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N-N* Interactions with Faddeev Equations

by

William A. Gale

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Thesis Director's signature:

[Signature]

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1. INTRODUCTION

The intent of this theoretical investigation is to examine states of two nucleons and a pion for bound states. The possibility of such, suggested to us by Phillips and Zupancic, is readily apparent upon consideration of the $N^*$ or $P_{33}$ resonance. This resonance dominates the $\pi-N$ scattering through medium energies where it saturates the unitarity limit in magnitude, and the Wigner limit in width. Examination of the various partial waves shows that the $P_{33}$ phase shift is considerably larger than any other through 800 Mev, and importantly dominates the $S_{31}$ and $P_{31}$ repulsive phase shifts. It makes the pion-nucleon interaction the best understood of strong interactions.

The motivation of this investigation is the possibility of a bound state of two nucleons and a pion in which intuitively the nucleons would exchange the pion, each of them interacting resonantly with it in turn. Such an object with isotopic spin $T = 2$ would have five charge states, $Q = -1, \ldots, 3$. Supposing the binding below threshold were less than a pion mass, the $Q = -1, +3$ charge states could only decay by weak interaction $\rightarrow e^\pm + \nu + 2N$ with $\beta$ energy $\sim 100$ Mev. For $Q = 0, 1, 2$ a change of $T$ by
one brings the object to a state of two nucleons; since electromagnetic interactions have $\Delta T = 1$ these states would decay electromagnetically, into gamma plus two nucleons. The production of a $T = 2$ object requires either simultaneous production of a high isospin object or a high isospin projectile or target. For example

$$\pi^+ + d \rightarrow X^2 + \pi^0 \quad (1.1)$$

$$\pi^+ + d \rightarrow X^2 + \gamma \quad (1.2)$$

or

$$p + d \rightarrow X^{0,2} + N^* \quad (1.3)$$

The experimental situation on the bound state is that the region is not reported. For example investigations such as those of Haddock$^2$ and Reay$^3$ sought a resonance and did not report data concerning a bound state.

The simplest model of the situation would be two potential wells of identical shape with parameters such that either alone would give a resonance of the position and width desired. We discuss such a simplified model in one dimensional separable formalism in appendix D. It appears that for modest separations of the two wells a bound state will indeed result.
A possible method of calculation is to calculate the Born amplitude from the diagram

\[ N^* \rightarrow N \]
\[ N \rightarrow N^* \]

(1.4)

and use this for input in the N/D formalism. This method has been worked out analytically in similar cases by Ball and Wong and others\(^4\). Only a simplified version of the equations was tractable enough for solution\(^5\). The difficulty with the procedure can already be found in the Born amplitude

\[ B = \frac{g^4}{4p^*} \ln \left( \frac{M^2 + M^2 \mu^2 - \mu^2 - 2p \cdot q + 2p^2}{M^2 + M^2 \mu^2 - \mu^2 - 2p \cdot q - 2p^2} \right) \]

\[ p^* = \sqrt{M^2 + p^2} \]

\[ q^* = \sqrt{M^2 + p^2} \]

(1.5)

which develops singularities besides the unitarity cut on the right side. Their origin can be found by increasing \(M^*_\) from \(M\). For \(M^*_\) = \(M\) we have the well-known nucleon-nucleon scattering with a cut starting at \(-\mu^2\) (\(\mu < M\)) and going to \(-\infty\). For \(M^*_\) = \(M + \epsilon\) this cut breaks into two
around \(-M^2\). As \(M_x\) is increased the cut coalesces into a pole at the origin and then expands into a cut on the right side. The accurate treatment of this dynamical cut on the right side necessitates the coupled integral equations given by Ball, Frazer and Nauenberg\(^6\), which are more formidable than the Faddeev equations.

The previous work upon which we base ours is that started by Faddeev's derivation of a mathematically correct theory of nonrelativistic three particle systems\(^7\). A critical citation of this literature will be found in a recent review by Duck\(^2\). Faddeev's original work resolved the difficulties of the three particle Lippmann-Schwinger equations, which do not have a compact kernel, (i.e. one which can be approximated by a series of separable terms) by separating the T-matrix into three parts such that the (matrix) kernel of the three resulting coupled integral equations is compact. The advantage of a compact kernel is that Fredholm theory guarantees that a unique solution to the integral equations exists. This solution is obtained in practice by approximating the integral by a finite sum reducing the problem to the solution of linear algebraic equations. This method does converge to the solution\(^8\).
Lovelace\textsuperscript{9}) has given justification for the use of a separable approximation to two particle scattering amplitudes, whose use reduces the two dimensional integral equations to one dimension which is practical for computer solution. Ahmadzadeh and Tjon\textsuperscript{10}) have given a three particle angular momentum reduction in the spinless case. We give the extension of this method to include spin. Hetherington and Schick\textsuperscript{11}) have solved such integral equations at positive energies where the kernels develop singularities (due to the vanishing of energy denominators) which cause numerical difficulties. Our application, searching for bound states and low energy resonances in the three particle system, only involves solutions at energies below the three particle threshold and we do not meet singular integral equations.

The relativistic problem has been investigated by several authors. Lovelace\textsuperscript{12}) has presented a practical computational scheme by forcing the cluster property into the propagators. The cluster property means that the three body propagator can be expressed in terms of the two body propagator and is a natural feature of the nonrelativistic theory. Forcing it upon the relativistic theory introduces extraneous singularities and loses time reversal invariance
as Omnes\textsuperscript{13}) points out from the viewpoint of an alternative theory not suitable for computation. We use the Lovelace scheme. It must be pointed out that all these versions of Faddeev equations have a fixed number of particles, and will become inapplicable when production processes assume importance. In other words relativistic dynamics are never considered; it is a three body potential model possibly with relativistic kinematics.

The separable approximation to the T-matrix can be made in a number of ways\textsuperscript{14}). We have mainly used the separable potential of Barbour and Schult\textsuperscript{15}) which generates a separable T-matrix certain to satisfy off shell unitarity\textsuperscript{9}). We examine the sensitivity of the results to this assumption by the use of a direct approximation to the t-matrix for a resonant interaction.

In seeking bound states, we rely on the standard results concerning analyticity of the T-matrix, in particular that the T-matrix has a pole at a negative energy corresponding to a bound state. This statement of the bound state condition and the equivalent statement that the Jost function, $F_0$, has a zero at a bound state are derived from the asymptotic form of the wave function\textsuperscript{16})
\( \varphi_x(k, x) \sim c_x \left[ F_x(k) e^{ikx} - F_x(-k) e^{-ikx} \right] \) \hspace{1cm} (1.6)

To have a normalizable wave function at \( k = ib \) (\( b > 0 \)) we must have \( \mathcal{F}_k(-ib) = 0 \) to obtain a decreasing exponential. Then since

\[ S_x(k) = \frac{F_x(k)}{F_x(-k)} = e^{2i\xi_x(k)} \] \hspace{1cm} (1.7)

and

\[ t_x(k) = -\frac{i}{2ik} \left( S_x(k) - 1 \right) \] \hspace{1cm} (1.8)

the S-matrix and T-matrix have poles at bound states. Furthermore, besides the formal expression of \( S \) as a ratio of Jost functions, it can be proved that \( \mathcal{F}_k(-k) \) is exactly the Fredholm denominator of an integral equation for \( t_k(k) \), which is itself the quantity approximated by the determinant of the algebraic equations solved numerically. Thus we actually calculate a determinant which we report as \( D_3 \), which is an approximation to the Jost function, and which we search for zeros. We are relying on the extension of the two body results quoted above from Regge, by Rubin, Sugar, and Tiktopoulos\(^{17}\).
Our general conclusion about the nucleon-nucleon-pion system is that the Peierls mechanism is in fact sufficient to produce no $B = 2$, $T = 2$ bound state and but only a broad resonance in a two-one configuration at energy four or five hundred Mev above the two nucleon threshold.

The principal investigations of the $NN\pi$ system have been by Doolen$^{18)}$ and Varma$^{19)}$. Investigations of the somewhat similar $NN\pi$ system have been made by many authors including Lovelace$^9)$ and Aaron$^{20)}$. Doolen investigated $\pi$-$D$ elastic scattering and found the multiple scattering terms to be quite small. This was in contrast to the work of Hetherington and Schick on $K$-$D$ elastic scattering. Doolen neglected spin and isospin. Varma has limited himself to $T = 0$, $S = 0$ scattering of two nucleons. Neither of these calculations considered relativistic effects. In our calculation we have included spin and isospin. We calculate mainly for negative energies to find which channels and which couplings are important before making a slight extension to positive energies. We have also made calculations using the relativistic model which has important differences. Both of the above calculations are free of arbitrary parameters, as is ours.
In the following section 2 we discuss kinematics and normalization conventions. We then give a discussion of the Faddeev equations including symmetry, spin, isospin, their numerical solution and interpretation in section 3. The results of the nonrelativistic potential models for $\mathcal{F} \leq 2$ and for some simplified cases are given in section 4. Section 5 contains a discussion of the relativistic potential model and its results. In section 6 we discuss model dependence. Possible extensions of this work are aired in section 7. The conclusions of our investigation are given in section 8. Appendices contain the details of derivations, the static limit, and the two well potential model.
2. KINEMATICS

In this section we present definitions for the purpose of noting the normalizations, which are unfortunately not uniform in the literature. We define momentum variables first, and their angular momentum decomposition. We give the nonrelativistic Green's function for three bodies, and finally we establish our conventions for two body systems.

First

\[ \langle \vec{r} | \vec{p} \rangle = e^{i \vec{p} \cdot \vec{r}} \]  \hspace{1cm} (2.1)

with

\[ \langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \]  \hspace{1cm} (2.2)

When considering three particles labeled \( (i, j, k) = (1, 2, 3) \) with momenta \( \vec{p}_i \) \( i = 1, 2, 3 \), we define

\[ \vec{k}_i = \frac{m_j \vec{p}_k - m_k \vec{p}_j}{m_j + m_k} \] \hspace{1cm} \((i, j, k) \text{ cyclic}\)  \hspace{1cm} (2.3)

which is the momentum of particle \( (k) \) in the cm system of \( (j, k) \). The center of mass motion described by \( \vec{P} = \Sigma \vec{P}_i \) can be solved separately and trivially, so we take \( \vec{P} = 0 = \Sigma \vec{P}_i \).

We can describe the resulting system by two independent vectors with a vector equation they must satisfy.
In particular $\vec{p}_i, \vec{k}_i$ form an independent pair because if there exist $\alpha, \beta$ not both zero such that

$$\alpha \vec{p}_i + \beta \vec{k}_i = 0 \quad (2.4)$$

then

$$(-\alpha + \beta \frac{m_j}{m_j + m_k}) \vec{p}_k - (\alpha + \beta \frac{m_k}{m_j + m_k}) \vec{p}_i = 0$$

so that $(\vec{p}_i, \vec{k}_i)$ are linearly independent whenever $(\vec{p}_i, \vec{p}_k)$ are. A state may be specified equally well by any of the four systems of momentum variables symbolized by

$$|\vec{p}, \vec{p}_i, \vec{p}_j, \vec{p}_3\rangle = |\vec{p}, \vec{k}_i\rangle = |\vec{p}_i, \vec{k}_j\rangle = |\vec{p}_3, \vec{k}_3\rangle \quad (2.5)$$

We note that the Jacobians

$$\frac{\partial \vec{p} \partial \vec{k}_i \partial \vec{p}}{\partial \vec{p}, \partial \vec{p}_i \partial \vec{p}_j} = 1 \quad (2.6)$$

Now we have the normalization

$$\langle \vec{p}, \vec{k} \mid \vec{p}', \vec{k}' \rangle = (2\pi)^6 \delta^3(\vec{p}' - \vec{p}) \delta^3(\vec{k}' - \vec{k}) \quad (2.7)$$

The angular momentum decomposition of these states is needed:

$$|l, m_l, k, m_k\rangle = \int d\Omega_l \int d\Omega_k \sum_{m_k} \langle \vec{p}, \vec{k}_l | m_l \rangle \langle \vec{k}, \vec{k}_k | m_k \rangle |\vec{p}, \vec{k}\rangle \quad (2.8)$$

or the more convenient description (since the total angular momentum and its projection are constants of the motion)
\[ \langle p \, k \, l \, j \, m_g \rangle_i = \sum_{m_i, m_l, m_j} \binom{m_j}{m_k \, m_l \, m_i} \langle p \, k \, m_j \, m_l \rangle_i \]  

(2.9)

where we have introduced the vector coupling coefficients.

The nonrelativistic three body Green's function, or propagator

\[ \mathcal{G}_3(s, \vec{p}_1, \vec{p}_2, \vec{p}_3) = \left( \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} - s \right)^{-1} \]  

(2.10)

But we have the equality

\[ \frac{p_1^2}{2(m_1 + m_2)} + \frac{k_1^2}{2(m_2 \cdot m_3)} \right) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \]  

(2.11)

Hence

\[ \mathcal{G}_i(s, \vec{p}_1) = \left[ \frac{k_1^2}{2(m_1 \cdot m_2 \cdot m_3)} - (s - \frac{m_1 + m_2 + m_3}{2m_1 \cdot m_2 \cdot m_3}) \right]^{-1} \]  

(2.12)

The linearity of \( p_1^2 \) in the expression, which allows it to be grouped with \( S \) is called the "cluster" property. The cluster property allows the two body T-matrix in three body space to be equated to a two body T-matrix in two body space evaluated at the available energy to the two body system. We review the two body formalism to establish our conventions with units such that \( 2m_r = \hbar = c = 1 \), where
\[ m_r = \frac{m_1 m_2}{m_1 + m_2} \]. We expand the two particle T-matrix

\[
\langle t_{k'} | t(k') | t_0 \rangle = 4\pi \sum_{l} (2l + 1) \frac{P_2(k'_l; P'_l)}{P_2(k; P)} \langle kl | t(k') | kl \rangle
\]  

(2.13)

with the normalization of the partial wave amplitude

\[
\langle kl | t(k') | kl \rangle = -\frac{\varepsilon^{iS(k)} \sin S(k)}{k}
\]  

(2.14)

and the corresponding Lippmann-Schwinger equation in partial wave expansion

\[
\langle kl | t(p') | kl \rangle = \langle kl | \nu | kl \rangle
\]

\[
-\frac{2\pi}{\alpha} \int_0^\infty dq G_\alpha(p^*, q^*) \langle kl | \nu | ql \rangle \langle ql | t(p') | kl \rangle
\]  

(2.15)

where

\[
G_\alpha(p^*, q^*) = \frac{1}{q^* - p^*}
\]  

(2.16)

The two body operator \( t_\perp \) in three body space is the operator describing the interaction of particles 2 and 3, with the particle 1 present but a non-interacting spectator. Since the particle 1 is not acted upon, it is convenient to use the representation \( 1 \) where we obtain a factor

\[
(2\pi)^3 \delta^3(\vec{p}' - \vec{p})
\]  

(2.17)
times a two particle T-matrix (calculated with three body propagator). Now we note that by the cluster property

\[ G_3(s, p_a) = G_3 \left( s - M_R \hat{p}_i^s, k_i^s \right) \]  

(2.18)

where

\[ M_R = \frac{m_i + m_\perp + m_\lambda}{2 m_i (m_\perp + m_\lambda)} \]  

(2.19)

Hence

\[
\langle \hat{p}', \hat{t}', \hat{t}, (s) \mid \hat{p}, \hat{t}, \hat{t} \rangle \\
= (2\pi)^3 \delta^3(\hat{p}' - \hat{p}) \left\langle \hat{t}', \left| t, (s-M_R \hat{p}_i^s) \right| \hat{t} \right\rangle \\
= (2\pi)^3 \delta^3(\hat{p}' - \hat{p}) 4 \pi \sum (2\lambda + 1) P_2(\hat{t}', \hat{k}_i^s) \langle k_\lambda | t, (s-M_R \hat{p}_i^s) | k_\lambda \rangle \\
= (2\pi)^3 (4\pi)^3 \delta^3(\hat{p}' - \hat{p}) \sum_{\lambda \mu} \gamma_{2\lambda, \lambda}^{\mu} (\hat{k}_i^s) \gamma_{\lambda}^\mu (\hat{k}_i^s) \langle k_\lambda | t, (s-M_R \hat{p}_i^s) | k_\lambda \rangle 
\]  

(2.20)
3. FADDEEV EQUATIONS

A. Derivation

The content of the Faddeev equations is best shown by diagrams. We symbolize the entire three body t-matrix \( T \) by

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\quad (3.1)
\]

We write the Faddeev equations in terms of those parts of \( T \) in which the pair \((i,j) = k\) interacts last. For instance \( T^1 \) is symbolized by

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\quad (3.2)
\]

where

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\quad (3.3)
\]

is the interaction of the pair \((2,3)\), with particle 1 present but not interacting, and is described by the two body t-matrix in three body Hilbert space \( T^1 \). The large blob in diagram (3.2) now represents the entire three body amplitude except that pair 1 does not interact last. The remaining possibilities are (a) no interaction (b) pair 2 interacts last (c) pair 3 interacts last. Thus
\[ T' = t_i - t_i G_3 T^2 - t_i G_3 T^3 \]  \hspace{1cm} (3.4)

where \( G_3 \) is the three body propagator, and all quantities depend on the energy \( S \).

An algebraic derivation writes the Lippmann-Schwinger equation for the three body t-matrix

\[ T = V - V G_3 T \]  \hspace{1cm} (3.5)

where \( V = V_1 + V_2 + V_3 \), with \( V_1 \) the interaction of pair \( i \). We decompose \( T \) into

\[ T = (T'' + T^2 + T^3) = (V_i + V_i + V_i) - (V_i + V_i + V_i) G_3 T \]  \hspace{1cm} (3.6)

which will be solved so long as

\[ T' = V_i - V_i G_3 (T'' + T^2 + T^3) \]  \hspace{1cm} \text{and cyclic}  \hspace{1cm} (3.7)

are all satisfied. Now we use the two body amplitude equation

\[ t_i = V_i - t_i G_3 V_i \]  \hspace{1cm} (3.8)

solving for

\[ V_i = (1 - t_i G_3)^{-1} t_i \]  \hspace{1cm} (3.9)
which we substitute back into (3.7)

\[ T' = (1 - t, G_j) \nu t' - (1 - t, G_j) \nu t, G_j (T' + T^2 + T^3) \]

or

\[ (1 - t, G_j) T' = t, - t, G_j (T' + T^2 + T^3) \]

From which, finally, we obtain

\[ T' = t_i - t, G_j T^2 - t, G_j T^3 \text{ and cyclic.} \]

We have proved the sufficiency of the solution of these equations for the solution of (3.5). The necessity of the decomposition is considerably more difficult, but was first resolved by Faddeev. He proved that the Lippmann-Schwinger kernel cannot be made compact by expanding the set of wave functions from those of a Hilbert space to those of a Banach space. Since this is one of two possible ways to solve a singular (= noncompact kernel) integral equation, the alternative method that some part of the problem be solved and incorporated into the whole has to be used. The two body t-matrices play a natural role here, since they are a solved part.
We have not proved here that the Lippmann-Schwinger kernel is not compact, nor that the Faddeev kernel is compact.

