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BEHAVIOR OF PLAIN CONCRETE UNDER VARIABLE LOAD HISTORIES

by

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ABSTRACT

BEHAVIOR OF PLAIN CONCRETE
UNDER VARIABLE LOAD HISTORIES

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In this investigation the effect of various load histories on the behavior of plain concrete was studied.

A series of short rectangular columns were tested. The shape of the columns and the loading arrangement were designed to simulate different stress conditions on actual structures.

A total of 44 specimens were tested under various concentric loading histories. Using these results, the characteristics of the stress-strain relation of plain concrete were studied and analytical expressions for these relationships were developed.

The stress-strain relations derived for concentric loading were used to predict the behavior of specimens under combined loading histories. In the combined load tests, the strain on one face of the rectangular specimen was zero and the strain on the other face was varied. A digital computer program for calculating the response under various combined loading histories was developed.

\[(A-1)\]
A total of 18 specimens were subjected to combined loading histories and compared to the analytical results. The differences between the stress-strain relations under concentric and eccentric loads were specifically studied.

On the basis of the test specimens studied, the following conclusions can be made;

1. **Concentric Loading Histories**

   a) Plain concrete possesses a unique envelope curve within which the stress-strain relations for all load histories will lie.

   b) The shakedown limit of plain concrete is dependent on the magnitude of the maximum stress and strain value of the previous loading. It is independent of the minimum stress level of the previous unloading cycle.

   c) Stress-strain curves starting from a point in the stress-strain domain generally are not unique and the maximum stress and strain value of the previous loading must be known to predict the behavior.

   d) The slow cycle fatigue limit of plain concrete will be approximately 63 percent of the standard cylinder strength.

(A-2)
2. Combined Loading Histories

a) The assumption of a linear strain gradient across a rectangular prism is valid. The observed behavior under cyclic combined loads confirmed this assumption.

b) The assumption of zero tensile capacity for reversed loadings will be nearly correct when compared with the actual behavior during loading cycles with high outside fiber strain values. The tensile capacity is practically destroyed when the outside fiber strain exceeds $1.6\varepsilon_0$; where $\varepsilon_0$ is the value of strain corresponding to ultimate stress under monotonic concentric loading.

c) The stress-strain relation of a fiber subjected to monotonically increasing combined loads will be nearly the same as the average stress-strain relation for a fiber subjected to concentric loading for outside fiber strains smaller than $1.20\varepsilon_0$. The peak values of stress-strain are the same under concentric and eccentric loading. For strains greater than $1.2\varepsilon_0$, stresses under combined loads are greater than those measured under concentric loads.

d) Plain concrete under combined load histories possesses a unique envelope for the relation between the
resultant force and the outside fiber strain. Failure occurs when the declining portion of this envelope is reached.

e) The slow cycle fatigue failure of plain concrete under strain gradients will occur under average combined stress values of about 51 percent of the cylinder strength.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Acknowledgments</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract.</td>
<td>A-1</td>
</tr>
<tr>
<td>List of Tables</td>
<td>IV</td>
</tr>
<tr>
<td>List of Figures</td>
<td>V</td>
</tr>
<tr>
<td>Notation and Definitions.</td>
<td>X</td>
</tr>
<tr>
<td>Chapter 1. Introduction.</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Object and Scope.</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Historical Background.</td>
<td>3</td>
</tr>
<tr>
<td>Chapter 2. Experimental Program.</td>
<td>9</td>
</tr>
<tr>
<td>2.1 Test Specimens.</td>
<td>9</td>
</tr>
<tr>
<td>2.2 Loading Frame.</td>
<td>10</td>
</tr>
<tr>
<td>2.3 Instrumentation.</td>
<td>12</td>
</tr>
<tr>
<td>Chapter 3. Behavior Under Axial Load.</td>
<td>14</td>
</tr>
<tr>
<td>3.1 Testing Program and Procedure.</td>
<td>14</td>
</tr>
<tr>
<td>3.2 Evaluation of the Test Results.</td>
<td>18</td>
</tr>
<tr>
<td>3.2a Monotonic Axial Loading to Failure.</td>
<td>18</td>
</tr>
<tr>
<td>3.2b Existence and Uniqueness of the Envelope Curve.</td>
<td>23</td>
</tr>
<tr>
<td>3.2c Investigation of the Shakedown Limit.</td>
<td>26</td>
</tr>
<tr>
<td>3.2d Effect of Nonrecoverable Strains on Behavior.</td>
<td>39</td>
</tr>
</tbody>
</table>

(i)
3.2e  Algebraic Expressions for Loading and Unloading Curves. .......... 45
3.2f  Cyclic Loading Between Constant Stress Levels. ................. 54
3.3  Concluding Remarks. ......................................... 60

Chapter 4. Behavior of Plain Concrete Subjected to Strain Gradients. ......... 62
4.1  Testing Program and Procedure. ................. 62
4.2  Evaluation of the Behavior Under Cyclic Loads Producing Strain Gradients. .64
4.2a  Assumptions. ........................................... 64
4.2b  Computed Stress Strain Relations. ................. 66
4.2c  Computer Program for Numerical Solution. ................. 70
4.3  Comparison of the Analytical and Experimental Results. ................. 72
4.3a  Virgin Curves Under Increasing Strain Gradients. ................. 72
4.3b  Examination of Envelope Curve for Various Loading Histories. .......... 78
4.3c  Cyclic Loading Producing Given Extreme Fiber Strain Increments. ....... 79
4.3d  Behavior Under Combined Load Cycles Between Given Values of $\alpha_1$. .......... 85
4.3e Investigation of Shakedown and Slow Cycle Fatigue Limits Under Strain Gradients. .......................... 86
4.4 Concluding Remarks. ......................... 90

Chapter 5. General Summary. .................... 93
5.1 Object and Scope. ............................. 93
5.2 Behavior of Test Specimens. ................ 93
5.2a Axial Loading. .............................. 94
5.2b Combined Loading. .......................... 96

List of References. ............................. 98

Tables

Figures
LIST OF TABLES

Table

3.1 Monotonically Increasing Axial Load Tests
3.2 Cyclic Loading to the Envelope Curve (Axial)
3.3 Constant Strain Increment Tests
3.4 Tests with Load Cycles Between Maximum and Minimum Stress Levels
3.5 Tests for Shakedown Limit (Axial)
3.6 Peak Values of Shakedown Curve for Some Tests Reaching the Envelope Curve
4.1 Monotonically Increasing Combined Load Tests
4.2 Combined Cyclic Loading Tests
LIST OF FIGURES

Figure
2.1 Test Specimen
2.2 Instrumented Axial Load Specimen
2.3 Instrumented Combined Loading Specimen
2.4 Loading Frame
2.5 Loading Frame
2.6 Instrumentation, Gage Locations
2.7 Instrumentation Diagrams
2.8 Deformation Transducer Mounted for Axial Loading Tests
2.9 Deformation Transducer Mounted for Combined Loading Tests
2.10 Specimen Placed Into the Loading Frame
2.11 A Typical Test Set-up
3.1 A Typical Failure Under Axial Cyclic Loading
3.2 Monotonic Axial Loading to Failure
3.3 Comparison of Smith-Young Curve to U. S. Bureau of Reclamation Tests (Ref. 33)
3.4 Experimental Points on the Envelope Curve
3.5 Comparison of Test AC2-10 with Smith-Young Envelope Curve
3.6 Comparison of Test AC2-09 with Envelope Curve
3.7 Comparison of Envelope Curves with Test AC4-10
3.8 Comparison of Envelope Curves with Test AC4-11
3.9 Comparison of Envelope Curves with Test AC4-13
3.10 Comparison of Test AC2-07 with Envelope Curve
Figure

3.11  Shakedown Limit
3.12  Experimental Shakedown Points
3.13  Experimental Points on Upper Shakedown Curve
3.14  Variation of the Shakedown Curve
3.15  Shakedown Limit for Tests with Constant Maximum Stress Level
3.16  Effect of the Minimum Stress Level on the Shakedown Curve
3.17  Effect of the Minimum Stress Level on the Shakedown Curve
3.18  Experimental Shakedown Points Compared to Analytical Formulation
3.19  Shakedown Points for Tests with Nonzero Stress Level
3.20  Idealized Shakedown Curves for Cyclic Loading with Constant Maximum Stress Level
3.21  Loading Curves
3.22  Unloading Curves
3.23  Relationship Between Shakedown Strains and Plastic Strains
3.24  Relationship Between Shakedown Strains and Plastic Strains
3.25  Variation of $S_S$-SP relation with $\beta$
3.26  Strains on the Envelope for Loading Curves From SP
3.27  Values of SP for Unloading Curves From Points on the Envelope (SE)
3.28  Comparison of Linear and Parabolic Approximations with Experimental Loading Curves

(vi)
3.29  Comparison of Linear and Parabolic Approximations with Unloading Curves
3.30  Analytical Loading and Unloading Curves
3.31  Comparison of Analytical and Test Results (AC2-09)
3.32  Comparison of Analytical and Test Results (AC2-05)
3.33  Comparison of Analytical and Test Results (AC2-06)
3.34  Comparison of Analytical and Test Results (AC4-10)
3.35  Comparison of Analytical Result Obtained in this Study and Test Result Reported by Sinha, Gerstle and Tulin
3.36  Comparison of Stress-Strain History Obtained in Ref. 3 and in this Research
3.37  Flow Chart for Stress Level Problem
3.38  Flow Chart for Strain Increment Problem
3.39  Comparison of Analytical and Test Results (AC2-05)
3.40  Comparison of Analytical and Test Results (AC4-10)
3.41  Comparison of Analytical and Test Results (AC4-11)
3.42  Comparison of Analytical and Test Results (AC4-13)
3.43  Number of Cycles to Failure for Tests with Constant $F_{\text{max}}$ Level ($F_{\text{min}} = 0.0$)
3.44  Computed Number of Cycles to Failure for Tests with Constant $F_{\text{max}}$ Level ($F_{\text{min}} = 0.0$)
3.45  Nomograph for Estimating the Number of Cycles for Failure under Repeated Constant Stress Levels
4.1  Study of the Strain Distribution (BC2-06)
4.2  Analytical Calculation of Resultant Force
4.3  Force Resultants on a Rectangular Section
4.4  Flow Chart for the Computation of the Stress Resultants

(vii)
Figure

4.5 Values of $\alpha_1, \alpha_2, \alpha_e$ (Virgin Loading)

4.6 Comparison of the Experimental and Computed $\alpha_1$ - Sco Relation

4.7 Comparison of the Experimental and Computed Values of $\alpha_2$

4.8 Stress-Strain Curves for Concentric and Eccentric Loading

4.9 $\alpha_1$ - Sco Relation Computed From Various Stress-Strain Curves

4.10 $\alpha_e$ - Sco Relation Computed From Various Stress-Strain Curves

4.11 Combined Loading Envelope Curves

4.12 Comparison of Measured and Computed Values of $\alpha_1$ - Sco (BC3-04)

4.13 Comparison of Measured and Computed Values of $\alpha_e$ - Sco (BC3-04)

4.14 Computed Stress Distribution for Test BC3-04

4.15 Comparison of Measured and Computed $\alpha_1$ - $S_{co}$ Relation (BC2-06)

4.16 Comparison of Measured and Computed Values of $\alpha_e$ - Sco (BC2-06)

4.17 Computed Stress Distribution for Test BC2-06

4.18 Comparison of Computed and Measured Values of $\alpha_1$ - Sco (BC3-03)

4.19 Comparison of Measured and Computed Values of $\alpha_1$ - Sco (BC3-02)

4.20 Comparison of Measured and Computed $\alpha_1$ - Sco (BC2-01)

4.21 Comparison of Measured and Computed $\alpha_e$ - Sco (BC2-01)

4.22 Computed Stress Distribution for Test BC2-01
Figure

4.23 Computed and Experimental Number of Cycles for Failure Under Constant $\alpha_{1}\text{max}$ Level

4.24 Comparison of Computed and Experimental $\alpha_{1}$ - Sco Relation (BC2-05)

4.25 Comparison of Computed and Experimental $\alpha_{1}$ - Sco Relation (BC4-02)

4.26 Computed Variation of Stress Distribution with Number of Cycles for $\alpha_{1}\text{max} = 0.63$

4.27 Variation of Shakedown Limit with Cyclic Loading (BC2-05)

4.28 Computed and Experimental Lower Shakedown Curve (BC2-05)
Envelope Curve = Locus of limiting stress-strain values in the stress-strain domain which cannot be exceeded by any loading curve without having an apparent failure in the concrete.

Shakedown Curve = The locus of the points where the reloading portion of any cycle crosses the unloading portion of previous cycle. Stresses above this limit will lead to additional strains (Section 3.2c).

Slow Cycle Fatigue Limit = Fatigue limit when acceleration and creep effects are not present.

Combined Loading = Any loading combination causing zero strain distribution at one face and nonzero strains on the opposite face of the cross section of a rectangular test specimen.

Strain Gradient = Nonuniform strain distribution.

b = Width of the rectangular section.

E = Eccentricity.

fc' = Ultimate strength of standard 6 in. x 12 in. cylinder.

fc = Concrete stress.
ult = Ultimate concrete stress.

h = Depth of the rectangular section.

\[ k_2 = 1 - \alpha_e \]

\[ k_1 k_3 = \alpha_e \]

X = Distance from the face where strains are zero.

\[ E = \frac{dF}{ds} \] = Tangent modulus of plain concrete (nondimensionalized).

F = fc/fc' = Stress ratio.

Fmax = Maximum stress ratio in a loading cycle.

Fmin = Minimum stress ratio in a loading cycle.

FS = Stress ratio on the shakedown curve.

FE = Stress ratio on the envelope curve.

\[ F = \frac{(Fmax + Fmin)}{2} \] = Average stress ratio.

Fci = The value of nondimensionalized stress at \( \eta_i \).

Agi = Gaussian Quadrature weighing coefficients.

M = Moment resultant on the rectangular section.

P = Axial force resultant on the rectangular section.

Ps = Axial load value on the combined load shakedown curve.
\[ S = \frac{\epsilon_c}{\epsilon_0} = \text{Strain ratio.} \]

\[ \text{Smax} = \text{Maximum strain ratio in a loading cycle.} \]

\[ \text{Smin} = \text{Minimum strain ratio in a loading cycle.} \]

\[ S_S = \text{Strain ratio on the shakedown curve.} \]

\[ SP = \frac{\epsilon_p}{\epsilon_0} = \text{Plastic strain ratio.} \]

\[ S_E = \text{Strain ratio on the envelope curve.} \]

\[ S_{CO} = \frac{\epsilon_{CO}}{\epsilon_0} = \text{Outside fiber strain ratio.} \]

\[ S_C = S_{CO} \cdot \eta = \text{Inside fiber strain ratio.} \]

\[ \alpha = \text{Strain ratio at the peak of shakedown curve.} \]

\[ \alpha_t = \frac{P}{bhfc'} = \text{Nondimensionalized axial force.} \]

\[ \alpha_2 = \frac{M}{bh^2fc'} = \text{Nondimensionalized moment.} \]

\[ \alpha_e = \frac{e}{h} = \text{Nondimensionalized eccentricity.} \]

\[ \alpha_{\text{max}} = \text{Maximum nondimensionalized axial force.} \]

\[ \alpha_{\text{min}} = \text{Minimum nondimensionalized axial force.} \]

\[ \alpha_{15} = \frac{P_S}{bhfc'} = \text{Nondimensionalized axial load value on the combined load shakedown curve.} \]

(xii)
$\beta =$ Stress ratio at the peak of the shakedown curve.

$\varepsilon =$ Strain

$\varepsilon_c =$ Concrete strain corresponding to $f_c$.

$\varepsilon_0 =$ Ultimate strain corresponding to $f_{c'}$.

$\varepsilon_p =$ Plastic strain

$\varepsilon_{cu} =$ Outside fiber strain.

$\eta =$ $x/h =$ Depth ratio.

$\sigma =$ Stress.
CHAPTER I - INTRODUCTION

1.1 OBJECT AND SCOPE

Object: In this investigation, the effect of various load histories on the behavior of plain concrete was studied. The factors governing the response of concrete under different load histories were evaluated. The limitations of existing methods and assumptions concerning concrete behavior were examined and the mechanism of the failure of plain concrete under various load histories was investigated.

At present, the response of concrete structures subjected to high amplitudes of repeated loads, such as earthquakes, is generally computed by assuming linear, bilinear or elastoplastic load displacement relations. The effect of the history of loading on the load deformation relation is neglected. Actually structures subjected to high amplitudes of disturbances may have a different behavior than what is assumed.

Recent research in this field suggests that the stress-strain relation of concrete is dependent on the amplitude and the number of load repetitions, and the behavior of the material may be altered by the loading history. Therefore it is necessary to establish certain limitations on the assumptions and applicability of the
existing procedures. As a result, advanced knowledge of the behavior of plain concrete under high amplitudes of repeated loads must be obtained. In addition the failure criteria for concrete subjected to various load histories must be established.

**Scope:** To investigate the behavior of plain concrete under various load histories, a series of short rectangular columns were tested. The shape of the columns and the loading arrangement were designed to simulate possible load conditions on actual structures.

A total of 44 specimens were tested under various concentric loading histories to establish the stress-strain relation of plain concrete. Using the results of these tests, the characteristics of the loading and unloading stress-strain relationships for plain concrete under concentric loads was studied and analytical expressions for these relationships were derived.

The algebraic equation for the stress-strain relation of plain concrete, obtained from concentric loading tests, was used to predict the behavior of specimens under combined loading histories. In the combined load tests the strain on one face was zero and the strain on the other face was varied. A digital computer program for calculating the response under
various combined loading histories was developed. A total of 19 specimens were subjected to combined loading histories and the results were compared with the predicted behavior obtained from the digital computer program.

The differences between the stress-strain relation under concentric and eccentric load histories were specifically studied and the parameters influencing the behavior were examined. Concepts relating to an envelope curve and the shakedown limits defining the failure criteria were developed. The slow cycle fatigue life under combined axial load and moment histories was examined.

