HURLEY, Jr., Francis Xavier, 1940-
A PROBLEM OF THREE-DIMENSIONAL HYPERSONIC BOUNDARY LAYER INTERACTION.

Rice University, Ph.D., 1968
Engineering, aeronautical

University Microfilms, Inc., Ann Arbor, Michigan
RICE UNIVERSITY

A PROBLEM OF THREE-DIMENSIONAL HYPERSONIC BOUNDARY LAYER INTERACTION

BY

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Thesis Director's Signature:

Houston, Texas
May, 1968
ACKNOWLEDGEMENTS

The author expresses sincere appreciation to the members of his Rice committee: Dr. A. J. Chapman, Dr. C. S. Burrus, and in particular to his advisor, Dr. F. A. Wierum, whose guidance was ever instructive and never obstructive.

Much thanks are due to the National Aeronautics and Space Administration, which supported this research and the author as a NASA trainee under NASA Grant NSG(T)9. Support was also provided by the Goldston Foundation.

Certainly the author should acknowledge the opportunities and aids that have come before: especially the help of his parents, and the fine undergraduate and master's level education at Elizabethtown College, Pennsylvania State University, and Princeton University.

The key event is having been born in this country.
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SYMBOLS

\(a\) = energy integral constant
\(A\) = free parameter of symmetry plane boundary layer growth
\(b\) = energy integral constant
\(B\) = constant of quasi-two-dimensional boundary layer growth
\(B'\) = modified boundary layer growth constant
\(C_H\) = coefficient of heat transfer, \(\left(\frac{\nu T}{\nu Y}\right)^{1/2} \cdot \frac{C_p}{2} (T_{STAG} - T_{∞}) U_{∞}\)
\(C_p\) = specific heat at constant pressure
\(C_{f_x}\) = coefficient of skin friction, \(\frac{\nu \frac{\partial u}{\partial Y}}{U_{∞}^2 R_{∞}}\)
\(F\) = function for center plane region boundary layer thickening
\(h\) = enthalpy
\(H\) = total enthalpy, \(h + \frac{1}{2} q^2\)
\(k\) = conductivity
\(K\) = hypersonic similarity parameter
\(m,n\) = general exponents
\(M\) = Mach number
\(n_i\) = component of unit surface normal
\(P\) = pressure
\(Pr\) = Prandtl number
\(q_i\) = velocity component
\(\bar{Q}\) = heat conduction vector
\(J^{R}\) = shorthand for equation residual, in iterative solution technique
\begin{align*}
R &= \text{gas constant} \\
Re_x &= \text{Reynolds number, } \frac{\rho u_x}{\mu} \\
s &= \text{effective boundary layer development length} \\
S &= \text{control surface area} \\
S_u &= \text{constant of } u\text{-velocity profile} \\
S_w &= \text{constant of } w\text{-velocity profile} \\
S_\rho &= \text{constant of density profile} \\
T &= \text{temperature} \\
u &= \text{chordwise velocity component} \\
u_{10} &= \text{free parameter of } u\text{-function} \\
U &= \text{free stream velocity} \\
v &= \text{vertical velocity component} \\
V &= \text{volume} \\
w &= \text{spanwise velocity component} \\
w_{12} &= \text{free parameter of } w\text{-function} \\
x &= \text{chordwise coordinate, measured from vertex origin} \\
y &= \text{coordinate normal to wing surface} \\
z &= \text{spanwise coordinate, measured from symmetry plane} \\
z_E &= \text{ray defining outboard boundary of symmetry plane region} \\
\gamma &= \text{specific heat ratio} \\
\delta &= \text{boundary layer thickness} \\
\delta_{ij} &= \text{Kronecker delta} \\
\Delta \xi &= \text{width dimension of center plane region, near vertex} \\
\Delta_\delta(\ ) &= \text{flux from outer flow stream} \\
\epsilon &= \text{surface streamline inflow angle} \\
\zeta &= \text{nondimensional spanwise coordinate} \\
\eta &= \text{nondimensional normal coordinate}
\end{align*}
\( \lambda \) = second coefficient at viscosity

\( \Lambda \) = sweep angle of leading edge

\( \Lambda_E \) = sweep angle of ray \( z_E \)

\( \mu \) = viscosity

\( \rho \) = density

\( \rho_1 \) = shorthand representation of \( \rho \)-function factor

\( \rho_{10} \) = free parameter of \( \rho \)-function

\( \bar{\sigma} \) = viscous stress tensor

\( \phi_i \) = shorthand for parameter value, in iterative solution technique

\( \chi \) = hypersonic interaction parameter

Subscripts

\( c \) = center plane

\( \xi \) = centerline

\( E \) = edge of symmetry plane region

\( i,j,k \) = general indices

\( \text{MIN} \) = minimum

\( W \) = wall surface

\( x \) = derivative with respect to \( x \) when \( z \) is constant

\( z \) = derivative with respect to \( z \) when \( x \) is constant

\( \delta \) = outer edge of boundary layer

\( 1 \) = upstream control surface

\( 2 \) = downstream control surface

\( 2D \) = quasi-two-dimensional case

\( \infty \) = free stream
INTRODUCTION

The Prandtl boundary layer concept is well established in fluid mechanics. The idea is that frictional forces may not be ignored at those points in the flow which are quite close to solid boundaries or obstacles. The fluid right at a foreign surface has zero velocity, and there is a thin frictional layer across which the speed varies from zero up to the "outer" flow value, $U_S$. See Fig. 1.

Often a boundary layer may be analyzed with the aid of the "two-dimensional" simplification, in which only the coordinates in the outer flow direction and in the perpendicular direction to the retarding surface are required. This might be the case for a long flat plate or for a long wing without sweep. Variations in the spanwise direction are considered negligible. The same solution holds for each spanwise location.

One may then subclassify two-dimensional problems according to speed or relative compressibility effects. At "low" speeds density may remain nearly constant, and the simplest, earliest-solved problem obtains (Ref. 1). At higher speeds, more quantities must be considered variable, and solutions are more difficult. Even so, much work has been done in this area (Ref. 2), and in fact compressible problems can be related to incompressible problems under certain assumptions. At very high speeds a new phenomenon - hypersonic viscous interaction - may appear. It becomes necessary to account for the fact that the boundary layer, though still rather thin, can alter the outer flow field and
pressures by effectively thickening the body. See Refs. 3 and 4, for example.

For a swept surface or for a body of irregular shape, the two-dimensional simplification may not be appropriate. Again the problems may be subclassified according to relative compressibility effects. Simplifications occur and various results from two-dimensional theory become available, if the coordinate system is aligned with the local outer flow stream direction and if it may be assumed that the relative flow turning, down through the boundary layer, is small enough to be treated as a perturbation. See Refs. 5 and 6 for such treatments of the incompressible and compressible-without-interaction problems respectively.

Finally then we get to the area of three-dimensional boundary layers in hypersonic viscous interaction. Perhaps the simplest case is that of the infinite span but swept flat plate wing. The viscous induced pressures are highest along the leading edge. Therefore boundary layer streamlines are pushed in an "inboard" direction. See Fig. 2a. Actually this problem could be regarded as quasi-two-dimensional, for there is still a direction in which all derivatives are zero: the infinite dimension, parallel to the leading edge. Solutions to this problem have been generated by at least three different workers, in Refs. 7, 8, and 9.

Perhaps the most logical next topic is the center plane region of the delta wing. With boundary layer fluid being deflected inboard from either side, streamlines should be bunched up and turned downstream in this area. See Fig. 2b. Probably the boundary layer
will be thickened and the viscous induced pressures will be elevated. Supporting experimental data is found in Refs. 10 and 11. The region is truly three-dimensional; i.e., derivatives are not zero in any direction. It seems worthwhile to study this problem, for the delta shape is of practical importance, and the boundary layer solution is what determines the heat transfer requirements as well as both drag (surface friction) and lift (pressure) forces.

Ref. 12 is one relevant treatment for the boundary layer in the center plane region of a hypersonic delta wing. The degree of complication is reduced by using the spanwise coordinate from the center plane as a perturbation parameter. Further, a particular surface contouring rather than the viscous interaction phenomenon is postulated to dominate the pressure field. This results in the appearance of a "similarity variable," and thus another reduction in degree of complication. Therefore the originally three-dimensional boundary layer equations are reduced to a set of coupled but ordinary differential equations. To solve them, extensive machine computations are yet required.

Ref. 13 appears to be the most advanced treatment of the problem investigated here, i.e., the flat delta wing with the pressure field dominated by viscous interaction. The starting point in the analysis of that paper is the same as will be used in the present work, and it may be stated as follows: the center plane region of interest is enclosed by a control surface, and balance equations for mass, momentum, and energy are written. These equations consist of terms which are integrals of fluxes through the bounding surfaces (like \( \iint \rho u^2 ds_i \)) and integrals of forces on the bounding surfaces (like \( \iint P ds_i \)).
From this point, there are various ways of seeking solutions that satisfy the overall balances. Without going into the details, it is noted that Ref. 13 examines the problem under a pair of special limit assumptions: Mach number very large and downstream distance from vertex very large. The resulting limit solution is not physically realistic, as it describes a very thick layer of slow-moving fluid which does not much resemble a boundary layer.

In the present work no such limit case will be studied. No universal parameters or similarity variables will appear. Instead, a specific, practical set of flight conditions will be postulated. Therefore it may be said that although the method is a general one, no general solution will be obtained; different flight conditions will always require new calculations.

Briefly, the method is as follows: functions are written for all the dynamic and thermodynamic quantities \((u, p, \text{etc.})\) within the region of interest, as certain three-dimensional polynomial forms containing strategically placed solution parameter coefficients to be found. Upon substituting same into the balance equations, the integrals can be carried out analytically. The resulting algebraic equations are solved for the solution parameters.

Since the integral balance forms rather than the differential equations of motion are used, the method is an approximate one. The two-dimensional analog is the von Karman-Pohlhausen technique, which has proved quite successful (as, Refs. 1, 14). Nevertheless there is no mathematical guarantee that a parameter solution set which is physically realistic will be obtained. Certainly it is important that the
postulated function forms satisfy all reasonable boundary conditions independently of the sought parameters.

The algebra is rather complicated. Iterative machine computations are required to find the parameter solution set. It turns out that, for the flight conditions specified here, there is prompt convergence to a satisfying solution.

That solution does describe a thickened boundary layer and (correspondingly) elevated pressures. Skin friction and heat transfer coefficients are reduced.

Throughout the development, account is kept of the various assumptions and limitations in the proposed model. In view of the acceptable character of the results for the example problem solved here, it is believed that the present method should give at least qualitatively accurate answers for a range of cases within those limitations. A systematic experimental study, for quantitative comparisons, does not seem to exist in the open literature at this time.
MASS, MOMENTUM, AND ENERGY BALANCES

In order to systematically derive the mass, momentum (vector), and energy integral balances for any control volume, we first write the differential equations "in divergence form:"

\[ \nabla \cdot \rho \vec{q} = 0 \]  
1.

\[ \nabla \cdot (\rho \vec{q} \vec{q}) + \nabla P - \nabla \cdot \vec{\sigma} = 0 \]  
2.

\[ \nabla \cdot (\rho \vec{q} \vec{H}) + \nabla \cdot \vec{Q} - \nabla \cdot (\vec{\sigma} \cdot \vec{q}) = 0 \]  
3.

The usual linear constitutive equations will be used:

\[ \vec{\sigma}_{ij} = \lambda \delta_{ij} q_{k,k} + \mu [q_{i,j} + q_{j,i}] \]  
4.

\[ \vec{Q} = -\kappa \nabla T \]  
5.

Disallowing any mass or (real gas) energy sources or sinks, Eqs. 1-3 hold at each point in the fluid, and the integral over any volume will still give zero for the right hand sides. If we then apply the vector theorem:

\[ \iiint_{V_o} (\ ) \cdot i \, dV = \iint_{S_o} (\ ) \cdot n_i \, dS \]  
6.

the three equations will become sets of area integrals describing fluxes through and forces on the several bounding surfaces of the chosen control volume.