B. Spinless Angular Momentum Reduction

We consider the minimal three body problem of a system in which two of the particles (1 and 2) do not interact with each other, but only with the third particle (3). Since any interaction between the two can be incorporated once the minimal problem is solvable (Pauli), this system offers all the essential features of the full problem. In the Faddeev equations, this amounts to setting $t_3 = 0$, hence $T^3 = 0$. Without loss of generality we discuss only the equation

$$T^1(s) = t_1(s) - t_1(s) G_3(s) T^2(s)$$

(3.13)

since the other coupled equation is obtained simply by interchanging the sub and superscripts 1 and 2. Since $T^1$ describes that part of $T$ in which pair (1) interacts last, the final state is best described in system 1. The initial state will be denoted $|I\rangle$ now, and discussed later. Since $T^2$ describes that part of $T$ in which pair (2) interacts last, the intermediate states will be described in system 2. Thus the angular momentum reduction will start from
\[ \langle p', k'_1 | t'_1(s) | I \rangle = \langle p', k'_1 | t_1(s) | I \rangle \]

\[ -\int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \langle p', k'_1 | t_1(s) | p_2 k_2 \rangle G_3(s, p_1) \langle p_2 k_2 | T^z(s) | I \rangle. \]

The real difficulty is that inside the integral, \( t_1(s) \) must be taken between a 1-bra and a 2-ket. We leave the details to the appendix A, noting the equation

\[ \langle p', k'_1, l' | T'(s) | I \rangle = \langle p', k'_1, l' | G^m g' y'_m | t_1(s) | I \rangle \]

\[ -\frac{2\beta}{\sqrt{p'}} \int_0^\infty d\tau \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} G_3(s, p_1) \sum_{\ell L} \sqrt{\tau} \langle k', l' | t_1(s - M_m (p'_2)) | k, l' \rangle \sum_{m'_L m'_x} \left( \begin{array}{ccc} L & \ell & \ell' \\ m'_{L'} & m'_{x'} & m'_{x} \end{array} \right) \left( \begin{array}{ccc} L & L' & L' \\ m_L & m_{x'} & m_{x} \end{array} \right) \langle p_2 k_2 l L | g' y'_m | T'(s) | I \rangle \]

with

\[ \beta = 1 + m_1 / m_2 \]

\[ \hat{p}_i \cdot \hat{k} = \beta / 2 p_1 k \left( M_1^2 p_1^2 - k^2 / \beta - k^2 \right) \]

\[ \hat{p}_i \cdot \hat{k} = 1 / 2 p_1 k \left( k_i^2 + p_i^2 - p_i^2 / \beta \right) \]

\[ \hat{p}_i \cdot \hat{k} = \beta / 2 p_1 k \left( k_i^2 + p_i^2 / \beta - p_i^2 \right) \]

\[ k_i^2 = k_i^2 + (\beta - 1) / \beta \left( p_i^2 - p_i^2 \right) \]
\begin{equation}
\langle \chi | T'(s) | \chi \rangle = \langle \chi | T_{t}(s) | \chi \rangle
- \int_{0}^{\infty} dp_{1} \int_{0}^{p_{1}} dp_{2} e^{i K_{1} \cdot \vec{P}_{1}} \left( \frac{2}{\pi} \right)^{2} \langle p_{1} k_{1} | T'_{t}(s-M_{R}) | k_{1} \bar{k}_{1} \rangle \langle k_{1} \bar{k}_{1} | T_{t}(s-M_{R}) | p_{2} \bar{p}_{2} \rangle \langle p_{2} \bar{p}_{2} | T^{\prime}(s) | \chi \rangle
\end{equation}

We are particularly interested in the case where particles 1 and 2 (but not 3) are identical. There are two facts to take into account: (a) the initial state must be appropriately symmetrized:

\begin{equation}
| a b c \rangle_{\pm} = \pm | b a c \rangle_{\pm} \equiv (\varepsilon)^{\varepsilon} | b a c \rangle_{\pm}
\end{equation}

where \( \varepsilon = 0 \) for Bosons, \( \varepsilon = 1 \) for Fermions. Clearly we have

\begin{equation}
\langle a b c | T'(s) | a' b' c' \rangle_{\pm} = \langle b a c | T^{\prime}(s) | b' a' c' \rangle_{\pm}
\end{equation}

which is obvious by diagrams

\begin{equation}
\begin{array}{c}
\text{1.} \quad a' \quad b' \\
\text{2.} \quad c' \quad b
\end{array}
\quad
\begin{array}{c}
\text{1.} \quad b \\
\text{2.} \quad a \quad c
\end{array}
\quad
\begin{array}{c}
\text{1.} \quad b' \quad b \\
\text{2.} \quad a' \quad c
\end{array}
\end{equation}

Given these two equations, we have

\[ 2 \langle b (c',a) | T^{2} | a' b' c' \rangle_{\pm} = \langle a b c | T^{2} | a' b' c' \rangle_{\pm} = \langle b a c | T' | b' a' c' \rangle_{\pm} \]
\[ = (-)^\varepsilon \langle b^{\prime} \bar{z} z \mid T' \mid \bar{a}^{\prime} b^{\prime} z' \rangle_\pm \]

\[ = (-)^\varepsilon \langle b^{\prime} (\bar{z}, z) \mid T' \mid \bar{a}^{\prime} b^{\prime} z' \rangle_\pm \] (3.21)

In other symbols,

\[ \langle \bar{p}, \bar{p}_r | T^z | \bar{a}^{\prime} b^{\prime} z' \rangle_\pm = (-)^\varepsilon \langle \bar{p}, \bar{p}_r | T' \mid \bar{a}^{\prime} b^{\prime} z' \rangle_\pm \] (3.22)

Then since

\[ \langle -\bar{p}_r | = \sum_{L m_L} Y_L^{m_L} (-\bar{p}_r) \langle k_L L m_L | \]

\[ = \sum_{L m_L} (-)^l Y_L^{m_L} (\hat{p}_r) \langle k_L L m_L | \] (3.23)

We have the following relation

\[ \langle \bar{p}, k, L L' m_{L'} | T^z(s) | \Delta \rangle_\pm = (-)^{\varepsilon + L} \langle \bar{p}, k, L L' m_{L'} | T'(s) | \Delta \rangle_\pm \] (3.24)

which can be substituted in equation (3.15) to give one integral equation in two dimensions. Up to this point the equations are exact, but on the large side for practical handling: If 20 points are required for each integral,
they will require 400 x 400 matrices, in uncoupled cases; to couple two states would require 800 x 800 matrices. We note with numerous authors that practical simplicity is obtained by assuming that the two body t-matrix is separable, that is

$$\langle k' L | t_i(s) | k L \rangle = \frac{V_o(k') V_o(k)}{D_o(s)}$$ (3.25)

(A sum of such terms can also be handled, but we have used one term separable t-matrices.) For simplicity we include a factor $\delta_{L \rightarrow 0}$, assuming a dominant channel exists. (We relax this assumption later: See coupled channels.) In this case

$$\langle p', k', L', L \gamma | T(s) | 1 \rangle \pm = V_o(k', \gamma) \langle p', L' L \gamma | T(s) | 1 \rangle$$ (3.26)

with

$$\langle p', L \gamma | T(s) | 1 \rangle \pm = \frac{\langle p', k' \gamma | T(s) | 1 \rangle \pm}{V_o(k')}$$

$$+ \frac{(\frac{s}{4} + \epsilon + L)}{D_o(s - k'^2 \rho)} \int_{0}^{\delta} \sum_{\Delta} \left( \frac{\rho \rho_{k'} \rho_{k}}{2} (p_{k} k_{\gamma}) \right) V_{o}(k) V_{o}(k) \right \}$$ (3.27)
This is a set of coupled, one dimensional linear integral equations for matrix elements of components of the three particle $T$ matrix $T^1$ between an initial state $|I\rangle$ (which need not be further specified) and a state (or channel or configuration) of the three particle system labeled by the quantum numbers $(\ell,L,J,M_J)$. We have already reduced the number of coupled equations by assuming a two particle interaction in only one partial wave $L_0$, and we can further reduce the complexity of the problem by neglecting the coupling between channels with different $\ell$ values to give us a single, one dimensional integral equation to be solved. We return to examine these assumptions under the section coupled channels.

C. Isospin

Isospin dependence in the kernel factors out, since we are dealing with strong interactions for which isospin is conserved. The projection of one isospin configuration on another is its total effect, which is a recoupling of a spinlike quantity, introducing a $6j$ coefficient.

Our states must now include isospin information

$$|\bar{\ell}_i \ell_i \rangle_c \rightarrow |\bar{\ell}_i \ell_i \rangle_c |t_1 t_2 t_3 t_{12} t_{13} t_{23}\rangle$$

(3.28)
\[
\begin{align*}
|t, t_1, t_2, t_3, t_1', t_2', t_3'\rangle &= \sum_{t_0, t_0'} \left| t, t_1, t_2, t_3, t_0, t_0' \right\rangle \left\langle t_1, t_2, t_3 \right| t_0 \rangle \\
&= \sum_{t_0, t_0'} \left( \begin{array}{c}
t_1 \\
t_2 \\
t_3 \\
t_0
\end{array} \right) \left( \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right) \\
&= \left( \begin{array}{c}
t_1 \\
t_2 \\
t_3 \\
t_0
\end{array} \right) \left( \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right) \\
&= \left( \begin{array}{c}
t_1 \\
t_2 \\
t_3 \\
t_0
\end{array} \right) \left( \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right)
\end{align*}
\]

(3.29)

since \(t_1, t_2, t_3, T, \) and \(T_3 \) are constants only \(t_{(i)} \) needs to be specified to identify the state \(|\vec{P}_1, \vec{k}_1\rangle_i \) completely.

In the derivation, the essentially new part is

\[
\begin{align*}
\langle \ldots t_1' \ldots t_{(s)} \ldots t_0 \rangle \\
&= \sum_{t_0, t_0'} \left( \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right) \left( \begin{array}{c}
t_1 \\
t_2 \\
t_3 \\
t_0
\end{array} \right) \left( \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right) \left( \begin{array}{c}
t_1 \\
t_2 \\
t_3 \\
t_0
\end{array} \right) \left( \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right) \\
&= \langle \ldots t_1' t_2' t_3' t_1' t_2' t_3' \ldots t_{(s)} \ldots t_1 t_2 t_3 t_1' t_2' t_3' \rangle \\
&= \langle \ldots t_{(s)} \ldots \rangle (-i)^{2T} \left[ t_{(0)}, t_{(0)} \right]^{\alpha} \left\{ \begin{array}{c}
t_1 \\
t_2 \\
t_3 \\
t_0
\end{array} \right\} \left\{ \begin{array}{c}
t_1' \\
t_2' \\
t_3' \\
t_0'
\end{array} \right\}
\end{align*}
\]

(3.30)
using Edmonds' 3.5.14 and 6.1.5.

For the \( NNN \) system with just the \( P_{33} \) resonance taken into account \( t_1 = t_2 = \frac{1}{2} \), \( t_3 = 1 \) and \( t'_1 = t'_2 = \frac{3}{2} \).

Hence the numerical factor becomes

\[
\begin{align*}
& 1 \quad \text{for} \quad T = 2 \\
& -\frac{1}{3} \quad \text{for} \quad T = 1 .
\end{align*}
\]

These isospin factors of 1 for the \( T = 2 \) channels and \(-\frac{1}{3}\) for the \( T = 1 \) channels are of critical importance, as we shall see, in determining the effect of the \( P_{33} \) pion-nucleon resonance as an \( NNN \) interaction mechanism. The derivation of the equations including spin is placed in Appendix B.

The result for \( NNN \) is to replace the factor

\[
\delta \Theta^L_{\ell \ell'}(p, k) = \sum_{\text{h}
\text{m}} \sqrt{2} \left( \frac{\ell' \ell \cdot \mathbf{Q}}{m_{l'} - m_{l}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} + \mathbf{K})}{m_{l} + m_{l'}} \right)^{1/2} \left( \frac{\ell' \ell \cdot (\mathbf{Q} - \mathbf{K})}{m_{l} - m_{l'}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} + \mathbf{K})}{m_{l} + m_{l'}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} - \mathbf{K})}{m_{l} - m_{l'}} \right)^{1/2}
\]

by

\[
\delta \Theta^L_{\ell \ell'}(p, k) = \sum_{\text{h}
\text{m}} \sqrt{2} \left( \frac{\ell' \ell \cdot \mathbf{Q}}{m_{l'} - m_{l}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} + \mathbf{K})}{m_{l} + m_{l'}} \right)^{1/2} \left( \frac{\ell' \ell \cdot (\mathbf{Q} - \mathbf{K})}{m_{l} - m_{l'}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} + \mathbf{K})}{m_{l} + m_{l'}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} - \mathbf{K})}{m_{l} - m_{l'}} \right)^{1/2}
\]

(3.31)

\[
\delta \Theta^L_{\ell \ell'}(p, k) = \sum_{\text{h}
\text{m}} \sqrt{2} \left( \frac{\ell' \ell \cdot \mathbf{Q}}{m_{l'} - m_{l}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} + \mathbf{K})}{m_{l} + m_{l'}} \right)^{1/2} \left( \frac{\ell' \ell \cdot (\mathbf{Q} - \mathbf{K})}{m_{l} - m_{l'}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} + \mathbf{K})}{m_{l} + m_{l'}} \right)^{1/2} \left( \frac{\ell \ell' \cdot (\mathbf{Q} - \mathbf{K})}{m_{l} - m_{l'}} \right)^{1/2}
\]

(3.32)
where we now decompose $|p_k\rangle$ into $|pj ls; kJLS; \ell m q_i\rangle$. A table of the values of these numbers is included in Appendix B, for the states investigated.

The solution of the equations is now numerically straightforward for non-positive energies. The kernel integral

$$K(p', p; s) = \frac{\beta}{\alpha D(s - M_k p'^2)} \int \frac{d^3k}{p'^2} \frac{v(k) v(k)}{M_k p'^2 + k^2 - s} \Theta(p, k)$$

is a finite integral without singularities and is done by Simpson's method. In practice, convergence is often obtained before the upper limit is reached, in which case the integral is terminated.

For uncoupled channels we can then abbreviate our equation

$$\mathcal{T}(p', s) = t(p', s) + \int_0^\infty d\rho \ K(p', p; s) \ T(p, s) \ .$$

We denote

$$p_i = (i - \frac{1}{2}) \Delta \rho \quad i = 1, 2, \ldots, N \quad (N = 20)$$

$$\mathcal{T}_i(s) = \mathcal{T}(p_i, s)$$

$$t_i(s) = t(p_i, s)$$

$$K_{ij}(s) = K(p_i, p_j, s)$$

$$
(3.35)$$
so that

\[ T_i(s) = \tau_i(s) + \sum \Delta p K_{ij}(s) T_j(s) \]  

or

\[ (\delta_{ij} - \Delta p K_{ij}(s)) T_j(s) = \tau_i(s) \]

which has a solution unless \( \text{Det} [\delta_{ij} - \Delta p K_{ij}(s)] = 0 \). We use rectangular quadrature here because (a) we found Simpson or Gaussian quadrature did not help in the numerical solution of the Lippmann-Schwinger equation, (b) the partial determinants \( D_n = \text{Det} [\delta_{ij} - \Delta p K_{ij}(s)]_{i,j \leq n} \) have the uniform physical meaning that the \( p \)-integral was terminated at \( p_n \); hence convergence can be checked, and furthermore each change of sign indicates a bound state below the energy being calculated.

As explained in the introduction the determinant that we calculate is an approximation to the Jost function. A zero at negative energies corresponds to a bound state. A zero of the real part corresponds to a resonance. We can see this condition on the determinant in a simple way also, assuming the pole in \( T \) marks a bound state: Since \( T \) has a single pole, and \( t \) does not, the residue \( T_R \) at the pole satisfies a homogeneous equation.
\[ T_R = K(s_R) T_R \]  
\[ (1 - K(s_R)) T_R = 0 \]  
\[ \text{hence} \quad \det [1 - K(s_R)] = 0 \]  
\[ \text{(3.37)} \]  
\[ \text{(3.38)} \]  
\[ \text{(3.39)} \]  

since \( T_R \) is not zero and a nontrivial solution to the equations (3.38) thus does exist.