1.2 HISTORICAL BACKGROUND

The effect of variable loads on concrete was first investigated at the end of the 19th century. The majority of the experimental and theoretical work was inspired by earlier research done on metals and was aimed at obtaining a fatigue stress level for various types of reinforced concrete members and cement products.

In 1898, Considere (7) and De Jouy (9) did some research on the fatigue of cement mortar. The first experimental work on plain concrete was reported by Van Ornum (34, 35) in 1903. From compression tests on 7 in. x 7 in. concrete cubes subjected to repeated 1/3
stresses ranging from zero to 95 percent of the cube strength, Van Ornum concluded that "the fatigue of concrete is caused as a result of gradual fracture which becomes complete under the repetition of load above certain stress levels". The ultimate strength of Van Ornum's specimens varied between 1200 and 1500 psi, considerably less than the high strength concrete used today. Another important phenomenon noted was the variation of the stress-strain relation with the number of load repetitions. However, no attempt was made to obtain an analytical relation between stress and strain.

For a period of 20 years after Van Ornum, various attempts were made to find the critical variables effecting the behavior of concrete under repeated loads. In 1926 Mehmel (25) defined a fatigue limit for plain concrete under compression. In this investigation, Mehmel introduced the effect of nonrecoverable deformations on the stress-strain relation. The fatigue limit was found to be 47 to 60 percent of the cylinder strength fc'. Mehmel noted that a specimen under repeated loads exceeding the fatigue limit never becomes stabilized with regard to deformation and these incremental deformations lead to the failure. Yashido's (37) tests in 1930 showed a decrease in the Poisson's ratio of concrete with increased number of cycles.
Graf and Brenner (13, 14) attempted to apply the Goodman diagrams, used for determining the fatigue strength of metals, to plain concrete.

During this time parallel experimental work was carried out on the fatigue of plain concrete under flexural loads. Clemmer (6) studied the behavior of highway pavement corners under repeated flexural loads using 6 in. x 6 in. x 36 in. beams. He obtained values of the fatigue limit for the beams ranging from 51 to 54 percent of the static failure stress in bending. A similar flexural fatigue range, 50 to 55 percent, was also found by Crepps (8) and Hatt (15, 16, 17, 18).

Kessler (22, 23) in 1955, conducted tests on standard bending specimens by subjecting them to repeated flexural loads at frequencies of 70, 230 and 400 cycles per minute. Kessler concluded that there was no significant fatigue limit for concrete in the order of millions of cycles. However, with a given number of cycles a stress level at which the concrete will fail could be obtained.

The research discussed in the preceding sections is limited to work in which the fatigue limit for the material was studied. The manner in which the failure was reached or the variation of the stress-strain,
load-deformation relations has not been discussed. With the exception of the Kessler's research, the load carrying capacity of concrete for a finite number of cycles was not considered.

Recently, Sinha, Gerstle and Tulin (3) reported tests of 6 in. x 12 in. standard compression cylinders used to formulate the stress-strain relations for monotonically increasing loads and for cyclic loads. The time for an individual cycle was about one minute. No tensile load was applied. The following conclusions were reported:

i) Concrete under compressive stress possesses an envelope curve, which may be considered unique and identical to the monotonically increasing compressive stress strain curve.

ii) Concrete possesses a shakedown limit such that "the stresses above this limit will lead to additional strains, whereas maximum stresses at or below this limit will cause the stress-strain history to go into a loop repeating the previous cycle without further permanent strain". This limit "is the locus of points where the reloading portion of any cycle crosses the unloading portion". It was also observed that the value of shakedown limit depended on the minimum stress in the cycle, i.e., the stress amplitude of the cycle.
After formulating the stress-strain curves for plain concrete subjected to axial loads, the moment curvature relation for singly and doubly reinforced concrete beams was computed assuming each concrete fiber in the compression zone of the beam behaved as it did under axial load (Ref. 2, 4). Comparison of analytical and test results showed some differences at high strains. The main inconsistency was in the measured and computed shakedown limit under flexure.

Aoyama (19) made tests on eccentrically loaded beams under repeated reversed loads. He assumed a bilinear stress-strain relation for concrete and steel and neglected the tensile capacity of the concrete. Using these relations, load-deflection relations were computed and compared to values measured in the tests. The load carrying capacities for initial loading and first reversal of load were in good agreement with the computed values, however, agreement was not as good when the load was reversed from a point beyond the yielding of the reinforcement. Aoyama concluded that better results could have been obtained if a more accurate stress-strain relation for concrete was used and the Baushinger effect on the steel was considered.

Shah, Winter and Sturman (30) reported tests of prismatic specimens subjected to repeated axial
compression showed that concrete has a shakedown limit of about 90 percent of ultimate capacity (minimum stress of zero). The shakedown limit appeared to be approximately equal to the critical load at which the volume of the concrete under compressive loading ceases to decrease and microcracking in the mortar sharply increases. The tests indicated that concrete is relatively insensitive to a few cycles of unusually high stresses. No visible damage to the ultimate capacity of the prisms was noted when the prisms were loaded 15 times to stresses near their ultimate capacity.

Most of the experimental work up-to-date was aimed at obtaining a fatigue stress level for concrete. The loadings were generally at high speeds. The effects of acceleration and speed on the behavior were generally not eliminated. With the exception of the research reported in References (3) and (30) no attempt was made to define the mechanism of the failure under repeated loads.
CHAPTER 2 - EXPERIMENTAL PROGRAM

2.1 TEST SPECIMENS

The specimens used in this investigation are shown in Figures 2.1, 2.2, 2.3. The dimensions are shown in Figure 2.1. The critical section dimensions were 3 by 5 in. Photographs of instrumented specimens are shown in Figures 2.2 and 2.3. Both ends of the specimens were flared and reinforced with #5 bars to confine failure to the critical section. Reinforcement was provided at the ends of the specimens to avoid problems resulting from vertical tensile cracking produced by distributed axial load applied at the ends as shown in Figure 2.1a, section (bg).

A preliminary elastic study showed that the flared ends and the reinforcement had no effect on the behavior of constant midsection of the specimens.

The concrete mix proportions were constant for all the test specimens. The concrete was a blend of Type III Portland cement and Colorado River sand and gravel of 3/4 in. maximum size. Three 6 in. x 12 in. compression cylinders were cast with each specimen and cured under the same conditions. Specimens were 7, 14, and 21 days old when tested and were cured in a moist room for 2, 5 and 7 days respectively. After removal from the moist
room, the specimens and control cylinders were stored in the laboratory until testing.

The concrete cylinder strengths varied between 3.5 and 5.0 ksi. Values of average fc' for three cylinders per test are listed in Tables 3.1 through 3.5 and 4.1 through 4.4.

2.2 LOADING FRAME

The stress-strain curve of plain concrete beyond ultimate is very unstable and the release of energy stored in the testing apparatus may decrease the accuracy of the strain measurements. A loading frame with high rigidity was constructed to reduce the effects of energy release beyond the ultimate capacity of the plain concrete. The frame is shown in Figures 2.4, 2.5. The frame was constructed so that axial or flexural loads could be applied simultaneously or separately. Axial load was applied with a 60 ton hydraulic ram connected to a hand operated high pressure oil pump. The load was transmitted to the specimen through a movable yoke resting on a spherical head on the ram. The movable yoke was pin connected to a rigid base plate which distributed the load to the end face of the test specimen. The axial force was monitored by a 100 Kip. load cell placed between the top of the specimen and the frame.
The load was transferred through an end plate on the specimen which was pin-connected to the load cell.

Horizontal load was applied through a manually operated screw-type mechanism. As the screw rotated, a horizontal thrust was developed against the column of the load frame. The load was monitored by a bending type 10 Kip. capacity load cell, which also functioned as a beam to distribute load to the two load points. Two movable yokes connected to the top and the bottom pinned-ends of the specimens load plates and provided the horizontal reaction for the specimen. In the axial load tests the horizontal load mechanism was also used to eliminate load eccentricities due to initial curvature or slight misalignment of the specimen in the frame. In effect, the specimen was a pin ended beam column subjected to axial load at the ends and equal lateral loads at the third points.

Specimens were placed between the top and bottom base plates and a quick setting, high strength gypsum cement was placed between the ends of the specimen and the base plates to ensure uniform load distribution. When the specimen was in place, the final distance between the pinned-ends was approximately 3 feet.
2.3 INSTRUMENTATION

Four 0.8 in. wire resistance strain gages were applied to the faces of the mid-height of the specimens. In concentric load tests the gages were applied on all four sides of the rectangular section. In eccentric specimens the gages were applied on the two faces subjected to maximum and minimum strains. Figures 2.6, 2.2, 2.3 show the locations of the strain gages on axial load and combined load specimens.

The strain gage output was fed through a gage control unit into an oscillograph recorder (Figures 2.9, 2.13). The axial and horizontal load cells were also connected to the recording unit, thus providing a parallel continuous record of strains and loads.

The deformation in a 6 1/2 in. length at the mid-height of the specimen was monitored by two differential transformer displacement transducers placed between two steel frames attached to the specimen with pointed set screws. The transducers were placed at the middle of the two opposite 5 in. wide faces for concentric load tests and at the two corners of the face where the non-zero strains were measured for the combined loading tests. The location of the displacement transducers is shown in Figures 2.10, 2.11. The transducers were used
to compare the average strains over a 6 1/2 in. length and local strains measured with the strain gages. The transducers made it possible to record strains which exceeded the range of the strain gages. The deflection transducer output was fed into an X-Y recorder along with the output of the axial load cell to obtain a load-deflection diagram.
CHAPTER 3 - BEHAVIOR UNDER AXIAL LOAD

3.1 TESTING PROGRAM AND PROCEDURE

A total of 44 test specimens described in the previous chapter and four 6" x 12" cylinders were tested to investigate the behavior of plain concrete under varied axial compressive load histories. Characteristics of the specimens and the loading histories can be classified into the following groups:

Group I: Monotonically Increasing Axial Load Tests

Monotonically increasing axial tests were conducted to obtain the ultimate strength of the specimens and to determine the strain at the ultimate strength ($\epsilon_u$). These tests were also used in investigating the existence and uniqueness of an envelope curve. The envelope is defined as the locus of limiting stress values in stress-strain domain which will not be exceeded by any loading curve. The existence of the envelope was examined by comparing the results of specimens subjected to monotonic loading to failure with the results of specimens subjected to various stress and strain histories. The specimens tested in Group 1 are listed in Table 3.1.

Axial load tests were conducted on groups of two or four specimens in which one specimen was always a monotonically increasing axial load test. In this way,
the maximum capacity and the strain at the maximum stress were determined for each group of specimens tested under various load histories.

In all of the pure axial load tests, the horizontal load resultant was zero. The loading rate was such that the ultimate capacity was reached in 3 to 5 minutes. In order to accurately record the strains when rapid changes were occurring, the recording speed of the oscillograph recorder was gradually increased as the strains became stress sensitive. This was especially true after the peak of the stress-strain curve was reached. The response of the strain gages was accurate up to strains of $4 \times 10^3$ in./in. Strains greater than 0.004 resulted in cracks passing through concrete gage location. In some specimens, the surface separated from the interior of the material due to excessive microcracking and rendered the strain gages ineffective. Average strains exceeding the peak of the stress-strain curve ($\varepsilon_0$) were more accurately measured using the differential transformer deflection transducers.

**Group 2: Axial Compressive Load Cycles to the Envelope Curve**

In these tests the stress-strain envelope was reached during each load cycle. Previous investigators have discussed the existence of an envelope curve for
the stress-strain properties of plain concrete (Reference 3). By comparing the stress-strain curves obtained from the tests in Group 1 with the curves in Group 2 the uniqueness of the envelope curve was examined. The tests in Group 2 were also used to investigate the shake-down limit. Characteristics of the specimens in Group 2 are listed in Table 3.2.

The axial load was increased to a level where the slope of the load-deformation curve approached zero. From this point the load was gradually released to zero, and a new load cycle was started. Using this procedure, the shape of the envelope curve up to strains that were 3 to 4 times the strain ($\varepsilon_\sigma$) at ultimate were obtained. Throughout testing, the horizontal load resultant was zero. Loading rates varied between 1 to 2 minutes per cycle. At this rate of loading, dynamic effects and creep were not significant.

**Group 3: Loading to Produce a Specified Strain Increase in Each Cycle**

Strain increment tests were conducted primarily to investigate the mechanism of the failure. Constant strain increment tests also served to check the formulation of the loading and the unloading curves and the existence of the envelope curve. Strain increments were controlled by reading the average strains from the
linear differential transformer transducer. The load was gradually released to zero when the desired strain was reached, thereby eliminating possible creep effects under sustained loads. A list of the tests in Group 3 is given in Table 3.3.

**Group 4: Repeated Loads Between Specified Maximum and Minimum Stress Levels**

Repeated axial loadings between specified stress levels were carried out to examine the mechanism of failure under stress levels below ultimate. The experiments were aimed at obtaining a rational explanation of the fatigue type of failures of concrete and investigating the main variables affecting the mode of failure. These series also aided in the development of diagrams similar to the Goodman diagrams which are commonly used to determine the fatigue limit of a metal under repeated constant stress levels. Some of the specimens were loaded to failure, i.e.; the specimen was not able to sustain the maximum value of the repeated stress. Specimens in which a large number of cycles would have been needed to reach failure were loaded to ultimate after a finite number of cycles had been applied. In this way it was possible to determine the decrease in the ultimate strength due to the application of the previous load cycles. Figure 3.1 shows a typical failure after excessive number of cycles. The specimens tested in
Group 4 are listed in Table 3.4.

**Group 5: Investigation of Shakedown Limit**

In this series, specimens were subjected to an excessive number of cycles at certain strain levels to examine the effect of number of cycles and the previous history on the location of the shakedown stress level and to investigate the uniqueness criterion. The specimens in Group 5 are listed in Table 3.5.

Since some of the tests were used to investigate the effects of several different load histories, it is impossible to classify them into a single group. As a result, these tests are shown in the tables for each of the groups for which they were applicable.

### 3.2 EVALUATION OF THE TEST RESULTS

#### 3.2 a Monotonic Axial Loading to Failure

The stress-strain relations for concrete under monotonically increasing axial load has been formulated by other investigators, (References 3, 10, 21, 28, 30). In this study, the stress-strain relation obtained experimentally for monotonically increasing axial load was compared to several stress-strain relations developed by previous investigators.

Monotonically increasing axial load test results were also compared to the envelope curves obtained for
specimens subjected to various load histories and the uniqueness of the envelope curve was examined.

Coordinates for stress and strain were nondimensionalized with respect to the ultimate strength $f_c'$ of a 6 in. x 12 in. control cylinder and strain ($\varepsilon_o$) corresponding to $f_c'$. The resulting nondimensionalized coordinates are $F = f_c/f_c'$ and $S = \varepsilon_c/\varepsilon_o$.

Table 3.1 shows values of $F_{max} = f_{c max}/f_c'$ and $\varepsilon_o$ obtained from the monotonically increasing axial load tests. A statistical analysis for values of $F_{max}$ gave a median of $.852$ and a mean of $.86$ with a standard deviation from the median of $.044$. For $\varepsilon_o$ the median was $1.68 \times 10^3$ and the mean was $1.71 \times 10^3$ with a standard variation from the median of $.14$. In the following analytical study a value of 0.85 for $F_{max}$ will be used and $\varepsilon_o$ will be taken as $1.71 \times 10^3$. In the mathematical formulation of the stress-strain curve for monotonic loading to failure, the peak of all the experimental stress-strain curves were assumed to have ordinates $F_{max} = .85$ and $S = \varepsilon_c/\varepsilon_o = 1.0$ and all other points on the experimental stress-strain curve were nondimensionalized with respect to these points. In Figure 3.2, the stress-strain curves obtained for 14 specimens subjected to monotonically increasing axial loads are represented by points extracted from the
curves at selected intervals.

The test results were compared with three analytic expressions suggested by previous authors. Smith and Young (26, 27) suggested the exponential curve;

\[ f_c = f_{ult} \frac{\varepsilon}{\varepsilon_0} e^{(1 - \varepsilon/\varepsilon_0)} \]

In nondimensionalized coordinates this expression becomes

\[ F = 0.35 S e^{(1 - S)} \]  
Eq. 3.1

and is shown in Figure 3.5 as curve I.

Hognestad's equation (18, 19) uses a parabola in combination with a straight line. For \( \varepsilon_c \leq \varepsilon_0 (S \leq 1.0) \) the stress-strain relation is;

\[ f_c = f_{cu} \left[ 2 \frac{\varepsilon}{\varepsilon_0} - \left( \frac{\varepsilon}{\varepsilon_0} \right)^2 \right] \]

In nondimensionalized coordinates, the expression becomes;

\[ F = 0.35 S (2 - S) \]  
Eq. 3.2

The stress-strain relation beyond ultimate is taken as a straight line with a negative slope. The ordinate of the ends of the line are \( \varepsilon_c = \varepsilon_0, f_c = f_{cu} \) and \( \varepsilon_c = 3.3 \times 10^{-3} \) \( f_c = 0.35 f_{cu} \).
Using the results of this investigation;
0.85 $f_{cu} = 0.85 (0.85 f'c) = 0.72 f'c$ and
$
\varepsilon_0 = 1.71 \times 10 \text{ in./in.}
$, the straight line is expressed
by

$$F = 0.957 - 0.107 S \quad \text{Eq. 3.2 a}$$

$$1.0 \leq S \leq 2.2$$

The curve developed by Hognestad is plotted in Figure 3.5 as curve II.

Desayi and Kirshnan (8) suggest the following relation:

$$f = \frac{E_s}{1 + (\varepsilon_0 / \varepsilon_o)^2}, \quad E = \frac{2 f_o}{\varepsilon_o}$$

Where $E$ is the initial tangent modulus of Elasticity and $f_o$, $\varepsilon_o$ are the values of stress and strain at the peak of the stress-strain curve. If $S = \varepsilon / \varepsilon_o$, $f_o = 0.85 f'c$ and $F = f / f'c$ are substituted into the expression above;

$$F = 0.85 S \frac{2}{1 + S^2} \quad \text{Eq. 3.3}$$

Equation 3.3 is plotted as curve III in Figure 3.5.

