $\pi$-manifolds have index zero. By (1) we conclude that $\hat{W}$ could not be a $\pi$-manifold. Note, however, that removing the interior of a small disk from $\hat{W}$ yields a bounded manifold with trivial tangent bundle, for instance $W$.

Let $f: W^{4k} \to D^{4k}$ be defined by the identity on the boundary, stretching a collar of $\partial W$ over $D^{4k}$, and sending the remainder of $W$ to a point. In other words, if $c: \partial W \times I \to W$ is an embedding satisfying $c(x, 0) = x$ for $x \in \partial W$, then $c(\partial W \times \{0\})$ is mapped by the identity, $c(\partial W \times I)$ is mapped onto $D^{4k}$, and $c(\partial W \times \{1\})$ is mapped to the point in int $D^{4k}$ where the part of $W$ outside the collar is sent. This gives a degree 1 map $f: (W, \partial) \to (D^{4k}, \partial D^{4k})$. 
Since $W$ is a $\pi$-manifold, $f$ is a tangential map, so we have a Novikov surgery problem in the bounded case. $H_1(f)$ is already an isomorphism below the middle dimension because $W$ is $(2k-1)$-connected, and \( \ker H_{2k}(f) = H_{2k}(W) \approx \bigoplus_{8r} \mathbb{Z} \). Since the index restricted to the kernel is just the index of $W$ which is $8r$, further surgery is not possible.

Now suppose $L^m$ is a closed, smooth, simply connected $\pi$-manifold. We shall see that the surgery problem \( f \times 1_L : W \times L \to D^{4k} \times L \) does have a solution. If $L$ has odd dimension, it is immediate from Theorems 8.1 and 8.2 that surgery can be done on $W \times L$ (leaving $\partial W \times L$ fixed) to make it homotopy equivalent to $D^{4k} \times L$. When the dimension of $L$ is even, we must refer to the product formulas for index and Kervaire obstructions obtained by Rourke [11]. The first formula states that if $m \equiv 0 \pmod{4}$, the (index) obstruction of $f \times 1_L$ is given by $I(f) \cdot I(L)$ where $I(f)$ is the index obstruction of the map $f$, which in this case is just $I(W)$. Since $L$ is a $\pi$-manifold, $I(L) = 0$ and the obstruction vanishes, so surgery is also possible in this case. There is an analogous formula for the Kervaire invariant obstruction of $f \times 1_L$ if $m \equiv 2 \pmod{4}$, $KI(f \times 1_L) = [I(f)]_2 \cdot KI(L)$. Since $I(f) = 8 \cdot r$, it is again possible to complete the surgery.

Now we change the surgery problem discussed above into a problem for closed manifolds. Let $N$ be a smooth closed $\pi$-manifold of dimension $4k$. Take a small disk $D^{4k}$ in $N$ and form the connected sum $N \# \bar{W}$ using this disk and the
disk which was attached to \( W \) to make \( \hat{W} \). Define \( g: N\#\hat{W} \to N \) by the identity on \( N\text{-int } D^{4k} \) and \( f \) on \( W \). (We could write \( g \) as \( 1_{N\#\hat{W}} \).) Since \( I(N\#\hat{W}) = I(N) + I(\hat{W}) = I(\hat{W}) \neq 0 \), \( N\#\hat{W} \) is not a \( \pi \)-manifold, so \( g \) is not a stable tangential equivalence. Since the product of two manifolds is a \( \pi \)-manifold if and only if both factors are, \( N \times L \) is a \( \pi \)-manifold while \( N\#\hat{W} \times L \) is not. Therefore \( g \times 1_L: N \# \hat{W} \times L \to N \times L \) is not a stable tangential equivalence. It turns out, however, that surgery is possible since \( (g| N\text{-int } D^{4k}) \times 1_L \) is a homotopy equivalence, allowing us to operate entirely in \( \text{int } W \times L \) where \( g \) is a tangential map. Another way of saying this is that all the homology kernel of \( g \) "resides in \( \text{int } W \times L \)" so that we may do the surgery there modulo the boundary \( \partial W \times L \) to make \( g \) a homotopy equivalence. We sum up this discussion in the following proposition.