The \( D_3 \)'s that we have obtained have all been very smooth without extra kinks they might have. Their asymptotic value is one, and they have just one maximum or minimum. The magnitude and location of the extremum characterizes the interaction, as does the one value at threshold, \( D_3(0) \). If \( D_3(0) < 0 \) there is a bound state. For small positive \( D_3(0) (\ll .3) \) there is probably a resonance. For \( D_3(0) \ll 1 \) the forces are still attractive. For \( D_3(0) \gg 1 \) the interaction is repulsive. It is this value at threshold which we typically quote as characterizing the whole interaction.
4. NONRELATIVISTIC MODEL

A. Barbour-Schult Potential

The Faddeev formalism we have presented reliessfor
input on a separable two body t matrix. We now turn to the
description of a model for the (3,3) resonance interaction
which is the strongest interaction between any pair of the
particles we are considering. With

$$\langle kL | t(s) | k'L \rangle = \frac{V_L(k) V_{L'}(k')}{D_L(s)} \quad (4.1)$$

we require for a resonance at energy $S_\pi = 160$ MeV that

$$\text{Re}D_L(S_\pi) = 0,$$ and that

$$\sigma \propto |\langle kL | t(s) | k'L \rangle|^2 \quad (4.2)$$

have full width at half maximum value $\Gamma = 120$ Mev. We also
note the following table of low energy phase shifts

<table>
<thead>
<tr>
<th>Pion Lab Energy Mev</th>
<th>cot$\delta_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>71</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>40</td>
<td>13</td>
</tr>
<tr>
<td>80</td>
<td>4.5</td>
</tr>
</tbody>
</table>

which we want to fit reasonably well. Here the relation of
\[ k \cot \delta = -\hbar \varepsilon t'' \]  

(4.3)

A separable \( t \) matrix can be obtained as an approximation to the \( t \) matrix of a local potential, for instance

\[ \langle k | t(p^2) | k' \rangle = \frac{\langle k | t(p^2) | p \rangle \langle k' | t(p) | p \rangle}{\langle p | t(p^2) | p \rangle} \]  

(4.4)

or as the exact solution for the non-local separable potential \( V(r,r') = \lambda V(r)V(r') \), in which case

\[ \langle k | t(p^2) | k' \rangle = \frac{\lambda V(k)V(k')}{1 + \lambda \sum_q \frac{\delta^2}{q^2 - p^2} V(q)} \]  

(4.5)

where

\[ \lambda V(k)V(k') = \langle k | V(k') \rangle = \int r^2 dr \, r''^2 dr'' \, \langle k | r \rangle \langle r | V(r') | r' \rangle \langle r' | k' \rangle \]  

(4.6)

\[ \langle r' | \ell | k \ell \rangle = i^\ell \int_r^{k} (kr) \]

In practice a simple form for \( V(k) \) is chosen. The simplest is \( V(k) = \frac{1}{k^2 + \beta^2} \), which has been used by Yamaguchi. With this potential

\[ \lambda D(p^2) = 1 + \frac{\lambda}{2\beta(\beta - ip)^2} \]  

(4.7)
\[ \lambda \text{Re} D(p^2) = \begin{cases} 
1 + \frac{\lambda (\beta^2 - p^2)}{2 \beta (\beta^2 + p^2)^{\frac{3}{2}}} & p^2 > 0 \\
1 + \frac{\lambda}{2 \beta (\beta^2 - q^2)^{\frac{3}{2}}} & p^2 < 0 \ p = iq, \ q > 0 \end{cases} \tag{4.8} \]

We find that if \( \text{Re} D \) has a zero for positive energy \( \beta^2 < p^2 \) (in order that \( \lambda < 0 \) which corresponds to attraction), then it has also a zero for \( -q^2 \).

\[ q^2 = \frac{\beta^2 + p^2}{\sqrt{\beta^2 - p^2}} - \beta \tag{4.9} \]

which is positive. In other words if this potential is attractive, it can have a bound state, but not a resonance. A somewhat less simple form is

\[ V(k) = \frac{1}{(k^2 + \beta^2)(k^2 + \beta^* x^2)} \tag{4.10} \]

where \( \beta \) is now complex, but \( V \) is still real by construction. This potential has been used by Barbour and Schultz\(^{15} \) and others. In evaluating \( D(p^2) \) we note that

\[ \Gamma(p^2, \beta^2, \alpha^2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{q^2 dq}{(q^2 - p^2)(q^2 + \beta^2)(q^2 + \alpha^2)} = \frac{1}{(\alpha + \beta)(\alpha - ip)(\beta - ip)} \tag{4.11} \]

(See Gradsteyn and Rysik\(^{21} \) p. 218)
Hence that

\[
D(p^*) = \frac{1}{\lambda} + \frac{2}{\pi} \int_0^\infty \frac{d\varrho}{(q^2 + p^2)(q^2 + \beta^2)(q^2 + \beta^2)}
\]

\[
= \frac{1}{\lambda} + \frac{1}{2\beta^*} \frac{\partial}{\partial \beta^*} \frac{1}{2\beta} \frac{\partial}{\partial \beta} \left\{ \mathcal{I} (p^*, \beta^*, p^{*2}) \right\}
\]

\[
= \frac{1}{\lambda} + \frac{-p^* - 4i\rho \mathcal{R} \beta + 3\beta^2 + \beta^4 + \beta^6}{16\beta^6 \mathcal{R} \beta^3 (\beta^* - i\rho)^2 (\beta - i\rho)^2}
\]

(4.12)

The above derivation is suitable for an S-wave interaction, but we need a p-wave interaction. Now the corresponding form is

\[
\psi(h^+) = \frac{\kappa}{(h^2 + \beta^2)(h^2 + \beta^{*2})}
\]

(4.13)

and

\[
D(p^*) = \frac{1}{\lambda} + \frac{1}{2\beta^*} \frac{\partial}{\partial \beta^*} \frac{1}{2\beta} \frac{\partial}{\partial \beta} \left\{ \mathcal{I} (p^* q^2) \frac{d}{dq} (q^2 + \beta^2)(q^2 + \beta^{*2}) \right\}
\]

\[
= \frac{1}{\lambda} + \frac{1}{2\beta^*} \frac{\partial}{\partial \beta^*} \frac{1}{2\beta} \frac{\partial}{\partial \beta} \left\{ \frac{-i\beta \beta^* + i\beta^* \beta^* - \beta \beta^*}{(\beta + \beta^*) (\beta^* - i\rho) (\beta - i\rho)} \right\}
\]

\[
= \frac{1}{\lambda} - \frac{p^* + 2i\rho (\beta^* + \beta) - \beta^2}{16 \mathcal{R} \beta^2 (\beta^* - i\rho)^2 (\beta - i\rho)^2}
\]

(4.14)

The potential we have chosen has three parameters: \( \lambda \), \( \mathcal{R} \beta \), \( \mathcal{M} \). We have followed this procedure to fix these parameters:
(1) assume $\text{Re } \beta$
(2) assume $\text{Im } \beta$
(3) calculate $\lambda$ such that $S^+_{11} = 160 \text{ Mev}$
(4) find $\text{Im } \beta$ such that $\Gamma = 120 \text{ Mev}$ by search
(5) find $\text{Re } \beta$ for which best fit to $\cot \delta$ is obtained.

We find $\text{Re } \beta = 2\mu$ ($\mu = \text{pion mass}$), $\text{Im } \beta = 3.68 \mu$

$$= 1.15$$

$$= 2.10$$

$\lambda = -112$.

(Recall our units set $\frac{M\mu}{M+\mu} = \frac{1}{2}$). The values of $\cot \delta$ are shown in graph 3 along with experimental values.

B. Simplified Cases

In dealing with an intricate problem, a standard technique is to construct the simplest possible model, then step by step add features of realism. Accordingly, the minimal problem we solve is as follows: Two identical particles of mass $M = 938 \text{ Mev}$ which do not interact with each other, have an $S$-wave resonance interaction of position 160 Mev, width 120 Mev, with a particle of mass $\mu = 140 \text{ Mev}$, $C = \infty$, spin = 0, isospin = 0. Is there a bound state?

We examine the $J = 0$ state, and find that if the identical particles are Bosons there will be a bound state of several hundred Mev. If they are Fermions, the opposite repulsive force will result.
The second problem left spin, isospin zero and non-relativistic kinematics, but took a p-wave resonance. The angular factor \( \mathcal{O}_{\ell \ell'}^{\mathcal{G} \mathcal{L} \mathcal{L}'} \) for \( \mathcal{G} = 1, \mathcal{L} = \mathcal{L}' = 1, \ell = \ell' = 0 \) which is now the case most similar to the previous one reduces to

\[
- \cos \frac{\theta_{k\ell} \ell}{k} = \mathcal{G}
\]  

(4.15)

The significance of the nearly unit magnitude is that the \( \mu \) is almost moving along the line between the two \( \mathcal{M}' \)'s, which is quite reasonable. The Boson case is still bound, now by only 100 Mev, while the Fermion case is repulsive. The type of force is seen to be strong enough to create a bound state. We thus feel that the full problem should be investigated in order that ambiguities of the simple model be removed, and also in order to investigate all low lying angular momentum states accurately.

Thus from this point we stepwise add isospin, spin, repulsive interactions, and relativistic propagators. The results are detailed in the following sections.

C. Uncoupled Channels - Results

We have calculated \( D_3 \) at negative energies for total isospin \( T = 1, 2 \); total angular momentum \( \mathcal{J} = 0^+, 1^+, 2^+ \); spectator orbital angular angular momentum \( \lambda = 0, 1, 2 \). The states
differing only in spectator angular momenta \( l, j \) must be coupled together, but we return to this under coupled channels. We found 5% numerical convergence for \( p_{\text{max}} = 12 \) \( (E_{\text{max}} = 36 \text{ Gev}) \), \( \Delta p = .6 \) (150 Mev/c), \( \Delta k = .2 \) (50 Mev/c). In other words the matrices we generated were 20 x 20, and the integrals to calculate the matrix elements were evaluated on a finer grid. The time to calculate one point was approximately .04 hour.

We display \( D_3(0) \) and \( D_3(-50 \text{ Mev}) \) in table 1 for all these values. We have also recorded the \( D_3(0) \) which the channel would have if it were (or is) attractive when \( T = 2 \). This gives a measure of relative strengths, the smaller values indicating greater strength, and a negative value sufficient strength to bind a three body state. The strongest interactions contributing to \( \mathcal{G}^P = 1^- , 2^- , 2^+ \) are repulsive. The strongest interactions contributing to \( \mathcal{G}^P = 0^\pm , 1^+, 2^- \) are attractive.

The only attractive channels of interest are \( \mathcal{G}^P = 0^+_L \). Here we have calculated points at positive energies by the expedient of giving the energy a small positive imaginary part. We display the result in graph 1. The resonance predictable from the negative energy results appears at approximately 170 Mev.
The channel $2^+$ is repulsive, but if it were attractive, it would be strong enough to bind a three body state. The same factors result in the case of spinless bosons, in which case the force is attractive. We call this our "Boson Model," and note that in the nonrelativistic case the Peierls mechanism is strong enough to bind a three body state. We have made relativistic considerations of this model to emphasize the importance of a correct relativistic theory.

In a number of the configurations, intuition based on the Pauli-Exclusion Principle accounts for the qualitative behavior (repulsion or attraction). For instance consider the $J^P = 2^+, L=0, J = \frac{1}{2}$ configuration. To obtain $T = 2$, the nucleon isospins must be "parallel," to obtain $g = 2$, their spins must be parallel in this case. Since the pion is light, the relative angular momentum of the two nucleons is nearly that of the spectator. Thus the $J^P = 2^+, L=0$ configuration should be repulsive, as it is.

A final point to be made is that the relative importance of various states will be reasonably independent of either kinematics (Green's function) or dynamics (potential). This is because the weighted cosines which enter depend only on the position in momentum space, and the relatively
most important region of momentum space will not be changed by physical models.

D. Coupled Channel Considerations

Coupled channels are described mathematically by coupled integral equations. They arise as we start to take into account the secondary effects. First, with just a one-term separable interaction in one partial wave we must for accuracy couple in the various possible states described by \( l, j \) but which still have given \( \phi \). The use of more than one term in the separable interaction results in coupled integral equations, but we have not found description of this sort necessary for the accuracy we desire. Only the strongest \( \pi\)-N interaction, the \( P_{3-3} \) resonance, has been used. The introduction of interaction in other channels (in particular the repulsive \( S_{31} \)) results in coupled equations. The introduction of the singlet deuteron as a description of the N-N interaction would produce more equations to couple. As a first approximation, a channel should be calculated in isolation, and weaker channels ignored. This can be wrong if the coupling is strong, and a second step is to couple each channel in turn. The channels which do not make a difference when coupled should then be ignored, while those which do will
<table>
<thead>
<tr>
<th>$J^P (\ell, j)$</th>
<th>$T = 1$</th>
<th>$T = 1$</th>
<th>$T = 2$</th>
<th>$T = 2$</th>
<th>$T = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$D_3 (0)$</td>
<td>$D_3 (-50\text{MeV})$</td>
<td>$D_3 (0)$</td>
<td>$D_3 (-50\text{MeV})$</td>
<td>$D_3 (0)$ if attractive</td>
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<tr>
<td>$0^- (1, \frac{3}{2})$</td>
<td>1.59</td>
<td>1.52</td>
<td>.11</td>
<td>.16</td>
<td>.11</td>
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<tr>
<td>$0^+ (2, \frac{3}{2})$</td>
<td>1.39</td>
<td>1.31</td>
<td>.30</td>
<td>.35</td>
<td>.30</td>
</tr>
<tr>
<td>$1^- (1, \frac{1}{2})$</td>
<td>1.13</td>
<td>1.14</td>
<td>.49</td>
<td>.51</td>
<td>.49</td>
</tr>
<tr>
<td>$(1, \frac{3}{2})$</td>
<td>.56</td>
<td>.59</td>
<td>3.89</td>
<td>3.51</td>
<td>.11</td>
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<tr>
<td>$1^+ (0, \frac{1}{2})$</td>
<td>1.21</td>
<td>1.19</td>
<td>.45</td>
<td>.50</td>
<td>.45</td>
</tr>
<tr>
<td>$(2, \frac{3}{2})$</td>
<td>1.01</td>
<td>1.01</td>
<td>.97</td>
<td>.96</td>
<td>.97</td>
</tr>
<tr>
<td>$(2, \frac{5}{2})$</td>
<td>1.14</td>
<td>1.13</td>
<td>.66</td>
<td>.68</td>
<td>.66</td>
</tr>
<tr>
<td>$2^- (1, \frac{1}{2})$</td>
<td>1.17</td>
<td>1.16</td>
<td>.59</td>
<td>.62</td>
<td>.59</td>
</tr>
<tr>
<td>$(1, \frac{3}{2})$</td>
<td>1.53</td>
<td>1.48</td>
<td>.17</td>
<td>.21</td>
<td>.17</td>
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<tr>
<td>$2^+ (0, \frac{1}{2})$</td>
<td>.53</td>
<td>.58</td>
<td>3.0</td>
<td>2.8</td>
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<td>.87</td>
<td>.87</td>
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<tr>
<td>$(2, \frac{5}{2})$</td>
<td>1.32</td>
<td>1.30</td>
<td>.36</td>
<td>.40</td>
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</table>
Caption: Graph 1: Re(D3) for J^P = 0^- as a function of energy. The N's mark the nonrelativistic curve and show the resonance near 70 Mev. Solid R's are calculated relativistic values, dotted R's are estimated.
Caption Graph 2: Re(D_3) for Boson Model. N's mark the nonrelativistic curve and show a bound state. R's mark the relativistic curve and show only a resonance.
Boson Model

$g = 1$

$l = 0$
Caption Graph 3: $\cot \delta_{33}$ as a function of laboratory energy. E's mark the experimental values, N's mark the values of the nonrelativistic model, R's mark the values of the relativistic model.
Cot $\delta_{33}$ for Models

Pion Laboratory Energy (MeV)

Note Scale Change
be retained. In actuality, we have not found that any of
the various possibilities significantly effect the denomina-
tor function calculated for $\mathfrak{g}^P = 1^+$. States Reached by $P_{33}$ Interaction. All the channels
with $\zeta \leq 2$ contributing to states of $\mathfrak{g} \leq 2$ have been calcu-
lated, and displayed in table 1. In each case the strongest
channel is taken as the first approximation to the state.
In the $\mathfrak{g}^P = 2^+$ state this first approximation $\zeta = 1, j = \frac{3}{2}$
indicates a resonance near 100 Mev. Thus the validity of
the approximation is of considerable interest. The next
strongest channel ($\zeta = 0, j = \frac{1}{2}$) has been included. The
result for $D_3$ (coupled) (0) is .09), while the product of
the individual determinants is .004. The latter figure is
the result that we would obtain if coupling were zero. We
see that the coupling is indeed weak. The coupling to
$\mathfrak{g} = \zeta, j = \frac{5}{2}$ has been estimated as weak also.
The same type of result has been obtained in the
boson model for $\mathfrak{g} = 1$, in which the attractive channel
$\zeta = 0$ provides a bound state, while the next strongest
channel is weak. The coupling is weak, and only per-
turbs the bound state.
When the coupling is weak, the coupled $D_3$ will be the product of the diagonal $D_3$'s, and the phenomenon (namely a zero) of interest in one of the diagonal $D_3$'s will be preserved, though a slowly varying multiplicative factor will now be present.

$S_{31}$ Interaction. An examination of the pion-nucleon phase shifts shows that the $S_{31}$ is the next strongest interaction of the pion and nucleon. Since the interaction is repulsive, we investigate its effect on three body phenomena. We can easily use the simpler Yamaguchi potential to represent the repulsive interaction. Here

$$V(k) = (k^2 + \beta^2)^{-1}, V(k, k') = \lambda V(k) V(k').$$

We fit the phase shift with $\lambda = 16$, $\beta = 2.8$, which describes a slightly stronger interaction than is actually present. We now have states for which $L = 0; J = \frac{1}{2}$. The only states contributing to $\varphi^p = 1^+$ have $l = 1; j = \frac{1}{2}, \frac{3}{2}$. The angular factors are equal in magnitude for the two diagonal states, but opposite in sign. The off diagonal elements are not small, being some three times the diagonal elements. The uncoupled diagonal channels give $D_3(0) = .93, 1.08$ respectively for $j = \frac{1}{2}, \frac{3}{2}$. Coupling together gives the extremely uninteresting $D_3(0) = .99$. Since we chose this interaction for its apparent strength, and its three body effects are null,
we did not push this question further.

The physical point seems to be that the particles will not remain nearby as the result of a repulsive interaction. The coupling to such interactions will be geometrically determined, rather than dynamically.

**Nucleon-Nucleon Interaction.** We have investigated the question of whether the $P_{33}$ resonance can, by itself, create a $T = 2$ bound state. Since we have reached a negative conclusion, we should consider the nucleon-nucleon interaction, which we have set to zero. In particular one would include the $T = 1$ interaction of two nucleons, dominated by the singlet deuteron.