A comparison of these curves with the test results
shows that curve II underestimates the stresses between $S = 0$ and 1 and overestimates the stresses when $S \geq 1.6$. Curve III constitutes a lower bound for stress between $S = 0$ and 1 and overestimates the stresses when $S \geq 2$. The Smith-Young equation (curve I) approximates the experimental results most accurately. The curve fits well for all values of $S$, whereas, the other expressions fit less well in certain regions.

Since the Smith-Young expression was in good agreement with axial load test results, it was used to approximate the monotonically increasing axial load curves obtained throughout the experiments.

Figure 3.3 shows a comparison of the Smith-Young stress-strain expression with the axial load tests reported by U. S. Bureau of Reclamation* on standard 6" x 12" cylinders of various ages. There is a fairly good agreement, especially for the specimens over 9 days of age.

*Curves 1, 2, 3 are taken from Figure 8 and curve 4 is taken from Figure 11, curve 4 in Reference No. 31. All curves have been nondimensionalized in Figure 3.3 of this report.
3.2b Existence and Uniqueness of the Envelope Curve

The envelope curve is defined as the locus of limiting stress values in the S-F domain which cannot be exceeded by any loading curve without having an apparent failure in the concrete. In the following discussion the existence and validity of the envelope curve will be examined.

The existence of an envelope curve for plain concrete can best be investigated by comparing test results with the Smith-Young curve which approximates the results of the monotonically increasing axial load tests.

The points shown in Figure 3.4 were obtained from the tests in Group 2. The points are the peak values of stress and strain in a given cycle in which the curve reached its limiting stress value. The agreement between these limiting points and the Smith-Young curve is good. This suggests the existence and the uniqueness of an envelope curve for plain concrete under compressive stresses. Points between $S = 1.0$ and $S = 1.5$ show a lower envelope than the Smith-Young curve. This can be explained by observing that the limiting value of stress corresponding to a given load cycle beyond $S = 1$ may be reached before the loading curve joins the envelope. Therefore it is fairly difficult to determine if a particular loading cycle has
reached the envelope or not. This can be seen in Figure 3.5 and 3.10 where some of the loading curves beyond $S = 1$ reach a limiting stress capacity before the envelope is reached. If the loading was continued in a given cycle, the curves would follow the path shown in dotted lines and join the envelope at a lower stress value than the limiting stress.

In Figure 3.6 a portion of the monotonically increasing axial load curve (AMI-04) beyond $S = 1$ is not in complete agreement with the Smith-Young curve. However, the results of test AMI-04 constitute an envelope for specimen AC2-09 which was subjected to cyclic repeated load of magnitude $.82$ $fc'$. Both specimens were cast from the same batch and tested at the same age. Specimen AC2-09 withstood 2 cycles of load to a stress $F_{\text{max}} = .82$. In the third cycle, the envelope curve was reached before $F_{\text{max}}$ reached $.82$. Cyclic loading was continued to investigate the envelope curve at greater strains.

Figures 3.7 and 3.8 show behavior under repeated loadings of magnitudes $.76$ $fc'$ and $.71$ $fc'$ respectively. In both of the experiments, failure was due to the accumulation of the strains up to the point where the stress-strain curve met or approached the envelope curve. Envelope curves obtained from the monotonic
tests to failure and from the Smith and Young equation are shown. It can be postulated that, when a strain on the envelope curve corresponding to the maximum stress level is reached, the stress-strain relationship follows the descending part of the envelope.

Test AC4-12, (Figure 3.9), shows the behavior of plain concrete subjected to repeated loads between .40 and .79 fc'. Comparison of test AC4-13 to the monotonically increasing axial load test AM1-13 shows a remarkable agreement between the envelope curves of the specimens. Failure occurred when the strain accumulated reached the limiting strain on the envelope.

Figure 3.10 shows another comparison of the envelope curves with the results of a cyclic load history. The strain history in test AC2-07 which was used to investigate the shakedown limit illustrates the validity of the envelope curve. While this specimen was subjected to a completely different loading history than the previous tests, the concrete response remained within the envelope. In Figure 3.10 only the Smith-Young curve is shown.

In all of the test results discussed herein, it was evident that plain concrete under compression possesses an envelope curve which constitutes an upper bound for all loading histories. This limiting curve is very
close to the monotonically increasing axial load curve which can be assumed as being a unique property of the material. Therefore, in subsequent analyses, the envelope curve will be taken as identical with the monotonically increasing axial load curve and will be represented by the Smith-Young expression which is;

\[ F = 0.35 S e^{(1 - S)} \]  
\[ \text{Eq. 3.1} \]

\[ \text{3.2 c Investigation of the Shakedown Limit} \]

The shakedown limit has been defined by previous investigators (3) as "the locus of the points where the reloading portion of any cycle crosses the unloading portion of previous cycle. Stresses above this limit will lead to additional strains, while the stresses below this limit will give no strain increments, and the stress-strain curves will go into a closed hysteresis loop."

Figure 3.11 shows the shakedown limit for cycles with zero minimum stress levels, obtained in test AC2-10. At points A through G, the reloading portion of the cycles starting from \( F = 0 \) intersects the unloading curves of the previous cycle giving the locus of points which constitute the shakedown limit for the load history applied. If the shakedown limit corresponding to a given loading history is unique, as
assumed by (Ref. 3), i.e. if the intersection of the loading curve with the previous unloading curve is unique, cycles in which the stresses are below this curve will not result in higher strains than the strains obtained at the same stress level in the previous unloading. For example, if the stress level corresponding to point G in Figure 3.11 is not exceeded the strains obtained in the previous unloading will never be exceeded and all the cycles with a maximum stress below the stress at G will go into a hysteresis loop inside HIGJH, causing no apparent damage to the material. This assumption is contradicted by test AC2-07 in Figure 3.10. If the shakedown limit is unique, point B which was obtained as the shakedown limit corresponding to loading AB should have coincided with point C corresponding to loading AC. The strain accumulation should have started after the stress level corresponding to point B was exceeded. In the test the strain accumulation started at a lower stress level than B. Similar tests with maximum stress levels below the envelope show that shakedown limits lower than the first shakedown limit can be obtained and points out some errors in the definition of the shakedown limit. Moreover in most of the previous investigations the effect of minimum and maximum stress levels was not clearly defined.

-27-
In the following investigation the behavior of the shakedown limit under various load histories will be investigated, and a general formulation will be obtained.

1. Shakedown Limit for Loading Histories with Zero Minimum Stress Level

The shakedown limits for 21 cyclic loading tests with zero minimum stress and variable maximum stress level are plotted as discrete points in Figure 3.12. There is a considerable scattering in the shakedown points. However, bounds on the distribution of the points can be obtained. The lower bound for the shakedown has a peak of about .63 fc'.

Figure 3.13 shows the shakedown points obtained from 10 tests in which the loading cycles were started from $F = 0$. and loaded to or very close to the envelope curve before unloading was started. These points have less scatter than the basic shakedown points in Figure 3.12 and suggest the existence of an upper shakedown limit for cycles in which the envelope is reached.

The reason for the scatter of the points in Figure 3.12 can be explained using the curves shown in Figures 3.14 and 3.15. Figure 3.14 shows the several shakedown points, and the shakedown curves approximating these
points, obtained from test AC2-07. The S-F history of test AC2-07 is plotted in Figure 3.10. This test consisted of 6 stages. In each stage the stress was first increased up to a point on the envelope and then reduced to zero. Then stresses were increased up to the intersection of the loading curve with the previous unloading curve to obtain the first shakedown point. The load was again released down to zero and reloaded up to the intersection with the first unloading curve to obtain the second shakedown point. This was continued until the shakedown point stabilized. It can be observed that during each stage the shakedown point was gradually pushed down. The magnitude of the reduction of the stress level decreased with the number of cycles until the intersection point stabilized at a certain stress level. Once this stabilization was roughly achieved, the stress was increased up to another point on the envelope curve and the same procedure was repeated. If the locus of points for the first, second, third..., etc. shakedown limits are drawn, a family of shakedown curves as shown in Figure 3.14 can be obtained. The results of this test and other similar ones show that the shakedown limits for unloading from and reloading to the envelope constitute the upper bound of the shakedown limits. As cycles
with lower stress levels are introduced the shakedown points are reduced to a lower limit where the accumulation of strains finally stabilizes. Therefore it is reasonable to define a series of shakedown curves (when $F_{\text{min}} = 0$) instead of defining a single shakedown curve and in addition defining the transition between curves.

In Figure 3.15 shakedown limits for 5 tests with cyclic loadings starting from zero and a constant maximum stress level are shown. An analysis of the test results obtained shows a definite relation between the maximum value of the applied stress and the location of the related shakedown curve. In test AC4-12, the maximum stress level (0.79 $f_{c}'$) was higher than the peak of the upper shakedown curve. In this case, the shakedown curve is fairly smooth and very close to upper bound. Test AC4-10 had a maximum stress level of 0.76 $f_{c}'$ which was close to the average peak of the upper shakedown curve. In this test the upper bound for the shakedown limit was followed until the accumulated strains came close to the shakedown limit. From this point the limit was gradually reduced and a curve close to a horizontal line was followed, until the upper shakedown limit was intersected.
This trend can also be seen in Test AC4-11 in which $F_{max}$ was $0.71 \text{ fc'}$ (considerably less than the peak of the upper shakedown limit). When the strains approached the point corresponding to $0.71 \text{ fc'}$ on the shakedown limit, strain accumulation decreased and the shakedown points were reduced to lower stress levels. Since $0.71 \text{ fc'}$ is greater than the lower shakedown limit, there is a continuation of the strain accumulation. The locus of the shakedown points after a certain value of strain formed a pattern close to a horizontal line until the declining portion of the upper shakedown limit was reached. From this point on the upper shakedown curve was followed and an increase in the strain accumulation was observed. Finally the loading curves joined the envelope and the specimen failed. The upper shakedown limit and the envelope for Test AC4-11 are not in complete agreement with the analytical formulation. However the experimental envelope curve for Test AC4-11 was higher than Smith-Young curve and the specimen failed when the S - F history reached this envelope (Figure 3.41).

Tests AC4-03 and AC4-01 both had maximum stress levels below the peak of lower shakedown limit. Shakedown points were reduced until the lower shakedown
limit was reached. At this point, the strains stabilized and the stress-strain diagram formed a closed hysteresis loop in which there was no apparent damage to the material.

2. Shakedown Limit for Loading Histories with Non-Zero Minimum Stress Level

The effect of the level of minimum stress is illustrated in Figure 3.16. In Figure 3.16, two tests on specimens from the same concrete were loaded to the same maximum stress level but the minimum stress levels were different. Test AC4-12 was subjected to cyclic loading between $F_{\text{max}} = .79$ and zero. Failure occurred after a certain number of cycles when the accumulated strains reached the envelope curve. Test AC4-13 was subjected to cycles with the same maximum stress level but with a minimum stress level of $.40 \text{ ft}$. Although the minimum stresses were different, the shakedown limit was the same for both tests.

The results of two other specimens subjected to the same maximum but to different minimum stress levels are shown in Figure 3.17. Within the loading range in which maximum stresses were the same, both specimens had the same shakedown limit. The evidence from these 4 tests and other similar tests shows that for minimum
stress levels below the shakedown limit, the minimum stress in a load cycle does not have a significant effect on the location of the shakedown point. Therefore, it can be assumed that the shakedown limit is unique and identical with the shakedown limit corresponding to a cycle with zero minimum stress level once the maximum stress and strain value of the previous loading cycle is known.

Observations made on the behavior of the shakedown limit can be summarized as follows;

i) All the tests in which the shakedown limit was investigated showed a definite relationship between stress and strain at the peak of the loading cycle and the location of the shakedown limit. The minimum stress in an unloading cycle does not have a significant effect on the location of the shakedown point.

ii) If the stress and the strain at the peak of the cycle is above the upper shakedown limit, the shakedown limit is very near the upper bound. If the stress strain peak is below the maximum stress of the upper shakedown limit, the shakedown point is reduced, and for constant maximum stress levels a pattern very close to a horizontal line is obtained.
iii) If the peak stress of the load cycle is higher than the maximum stress of the lower shakedown limit, repeated cycles will gradually produce failure as the accumulated strains approach the envelope. If the peak of the load cycles is less than the maximum stress of the lower shakedown limit, the shakedown point will eventually reach the lower shakedown curve, and at this point, the strains will stabilize resulting in a closed hysteresis loop for subsequent load cycles.

3. Development of Algebraic Equations for Shakedown Limit

In the following discussion, algebraic expressions for the shakedown behavior of plain concrete will be developed in the same terms as were discussed in the previous section. Expressions for the upper and the lower shakedown limits and the transition between these curves will be developed.

Values of stress and strain at the peak of the shakedown limit for cyclic loading tests reaching the envelope curve are given in Table 3.6. Examination of the maximum stress and strain at the peak of the upper shakedown curve gives mean values of $F_{\text{max}} = 0.76$ and $S_{\text{max}} = .92$. From Figure 3.12 the approximate peak values of the lower bound shakedown limit are $F_{\text{max}} = .63$ and $S_{\text{max}} = .80$. 

-34-
From Figure 3.14 it can be seen that as the peak stress of a shakedown curve is reduced this also results in a reduced peak strain. With these values the family of shakedown curves may be expressed as a function similar to envelope curve:

\[ F_s = \beta \frac{S}{\alpha} e^{(1 - S/\alpha)} \]  \hspace{1cm} \text{Eq. 3.2}

where \(0.80 \leq \alpha \leq 0.90\)
\(0.63 \leq \beta \leq 0.76\)

In equation 3.2 \(\alpha\) shows the strain ratio and \(\beta\) the stress ratio at the peak of a shakedown curve. It has been observed that a reduction in the value of the peak stress of a given shakedown curve also results in a reduced peak strain. If \(\alpha\) is assumed to be a linear function of \(\beta\) in the region in which \(\alpha\) and \(\beta\) are considered, the following expression is obtained.

\[ \alpha = 0.315 + 0.77\beta \]  \hspace{1cm} \text{Eq. 3.3}

\(0.63 \leq \beta \leq 0.76\)

The approximation for the shakedown limit reduces to;

\[ F_s = \beta \frac{S}{0.315 + 0.77\beta} e^{(1 - \frac{S}{0.315 + 0.77\beta})} \]  \hspace{1cm} \text{Eq. 3.2 b}

Equation 3.2b approximates the upper shakedown limit when \(\beta = .76\) and the lower bound shakedown limit when \(\beta = .63\). Equation 3.2b is plotted and compared to the experimental shakedown points for cycles with zero minimum stress levels in Figure 3.18. The same
comparison is made for shakedown points with nonzero minimum stress levels in Figure 3.19.

For a loading cycle in which the coordinates of the peak are P (Fmax, Smax), the shakedown point is calculated as follows:

i) **If P is Above the Upper Shakedown Limit**
   
   \( \beta = .76 \):
   
The shakedown limit is taken as corresponding to the curve for \( \beta = .76 \).

ii) **If P is Between the Upper and the Lower Shakedown Curves**

   From the tests shown in Figures 3.10 and 3.15 and similar tests, it was observed that the vertical distance between point P and the corresponding shakedown limit gradually decreased to zero as the shakedown point reached the lower shakedown limit. The vertical distance between P and the corresponding shakedown limit can be expressed as a fraction of the vertical distance between P and the lower shakedown limit. This ratio was evaluated for all the tests in which P was between the upper and lower bounds. It was found that the corresponding shakedown points were below P at a vertical distance of about 5% of the distance between P and the lower shakedown limit. A reduction in the value of \( \beta \) in Equation 3.2b results in a proportional vertical reduction in all the
shakedown points corresponding to $\beta$. The distance from $P$ to the corresponding shakedown limit is 0.05 of the difference between the value of the stresses at $P$ ($\beta_p$) and $\beta_o$ which is the $\beta$ value corresponding to the lower shakedown limit. Therefore, to compute $\beta$ the following steps are applicable:

1) Compute the value of $\beta_p$ for the shakedown curve passing through $P$.

2) Compute $\beta$ for shakedown limit for $P$,

$$\beta = \beta_p - 0.05 (\beta_p - 0.63)$$  \hspace{1cm} \text{Eq. 3.3}

$$= 0.95 \beta_p + 0.0315$$

iii) **If the Point P is Below the Lower Shakedown Limit ($\beta = 0.63$)**

The shakedown point will be $P$. The cycles will be closed hysteresis loops causing no strain increments with respect to previous cycles.

Using the criteria expressed, some of the shakedown limit patterns that result under cyclic loading with constant maximum stress levels are shown in Figure 3.20. For example, the loading history with $F_{\text{max}} = 0.70$ follows the shakedown path AEHO and fails when the loading curve approaches the envelope. With $F_{\text{max}} = 0.50$
the shakedown curve follows ABP and the strains stabilize at point P.

It is also important to note that with $\beta = 0.63$, the slow cycle fatigue limit for plain concrete is obtained. Under compressive loadings on plain concrete in which acceleration and creep effects are not important no failure will be caused if $F_{max}$ is below 0.63. Maximum stresses below 0.63 $f_c'$ will eventually lead to stabilization of strains and the formation of closed stress-strain hysteresis loops.

In this section an approximate formulation for the behavior of the shakedown limit was obtained. This formulation outlines the mechanism and the behavior of shakedown points and reasonably approximates the observed behavior. Contradictory to the results reported in Reference 3, the observed loading and the unloading curves are not unique with respect to a point in the $S - F$ domain. In addition, the location of the shakedown point does not depend on the stress amplitude of the cycle.

In tests shown in Figures 3.16 and 3.17 and similar tests, it was observed that the location of the shakedown point is independent of the minimum stress level in a cycle. The only exception is in the case
where reloading occurs before the shakedown level is reached. If this happens the reloading curve will start at the point from which unloading stopped and strains will increase immediately.

3.2d Effect of Nonrecoverable Strains on Behavior

Nonrecoverable or plastic strains are defined as the strains corresponding to a zero stress level on the loading or unloading stress-strain curve. For example Figure 3.11 shows a plastic strain of $0.76 \varepsilon_0$ or a plastic strain ratio of 0.76 corresponding to loading curve HIG and the unloading curve GJH.