Before choosing the control volume and thus specifying the problem to be solved, some general background comments are in order.

"Hypersonic viscous interaction" occurs if the outer inviscid flow field (especially the pressure) is measurably affected by the
presence of the boundary layer. The interaction is termed "strong" rather than "weak" if the boundary layer is itself modified significantly from its no-interaction solution. See Ref. 3. Strong interaction obtains when \( \overline{\chi} \gg 1 \), where:

\[
\overline{\chi} \equiv \frac{M_{\infty}^3}{V_{Re_x}} \sim M_{\infty} \frac{dS}{dx} \equiv K
\]

7.

Consequently it can be said that one must assume strong interaction if either

1. \( M_{\infty} \) is very large, or
2. A region near the leading edge is being examined, and \( \frac{dS}{dx} \) is not small.

Extra difficulties usually arise if the interaction is strong. In the present treatment they would appear not because of the feedback coupling, but because of the following reasons. If \( M_{\infty} \) is very large, excessively high temperatures and therefore chemical activity probably will occur within the boundary layer. Furthermore, it turns out to be much more difficult to approximate with simple polynomials the sort of temperature or density profiles that occur. If \( \frac{dS}{dx} \) is not small, certain small angle approximations that will be algebraically crucial in the formulation are disallowed.

In the present treatment, then, it will be assumed that the speed is only moderately hypersonic, and also that \( K < \frac{4}{\gamma + 1} \). This will allow the use of the "small K" tangent wedge formulas, and will defeat the difficulties mentioned above. The two way coupling between the outer and inner (boundary layer) flows, now rather slight, will be kept, however.
Fortunately there are some arguments against having to rank the strong interaction problem high in importance. Likely reentry vehicle speed/altitude trajectories are such as to confine strong interaction to regions very near the leading edge, compared with a typical length dimension. Slight blunting or slight surface inclination also smoothes strong interaction.

It is possible to do a qualitative, order of magnitude analysis of what must occur at the vertex (strong interaction) of the flat delta wing. The reader is referred to Appendix A.

For the "weak" interaction study in the present paper, the control volume as pictured in Fig. 3 is used. Its planform is trapezoidal, with the surface $S_1$ far enough downstream to be outside of the strong interaction region. The bounding ray is $z_E = x / \tan \Lambda_E$. Ref. 15 indicates that in problems of this general nature, the exact choice of $\Lambda_E$ is not very important, as long as the whole Mach cone from the vertex is enclosed. The outboard, quasi-two-dimensional solution, i.e. at all $z > x / \tan \Lambda_E$, is assumed to be completely known.

Therefore the volume integrals over Eqs. 1 - 3 are taken, and Eq. 6 is used. The usual boundary layer approximations are made:

\[

\begin{align*}
1 & \approx 1, \omega \gg 1, v \\
\left| \frac{\partial}{\partial y} \right| & \gg \left| \frac{\partial}{\partial x} \right|, \left| \frac{\partial}{\partial y} \right| > \left| \frac{\partial}{\partial z} \right| \quad \text{at least at } z = 0 \text{ and } z = z_E \\
\left| \frac{\delta}{\delta x} \right|, \left| \frac{\delta}{\delta z} \right| & \ll 1 \quad \text{at least at } z = 0 \text{ and } z = z_E \\

P & \approx f_n (y) \\
\end{align*}
\]

\[\text{8.}\]
Quite a few terms have to be examined, but the final balance equations turn out to be:

\[
\begin{align*}
\left\{ \left( \int \rho u \, dS_i + \int \rho u \, dS_2 + \int \rho \left[ u \cos \Lambda_E - w \sin \Lambda_E \right] \, dS_E \right) 
+ \int \rho w \, dS_c \right\} \Delta_B \text{ mass flux} &= 0,
\end{align*}
\]

\[
\begin{align*}
\left\{ \left( \int \rho u^2 \, dS_i + \int \rho u^2 \, dS_2 - \int \rho u \left[ u \cos \Lambda_E - w \sin \Lambda_E \right] \, dS_E \right) 
- \int P \, dS_i + \int P \, dS_2 - \int P \cos \Lambda_E \, dS_E - \int P \delta_\xi \, dS_x \right\}
\left( \int \rho u w \, dS_c \right) + \int \mu \frac{\partial u}{\partial y} \, dS_w - \Delta_B \text{ x-momentum flux} &= 0,
\end{align*}
\]

\[
\begin{align*}
\left\{ \left( \int \rho u \, dS_i + \int \rho u \, dS_2 - \int \rho w \left[ u \cos \Lambda_E - w \sin \Lambda_E \right] \, dS_E \right) 
+ \int P \sin \Lambda_E \, dS_E - \int P \, dS_c - \int P \delta_\xi \, dS_x \right\}
\left( \int \rho w^2 \, dS_c \right) + \int \mu \frac{\partial w}{\partial y} \, dS_w + \Delta_B \text{ z-momentum flux} &= 0.
\end{align*}
\]

\[
\begin{align*}
\left\{ \left( \int \rho u \left[ h + \frac{1}{2} u^2 + \frac{1}{2} w^2 \right] \, dS_i + \int \rho u \left[ h + \frac{1}{2} u^2 + \frac{1}{2} w^2 \right] \, dS_2 
- \int \rho \left[ h + \frac{1}{2} u^2 + \frac{1}{2} w^2 \right] \left[ u \cos \Lambda_E - w \sin \Lambda_E \right] \, dS_E \right) 
\left( \int \rho \left[ h + \frac{1}{2} u^2 + \frac{1}{2} w^2 \right] \, dS_c \right) + \int k \frac{\partial T}{\partial y} \, dS_w \right\} \Delta_B \left[ h + \frac{1}{2} u^2 + \frac{1}{2} w^2 \right] \text{ flux} &= 0.
\end{align*}
\]
A few other things should be noted here. The $\Delta \delta(\ )$ flux terms represent the mass, momentum, and energy contributions from the outer flow; as, the usual "growth rates." In the present treatment it will be assumed that the three-dimensional, symmetry plane effect is moderate rather than catastrophic. Then the $\Delta \delta(\ )$ flux quantities will be taken directly from the quasi-two-dimensional problem, i.e. the problem in which the outboard solution is extrapolated through the center plane region. It is only in this case that the $\delta S_c$ integrals which have been set off by parentheses are not equal to zero, i.e. $w(x, y, z = 0) = 0$ is not imposed. Incidentally, in the limit case of Ref. 13, the flux terms from the outer flow were negligible altogether.

Also, it is noted that the density profile but not the temperature profile appears in the equations, as long as we use the perfect gas relations:

$$\rho h = C_p \rho T = \frac{C_p}{R} P = \frac{\gamma}{\gamma - 1} P(x, z \text{ only})$$

13.

This turns out to be algebraically important.
POSTULATED FUNCTIONS

In two-dimensional problems it is said that a \( u(x,y) \) solution is "similar" if \( \frac{u(x,y)}{U_S(x)} \) is a function only of \( \frac{y}{\delta(x)} \equiv \eta \), where \( \delta(x) \) is usually part of the solution and \( U_S(x) \) is usually established beforehand. See Ref. 1. In the present formulation of the center plane region problem, it will be postulated that this holds along each ray \( \frac{Z}{Z_E(x)} \equiv \zeta = \text{constant}, \text{ separately}. \) The manner in which \( u, \, w, \) and \( \rho \) vary from ray to ray as the center plane is approached, as well as the modification of boundary layer thickness \( \delta(x,z) \), is to be determined by the solutions for the free parameters.

At \( \zeta = 1 \), the center plane region solution is to join smoothly with the outboard, quasi-two-dimensional solution. At the symmetry plane, \( \zeta = 0 \), the condition will be imposed that all \( \zeta \)-derivatives equal zero, for it is expected that there should be no "sharp corners" in any of the distributions there.

It will be appreciated that there is an unlimited number of ways in which the four functions \( u, \, w, \rho, \) and \( \delta \) can be constructed incorporating four strategically placed free parameters. At this point algebraic considerations enter the picture. A look at the integrals in the four balance equations (9-12) indicates the practical difficulties that can arise in carrying them out, especially when it is remembered that the current degree of outer flow compression by \( \delta(x,z) \) determines the pressure as well as the density scale \( \rho_\delta \). It has been concluded that simple polynomial forms should be used wherever possible.
The $\delta(x,z)$ boundary layer thickness function is constructed as follows. In the quasi-two-dimensional region, i.e. for $\zeta > 1$, it is expected that the thickness is given by the usual strip theory result:

$$\delta_{2D} = B s^{\frac{1}{2}} = B \left[ x - \frac{x}{\tan \phi} \right]^{\frac{1}{2}}$$

where $B$ is known. If there were no bunching, three-dimensional effect at the symmetry plane $z = 0$, the answer there would be $\delta = B x^{\frac{1}{2}}$; instead we postulate the average growth pattern:

$$\delta \left( x, \zeta = 0 \right) = A x^{\frac{1}{2}}$$

where $A$ is a free parameter to be determined, and it is expected to be greater than $B$. In between $\zeta = 0$ and $\zeta = 1$ it is desired to have a smooth thickness variation, say as indicated in Fig. 4a. After some algebra it turns out that a satisfactory function is:

$$\delta \left( x, \zeta \right) = A x^{\frac{1}{2}} \left[ 1 - \frac{A-B'}{A} \tan \phi \left( \frac{\zeta}{x} \right) \right]$$

where

$$B' = B \left[ 1 - \frac{\tan \phi}{\tan \phi' \tan \phi} \right]^{\frac{1}{2}}$$

$$F \left( \zeta \right) = \left( 3 - S_8 \right) \zeta^2 + \left( S_8 - 2 \right) \zeta^3$$

$$S_8 = \frac{1}{2} \frac{B'}{A-B'} \frac{\tan \phi}{\tan \phi' - \tan \phi}$$

Because of the complicated way in which the free parameters like $A$ enter the various integrals, and because only physically realistic solutions are to be sought, it is important to bracket such parameter solutions beforehand. Sketching the different possibilities for the cubic $F(\zeta)$ and relating the geometry to the derivatives, it is found that $\delta(x,\zeta)$ is monotonic in $\zeta$, and is largest at $\zeta = 0$, i.e. as pictured in Fig. 4a, if and only if $A$ is bracketed through
The outer flow properties are automatically determined by $\delta (x, \zeta)$ through the small $K$ tangent wedge approximations, i.e. Eq. 7 and

\begin{align*}
\frac{P}{P_\infty} &= 1 + \gamma K + O(K^2) \\
\frac{P_e}{P_\infty} &= 1 + K + O(K^2) \\
\frac{T_e}{T_\infty} &= 1 + (\gamma - 1) K + O(K^2) \\
\frac{U_e}{U_\infty} &= 1 - O(K^2)
\end{align*}

For example, see Ref. 16.