We have not included this interaction because (a) the region of importance in nucleon energies is up to 250 Mev, so we feel physically that the low energy attraction of the singlet will be of less overall importance than the $P_{33}$ resonance, and (b) in his similar calculation, Doolen regards the $\pi$-$N$ forces as more important than the $N$-$N$ forces\(^{23}\).
5. RELATIVISTIC MODEL

The energies for which we quote results would not demand a correct relativistic model. The integral equation that is solved to obtain these results however involves highly relativistic regions. The region of "importance," i.e. the region with largest kernel elements is bounded by the order of magnitude of 1 Gev nonrelativistic pion energy. The region of "convergence" (i.e. the area where one more step produces percentage changes as 10%, but no new qualitative features) extends to 36 Gev.

It is unfortunate that no completely acceptable relativistic theory exists. This is true even for our limited aim of relativistic kinematics without relativistic dynamics (we will have no place for production of particles). The trouble is due to the lack of a natural "cluster" property expressed by equations 2.12, 2.18 in the nonrelativistic case. This means that the 3-body propagator, in the Lippmann-Schwinger equations expressing the interaction of two particles out of three, can be expressed directly in terms of the propagator of the two bodies by themselves because the energy of the interacting particles enters linearly.
The straightforward generalization of $G_3^{NR}$ is

$$G_3^R = \left( \omega + \omega_z + \omega_j - \mathcal{Z} \right)^{-1} \tag{5.1}$$

where $\omega_i = \sqrt{m_i^2 + p_i^2}$.

and $\mathcal{Z}$ is the total energy of the three body system.

We now have

$$G_3^R = \left( \sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + k_1^2} + \sqrt{m_3^2 + k_1^2} - \mathcal{Z} \right)^{-1} \tag{5.2}$$

where $\mathcal{Z}_1 = \sqrt{m_2^2 + k_1^2} + \sqrt{m_3^2 + k_1^2}$ is the total energy of the $(2,3)$ pair, and it no longer enters $G_3$ linearly.

Lovelace\textsuperscript{12} has solved the Bethe-Salpeter equation with separable potentials, and found that the correct generalization of $G_3$ is

$$G_3(\mathcal{Z}) = \frac{2 M^M}{\omega_1 \omega_2} \frac{(\omega_1 + \omega_2 + \omega_3)}{(\omega_1 + \omega_2 + \omega_3 - \mathcal{Z})} \tag{5.3}$$

and forces the cluster property by using in the two body Lippmann-Schwinger equation in three body space

$$G_3(\mathcal{Z}) = \frac{2 M^M}{\omega_1 \omega_3} \frac{(\omega_2 + \omega_3)}{(\omega_2 + \omega_3 - \mathcal{Z})(\omega_2 + \omega_3 - \mathcal{Z} - p_1^2)} \tag{5.4}$$

The introduction of the cluster property has the following effects (a) introduction of a strong singularity at $\mathcal{Z} = 0$, 
(b) destruction of time reversal invariance\textsuperscript{13}). It is the only way to calculate though, and we follow it. We have used this with \( V(k^2) \) given by the Barbour-Schult form again, though we need to refit the parameters. We now obtain our best fit for \( \beta = 3\mu + i\ 3.18\mu, \lambda = -928 \). The evaluation of \( D_2(w^2) \) is accomplished by numerical integration which is itself rather lengthy. Hence we fit the values obtained for \( D_2(w^2) \) to their asymptotic form below threshold

\[
\lambda D_2(w) = 1 - \frac{1}{\alpha + \beta w^2}
\]

(5.5)

We obtain values \( \alpha = 2.44, \beta = -0.0549 \). The values of \( \cot \delta_{33} \) are also shown in graph 3 along with the experimental values, and nonrelativistic model. This results, for the Boson model, in the important change from bound state to resonance. Exactly we pass from \( D_3(0) = -0.12 \) to \( D_3(0) = +0.23 \), with a negative slope suggesting a zero before +150 Mev.

For the realistic cases \( j = 0^+, 2^-, j = \frac{3}{2} \) we now obtain

\[
D_3(-50) = 0.29 \quad 0.58
\]

\[
D_3(0) = 0.26 \quad 0.52
\]
which seems considerably weaker than the nonrelativistic case. The real \( D_3 \) may cross zero about 300 Mev. The turning point for values obtained for real \( D_3 \) in the nonrelativistic model suggest that a zero would not be obtained if the relativistic calculation were done above threshold. The calculation could be extended easily for the nonrelativistic case because we had an analytic form for \( D_2 \), which is not the present case.

The effects of \( G_3 \) on the calculation were dramatic. (1) The individual matrix elements of the kernel were slightly smaller in magnitude in the region of importance. It is this effect, rather than the second, below, which leads to the above reported lessening of strength of interaction; (2) The convergence was much more rapid so that the practical upper limit of the integral equation could be reduced by a factor of two, in turn cutting the number of mesh points from 20 to 10.
6. MODEL INDEPENDENCE

Since we used the asymptotic form of $D_2$ in the relativistic calculation in any case, we considered the following extremely crude representation of the interaction. A straight line from the $D_2$ (threshold) through the resonance energy was drawn. The asymptotic negative energy form

$$l - \frac{1}{\alpha + c\omega^2} \quad \text{(relativistic)} \quad (6.1)$$

$$l - \frac{1}{\alpha + c p^2} \quad \text{(nonrelativistic)} \quad (6.2)$$

was matched in value and slope at threshold. The $V(k^2)$ was taken as a sharp cutoff function with magnitude chosen to give the correct scattering length. The value of the cutoff was chosen as $k^2 = 9$, or nonrelativistic pion energy 2 Gev, planning to include the region we had observed to be important. $D_2$ (threshold) was taken to be the Chew-Low value of .2, and varied by $\pm 50\%$.

The results are quite comparable to the results of the Barbour-Schult model. We quote only the Boson model case, giving results in table 2. The nonrelativistic model gives binding for the entire range of $D_2$ (threshold) con-
sidered, the relativistic model gives possible resonances for the entire range.

The variation of $D_3(0)$ with $D_2(0)$ is opposite to that of the Barbour-Schult model. In the latter an increase in $D_2(0)$ is accomplished (for given resonance position and width) by decreasing $\text{Re} \, \beta$, and results in a weaker $D_3(0)$. 
### Table 2

<table>
<thead>
<tr>
<th>Nonrelativistic</th>
<th>Interpolated Binding Energy Mev</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2(0)$</td>
<td>$D_3(0)$</td>
</tr>
<tr>
<td>.3</td>
<td>-.12</td>
</tr>
<tr>
<td>.25</td>
<td>-.10</td>
</tr>
<tr>
<td>.2</td>
<td>-.08</td>
</tr>
<tr>
<td>.15</td>
<td>-.05</td>
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</table>

<table>
<thead>
<tr>
<th>Relativistic</th>
<th>Extrapolated Resonance Energy Mev</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2(M+\mu)$</td>
<td>$D_3(2M+\mu=t)$</td>
</tr>
<tr>
<td>.3</td>
<td>.09</td>
</tr>
<tr>
<td>.25</td>
<td>.12</td>
</tr>
<tr>
<td>.2</td>
<td>.15</td>
</tr>
<tr>
<td>.15</td>
<td>.18</td>
</tr>
</tbody>
</table>
7. EXTENSIONS

We feel that a thesis is incomplete without a consideration of the problems surrounding the particular problem discussed. This serves to indicate problems tractable by similar measures and problems to which an understanding of the material presented here can be useful. We also discuss the immediate weaknesses of the calculation.

The immediate extensions to this work would be to check exactly the importance of inclusion of nucleon-nucleon forces via the singlet deuteron, and the extension of the relativistic calculation to positive energies in a crude fashion. Another extension would be an accurate means of handling the positive energy singularities.

A similar problem, in nuclear physics, is how well Be$^9$ can be described as two alpha particles and a nucleon. Similar to the $P_{33}$ resonance of this calculation, would be the He$^5$ to describe the nucleon-alpha interaction. In this case the alpha-alpha interaction would have to be included. The Boson model approximates this situation to some extent, and we note that in that case we did find a non-relativistic bound state. Since the nuclear three body problems are correctly handled with nonrelativistic dynamics,
this is the correct analogy rather than the relativistic resonance. Another system depending for binding on resonances is the $C^{12} = 3\alpha$ system discussed by McMahan\textsuperscript{24} and Wong\textsuperscript{25}). The system analogous to $N\pi\pi$ is $Li^6$ discussed by Shanley\textsuperscript{26}).

An important extension of this model is to include besides the $P_{33}$ resonance, only a $P_{11}$ bound state, with binding energy equal to the pion mass. The latter would be identified as a nucleon. The questions of interest are: First the self-consistency, to be checked in the scattering of a pion from the pion-nucleon bound state; and second the question of how much of the two nucleon phenomena are explained by these interactions alone, especially whether the deuteron is obtained. Aaron\textsuperscript{20}) has discussed the self-consistency ($N\pi\pi$) domain in a similar model, but without recoil. In this problem the $P_{11}$ state is most important, but the $P_{33}$ interaction is necessary. This is the only sense in which production processes can be handled by the Faddeev equations, and the accuracy of the results would be of considerable interest.
8. CONCLUSIONS

We conclude that the resonant pion exchange mechanism, which might be thought intuitively to provide a very strong attractive interaction between two nucleons, is in fact insufficient to produce a $B = 2$, $T = 2$ bound state and probably produces only a broad resonance in just $\pi\pi\pi$ configurations, at an energy far above threshold of the two nucleon system.

We have explicitly calculated the three body denominator functions for all cases with $\mathcal{J} \leq 2$, in which a bound state would appear as a zero, and find no zeros. We do not expect $\mathcal{J} > 2$ to have as strong forces as the cases we have calculated because of the multiplicity of terms in the angular factors.

There are possibly $\mathcal{J} = 0, 2^+$ resonances. We find such resonances present in our nonrelativistic model. We have not extended the relativistic calculation to positive energies, but we suspect from the smallness of the (negative) minimum for the nonrelativistic case, that the relativistic calculation will show a pearizence minimum, hence no zero, and at best a very broad resonance.
We have checked the dependence of our model on the two body input. We find that the accurate treatment of three body kinematics and dynamics is of greater importance than the details of two body dynamics. A quite crude two body t matrix gives the same qualitative behavior as our more elaborate separable potential model. This is also notably true in the relativistic case.

We are in a position to comment on the relative importance of inclusion of spin, isospin, relativity, and various interactions besides the pion-nucleon $P_{33}$ resonance. Isospin is simply an overall factor in the kernel and easily included. Its exclusion would lead to serious error in the $T = 1$ states of this calculation. The inclusion of spin is essential. The effect of relativity is a significant weakening of the interaction, but its effect is uniform over the various diagonal channels. Hence the nonrelativistic calculation is not accurate, but should be done first to find what is important. Couplings checked between channels of the same total $J^P$ were found to be weak, but this is a matter to check in each case. The inclusion of a repulsive two body interaction channel between pion and nucleon was found to be insignificant in its three body effect. Its action could be simulated equally well by a no interaction
diagonal channel coupled as the kinematics require. The effect of the nucleon-nucleon interaction is estimated to be small.
APPENDIX A.

As discussed in the text, we start from

\[ \langle \mathbf{p}', \mathbf{k}' | T'(s) | I \rangle = \langle \mathbf{p}, \mathbf{k} | t(s) | I \rangle \]

\[ - \int d^3 p_z d^3 k_z \langle \mathbf{p}, \mathbf{k} | t(s) | \mathbf{p}_z \mathbf{k}_z \rangle G_z(s, p_z, k_z) \langle \mathbf{p}_z \mathbf{k}_z | T^z(s) | I \rangle. \]

(A.1)

The same analysis will apply to \( \langle \mathbf{p}_1, \mathbf{k}_1 | I \rangle \) in each term of the equation, and we can write

\[ \langle \mathbf{p}, \mathbf{k} | \mathbf{e}, \mathbf{L}' \mathbf{g}' \mathbf{m}_g | T'(s) | I \rangle = \langle \mathbf{p}, \mathbf{k} | \mathbf{e}, \mathbf{L}' \mathbf{g}' \mathbf{m}_g | t(s) | I \rangle \]

\[ - \int \frac{1}{(2\pi)^6} d\Omega_{\mathbf{p}_z} d\Omega_{\mathbf{k}_z} \langle \mathbf{p}_z, \mathbf{k}_z | \mathbf{e}, \mathbf{L}' \mathbf{g}' \mathbf{m}_g | t(s) | \mathbf{p}_z \mathbf{k}_z \rangle \]

\[ \sum_{\lambda, m_\lambda, m_{\lambda'}} \mathcal{Y}^\dagger_{\lambda m_\lambda} (\mathbf{p}_z) \mathcal{Y}^{m_{\lambda'}}_{\lambda m_{\lambda'}} (\mathbf{k}_z) \chi \langle \mathbf{p}_z \mathbf{k}_z | J' m_{\lambda'} | J m_\lambda | T^z(s) | I \rangle. \]

(A.2)

The angular integrals involve only the spherical harmonics and are easily done, yielding \( \delta \) functions which allow the sums on \( \lambda, M_\lambda, \Lambda, \) and \( M_\Lambda \) to be done. The \( M_\ell \) and \( M_L \) sums then involve only the two vector coupling coefficients, which give \( \delta \) functions for \( J', \mathcal{G} \) and \( M_j, M_g \). Thus
\[ \begin{align*}
\langle p', k', e' l' \gamma' m_g' | T(s) | I \rangle &= \langle p', k', e' l' \gamma' m_g' | t_{i(s)} | I \rangle \\
- \int \frac{d p_1}{(2 \pi)^5} \frac{d k_1}{(2 \pi)^5} \sum_{g' m_g'} G_3(s, p_1) \langle p', k', e' l' \gamma' m_g' | t_{i(s)} | p_1, k_1, e, m_g \rangle \\
&\times \langle p, k, e, l, \gamma, m_g | T'_{i(s)} | I \rangle 
\end{align*} \]

(A.3)

The difficulties have been rolled into \( \langle t_1 l_1 \rangle \). We return to the vector representation, in order to invoke \( |\vec{p}_2 \times k_2 \rangle_2 = |\vec{p}_1 \times k_1 \rangle_1 \) and equation (2.20) discussed in kinematics.

\[ \begin{align*}
\langle t, l \rangle &= \sum_{m_2, m_1} \left( \frac{m_e}{m_2, m_1} \right) \left( \frac{m_e'}{m_1, m_2'} \right) \int d \Omega_{p_1} d \Omega_{k_1} \sum_{\lambda, \mu} Y_{\lambda}^{m_e}(\vec{k}_1) Y_{\lambda}^{m_e'}(\vec{p}_1) Y_{\lambda}^{m_{e'}}(\vec{k}_1) Y_{\lambda}^{m_{e}}(\vec{p}_1) \\
&\times (2\pi)^3 (4\pi)^2 \sum_{\lambda, \mu} \frac{2 \delta(p_1^{\times} - p_2^{\times})}{p_1^{\times}} \delta(\Omega_{p_1} - \Omega_{p_1}) \\
&\times Y_{\lambda}^{\mu}(\vec{k}_1) Y_{\lambda}^{\mu*}(\vec{k}_1) \langle k, \lambda | t_{i(s - m_2, p_1^{\times})} | k, \lambda \rangle 
\end{align*} \]

(A.4)

The \( d \Omega_{p_1} \) integral is done using the delta function; the \( d \Omega_{k_1} \) integral involves only spherical harmonics and allows the sums on \( \lambda, \mu \) to be performed.
\[ \langle \ell, \lambda \rangle = \frac{2^6 \pi^5}{p_i^5} \sum_{\{m\}} \int d\Omega_{p_i} \int d\Omega_{k_0} \left( \begin{array}{ccc} m_e & m_e & m_g \\ \ell_i & \ell & \ell' \end{array} \right) \left( \begin{array}{ccc} m_e' & m_e' & m_g' \\ \ell_i' & \ell' & \ell' \end{array} \right) \]

\[ Y_{\ell_i'}^{m_g'}(\hat{p}_i) Y_{\ell_i}^{m_e}(\hat{p}_i) Y_{\ell}^{m_g}(\hat{k}_0) Y_{\ell'}^{m_e'}(\hat{k}_0') \sum (p, p) \langle k, l' | t, (s-M_p p) | k', l' \rangle \]

\[ \langle \ell', \lambda' | t, (s-M_p p') | k, \ell \rangle \]

(A.5)

The four angular integrals remaining can be done exactly, yielding \( \frac{1}{\hat{g}} \delta_{g' g} \delta_{M_y M_y} \) in the process. We will establish this after following the rest of the proof to the end. We take \( k_2 \) as the direction of quantization, obtaining then \( 4\pi \) for the \( d\Omega_{k_2} \) contribution, \( 2\pi \) from \( d\phi_{p_2 k_2} \), leaving \( d\mu_{p_2 k_2} = \cos \theta_{p_2 k_2} \).

\[ \langle \ell, \lambda \rangle = \frac{2^6 \pi^5}{p_i^5} \sum_{\{m\}} \int d\Omega_{p_2} \left( \begin{array}{ccc} L & L & \ell_i \end{array} \right) \left( \begin{array}{ccc} L' & L' & \ell' \end{array} \right) \]

\[ \delta_{g' g} \delta_{m_g m_g'} Y_{\ell_i'}^{m_g'}(\hat{p}_i) Y_{\ell_i}^{m_g}(\hat{p}_i) \sum_{\delta_{m_0}} \left( \begin{array}{ccc} L & L & \ell \end{array} \right) \]

\[ Y_{\ell}^{m_g}(\hat{k}_0) Y_{\ell'}^{m_g'}(\hat{k}_0') \sum (p, p') \langle k, l' | t, (s-M_p p') | k', l' \rangle \]

(A.6)

We have used the exact \( \frac{1}{\hat{g}} \) in going to \( 3j \) symbols from vector coupling coefficients. The delta function allows this final integral to be done, expressing everything as functions of
Since
\[ p_i = \frac{k_i}{m_i} - \frac{m_i}{m_i + m}, \quad p_i = \frac{2m_m^2}{(m_i + m)^2} p_i \]
\[ (A.7) \]

\[ p_i^2 = k_i^2 + \frac{1}{\beta} p_i^2 - \frac{2}{\beta} k_i p_i \cos \theta_{k_i p_i} = p_i' \]
\[ (A.8) \]

and it follows that
\[ k_i = k_i^2 + \frac{(\beta - 1)}{\beta} (p_i^2 - p_i') \]
\[ (A.9) \]

where \[ \beta = 1 + \frac{m_i}{m} \]