In the classical approach to the stress-strain relation for materials in the plastic range, loading and unloading curves are generally considered to coincide with one another and to parallel the initial loading curve regardless of the value of the plastic strain. This approximation is quite accurate for a homogeneous material such as structural steel.

In concrete and similar materials, experimental studies show that the curves do not coincide and are not parallel to the initial loading curve but are changed by the loading history. Figure 3.11 shows the loading and the unloading curves observed in Test AC2-10. The average slope of the loading and unloading
curves is inversely proportional to the plastic strain ratio. Loading curves starting at low plastic strains have almost constant slopes up to the corresponding shakedown limit. After the stresses exceed the shake-down stress, the rate of increase in strains is accelerated forming a curve with a decreasing slope. The change in slope can be attributed to an increase in microcracking. This phenomena was also reported in References 30 and 31. Loading curves starting at high plastic strain ratios have a point of inflection in the stress-strain curve. However, this effect is negligible at moderately high values of plastic strain ratios.

Unloading curves in Figure 3.11 are smoother than the loading curves. The average slopes of the unloading curves are also inversely proportional to the plastic strain ratio. The curves have decreasing slopes with decreasing stresses.

The observed changes in the shapes of the stress-strain curves with increasing plastic strains suggests a relationship between the plastic strain ratio and the nature of the loading and unloading curves.

Figure 3.21 shows the loading curves obtained from a number of test specimens subjected to various load histories. In all cases the minimum stress in each
cycle was zero. Each group of curves is plotted from the same approximate plastic strain ratio. In all cases a definite decrease in the average elastic modulus of the loading curves with an increase in the plastic strain ratio was observed. Loading curves for additional tests with different maximum stress levels and loading histories were approximately the same provided reloading was started from the same plastic strain ratio.

In Figure 3.22 unloading curves starting at or very close to the envelope curve and ending at the same plastic strain ratio are plotted. The unloading curves are approximately the same shape if the unloading starts from the envelope and ends at the same plastic strain ratio.

On the basis of results shown in Figures 3.21 and 3.22 the plastic strain ratio is one of the major variables affecting the shape of the stress-strain relation. In the following sections, the relation between the plastic strain ratio and the parameters which describe the behavior will be discussed. The pertinent parameters are the shakedown limit, the shapes of loading and the unloading curves and the points at which loading curves intersect the envelope.

The tests show a definite relation between the plastic strain ratio SP and the corresponding strain.
ratio \( S_S \) which lies on the shakedown limit curve. Figure 3.23 shows a plot of the relation between the plastic strain ratio \( SP \) and the strain ratio \( S_S \) at the shakedown limit for loading and the unloading curves which pass through the same plastic strain ratio. Points shown were taken from tests in which the maximum stress was above the upper shakedown limit. (\( \beta = .76 \)). A least squares approximation for a parabola passing through the origin yields the relation.

\[
SP = 0.160 \ S_S^2 + 0.133 \ S_S \\
\text{Eq. 3.4 a}
\]

If the peak stress in the cycle is below the upper shakedown curve, the shakedown point will be altered. Figure 3.24 shows the plastic strain ratios corresponding to the lower shakedown limits obtained from several tests in which the shakedown limit was reduced with additional cycles until a stable hysteresis loop was obtained. (Test AC2-07, Figure 3.10; Tests AC4-01 and AC4-03, Figure 3.15.) A least squares approximation for the relation between \( SP \) and the corresponding strain ratio on the lower shakedown limit (\( \beta = .63 \)) was made. In the least squares approximation, the relation was assumed to be a multiple of Equation 3.4a.

\[
SP = C \ (0.160 \ S_S^2 + 0.133 \ S_S)
\]
The value of C was found to be $= 1.13$, so that

$$SP = 1.13 \left( 0.160 S_s^2 + 0.133 S_s \right)$$  \hspace{1cm} \text{Eq. 3.4 b}

This curve is compared with experimental values in Figure 2.4.

Equations 3.4 a and b are bounds for loading cycles in which the shakedown points fall between the shakedown curves for values of $\beta$ of 0.63 and 0.76. Figure 3.25 shows the shakedown points versus the corresponding plastic strain ratios for tests AC2-07 and AC2-08 which illustrate the effect of a reduced shakedown limit on $S_s$-SP relation. The stress-strain curves for Test AC2-07 are shown in Figure 3.10 and for Test AC2-08, a similar loading pattern was observed. When the peak stress in the loading cycle is above the upper shakedown limit, the value of $S_s$ corresponding to a given value of SP is higher than when the peak stresses are below the upper shakedown curve. As a result the curves for the $S_s$-SP relation are shifted to the left for load cycles below the upper shakedown curve. The limiting and intermediate curves for the SP-$S_s$ relation in Tests AC2-07 and AC2-08 are shown in Figure 3.25.
A simplifying assumption can be made by assuming a linear variation in the \( S_s - \text{SP} \) curves from \( \beta = .76 \) to \( \beta = .63 \). If;

\[
\text{SP} = C \left( .160 \ S_s^2 + .133 \ S_s \right) \quad \text{and},
\]

\[
C = 1.00 \quad \text{when} \quad \beta = .76; \quad C = 1.13 \quad \text{when} \quad \beta = .63,
\]

the resulting equation is;

\[
\text{SP} = (1.76 - \beta) \times \left( .160 \ S_s^2 + .133 \ S_s \right) \quad \text{Eq. 3.4c}
\]

where \( .63 \leq \beta \leq .76 \).

In this form, the \( \text{SP}-S_s \) relation is not only a function of \( \text{SP} \) but also a function of the point where the previous unloading was initiated. The effect of \( \beta \) on Equation 3.4c is not as dominant as the effect of \( \text{SP} \).

The relation between the plastic strain ratio \( \text{SP} \) and the point where a given loading curve starting from \( \text{SP} \) intersects the envelope was also investigated. Since the equation for the envelope curve has already been formulated it is sufficient to find the relation between \( \text{SP} \) and the strain ratio \( S_E \) where the loading curve joins the envelope. In Figure 3.26 the strain ratio \( S_E \) at which a particular loading curve (starting from zero stress) joins the envelope is plotted against the plastic strain ratio \( \text{SP} \) from which loading started. A least squares approximation for a parabola passing through the origin gives the relation,
\[ SP = 0.093 \, S_E^2 + 0.091 \, S_E \quad \text{Eq. 3.5} \]

An \( S_E - SP \) relation can also be developed for unloading curves starting from the envelope curve. Figure 3.27 shows the strains (\( S_E \)) on the envelope from which the unloading curve started plotted against the plastic strain ratio (\( SP \)) after zero stress was reached. A least squares approximation for a parabola passing through the origin gives the relation;

\[ SP = 0.145 \, S_E^2 + 0.127 \, S_E \quad \text{Eq. 3.6} \]

In summary, the relations developed in this section illustrate the correlation between the plastic strain ratio and the shape of the loading and the unloading curves. These expressions will be used in the following section to obtain a general equation for the loading and unloading stress-strain curves.

3.2.e Algebraic Expressions for Loading and Unloading Curves

As an initial step, the type of curve that best approximates the shape of the loading and unloading curves is examined. Figure 3.28 shows loading curves
obtained from Test AC2-10 compared with linear and parabolic least squares approximations for the same curves.

In the linear approximation, only points below the shakedown curve were used to eliminate the highly nonlinear nature of the curves beyond the shakedown limit. The linear least squares approximation is in excellent agreement with the experimental loading curves up to the shakedown limit. This agreement suggests the possibility of considering the loading curves as straight lines up to the shakedown limit of the plain concrete. Experimental results obtained in Reference (30) on flared end specimens under repeated compressive loads showed that microcracking increased markedly beyond the shakedown limit. In Reference (30) it was concluded that the microcracking causes a decrease in the area of the effective section and produces a nonlinear stress-strain relation. This pattern of behavior was also observed in all of the tests reported herein. At the shakedown limit the stress-strain relation invariably became highly nonlinear.

A least squares approximation for a parabola for the loading curves of Test AC2-10 included all the points from zero stress up to the envelope curve.
Below the shakedown limit the parabolic loading curves do not approximate the experimental curves as accurately as did the linear approximations. However, better agreement with the experimental curves is obtained for stresses beyond the shakedown limit and the intersection with the envelope is very closely approximated.

Figure 3.29 shows the unloading curves obtained from Test AC2-10 compared with linear and parabolic least squares approximations. In this case the parabola approximates the experimental curve much better than the straight line.

Better approximations with higher order or transcendental curves might have been obtained. However, when considering the accuracy of the experimental results obtained, the advantages of a simple stress-strain relation outweighs the small gain in accuracy which would be derived by higher order approximations. Expressing the nondimensionalized stress as a second degree function of the nondimensionalized strain results in reasonable agreement with the experimental loading curves and an excellent agreement with the unloading curves.

i) Loading Curves

The algebraic expressions of the loading curves starting from points below the shakedown limit will be
taken as second degree parabolas passing through the three points discussed below:

1) Point at which reloading is started \((S_{\text{min}}, F_{\text{min}})\).

2) Shakedown point \((S_{s}, F_{s})\), where the shakedown stress is determined in the following manner;

The value of plastic strain ratio \((SP)\) corresponding to previous unloading is found. Once the value of SP is defined, the strain at which the subsequent reloading cycle intersects the shakedown curve is expressed as follows:

\[
SP = (1.76 - \beta)(0.160 S_{s}^{2} + 0.133 S_{s}) \quad \text{Eq. 3.4c}
\]

and, \(0.63 \leq \beta \leq 0.76\). Computation of \(\beta\) was discussed in Sec. 3.2c. The stress corresponding to \(S_{s}\) is found using the equation

\[
F_{s} = \frac{\beta}{0.315 + 0.77} S_{s} e^{\left(\frac{S_{s}}{0.315 + 0.77 \beta}\right)} \quad \text{Eq. 3.2b}
\]

3) Point at which the loading curve intersects or becomes tangent to the envelope curve \((S_{E}, F_{E})\). As was discussed before, the value of \(S_{E}\) is defined as a function of SP from the previous unloading cycle and is expressed as
follows:

\[ S_E = 0.239 + 10.75 \ SP - 0.489 \]  \hspace{1cm} \text{Eq. 3.5}

The value of \( F_E \) corresponding to \( S_E \) is determined from the equation for the envelope, where:

\[ F_E = 0.85 \ S_E e^\left(1 - S_E\right) \]  \hspace{1cm} \text{Eq. 3.1}

The second degree parabola passing through these three points has the form:

\[ F = \frac{(S - S_S)(S - S_E)}{(S_{\min} - S_S)(S_{\min} - S_E)} F_{\min} + \frac{(S - S_{\min})(S - S_E)}{(S_S - S_{\min})(S_S - S_E)} F_S + \frac{(S - S_S)(S - S_{\min})}{(S_E - S_S)(S_{\min} - S_E)} F_E \]  \hspace{1cm} \text{Eq. 3.7}

As a special case, if the starting point \((F_{\min}, S_{\min})\) is above the shakedown curve the parabolic curve;

\[ F = \frac{(S - S_S)(S - SP)}{(S_E - S_S)(S_E - SP)} F_E + \frac{(S - SP)(S - S_E)}{(S_S - SP)(S_S - S_E)} F_S \]  \hspace{1cm} \text{Eq. 3.8}

will be used.

The value of \( SP \) in Eq. 3.8 will be found by trial and error as the value that will make the second degree parabola 3.8 pass through starting point \((S_{\min}, F_{\min})\) when Equations 3.4c, 3.2b, 3.5 and 3.1 are satisfied. Equation 3.7 is curve DCG and Equation 3.8 is curve
AH in Figure 3.30.

ii) Unloading Curves

The algebraic expressions for the unloading curves starting from the points above the lower shakedown limit will be taken as second degree parabolas passing through the following three points:

1) Plastic strain ratio (SP, 0).
2) Shakedown point \((S_S, F_S)\), where the shakedown stress is determined following the same steps used for the loading curves.
3) Point at which unloading curve intersects the envelope curve \((S_E, F_E)\). Where \(S_E\) is defined as a function of SP as follows:

\[
S_E = \sqrt{0.186 + 6.896 \text{SP}} - 0.431 \quad \text{Eq. 3.6}
\]

The value of \(F_E\) corresponding to \(S_E\) is determined from the equation for the envelope where:

\[
F_E = 0.85 [e^{(1 - S_E)}] S_E \quad \text{Eq. 3.1}
\]

The parabola passing through these three points can be written as:

\[
F = \frac{(S - S_S)(S - SP)}{(S_E - S_S)(S_E - SP)} F_E + \frac{(S - SP)(S - S_E)}{(S_S - SP)(S - S_E)} F_S \quad \text{Eq. 3.9}
\]
The value of plastic strain ratio is found by trial and error, such that curve 3.8 will pass through point (Smax, Fmax) where the unloading was started, when equations 3.4c, 3.2b, 3.6 and 3.1 are used.

The investigation of the shakedown limit in section 3.2c showed that any unloading curve from a point below the lower shakedown limit will not result in any plastic strain accumulation. Therefore any unloading curve below the lower shakedown limit should end at the plastic strain ratio corresponding to the previous unloading. The form of the unloading curve starting from a point below the lower shakedown limit will be taken as a parabola passing through the following points:

1) Starting point (Fmax, Smax)
2) Plastic strain ratio SP corresponding to previous unloading and tangent to the previous unloading curve at point (SP, 0).

The second degree parabola satisfying the above conditions can be expressed as

\[
F = \frac{(S-SP)^2}{(S_{\text{max}}-SP)^2} \cdot F_{\text{max}} + \frac{(S_0-SP)F_E}{(S_E-SP)(S_S-S_E)} \cdot \left[ (S-SP)^2 - \frac{(S-SP)^2}{(S_{\text{max}}-SP)^2} (S_{\text{max}}-SP) \right] \\
(S_E, F_E) \text{ and } (S_S, F_S) \text{ can be found by using the equations 3.6, 3.1, 3.4c and 3.2b respectively.}
\]

-51-
Equation 3.9 is curve IE and Equation 3.10 is curve FE in Fig. 3.30.

In the following sections the comparison of the analytical loading and unloading curves with experimental results under various load histories will be made.

Figures 3.31, 3.32, 3.33 and 3.34 show comparison between analytic and experimental loading and unloading curves obtained from four tests. The tests consisted of cycles between zero stress and the envelope curve (β = .76). Since the analytical curves are dependent on SP, the values of SP from the experimental curves are used in the computations. The curves are in good agreement at low values of SP. As SP gets larger, the loading curves develop points of inflection and the resulting change in curvature is impossible to duplicate with a second degree parabola. However, the parabola compares very well at high stress ratios, especially at points above the shakedown limit.

Figure 3.35 shows the stress-strain curves under cyclic loading between zero stress level and the envelope curve reported by Sinha, Gerstle and Tulin (3). The results reported are nondimensionalized for a peak stress on the envelope of .85 fc' and a peak strain $= 2.64 \times 10^{-3}$ in./in. as determined from the results
in Reference 3.

A comparison of this test with the analytical solution developed in this study indicates reasonable agreement. The shakedown limits and the points at which the loading curves joined the envelope are approximated well.

In figure 3.36 a typical loading history used in Reference 3 is shown. The resulting stress-strain relation predicted in Reference 3 is compared with the analytical stress-strain relation for the same load history developed in this research. Results of the first and second cycle of loading, up to point E are nearly identical. The main difference between the two approaches is in the third cycle after point E. Using the method developed herein, the loading curve from E passes through the shakedown limit corresponding to an unloading curve from D to $F = 0$, if D is below the shakedown limit. Therefore in general the loading curves are not unique with respect to a given point in the S, F domain. In addition, there will not be any strain accumulation in the loading curve from E until the stress level corresponding to the shakedown limit is exceeded (Point D). As a result the peak F of the loading curve will fall to the left of the previous unloading curve DE. The unloading curve from F will
return to the plastic strain ratio at C. Using the method developed in Reference 3, the third cycle produces additional strains at given stress levels and returns to a higher plastic strain ratio at point G than would have been obtained if the unloading curve DE was continued to zero stress.

3.2 f Cyclic Loading Between Constant Stress Levels

A computer program was developed for studying the behavior of plain concrete under cyclic loadings varying between two constant stress levels or producing given strain increments at each cycle. Figures 3.37 and 3.38 show the flow chart of the program.

The type of the problem is determined by the value of G. If G is positive the constant stress level program is followed;

i) The envelope is followed up to Fmax. Smax is found by trial and error. The stress-strain relation up to Fmax is printed out.

ii) The unloading curve starting from Fmax is formulated by finding SP as the plastic strain ratio corresponding to Smax on the envelope from equation 3.6.

iii) The value of Smin corresponding to Fmin is found by trial and error. The stress-strain relation from Fmax to Fmin is printed out.
iv) The loading curve starting from point \((F_{\text{min}}, S_{\text{min}})\) is formulated by using the applicable equation 3.7 and 3.8.

v) The value of \(S_{\text{max}}\) corresponding to \(F_{\text{max}}\) is found by trial and error. The stress-strain relation from \(F_{\text{min}}\) to \(F_{\text{max}}\) is printed out.

vi) The unloading curve starting from \((F_{\text{max}}, S_{\text{max}})\) is formulated by using the applicable equation 3.9 or 3.10.

vii) The program is sent back to (ii) to start a new cycle (junction point 2 in Figure 3.37).

The program continues to cycle between two given stress levels until:

1) The strain increase is limited by the lower shakedown limit and the stress-strain curves go into a closed hysteresis loop, or;

2) The strain increase is limited by the envelope curve, and failure follows.

In each case the number of cycles required for stabilization or failure is printed out.

If the value of \(G\) is negative the solution of the problem with constant strain increments is started. In this case the method is almost the same as the one used in the constant stress level problem. Instead of loading to a maximum stress level, loading is continued until a
given strain increment is reached (Fig. 3.38).