An $\eta$-profile of the function $u(x,y,z)$ is expected to be somewhat as pictured in Fig. 4b. The usual no-slip condition is imposed, and the profile slope is set equal to zero at the top of the boundary layer, i.e. at $\eta = 1$, where $\frac{u}{U_\infty} \equiv 1$. Furthermore the function is constructed so as to have a zero $\zeta$-derivative at $\zeta = 0$, and to match the outboard region profile specified by $\frac{d u}{d \eta} (\eta = 0, \zeta = 1) \equiv U_e$. The simplest polynomial product which satisfies all these boundary conditions is:

$$
\frac{u}{U_\infty} = 3 \eta^2 - 2 \eta^3 + [\eta - 2 \eta^2 + \eta^3] \left[ u_o + (S_u - u_o) \zeta \right]
$$

The free parameter is $u_o = \frac{d u}{d \eta} (\eta = 0, \zeta = 0)$, and so its solution will indicate how much fatter or thinner the $u$-profile gets as one moves into the center plane region. If reversals or overshoots in the profile are considered unlikely (though not impossible), then solutions $0 < u_0 < 3$ are sought.
The function \( w(x,y,z) \) is expected to exhibit \( \eta \)-profiles as sketched in Fig. 4c. So it is constructed as above, except that one now imposes \( \frac{w}{U_\infty}(\eta = 1, \text{ all } \zeta) = 0 \) as well as \( \frac{w}{U_\infty}(\zeta = 0, \text{ all } \eta) = 0 \) (center plane impenetrability, because of symmetry). The result is:

\[
\frac{w}{U_\infty} = \left[ \eta - 2\eta^2 + \eta^3 \right] \left[ \nu_{12} \zeta^2 + \left( S_w - w_{12} \right) \zeta^3 \right]
\]

That the present treatment is a "gross" or "average" one is shown clearly by the Eq. 20, for \( \frac{\partial w}{\partial \eta}(\eta = 0, \zeta = 1) = S_w \), independently of \( x \). That is, the single given quantity \( S_w(<0) \) is supposed to represent the average intensity of the feeding span flow for the whole region. The free parameter \( w_{12} \) is an indication of profile shape changes, just as \( u_{10} \) above. If the span flow \( w \) is to be directed inboard through the whole center plane region, then one seeks solutions \( w_{12} < 0 \).

Construction of a density function \( \rho(x,y,z) \) is a somewhat more difficult proposition. Again it is desired to achieve a reasonably uncomplicated polynomial form, with a free parameter. Density profiles however are much more sensitive to the particular boundary conditions than are velocity profiles. For one thing, \( \rho(x,\zeta,\eta = 1) = \rho_\infty \) should not be taken as \( \rho_\infty \) if interaction is thought to be significant. See Eq. 18. More importantly, the hypersonic, cooled-surface boundary layer does not have a monotonic density profile. See Fig. 4d. The local outer flow Mach number \( M_\infty \) is very important in determining the relative dissipation, heating, and rarefaction of streamlines around the middle portion of the boundary layer, i.e. the extent to which the density and temperature profiles are "bowed." The coupling is quite complicated,
with temperature boundary conditions and velocity profile shapes also involved.

Therefore, two things are necessary at this stage in the formulation of the model: a particular set of flight conditions must be specified, and an approximate description of the manner in which the current degree of displacement compression influences the profile shape, for those flight conditions, must be derived. This is done below. It should be realized then that answers for other flight conditions will always require some new density profile analysis, rather than just substitution of new input parameters like \( B, S_u, \) etc. (See also Appendix D.)

The flight conditions chosen here are:

\[
\begin{align*}
\text{altitude} & \approx 150,000 \text{ feet} & M_\infty & = 7 & T_w & = 3 & \gamma & = 1.4 \\
\alpha & = 60^\circ & \alpha_E & = 78^\circ & x_1 & = 5 \text{ feet} & x_2 & = 15 \text{ feet} \\
B & = 0.034(\text{feet})^{-\frac{1}{2}} & S_u & = 1.45 & S_w & = -0.39 \text{ and } -0.838 & S_\rho & = -0.95(\text{below})
\end{align*}
\]

Using the above, as well as the common hypersonic linear \( u \)-velocity profile, it is possible to write the Crocco energy integral:

\[
C_p T + \frac{1}{2} u^2 = au + b
\]

as a polynomial relation of the form:

\[
\frac{T}{T_0} = \frac{\rho \dot{\rho}}{\rho} = f \left[ \eta, \frac{\rho_0}{\rho_w} \right]
\]

Taking the \( \eta \)-derivative of Eq. 23 and setting it equal to zero gives a description of the effect of the parameter \( \frac{\rho_0}{\rho_w} \) on the location
and magnitude of $p_{\text{MIN}}/p_8$. This has been done, and the results are described approximately by:

$$\eta_{p\text{en}} = .40$$

$$\frac{p_{\text{MIN}}}{p_8} = .02 + .61 \frac{p_w}{p_8}, \quad \text{where} \quad \frac{p_w}{p_8} = \frac{T_8}{T_{\infty}} \cdot \frac{T_{\infty}}{T_w}$$

Eq. 24 is the desired statement of the effect of current degree of displacement compression $\left(\frac{T_8}{T_{\infty}}\right)$ on $\rho$-profile shaping $\left(\eta_{p\text{MIN}}, p_{\text{MIN}}\right)$.

Certainly the above energy integral is not rigorously justified. In fact it neglects not only that $Pr \neq 1$ and $C_p \neq$ constant, but it also disregards the presence of a spanwise velocity component $w$ and a nonuniform pressure field. Furthermore Eq. 19 provides that $u/U_{\infty} \neq \eta$ in general. Let it be stated only that there is some evidence for believing that such energy integrals have a larger range of validity than the assumptions would suggest (Ref. 2). Furthermore all static pressures in the present problem will be small relative to the free stream dynamic pressure. Also $w^2$ will be quite a bit smaller than $u^2$ at most points, and the $u$-profile will probably become more linear as the plane of symmetry is approached.

The boundary conditions to be imposed on the $\rho(x,y,z)$ to be constructed are collected:

$$\frac{\rho}{\rho_8} = \frac{\rho_w}{\rho_8}, \quad \text{at} \quad \eta = 0$$

$$\frac{\rho}{\rho_8} = 1 \quad \text{and} \quad \frac{\partial \rho}{\partial \eta} = 0, \quad \text{at} \quad \eta = 1.$$
\[
\frac{\rho}{\rho_b} = 0.02 + 0.61 \frac{\rho_w}{\rho_b} \quad \text{and} \quad \frac{d}{d\eta} \frac{\rho}{\rho_b} = 0, \quad \text{at} \quad \eta = 0.40
\]

\[
\frac{d}{d\zeta} \left( \frac{\rho}{\rho_b} \right) = 0, \quad \text{at} \quad \zeta = 0
\]

\[
\frac{d}{d\eta} \left( \frac{\rho}{\rho_b} \right) \bigg|_{(\eta = 0, \ \zeta = 1)} = S_{\rho}
\]

A polynomial function of minimal complexity which satisfies all these conditions is:

\[
\frac{\rho}{\rho_b} = \frac{\rho_w}{\rho_b} + \eta \rho_i + \eta^2 \left[ 3.393 - 14.400 \frac{\rho_w}{\rho_b} - 7.000 \rho_i \right]
\]

\[
+ \eta^3 \left[ -18.180 + 51.175 \frac{\rho_w}{\rho_b} + 17.250 \rho_i \right]
\]

\[
+ \eta^4 \left[ 31.180 - 64.150 \frac{\rho_w}{\rho_b} - 17.500 \rho_i \right]
\]

\[
+ \eta^5 \left[ -15.393 + 26.375 \frac{\rho_w}{\rho_b} + 6.250 \rho_i \right]
\]

\[
\text{where} \quad \rho_i = \rho \bigg|_{10} + [S_{\rho} - \rho \bigg|_{10}] \zeta^2
\]

The numerical coefficients in Eq. 26 will change when flight conditions are changed or when a different method for density function analysis is used. Although it is not obvious at this point, a polynomial form as written above turns out to be algebraically preferable to direct use of an energy integral like Eq. 23.

The quantity \( \rho \bigg|_{10} \) is a free parameter whose solution is sought, and its significance is easily seen by noting that:

\[
\frac{d}{d\eta} \left( \frac{\rho}{\rho_b} \right) \bigg|_{(\eta = 0, \ \zeta = 0)} = \rho_0
\]
Actually there are quite a few possible profile shapes for $\frac{\rho}{\rho_0}$ as given by Eq. 26, depending on the value of $\rho_{10}$. Sketching them, and relating the sketches to the derivatives, it is easily shown that shapes like Fig. 4d require a $\rho_{10}$ solution in the range:

$$-1.84 - 2.92 \frac{\rho_w}{\rho_0} < \rho_{10} < 0$$

It is now possible to substitute the postulated functions (Eqs. 16, 19, 20, 26) into the balance relations (Eqs. 9-12), using also the various auxiliary information (Eqs. 7, 18, 21, etc.). Then one obtains four complicated algebraic equations in the four unknowns: $A, u_{10}, w_{12}, \rho_{10}$. Prior calculation of the $\Delta_5(\cdot)$ flux terms must be accomplished by substitution of the completely specified quasi-two-dimensional functions, which are:

$$\delta_{2D} = B x^\frac{1}{2} \left[ 1 - \zeta \frac{\tan A}{\tan A_0} \right]^{\frac{1}{2}}$$

$$\frac{u_{2D}}{U_\infty} = 3 \eta^2 - 2 \eta^3 + S_u \left[ \eta - 2 \eta^2 + \eta^3 \right]$$

$$\frac{w_{2D}}{U_\infty} = S_w \left[ \eta - 2 \eta^2 + \eta^3 \right]$$

$$\frac{\rho_{2D}}{\rho_0} = \text{Eq. 26, but } \rho, = S_\rho$$

27.
ALGEBRAIC AND NUMERICAL CONSIDERATIONS

The results of substituting the postulated functions into the balance relations (Eqs. 9-12) are listed in Appendix B. It is noted that many pages are required to do this. Incidentally, the use of polynomial forms much reduces the likelihood of algebraic errors, for certain patterns and symmetries in coefficients and exponents persist through the manipulations.

Included in these expressions are some very complicated integrals over the spanwise coordinate $\zeta$. In principle these can also be carried out by hand, thus giving even longer algebraic expressions. This is unnecessary and inefficient. Instead one realizes that machine evaluation of the various terms is required in seeking parameter solution sets, and so the $\zeta$-integrals might as well be machine-computed through a Simpson's rule subroutine.

Essentially then, the present approach has reduced the task of solving the several coupled, partial, three-dimensional fluid equations to this: substitute trial parameter solution sets until four complicated algebraic expressions come out to be zero.

This may be done in a systematic iterative manner through the use of the (multi-variable) Newton-Raphson technique. It is summarized as follows, including the improvement discussed in Ref. 17.

Let the term sums in the mass, $x$-momentum, $z$-momentum, and energy equations be represented by $J_R, j = 1,2,3,4$, (i.e. "residuals" that will reduce to zero when the correct solution parameters have been
used). Denote these parameters \( (A, u_{10}, w_{12}, \rho_{10}) \) as \( \phi_i, \ i = 1, 2, 3, 4. \)

Subscripts on the \( jR \) residuals mean differentiation and superscripts mean iterate level. Then, the algorithm for the correcting \( \Delta \phi_i \) set, which will provide the new \( \phi_i^{(n+1)} = \phi_i^{(n)} + \Delta \phi_i \) set, is:

\[
-jR(n) = \sum_{i=1}^{4} \Delta \phi_i \left[ jR_{\phi_i}^{(n)} + \frac{1}{2} \sum_{k=1}^{4} jR_{\phi_k \phi_k}^{(n)} \cdot \Delta \phi_k' \right]
\]

\( j = 1, 2, 3, 4 \) \hfill 28.

where

\[
-jR(n) = \sum_{k=1}^{4} \Delta \phi_k' jR_{\phi_k}^{(n)} \quad j = 1, 2, 3, 4 \quad 29.
\]

The derivatives or "variations" \( jR \phi_i \) are of course the finite difference quotients that result from evaluating residuals at slightly different \( \phi_i \) inputs (as, changes of one per cent). To get the \( \Delta \phi_i' \) and then the \( \Delta \phi_i \), only simple \( (4 \times 4) \) matrix inversions are required.

For the two problems defined by the flight conditions of Eq. 21, quick convergence to parameter solution sets occurred, with the use of Eq. 29 only. That is, the computation of the second residual derivatives \( jR_{\phi_i \phi_k} \) and the corrections from Eq. 28 (or Ref. 17) proved quite unnecessary. This may not always be the case; use of the full algorithm generally defeats "jumping past" solutions and improves convergence speed.