Furthermore
\[ \Delta \left( p_i' - p_i \right) = \frac{\beta}{2 k_i p_i} \Delta \left( \frac{\beta}{2 k_i p_i} (p_i' - k_i^2 - \frac{p_i^2}{\beta}) \right) \]
\[ (A.10) \]

so long as
\[ -1 \leq \frac{\beta p_i' - \beta k_i^2 - \frac{1}{\beta} p_i^2}{2 k_i p_i} \leq 1 \]
\[ (A.11) \]

which is equivalent to
\[ (k_i - \frac{p_i^2}{\beta})^2 \leq p_i' \leq (k_i + \frac{p_i^2}{\beta}) \]
\[ (A.12) \]

This must be expressed as a condition on \( k_2 \), however. We write the following possible cases of the above inequality
\begin{align*}
k^*_z - \frac{p^*_z}{\beta} & \leq p'_i \quad \quad p'_i \leq k^*_z + \frac{p^*_z}{\beta} \\
- k^*_z + \frac{p^*_z}{\beta} & \leq p'_i \quad \quad (p'_i \text{ is positive})
\end{align*}

These give
\begin{align*}
k^*_z & \leq p'_i + \frac{p^*_z}{\beta} \\
k^*_z & \geq p'_i - \frac{p^*_z}{\beta} \\
k^*_z & \geq \frac{p^*_z}{\beta} - p'_i
\end{align*}
and

Or altogether
\begin{equation}
| p'_i - \frac{p^*_z}{\beta} | \leq k^*_z \leq p'_i + \frac{p^*_z}{\beta}
\end{equation}

To find $\cos \Theta_{k^*_z p'_i}$:
\begin{align*}
k^*_z - \bar{p}'_i &= \frac{1}{\beta} \bar{p}_z \\
k^*_z + p^*_z - 2 k^*_z p^*_z \cos \Theta_{k^*_z p^*_z} &= p^*_z / \beta^2
\end{align*}

\begin{equation}
\cos \Theta_{k^*_z p^*_z} = \frac{1}{2 k^*_z p^*_z} \left\{ k^*_z + p^*_z - \frac{p^*_z}{\beta^2} \right\}
\end{equation}

Likewise
\begin{equation}
\cos \Theta_{k^*_z k^*_z} = \frac{\beta^2}{2 k^*_z k^*_z} \left\{ M^2 p^*_z - \frac{1}{\beta} k^*_z - k^*_z \right\}
\end{equation}

and
\begin{equation}
\cos \Theta_{k^*_z p'_i} = \frac{\beta}{2 k^*_z p'_i} \left\{ k^*_z + \frac{p^*_z}{\beta^2} - p'_i \right\}
\end{equation}

Thus we obtain
\begin{equation}
\langle p', k', \ell', \ell', g', m_g', | T'(s) | I \rangle = \langle p', k', \ell', \ell', g', m_g', | t'(s) | I \rangle
\end{equation}

\begin{equation}
- \frac{2^{\frac{N}{2}} \beta}{p_i} \int_0^{\infty} \int_0^1 \sum_{i' k} \frac{p'_i}{p_i} \int_{k'_k} G_j(s, p_i) \sum_{L, L'} \langle k, L' | t(s - m_k k') | k, L \rangle
\end{equation}

\begin{equation}
| k, L \rangle \sum_{m_i} \left( \begin{array}{ccc} L & \ell & g' \\ 0 & m_e & m_e' \\
\end{array} \right) \left( \begin{array}{ccc} \ell & L' & g' \\ m_e & m_e' & m_e' \\
\end{array} \right)
\end{equation}

\begin{equation}
Y_{\ell'}^{m'_{\ell'}}(\hat{p}, \hat{r}_1) Y_{\ell}^{m_{\ell}}(\hat{r}_2) Y_{\ell}^{m_{\ell}}(\hat{r}_3) Y_{\ell'}^{m'_{\ell'}}(\hat{r}_4)
\end{equation}

\begin{equation}
\langle p, k, \ell, \ell, g, m_g | T^{-1}(s) | I \rangle
\end{equation}

(A.18)

as stated in the text.

We now treat

\begin{equation}
I_n = \sum_{m_i} \int d \sum_{r_{12}} (m_e m_e' | m_e' m_e | m_{\ell} m_{\ell} | m_{\ell} m_{\ell}')
\end{equation}

\begin{equation}
\sum (p_i^2 - p_i'^2) Y_{\ell'}^{m'_{\ell'}}(\hat{p}) Y_{\ell}^{m_{\ell}}(\hat{r}_1) Y_{\ell}^{m_{\ell}}(\hat{r}_2) Y_{\ell'}^{m'_{\ell'}}(\hat{r}_3)
\end{equation}

(A.19)

less cavalierly, noting that

\begin{equation}
Y_{\ell}^{m}(\theta_q, q_q) = \sum_{n} D_{nm}^{(c)} (\ell, \theta_k, q_k) Y_{\ell}^{n}(\theta_{\ell, q_{\ell}}, 0)
\end{equation}

(A.20)

and that the sense of \( \hat{p}_2 \) is opposite that of \( \hat{p}_1 \) and \( \hat{r}_1 \).

All \( \mathcal{D} \)'s are understood to be evaluated at \( w \). Then
\[ I_4 = \sum_{\{m\}} \int d\omega \int \mu_{\mathbf{p}, \mathbf{k}} \left( m_e m_l \left| m_{\mathbf{p}}, m_{\mathbf{k}} \right) \left( m_{e'} m_{l'} \left| m_{\mathbf{p}'}, m_{\mathbf{k}'} \right) \delta (p_i^2 - p_i^2) \right. \]

\[ \sum_{\{n\}} \int n_{e,m} \int n_{e', m'} \int n_{e''}, m'' \int n_{e'''}, m''' \int n_{e''''}, m'''' \int n_{e'''''}, m''''' \delta \left( \theta_{\mathbf{p}, \mathbf{k}, 0} \right) \delta \left( \theta_{\mathbf{p}', \mathbf{k}', 0} \right) \delta \left( \theta_{\mathbf{p}'', \mathbf{k}'', 0} \right) \delta \left( \theta_{\mathbf{p}''', \mathbf{k}''', 0} \right) \delta \left( \theta_{\mathbf{p}''''}, \mathbf{k}'''' \right) \delta \left( \theta_{\mathbf{p}'''', \mathbf{k}'''', 0} \right) \]

(A.21)

Consider \[ I_3 \{n\} = \sum_{\{m\}} \int d\omega \left( \frac{m_e m_l \left| m_{\mathbf{p}}, m_{\mathbf{k}} \right) \left( m_{e'} m_{l'} \left| m_{\mathbf{p}'}, m_{\mathbf{k}'} \right) \delta \left( \theta_{\mathbf{p}, \mathbf{k}, 0} \right) \delta \left( \theta_{\mathbf{p}', \mathbf{k}', 0} \right) \delta \left( \theta_{\mathbf{p}'', \mathbf{k}'', 0} \right) \delta \left( \theta_{\mathbf{p}''', \mathbf{k}'''}, m_{\mathbf{p}'''}, m_{\mathbf{k}'''} \right) \right. \]

\[ \int \left( \begin{array}{c} m_e \ m_l \ m_{\mathbf{p}} \ m_{\mathbf{k}} \ m_{e'} \ m_{l'} \ m_{\mathbf{p}'} \ m_{\mathbf{k}'} \ m_{e''} \ m_{l''} \ m_{\mathbf{p}''} \ m_{\mathbf{k}''} \ m_{e'''} \ m_{l'''} \ m_{\mathbf{p}'''} \ m_{\mathbf{k}'''}. \end{array} \right) \]

(A.22)

where we have used equations (3.7.3) and (4.3.2) of Edmonds\(^{27}\). We now use equation (4.6.1) to perform the \(dw\) integral and then sum \(j'\nu'\mu'\).
\[ I_{3\gamma} = g \alpha^2 \sum_{j=1}^{3} \left[ \hat{q} \hat{j} \right]^{m \nu} (e' e' g') (m_x, m_z) \left( \begin{array}{cc} e' & e' \\ m_x & m_z \end{array} \right) \left( \begin{array}{cc} e' & e' \\ m_x & m_z \end{array} \right) \]

Now use Edmonds (3.7.8) twice

\[ = g \alpha^2 \delta_{gg'} \delta_{m_x m_y} \left( - \right)^{L + L' - L'} \left( \begin{array}{cc} e' & e' \\ m_x & m_z \end{array} \right) \left( \begin{array}{cc} e' & e' \\ m_x & m_z \end{array} \right) \left( \begin{array}{cc} e' & e' \\ m_x & m_z \end{array} \right) \]

\[ = g \alpha^2 \delta_{gg'} \delta_{m_x m_y} \left( n_0, n_0 \right) \left( n_0, n_0 \right) \]

(A.23)

which gives

\[ I_{4} = g \alpha^2 \sqrt{\frac{1}{4 \alpha^2}} \sum_{m_z} \int d\mu_{p,k} \delta(p'^z - p^z) Y_{e'}^{m_z} \left( \Theta_{p,k} \right) Y_{e'}^{m_z} \left( \Theta_{p,k} \right) \]

\[ Y^{m_z} \left( \Theta_{k,k} \right) \left( \begin{array}{cc} e' & e' \\ m_z & m_z \end{array} \right) \left( \begin{array}{cc} e' & e' \\ m_z & m_z \end{array} \right) \]

(A.24)

From which point, now reached exactly, the argument continues as before. As stated the coarse averaging process erred only by \( \frac{1}{2} \), and the angular momentum conserving \( \delta' \)'s.
APPENDIX B.

The derivation of partial wave Faddeev equations including spin is made more complex than the spinless case by (1) introduction of spinors, (2) the extra algebra to reach \( j \) symbols. In the interest of keeping spinors and scalars straight we note that the scalar

\[
|\bar{\psi}\rangle = \sum_{Jm_J} q_{JLS}^{m_J} (\hat{\rho}) \langle p JLS m_J | \tag{B.1}
\]

where \( |pJLSm_J\rangle \) is a spinor, and

\[
q_{JLS}^{m_J} (\hat{\rho}) = \sum_{m_s m_L} \left( \begin{array}{c} m_s \ m_L \end{array} \right) \ Y_L^{m_L} (\hat{\rho}) \ Y_S^{m_s} \tag{B.2}
\]

is a spinor, with conjugate spinor \( q_{JLS}^{m_J*} (\hat{\rho}) \).

The analogue of (A3) is now

\[
\langle p'_j k'_J l'_j s'_j J' L'S' \bar{\rho}' m'_q' | T'(s) | I \rangle = \langle p'_j k'_J l'_j s'_j J' L'S' \bar{\rho}' m'_q' | t_i (s) | I \rangle - \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} \ G_i (s', p^i) \sum_{J'L} \langle \begin{array}{c} s' \ m'_q' \end{array} | p'_j k'_J l'_j s'_j J' L'S' \bar{\rho}' m'_q' | \nonumber
\]

\[
t_i (s) | p_j k_j J L S \bar{\rho} m_q \rangle \langle p_j k_j J L S \bar{\rho} m_q | T^2 (s) | I \rangle \tag{B.3}
\]

Instead of (A6), we obtain by exactly analogous procedure:
\[ \langle t_1 \rangle_s = \frac{2n e^{-}}{p'^{-}} \sum_{m_j} \int \frac{d^4 p}{p^2} \delta (p'' - p') \langle k', l', j'| t(s - M)^{-} \rangle |k, l, j' \rangle \]

\[ = \frac{g m_j'}{\mathcal{J}_{J, L}^*} (\Theta, k, q') \frac{m_j}{\mathcal{J}_{J', L'}^*} (\Theta, k, q) \mathcal{J}_{J, L} (\Theta, p, q) \]

\[ \delta_{J, J'} \delta_{m_j, m_j'} \delta_{m_j, m_j'} \]

\[ \delta_{j, j'} \delta_{m_j, m_j'} \]

(B.4)

From here the investigation of

\[ \Theta = \sum_{m_j} \frac{1}{g^2} \left( \begin{array}{cc} m_j & m_j' \\ j & j' \end{array} \right) \left( \begin{array}{c} m_j' \\ j' \end{array} \right) \left( \begin{array}{cc} m_j & m_j' \\ j & j' \end{array} \right) \mathcal{J}_{J, L} (\Theta, k, q) \mathcal{J}_{J', L'} (\Theta, k, q) \mathcal{J}_{J, L}^* (\Theta, p, q) \]

(B.5)

is what is essentially different. The order of the \( \mathcal{J} \)'s is important, for the contraction of spinors with conjugate spinors gives us \( \delta_s \)'s \( \delta_s \)'s.
\[ 0 = \sum_{j_0} \frac{1}{g} \begin{pmatrix} m_{j'} & m_{j_0} & m_{j'_f} & m_{j'_0} \\ f_j & j & J & J' \\ m_s & m_e & m_s' & m_e' \end{pmatrix} \begin{pmatrix} m_{j} & m_{j_0} & m_{j_f} & m_{j_0} \\ f_j & j & J & J' \\ m_s & m_e & m_s' & m_e' \end{pmatrix} \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \delta_{m,0} \]

(B.6)

\[ = \sum_{j_0} \frac{[\Gamma^{j'} \Gamma_j \Gamma_j' \Gamma_{j_0}]}{g} (-1)^{J'+J'-J-J} \delta_{J',J} \delta_{s,s} \delta_{L',L} \]

(B.7)

This we multiply by the identities

\[ 1 = \sum_{\lambda, \lambda', \lambda''} \frac{1}{m_{x'}} \begin{pmatrix} \lambda & \lambda' & \lambda'' \\ \lambda' & \lambda'' & \lambda' \end{pmatrix} \begin{pmatrix} \lambda & \lambda' & \lambda'' \\ \lambda' & \lambda'' & \lambda' \end{pmatrix} \text{ (Ed 3.7.7)} \]

\[ 1 = \sum_{\lambda, \lambda', \lambda''} \frac{1}{m_{y'}} \begin{pmatrix} \lambda & \lambda' & \lambda'' \\ \lambda' & \lambda'' & \lambda' \end{pmatrix} \begin{pmatrix} \lambda & \lambda' & \lambda'' \\ \lambda' & \lambda'' & \lambda' \end{pmatrix} \]
and note that
\[
\sum_{\{m_i', m_j\}} \left\{ \begin{array}{l} s' \ v' \ j' \ v' \\ m_i' \ m_j' \ \end{array} \right\} \left( \begin{array}{ccc} l' & i' & j' \\ m_i & m_j & m_i' \end{array} \right) \left( \begin{array}{ccc} j' \ v' \ m_j' \end{array} \right) \left( \begin{array}{ccc} J' \ v' \ m_j' \end{array} \right)
\]

\[
= \left\{ \begin{array}{l} s' \ v' \ j' \\ i' \ j' \\ J' \ v' \\ \end{array} \right\}
\]

(Ed 6.4.4)

so that
\[
\Theta = \sum_{\{m_i\}} \left\{ \begin{array}{l} s' \ v' \ j' \\ \end{array} \right\} \left( \begin{array}{ccc} l' & i' & j' \\ m_i & m_j & m_i' \end{array} \right) \left( \begin{array}{ccc} a \ l \ j \\ m_a & m_e & -m_j' \end{array} \right) \left( \begin{array}{ccc} J' \ v' \ m_j' \\ 0 & m_e & -m_j' \end{array} \right)
\]

\[
(\mathbf{Y})^4
\]

(B.8)

repeating the process gives
\[
\Theta = \sqrt{\frac{e^{-J}}{2\pi \hbar c}} \sum_{\{m_i'\}} \left\{ \begin{array}{l} s' \ v' \ j' \\ \end{array} \right\} \left( \begin{array}{ccc} l' & i' & j' \\ m_i & m_j & m_i' \end{array} \right) \left( \begin{array}{ccc} a' \ l' \ j' \\ m_a' & m_e' & -m_j' \end{array} \right) \left( \begin{array}{ccc} J' \ v' \ m_j' \\ 0 & m_e' & -m_j' \end{array} \right)
\]

\[
Y_{L'}^{m_e'} \left( \Theta_{p, k_y} \right) Y_{L'}^{m_e} \left( \Theta_{p, k_y} \right) Y_{L'}^{m_e} \left( \Theta_{k_x} \right)
\]

(B.9)
The sign is made more convenient by noting \((-)^{L+L'+J+J'} = +1\) by parity, and by multiplying by \(+1 = (-)^{2S+2J+2S+2J}\), resulting in a sign of

\[ (-)^J + J + J' \]

We note that if \(S = \beta = 0\)

\[
\sum_{II'} \frac{\hat{I} I'}{[2^I 2^I']^n} \begin{bmatrix} 0 & L' & L' \\ L' & L & L' \\ L & L & L' \end{bmatrix} \begin{bmatrix} 0 & L' & L' \\ L & L & L' \\ L & L & L' \end{bmatrix} \]

\[= \sum_{II'} \frac{\hat{I} I'}{[2^I 2^I']^n} \begin{bmatrix} L' & L & L' \\ L' & L & L' \\ L & L & L' \end{bmatrix} \begin{bmatrix} L' & L & L' \\ L & L & L' \\ L & L & L' \end{bmatrix} \]

\[= \sum_{II'} \delta_{I'I'} \begin{bmatrix} L' & L & L' \\ L & L & L' \end{bmatrix} \begin{bmatrix} L' & L & L' \\ L & L & L' \end{bmatrix} \]

\[= \delta_{I'I'} \]  

(Ed. 6.2.9)

and we are left with \((-)^{L+L'+\beta} = (-)^{L'+L'+\beta}\) with which we can make the order of the first 3-j coefficient agree with the spinless case. We thus obtain the correct spinless limit. So long as both spins are \(\frac{1}{2}\), the \(\beta\)-j coefficients can be evaluated using Edmonds (6.4.15).
Table 3

<table>
<thead>
<tr>
<th>$\ell = 2$</th>
<th>$j = \frac{3}{2}$</th>
<th>$j = \frac{5}{2}$</th>
<th>$j = \frac{3}{2}$</th>
<th>$j = \frac{3}{2}$</th>
<th>$j = \frac{5}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[011]</td>
<td>$-0.1392$</td>
<td>$-0.0071$</td>
<td>$+0.75$</td>
<td>$-0.65$</td>
<td>$+0.15$</td>
</tr>
<tr>
<td>[101]</td>
<td>$0.0108$</td>
<td>$0.6071$</td>
<td>$1.5$</td>
<td>$-0.7$</td>
<td>$0.45$</td>
</tr>
<tr>
<td>[121]</td>
<td>$0.2892$</td>
<td>$0.6214$</td>
<td>$0$</td>
<td>$0.60$</td>
<td>$0.15$</td>
</tr>
<tr>
<td>[211]</td>
<td>$-0.1608$</td>
<td>$-1.2214$</td>
<td>$-2.25$</td>
<td>$+0.75$</td>
<td>$-0.75$</td>
</tr>
<tr>
<td>[220]</td>
<td>$-0.5784$</td>
<td>$-0.5643$</td>
<td>$0$</td>
<td>$-0.60$</td>
<td>$-0.15$</td>
</tr>
<tr>
<td>[110]</td>
<td>$+1.4136$</td>
<td>$-1.4286$</td>
<td>$-4.5$</td>
<td>$+2.7$</td>
<td>$-1.2$</td>
</tr>
<tr>
<td>[000]</td>
<td>$-0.0928$</td>
<td>$0.2214$</td>
<td>$0.5$</td>
<td>$-0.2333$</td>
<td>$0.15$</td>
</tr>
</tbody>
</table>

The symbol $[\ell \ell \ell]$ means $y_{\ell}^{\ell} (\Theta_{p_1 k_2}) y_{\ell_2}^{\ell_2} (\Theta_{p_2 k_2}) y_{\ell_3}^{\ell_3} (\Theta_{k_1 k_2})$

where $y_{\ell}^{\ell} (\Theta)$ is the unnormalized trigonometric dependence of $y_{\ell}^{\ell}$ on $\Theta$. 
THE STATIC LIMIT - APPENDIX C.