**Prop. 9.2.** Suppose \( N^{4k} \) is a closed smooth \( \pi \)-manifold and \( L^m \) is a closed, smooth, simply connected \( \pi \)-manifold. Then there is a manifold \( M^{4k+m} \), homotopy equivalent to \( N \times L \), obtained by surgery on \( (1_{N\#\hat{W}} \times 1_L): N\#\hat{W} \times L \to N \times L \).

**Proof.** Embed \( N\#\hat{W} \times L \) in \( S^{4k+m+N} \) so that \( N\text{-int } D^{4k} \times L \) lies in one hemisphere, \( W \times L \) lies in the other hemisphere, and \( \partial W \times L = \partial(N\text{-int } D^{4k} \times L) \) lies in the equator sphere \( S^{4k+m+N-1} \). Take the integer \( N \) sufficiently large to obtain the stable normal bundle \( \nu_N \) of \( N\#\hat{W} \times L \) in \( S^{4k+m+N} \). Now \( \nu_N \) is not trivial, but restricted to each of \( N\text{-int } D^{4k} \times L \) and \( W \times L \) it is trivial. In particular \( f \times 1_L: W \times L \to D^{4k} \times L \) is covered by a bundle map of normal bundles. Since \( f \times 1_L \) is
a homotopy equivalence on the boundary and its surgery
obstruction vanishes, surgery can be done on $W \times L$, leaving
the boundary fixed, to obtain a manifold $\hat{M}^{4k+m}$ and a
homotopy equivalence $\hat{f}: \hat{M} \to D^4 \times L$, which like $f \times 1_L$
is the identity on the boundary. In fact $\hat{M}$ is framed cobordant
to $W \times L$ and the framing on the boundary remains unchanged.
This procedure kills all the homology kernel of $(\mathbb{1}
\# f) \times 1_L$ so we get a homotopy equivalence $\hat{g}:
(N \text{-int } D^{4k} \times L) \cup \hat{M} \to N \times L$
$\partial \hat{M}$
and a cobordism between $(N \# W) \times L$ and $M = (N \text{-int } D^{4k} \times L) \cup \hat{M}$.

\textbf{Q.E.D.}

By Novikov's results [9] if $m$ is even, the solution $M$
to the surgery problem is unique up to diffeomorphism. If
$m$ is odd, the solution is unique up to connected sum with a
homotopy sphere and thus is unique up to PL homeomorphism.
Since we shall be primarily interested in the topological
type of the solution, we shall not concern ourselves with
this ambiguity.

10. \textbf{Further properties of the surgery solution.}
We now show that the manifold $M$ which is obtained by surgery
on $N \# W \times L$ is not a $\pi$-manifold. To do this we need two lemmas.

\textbf{Lemma 10.1} Suppose $M$ is a smooth closed manifold and
$M = M_1 \cup_T M_2$ where $T$ is a closed, codimension 1 submanifold of
$M$ and $\partial M_1 = \partial M_2 = T$. For any map $\alpha: T \to N$ let $M_1 \times D^N \cup \alpha
M_2 \times D^N$ be the bundle obtained by using $\alpha$ to piece together
the trivial bundles over $M_i$ and $M_2$ as follows: $(x,y) \sim (x,\alpha(x) \cdot y)$
where $x \in T$ and $\alpha(x) \cdot y$ denotes the action of $O_N$ on $D^N$. Then
$\alpha$ determines a trivial bundle over $M$ if and only if $\alpha = h_2 \cdot h_1$
where $h_1 : T \rightarrow O_N$ extends over $M_i$ ($i = 1,2$) and the multiplication
indicated is induced from $O_N$.

**Proof.** Denote by $e$ the map from $T$ to $O_N$ which is
constantly the identity element of $O_N$. Also notice that $h_1$ extends
over $M_i$ if and only if $h_1^{-1}$ does $(h_1^{-1}(x) = (h_1(x))^{-1}, x \in T)$.