Appendix C presents some sample numerical output, showing the relative sizes of the terms and the speed of convergence to solutions.
Computer time is not at all excessive, as evaluation of the four residuals for one parameter input set requires less than half a minute on the 7040.
RESULTS AND DISCUSSION

For the pair of problems specified by Eq. 21, the free parameter answers turn out to be:

Small spanflow problem, \( S_w = -.39 \):

\[
A = 0.0400 \quad u_{10} = 1.084 \quad w_{12} = -4.00 \quad \rho_{10} = -.933
\]

Large spanflow problem, \( S_w = -.838 \):

\[
A = 0.0489 \quad u_{10} = 0.552 \quad w_{12} = -10.08 \quad \rho_{10} = -.950
\]

Substitution of these numerical values into the postulated functions (Eqs. 16, 19, 20, and 26) gives the complete, three-dimensional centerplane region solution, within the framework and assumptions of the present model. Plots of the important fluid mechanical quantities (pressure, skin friction, etc.) may be constructed and compared with the quasi-two-dimensional extrapolations (from Eq. 27). These are Figs. 5-16.

The results, qualitatively, are easy to summarize. The side inflow of mass, momentum, and energy (which cannot exit by way of the impenetrable symmetry plane) can only be accommodated if there is a thickening of the boundary layer. Therefore \( A \) turns out to be greater than \( B \). The increased displacement implies higher viscous-induced pressures. Skin friction and heat transfer coefficients decrease, for these vary directly with the velocity and temperature gradients, which vary inversely with the thickness dimension. Since \( u_{10} \) turns out to be less than \( S_u \), the surface velocity gradients near the center plane are reduced still further. That
is, there exists a tendency for the colliding, inboard flows to cause separation. As long as separation has not occurred, the surface streamline at the center plane must be directed "downstream", for \( w( \zeta = 0, \eta = 0 ) = 0 \) has been imposed. Density profiles change little, since \( (\rho - \rho_0) \) turns out to be small.

The above effects are seen from the plotted results to be magnified as the average inflow parameter \( |S_W| \) is increased. The large spanflow input represents a rather substantial surface streamline inflow angle at \( S_E \):

\[
\epsilon_E = \tan^{-1} \left( \lim_{\eta \to 0} \frac{\int \frac{W}{u_\infty} (\zeta = 1, \eta) \, d\eta}{\lim_{\eta \to 0} \int \frac{u}{u_\infty} (\zeta = 1, \eta) \, d\eta} \right) \tag{30}
\]

\[
\epsilon_E = \tan^{-1} \left( -\frac{S_W}{S_u} = 30^\circ \right), \text{ large spanflow.} \tag{31}
\]

It might be argued that this \( \epsilon_E \) is somewhat large for the present formulation or even for practical problems. The calculated results for this case predict a symmetry plane boundary layer of roughly half again its quasi-two-dimensional thickness. The average pressure coefficient through the region is increased by about the same proportion. The heat transfer coefficient decreases by about the same amount, and the skin friction coefficient decreases by a factor of three.

The calculated effects of three-dimensionality are more modest in the small spanflow case, which corresponds to an \( \epsilon_E \) of \( 15^\circ \). The reader is referred to Figs. 5-16 for the detailed results.
Once again it is noted that application of the present method to different flight conditions will require some extra calculations, and Appendix D should be considered.
REFERENCES


FIGURE 1

The Boundary Layer Concept

FIGURE 2

Surface Streamlines upon Swept Surfaces in Hypersonic Interaction
FIGURE 4
Sketches of Boundary Layer Properties
FIGURE 5
Boundary Layer Thickness in Center Plane Region;
Small Span Flow

FIGURE 6
Viscous Induced Pressure in Center Plane Region at x = 10 feet;
Small Span Flow
FIGURE 7
Chordwise Velocity Component in Center Plane Region; Small Span Flow
Spanwise Velocity Component in Center Plane Region; Small Span Flow
FIGURE 9
Skin Friction Coefficient in Center Plane Region;
Small Span Flow

FIGURE 10
Heat Transfer Coefficient in Center Plane Region;
Small Span Flow
FIGURE 11
Surface Streamline Geometry in Center Plane Region;
Small Span Flow
FIGURE 12
Boundary Layer Thickness in Center Plane Region; Large Span Flow

FIGURE 13
Viscous Induced Pressure in Center Plane Region at x = 10 feet; Large Span Flow
FIGURE 14
Chordwise Velocity Component in Center Plane Region; Large Span Flow

\( \frac{u(\eta, z = 0)}{U_\infty} \)
Spanwise Velocity Component in Center Plane Region; Large Span Flow
FIGURE 16
Skin Friction Coefficient in Center Plane Region;
Large Span Flow

FIGURE 17
Heat Transfer Coefficient in Center Plane Region;
Large Span Flow
FIGURE 18
Surface Streamline Geometry in Center Plane Region;
Large Span Flow
APPENDIX A

APPROXIMATE TREATMENT OF VERTEX REGION

Three-dimensional strong interaction must exist in the vicinity of the vertex of a flat delta wing. It is possible to infer that as $x \rightarrow 0$ the center plane region is bounded by rays and its boundary layer does grow according to a $\frac{3}{4}$ power law, the same as the more outboard boundary layer.

The $x$-dependence of the quantities in the outboard, quasi-two-dimensional boundary layer may be expressed as:

$$\delta_{2D} \sim x^{\frac{3}{4}}$$

$$p_{2D} \sim \left( \frac{\delta_{2D}}{x} \right)^2 \sim x^{-\frac{1}{2}} \quad \text{(large K approximation)}$$

$$\rho_{2D} \sim p_{2D} \sim x^{-\frac{1}{2}} \quad \text{(using state equation } P = \rho RT \text{ and saying } T_{2D} \sim T_{STAG})$$

Then, with $n$ and $m$ to be determined, we use for the centerline $\phi$ symmetry plane region:

$$\delta_{\phi} \sim x^n$$

$$\Delta_{\phi} \equiv \text{width } \sim x^m$$

$$p_{\phi} \sim \left( \frac{\delta_{\phi}}{x} \right)^2 \sim x^{2n-2}$$

$$\rho_{\phi} \sim \rho_{\phi} \sim x^{2n-2}$$

All velocities are $\sim U_\infty \sim x^o$. 


At this point, one examines the character of all terms contributing to the mass and momentum balances for the small $\xi$ region. It is hoped that a solution for $n$ and $m$ will be obtainable by requiring that no single term goes to zero more slowly than all the other terms go to zero as $x$ goes to zero, in each balance.

For mass conservation:

1. Mass influx from side $\sim \rho_{20} \cdot U \cdot x \cdot \delta_{20} \sim x^{5}$

2. Mass influx from external stream $\sim \rho_{0} \cdot U \cdot x \cdot \Delta \xi \cdot \frac{\delta_{0}}{x} \sim x^{3n + m - 2}$

3. Mass flux at the station $x \sim \rho_{0} \cdot U \cdot \delta_{0} \cdot \Delta \xi \sim x^{3n + m - 2}$

For $\xi$-momentum conservation:

1. Momentum influx from side $\sim \rho_{20} \cdot U^{2} \cdot x \cdot \delta_{20} \sim x^{5}$

2. Momentum influx from external stream $\sim \rho_{0} \cdot U^{2} \cdot x \cdot \Delta \xi \cdot \frac{\delta_{0}}{x} \sim x^{3n + m - 2}$

3. Momentum flux at the station $x \sim \rho_{0} \cdot U^{2} \cdot \delta_{0} \cdot \Delta \xi \sim x^{3n + m - 2}$

4. Pressure force on outer surface, $\xi$-component $\sim P_{20} \cdot x \cdot \Delta \xi \cdot \frac{\delta_{0} \delta_{0}}{x} \sim x^{3n + m - 2}$

5. Pressure force on side surface, $\xi$-component $\sim P_{20} \cdot x \cdot P_{20} \cdot \Delta \xi \cdot \frac{\delta_{0} \delta_{0}}{x} \sim x^{m + \frac{5}{4}}$

6. Pressure force at the station $x \sim \rho_{0} \cdot \delta_{0} \cdot \Delta \xi \sim x^{3n + m - 2}$

7. Surface stress viscous force $\sim \mu \cdot \frac{U_{\xi}}{\delta_{0}} \cdot x \cdot \Delta \xi \sim x^{-n + m - 1}$

It turns out that

$$3n + m - 2 = -n + m + 1 = m + \frac{5}{4} = \frac{5}{4}$$

is satisfied by

$$n = \frac{3}{4}, \quad m = 1$$

as anticipated.
APPENDIX B - ALGEBRAIC LISTINGS

SHORTHAND DEFINITIONS

\[ a_1 = \frac{1}{4} - \frac{4}{5} + \frac{6}{6} - \frac{4}{7} + \frac{1}{8} \]

\[ a_2 = \frac{1}{5} - \frac{1}{6} + \frac{6}{7} - \frac{6}{8} + \frac{1}{9} \]

\[ a_3 = \frac{1}{6} - \frac{4}{7} + \frac{6}{8} - \frac{4}{9} + \frac{1}{10} \]

\[ a_4 = \frac{1}{7} - \frac{4}{8} + \frac{6}{9} - \frac{4}{10} + \frac{1}{11} \]

\[ a_5 = \frac{1}{8} - \frac{4}{9} + \frac{6}{10} - \frac{4}{11} + \frac{1}{12} \]

\[ a_6 = \frac{1}{9} - \frac{4}{10} + \frac{6}{11} - \frac{4}{12} + \frac{1}{13} \]

\[ a_7 = \frac{1}{10} - \frac{4}{11} + \frac{6}{12} - \frac{4}{13} + \frac{1}{14} \]

\[ a_8 = \frac{1}{11} - \frac{4}{12} + \frac{6}{13} - \frac{4}{14} + \frac{1}{15} \]

\[ b_1 = \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \]

\[ b_2 = \frac{1}{5} - \frac{3}{6} + \frac{1}{7} \]
\[
\begin{align*}
  b_3 &= \frac{1}{6} - \frac{3}{7} + \frac{1}{8} \\
  b_4 &= \frac{1}{7} - \frac{3}{8} + \frac{1}{9} \\
  b_5 &= \frac{1}{8} - \frac{3}{9} + \frac{1}{10} \\
  b_6 &= \frac{1}{9} - \frac{3}{10} + \frac{1}{11} \\
  b_7 &= \frac{1}{10} - \frac{3}{11} + \frac{1}{12} \\
  b_8 &= \frac{1}{11} - \frac{3}{12} + \frac{1}{13} \\
  b_9 &= \frac{1}{12} - \frac{3}{13} + \frac{1}{14} \\
  b_{10} &= \frac{1}{13} - \frac{3}{14} + \frac{1}{15}
\end{align*}
\]

\[
\begin{align*}
  C_1 &= \frac{1}{4} - \frac{6}{5} + \frac{15}{6} - \frac{20}{7} + \frac{15}{8} - \frac{6}{9} + \frac{1}{10}\\
  C_2 &= \frac{1}{5} - \frac{6}{6} + \frac{15}{7} - \frac{20}{8} + \frac{15}{9} - \frac{6}{10} + \frac{1}{11}\\
  C_3 &= \frac{1}{6} - \frac{6}{7} + \frac{15}{8} - \frac{20}{9} + \frac{15}{10} - \frac{6}{11} + \frac{1}{12}\\
  C_4 &= \frac{1}{7} - \frac{6}{8} + \frac{15}{9} - \frac{20}{10} + \frac{15}{11} - \frac{6}{12} + \frac{1}{13}\\
  C_5 &= \frac{1}{8} - \frac{6}{9} + \frac{15}{10} - \frac{20}{11} + \frac{15}{12} - \frac{6}{13} + \frac{1}{14}\\
  C_6 &= \frac{1}{9} - \frac{6}{10} + \frac{15}{11} - \frac{20}{12} + \frac{15}{13} - \frac{6}{14} + \frac{1}{15}
\end{align*}
\]
\[ d_1 = \frac{1}{3} - \frac{4}{7} + \frac{6}{5} - \frac{4}{6} + \frac{1}{7} \]