In the static limit $M \to \infty$ (or $\mu/M \to 0$), and the kernel is no longer $L^2$.

$$K(p', p; s) = \int_{-\pi}^{\pi} \frac{d\theta_k}{2\pi} \frac{k^x}{2M(\mu+M)} p^x + k^x - s \nu(k) \nu(k) \Theta(s, \theta_k, \theta_{k'})$$

(C.1)

In the static limit

$$\beta = 1 + \frac{M}{\mu} \to 1$$

$$M_R = \frac{2M + M}{2M(\mu+M)} \to 0$$

$$k_i^2 = k^2 + M_R (p^i - p'^i) \to k^2$$

$$\cos \Theta_{kk} = \frac{\beta}{2k_k} \left\{ M_R^2 p^3 - k^x/k^3 - k_i^2 \right\} \to 0$$

$$\cos \Theta_{pk} = \frac{1}{2k_p} \left\{ k^x + p^x - \frac{p_i^x}{k^3} \right\} \to \frac{k^x + p_i^x - p^x}{2k_p}$$

$$\cos \Theta_{kR} = \frac{1}{2k_p} \left\{ k^x + \frac{p^x}{k^3} - p'^x \right\} \to \frac{k^x + p_i^x - p'^x}{2k_p}$$

(C.2)

Along the diagonal for instance,

$$K(p, p; s) = \int_0^{2\pi} \frac{k^x}{k^2 - s} \nu^2(k) \Theta(s, \theta_k, \theta_{k'})$$

$$\to \Theta(0, 0, 0) \int_0^{2\pi} \frac{k^x}{k^2 - s} \nu^2(k) \neq 0$$

(C.3)
where \( \Phi (0,0,0) \) is not zero. Let us consider spinless particles interacting in S waves, for which \( \Phi = 1 \). Then in the static limit

\[
K(p, p'; s) = \int \frac{d^4 k}{|p' - p|} \frac{1}{k^2 - s} \nu^2(k)
\]

(C.4)

Consider this for large \( S \) such that the potential is the most important term, and consider the sharp cutoff potential,

\[
\nu(k) = \begin{cases} 
1 & k \leq k_c \\
0 & k > k_c
\end{cases}
\]

(C.5)

then

\[
K(p, p'; s) = \begin{cases} 
- \frac{1}{s} \left( k_c^2 - (p - p')^2 \right) & |p - p'| \leq k_c \\
0 & |p - p'| > k_c
\end{cases}
\]

(C.6)

In other words the kernel does not fall to zero immediately off the diagonal, and the integral of its square will be infinite. A rigorous proof should be able to be worked out for small \( s \) and a class of reasonable \( \nu \)'s, but this should demonstrate the nature of the difficulty encountered when the diagonal elements themselves do not go to a zero limit.
APPENDIX D.

In this appendix we examine the qualitative features of a static two well potential in one dimension. In one dimension the Lippmann-Schwinger equation reads

\[
\langle k | t(p') | k' \rangle = \langle k | V | k' \rangle - \int dq \langle k | V | q \rangle \frac{1}{q - p'} \langle q | t(p') | k' \rangle
\]

(D.1)

and with

\[
\langle k | V | k' \rangle = \lambda \ g(k) g(k')
\]

\[
\langle k | t(p') | k' \rangle = \frac{\lambda \ g(k) g(k')}{1 + \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} dq \frac{1}{q - p'} \ g'(q)}
\]

(D.2)

Now given a suitable potential form \( f(x) \) we can calculate

\[
\frac{1}{\lambda} = - \Re \ \frac{i}{2\pi} \int_{-\infty}^{\infty} dq \frac{f(q)}{q - p'} \ g''(q)
\]

(D.3)

where

\[
g(q) = \int_{-\infty}^{\infty} dx \ e^{iqx} \ f(x)
\]

is the corresponding potential form in momentum space. Now when we displace \( f(x) \) to \( f(x + x_0) \) and add \( f(x - x_0) \), the momentum transform is

\[
g'(q) = 2 \cos q x_0 \ g(q)
\]

(D.4)
For $x_o = 0$ this increases the potential strength parameter $\lambda$ by four without changing the shape. This is a very large increase and would result in a strongly bound state. As we increase $x_o$ the binding will decrease. So long as the main contribution to the original integral occurs between $\pm k_c$ we would expect binding to remain so long as

$$k_c x_o < \frac{\pi}{k_c}$$

$$2 x_o < \frac{\pi}{k_c} \sim \pi \times \text{range} \quad (D.5)$$

where $2x_o$ is the separation of the two potential wells and $\frac{1}{k_c}$ is roughly the range of the potential. (We have estimated $\cos^2 k_o q \sim \frac{1}{2}$, from 0 to $\frac{\pi}{2}$, and consider a factor of two will still bind the model.) Thus the two potential wells can be separated by several times the range.
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REFERENCES


13. R. L. Omnes, Proceedings of Conference on Three-Particle Scattering in Quantum Mechanics, held at Texas A & M University, April 1968. Hereinafter referred to as "A & M Conference (1968)."
17. M. H. Rubin, R. L. Sugar and G. Tiktopoulos, Phys. Rev. 146, 1130 (1966);
     Phys. Rev. 159, 1348 (1967);
23. G. Doolen, private communication.
Determination of the Quantum State by Measurements

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A question raised by Pauli as to whether or not the probability densities in space and momentum determine the wave function is answered negatively. The assertion by Kemble that the probability density and its time derivative determine the wave function is shown to be not generally true. It is shown that measurement of the probability density and the probability current determine the wave function of a spinless particle. The measurement of the probability current is discussed. Measurements which determine the spin state and the density matrix of a mixture are also considered.

I. INTRODUCTION

In this paper we are concerned with what measurements may be made on an ensemble of systems representing the same quantum state, to determine that state. In the first two sections, we consider the case in which the system is in a pure quantum state, and in the third section we consider mixtures.

This is a matter of some interest, yet we have found very little discussion of it in the literature. Pauli, in his famous ‘Handbuch’ article,1 says that it remains an open question (this still appears in the reprinted version) whether or not the distribution function in r, \( \rho(r) = |\psi(r)|^2 \), and the distribution function in momentum, \( \rho(p) = \int \psi(r) \times \exp(-ip \cdot r) dr \), serve to determine \( \psi(r) \).2 That \( \rho(p) \) does not determine \( \psi \) generally is easily seen. For example, if we take

\[ \psi(r) = f(r) P_{\pi \theta}(\theta) \cos \phi, \]

\[ \rho(r) \] and \( \rho(p) \) are obviously independent of the sign of \( m \).

Kemble, in his well-known text book,3 considers this problem, but he errs in extending considerations of Feenberg on one-dimensional motion to higher dimensions. He makes the statement4 that \( \rho(r) \) and \( \rho(p) \) are sufficient to determine \( \psi \). That this statement is not generally true is also clear from the example (1), because, if \( f(r) \) is real, \( \rho(r) = 0 \) and \( \rho(r) \) is independent of the sign of \( m \).

It is immediately apparent that in order to determine \( \psi \) it is sufficient to determine the probability density \( \rho(r) \) and the probability current \( j(r) \) by measurements

\[ j(r) = (\hbar/2m) (\vec{P} \nabla \psi - \psi \nabla \vec{P}) = \rho(r) \nabla S(r) m^{-1}. \]

Equations (2)–(4) serve to determine \( \psi \) (to within a constant phase) if \( \rho \) and \( j \) are determined by measurements.

Although the Born relation [Eq. (3)] is fundamental for the physical interpretation of the mathematical formalism of quantum mechanics, the measurements required to determine the probability current have received relatively little discussion. A gedanken experiment to determine \( j(r) \) by measuring the average velocity of the particle at a point, \( \langle v(r) \rangle = \nabla S(r)/m \), is discussed in Appendix A.

There is no difficulty in principle in extending this method for determining \( \psi \) to a system containing several particles: one determines \( \rho(r_1, \ldots, r_n) \) and \( j(r_1, \ldots, r_n) = \rho(r_1, \ldots, r_n) \nabla S(r_1, \ldots, r_n)/m_t \).

II. SPIN STATE

In the preceding section, we neglected spin. If the particle has spin and if the spin state depends upon \( r \) then the analysis of that section becomes somewhat more complicated, but in an uninteresting way. It is of some interest, however, to give brief consideration to the determination of the spin state when spin-orbit coupling can be neglected. This can be done (in principle) by means of the modifield Stern-Gerlach apparatus.

† National Science Foundation Fellow.
2 For simplicity, we consider the state of a single particle. All of our considerations are nonrelativistic, and are based on the Copenhagen interpretation of the existence of classically described measuring instruments.
4 One might be concerned about how to determine the relative signs of \( f \) on the two sides of a nodal surface of \( f \) when \( \psi \) is real, say. The relative sign is negative (if \( \psi \) is real), or more generally \( \psi \) changes sign when passing through a nodal surface since its normal derivative is continuous and nonzero.
ratus as described in Feynman's lectures, which we call a Feynman filter. We refer the reader to Ref. 5 for detailed explanation. Briefly, the apparatus consists of three Stern-Gerlach magnets in series. The two end ones are identical and of the same polarity, whereas the middle one is twice as long as an end one and of the opposite polarity. A parallel beam of particles entering along the axis of the apparatus is separated into $(2S+1)$ spatially separated beams within the apparatus ($S =$ spin) and will be brought back into an unseparated parallel beam upon leaving the apparatus. The advance in phase along each path in the apparatus is the same, so that, for example, if a particle entering the apparatus is in an eigenstate of an arbitrary component of the spin, it will be in that same spin state upon leaving the apparatus. The apparatus is also provided with gates which may be opened or closed, passing or stopping chosen separated beams within the apparatus.

Let $S$ be the spin of the particle and let $|\frac{\pi}{2}, m\rangle$ be an eigenstate of the $\frac{\pi}{2}$ component of the spin with eigenvalue $m$, 

$$\frac{\pi}{2} \cdot S |\frac{\pi}{2}, m\rangle = m |\frac{\pi}{2}, m\rangle.$$  

(5)

An arbitrary spin state may be written 

$$|\psi\rangle = \sum_{m=-S}^{S} C_m |\frac{\pi}{2}, m\rangle ,$$  

(6)

$|\psi\rangle$ is determined only to within a phase factor, of course, and fixing its normalization leaves $2S$ moduli and $2S$ (relative) phases of the $C$s to be determined by measurement. This determination can be done very simply (in principle) with Feynman filters.

Let the Stern-Gerlach magnets separate the $|\frac{\pi}{2}, m\rangle$ states internally; then closing the gates and counting the particles collected at each gate gives us the $|C_m(\frac{\pi}{2})|$. To determine the relative phase of $C_m(\frac{\pi}{2})$ and $C_n(\frac{\pi}{2})$, say, we close all the gates except those corresponding to $S_z = m$, $n$. The spin state of the particles passing the filter is then 

$$|\psi\rangle = e^{i\phi} (|\frac{\pi}{2}, m\rangle\langle\frac{\pi}{2}, m| + |\frac{\pi}{2}, m\rangle\langle\frac{\pi}{2}, n|) |\psi\rangle = C_m(\frac{\pi}{2}) |\frac{\pi}{2}, m\rangle + C_n(\frac{\pi}{2}) |\frac{\pi}{2}, n\rangle$$  

(7)

except for a normalization factor and a common phase factor. We now analyze the spins of the resulting particles in the $x$-$y$ plane and determine the relative phase of $C_m$ and $C_n$. If particles in state (7) are analyzed by a Stern-Gerlach magnet which separates the components of $S \cdot \frac{\pi}{2}$, where $\frac{\pi}{2}$ is in the $x$-$y$ plane and makes an angle $\phi$ with the $x$ axis, then with the standard choice of phases$^7$

$$|\langle \frac{\pi}{2}, \phi |\psi\rangle|^2 = |d_{pm} \frac{\pi}{2}(2S)C_m(\frac{\pi}{2})|^2 + |d_{pn} \frac{\pi}{2}(2S)C_n(\frac{\pi}{2})|^2 + 2 |C_m(\frac{\pi}{2})C_n(\frac{\pi}{2})| d_{pm} \frac{\pi}{2} d_{pn} \frac{\pi}{2} \cos (m - n) \phi + \theta_m - \theta_n ,$$  

(8)

where $\theta_m$ and $\theta_n$ are the phases of $C_m$ and $C_n$ and the $d$'s are the rotation matrices.$^7$ By choosing $(m - n)\phi = 0, -\frac{1}{2}\pi$, say, we determine $\theta_m = \theta_n$.

Of course, there are many other ways to determine the spin state. It can be shown that it generally is not sufficient to determine the distribution functions of $S_z, S_y$, and $S_x$ [an insufficiency somewhat similar to that of the determination of $\rho(x)$ and $\rho(p)$ for the space part of the wave function]; however, as Fano$^8$ has pointed out, the determination of the mean values of the non-vanishing multipole components of the spin distribution is sufficient to determine the spin state.

### III. Mixtures

In the preceding sections, we supposed that we had a system in a pure quantum state, and we discussed means of determining this state by measurements carried out on an ensemble of systems all in the same quantum state; a concrete ensemble generally does not consist of systems all in the same quantum state; it is a mixture, and the state of the ensemble is represented by a density matrix.$^9$ We discuss measurements on the ensemble which may serve to determine the state of the mixture, first for the spin, then for the cm motion.

**Spin.** The density matrix for the spins may be written 

$$\rho = \sum_{n, m} \rho_{nm} |\frac{\pi}{2}, n\rangle\langle\frac{\pi}{2}, m|$$  

(9)

in terms of the eigenstates of $S \cdot \frac{\pi}{2}$, Eq. (5), where the matrix $\rho_{nm}$ is Hermitian, has positive eigenvalues, and has unit trace.$^9$ The sums in (9) run from $-S$ to $S$. For a pure state $\rho_{nn} = C(\frac{\pi}{2})^2 C_n(\frac{\pi}{2})$ [Eq. (6)] and only $4S(2S+1)$ real numbers are required to determine the state of the ensemble. In the general case, however, $4S(2S+1)$ real numbers are required to determine the state.$^{10}$

By means of Stern-Gerlach magnets and Feynman filters, the $\rho_{nm}$ are determined by the methods discussed in the last section; $\rho_{nm}$ is the probability of finding $S_z$ having the value $n$. If in the Feynman filter discussed in the last section only the gates $n$ and $m$ are left open, then the density matrix for those systems which pass through the filter is$^9$

$$\rho' = \sigma \rho \sigma / Tr(\sigma \rho) ,$$  

(10)

where 

$$\sigma = |\frac{\pi}{2}, n\rangle\langle\frac{\pi}{2}, n| + |\frac{\pi}{2}, m\rangle\langle\frac{\pi}{2}, m| .$$  

(11)

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$^8$ V. Fano, Rev. Mod. Phys. 29, 74 (1957).


$^{10}$ A particle of spin $S$ may have nonvanishing multipole moments up to order $2S^2$. The average values of the $4S(2S+1)$ multipole components determine the spin state (see Fano, Ref. 8).
Regularization and Peratization of Singular Potentials

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We prove that the commonly used regularizations for singular potentials are successful. This means that one can investigate the peratization properties with confidence. A general argument for the success of peratization, as an approximation procedure, is presented. The class of failures of peratization is extended to a series of arbitrarily weakly singular perturbations on the inverse fourth potential. The two results present an unresolved contradiction; some resolutions are considered.

I. INTRODUCTION

The technique of peratization is designed to give meaning to a series in which each term is a divergent function of some parameter. This approximation has been studied in the calculation of the scattering lengths of singular potentials. We restrict the problem in several ways. The potential must have its leading singularity repulsive. At infinite distance we require that the potential fall off more rapidly than inverse cubic. We consider only the zero-energy problem. Due to the bound states which exist for any negative coupling constant, the radius of convergence of expansions in the coupling constant must be zero. To overcome this a regulated potential is introduced. A regulated potential \( V(g, r, \alpha) \) satisfies

\[ (1) \quad V(g, r, \alpha) \text{ is nonsingular for } \alpha > 0, \]
\[ (2) \quad V(g, r, 0) = V(g, r). \]

The most common regularizations are

\[ V(g, r, \alpha) = \delta(r - \alpha)V(g, r), \]
\[ V(g, r, \alpha) = V(g, r, +\alpha). \]

We prove in the following section that the two common regularizations are successful. We then present a general argument for peratization, with a specific example, and a class of counterexamples.