Figures 3.39, 3.40 and 3.41 show the predicted and measured stress-strain relations and the number of load cycles applied before the concrete failed for three different maximum stress levels (Fmin = 0). The observed number of cycles to failure under a given stress-strain history is very close to what is predicted by the analytical solution. The analytical solution is especially accurate at high values of Fmax. Test AC4-11 shown in Figure 3.41 was subjected to the lowest value of Fmax (.71) and failed after 226 cycles. Failure was predicted in the 149th cycle by the analytical solution. This difference can be attributed to the approximations made in the formulation. For example, analytical loading curves from high values of SP are not as accurate as the loading curves from low values of SP. The approximation of the manner of variation in shakedown points with β is not very accurate at low stress values. The analytical loading and unloading curves create some limit problems when the starting points of loading and unloading curves are close to the shakedown limit which is a transition zone between two types of formulas used. These errors may explain the difference in the number of cycles for failure computed analytically and the experimental results for low values of Fmax.
Figure 3.42 shows the comparison of Test AC4-13 with the analytical solution. The stress-strain histories are nearly identical. The calculated failure at 34 cycles is close to the experimental value of 28.

Test AC2-12 had the same maximum stress level as Test AC2-13 but a zero minimum stress level. Failure was reached during 17th cycle which is 11 cycles less than Test AC2-13. The difference in the number of cycles to failure with the minimum stress level is probably due to the fact that a loading curve starting from a higher stress level passes through the same shakedown limit as a loading curve starting from zero stress. As a result, the loading curves have a higher slope which increases the fatigue strength of the material. This condition is also approximated by the parabolic loading curves used in this research. The analytical solution for the load history in Test AC4-12 predicted failure in the 10th cycle, which is less than the number of cycles computed for the load history of Test AC4-13.

If the uniqueness of loading curves with respect to a point in S, F domain had been assumed, the failure pattern would have been as shown in Figure 3.42 c. The number of cycles required to produce failure in this case would be 3 since the strain accumulation is overestimated significantly.
An increase in the number of cycles to failure under a given maximum stress level with an increase in the minimum stress level was observed in all the tests with nonzero stress levels.

In this section a comparison between the analytical formulation derived and the experimental results obtained from tests under constant stress levels has been made. The results agree very favorably with the experimental observations showing that the formulation accurately represents the actual behavior of plain concrete under cyclic loading between constant minimum and maximum stress levels.

In the following section the analytical formulation and the computer program developed were used to compute the necessary quantities governing the slow cycles fatigue life of the concrete. In particular, the fatigue strength under a given number of cycles between constant stress levels was investigated.

Figure 3.43 shows the number of cycles to failure under different maximum stresses with $F_{\text{min}} = 0$. Both measured and computed values are plotted. Figure 3.44 shows the number of cycles to failure with nonzero minimum stress levels. The computed values of maximum stress level required to obtain failure in a given
number of cycles under a constant average stress level \( \bar{F} = \frac{F_{\text{max}} + F_{\text{min}}}{2} \) are plotted for different values of \( \bar{F} \).

Figure 3.45 shows the nomograph for the approximation of the number of cycles required for failure under loads varying between constant maximum and minimum stress levels. This nomograph is a special case of the Goodman diagrams used for the determination of the fatigue strength of materials under repeated loads. One notable difference between Figure 3.45 and an ordinary Goodman diagram is that Figure 3.45 predicts the failure under a finite number of load repetitions while the Goodman diagrams are used for failure under an infinite number of cycles (usual fatigue limit). To use Figure 3.45 the average stress \( \bar{F} = \frac{F_{\text{max}} + F_{\text{min}}}{2} \) is computed and the intersection point B of the horizontal line \( \bar{F} \) with line CD (inclined at 45 degrees). This is shown as AG in Fig. 3.45. Point E is the reference point which should be compared to the family of contour curves plotted on the nomograph to obtain the number of cycles required for failure. The distance EB is equal to \( (F_{\text{max}} - F_{\text{min}})/2 \).

For example, the number of cycles required to obtain failure under a slow cyclic loading varying between the stress levels \( F_{\text{max}} = .75 \; \text{fc}' \), \( F_{\text{min}} = .45 \; \text{fc}' \) can be computed.

-59-
a) Compute $\bar{F} = \frac{F_{\text{max}} + F_{\text{min}}}{2} = \frac{.75 + .45}{2} = .60$

Obtain point X corresponding to the intersection of a horizontal line at $\bar{F} = .60$ with line CD.

b) Find point Y which is a positive vertical distance of $(F_{\text{max}} - F_{\text{min}})/2$ from X and in this case is equal to $(0.75 - 0.45)/2 = 0.15$.

c) Comparing the contours in Figure 3.45 to point X about 250 cycles are required to produce a failure in the plain concrete.

3.3 CONCLUDING REMARKS

The behavior of plain concrete under compressive load histories were studied. The following conclusions were obtained;

i) Experimental results showed that plain concrete possesses an envelope curve which is unique and may be considered to coincide with the monotonically increasing axial load stress-strain curve. Failure occurs when a given load cycle approaches the envelope curve.

ii) The shakedown limit of plain concrete is mainly dependent on the magnitude of the maximum stress and strain value of the previous load cycle. It is independent of the minimum stress value during a cycle. As a result the shakedown limit for loading from nonzero stress levels is unique and the same as the shakedown limit corresponding to the load cycles starting from
\( F_{\text{min}} = 0. \)

iii) Examination of the location of the shakedown shows that the probable slow cycle fatigue limit of plain concrete can be expected at a stress level of about \( 0.63 \times f_{c'} \) which is the peak of the lower shakedown limit. This limit is independent of the minimum stress levels in the cycles.

iv) Loading and unloading curves starting from a point \( P \) in the \( S, F \) domain (not on the envelope) are not unique, and the maximum stress-strain value of the previous loading must be known to predict the behavior. The assumption of uniqueness of loading and unloading curves will generally lead to erroneous results, especially for load cycles starting below the shakedown limit.

An approximate formulation of the envelope curve, upper and lower shakedown limits, loading and unloading curves was obtained. These formulations compare well with the experimental results. The formulation gives a general explanation of failure under slow cycle repeated concentric loads which compares very favorably with the test results.
CHAPTER 4 - BEHAVIOR OF PLAIN CONCRETE SUBJECTED TO STRAIN GRADIENTS

4.1 TESTING PROGRAM AND PROCEDURE

To produce a strain gradient across the column sections, 19 specimens were subjected to a combination of axial load and flexure. The loading procedure was discussed in Section 2.2.

The value of strain at one face of the specimen was kept zero while the strain on the other face was varied. The resultant stresses approximated the compression block of a rectangular reinforced concrete beam. The specimens can be divided into two groups.

Group 1 Investigation of the Virgin Curve (Monotonic Loading to Failure)

Six tests were conducted to obtain the magnitude and the location of the stress resultant under monotonically increasing combined loads. The results of these tests were used for comparing the stress-strain relation of plain concrete under a strain gradient to the average stress-strain relation under concentric loading. Each specimen in this group was a companion specimen for a group subjected to cyclic loading. In this manner, the existence and uniqueness of an envelope curve for the combined load tests was studied. Specimens
tested in Group 1 are listed in Table 4.1.

**Group 2 Cyclic Combined Loading Tests**

13 Tests were carried out to compare the predicted and observed behavior of plain concrete under combined cyclic loads. The stress-strain diagram was predicted using the algebraic expression developed for axial load tests. The concrete fibers were assumed to act as under axial load.

Some of these tests had a given outside fiber strain variation. In these tests either a given strain pattern was followed, or constant strain increments were imposed on one face of the specimen while the strain on the opposite face was kept zero. The average strain on one face of the specimen was monitored on an X-Y recorder via two linear differential transformer transducers connected in series. Unloading or reloading was started when the required strain value was reached.

Three specimens were subjected to cyclic loading between maximum and zero axial load levels. The strain in one face of the specimen remained zero throughout loading.

Constant strain increment tests and cyclic loading tests between given axial load levels were used to investigate the failure mechanism of plain concrete under
combined loading. These tests also served to investigate the existence and uniqueness of the envelope curve under combined loads. The shakedown limit was also studied. Characteristics of the tests in Group 2 are listed in Table 4.2.

4.2 EVALUATION OF THE BEHAVIOR UNDER CYCLIC LOADS PRODUCING STRAIN GRADIENTS

4.2a Assumptions

In the following discussion the simplifying assumptions listed below were checked experimentally in evaluating the behavior of plain concrete under load histories producing strain gradients.

i) Strains in a rectangular cross sections vary linearly with the depth of the section from a value of $\varepsilon_{c0}$ on one face to a value of zero on the other.

ii) Plain concrete has zero tensile stress capacity under strains smaller than the plastic strain ratio of plain concrete.

iii) Each concrete fiber in the section behaves in accordance with the stress-strain relation developed in Chapter 3 for plain concrete under axial load (uniform strain across the section).
Assumption (i) was checked in Test BC2-06. Values of strains were measured at various locations (Fig. 4.1a) using 8 resistance strain gages. The loading history is plotted in Fig. 4.1b and the recorded strains are shown in Fig. 4.1c. The points shown are averages of the two strain gage readings on opposite sides of the specimen. The results show that a linear strain assumption is justified and does not introduce any significant error in the calculation.

Assumption (ii) is made in most calculations and practical design methods used for concrete structures. Neglecting the tensile capacity introduces some error in the calculations. If the tensile capacity is zero, there can be no residual stress distribution in the section when the axial and flexural loads are reduced to zero. As a result, there are some disagreements between calculated and experimental results. However, at extremely high external fiber strains where the tensile capacity is destroyed due to intensive compressive microcracking, the assumption is essentially correct.

Assumption (iii) is not as easy to check experimentally as assumption (i) due to the impossibility of direct stress measurements throughout the section. Some authors reported that the existence of a strain gradient in the section introduces a change in the stress-strain
relation of the plain concrete. Shah, Sturman and Winter (30) concluded, from several monotonically increasing eccentric and concentric loading tests on concrete prisms, that the ultimate stress and the strain capacity of plain concrete increases by introduction of strain gradients. In Reference 30 a decrease in the rate of microcracking of the concrete mortar was observed when the strain gradient was increased. However, in this study each point in the section will be assumed to behave as observed under concentric load. The validity of the assumption will be checked on the basis of a comparison of the analytical and the experimental results.

Using the above assumptions, a general formulation of the behavior of plain concrete under combined loading histories is derived.

4.2 b Computed Stress-Strain Relations

The independent variable of the formulation is taken as the external fiber strain ratio $S_{co} = \varepsilon_{co}/\varepsilon_o$ where $\varepsilon_{co}$ is the strain at the face and $\varepsilon_o$ is the strain at maximum stress. Using assumption (i) the strain ratio at a point C (Figure 4.2) located a distance $x$ from the face on which the strain is zero, can be calculated as;

$$S_o = S_{co} \cdot \frac{x}{h} = S_{co} \cdot \eta$$

Eq. 4.1
where $\eta = x/h$. The stresses corresponding to these strains can be calculated using the expressions developed in Chapter 3.

Figure 4.2 shows the change in the stress distribution on Section ABCD with a given strain history. Point D on the extreme face is subjected to the strain history $D, D_1, D_0, D, D_2, D_3$, (Figure 4.2b). Point C inside the section follows a proportional strain history ($Sc = \eta . S_{C0}$). This results in the stress strain curve $A, C_1, C_0, A, C_2, C_3$ for point C (Figure 4.2a). Stress-strain values corresponding to every point inside the section are obtained in the same manner. The resultant force and eccentricity can be computed by integrating the stresses over the area (Figure 4.2c). In order to explain the results of the combined loading tests, the following quantities are defined:

- $P$ = Axial force resultant on the section,
- $M$ = Bending moment required to produce strain gradient (Figure 4.3) when $P$ is acting at the mid-section.
- $e$ = Eccentricity of force $P$ to yield a moment equal to $M$.

Quantities $P$, $M$ and $e$ can be calculated analytically as follows;
\[ S_c = S_{c0} \cdot \eta \quad , \quad F_c = F(S_c) \cdot f_c \]
\[ dP = f(S_c) \cdot b \cdot dx = F(S_c) \cdot b \cdot h \cdot f_c' \cdot d\eta \]
\[ P = h \cdot b \cdot f_c \cdot \int_0^1 F(S_c) \cdot d\eta \]  \hspace{1cm} \text{Eq. 4.2 a}

If \( \alpha_1 \) is defined as the nondimensionalized axial force
\[ \alpha_1 = \frac{P}{bhf_c} = \int_0^1 F(S_c) \cdot d\eta \]  \hspace{1cm} \text{Eq. 4.3}

\[ dM = x \cdot dP = f(S_c) \cdot b \cdot dx = b \cdot h^2 \cdot f_c' \cdot F(S_c) \cdot \eta \cdot d\eta \]
\[ M = b \cdot h \cdot f_c' \cdot \int_0^1 F(S_c) \cdot \eta \cdot d\eta \]  \hspace{1cm} \text{Eq. 4.2 b}

If \( \alpha_2 \) is the nondimensionalized moment
\[ \alpha_2 = \frac{M}{bh^2 f_c} = \int_0^1 F(S_c) \cdot \eta \cdot d\eta \]  \hspace{1cm} \text{Eq. 4.4}
\[ e = \frac{b fh^2 f_c}{\alpha_1 bh^2 f_c} = \frac{\alpha_2}{\alpha_1} \cdot h \]  \hspace{1cm} \text{Eq. 4.2 c}

The variable \( \alpha_e \) is defined as the nondimensionalized eccentricity, where
\[ \alpha_e = \frac{e}{h} = \frac{\alpha_2}{\alpha_1} \]  \hspace{1cm} \text{Eq. 4.5}
If the loading or the strain history for the extreme fiber is given values of $Sc$ at any load or strain value can be calculated using the algebraic expressions developed in Chapter 3. The nature of the expressions allows the calculation of the stress distribution in the section as discrete stresses at given points. Therefore it is preferable to use a numerical method for the integration of the stresses across the section.

Calculation of the area of the stress block was done numerically using an eight point Gaussian quadrature formula, which is convenient for the integration of smooth curves. The values of $\alpha_1, \alpha_2, \alpha_e$ can be written in terms of the Gaussian quadrature, as;

$$
\begin{align*}
\alpha_1 &= \sum_{i=1}^{8} F_{ci} \cdot Ag_i \\
\alpha_2 &= \sum_{i=1}^{8} F_{ci} \cdot \eta_{1i} \cdot Ag_i \\
\alpha_e &= \frac{e}{h} = \frac{\alpha_2}{\alpha_1}
\end{align*}
$$

Eq. 4.6

where;

$\eta_{1i}$ = The values of the points where $F_{ci}$ is evaluated.
$F_{ci}$ = The value of nondimensionalized stress at $\eta_{1i}$
$Ag_i$ = 8 point Gaussian quadrature weighing coefficients.
4.2c Computer Program for Numerical Solution

A computer program for the numerical evaluation of $\alpha_1, \alpha_2, \alpha_e$ values for:

i) Variations in the extreme fiber strain increment, and

ii) For cyclic loading between constant $\alpha_1^{\text{max}}$ and $\alpha_1^{\text{min}}$ levels was written. A flow chart of the program is shown in Figure 4.4.

The type of problem to be solved is determined from the value of $G$. If $G$ is positive the program solves a cyclic loading problem between constant $\alpha_1^{\text{max}}$ and $\alpha_1^{\text{min}}$ levels. At the start the stress-strain relation for each point inside the section is on the envelope curve up to the point where the integral in Eq. 4.3 reaches the value of $\alpha_1^{\text{max}}$. The value of Sco corresponding to $\alpha_1^{\text{max}}$ is found by numerical iteration and $\alpha_1, \alpha_2, \alpha_e$ versus Sco values are printed out for several points. When $\alpha_1^{\text{max}}$ is reached unloading computations are started. The maximum Sc and Fc value of each fiber are used to calculate the shakedown limit and the plastic strain ratio for the fiber considered. Using the shakedown and SP values, an Fc (Sc) relation is obtained for each point. From this relation, the outside fiber strain corresponding to $\alpha_1^{\text{min}}$ is calculated using the numerical integration technique and Eq. 4.3. Then $\alpha_1, \alpha_2, \alpha_e$ versus Sco values
are calculated and printed out for values of Sco between $\alpha_1^{\max}$ and $\alpha_1^{\min}$. When the value of $\alpha_1^{\min}$ is reached, reloading calculations are started. The $F_c$ ($Sc$) value for each fiber is found from the maximum ($F_c$, $Sc$) values of the previous loading and $SP$ is determined as the strain at zero stress if the previous unloading was carried to that level. Using the $F_c$ ($Sc$) relation for each point, the maximum value of $\alpha_1$ that can be reached in the loading is found. This value is compared with the value of $\alpha_1^{\max}$ applied externally. If the value of the maximum $\alpha_1$ that can be obtained in a given loading is less than $\alpha_1^{\max}$, the program prints out the number of cycles required to reach the failure and the final $\alpha_1, \alpha_2, \alpha_e$ versus Sco history. If the loading can reach values higher than $\alpha_1^{\max}$, the value of Sco corresponding to $\alpha_1^{\max}$ is found, and the values $\alpha_1, \alpha_2, \alpha_e$ for several values of Sco between $\alpha_1^{\min}$ and $\alpha_1^{\max}$ are calculated. The program will then return to junction (1) and a similar path will be followed until failure is obtained.

If the value of $G$ is negative, the program will solve the problem of cyclic loading with incremental outside fiber strains. The solution of the incremental extreme fiber strain problem is similar to the constant $\alpha_1^{\max}$ case. The only difference is that the loading and unloading curves do not start and end at given $\alpha_1^{\max}$ and $\alpha_1^{\min}$ values. Extreme fiber strains are increased
by a certain strain increment over the strain obtained in the previous loading cycle. Unloading is continued to zero strain or to a given $\alpha_{\text{min}}$ or $\text{S}_{\text{comin}}$ level.

4.3 COMPARISON OF THE ANALYTICAL AND THE EXPERIMENTAL RESULTS

4.3a Virgin Curves Under Increasing Strain Gradient

The virgin curves under combined loading were obtained for the variables $\alpha_1, \alpha_2, \alpha_e$ as functions of $\text{S}_{\text{co}}$ for monotonically increasing load to failure.