\[ d_i = a_{i-1}, \quad d_2 \text{ through } d_6 \]

\[ e_1 = \frac{1}{3} - \frac{3}{4} + \frac{1}{5} \]

\[ e_i = b_{i-1}, \quad e_2 \text{ through } e_6 \]

\[ g_1 = \frac{1}{4} - \frac{2}{3} + \frac{4}{7} \]

\[ g_i = e_{i-1}, \quad g_2 \text{ through } g_6 \]

\[
\begin{align*}
\mathcal{f}(3D) &= A \left[ 1 - \frac{A-B'}{A} \left( (3-S_0) s^2 + (S_0-2) s^3 \right) \right] \\
\mathcal{f}(2D) &= B \left[ 1 - s \frac{\tan A}{\tan A_e} \right]^2 \\
\mathcal{f}_k &= \mathcal{f} - 2 \mathcal{f}' \\
K &= \frac{1}{2} x^{-\frac{1}{2}} M_o \mathcal{f}_k \\
\frac{\rho_\delta}{\rho_\infty} &= 1 + K + 0.4 K^2
\end{align*}
\]
\[
\frac{p_w}{p_0} = \frac{T_{\infty}}{T_w} \left[ 1 + .4K + .04K^2 \right]
\]

\[\rho_1 (3D) = \rho_0 + \left[ S_\rho - \rho_0 \right] \gamma^2\]

\[\rho_1 (2D) = S_\rho\]

\[F_{\rho_1} = 3.393 - 14.400 \frac{p_w}{p_0} - 7.000 \rho\]

\[F_{\rho_3} = -18.180 + 51.175 \frac{p_w}{p_0} + 17.250 \rho\]

\[F_{\rho_4} = 31.180 - 64.150 \frac{p_w}{p_0} - 17.500 \rho\]

\[F_{\rho_5} = -15.393 + 26.375 \frac{p_w}{p_0} + 6.250 \rho\]

\[C_{21} = 3.393 - 7.000 \rho - 14.400 \frac{T_{\infty}}{T_w}\]

\[C_{31} = -18.180 + 17.250 \rho + 51.175 \frac{T_{\infty}}{T_w}\]

\[C_{41} = 31.180 - 17.500 \rho - 64.150 \frac{T_{\infty}}{T_w}\]

\[C_{51} = -15.393 + 6.250 \rho + 26.375 \frac{T_{\infty}}{T_w}\]

\[C_{22} = -2.88 (M_{\infty} f_{K}) \frac{T_{\infty}}{T_w}\]

\[C_{23} = -1.44 (M_{\infty} f_{K})^2 \frac{T_{\infty}}{T_w}\]
\[ C_{32} = 10.235 \left( M_{\infty f_k} \right) \frac{T_0}{T_w} \quad C_{33} = 0.51175 \left( M_{\infty f_k} \right)^2 \frac{T_0}{T_w} \]

\[ C_{42} = -12.83 \left( M_{\infty f_k} \right) \frac{T_0}{T_w} \quad C_{43} = -0.645 \left( M_{\infty f_k} \right)^2 \frac{T_0}{T_w} \]

\[ C_{52} = 5.275 \left( M_{\infty f_k} \right) \frac{T_0}{T_w} \quad C_{53} = 0.26375 \left( M_{\infty f_k} \right)^2 \frac{T_0}{T_w} \]

\[ XRNTW = \frac{2}{3} \frac{T_0}{T_w} f \left( x_2^{\frac{3}{2}} - x_1^{\frac{3}{2}} \right) + 0.7 \frac{T_0}{T_w} f(M_{\infty f_k}) (x_2 - x_1) \]

\[ + 0.42 \frac{T_0}{T_w} f \left( M_{\infty f_k} \right)^2 (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) + 0.025 \frac{T_0}{T_w} f(M_{\infty f_k})^3 \ln \frac{x_2}{x_1} \]

\[ - 0.002 \frac{T_0}{T_w} f \left( M_{\infty f_k} \right)^4 (x_2^{-\frac{1}{2}} - x_1^{-\frac{1}{2}}) \]

\[ XRNT1 = \frac{2}{3} \rho \frac{f}{f} \left( x_2^{\frac{3}{2}} - x_1^{\frac{3}{2}} \right) + 0.5 \rho \frac{f}{f} \left( M_{\infty f_k} \right) (x_2 - x_1) \]

\[ + 0.2 \rho \frac{f}{f} \left( M_{\infty f_k} \right)^2 (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) \]

\[ XRNT_i = \frac{3}{3} C_i f \left( x_2^{\frac{3}{2}} - x_1^{\frac{3}{2}} \right) + \left( C_{i2} + 0.5 C_i \left( M_{\infty f_k} \right) \right) f \left( x_2 - x_1 \right) \]

\[ + \left( C_{i3} + C_{i2} \left( M_{\infty f_k} \right) + 0.2 C_i \left( M_{\infty f_k} \right)^2 \right) f \left( x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}} \right) \]

\[ + \left( 0.5 C_{i3} \left( M_{\infty f_k} \right) + 0.1 C_{i2} \left( M_{\infty f_k} \right)^2 \right) f \ln \frac{x_2}{x_1} \]

\[ - 0.2 C_{i3} \left( M_{\infty f_k} \right)^2 f \left( x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}} \right), \quad i = 2, 3, 4, 5 \]
\[ F_\mu (3D) = u_\mu + (s_\mu - u_\mu) j^3 \]

\[ F_\mu (2D) = s_\mu \]

\[ G_{\mu 2} = 3 - 2 F_\mu \]

\[ G_{\mu 3} = -2 + F_\mu \]

\[ F_w (3D) = \omega_1 j^2 + (s_w - \omega_1) j^3 \]

\[ F_w (2D) = s_w \]

\[ FNU1 = \frac{1}{5} F_\mu + \frac{1}{3} G_{\mu 2} + \frac{1}{6} G_{\mu 3} \]

\[ FNU2 = \frac{1}{3} F_\mu + \frac{1}{4} G_{\mu 2} + \frac{1}{5} G_{\mu 3} \]

\[ FNU3 = \frac{1}{4} F_\mu + \frac{1}{5} G_{\mu 2} + \frac{1}{6} G_{\mu 3} \]

\[ FNU4 = \frac{1}{5} F_\mu + \frac{1}{6} G_{\mu 2} + \frac{1}{7} G_{\mu 3} \]

\[ FNU5 = \frac{1}{6} F_\mu + \frac{1}{7} G_{\mu 2} + \frac{1}{8} G_{\mu 3} \]

\[ FNU6 = \frac{1}{7} F_\mu + \frac{1}{8} G_{\mu 2} + \frac{1}{9} G_{\mu 3} \]

\[ FNU7 = \frac{1}{3} F_\mu + \frac{3}{4} F_\mu G_{\mu 2} + \frac{3}{5} F_\mu G_{\mu 3} + \frac{1}{3} G_{\mu 2} + \frac{2}{5} \xi G_{\mu 2} G_{\mu 3} + \frac{1}{5} G_{\mu 3}^2 \]
\[
\text{FNU8} = \frac{1}{4} F_u^2 + \frac{3}{5} F_u G_{u2} + \frac{3}{6} F_u G_{u3} + \frac{1}{6} G_{u2}^3 + \frac{2}{7} G_{u3} G_{u4} + \frac{1}{5} G_{u3}^2
\]

\[
\text{FNU9} = \frac{1}{5} F_u^2 + \frac{3}{6} F_u G_{u2} + \frac{2}{7} F_u G_{u3} + \frac{1}{6} G_{u2}^3 + \frac{3}{8} G_{u2} G_{u3} + G_{u3}^2
\]

\[
\text{FNU10} = \frac{1}{6} F_u^2 + \frac{3}{7} F_u G_{u2} + \frac{3}{8} F_u G_{u3} + \frac{1}{6} G_{u2}^3 + \frac{3}{9} G_{u2} G_{u3} + \frac{1}{6} G_{u3}^2
\]

\[
\text{FNU11} = \frac{1}{7} F_u^2 + \frac{2}{8} F_u G_{u2} + \frac{3}{9} F_u G_{u3} + \frac{1}{6} G_{u2}^3 + \frac{3}{10} G_{u2} G_{u3} + \frac{1}{8} G_{u3}^2
\]

\[
\text{FNU12} = \frac{1}{8} F_u^2 + \frac{3}{9} F_u G_{u2} + \frac{3}{10} F_u G_{u3} + \frac{1}{6} G_{u2}^3 + \frac{3}{11} G_{u2} G_{u3} + \frac{1}{12} G_{u3}^2
\]

\[
\text{FNU13} = e_1 F_u + e_2 G_{u2} + e_3 G_{u3}
\]

\[
\text{FNU14} = e_2 F_u + e_3 G_{u2} + e_4 G_{u3}
\]

\[
\text{FNU15} = e_3 F_u + e_4 G_{u2} + e_5 G_{u3}
\]

\[
\text{FNU16} = e_4 F_u + e_5 G_{u2} + e_6 G_{u3}
\]

\[
\text{FNU17} = e_5 F_u + e_6 G_{u2} + e_7 G_{u3}
\]

\[
\text{FNU18} = e_6 F_u + e_7 G_{u2} + e_8 G_{u3}
\]

\[
\text{FNU19} = \frac{1}{4} F_u^3 + \frac{3}{5} F_u^2 G_{u2} + \frac{3}{6} F_u^2 G_{u3} + \frac{3}{7} F_u G_{u2}^3 + \frac{6}{7} F_u G_{u3} G_{u4} + \frac{1}{6} G_{u2}^3 + \frac{3}{8} G_{u2} G_{u3} + \frac{1}{6} G_{u3}^2
\]

\[
\text{FNU20} = \frac{1}{5} F_u^3 + \frac{3}{6} F_u^2 G_{u2} + \frac{3}{7} F_u^2 G_{u3} + \frac{3}{8} F_u G_{u2}^3 + \frac{1}{10} F_u G_{u4} G_{u3} + \frac{1}{8} G_{u2}^3 + \frac{3}{10} F_u^2 G_{u3} + \frac{3}{11} G_{u2}^3 + \frac{1}{12} G_{u3}^2
\]
\[\text{FNU21} = \frac{1}{6} F_u^3 + \frac{3}{7} F_u^2 G_{u_2} + \frac{3}{8} F_u G_{u_2}^2 + \frac{3}{7} F_u G_{u_3}^2 + \frac{6}{7} F_u G_{u_2} G_{u_3} + \frac{1}{9} G_{u_2}^3 + \frac{3}{10} F_u G_{u_3}^2 + \frac{3}{12} G_{u_2} G_{u_3}^2 + \frac{1}{12} G_{u_3}^3\]

\[\text{FNU22} = \frac{1}{7} F_u^3 + \frac{3}{8} F_u^2 G_{u_2} + \frac{3}{9} F_u^2 G_{u_2} + \frac{3}{9} F_u G_{u_2}^2 + \frac{6}{10} F_u G_{u_3} G_{u_3} + \frac{1}{10} G_{u_2}^3 + \frac{3}{11} F_u G_{u_3}^2 + \frac{3}{11} G_{u_2}^2 G_{u_3} + \frac{1}{13} G_{u_2} G_{u_3}^2 + \frac{1}{13} G_{u_3}^3\]