II. REGULARIZATION

Since \( V(g, r, \alpha) \) is nonsingular for each \( \alpha > 0 \), there is for each \( \alpha \), an analytic expression for the scattering length \( A(\alpha, g) = \sum a_i(\alpha)g^n \). It may be possible to sum the series to obtain \( A(\alpha, g) \). If now \( \lim A(\alpha, g) = A(g) \), then the process of regularization has succeeded. The common regularizations have not been known to fail, but there have been no adequate theorems of sufficient conditions or necessary conditions.\(^1\) Some regularizations have been invented which fail.\(^2\) We consider the regularization of (3) first.

We write the solutions to the unregulated Schrödinger equation as \( \psi_r(r) = r\Phi_r(r) \), the regular solution

\(^1\) N. N. Khuri and A. Pais, Rev. Mod. Phys. 36, 590 (1964).
\(^2\) F. Calogero, Phys. Rev. 139, B602 (1965).
\(^3\) M. Cornille, Nuovo Cimento 38, 1243 (1965).
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and \( \psi_\varepsilon(r) = r \Phi_\varepsilon(r) \), the singular solution with

\[
\lim_{r \to 0} \frac{\psi_\varepsilon(r)}{\psi_\varepsilon(r)} = 0.
\]

We know that the asymptotic behavior of the solution is \( \psi_\varepsilon(r) \sim N \cdot (r + A) \), hence there is an expansion about infinity and

\[
A = \Phi'_\varepsilon(\infty)/\Phi_\varepsilon(\infty).
\]

We then turn to the regulated problem which is solved

\[
\psi(r, \alpha) = \psi_\varepsilon(\alpha)\psi_\varepsilon(r + \alpha) - \psi_\varepsilon(\alpha)\psi_\varepsilon(r + \alpha)
\]

\[
\sim \psi_\varepsilon(\alpha)\Phi'_\varepsilon(\infty) - \psi_\varepsilon(\alpha)\Phi_\varepsilon(\infty)
\]

\[
+ [\psi_\varepsilon(\alpha)\Phi'_\varepsilon(\infty) - \psi_\varepsilon(\alpha)\Phi_\varepsilon(\infty)].
\]

Since the coefficient of \( r \) must be finite, and \( \Phi'_\varepsilon(\infty) \) is defined, \( \Phi_\varepsilon(\infty) \) is also defined; and again since the scattering length exists for the regulated potential, \( \Phi'_\varepsilon(\infty) \) exists. Hence

\[
A(\alpha) = \frac{\psi_\varepsilon(\alpha)/\psi_\varepsilon(\alpha)\Phi'_\varepsilon(\infty) - \Phi'_\varepsilon(\infty)}{\psi_\varepsilon(\alpha)/\psi_\varepsilon(\alpha)\Phi_\varepsilon(\infty) - \Phi_\varepsilon(\infty)} + \alpha.
\]

Then as \( \alpha \to 0 \) the correct scattering length is regained; hence the regularization procedure \( V(g, r, \alpha) = V(g, r + \alpha) \) has succeeded for any singular potential. In a similar fashion, we obtain for the regularization

\[
A(g, \phi) = \frac{\Phi'_\varepsilon(\alpha)\Phi_\varepsilon(\infty) - \Phi'_\varepsilon(\alpha)\Phi'_\varepsilon(\infty)}{\Phi_\varepsilon(\alpha)\Phi_\varepsilon(\infty) - \Phi_\varepsilon(\alpha)\Phi_\varepsilon(\infty)}.
\]

Now if one function is more singular than another, its derivative will be more singular than that of the other. Thus the \( \varepsilon \) regularization also works.

III. PERATIZATION

For a series, each of whose terms diverges as a function of a parameter, the first peratization approximation is made by summing the leading singularity as a function of \( \alpha \) in each order of \( g \), then taking the limit of the resulting expression. The second peratization approximation is made by summing the two greatest singularities, etc. We term a series, each of whose terms diverges as a function of a parameter, peratizable if in each order of approximation, the sum is finite, and the approximation is improved.

The following theorem can be proved: The sum or product of an analytic function with the peratizable representation of another function is again peratizable. The same result no longer applies in general if both functions are in a peratizable representation. Conditions on the orders of the singularities can be set up which allow this, in particular if the singularity structure is the same, then the sums and products of two peratizable functions are peratizable. We now extend the argument given previously for regularization.

We recall Eq. (8) and suppose that the sum of the leading singularities in each order for \( [\psi_\varepsilon(\alpha)/\psi_\varepsilon(\alpha)] \) diverges as \( \alpha \to 0 \) (which, considering its behavior as \( \alpha \to 0 \), means that it can be peratized). If we perform all the multiplications and additions before taking leading singularities, then some of the nonsingular terms of \( \Phi'_\varepsilon \) or \( \Phi_\varepsilon \) may be neglected in their order of \( g \).

As long as only a finite number of singular terms occurs in each order of \( g \), we recover this information, in successive approximations. We expect to be able to divide the two peratizable functions in the numerator and denominator because of their similarity of singularities, though after multiplication they may no longer be associated with the same order of \( g \).

A number of examples of solvable cases have been given previously in each case the argument can be followed explicitly. As a definite example we consider here the potential \( g(r^5 + \phi r^4) \), which is similar to the case studied by Pais and Wu, but solvable in readily visualized functions:

\[
\psi(r) = r \text{Ai}(g^{\frac{1}{5}}(r^{rac{1}{5}} + \phi)),
\]

where \( \text{Ai} \) and \( \text{Bi} \) are independent solutions of the Airy equation. The scattering length is

\[
A(g, \phi) = g^{\frac{1}{5}} \text{Ai}'(g^{\frac{1}{5}}\phi)/\text{Ai}(g^{\frac{1}{5}}\phi).
\]

The scattering length for the \( \varepsilon \) regulated potential is

\[
\text{Bi}'(g^{\frac{1}{5}}\phi) \text{Ai}'(g^{\frac{1}{5}}\phi)
\]

\[
A(g, \phi) = \gamma \frac{\text{Bi}'(g^{\frac{1}{5}}\phi) \text{Ai}'(g^{\frac{1}{5}}\phi)}{\text{Bi}'(g^{\frac{1}{5}}\phi) \text{Ai}(g^{\frac{1}{5}}\phi)} - \text{Ai}'(g^{\frac{1}{5}}\phi) \text{Bi}(g^{\frac{1}{5}}\phi),
\]

where \( \gamma = [g^{\frac{1}{5}}] \). As \( \alpha \to 0 \), \( A(g, \phi, \alpha) \to A(g, \phi) \). The summation of the leading singularities of the directly calculated Born series gives

\[
\gamma \text{Bi}'(g^{\frac{1}{5}}\phi) \text{Ai}'(0) \to \text{Ai}'(g^{\frac{1}{5}}\phi) \text{Bi}'(0),
\]

as can be most readily checked by Eqs. (10.4.2) of Ref. 7. This gives a limit of \( \gamma \text{Ai}'(0)/\text{Ai}(0) \), the scattering length of \( g r^5 \).

The answer is no longer exact; however, for small \( \phi \), positive or negative, it is excellent approximation.

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The summation of the second terms gives a term linear in $\phi$, or the slope of $A$ near $\phi = 0$ which improves the approximation. The region in which the power series in $\phi$ for $A$ converge is the circle about the origin with the radius equal to the distance of the nearest singularity: $|\phi| < g^\frac{1}{2} 2.338$. We see that this is exactly the nearest zero energy bound state.

Calogero and Casassandro\textsuperscript{a} considered a potential with an exponential singularity. Their results can be similarly interpreted.

The evidence is not so complete, nor so convincing for the weak singularities multiplied by an inverse power.\textsuperscript{a,b} We examine the Born series of $g r^{-\beta} \varphi(r)$, where $\varphi(+\epsilon) > 0$, $r^{\pm 1} \varphi(r) \sim r^{\pm 1}$, and $r \varphi'(r)$ is less singular than $\varphi(r)$. Examples of such functions are $[\ln (1/r)]^\beta, \beta > 1; \ln \ln (r + 1); \ln \ln \ln (r + \epsilon)]$; giving a series of extremely weak singularities. The leading singularities of the $\delta$ regulated Born series are

$$-A(\alpha) = g \frac{\varphi(\alpha)}{\alpha} - \frac{1}{2} g^2 \frac{\varphi'(\alpha)}{\alpha^2} + \frac{3}{8} g^3 \frac{\varphi''(\alpha)}{\alpha^3} - \cdots.$$  

(14)

This is established by one integration by parts, the residual integral being less singular than the product removed. Thus

$$A(\alpha) = -g^\frac{3}{2} \varphi^3(\alpha) \tanh [(g \varphi(\alpha)/\alpha)^\frac{1}{2}] g_0 - g^\frac{1}{2} \varphi(\alpha),$$  

(15)

which lacks a limit in the cases cited.

Unfortunately the cases cited are not solvable so that the breakdown of the general argument presented


cannot be localized. We note with Cornille\textsuperscript{a} that the expansion above for the hyperbolic tangent does not have an infinite radius of convergence, and the continuation assumed may well be incorrect for small $\alpha$. Secondly, the essential quality of $\varphi(r)$ above is the weakness of its singularity; Wu\textsuperscript{a} has suggested a reinterpretation of the cutoff parameter in one such case, and the suggestion is easily generalized for such weak perturbations.

IV. CONCLUSION

The success of the peratization program assumes the previous success of the regularization process. We have proved that the two common regularizations are successful.

Building on this result we have shown why we expect the peratization to be successful. We have, on the other hand, extended the class of potentials for which peratization appears to fail. These are a series of arbitrarily weak singular perturbations on an inverse fourth power potential.

We expect the peratization program to be valid as an approximation procedure at least for potentials without singularities weaker than a pole. In the exceptional cases cited, an additional prescription due to Wu gives an approximate answer.

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PERATIZATION OF THE LOGARITHMICALLY SINGULAR POTENTIAL

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Peratization of the Logarithmically Singular Potential

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The potential of the form \(g^{-\alpha} \ln^a r - g\alpha^{-\beta} \delta(r - r_0)\) is shown explicitly to be regularizable by an infinitesimal cutoff. It is also shown that, when a modified sense of the peratization technique is applied to the regulated scattering length, the correct answer is regained.

1. INTRODUCTION

The technique of peratization is designed to give meaning to a series in which each term is a divergent function of some parameter. It has been applied to the calculation of the scattering length of various repulsive singular potentials.

For a repulsive singular potential \(gV(r)\), the scattering length as a function of the coupling constant \(g\) has an essential singularity at the origin, and is well defined nearby, since the potential is repulsive. Hence, an attempt to expand the scattering length in a Born series is frustrated by infinite integrals. To overcome this, one introduces a one-parameter family of regular potentials, \(V(r, \alpha)\), such that \(V(r, 0) = V(r)\). A series for the scattering length in the coupling constant then exists for each value of \(\alpha\); however, as to be expected, each term diverges as \(\alpha \to 0\). If the series can be summed, and the limit \(\alpha \to 0\) then be taken, the process of regularization is said to have been applied. If the coefficient of a given power of the coupling constant consists of more than one term in \(\alpha\), the prescription of peratization can be applied by discarding all but the most singular term, then summing the resulting series, and finally letting \(\alpha \to 0\).

Potentials with singularities of pole, exponential, and logarithmic type have been investigated. Regularization succeeds with poles, but there is no possibility of peratization since only one term in \(\alpha\) exists in each order of the coupling constant. Calogero\(^9\), pointing out the difficulties in any general argument for regularization, has shown that one can choose \(V(r, \alpha)\) complicated enough so that the limit \(\alpha \to 0\) does not give the right answer. For straightforward ways of choosing \(V(r, \alpha)\), it has been possible to show in each case that the limit does give the right answer. Pais and Wu\(^10\) have shown that the result as used by Khuri and Pais\(^8\) and others\(^4,7\) is valid in general for singular potentials, resting on a cutoff intrinsic to the general scattering formalism. (A discussion of regularization is found in the Appendix.) An exponentially singular potential has been solved directly\(^4,8\) by the intrinsic cutoff method, and by regularization and peratization in a successful manner.

The scattering lengths due to a pair of regularized logarithmic potentials have been expanded in the Born series, and are found to yield nonfinite answers upon peratization, unless all order singularities are sequentially summed.\(^8\) The regularization used in both cases was \(V(r, \alpha) = \delta(\alpha - r)V(r)\), where \(\alpha = 1\) if \(x < 0\), \(\alpha = 0\) if \(x > 0\). Since the scattering length (the existence of which is certain) could not be written down explicitly for comparison, the result cannot be clearly interpreted: either the peratization or the regularization (upon which the peratization was built) might have failed. The slightly modified potential \((g^{-\alpha} \ln^a r - g\alpha^{-\beta} \delta(r - r_0))\) still has a leading logarithmic singularity in \(r\), but has been solved directly and by the intrinsic cutoff method.\(^7\) We introduce for this potential the same regularization \(V(r, \alpha) = \delta(\alpha - r)V(r)\), and show explicitly that the regularization is successful. We then apply a peratization argument and show that, as in a previous case,\(^8\) the retention of two leading singularities is necessary to regain the correct answer.

2. SOLUTION

The equation

\(\frac{d^2 \psi}{dr^2} = (g^{-\alpha} \ln^a r - g\alpha^{-\beta})\psi\) \hspace{1cm} (1)

has the two general solutions \( y_1(r) = r \exp[F(r)] \)
and \( y_2(r) = y_1(r)L(r) \), where
\[
F(r) = g^3r^2(\ln r + 1)
\]  
(2)
and
\[
L(r) = \int_1^r x^{-2} \exp(-2F(x)) \, dx.
\]  
(3)
The \( \xi \) is arbitrary and, in the following, we choose \( \xi = r_0 \). As usual, we limit our consideration to zero-energy s-wave scattering, so that, for the range \( 0 \leq r \leq r_0 \), Eq. (1) is the Schrödinger equation with the regular solution \( \psi(r) = y_1(r) \). For the range \( r > r_0 \), the Schrödinger equation is \( D^2\psi = 0 \), with the solution \( \psi(r) = br + ba \). By equating the logarithmic derivatives at \( r = r_0 \), we find \( a \), the scattering length, to be
\[
a = g^3 \ln r_0(1 - b^2r_0^{-1} \ln r_0)^{-1}.
\]  
(4)

3. REGULARIZATION

We introduce the one-parameter family of potentials
\[
V(r, \alpha) = \vartheta(\alpha - r)(gr^{-4} \ln^2 r - g^3r^{-3})\vartheta(r - r_0),
\]
so that the Schrödinger equation is
\[
\begin{bmatrix}
0, & 0 \leq r \leq \alpha, \\
(gr^{-4} \ln^2 r - g^3r^{-3})\psi(r, \alpha), & \alpha < r \leq r_0, \\
0, & r_0 < r < \infty
\end{bmatrix}
\]  
\[
\frac{d^2\psi(r, \alpha)}{dr^2} = \begin{cases}
0, & 0 \leq r \leq \alpha, \\
(gr^{-4} \ln^2 r - g^3r^{-3})\psi(r, \alpha), & \alpha < r \leq r_0, \\
0, & r_0 < r < \infty
\end{cases}
\]  
(5)
with the solution
\[
\psi(r, \alpha) = \begin{cases}
\vartheta \alpha + d, & 0 \leq r \leq \alpha, \\
A(\alpha)y_1(r) + B(\alpha)y_2(r), & \alpha < r \leq r_0, \\
b(\alpha)r + b(\alpha)a, & r_0 < r < \infty
\end{cases}
\]  
(6)
In order for \( \psi(0, \alpha) \) to be zero, we set \( d = 0 \) and \( c = 1 \). By joining the solutions at \( \alpha \) and at \( r_0 \), we find
\[
B(\alpha) = -a^2F'(\alpha) \exp[F(\alpha)],
\]
\[
A(\alpha) = \exp[-F(\alpha)] + a^2F'(\alpha) \exp[F(\alpha)]L(\alpha),
\]
and
\[
a(\alpha) = -\frac{A(\alpha)}{A(\alpha)[1 + r_0F'(r_0)]} + B(\alpha) \exp[-F(\alpha)]
\]
\[= -\frac{A(\alpha)[r_0F'(r_0) + B'(\alpha)]}{A(\alpha)[1 + r_0F'(r_0)] + B'(\alpha)},
\]  
(8)
where
\[
A'(\alpha) = 1 + a^2F'(\alpha) \exp[2F(\alpha)]L(\alpha)
\]
and
\[
B'(\alpha) = a^2F'(\alpha) \exp[2F(\alpha) - 2F(r_0)].
\]
Now \( B'(\alpha) \to 0 \) exponentially, and, by the theorem of the mean, we find that \( \exp[2F(\alpha)]L(\alpha) \) does likewise as \( \alpha \to 0 \). Then
\[
a(0) = \lim_{\alpha \to 0} a(\alpha) = -\frac{r_0^2F'(r_0)}{1 + r_0F'(r_0)} = \frac{g^3 \ln r_0}{1 - g^3r_0^{-1} \ln r_0}.
\]  
(9)
Thus we see explicitly that regularization has succeeded.

4. PERATIZATION

Having divided through numerator and denominator by \( \exp[-F(\alpha)] \), we have gotten \( a(\alpha) \) in a closed form, where it has no singular terms remaining for a constant \( g \). The application of peratization is purely pedagogical but possible, since we observe singular terms by expanding. We must expand both numerator and denominator in terms of \( g \), then keep only the most singular (as a function of \( \alpha \)) term in each order of \( g \). This does not work, because the terms of interest to us are not singular in \( \alpha \), and they are combined additively in their order in \( g \) with singular terms. Exactly, in the denominator, we would discard \( g^3r_0^{-1} \ln r_0 \) so that we would obtain
\[
a = g^3 \ln r_0.
\]  
(10)
If we retain, as did Calogero and Cassandro,\(^6\) two leading terms, we obtain
\[
a(\alpha) = -\frac{A''(\alpha)(\alpha) \ln r_0 + B''(\alpha)}{A'(\alpha)(1 + r_0^2 \ln r_0) + B'(\alpha)}
\]  
(11)
with
\[
A''(\alpha) = 1 + g^3(\alpha) \ln \alpha \exp(2g^3(\alpha) \ln \alpha)
\]
and
\[
B''(\alpha) = g^3 \ln \alpha \exp(2g^3(\alpha) \ln \alpha),
\]
so that \( A''(\alpha) \to 1 \), \( B''(\alpha) \to 0 \), and \( a(\alpha) \to a \), regaining the correct scattering length.