Suppose $\alpha$ determines a trivial bundle. Then we have
a bundle map

\[
\begin{array}{c}
M_1 \times D^N \\ \cup \\ M_2 \times D^N \\
\alpha \\ h \\
M_1 \times D^N \\ \cup \\ e \\ M_2 \times D^N
\end{array}
\]

Restricting to $M_1 \times D^N$, $h$ defines $h_1 : M_1 \rightarrow O_N$ by $h_1(x) \cdot y = h(x,y)$ for $x \in M_1$, $y \in D^N$. Since $h_1$ and $h_2$ are consistent
with the identifications on $T \times D^N$, $h_1|T = h_2|T \cdot \alpha$. Thus
$\alpha = h_2^{-1}|T \cdot h_1|T$.

Suppose, conversely, that we have $h_1$ and $h_2$ satisfying
$\alpha = h_2^{-1} \cdot h_1$, and $h_1$ is actually defined on all of $M_1$. Then
$h_1$ and $h_2$ can be pieced together to give a bundle map
Def. 10.1. A map \( h: (X, A) \longrightarrow (Y, B) \) is a homotopy equivalence of pairs provided there is a map \( g: (Y, B) \longrightarrow (X, A) \) such that \( g \circ h = 1_X \) and \( h \circ g = 1_Y \) where the homotopies carry \( A \) into \( A \) (or \( B \) into \( B \)) at each stage. In particular \( h \mid A \) will be a homotopy equivalence with homotopy inverse \( g \mid B \).

Lemma 10.2. Suppose \( M \) and \( N \) are topological manifolds with boundary, \( \partial M = \partial N = T \), and \( h: (M, T) \longrightarrow (N, T) \) is a homotopy equivalence such that \( h \mid T = 1_T \). Then there exists a homotopy inverse \( g \) for \( h \) such that \( g \mid T = 1_T \).

We omit the proof, which is an easy consequence of the homotopy extension property for \( (N, T) \).

We shall give two different proofs of the following important proposition since each provides some insight into it.

Prop. 10.3. The manifold \( M^{4k+m} \) obtained by surgery on \( (1_N \# f) \times 1_L: N \# \hat{w} \times L \longrightarrow N \times L \) is not a \( \pi \)-manifold.

Proof 1. We show that the stable normal bundle of \( M \) is non-trivial. Embed \( N \# \hat{w} \times L \) in \( S^{4k+m+N} \) as in the proof of Prop. 9.3. Select trivializations of the stable normal bundle \( v_N \) restricted to \( N \)-int \( D^{4k} \times L \) and restricted to \( W \times L \). With these trivializations fixed the bundle isomorphism class of \( v_N \) is determined by a map \( a: S^{4k-1} \times L \longrightarrow O_N \). Since the
framing on the boundary $\partial W \times L$ remains fixed, the normal bundle $\nu_N'$ of the manifold $\tilde{M}$ is determined by the same $\alpha$.

Suppose $\alpha$ determines a trivial bundle after surgery.

Then by Lemma 10.1 $\alpha = h_2^{-1} \cdot h_1$ where $h_2$ extends over $N \text{-int} D^{4k} \times L$ and $h_1$ extends over $\tilde{M}$, the result of surgery on $f \times 1_L: W \times L \rightarrow D^{4k} \times L$. After surgery we have a homotopy equivalence $\tilde{\gamma}: \tilde{M} \rightarrow D^{4k} \times L$ which is the identity on the boundary. By Lemma 10.2 we can find $\tilde{g}: D^{4k} \times L \rightarrow \tilde{M}$, a homotopy equivalence which is the identity on the boundary.

Now $\tilde{g} \circ f \times 1_L : W \times L \rightarrow \tilde{M}$ is the identity on the boundary so that if $h_1$ extends over $\tilde{M}$ by $\tilde{h}_1$, it extends over $W \times L$ by $\tilde{g} \circ \tilde{h}_1 \circ f \times 1_L$. Thus $\alpha = h_2^{-1} \cdot h_1$ where $h_2$ extends over $N \text{-int} D^{4k} \times L$ and $h_1$ extends over $W \times L$. Lemma 10.1 then implies that $\alpha$ determines a trivial bundle, contradicting the fact that $\nu_N$ is non-trivial. Hence $\nu_N'$ is non-trivial.