\[\text{FNU23} = \frac{1}{8} F_u^3 + \frac{3}{9} F_u^2 G_{u_2} + \frac{3}{10} F_u^2 G_{u_3} + \frac{3}{10} F_u G_{u_2}^2 + \frac{6}{11} F_u G_{u_3} G_{u_3} + \frac{1}{11} G_{u_2}^3 + \frac{2}{12} F_u G_{u_3}^2 + \frac{3}{12} G_{u_2}^2 G_{u_3} + \frac{1}{14} G_{u_2} G_{u_3}^2 + \frac{1}{14} G_{u_3}^3\]

\[\text{FNU24} = \frac{1}{9} F_u^3 + \frac{3}{10} F_u^2 G_{u_2} + \frac{3}{11} F_u^2 G_{u_3} + \frac{3}{11} F_u G_{u_2}^2 + \frac{6}{12} F_u G_{u_3} G_{u_3} + \frac{1}{12} G_{u_2}^3 + \frac{3}{13} F_u G_{u_3}^2 + \frac{3}{13} G_{u_2}^2 G_{u_3} + \frac{1}{15} G_{u_2} G_{u_3}^2 + \frac{1}{15} G_{u_3}^3\]

\[\text{FNU25} = a_1 F_u + a_2 G_{u_2} + a_3 G_{u_3}\]

\[\text{FNU26} = a_1 F_u + a_2 G_{u_2} + a_4 G_{u_3}\]

\[\text{FNU27} = a_3 F_u + a_4 G_{u_2} + a_5 G_{u_3}\]

\[\text{FNU28} = a_4 F_u + a_5 G_{u_2} + a_6 G_{u_3}\]

\[\text{FNU29} = a_5 F_u + a_6 G_{u_2} + a_7 G_{u_3}\]

\[\text{FNU30} = a_6 F_u + a_7 G_{u_2} + a_8 G_{u_3}\]
\[ FNU31 = b_1 F_u^2 + 2b_2 F_u G_{u2} + 2b_3 F_u G_{u3} + b_3 G_{u2}^2 + 2b_4 G_{u2} G_{u3} + b_5 G_{u3}^2 \]
\[ FNU32 = b_2 F_u^2 + 2b_3 F_u G_{u2} + 2b_4 F_u G_{u3} + b_4 G_{u2}^2 + 2b_5 G_{u2} G_{u3} + b_6 G_{u3}^2 \]
\[ FNU33 = b_3 F_u^2 + 2b_4 F_u G_{u2} + 2b_5 F_u G_{u3} + b_5 G_{u2}^2 + 2b_6 G_{u2} G_{u3} + b_7 G_{u3}^2 \]
\[ FNU34 = b_4 F_u^2 + 2b_5 F_u G_{u2} + 2b_6 F_u G_{u3} + b_6 G_{u2}^2 + 2b_7 G_{u2} G_{u3} + b_8 G_{u3}^2 \]
\[ FNU35 = b_5 F_u^2 + 2b_6 F_u G_{u2} + 2b_7 F_u G_{u3} + b_7 G_{u2}^2 + 2b_8 G_{u2} G_{u3} + b_9 G_{u3}^2 \]
\[ FNU36 = b_6 F_u^2 + 2b_7 F_u G_{u2} + 2b_8 F_u G_{u3} + b_8 G_{u2}^2 + 2b_9 G_{u2} G_{u3} + b_{10} G_{u3}^2 \]
\[ T_1 = \frac{1}{\rho_\infty U_\infty z_\varepsilon (x_i) \delta_{2d\xi}} \int \int \rho u dS_i \]

\[ = \frac{1}{B} \int_0^{x_2} \frac{\rho_s}{\rho_\infty} f \left[ \frac{\rho_s}{\rho_s} FNU1 + \rho_s FNU2 + F_{1s} FNU3 \right. \]
\[ + F_{pS} FNU4 + F_{pS} FNU5 + F_{pS} FNU6 \left. \right] \Bigg|_{x=x_i} \, d\xi \]

\[ T_2 = \frac{-1}{\rho_\infty U_\infty z_\varepsilon (x_i) \delta_{2d\xi}} \int \int \rho u dS_2 \]

\[ = -\left( \frac{x_2}{x_1} \right)^{3/2} \frac{1}{B} \int_0^{x_2} \frac{\rho_s}{\rho_\infty} f \left[ \text{as in } T_1 \right] \Bigg|_{x=x_2} \, d\xi \]

\[ T_3 = \frac{1}{\rho_\infty U_\infty z_\varepsilon (x_i) \delta_{2d\xi}} \int \int \rho u \cos \Lambda_\varepsilon dS_3 \]

\[ = \frac{1}{Bx_l^2} \left[ XRNTW \cdot FNU1 + XRNT1 \cdot FNU2 + XRNT2 \cdot FNU3 \right. \]
\[ + XRNT3 \cdot FNU4 + XRNT4 \cdot FNU5 + XRNT5 \cdot FNU6 \left. \right|_{x=1} \]
\[ T_k = \frac{-1}{\rho_0 U_\infty Z_E(x_i) \delta(x_i)} \int \int \rho w \sin \Lambda_E \, dS_E \]
\[ = -\frac{1}{B x_{i,2}^{\frac{3}{2}}} \frac{\sin^2 \Lambda_E}{\cos \Lambda_E} \left[ \left[ g_i \text{XRNTW} + g_2 \text{XRNT1} \right. \right. \]
\[ + g_3 \text{XRNT2} + g_4 \text{XRNT3} + g_5 \text{XRNT4} + g_6 \text{XRNT5} \left. \left. \right|_{x=x_1} \right) \]
\[ T_5 = \frac{1}{\rho_0 U_\infty Z_E(x_i) \delta(x_i)} \int \int \rho w \, dS_c \]
\[ = \frac{1}{B x_{i,2}^{\frac{3}{2}}} \frac{\sin^2 \Lambda_E}{\cos \Lambda_E} \left[ \text{as in } T_4 \right] \left. \right|_{\theta=0} \]
\[ T_6 = \frac{\Delta \text{mass flux}}{\rho_0 U_\infty Z_E(x_i) \delta(x_i)} = -\sum_{i=1}^{5} T_i \quad \text{(2D)} \]
\[ T_7 = \frac{-1}{\rho_0 U_\infty^2 Z_E(x_i) \delta(x_i)} \int \int \rho u^2 \, dS_i \]
\[ = -\frac{1}{B} \int_0^{1} \int_{\rho_0}^{\infty} \left[ \frac{\rho w}{\rho_0} F_{\text{NU}7} + \rho F_{\text{NU}8} + F_{\text{NU}9} \right. \]
\[ + F_{\rho_3} F_{\text{NU}10} + F_{\rho_4} F_{\text{NU}11} + F_{\rho_5} F_{\text{NU}12} \left. \right|_{x=x_1} \, d\rho \]
\[ T_8 = \frac{1}{\rho \bar{U}^2 \bar{Z}_e(x_1, \bar{x}_1)} \int \int \rho u^2 \, dS \\
= (\frac{x_1^2}{x_1}) \frac{3}{B} \int \int \rho \bar{U}^2 \bar{Z}_e(x_1, \bar{x}_1) \phi \left[ \text{as in } T_7 \right] \, d\bar{y} \quad \bar{x} = x_2 \]

\[ T_9 = \frac{-1}{\rho \bar{U}^2 \bar{Z}_e(x_1, \bar{x}_1)} \int \int \rho u^2 \cos \lambda_e \, dS_e \]

\[ = - \frac{1}{Bx_1^{3/2}} \left[ xRNTW \cdot FNU7 + xRNT1 \cdot FNU8 + xRNT2 \cdot FNU9 + xRNT3 \cdot FNU10 + xRNT4 \cdot FNU11 + xRNT5 \cdot FNU12 \right] \int_{y=1}^{y=1} \]

\[ T_{10} = \frac{1}{\rho \bar{U}^2 \bar{Z}_e(x_1, \bar{x}_1)} \int \int \rho w \sin \lambda_e \, dS_e \]

\[ = \frac{1}{Bx_1^{3/2}} \frac{\sin \lambda_e}{\cos \lambda_e} \int \int F_w \left[ xRNTW \cdot FNU13 + xRNT1 \cdot FNU14 + xRNT2 \cdot FNU15 + xRNT3 \cdot FNU16 + xRNT4 \cdot FNU17 + xRNT5 \cdot FNU18 \right] \int_{y=1}^{y=1} \]

\[ T_{11} = \frac{-1}{\rho \bar{U}^2 \bar{Z}_e(x_1, \bar{x}_1)} \int \int P \, dS_1 \]

\[ = - \frac{1}{4\bar{M}_e M_0^2 Bx_1^{3/2}} \left[ x_1 \int f \, d\bar{y} + M_0 x_1 \int f f_k \, d\bar{y} + 0.84 (\frac{1}{2} M_0) \int f f_k \, d\bar{y} + 0.252 (\frac{1}{2} M_0)^2 \int f f_k \, d\bar{y} \right] \]
\[ T_{12} = \frac{1}{\rho_{\infty} U_{\infty}^2 \frac{\pi}{4} \frac{x_1}{d_{k_1}}} \int \int P \, dS \]
\[ = \frac{1}{1.4 M_{\infty}^2 B X_{1/2}^{2/3}} \left[ x_2^{2/3} \int_0^l f dS + 1.4 (\frac{1}{2} M_{\infty}) x_2 \int_0^l f f_k dS \right. \\
+ .84 \left( \frac{1}{2} M_{\infty} \right)^2 x_2^{1/3} \int_0^l f f_k^2 dS \left. + .252 \left( \frac{1}{2} M_{\infty} \right)^3 \int_0^l f f_k^3 dS \right] \]

\[ \begin{align*}
T_{13} &= \frac{-1}{\rho_{\infty} U_{\infty}^2 \frac{x_2}{d_{k_1}}} \int \int P \cos \Lambda_\infty \, dS \\
&= -\frac{B'}{1.4 M_{\infty}^2 B X_{1/2}^{2/3}} \left[ \frac{2}{3} (x_2^{2/3} - x_1^{2/3}) + 1.4 \left( \frac{1}{2} M_{\infty} f \right) (x_2 - x_1) \\
&+ 1.68 \left( \frac{1}{2} M_{\infty} f \right)^2 (x_2^{1/3} - x_1^{1/3}) + .252 \left( \frac{1}{2} M_{\infty} f \right)^3 \ln \frac{x_2}{x_1} \right] \bigg|_{y=1} \\
\end{align*} \]

\[ \begin{align*}
T_{14} &= \frac{-1}{\rho_{\infty} U_{\infty}^2 \frac{x_2}{d_{k_1}}} \int \int P \, dS \\
&= \frac{1}{2.8 M_{\infty}^2 B X_{1/2}^{2/3}} \left[ \frac{3}{5} (x_2^{2/3} - x_1^{2/3}) \int_0^l f dS + 1.4 (\frac{1}{2} M_{\infty}) (x_2 - x_1) \int_0^l f \, f_k dS \right. \\
&+ 1.68 \left( \frac{1}{2} M_{\infty} \right)^2 (x_2^{1/3} - x_1^{1/3}) \int_0^l f f_k^2 dS \left. + .252 \left( \frac{1}{2} M_{\infty} \right)^3 \ln \frac{x_2}{x_1} \int_0^l f f_k^3 dS \right] \\
\end{align*} \]