In this special case, if the equation for the monotonically increasing concentric load curve (uniform strain) is taken as;

$$F = 0.85 \text{Se} (1 - S)$$

Eq. 3.1

one can compute;

$$\alpha_1 = \frac{0.85}{\text{S}_{\text{co}}} \left[ e - (1 + \text{S}_{\text{co}}) e \left( 1 - \text{S}_{\text{co}} \right) \right]$$

Eq. 4.7

$$\alpha_2 = \frac{0.85}{\text{S}_{\text{co}}^2} \left[ 2e - (\text{S}_{\text{co}}^2 + 2\text{S}_{\text{co}}) e \left( 1 - \text{S}_{\text{co}} \right) \right]$$

Eq. 4.8

$$\alpha_e = \frac{1}{\text{S}_{\text{co}}} \left[ 2 - \frac{\text{S}_{\text{co}}^2}{e^{\text{S}_{\text{co}} - (1 + \text{S}_{\text{co}})}} \right]$$

Eq. 4.9

where $\text{S}_{\text{co}}$ is the strain ratio of the outside fiber.

Note also that

$$k_2 = 1 - \alpha_e$$

Eq. 4.10

$$k_1 \cdot k_3 = \alpha_1$$

-72-
The parameters $k_1$, $k_2$ and $k_3$ are used in design to determine the magnitude and the location of the force result for the compression block of a reinforced concrete beam.

Figure 4.5 shows the values of $\alpha_1$, $\alpha_2$, $\alpha_6$ that will be obtained as a function of $\text{Sco}$ for monotonic loading to failure from Eqs. 4.7, 4.8 and 4.9.

Figures 4.6 and 4.7 show the experimental results obtained from 6 virgin combined loading tests, compared with the computed results. The computed values obtained for $\alpha_1$ and $\alpha_2$ are in good agreement with the experimental results for $\text{Sco} \leq 1.2$. For $\text{Sco} \geq 1.2$ the experimental results show consistently higher values of $\alpha_1$ and $\alpha_2$. The parameter $\alpha_6$ is approximated fairly well for all values of $\text{Sco}$.

Referring to Figure 3.6, it can be seen that the Smith-Young curve approximates the average stresses as a function of average strains. If the concrete under strain gradients followed the same average stress-strain curve, it would be expected that some points will fall below the $\alpha_1$ curve in Figure 4.6. Since the linear stress distribution was checked (Figure 4.1c) and the assumption of zero tensile capacity has no effect on the virgin curve, there remains only the possibility of a difference between the stress-strain curves under concentric and
eccentric loads.

Figure 4.6 shows that the agreement between the experimental and the analytical curves for $\phi_1$ is excellent until $\text{Sco}$ reaches 1.2. This suggests that the difference between the stress-strain curves starts after the peak of the stress-strain curve under concentric load is reached. This difference can be explained by observing that under axial load the average stress-strain curve after the peak is reached is highly unstable due to the negative value of the tangent modulus of the plain concrete section. Very small changes in the stress distribution cause great changes in the value of strain at a given point due to the instability of strains with respect to stresses. Some nonuniformity of the stress distribution is likely to occur after the peak. Even though the value of average stress is not changed appreciably and the stress distribution is nearly linear throughout the section, the average strain will be considerably less than the actual strain value at a given point. As a result, the average stress-strain relation for the material is lower than when the strain distribution is exactly uniform.

If the material is subjected to a loading history producing strain gradients and the strain on one face is held at zero while the strain on the other face is varied, the system is more stable at high strain ratios than a member under concentric load. Some portion of the section
may reach a maximum stress capacity, but the remainder of the section continues to have a positive tangent modulus. As a result the section is more stable and less sensitive to slight differences in the stress distribution. This enables the material to develop higher average strains. The Smith-Young curve for the average stress-strain relation under concentric loads gives results very close to the actual behavior. For monotonically increasing combined load tests, conservative results will be obtained.

An investigation for the shape of the stress-strain curve beyond $S > 1.1$ shows that if the declining part of the Smith-Young curve for $S > 1.1$ is considered to be a straight line tangent to the curve at $S = 1.1$, the computed and measured stress-strain curves compare more favorably. With this adjustment, the stress-strain relations for axial load become:

$$F(S) = 0.85 \, S \, e^{(1 - S)}$$

if $0 < S \leq 1.1$

$$F(S) = 0.931 - 0.077 \, S$$

if $S > 1.1$

Eq. 4.10

The modified Smith-Young curve (Eq. 4.10) is plotted in Figure 4.8 and compared to the stress-strain curve used for concentric loading. The values of $\alpha_1, \alpha_2, \alpha_c$
obtained as a function of Sco using values of F(s) obtained from modified Smith-Young curve are shown in Figs. 4.6 and 4.7 and compared with the experimental results obtained from the 6 tests subjected to monotonic loading to failure.

Shah, Sturman and Winter (30) reported a decrease in the rate of microcracking and a corresponding increase for the maximum values of the stress-strain curve with flexural strain gradients. They concluded that the peak of the stress-strain curve is shifted. The reported results showed that the strain at the peak was about 50% greater than under axial load and the stress was about 20% higher than the peak stress values for concentric loading. The stress-strain relation used in Ref. 30 for concentric loading was as follows:

\[ \sigma = 4.0 \times 10^3 \varepsilon - 517.0 \times 10^3 \varepsilon^{1.9} \quad \text{Eq. 4.11a} \]

and for eccentric loading,

\[ \sigma = 4.13 \times 10^3 \varepsilon - 80.4 \times 10^3 \varepsilon^{1.6} \quad \text{Eq. 4.12a} \]

where

\[ \sigma = \text{stress, ksi.} \]

\[ \varepsilon = \text{strain, in./in.} \]
Equation 4.11a has peak values of

\[ \varepsilon_0 = 2.23 \times 10^3 \text{ in./in.} \]
\[ f'_c = 4.186 \text{ ksi.} \]

If Eqs. 4.11a and 4.12 are nondimensionalized with respect to the peak values;

\[ S = \varepsilon / \varepsilon_0 \quad ; \quad F = 0.85 \sigma / f'_c \]

or,

\[ \varepsilon = \varepsilon_0 S \quad \quad \sigma = F \cdot f'_c / 0.85 \]

The equations become;

\[ F = 1.811 S - 0.961 S^{1.9} \quad \quad \text{Eq. 4.11b} \]

for concentric loading,

\[ F = 1.871 S - 0.934 S^{1.6} \quad \quad \text{Eq. 4.12b} \]

for eccentric loading.

Figure 4.8 shows the comparison of Equations 4.11b and 4.12b with the Smith-Young curve used in Chapter 3 and the modified Smith-Young curve. The stress-strain relation used by Shah, et al. for concentrically loaded specimens is almost symmetric with respect to the strain at the peak and yields zero stress at a strain of \( 4.49 \times 10^3 \text{ in./in.} \) or \( S = 2.01 \). The stress values obtained in this research do not exhibit this behavior. The declining part of Equation 4.11b does not represent the actual behavior of the plain concrete under concentric
loading. The Smith-Young equation predicts a stress ratio of .620 when $S = 2.01$. It is evident that the points used in Reference 30 for the least squares approximation to obtain Equation 4.11a were less than or equal to the peak values of the stress-strain curve under concentric loading. As a result the true nature of the declining portion of the curve was not established.

Figures 4.9 and 4.10 show the computed values of $\alpha_1'$ and $\alpha_2'$ if various suggested equations are used for concrete under axial load. All formulas, with the exception of the expressions suggested by Reference 30, are in a good agreement with the experimental results obtained in this study. All expressions, except the ones suggested by Reference 30, yield conservative values for $\alpha_1'$ and a nearly constant value for $\alpha_1'$ at high values of Sco.

4.3b Examination of Envelope Curve for Various Loading Histories

The assumption concerning the existence and uniqueness of the envelope curve for plain concrete under concentric load was checked experimentally in Chapter 3. If the stress-strain relation is unaffected by a strain gradient the uniqueness and existence of an envelope curve for the $\alpha_1'$-Sco relation under combined loads should hold true.

Figure 4.11 shows the points obtained at or very near the peak of the $\alpha_1'$-Sco curve for 9 specimens

-78-
subjected to various cycles. The results are compared with the $\alpha_1$- Sco relation derived from the modified Smith-Young curve (Equation 4.10) and the $\alpha_1$- Sco relation obtained from the Smith-Young curve. The modified curve constitutes an upper bound on the experimental results and strengthens the assumption regarding the existence and the uniqueness of an envelope for the $\alpha_1$- Sco relation. Results derived from the Smith-Young equation are in a good agreement for $S_{c0} \leq 1.6$. For larger values of Sco, the equation forms a lower bound for the experimental results. In the following sections the Smith-Young equation (Eq. 3.1) will be used for the envelope curve since it is quite accurate, and conservative for $S_{c0} \leq 1.6$.

4.3 C Cyclic Loading Producing Given Extreme Fiber Strain Increments

Figure 4.12 shows the $\alpha_1$- Sco relation obtained from Test BC3-04 compared with the predicted relation based on the assumptions and equations in Section 4.2. The agreement between experimental and analytical results is good. There are some differences in the unloading curves for values of Sco approaching zero. This is due to neglecting the tensile capacity of the plain concrete. As the strain ratio at which unloading starts is increased, extensive compressive microcracking destroys the tensile capacity and the agreement between the experimental and analytical values is improved. At high values
of Sco the predicted values of $\alpha_{\text{th}}$ (close to the envelope curve) are less than measured values as a result of the use of the Smith-Young curve.

Figure 4.13 shows a comparison between the analytical and the experimental $\alpha_{\text{e}}$ - Sco relation obtained for Test BC3-04. The major differences between the analytical and the experimental results is observed at strains below the plastic strain ratio corresponding to the maximum value of Sco reached in a given load cycle. This is again due to neglecting the tensile capacity of plain concrete at strains below the plastic strain ratio. At low values of Sco, the value of $\alpha_{\text{e}}$ is very sensitive to changes in the strain ratio (Sco).

Figure 4.14 shows the computed stress distribution at various points in the strain history of Test BC3-04. The roman numerals indicate the points on the loading history (Fig. 4.12) where the stress distribution is plotted. Dashed lines show the stress distribution if all the fibers inside the section followed the monotonically increasing concentric load stress-strain curve during both the loading and unloading portions of the load cycles. There is a continuous change in the form of the "Stress Block" and the stress distribution at a given strain value is not unique.

Figure 4.15 shows a comparison of measured and computed $\alpha_{\text{t}}$ - Sco relations for Test BC2-06. The
agreement between the analytical and the experimental results at initial loading and unloading is excellent. Thereafter the agreement is not as good as for Test BC3-04. The $\alpha_1$ values, below the intersection point of the loading and unloading curves in the second and the third loadings, are higher than the experimental curves. This difference can be attributed to the points of inflection observed in the loading curves of plain concrete under concentric loadings at high values of strains. The analytical stress-strain curves developed do not approximate this change in the slope of the loading curves and give higher stress ratios for a given strain ratio than were measured. The $\alpha_1$ values above the intersection point of the loading and unloading curves are lower than the experimental curves. This difference is a result of using Smith-Young curve as the envelope for the stress-strain relation which gives conservative values of stress for a given strain value.

Figure 4.16 shows a comparison of the analytical and experimental $\alpha_\varepsilon$-Sco relation for Test BC2-06. The pattern is similar to that for Test BC3-04 (Figure 4.13) but the analytical curves generally gives higher $\alpha_\varepsilon$ values for a given Sco. Figure 4.17 shows the computed stress distribution at various points (Figure 4.15) in the strain history of Test BC2-06.
Comparison of Tests BC3-04 and BC2-06 (Figures 4.12, 4.15) shows that the agreement between the analytical and the experimental results are extremely good up to the values of external fiber strain of $1.2 \varepsilon_0$. Beyond this strain ratio the computer values are conservative.

Figures 4.18 and 4.19 shows the comparison of the experimental and analytical $\alpha'_1$- Sco relation for two cyclic loading tests. In Test BC3-03 (Figure 4.18) strain increment of $0.5 \varepsilon_0$ were added during each cycle after an initial strain of $0.7 \varepsilon_0$ was reached in the first cycle. The experimental results show excellent agreement with the analytical results at low values of Sco. At high values of Sco the agreement is also good, but the computed maximum stress in each cycle is less than the experimental maximum stress.

In Test BC3-02 of Figure 4.19 strain increments $.2 \varepsilon_0$ were added during each cycle after an initial strain of $.75 \varepsilon_0$ was reached in the first cycle. The experimental results are generally in good agreement with the analytical results. The values of $\alpha'_1$ at the peak of each cycle were approximately same. The experimental loading curves had higher slopes at the peak. If the loading had been continued, higher values of $\alpha'_1$ would have been observed experimentally.

The results of Tests BC3-02 and BC3-03 shows that there is a residual stress distribution in the section
when the moment and the axial load are reduced to zero. This is evidenced by the existence of an external fiber strain Sco when $\alpha_1$ and $\alpha_2$ are zero. If the concrete had zero tensile capacity, the value of Sco should have been zero at the same time that $\alpha_1$ and $\alpha_2$ read zero. Shoving the concrete has a residual stress distribution in cycles where Sco is small (see Fig. 4.12, 4.15, 4.18 and 4.19). As the value of Sco increases extensive micro-cracking under compression reduces the tensile capacity and the residual stress distribution diminishes. This gradual destruction of the tensile capacity can be seen in Test BC3-04 in Figure 4.12. The value of Sco at the end of first cycle was 0.182. This value decreased to .145 at the end of the second cycle, showing a decrease with increasing Sco. This behavior was also observed in Tests BC3-03 and BC2-02 shown in Figures 4.18 and 4.19. These tests show that the tensile capacity in the section was almost completely destroyed after an external strain value of $1.6\varepsilon_0$ was reached.

Destruction of the tensile capacity with increasing plastic strain ratio (SP) can also be observed in the unloading curves for concentric loading (Chapter 3). For low values of SP the curves have positive tangent moduli when $F = 0$ indicating the existence of some tensile capacity. As SP increases the unloading curves
become tangent to the $S$ axis indicating a diminution of the tensile capacity.

Test BC2-01 was carried out to investigate the effect of neglecting the tensile capacity on the analytical computation of the $\alpha$ parameters. Unloading was stopped when the Sco value corresponding to the plastic strain ratio of the external fiber was reached. In this way all the strains inside the section were kept above the plastic strain ratio corresponding to a given fiber. This eliminated the effect of the zero tensile capacity assumption on the analytical computation. Arbitrary strain increments were added in each loading cycle.

The agreement between observed and computed $\alpha_1$ values is excellent up to Sco = 1.7 (Figure 4.20). Thereafter the analytical results are conservative; but the agreement is very good for points below the intersection of the loading and the unloading curves.

Figure 4.21 shows the $\alpha_e$-Sco relation for the same test. The curves are generally in good agreement. There is some variation in the unloading portion of first two cycles. This variation is caused by the approximation of a higher SP value for the external fiber in the analytical results. This approximation produced the sudden decrease in the analytical $\alpha_e$ curve at the end of the unloading portion of the cycle.

-84-
Figure 4.22 shows the computed stress distribution in the section at various points in the loading history of Test BC2-01 (see Figure 4.20).

4.3.3 Behavior Under Combined Load Cycles Between Given Values of $\alpha_1$

Figure 4.23 shows the number of cycles required to produce failure under cyclic loading with constant $\alpha_1\text{max}$ and $\alpha_1\text{min} = 0$, for 3 tests compared with computed values. The analytical curves are conservative with respect to the experimental results due to the differences in the declining portion of stress-strain relation of plain concrete under combined loading which were discussed before. Because of these differences in the stress-strain curves for values of $Sco \approx 1.2$, it may be more realistic to compare the results from specimens subjected to different $\alpha_1$ levels but having the same number of cycles to failure.

Figure 4.24 shows the results of Tests BC2-05 which failed after 4.5 cycles when subjected to a maximum $\alpha_1$ level of 0.70. The analytical solution for $\alpha_1\text{max} = .67$ predicted a failure after 4.5 cycles. The failure mechanism for the analytical solution and Test BC2-05 are the same, but the differences in the actual envelope curve for the concrete and the one used in the analytical solutions produce different rates of strain accumulation.
Therefore the value of strain at failure is not in close agreement with the experimental results. However the stress level producing the failure is nearly the same in both cases.

Figure 4.25 shows the comparison of Test BC2-05 which failed after 22 cycles when subjected to a maximum $\alpha_1$ level of 0.65. The analytical solution for $\alpha_1 \text{max} = 0.63$ predicted the failure in 22 cycles. The failure patterns are again similar but the strain increments and the failure strain is less than the experimental values.

Figure 4.26 shows the variation of computed stress distributions at $\alpha_1 \text{max} = 0.63$ ($\alpha_1 \text{min} = 0.0$) for various load cycles. The cycles for which the stress distributions are plotted are identified in Figure 4.25.

4.3 e Investigation of Shakedown and Slow Cycle Fatigue Limits Under Strain Gradients

The shakedown limit for a rectangular section under combined loading is defined herein as the locus of points where the reloading portion of any $\alpha_1$-Sco curve intersects the unloading portion of the previous cycle.

Figure 4.27 shows the $\alpha_1$-Sco curves obtained from Test BC2-05 which was programmed to investigate the variation of the combined load shakedown limit and to examine
the fatigue behavior of concrete subjected to strain gradients. The basic loading pattern of Test BC2-05 is similar to the pattern used for the investigation of the shakedown limit in Chapter 3. In each stage the value of $\alpha_1$ was increased to a value near or on the envelope. After unloading to $\alpha_1 = 0.0$, the specimen was reloaded to the intersection with the previous unloading curve (point a) to obtain the first shakedown limit and unloaded to $\alpha_1 = 0.0$. This pattern was continued with a gradual decrease in the shakedown limit with each cycle. If this cycling procedure continued indefinitely, the shakedown point stabilized and a closed $\alpha_1$- $\sigma_0$ hysteresis loop formed. In Test BC2-05, intermediate cycling was stopped when the shakedown point was nearly stabilized. Thereafter a new stage was started by loading to another point on the virgin combined load curve. The locus of the combined load shakedown points below which a closed hysteresis loop formed gives the lower combined load shakedown limit.