\textbf{Q.E.D.}

\textbf{Proof 2.} After surgery we have a homotopy equivalence $f: M \rightarrow N \times L$ and a cobordism $Z$ between $M$ and $N \# \hat{W} \times L$ together with a map $F: Z \rightarrow N \times L$ which restricts to $\tilde{\gamma}$ on $M$ and $(1_N \# f) \times 1_L$ on $N \# \hat{W} \times L$. If $*$ is a point of $L$, $(1_N \# f) \times 1_L$ is transverse regular with respect to $N \times *$.

Change $f$ by a small homotopy to make it transverse regular with respect to $N \times *$.

Finally, leaving $(1_N \# f) \times 1_L$ and $\tilde{\gamma}$ fixed, make $F$ transverse regular with respect to $N \times *$ to obtain the oriented cobordism $F^{-1}(N \times *)$ between $N \# \hat{W}$ and $S = \tilde{\gamma}^{-1}(N \times *)$. 
Because $N\#\tilde{W}$ and $S$ are oriented cobordant, $I(S) = I(N\#\tilde{W}) \neq 0$.

We have the usual equivalence of bundles $\tau(M)|S \cong \tau(S) \oplus \nu(S \subset M)$.

Since $f$ is transverse regular with respect to $N \times *$ and

$\nu(N \times * \subset N \times L)$ is trivial, $\nu(S \subset M)$ is trivial. Thus if

$\tau(M)|S$ were stable trivial, $\tau(S)$ would be stably trivial, contradicting $I(S) \neq 0$. Therefore $\tau(M)|S$ is not stably trivial and consequently $\tau(M)$ is not stably trivial.

Q.E.D.

**Prop. 10.4.** $M$ is not homeomorphic to $N \times L$.

**Proof.** Suppose $h: M \to N \times L$ is a homeomorphism. Denote by $p_j(M)$ the $j^{th}$ Pontrjagin class of $M$ and by $p_j(M_j;\mathbb{Q})$ the $j^{th}$ rational Pontrjagin class of $M$. In proof 2 of **Prop. 10.3** it was shown that $M^{4k+m}$ contains a closed submanifold $S$ of dimension $4k$ and index $8r$, i.e., $I(S) = I(N\#\tilde{W})$. If $i: S \to M$ is inclusion, we have from Theorem 5.4 $8 \cdot r = <L_k(p_1(S),...,p_k(S)), [S]> = <L_k(i^*p_1(M),...,i^*p_k(M)), [S]> = <L_k(p_1(M),...,p_k(M), i_*[S])>$. Now we may replace $p_j(M)$ by $p_j(M;\mathbb{Q})$ since any torsion evaluated on the orientation class is zero. By the topological invariance of rational Pontrjagin classes, $p_j(M;\mathbb{Q}) = h^*(p_j(N \times L; \mathbb{Q}))$.

(A proof of this famous conjecture combines [13] and [8]. It was first obtained by Novikov [10].) But $p_j(N \times L;\mathbb{Q}) = 0$ for every $j$ because $N \times L$ is a $\pi$-manifold. Therefore $p_j(M;\mathbb{Q}) = 0$ for every $j$, a contradiction. Note that it is immediate from **Prop. 10.3** that $M$ is not diffeomorphic to $N \times L$.

11. **The central theorem.** We are now ready to state and prove the principal theorem of this paper.
Theorem 11.1 Suppose $N$ is a smooth closed $\pi$-manifold of dimension $4k$ ($k \geq 2$) and $L$ is a smooth, closed, simply connected $\pi$-manifold. Then there is a countable sequence of smooth closed manifolds $\{M_i\}$ having the following properties:
1. No $M_i$ is a $\pi$-manifold,
2. Each $M_i$ is a homotopy equivalent but not homeomorphic to $N \times L$.
3. $M_i$ is not homeomorphic to $M_j$ if $i \neq j$.