\[ \begin{align*}
T_{15} &= \frac{-1}{\rho_{\infty} U_{\infty}^2 \frac{x_2}{d_{k_1}}} \int \int \rho u w \, dS \\
&= -\frac{1}{B X_{1/2}^{2/3} \cos \Lambda_\infty} \int \frac{x_2 dS}{w} \left[ \text{as in } T_{10} \right] \bigg|_{y=0} \\
\end{align*} \]
\begin{align*}
T_{16} &= \frac{1}{\rho_{0} U_{\infty}^2 z_{E}(x_{i}) \delta(x_{i})} \iint \mu \frac{d\nu}{r_{y}} dS_{w} \\
T_{16}^{(3D)} &= \frac{2}{3} \frac{1}{B_{x_{i}}^{\frac{3}{2}}} \rho_{0} U_{\infty}^2 \left( \frac{U_{w}}{U_{\infty}} \right) \left( x_{2}^{\frac{3}{2}} - x_{1}^{\frac{3}{2}} \right) \left[ \left( S_{u} - U_{10} \right) \int_{0}^{1} \frac{d\gamma}{f} \right] \\
T_{16}^{(2D)} &= \frac{2}{3} \frac{1}{B_{x_{i}}^{\frac{3}{2}}} \rho_{0} U_{\infty}^2 \left( x_{2}^{\frac{3}{2}} - x_{1}^{\frac{3}{2}} \right) S_{w} \int_{0}^{1} \frac{d\gamma}{f} \\
T_{17} &= - \frac{\Delta_{x} \chi \cdot \text{momentum flux}}{\rho_{0} U_{\infty}^2 z_{E}(x_{i}) \delta(x_{i})} = - \sum_{i=7}^{16} T_{i}^{(2D)} \\
T_{18} &= -\frac{1}{\rho_{0} U_{\infty}^2 z_{E}(x_{i}) \delta(x_{i})} \iint \rho \mu u w dS_{w} \\
&= -\frac{1}{B} \int_{0}^{1} \frac{d\gamma}{f_{\gamma}} f \cdot F_{w} \left[ \frac{\rho_{w}}{\rho_{0}} \frac{F_{NU13} + \rho_{0} F_{NU14}}{\rho_{2} F_{NU15} + \rho_{3} F_{NU16} + \rho_{4} F_{NU17} + \rho_{5} F_{NU18}} \right] d\gamma \\
T_{19} &= \frac{1}{\rho_{0} U_{\infty}^2 z_{E}(x_{i}) \delta(x_{i})} \iint \rho \mu u w dS_{2} \\
&= \frac{1}{B} \left( \frac{x_{2}}{x_{1}} \right)^{\frac{3}{2}} \int_{0}^{1} \frac{d\gamma}{f_{\gamma}} f \cdot F_{w} \left[ \text{as in } T_{18} \right] d\gamma \\
T_{20} &= \frac{-1}{\rho_{0} U_{\infty}^2 z_{E}(x_{i}) \delta(x_{i})} \iint \rho \mu u w \cos \lambda_{E} dS_{E} \\
&= -\frac{\cos \lambda_{E}}{\sin \lambda_{E}} T_{10}
\end{align*}
\[
T_{21} = \frac{1}{\rho_\infty C_{\infty}^2 z (x_i) \cdot \delta (x_i)} \int \frac{\rho w^2 \sin \Lambda E dS_E}{\delta_0 E}
\]

\[
= \frac{\sin \Lambda E}{\cos \Lambda E} \left[ d_1 \chi_{\nu} + d_2 \chi_{\nu} + d_3 \chi_{\nu} + d_4 \chi_{\nu} + d_5 \chi_{\nu}ight] \left| x = 1 \right.
\]

\[
T_{22} = \frac{1}{\rho_\infty U_{\infty}^2 z (x_i) \cdot \delta (x_i)} \int \frac{\rho \sin \Lambda E dS_E}{\delta_0 E}
\]

\[
= \frac{B'}{B_{x_i}^{3/2}} \frac{1}{1.4 M_{\infty}^2} \sin \Lambda E \left[ \frac{3}{3} (x_{2}^{3/2} - x_{1}^{3/2}) + 1.4 (M_{\infty} f_k) (x_i x_k) + 1.68 (\frac{1}{2} M_{\infty} f_k)^2 (x_{2}^{1/2} - x_{1}^{1/2}) + 2.52 (\frac{1}{2} M_{\infty} f_k)^3 \ln \frac{x_{2}}{x_{1}} \right] \left| x = 0 \right.
\]

\[
T_{23} = -\frac{1}{\rho_\infty U_{\infty}^2 z (x_i) \cdot \delta (x_i)} \int \frac{P dS_C}{\delta_0 E}
\]

\[
= -\frac{C'}{B_{x_i}^{3/2}} \frac{1}{1.4 M_{\infty}^2} \sin \Lambda E \left[ as \: in \: T_{22} \right] \left| x = 0 \right.
\]

\[
T_{24} = -\frac{1}{\rho_\infty U_{\infty}^2 z (x_i) \cdot \delta (x_i)} \int \frac{P dS_0 dS_0}{\delta_0 E}
\]

\[
= -\frac{1}{B_{x_i}^{3/2}} \frac{1}{1.4 M_{\infty}^2} \sin \Lambda E \left[ \frac{3}{3} (x_{2}^{3/2} - x_{1}^{3/2}) \int \frac{\rho f dS_0}{\delta_0 E} + 1.4 (M_{\infty} f_k) (x_i x_k) \int \frac{\rho f_k dS_0}{\delta_0 E} + 1.68 (\frac{1}{2} M_{\infty} f_k)^2 (x_{2}^{1/2} - x_{1}^{1/2}) \int \frac{\rho f_k dS_0}{\delta_0 E} + 2.52 (\frac{1}{2} M_{\infty} f_k)^3 \ln \frac{x_{2}}{x_{1}} \int \frac{\rho f_k dS_0}{\delta_0 E} \right]
\]
\[
T_{25} = \frac{-1}{\rho_0 U_\infty^2 z_e(x_i) \delta(x_i)} \iint \rho u^2 dS_e
= -\frac{F_{w}}{B X_i^{3/2}} \sin \Lambda_e \left[ \text{as in } T_{21} \right] \bigg|_{\psi=0}
\]
\[
T_{26} = \frac{1}{\rho_0 U_\infty^2 z_e(x_i) \delta(x_i)} \iint \rho \frac{d\psi}{d\xi} dS_w
\]
\[
T_{26} \ (3D) = \frac{3}{3} \frac{\mu_w U_\infty}{\rho_0 U_\infty^2 B X_i^{3/2}} (x_2^{3/2} - x_1^{3/2}) \left[ \omega_{w1} \int_0^1 \frac{\psi_0^2 d\psi}{f} 
+ (S_w - \omega_{w1}) \int_0^1 \frac{\psi_0^2 d\psi}{f} \right]
\]
\[
T_{26} \ (2D) = \frac{3}{3} \frac{\mu_w U_\infty}{\rho_0 U_\infty^2 B X_i^{3/2}} (x_2^{3/2} - x_1^{3/2}) S_w \int_0^1 \frac{d\psi}{f}
\]
\[
T_{27} = \frac{\Delta_s z \text{-momentum flux}}{\rho_0 U_\infty^2 z_e(x_i) \delta(x_i)} = -\sum_{i=18}^{26} T_i \ (2D)
\]
\[
T_{28} = \frac{-1}{\frac{1}{2} \rho_0 U_\infty^3 z_e(x_i) \delta(x_i)} \iint \rho u h dS_i
= -\frac{1}{B} \frac{5}{M_\infty} \left[ \int_0^1 f \left( \frac{1}{2} + \frac{1}{2} F_u \right) d\psi 
+ 1.4 \left( \frac{1}{2} M_\infty \right) x_i^{-\frac{1}{2}} \int_0^1 f f_k \left( \frac{1}{2} + \frac{1}{2} F_u \right) d\psi 
+ 0.84 \left( \frac{1}{2} M_\infty \right)^2 x_i^{-1} \int_0^1 f f_k^2 \left( \frac{1}{2} + \frac{1}{2} F_u \right) d\psi 
+ 0.252 \left( \frac{1}{2} M_\infty \right)^3 x_i^{-\frac{3}{2}} \int_0^1 f f_k^3 \left( \frac{1}{2} + \frac{1}{2} F_u \right) d\psi \right]
\]
\[ T_{29} = \frac{-1}{\frac{1}{2} \rho_0 U_\infty^2 z_E(x_i) \delta(x_i)} \iint_{20} \rho u \frac{1}{2} u^3 \, ds \]

\[ = -\frac{1}{B} \int_0^{\frac{\partial}{\rho_0}} f \cdot \left[ \frac{\rho^2}{F_{\text{N19}}} + \rho, F_{\text{N20}} + \frac{F_{\text{N17}}}{\rho^2} \right] \left. \delta(x_i) \right|_{x=x_i} \, \, dF \]

\[ \frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \frac{\partial}{\rho_0} \iint_{20} \rho u \frac{1}{2} u^2 \, ds \]

\[ = -\frac{1}{B} \int_0^{\frac{\partial}{\rho_0}} f \cdot F^2 \left[ \frac{\rho^2}{F_{\text{N25}}} + \rho, F_{\text{N26}} + \frac{F_{\text{N27}}}{\rho^2} \right] \left. \delta(x_i) \right|_{x=x_i} \, \, dF \]

\[ \frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \frac{\partial}{\rho_0} \iint_{20} \rho u h \, ds \]

\[ = \frac{1}{B} \frac{5}{M_\infty^2} \left( \frac{x_2}{x_1} \right)^{\frac{3}{2}} \left[ \iint_0^{x_2} f \cdot \left[ \frac{1}{2} + \frac{1}{2} F_u \right] \, ds \right] + 1.4 \left( \frac{1}{2} M_\infty \right) x_2^{-\frac{1}{2}} \iint_0^{x_2} f f_k \left[ \frac{1}{2} + \frac{1}{2} F_u \right] \, ds \]

\[ + 0.84 \left( \frac{1}{2} M_\infty \right)^2 x_2^{-1} \iint_0^{x_2} f f_k^2 \left[ \frac{1}{2} + \frac{1}{2} F_u \right] \, ds \]

\[ + 0.252 \left( \frac{1}{2} M_\infty \right)^3 x_2^{-\frac{3}{2}} \iint_0^{x_2} f f_k^3 \left[ \frac{1}{2} + \frac{1}{2} F_u \right] \, ds \]

\[ T_{32} = \frac{1}{\frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \delta(x_i)} \iint_{20} \rho u \frac{1}{2} u^2 \, ds \]

\[ = \frac{1}{B} \left( \frac{x_2}{x_1} \right)^{\frac{3}{2}} \int_0^{\frac{\partial}{\rho_0}} f \cdot \left[ \text{as in } T_{29} \right] \left. \delta(x_i) \right|_{x=x_i} \, \, dF \]
\[ T_{33} = \frac{1}{\frac{1}{2} \rho_\infty U_\infty^3 \delta(x_i) \delta(x_i)} \iint \rho u \frac{1}{2} w^2 dS_E \]

\[ = \frac{1}{B x_i^3} \int_0^{\Delta t} \int \frac{F w^2}{\rho_\infty} [\text{as in } T_{30}] \bigg|_{x = x_2} dS_E \]

\[ T_{34} = \frac{-1}{\frac{1}{2} \rho_\infty U_\infty^3 \delta(x_i) \delta(x_i)} \iint \rho u \cos \lambda_E \frac{1}{2} \ln \frac{x_3}{x_2} dS_E \]

\[ = \frac{-1}{B x_i^{3\frac{1}{2}}} \frac{5}{M_\infty^3} F \left[ \frac{\frac{3}{2} + \frac{1}{2} F - \delta(x_2 - x_1)}{\frac{3}{2} (x_2^3 - x_1^3) + 1.4 (\frac{1}{2} M_\infty k)(x_2 - x_1)} \right] \bigg|_{y = 1} \]

\[ T_{35} = \frac{-1}{\frac{1}{2} \rho_\infty U_\infty^3 \delta(x_i) \delta(x_i)} \iint \rho u \cos \lambda_E \frac{1}{2} u^2 dS_E \]

\[ = \frac{-1}{B x_i^{3\frac{1}{2}}} \left[ \text{XRNTW} \cdot \text{FNU19} + \text{XRNT1} \cdot \text{FNU20} + \text{XRNT2} \cdot \text{FNU21} + \text{XRNT3} \cdot \text{FNU22} + \text{XRNT4} \cdot \text{FNU23} + \text{XRNT5} \cdot \text{FNU24} \right] \bigg|_{y = 1} \]