The analytical form of the lower combined load shakedown limit was obtained by considering a loading history with an infinite number of cycles between zero and a constant $S_{CO\max}$ value. The value of strain ratio corresponding to $S_{CO\max}$ will be $S_{Cmax} = \gamma \times S_{CO\max}$ for a fiber inside the section. In the first loading the
stress ratio corresponding to \( S_{c_{\text{max}}} \) will be on the envelope curve. This stress value will gradually decrease as more cycles with constant \( S_{c_{\text{max}}} \) are added (see Test AC2-07, Figure 3.10). The value of stress will finally stabilize when the point on the lower shakedown curve corresponding to \( S_{c_{\text{max}}} \) is reached. A stable \( \alpha_{1} \) value corresponding to a given \( S_{c_{\text{max}}} \) is reached when all fibers attain a stable \( \alpha_{1} \) value corresponding to \( S_{c_{\text{max}}} \). As a result the stress distribution will be on the lower shakedown curve when a stable \( \alpha_{1} \) value is obtained.

The analytical expression for the lower bound combined load shakedown curve is obtained by integrating the lower shakedown curve formulated in Chapter 3.

\[
\alpha_{1s} = \frac{p_{s}}{bhf_{d}} = \int_{0}^{1} F_{s}(s_{d}) \, d\eta \quad \text{Eq. 4.13a}
\]

where,

\( \alpha_{1s} = \text{Value of } \alpha_{1} \text{ on the lower combined load shakedown curve corresponding to } S_{c_{0}}. \)

\( P_{s} = \text{Value of Axial load on the combined load shakedown curve.} \)

\( F_{s} = \text{Shakedown stress ratio corresponding to } S_{s}. \)

\( F_{s} \) is obtained when \( \beta = .63 \) in Eq. 3.2b, i.e.,

\[
F_{s}(s_{d}) = 0.788 \, S_{s} \, e^{(1-1.25 \, S_{s})}
\]
Performing the integration in Eq. 4.13a, the equation for \( \alpha_{1s} \) is:

\[
\alpha_{1s} = \frac{0.509}{S_{co}} \left[ e - (1.25S_{co} + 1) e^{(1 - 1.25S_{co})} \right] \quad \text{Eq. 4.13b}
\]

In Figure 4.28, Eq. 4.13b is plotted and compared to experimental results obtained from Test BC2-05. It can be expected on the basis of results from Test BC2-05 that the experimental loading curves would stabilize at the computed lower bound if the cyclic loading in each stage was continued indefinitely.

Equation 4.13b predicts a slow cycle fatigue limit of \( \alpha_1 = 0.51 \) for plain concrete under combined loading. Cyclic loadings with \( \alpha_{1\text{max}} \) lower than 0.51 will form a closed \( S_{co} - \alpha_1 \) loop when the value of \( S_{co} \) corresponding to \( \alpha_{1\text{max}} \) on the lower combined load shakedown limit is reached. This value corresponds to 74% of the theoretical maximum \( \alpha_1 \) capacity of the concrete on the envelope curve. It is interesting to note that the predicted slow cycle fatigue limit for plain concrete under concentric loading is 0.63 \( fc' \) which is 74% of the theoretical concentric load capacity of 0.85 \( fc' \).

The shakedown points obtained analytically for combined loading were generally in good agreement with the experimental results (see Figures 4.12, 4.15, 4.18, 4.19 and 4.20).
4.4 CONCLUDING REMARKS

For the type of loading and specimens used in the investigation of plain concrete subjected to strain gradients, the following conclusions can be made;

i) The assumption of a linear strain distribution throughout the cross section of the rectangular prism approximates the observed behavior.

ii) The assumption of zero tensile capacity for specimens loaded in cycles will affect the computed stress strain curves when the loading cycles have low maximum Sco values. However, the assumption of zero tensile capacity below the plastic strain will be very close to the real behavior for loading cycles with high maximum Sco values. The residual stress distribution upon unloading will gradually disappear with increasing values of Sco as the tensile capacity is reduced by extensive microcracking. The tensile capacity of the cross section is practically destroyed when the compressive outside fiber strain exceeds 1.6 ε₀.

iii) The stress-strain relation of a given fiber in a section subjected to monotonically increasing loads can be taken as the average stress-strain relation for a fiber subjected to concentric loading
if \( S_{oo} \leq 1.2\sigma_o \). The peak values of stress and strain are the same under both concentric and eccentric loads. Beyond this peak the stress-strain curve for combined loads lies above the one for axial loads. The difference in the declining part of the stress-strain curves is not due to a change in the internal structure of the material, but due to a consideration of the stability of the section.

Concentric loading specimens have zero stiffness \((\frac{dF}{dS} = 0.0)\) when \( \varepsilon_o \) is reached. Beyond this point the section is very unstable and there is a difference between the average and the maximum strains in the section. This produces an average stress-strain relation which is below that for some individual fibers in the section.

Specimens subjected to strain gradients have positive stiffness \((\frac{d\varepsilon}{dS_{oo}} > 0.0)\) when \( \varepsilon_o \) is reached. Stability of the section allows a uniform strain variation and the difference between the average and maximum strain values in the lateral direction is insignificant. This results in a stress-strain curve which lies above stress-strain curve for the concentric loading specimens.

iv) Plain concrete under combined load histories possesses a unique envelope for \( \alpha_1-S_{oo} \) relation. Failure is reached under constant or variable \( \alpha_1 \) levels when the
maximum value of $\chi_1$ and Sco lies on or near the declining part of the envelope curve.

v) The location of the lower shakedown limit for plain concrete under combined loading will be very close to the analytical lower bound obtained using the concentric loading test results. On this basis the plain concrete subjected to combined loads will have a slow cycle fatigue limit of $\chi_1 = .51$ which is about 74% of the theoretical virgin capacity.
CHAPTER 5 SUMMARY

5.1 Object and Scope

In this investigation the effect of various load histories on the behavior of plain concrete was studied. The mechanism of failure under various load histories was investigated.

A series of short rectangular columns were tested. The shape of the columns and the loading arrangement were designed to simulate different stress conditions on actual structures.

A total of 44 specimens were tested under various concentric loading histories. Using these results the characteristics of the stress-strain relation of plain concrete were studied and analytical expressions for these relationships were developed.

The stress-strain relations derived for concentric loading were used to predict the behavior of specimens under combined loading histories. Under combined loading the strain on one face of the rectangular specimen was zero and the strain on the other face was varied.

A total of 19 specimens were then subjected to combined loading histories and the results were compared with the analytical results. The differences between the
stress-strain relations under concentric and eccentric loads were studied.

5.2 Behavior of Test Specimens

5.2a Axial Loading

In each series of axial loading tests, one of the specimens was subjected to monotonic loading to obtain the ultimate stress and strain capacity and the stress-strain relation. Coordinates of stress and strain were nondimensionalized with respect to the ultimate strength $f_c'$ and the strain corresponding to $f_c'$ of a 6 in x 12 in control cylinder. The resulting nondimensionalized coordinates were; $f = f_c/f_c'$, and $S = \varepsilon_c/\varepsilon_0$. The monotonic axial load test results were compared with expressions suggested by previous authors (Fig. 3.5). The expression which was in best agreement with the test results was one suggested by Smith and Young (Eq. 3.1).

The existence of an envelope curve for plain concrete is discussed in section 3.2b. Comparing the monotonic loading stress-strain curve (approximated by Smith-Young curve) with the limiting stress-strain values of the cyclic loading curves (Fig. 3.4), the existence and uniqueness of an envelope curve for the stress-strain relation of plain concrete was established.
The shakedown limit of plain concrete was discussed in section 3.2c. The shakedown limits for loading histories with zero and nonzero minimum stress levels were compared. (See Figure 3.16.) On the basis of the comparison, the shakedown limit of plain concrete may be considered to be dependent only on the magnitude of the maximum stress and strain value of the previous loading. Test results showed that the shakedown limits for unloading from and reloading to the envelope constitute an upper shakedown limit. As cycles with lower maximum stress levels were introduced, the shakedown limits were reduced and formed a lower bound where the strain accumulation stabilized and approached zero. Using these test results, algebraic equations of the upper and lower shakedown limits were developed.

In section 3.2d the effect of nonrecoverable strains, or plastic strains, on the stress-strain relationship was studied. The tests showed a definite relation between the plastic strain ratio SP ($\epsilon_p/\epsilon_0$) and the strain values where the loading or unloading curves intersected the shakedown and envelope curves.

Algebraic expressions for loading and unloading curves were developed as functions of the plastic strain ratio and the shakedown and envelope curves in section 3.2e. The algebraic expressions for the loading and unloading compared favorably with test results (see Figures 3.31, 32, 33, 34).
A computer program was developed for studying the behavior of plain concrete under cyclic loadings varying between two constant stress levels or producing given strain increments during each cycle. (Section 3.2f.) A nomograph giving the number of loading cycles for failure under repeated stress levels was developed. (Fig. 3.45.)

5.2b Combined Loading

In Chapter 4 a computer program was developed to calculate the moment and axial load resultants under combined loads using the stress-strain relations developed in Chapter 3.

Using the results obtained from the computer solution, the nondimensionalized axial load ($\chi' = P/bh f_c'$) and nondimensionalized eccentricity ($\chi'_e = e/h$) were compared with the test results. This comparison showed that the stress-strain relation of a fiber subjected to monotonically increasing combined loads was nearly the same as the average stress-strain relation for a fiber subjected to concentric loading if the outside fiber strains were less than $1.2\varepsilon_0$. The peak values of the stress-strain curve were the same under concentric and eccentric loading. For strains greater than $1.2\varepsilon_0$, stresses under combined loading were greater than those measured under concentric loads.

In this study analytical expressions for the loading
and unloading curves of plain concrete under various loading histories have been obtained. These expressions are functions of the ultimate stress and strain values of standard 6 in. x 12 in. control cylinders and a given loading history. Using these expressions, the response of plain concrete subjected to varying load histories can be predicted.
LIST OF REFERENCES

1. ACI Bibliography No. 3, "Fatigue of Concrete" (Annotated)
   ACI Com. 215 (Fatigue Com.) 1960.


25. Mehmel A., "Investigations on the Effect of Frequently Repeated Stress on the Elasticity under Compression and Compressive Strength of Concrete" (German), Mitteilungen, Institut für Beton und Eisenbeton an der Technische Hochschule, Karlsruhe, pp. 74, Verlag Julius Springer, Berlin.


<table>
<thead>
<tr>
<th>Test No.</th>
<th>Age (days)</th>
<th>Humid Cure (days)</th>
<th>fc' (ksi)</th>
<th>F_{max} = \frac{f_{cmax}}{fc'}</th>
<th>\epsilon_0 \times 10^3 (in./in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AM1-01</td>
<td>14</td>
<td>5</td>
<td>4.51</td>
<td>.860</td>
<td>2.07</td>
</tr>
<tr>
<td>AM1-02</td>
<td>21</td>
<td>5</td>
<td>4.51</td>
<td>.870</td>
<td>1.55</td>
</tr>
<tr>
<td>AM1-03</td>
<td>21</td>
<td>7</td>
<td>4.50</td>
<td>.875</td>
<td>1.70</td>
</tr>
<tr>
<td>AM1-04</td>
<td>7</td>
<td>2</td>
<td>3.37</td>
<td>.850</td>
<td>1.56</td>
</tr>
<tr>
<td>AM1-05</td>
<td>7</td>
<td>2</td>
<td>4.55</td>
<td>.850</td>
<td>1.61</td>
</tr>
<tr>
<td>AM1-06</td>
<td>14</td>
<td>2</td>
<td>3.95</td>
<td>.840</td>
<td>1.70</td>
</tr>
<tr>
<td>AM1-07</td>
<td>7</td>
<td>2</td>
<td>3.41</td>
<td>.900</td>
<td>1.64</td>
</tr>
<tr>
<td>AM1-08</td>
<td>7</td>
<td>2</td>
<td>4.15</td>
<td>.850</td>
<td>1.85</td>
</tr>
<tr>
<td>AM1-09</td>
<td>14</td>
<td>5</td>
<td>4.18</td>
<td>.935</td>
<td>1.65</td>
</tr>
<tr>
<td>AM1-10</td>
<td>7</td>
<td>2</td>
<td>3.55</td>
<td>.930</td>
<td>1.75</td>
</tr>
<tr>
<td>AM1-11</td>
<td>21</td>
<td>7</td>
<td>4.33</td>
<td>.780</td>
<td>1.63</td>
</tr>
<tr>
<td>AM1-12</td>
<td>7</td>
<td>—</td>
<td>3.58</td>
<td>.830</td>
<td>1.68</td>
</tr>
<tr>
<td>AM1-13</td>
<td>7</td>
<td>7</td>
<td>3.63</td>
<td>.810</td>
<td>1.79</td>
</tr>
<tr>
<td>AM1-14*</td>
<td>21</td>
<td>2</td>
<td>4.59</td>
<td>—</td>
<td>1.67</td>
</tr>
</tbody>
</table>

**TABLE 3.1 MONOTONICALLY INCREASING AXIAL LOAD TESTS**

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Age (days)</th>
<th>Humid Cure (days)</th>
<th>fc' (ksi)</th>
<th>No. of Cycles to Envelope</th>
<th>Remarks (Also used for)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC2-01</td>
<td>14</td>
<td>5</td>
<td>3.77</td>
<td>3</td>
<td>Shakedown</td>
</tr>
<tr>
<td>AC2-02</td>
<td>42</td>
<td>27</td>
<td>4.60</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>AC2-03</td>
<td>21</td>
<td>7</td>
<td>4.87</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>AC2-04</td>
<td>7</td>
<td>2</td>
<td>3.65</td>
<td>5</td>
<td>Strain inc.</td>
</tr>
<tr>
<td>AC2-05</td>
<td>7</td>
<td>2</td>
<td>3.65</td>
<td>3</td>
<td>Stress levels</td>
</tr>
<tr>
<td>AC2-06</td>
<td>7</td>
<td>2</td>
<td>3.65</td>
<td>6</td>
<td>—</td>
</tr>
<tr>
<td>AC2-07</td>
<td>14</td>
<td>2</td>
<td>4.15</td>
<td>7</td>
<td>Shakedown</td>
</tr>
<tr>
<td>AC2-08</td>
<td>14</td>
<td>2</td>
<td>4.15</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>AC2-09</td>
<td>7</td>
<td>2</td>
<td>3.37</td>
<td>6</td>
<td>Strain inc.</td>
</tr>
<tr>
<td>AC2-10</td>
<td>14</td>
<td>4</td>
<td>5.01</td>
<td>7</td>
<td>—</td>
</tr>
<tr>
<td>AC2-11*</td>
<td>21</td>
<td>7</td>
<td>4.62</td>
<td>3</td>
<td>—</td>
</tr>
<tr>
<td>AC2-12*</td>
<td>21</td>
<td>7</td>
<td>4.76</td>
<td>9</td>
<td>—</td>
</tr>
<tr>
<td>AC2-13*</td>
<td>21</td>
<td>7</td>
<td>4.81</td>
<td>20</td>
<td>—</td>
</tr>
</tbody>
</table>

**TABLE 3.2 CYCLIC LOADINGS TO THE ENVELOPE CURVE (AXIAL)**

(*) Cylinder tests
<p>| TABLE 3.3 | CONSTANT STRAIN INCREMENT TESTS |
|-----------------|-----------------|----------------|----------------|
| Load to 1.5 x 10^3 then Elave 5 x 10^3 increments | 3.65 | 5 | 10 | AC3-06 |
| Add 1 x 10^3 increments from the start | 3.77 | 2 | 7 | AC3-08 |
| Add 1 x 10^3 increments from the start | 5.01 | 5 | 14 | AC3-07 |
| Load to 1.5 x 10^3 then Elave 1 x 10^3 increments | 4.66 | 5 | 21 | AC3-05 |
| Load to 1.5 x 10^3 then Elave 0.5 x 10^3 increments | 4.53 | 5 | 21 | AC3-04 |
| Load to 1.5 x 10^3 then Elave 0.25 x 10^3 increments | 4.52 | 3 | 21 | AC3-03 |
| Load up to 2 x 10^3 then Elave 0.25 x 10^3 increments | 4.38 | 4 | 21 | AC3-02 |
| Strain increments each cycle | 28 | 28 | AC3-01 |
| Magnitude of Type of the Strain Increment (kSI) | 21 | 21 | Test No. |
| Humidity Cure (days) | 21 | 21 | |
| Age (days) | 21 | 21 | |</p>
<table>
<thead>
<tr>
<th>Test No.</th>
<th>Age (days)</th>
<th>Humid Cure (days)</th>
<th>fc' (ksi)</th>
<th>Fmax.</th>
<th>Fmin.</th>
<th>No. of Loadings</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC4-01 (s)</td>
<td>7</td>
<td>2</td>
<td>3.65</td>
<td>.59</td>
<td>.00</td>
<td>11</td>
</tr>
<tr>
<td>AC4-02*</td>
<td>7</td>
<td>2</td>
<td>3.65</td>
<td>.65</td>
<td>.00</td>
<td>11</td>
</tr>
<tr>
<td>AC4-03 (s)</td>
<td>7</td>
<td>2</td>
<td>4.04</td>
<td>.64</td>
<td>.00</td>
<td>11</td>
</tr>
<tr>
<td>AC4-04*</td>
<td>7</td>
<td>2</td>
<td>4.04</td>
<td>.73</td>
<td>.00</td>
<td>12</td>
</tr>
<tr>
<td>AC4-05*</td>
<td>14</td>
<td>5</td>
<td>4.03</td>
<td>.77</td>
<td>.70</td>
<td>15</td>
</tr>
<tr>
<td>AC4-06*</td>
<td>14</td>
<td>5</td>
<td>4.03</td>
<td>.75</td>
<td>.56</td>
<td>18</td>
</tr>
<tr>
<td>AC4-07*</td>
<td>14</td>
<td>2</td>
<td>4.53</td>
<td>.72</td>
<td>.64</td>
<td>13</td>
</tr>
<tr>
<td>AC4-08*</td>
<td>14</td>
<td>2</td>
<td>4.53</td>
<td>.76</td>
<td>.47</td>
<td>11</td>
</tr>
<tr>
<td>AC4-09</td>
<td>7</td>
<td>5</td>
<td>4.55</td>
<td>.81</td>
<td>.00</td>
<td>4</td>
</tr>
<tr>
<td>AC4-10</td>
<td>21</td>
<td>2</td>
<td>3.95</td>
<td>.76</td>
<td>.00</td>
<td>21</td>
</tr>
<tr>
<td>AC4-11</td>
<td>7</td>
<td>2</td>
<td>3.63</td>
<td>.71</td>
<td>.00</td>
<td>226</td>
</tr>
<tr>
<td>AC4-12</td>
<td>7</td>
<td>2</td>
<td>3.76</td>
<td>.79</td>
<td>.00</td>
<td>17</td>
</tr>
<tr>
<td>AC4-13</td>
<td>7</td>
<td>2</td>
<td>3.76</td>
<td>.70</td>
<td>.40</td>
<td>28</td>
</tr>
<tr>
<td>AC2-05</td>
<td>7</td>
<td>2</td>
<td>3.65</td>
<td>.84</td>
<td>.00</td>
<td>3</td>
</tr>
</tbody>
</table>

(*) Loaded before failure to see the decrease in the ultimate capacity.