Proof. Define $W_i = \overbrace{W \ldots W}^i$ where $W$ is defined in Section 9. Then $W_i$ satisfies Prop. 9.1 except that $I(W_i) = 8ir$, and all the preceding discussion about $W$ applies also to $W_i$.

Let $M_i$ be the result of doing surgery on $(1_N \# f) \times 1_L : N \# W_i \times L \longrightarrow N \times L$ to make it a homotopy equivalence (Prop. 9.3).

By Prop. 10.3 and 10.4, (1) and (2) hold. It remains to establish (3). This is done by generalizing proof 2 of Prop. 10.3 and the proof of Prop. 10.4.

Suppose there exists a homeomorphism $h : M_j \longrightarrow M_i$, and $i \neq j$, say $i > j$. (For the remainder of this paragraph $t = i, j$.)

Let $f_t : M_t \longrightarrow N \times L$ be a homotopy equivalence which is transverse regular with respect to $N \times \ast$ so that $f_t^{-1}(N \times \ast) = S_t$ where $I(S_t) = 8rt$. By transverse regularity we also have that $\nu(S_t \subset M_t)$ is trivial and hence $\tau(S_t) = \tau(M_t)|S_t$. Let $\delta_t : S_t \longrightarrow M_t$ be inclusion. Then by Theorem 5.4 $<L_k(p_1(M_t; Q), \ldots, p_k(M_t; Q)), \delta_t[S_t]> = I(S_t)$. As remarked previously, it is sufficient to use rational classes in this formula since torsion evaluated on an orientation class is zero. (Henceforth
we abbreviate \( L_k(p_1(x;\mathbb{Q}), \ldots, p_k(x;\mathbb{Q})) \) by \( L_k(x) \). Since 
\( \varphi_t^{-1}(N \times \ast) = S_t, \) it follows that 
\( \varphi_t^{*} \beta^i_{k} [S_t] = [N \times \ast] = [N] \otimes 1; \) so if \( \varphi_j^{-1} \) is a homotopy inverse for \( \varphi_j \), 
\( \varphi_j^{*} \beta^i_{j} [S_j] = \beta^j_{\ast} [S_j]. \) Thus \( I(S_j) = \langle L_k(M_j), \varphi_j^{-1} \beta^i_{j} [S_j] \rangle = \langle L_k(M_j), \beta^i_{j} [S_j] \rangle \) by the topological invariance of rational Pontryagin classes.

Define a bundle \( \xi \) over \( N \times L \) by \( \varphi_1^{-1}(\tau(M_1)). \) This means that \( \tau(M_1) = \varphi_1^{*}(\xi). \) Since \( \varphi_1 \) is the identity on \( N \)-int \( D^{4k} \times L \) and \( \tau(M_1) \mid N \)-int \( D^{4k} \times L \) is trivial, it follows that \( \xi \mid N \)-int \( D^{4k} \times L \) is trivial. Let \( i: N \)-int \( D^{4k} \times L \longrightarrow N \times L \) be inclusion.

Then if \( x \otimes y \in H_{\ast}(N \times L) \) and \( \dim x < 4k, x \otimes y \in \text{image } i_{\ast}, \) say \( x \otimes y = i_{\ast} z. \) Thus

\[
\langle L_k(\xi), x \otimes y \rangle = \langle L_k(i_{\ast} \xi), z \rangle = 0 \text{ since } i_{\ast} \xi \text{ is trivial. This shows that if } \gamma_{\ast} \in H_{\ast}(N \times L), \text{ then } \langle L_k(\xi), \gamma_{\ast} \rangle \text{ is given by the product of the coefficient of } [N] \otimes 1 \text{ in } \gamma_{\ast} \text{ and } w_{\ast} = \langle L_k(\xi), [N] \otimes 1 \rangle.
\]