\[ T_{36} = \frac{-1}{\frac{1}{2} \rho_\infty U_\infty^3 \delta(x_i) \delta(x_i)} \iint \rho u \cos \lambda_E \frac{1}{2} w^2 dS_E \]

\[ = \frac{-1}{B x_i^{3\frac{1}{2}}} \int \frac{F}{w^2} [\text{XRNTW} \cdot \text{FNU25} + \text{XRNT1} \cdot \text{FNU26} + \text{XRNT2} \cdot \text{FNU27} + \text{XRNT3} \cdot \text{FNU28} + \text{XRNT4} \cdot \text{FNU29} + \text{XRNT5} \cdot \text{FNU30}] \bigg|_{y = 1} \]
\[ T_{30} = \frac{1}{\frac{1}{2} \rho_\infty U_\infty^3 z_e(x_i) \delta(x_i)} \int \int \rho \omega \sin \Lambda_e \ \frac{1}{2} \omega dS_e \]

\[ = \frac{1}{Bx_{h1}^{2/3} M_\infty^2 \cos \Lambda_e} \int \frac{F_\omega}{2} \left[ \frac{2}{3} (x_2^3 - x_1^3) + 1.4 \left( \frac{M_\infty f_i}{M_\infty f_k} \right) (x_2 - x_1) 
+ 1.68 \left( \frac{M_\infty f_k}{M_\infty k} \right)^2 (x_2^2 - x_1^2) + 2.5 \left( \frac{M_\infty f_i}{M_\infty k} \right)^3 \frac{x_2^3 - x_1^3}{x_2 - x_1} \right] \bigg|_{y=1} \]

\[ T_{38} = \frac{1}{\frac{1}{2} \rho_\infty U_\infty^3 z_e(x_i) \delta(x_i)} \int \int \rho \omega \sin \Lambda_e \ \frac{1}{2} \omega dS_e \]

\[ = \frac{1}{Bx_{h1}^{2/3} \cos \Lambda_e} F_\omega \int \left[ \frac{XRNTW \cdot FNU31 + XRNT1 \cdot FNU32}{XRTN2 \cdot FNU33 + XRNT3 \cdot FNU34 + XRNT4 \cdot FNU35 + XRNT5 \cdot FNU36} \right] \bigg|_{y=1} \]

\[ T_{39} = \frac{1}{\frac{1}{2} \rho_\infty U_\infty^3 z_e(x_i) \delta(x_i)} \int \int \rho \omega \sin \Lambda_e \ \frac{1}{2} \omega dS_e \]

\[ = \frac{1}{Bx_{h1}^{2/3} \cos \Lambda_e} F_\omega \int \left[ C_1 XRNTW + C_2 XRNT1 + C_3 XRNT2 + C_4 XRNT3 + C_5 XRNT4 + C_6 XRNT5 \right] \bigg|_{y=1} \]

\[ T_{40} = \frac{-1}{\frac{1}{2} \rho_\infty U_\infty^3 z_e(x_i) \delta(x_i)} \int \int \rho \omega h \ dS_e \]

\[ = -\frac{1}{Bx_{h1}^{2/3} M_\infty^2 \cos \Lambda_e} \frac{F_\omega}{12} \left[ \text{as \ in \ } T_{37} \right] \bigg|_{y=0} \]
\[ T_{41} = \frac{-1}{\frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \delta(x_i) \frac{2d_E}{2d_E}} \iint \rho u r \frac{1}{2} u^2 r dS_c \]

\[ = -\frac{1}{B X_1^{3/2} \cos \Lambda_E} F_w \left[ \text{as in } T_{38} \right] \bigg|_{y=0} \]

\[ T_{42} = \frac{-1}{\frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \delta(x_i) \frac{2d_E}{2d_E}} \iint \rho u r \frac{1}{2} u^2 r dS_c \]

\[ = -\frac{1}{B X_1^{3/2} \cos \Lambda_E} F_w \left[ \text{as in } T_{39} \right] \bigg|_{y=0} \]

\[ T_{43} = \frac{1}{\frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \delta(x_i) \frac{2d_E}{2d_E}} \iint k \frac{dT}{\partial_y} dS_w \]

\[ T_{43} (3D) = \frac{2k_w T_{\infty}}{\rho_0 U_\infty^{3/2} B X_1^{3/2}} \left( \frac{T_w}{T_{\infty}} \right)^2 \left[ -\frac{2}{3} \rho_0 (\chi_2^{3/2} - \chi_1^{3/2}) \int_0^1 \frac{d\beta}{f} \right. \]

\[ + 4 \rho_0 \left( \frac{1}{2} M_\infty \right) (\chi_2 - \chi_1) \int_0^1 \frac{F_k d\beta}{f} - 24 \rho_0 \left( \frac{1}{2} M_\infty \right)^2 (\chi_2^{1/2} - \chi_1^{1/2}) \int_0^1 \frac{F_k^2 d\beta}{f} \]

\[ - \frac{2}{3} \left( S - P_0 \right) (\chi_2^{3/2} - \chi_1^{3/2}) \int_0^1 \frac{F_k^2 d\beta}{f} \]

\[ + 4 \left( S - P_0 \right) \left( \frac{1}{2} M_\infty \right) (\chi_2 - \chi_1) \int_0^1 \frac{F_k^2 d\beta}{f} \]

\[ - 24 \left( S - P_0 \right) \left( \frac{1}{2} M_\infty \right)^2 (\chi_2^{1/2} - \chi_1^{1/2}) \int_0^1 \frac{F_k^2 d\beta}{f} \left] \right. \]

\[ T_{43} (2D) = \frac{2k_w T_{\infty}}{\rho_0 U_\infty^{3/2} B X_1^{3/2}} \left( \frac{T_w}{T_{\infty}} \right)^2 \int_0^1 \frac{d\beta}{f} \left[ -\frac{2}{3} (\chi_2^{3/2} - \chi_1^{3/2}) \int_0^1 \frac{d\beta}{f} \right. \]

\[ + 4 \left( \frac{1}{2} M_\infty \right) (\chi_2 - \chi_1) \int_0^1 \frac{F_k d\beta}{f} - 24 \left( \frac{1}{2} M_\infty \right)^2 (\chi_2^{1/2} - \chi_1^{1/2}) \int_0^1 \frac{F_k^2 d\beta}{f} \left] \right. \]

\[ T_{44} = -\frac{\Delta s}{\frac{1}{2} \rho_0 U_\infty^3 z_E(x_i) \delta(x_i) \frac{2d_E}{2d_E}} \iint \frac{H_{\text{flux}}}{S(x_i)} = -\sum_{i=28}^{\mu^3} T_i (2D). \]
APPENDIX C

NUMERICAL OUTPUT

The computer results summarized below demonstrate the successive reduction of residuals and therefore the convergence to the free parameter solution set. Eq. 29 was used to calculate the parameter input changes at each stage.

Small Span Flow Problem

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>A = 0.040</td>
<td>R₁ = 0.00728149</td>
</tr>
<tr>
<td>u₁₀ = 1.0</td>
<td>R₂ = -0.01142522</td>
</tr>
<tr>
<td>w₁₂ = -3.0</td>
<td>R₃ = 0.01102459</td>
</tr>
<tr>
<td>ρ₁₀ = -0.950</td>
<td>R₄ = -0.00958534</td>
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<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>A = 0.04001860</td>
<td>R₁ = -0.00000754</td>
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<tr>
<td>u₁₀ = 1.08410101</td>
<td>R₂ = 0.00004549</td>
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<td>w₁₂ = -4.00306398</td>
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<td>ρ₁₀ = -0.93408170</td>
<td>R₄ = 0.00012408</td>
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<table>
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<th>Output</th>
</tr>
</thead>
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<td>A = 0.04001820</td>
<td>R₁ = -0.00000051</td>
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<td>u₁₀ = 1.08353800</td>
<td>R₂ = 0.00000048</td>
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<tr>
<td>w₁₂ = -3.99860999</td>
<td>R₃ = -0.00000000</td>
</tr>
<tr>
<td>ρ₁₀ = -0.9328333</td>
<td>R₄ = 0.00000056</td>
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</table>
Therefore the answers are taken to be:

\[ A = 0.0400 \]
\[ u_{10} = 1.084 \]
\[ w_{12} = -4.00 \]
\[ \rho_{10} = -0.933 \]

For the final input set, the separate terms (as listed in Appendix B) come out to be:

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<tr>
<th>Term number</th>
<th>Value</th>
<th>Term number</th>
<th>Value</th>
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<tbody>
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<td>39</td>
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<td>-0.61707795</td>
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**Large Span Flow Problem**

**Input**

\[
\begin{align*}
A &= 0.04080 \\
u_{10} &= 1.5950 \\
w_{12} &= -0.4290 \\
\rho_{10} &= -1.0450
\end{align*}
\]

**Output**

\[
\begin{align*}
R_1 &= 0.09513438 \\
R_2 &= -0.01074666 \\
R_3 &= 0.10167959 \\
R_4 &= -0.04002621
\end{align*}
\]

**Input**

\[
\begin{align*}
A &= 0.04810 \\
u_{10} &= 0.681960 \\
w_{12} &= -9.593660 \\
\rho_{10} &= -1.000460
\end{align*}
\]

**Output**

\[
\begin{align*}
R_1 &= 0.00852852 \\
R_2 &= 0.00057422 \\
R_3 &= 0.00447973 \\
R_4 &= 0.00048120
\end{align*}
\]

**Input**

\[
\begin{align*}
A &= 0.04893800 \\
u_{10} &= 0.55160 \\
w_{12} &= -10.07869005 \\
\rho_{10} &= -0.94980
\end{align*}
\]

**Output**

\[
\begin{align*}
R_1 &= 0.00016976 \\
R_2 &= 0.00002039 \\
R_3 &= 0.00004084 \\
R_4 &= -0.00000152
\end{align*}
\]
It is seen that even an extremely poor first guess \( w_{12} \) has not prevented quick convergence. The answers are taken to be:

\[
\begin{align*}
A &= .0489 \\
u_{10} &= .552 \\
w_{12} &= -10.08 \\
\rho_{10} &= - .950
\end{align*}
\]

For the final input set, the separate terms come out to be:

<table>
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APPENDIX D

GENERALIZATION AND IMPROVEMENT

It has been noted that a new set of flight conditions will require a new density profile study, before the new solutions can be generated. If (say) an approximate Crocco integral analysis is used again, this will result in:

1) Different numerical coefficients in Eq. 24 for $\rho_{\text{MIN}}$ and $\eta_{\rho_{\text{MIN}}}$

2) Different numerical coefficients in Eq. 26 for the $\frac{\rho}{S}$ function.

3) Different numerical coefficients in the $F_{\rho_i}$ and $C_{ij}$ quantities listed in Appendix B.

Alternatively, different functional forms altogether (i.e., not the simple polynomials) could be postulated, in which event all of the algebra would have to be redone. It is not suggested that the mathematical model employed in this paper is necessarily the best. In fact, for Mach numbers greater than about ten, the density profile is not properly described by a simple polynomial in $\eta$. High Mach number implies high dissipation which implies a high negative wall gradient for density (as, $S_{\rho}$). The simple polynomial, when plotted, then exhibits a section of negative density.

In principle the present integral method can be successively improved by using postulated functions with more free parameters. At the same time more equations must be generated, by taking "higher
moments." The original differential equations (Eqs. 1-3) remain true when they are multiplied through by any quantity or function, known or unknown (such as $u, u^2, \rho$, etc.). The equality is still undisturbed when a volume integral is taken. Thus any number of integral/ algebraic equations can be generated, and the easily interpreted set employed here (mass, momentum, and energy balances) may be considered a special sub-case. Using a larger number of free parameters and accompanying equations increases the likelihood of getting "accuracy in the small," such as would be obtained from the original differential equations if they could be solved directly.