(s) Strains stabilized no failure was obtainable.

**TABLE 3.4 TESTS WITH LOAD CYCLES BETWEEN MAXIMUM AND MINIMUM STRESS LEVELS**
<table>
<thead>
<tr>
<th>Test No.</th>
<th>Age (days)</th>
<th>Humid Cure (days)</th>
<th>$f_{c'}$ (ksi)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC2-01</td>
<td>14</td>
<td>5</td>
<td>3.77</td>
<td>Also used in group 2</td>
</tr>
<tr>
<td>AC2-02</td>
<td>42</td>
<td>27</td>
<td>4.60</td>
<td>Also used in group 2</td>
</tr>
<tr>
<td>AC2-07</td>
<td>14</td>
<td>2</td>
<td>4.15</td>
<td>Also used in group 2</td>
</tr>
<tr>
<td>AC2-08</td>
<td>14</td>
<td>2</td>
<td>4.15</td>
<td>Also used in group 2</td>
</tr>
</tbody>
</table>

**TABLE 3.5 TESTS FOR SHAKEDOWN LIMIT (AXIAL)**

<table>
<thead>
<tr>
<th>Test No.</th>
<th>$F_s$ max</th>
<th>$S_s$ ($F_s$ max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC3-03</td>
<td>0.769</td>
<td>0.860</td>
</tr>
<tr>
<td>AC4-01</td>
<td>0.777</td>
<td>0.800</td>
</tr>
<tr>
<td>AC2-06</td>
<td>0.747</td>
<td>1.140</td>
</tr>
<tr>
<td>AC2-09</td>
<td>0.739</td>
<td>1.090</td>
</tr>
<tr>
<td>AC2-10</td>
<td>0.778</td>
<td>0.840</td>
</tr>
<tr>
<td>AC2-11</td>
<td>0.786</td>
<td>0.850</td>
</tr>
<tr>
<td>AC2-13</td>
<td>0.775</td>
<td>0.920</td>
</tr>
<tr>
<td>AC2-12</td>
<td>0.786</td>
<td>0.950</td>
</tr>
</tbody>
</table>

**TABLE 3.6 PEAK VALUES OF SHAKEDOWN CURVE FOR SOME TESTS REACHING TO THE ENVELOPE CURVE**
<table>
<thead>
<tr>
<th>Test No.</th>
<th>$f_c'$ (ksi)</th>
<th>$\varepsilon_0 \times 10^{-3}$ in/in</th>
<th>$\alpha_{1\text{max}}$</th>
<th>$S_{c_{\text{max}}} = \varepsilon_{\text{comax}}/\varepsilon_0$</th>
<th>Age (days)</th>
<th>Humid Cure Length (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BM1-01</td>
<td>3.91</td>
<td>1.57 $\times 10^3$</td>
<td>0.736</td>
<td>2.261</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>BM1-02</td>
<td>4.15</td>
<td>1.90 $\times 10^3$</td>
<td>0.728</td>
<td>1.785</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>BM1-03</td>
<td>4.18</td>
<td>1.55 $\times 10^3$</td>
<td>0.736</td>
<td>2.141</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>BM1-04</td>
<td>3.55</td>
<td>1.80 $\times 10^3$</td>
<td>0.690</td>
<td>2.011</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>BM1-05</td>
<td>4.33</td>
<td>1.76 $\times 10^3$</td>
<td>0.760</td>
<td>2.800</td>
<td>17</td>
<td>7</td>
</tr>
<tr>
<td>BM1-06</td>
<td>3.55</td>
<td>1.80 $\times 10^3$</td>
<td>0.735</td>
<td>2.48</td>
<td>7</td>
<td>2</td>
</tr>
</tbody>
</table>

**TABLE 4.1** MONOTONICALLY INCREASING COMBINED LOAD TESTS (VIRGIN CURVE)
<table>
<thead>
<tr>
<th>Test No.</th>
<th>$f_{c}'$ (ksi)</th>
<th>$e_{0} \times 10^{-3}$</th>
<th>Age (days)</th>
<th>Humid Cure Length (days)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC2-01</td>
<td>3.91</td>
<td>1.70</td>
<td>7</td>
<td>2</td>
<td>Unloadings done up to zero stress in the external fiber</td>
</tr>
<tr>
<td>BC2-02</td>
<td>4.18</td>
<td>1.55</td>
<td>14</td>
<td>5</td>
<td>Investigation of the combined load shakedown limit</td>
</tr>
<tr>
<td>BC2-03</td>
<td>3.55</td>
<td>1.70</td>
<td>7</td>
<td>2</td>
<td>3 different $\alpha_{\text{max}}$ levels applied before failure</td>
</tr>
<tr>
<td>BC2-04</td>
<td>3.58</td>
<td>1.72</td>
<td>7</td>
<td>2</td>
<td>Failure with constant $\alpha$ increments</td>
</tr>
<tr>
<td>BC2-05</td>
<td>3.58</td>
<td>1.72</td>
<td>7</td>
<td>2</td>
<td>Investigate the combined load shakedown limit</td>
</tr>
<tr>
<td>BC2-06</td>
<td>3.58</td>
<td>1.72</td>
<td>7</td>
<td>2</td>
<td>Investigate the linear strain variation in the section</td>
</tr>
<tr>
<td>BC3-01</td>
<td>3.91</td>
<td>1.57</td>
<td>7</td>
<td>2</td>
<td>Give $\Delta \alpha = 1.0$ strain increments after $\alpha = 1.0$</td>
</tr>
<tr>
<td>BC3-02</td>
<td>4.15</td>
<td>1.80</td>
<td>7</td>
<td>2</td>
<td>Give $\Delta \alpha = 0.20$ strain increments after $\alpha = 0.75$</td>
</tr>
<tr>
<td>BC3-03</td>
<td>4.15</td>
<td>1.80</td>
<td>7</td>
<td>2</td>
<td>Give $\Delta \alpha = 0.50$ strain increments after $\alpha = 0.75$</td>
</tr>
<tr>
<td>BC3-04</td>
<td>4.18</td>
<td>1.55</td>
<td>14</td>
<td>5</td>
<td>Failed after 22 cycles with $\alpha_{\text{max}} = 0.65, \alpha_{\text{min}} = 0.0$</td>
</tr>
<tr>
<td>BC4-01</td>
<td>4.33</td>
<td>1.76</td>
<td>17</td>
<td>5</td>
<td>Failed after 5 cycles with $\alpha_{\text{max}} = 0.70, \alpha_{\text{min}} = 0.0$</td>
</tr>
<tr>
<td>BC4-02</td>
<td>4.33</td>
<td>1.76</td>
<td>17</td>
<td>5</td>
<td>Failed after 62 cycles with $\alpha_{\text{max}} = 0.63, \alpha_{\text{min}} = 0.0$</td>
</tr>
</tbody>
</table>

**TABLE 4.2 COMBINED CYCLIC LOADING TESTS**
(a) Side Elev.  (b) Front Elevation.

FIG. 2.4 LOADING FRAME.
FIG. 2.6 INSTRUMENTATION - GAGE LOCATIONS.

(a) Axial Load Specimen.

(b) Combined Load Specimen.

FIG. 2.7 INSTRUMENTATION DIAGRAM.
FIG. 2.10 SPECIMEN PLACED INTO THE LOADING FRAME
FIG. 2.11  A TYPICAL TEST SET-UP
FIG. 3.1 A TYPICAL FAILURE UNDER AXIAL CYCLIC LOADING
FIG. 3.3 COMPARISON OF SMITH-YOUNG CURVE TO U.S. BUREAU OF RECLAMATION TESTS (REF. 33).
FIG 3.4 EXPERIMENTAL POINTS ON THE ENVELOPE CURVE.
FIG. 3.5 COMPARISON OF TEST AC2-10 WITH SMITH-YOUNG ENVELOPE CURVE

ENVELOPE (Smith-Yyoung)

TEST AC2-10

$S = ec/e_0$

$F = f_c / f_0$
FIG. 3.10 COMPARISON OF TEST AC2-07 WITH ENVELOPE CURVE
FIG. 3.13 EXPERIMENTAL POINTS ON UPPER SHAKE-DOWN CURVE.
FIG. 3.14 VARIATION OF THE SHAKEDOWN CURVE.
FIG. 3.15 SHAKEDOWN LIMIT FOR TESTS WITH CONSTANT MAXIMUM STRESS LEVEL.
FIG. 3.16 EFFECT OF THE MINIMUM STRESS LEVEL ON THE SHAKEDOWN CURVE.
\[ F_s = \frac{\beta_s \cdot \sigma_s / \alpha_s}{1 - \alpha_s} \]

ENVELOPE (Smith-Young)

UPPER S.D.L. (Analytical)

LOWER S.D.L. (Analytical)

FIG. 3.18

EXPERIMENTAL SHAKEDOWN POINTS COMPARED TO ANALYTICAL FORMULATION
FIG. 3.20 IDEALIZED SHAKEDOWN CURVES FOR CYCLIC LOADING WITH CONSTANT MAXIMUM STRESS LEVEL.
$SP = 0.160S_p^2 + 0.133S_s$

For load cycles with stresses greater than upper shakedown curve. ($\beta = 0.76$)

**FIG. 3.23 RELATIONSHIP BETWEEN SHAKEDOWN STRAINS AND PLASTIC STRAINS.**
$SP = 1.13(0.160S_s^2 + 0.133S_s)$

For loading cycles with stress levels less than upper shakedown curve. ($\beta = 0.63$)
FIG. 3.25 VARIATION OF S-SP RELATION WITH $\beta$

LOVEST SD. CURVE IN THE TEST

UPPER SD. CURVE

TEST AC2-07
Fig. 3.26 Strains on the envelope for loading curves from SP.

SP = 0.093 S_E^2 + 0.91 S_E
FIG. 3.27 VALUES OF SP FOR UNLOADING CURVES FROM POINTS ON THE ENVELOPE ($S_E$)

\[ SP = 0.145S_E^2 + 0.127S_E^* \]
FIG. 3.28 COMPARISON OF LINEAR AND PARABOLIC APPROXIMATIONS WITH EXPERIMENTAL LOADING CURVES.
FIG. 3.29 COMPARISON OF LINEAR AND PARABOLIC APPROXIMATIONS WITH UNLOADING CURVES.
FIG. 3.30 ANALYTICAL LOADING AND UNLOADING CURVES
FIG. 3.32 COMPARISON OF ANALYTICAL AND TEST RESULTS (AC2-05)

E = \frac{1}{1.7

S = \frac{c_c}{c_o}

Experimental

Analytical

AC2-05
FIG. 3.33 COMPARISON OF ANALYTICAL AND TEST RESULTS. (AC2-06)
FIG. 3.34 COMPARISON OF ANALYTICAL AND TEST RESULTS. (AC4-10)
FIG. 3.35 COMPARISON OF ANALYTICAL RESULT OBTAINED IN THIS STUDY AND TEST RESULT REPORTED BY SINHA GERSTLE & TULIN
FIG. 3.36 COMPARISON OF STRESS STRAIN HISTORY OBTAINED IN REF. 3 AND IN THIS RESEARCH.
FIG. 3.37 FLOW CHART FOR STRESS LEVEL PROBLEM.
FIG. 3.38 FLOW CHART FOR STRAIN INCREMENT PROBLEM.
FIG. 3.39 COMPARISON OF ANALYTICAL AND TEST RESULTS (AC2-05)
$F_{\text{max}} = 0.77$
$F_{\text{min}} = 0.00$

FIGURE 3.40 COMPARISON OF ANALYTICAL AND TEST RESULTS. (AC4 - 10)
$F_{\text{max}} = 0.71$
$F_{\text{min}} = 0.00$

(a) Experimental
(b) Analytical

FIG. 3.41 COMPARISON OF ANALYTICAL AND TEST RESULTS (AC4-11)
$F_{\text{max}} = 0.79$
$F_{\text{min}} = 0.40$

(c) Assumption of uniqueness of the loading and unloading curves.

AC4-13
Failed in 28\textsuperscript{th} loading

Failure predicted in 34\textsuperscript{th} cycle

FIG. 3.42 COMPARISON OF ANALYTICAL AND TEST RESULTS (AC4-13)
FIG. 3.43  NUMBER OF CYCLES TO FAILURE FOR TESTS WITH CONSTANT $F_{\text{max}}$ LEVEL. ($F_{\text{min}} = 0.00$)
FIG. 3.45 NOMOGRAPH FOR ESTIMATING THE NUMBER OF CYCLES FOR FAILURE UNDER REPEATED CONSTANT STRESS LEVELS.
FIG. 4.2 ANALYTICAL CALCULATION OF RESULTANT FORCE
(a) SECTION

(b) STRAIN DISTRIBUTION

\[ S_c = \eta \cdot S_{co} \]

(c) STRESS DISTRIBUTION

\[ F_c = F_c(S_c, \text{History}) \]

(d) RESULTANTS

\[ P = \alpha_1 b h f_c' \]

\[ M = \alpha_2 b h^2 f_c' \]

\[ \theta = \alpha_3 h \]

FIG. 4.3 FORCE RESULTANTS ON A RECTANGULAR SECTION
FIG. 4.4 FLOW CHART FOR THE COMPUTATION OF THE STRESS RESULTANTS.
FIG. 4.7 COMPARISON OF THE EXPERIMENTAL AND COMPUTED VALUES OF $\alpha_2$. 

Experimental Readings For $\alpha_{e}$

$\alpha_e$ (Eq. 4.10)

$\alpha_2$ (Eq. 4.10)

$\alpha_2$ (Eq. 4.9)

$k_2 = 1 - \alpha_e$
FIG. 4.8 STRESS STRAIN CURVES FOR CONCENTRIC AND ECCENTRIC LOADING.

SHAH (Eq. 4.12b)

S = \varepsilon_c / \varepsilon_0

F_s = 0.931 - 0.077S

(Eq. 4.10)

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\frac{\varepsilon_c}{\varepsilon_0} = S

\frac{F_s}{F_c} = F
Figure 4.9 $\alpha_1 - S_{co}$ relation computed from various stress-strain curves.
FIG. 4.10 $\alpha_{1}-S_{co}$ RELATION COMPUTED FROM VARIOUS STRESS-STRAIN CURVES.
FIG. 4.11 COMBINED LOADING ENVELOPE CURVES.
Figure 4.12: Comparison of Measured and Computed Values of $X_{1,e_0}$ (BC3-04) vs $S_c e_0/e_0$.
Fig. 4.13 Comparison of measured and computed values of $\alpha_1 - S_{C_0} (BC3-04)$
FIG. 4.14 COMPUTED STRESS DISTRIBUTION FOR TEST BC3-04
FIG. 4.15 COMPARISON OF MEASURED AND COMPUTED $\alpha_1 - S_c = \sigma / \sigma_0$ RELATION (BC2-06)
FIG. 4.17 COMPUTED STRESS DISTRIBUTION FOR TEST BC2-06
FIG. 4.22 COMPUTED STRESS DISTRIBUTION FOR TEST BC2-01
Fig. 4.23: Computed and experimental number of cycles for failure under constant $\alpha_{1\text{max}}$ level.
FIG. 4.24 COMPARISON OF COMPUTED AND EXPERIMENTAL \( \alpha_1 - S_{co} \) RELATION (BC4-02)
FIG. 4.25 COMPARISON OF COMPUTED AND EXPERIMENTAL $\alpha_1 - S_{co}$ RELATION. (BC4-02)
1st Loading

$S_{ce} = 1.020$
$F_{ce} = 0.647$
$\alpha_1 = 0.630$
$\alpha_2 = 0.602$

5th Loading

$S_{ce} = 1.393$
$F_{ce} = 0.719$
$\alpha_1 = 0.630$
$\alpha_2 = 0.577$

6th Loading

$S_{ce} = 1.441$
$F_{ce} = 0.701$
$\alpha_1 = 0.630$
$\alpha_2 = 0.571$

7th Loading

$S_{ce} = 1.483$
$F_{ce} = 0.685$
$\alpha_1 = 0.630$
$\alpha_2 = 0.570$

20th Loading

$S_{ce} = 2.813$
$F_{ce} = 0.482$
$\alpha_1 = 0.650$
$\alpha_2 = 0.522$

21st Loading

$S_{ce} = 2.550$
$F_{ce} = 0.440$
$\alpha_1 = 0.630$
$\alpha_2 = 0.506$

Last loading

$S_{ce} = 3.000$
$F_{ce} = 0.346$
$\alpha_1 = 0.666$
$\alpha_2 = 0.480$

Last loading

$S_{ce} = 3.600$
$F_{ce} = 0.227$
$\alpha_1 = 0.561$
$\alpha_2 = 0.413$

FIG. 4.26 COMPUTED VARIATION OF STRESS DISTRIBUTION WITH NUMBER OF CYCLES FOR $\alpha_{1m0} = 0.63$
Fig. 4.28 Computed and experimental lower shakedown curves (BC2-05)