Let \( g \) be the homotopy equivalence defined by 
\( \text{ho} \varphi_1^{-1} \circ \varphi_1 : M_1 \longrightarrow M_1. \) Then we have already shown

\[
\langle L_k(M_1), g_{\ast} \beta^i_{k} [S_1] \rangle = I(S_j) = j/i I(S_i). \quad \text{But } I(S_i) = \langle L_k(M_i), \beta^i_{k} [S_i] \rangle = \langle L_k(\xi), \varphi_1^{*} \beta^i_{1} [S_1] \rangle = \langle L_k(\xi), [N] \otimes 1 \rangle = w_{\ast}. \quad \text{However } \langle L_k(\xi), (g_{\ast} \varphi_1^{-1})_{\ast} [N] \otimes 1 \rangle
\]

\[
= \langle L_k(M_1), g_{\ast} \varphi_1^{-1} [N] \otimes 1 \rangle = L_k(M_1), g_{\ast} \beta^i_{k} [S_1] \rangle = I(S_j) = j/i w_{\ast}.
\]

Hence the coefficient of \( [N] \otimes 1 \) in \( (g \circ \varphi_1^{-1})_{\ast} [N] \otimes 1 \) is \( j/i, \) which is not an integer since \( i > j. \) This contradicts the fact that any induced map on rational homology must send integral classes to integral classes.

Q.E.D.
12. **Applications to Lie groups and $H$-manifolds.**

**Def. 12.1.** An $H$-manifold $M$ is closed, orientable topological manifold with a continuous multiplication $\mu: M \times M \longrightarrow M$ such that (1) $\mu \circ 1_M \times \mu = \mu \circ \mu \times 1_M$ (homotopy associativity) and (2) there exists $\ast \in M$ satisfying $\mu \circ 1_M \times \ast \circ \Delta = \mu \circ \ast \times 1_M \circ \Delta$ where $\Delta: M \longrightarrow M \times M$ is the diagonal map (existence of a homotopy identity).

It is clear, for example, that every compact Lie group is an $H$-manifold.

**Prop. 12.1.** If $M_1$ and $M_2$ are $H$-manifolds, $M_1 \times M_2$ is an $H$-manifold.

**Proof.** If $\mu_1$ and $\mu_2$ are respective multiplications, define the multiplication on $M_1 \times M_2$ by $M_1 \times M_2 \times M_1 \times M_2 \overset{\mu_1 \times \mu_2}{\longrightarrow} M_1 \times M_1 \times M_2 \times M_2 \overset{\iota}{\longrightarrow} M_1 \times M_2$ where $\iota$ interchanges the second and third factors.

**Prop. 12.2.** If $M$ is an $H$-manifold and $M'$ is homotopy equivalent to $M$, then $M'$ is also an $H$-manifold.

**Proof.** Take $\mu' = g \circ \mu \circ f \times f: M' \times M' \longrightarrow M'$ where $f: M' \longrightarrow M$ is a homotopy equivalence and $g$ is a homotopy inverse for $f$.

Properties (1) and (2) are easily verified.

**Theorem 12.2.** Suppose $N^{4k}$ and $L^m$ are smooth $H$-manifolds ($k \geq 2$), both $N$ and $L$ are $\pi$-manifolds, and $L$ is simply connected. Then there exists a sequence of topologically distinct smooth $H$-manifolds $\{M_1\}$ satisfying (1) no $M_1$ is a $\pi$-manifold, (2) each $M_1$ is homotopy equivalent but not homeomorphic to $N \times L$.
Proof. This follows immediately from Theorem 11.1 and Prop. 12.2.

Q.E.D.

In the case of Lie groups we can obtain a more interesting theorem by requiring $N$ also to be simply connected.

Theorem 12.3. Suppose $N^{4k}$ and $L^m$ are simply connected Lie groups (k≥2). Then there is a countable sequence of topologically distinct H-manifolds $\{M_1\}$ satisfying (1) no $M_1$ is a π-manifold, (2) each $M_1$ is homotopy equivalent to $N \times L$ but not homeomorphic to any Lie group.

Proof. Since every Lie group is a π-manifold, Theorem 11.1 and Prop. 12.2 apply. H. Sheerer has proved [12] that homotopy equivalent, compact, simply connected Lie groups are isomorphic, so if $M_1$ were homeomorphic to any Lie group, it would be homeomorphic to $N \times L$, contradicting Theorem 11.1.

Q.E.D.
Bibliography


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