5. DISCUSSION

We have examined the scattering length of a particular potential with a leading logarithmic singularity, and found that the regularization procedure works. We have also shown why a peratization procedure retaining two leading singularities is needed. The main difference in this potential and the previously investigated logarithmic potentials
(where peratization failed) is that the former is already nonanalytic in $g$. The Born series for this solvable potential may be investigated to help resolve the conflict.

In conclusion, the first-order peratization (keep only the first leading singularity) has been shown to fail in some cases. A second-order peratization worked in solvable cases.

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APPENDIX

The regularization argument of Khuri and Pais has given the scattering length $a = \lim_{\sigma \to 0} \lim_{\sigma \to 0} f(\alpha, \sigma)$ [where $f(\alpha, \sigma)$ represents the expression on the right-hand side of their formula 2.17]. Pais and Wu have shown that the scattering length of a singular potential is the limit of a ratio of Jost functions. By substituting the corresponding notations, this ratio reduces to $f(0, \sigma)$, with $\sigma$ now identified as the intrinsic cutoff. Thus, in general,

$$a = \lim_{\sigma \to 0} \lim_{\sigma \to 0} f(\alpha, \sigma),$$

which computation method we refer to as the intrinsic cutoff method.

Thus, the problem of regularization can be stated as that of finding the necessary and the sufficient conditions for this interchange to be valid. Calogero has given an example for which it is invalid. This may be a fruitful place to look for necessary conditions, but for practical verification one appeals more readily to the original condition for regularization: $a(g, 0) = \lim_{\alpha \to 0} a(g, \alpha)$. Using the latter formulation, we can state stringent sufficient conditions. If $V(r, \alpha)$ is analytic in $g$ in a neighborhood of the origin, and continuous in $r, \alpha$ (both positive), then the (zero-energy s-wave) wavefunction behaves likewise. Since the scattering length $a(g, \alpha) = \lim_{\sigma \to 0} (\psi(g, r, \alpha) - r)$ exists, it is continuous in $\sigma$, for $\alpha$ positive. Finally, since $(ag, 0)$ exists on physical grounds, $a(g, 0) = \lim_{\sigma \to 0} (ag, \alpha)$.

The regularization $V(g, r, \alpha) = V(g, r + \alpha)$ satisfies the conditions given if the potential $V(g, r)$ is already analytic in $g$, continuous in $r$. The conditions are by no means necessary; indeed, the widely used $g$ regularization violates these conditions though it still works in many cases. In particular, the regularized potential considered in the body of this paper does not satisfy any of the conditions, and yet regularization succeeds.

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PROTON-PROTON BREMSSTRAHLUNG

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Abstract: We calculate proton-proton Bremsstrahlung cross sections using the Tabakin separable potentials to generate off-shell nucleon-nucleon $t$-matrix elements in low partial waves and off-shell one-pion-exchange amplitudes for the higher partial waves. Our results are in good agreement with the experimental values reported by Warner, Halbert, Gottschalk and Thorndike at proton bombarding energies of 48, 62, 158 and 204 MeV, but are at the lower limit of the experimental results of Warner at 30 MeV and of Slaus at 46 MeV. The calculated cross sections in the Harvard geometry decrease monotonically to zero at zero proton energy. We have estimated the effect of rescattering corrections and of non-local corrections to the Bremsstrahlung amplitude and find these terms to be negligible for proton-proton Bremsstrahlung. Our results are insensitive to the choice of Tabakin's "soft-core" or "hard-shell" $S_0$ potential, and are insensitive to small changes in the P-wave potentials.

1. INTRODUCTION

The problem of proton-proton Bremsstrahlung has great significance as a test of proposed models of the nucleon-nucleon interactions [1]. A number of nuclear potentials have been constructed [2] which all give comparable fits [3] to the experimentally determined phase shifts [4] and which all have, therefore, more or less the same asymptotic scattering wave functions in the region of space outside a nucleon-nucleon interaction radius, for the range of energies ($E_{lab}$ = 0 to 340 MeV) for which the phase shifts are known. In order to select a "best" nuclear force model from models which have the same phase shifts (that is, the same asymptotic wave functions) it is necessary to test their predictions for the nucleon-nucleon wave functions within the interaction radius. Since the elastic scattering process only determines the asymptotic form of the wave function we must look to a variety of reactions which hopefully are sensitive to short range features of the nucleon-nucleon wave functions. The problem can be expressed as well in terms of the $t$-matrix formulation of scattering rather than in terms of the scattering wave function. In these terms, the elastic scattering amplitude is an on-shell matrix element of the nucleon-nucleon $t$-matrix and the equivalent potentials.

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have the same on-shell $t$-matrix elements. Reactions other than elastic scattering depend upon off-shell $t$-matrix elements and here again the hope is that equivalent potentials can be distinguished by their off-shell $t$-matrix elements. With the exception of deuteron photo-disintegration and nucleon-nucleon Bremsstrahlung all these reactions — nucleon-deuteron scattering, nuclear matter binding energies, radiative pion capture by a deuteron, etc. — are immediately complicated by the dynamical problems of three or more strongly interacting particles, which invariably necessitate some questionable approximations vitiating any definite conclusions. It is, then, clearly important to have a precise understanding of proton-proton Bremsstrahlung as a test of nuclear potentials.

A number of extremely beautiful $p-p-\gamma$ experiments have been done. In particular, the experiments of Warner [5] at an incident proton energy of 48 MeV, of Gottschalk et al. [6] at a proton energy of 158 MeV, and, subsequently, experiments by Slaus et al. [7] at 46 MeV, of Rothe et al. [8] at 204 MeV, of Halbert [9] at 62 MeV, and of Thompson et al. [10] at 30 MeV have been reported. The experiments are of two types. There is the so-called Harvard geometry (fig. 1) in which the two protons are detected in coincidence by counters making equal but opposite angles to the incident beam and usually coplanar with it. This geometry has been used by all the above authors [5-10] in measuring a double differential cross section $d^2\sigma/d\Omega_1d\Omega_2$. In addition, Thorndike [8] has made measurements with a single photon counter perpendicular to the incident beam and measured another cross section $d^2\sigma/d\Omega\_1dE\_\gamma^\prime$.

On the theoretical side, the first attempt at a detailed calculation of the proton-proton Bremsstrahlung cross section was made by Sobel [11], and by Sobel and Cromer [11]. Unfortunately, this early work contained a number of damaging errors which have been uncovered in particular by Signell and Marker*, but which at the time indicated a large discrepancy between the theoretically predicted and experimentally measured cross sections. For example, the calculations of Sobel and of Sobel and Cromer were at one time some twenty times and again, after supplying a missing factor in the calculations, some ten times greater than Warner's results at 48 MeV, and some four times greater than Gottschalk's results at 158 MeV. The apparent discrepancy lead to a variety of speculations, in the face of contrary evidence. The most radical speculation was that potentials which fit the experimentally determined on-shell $t$-matrix elements over a wide range of energy had failed completely to reproduce the off-shell $t$-matrix elements required in a Bremsstrahlung calculation. This speculation was objected to on the grounds that these potentials lead to qualitatively successful results in deuteron photo-disintegration calculations and in nuclear matter calculations, both of which require off-shell $t$-matrix elements. There was also the speculation that the

*Signell and Marker [12] listed five errors that they uncovered in the work of Sobel and Cromer [11] including (i) the wrong sign of the proton magnetic moment (ii) the wrong normalization of the vector potential, and (iii) an average instead of a summation over photon polarizations. Correction of these errors has not brought their calculations into agreement with our results but results in a Bremsstrahlung cross section an order of magnitude larger than ours at 48 MeV and smaller than ours at 204 MeV by about a factor of two.
p-p-γ process is a much more sensitive test of the nucleon-nucleon potentials than, for example, deuterium photo-disintegration, because of the close cancellations in the electric dipole matrix element. The proton-proton system is an equal-mass, equal-charge pair for which the center-of-mass and center-of-charge coincide and for which, in the limit of zero proton momentum, the electric dipole matrix element cancels to zero. It was also suggested that the Coulomb interaction between the protons, which had been neglected, was giving a large destructive interference with the nucleon-nucleon interaction. A destructive interference would reduce the theoretical prediction and might have brought it into agreement with the experiments. There was the suggestion that meson current effects were unusually important in this system and that it was not a good approximation to treat the electromagnetic interaction of the protons in a quasi-static limit. Intractable rescattering or double scattering terms in the Bremsstrahlung amplitude, which had been neglected in ref. [11] (and have been neglected in subsequent work including our own), were suspected of making large contributions.

There was the possibility that finite experimental resolution might explain the fact that the experimentally measured cross sections were so much smaller than the theoretical predictions. The original calculations for the Harvard geometry were only for coplanar proton counters coplanar with the incident beam and did not take account of finite resolution. If the Bremsstrahlung cross sections fell off sufficiently rapidly out of the plane, then the experimental cross sections would be reduced from the in-plane theoretical values. There was even the suspicion that the experiments themselves were in error, which would have been more easy to believe if the experimental cross sections had been larger than the theoretical predictions, due to counting of spurious coincidences.

This controversial situation led to further attempts to calculate the Bremsstrahlung cross sections. The first such calculation was that of Ueda [13] who abandoned the potential scattering model entirely and used a one-pion-exchange amplitude to describe the proton-proton scattering, and phenomenological pion photoproduction amplitudes to describe the (pγ → γp) vertex. This model can only hope to describe the Bremsstrahlung experiments at the highest energies but did, in fact, agree qualitatively with...
Thorndike's experiments at 204 MeV. Our own work [14], a preliminary
version of the calculations to be reported in detail in this paper, was the
first to show that the p-p-γ experiments could be understood on the basis
of a potential model of the proton-proton interaction. Refinements of our
work, with correction of one glaring but not damaging error * included in
in ref. [14], have been reported elsewhere [15, 16]. They substantiate the
conclusion of ref. [14], that a quantitative agreement with the experiments
[5-10] on proton-proton Bremsstrahlung can be obtained with a potential
model of the strong interactions and no obvious necessity to include anything
more than the quasi-static coupling of the electromagnetic field with point
protons.

Since that time a number of calculations have been undertaken and pre-
liminary results reported. These calculations include work by Sobel and
Cromer [17], Signell and Marker [18], Strobel and Ericson [19], Heller
and Rich [20], Brolley and Morrison [21], and Brown [22]. To the best of
our knowledge these calculations were incomplete at the time of this writing.

The calculations reported here and in refs. [14, 16] employ non-local
separable potentials constructed by Tabakin [23] to describe the proton-proton
interaction. We do not claim any more significance for these potentials than
their author did [24]. They are an extremely convenient calculational device
and provide an impressive fit to the experimental phase shifts, although not
one that can be compared to the fits obtained by local potentials [3] which
are the products of much more rigorous parameter searches than are the
Tabakin potentials which could, perhaps, be improved. These separable
potentials reduce the Lippmann-Schwinger integral equation [25] defining
off-shell matrix elements of the nucleon-nucleon t-matrix, to linear alge-
braic equations which can be solved analytically to give simple algebraic
expressions for the off-shell matrix elements. These explicit solutions
greatly reduce the complexity of our numerical calculations which would
otherwise involve a numerical solution of Schrödinger's equation or the equi-
valent Lippmann-Schwinger integral equation.

In addition to calculations of p-p-γ cross sections for all reported exper-
imental situations, we have done some preliminary n-p-γ calculations. Al-
though they are not of the same immediate interest since they are so inac-
cessible experimentally, they do possess some interesting features.

Following this introduction our paper is divided into seven more sections
containing:
2. A derivation of the Bremsstrahlung matrix elements and required
cross sections.
4. A comparison of our calculations with the existing experimental re-
sults.

* Ref. [14] contained an error in the sign of the proton magnetic moment. Since the
interference between electric and magnetic terms in the Bremsstrahlung cross sec-
tion is so small this error had little effect. The main differences between the pres-
et calculations and those reported in ref. [14] are the inclusion of off-shell effects
in the one-pion-exchange amplitudes and an improved numerical integration over
outgoing proton energy.
5. A discussion of the low-energy behaviour of the p-p-γ cross sections calculated ignoring Coulomb effects as a check on the full calculation.

6. An estimate of terms neglected in the above calculations, namely a rescattering term and a term arising through the velocity dependence of the nuclear forces.

7. A discussion of the n-p-γ cross sections.

8. Finally, conclusions and a discussion of the sensitivity of the Bremsstrahlung cross sections to the form of the potentials used.

After the main body of the text we have included appendices containing discussions of (A) the kinematics of proton-proton Bremsstrahlung as well as cross section formulae and factors taking account of the Pauli principle, (B) the Tabakin potentials and ℓ-matrix elements, (C) the off-shell one-pion-exchange amplitudes used for partial waves with orbital angular momentum greater than two.

2. BREMSSTRAHLUNG MATRIX ELEMENTS

A DWBA which treats the strong nucleon-nucleon potential exactly and the electromagnetic interaction of individual point nucleons correctly to first order in the fine structure constant, yields for the Bremsstrahlung amplitude [1]

$$ M = \langle \psi_i^{(-)} , \psi_f^{(+)} \rangle . $$

Here, $\psi_i^{(+)}$ is the wave function for the two nucleons with energy $E_i$ in the initial state which is an eigenstate of the full nucleon-nucleon Hamiltonian, and satisfies the boundary condition that it is asymptotic to an incident plane wave and outgoing spherical waves, and $\psi_f^{(-)}$ is the final state wave function for the two nucleons with energy $E_f$, and is asymptotic to an incident plane wave and incoming spherical waves. The operator $V_{em}$ corresponds to photon emission by a nucleon, to which we will return in a moment. The scattering wave functions satisfy the Lippmann-Schwinger equation [26]

$$ \psi_f^{(+)}(E_f) = [1 - G_0(E_f + i\eta) t(E_f + i\eta)] \phi_i(E_i) , $$

where eventually $\eta \to 0^+$ to guarantee the outgoing wave boundary condition. In eq. (2), $\phi_i(E_i)$ is an incident plane wave for the nucleons of energy $E_i$.

We leave the spin and isospin variables of the two-nucleon wave function implicit and we shall specify the normalization convention to be used, later.

The free two-particle Green function $G_0$ is given by

$$ G_0(E + i\eta) = [H_0 - (E + i\eta)]^{-1} $$

with $H_0$ the kinetic energy operator for the two particles. The two-particle ℓ-matrix appearing in eq. (2) satisfies a Lippmann-Schwinger equation

$$ t(E + i\eta) = V [1 - G_0(E + i\eta) t(E + i\eta)] $$

$$ = [1 - t(E + i\eta) G_0(E + i\eta)] V , $$

(4)
where $V$ is the potential acting between the two nucleons. As in eq. (2), we have left the spin-isospin variables in eq. (4) implicit. Similarly $\psi_f^{(-)}(E_f)$ satisfies

$$\psi_f^{(-)}(E_f) = [1 - G_0(E_f - i\eta) / (E_f - i\eta)] \phi_f(E_f)$$

(5)

with $(-i\eta)$ and $\eta \to 0^+$ to guarantee the incoming spherical wave boundary condition, and $\phi_f$ is a plane wave for the two nucleons of energy $E_f$. In the matrix element eq. (1), the Hermitian conjugate of $\psi_f^{(-)}$ is required,

$$\{\psi_f^{(-)}(E_f)\}^\dagger = \phi_f(E_f)^\dagger [1 - i(E_f - i\eta) G_0(E_f - i\eta)]$$

(6)

From eq. (4), since $V$ is Hermitian and since $G_0(E_f - i\eta)^\dagger = G_0(E_f + i\eta)$ we see that

$$i(E_f - i\eta)^\dagger = i(E_f + i\eta)$$

(7)

In terms of matrix elements between initial and final plane-wave states, the Bremsstrahlung amplitude eq. (1) can be expressed, using eqs. (2), (6) and (7), as

$$M = \langle \phi_f(E_f)^\prime, T \phi_f(E_f) \rangle$$

(8)

with

$$T = V_{em} - i(E_f) G_0(E_f) V_{em} - V_{em} G_0(E_f) i(E_f)$$

$$+ i(E_f) G_0(E_f) V_{em} G_0(E_f) i(E_f)$$

(9)

where the $(-i\eta)$ is understood.

The terms in $T$ can be represented by the diagrams of fig. 2. Vertical lines represent nucleons, blobs connecting two vertical lines represent nucleon-nucleon scattering through the $t$-matrix, and the wavy line connected to a particle line represents photon emission by a nucleon.

Diagrams (2a) correspond to the first term in eq. (9) which is the amplitude for the spontaneous emission of a photon by non-interacting nucleons. These processes are kinematically forbidden and do not enter our calculations. Diagrams (2b) correspond to the second term in eq. (9) and represent emission of a photon by one of the nucleons followed by a nucleon-nucleon scattering through the full $t$-matrix. Diagrams (2c) correspond to nucleon-nucleon scattering followed by photon emission, the third term in eq. (9). Diagrams (2d) correspond to the fourth terms in eq. (9) and are the amplitudes for a nucleon-nucleon scattering followed by photon emission followed by a rescattering. The diagrams (2d) are termed the rescattering diagrams. Diagram (2e) represents photon emission processes not included in eq. (9), which arise because of modifications of the interaction of the two nucleons with the electromagnetic field due to their strong interaction. Such processes can arise through electromagnetic couplings required to keep a theory with velocity-dependent potentials gauge invariant, or, more fundamentally, they would be due to photon emission by meson exchange currents during the nucleon-nucleon scattering process.
PROTON-PROTON BREMSSTRAHLUNG

The amplitude for emission of a photon of momentum \( k \) and unit polarization vector \( \hat{\varepsilon} \), by a point proton of momentum \( r \) is well known [27] (but was used with the wrong sign for the magnetic moment term in refs. [11,14]). It is the matrix element between initial and final proton Pauli spinors of the operator

\[
V_{\text{em}} = -e \{ r - \frac{1}{2} i \mu \sigma \times k \} \cdot \hat{\varepsilon},
\]

(10)

where \( e^2 = 4 \pi \frac{1}{137} \) in units where \( \hbar = c = \) nucleon mass = 1 which we use. Also \( \mu \) is the proton magnetic moment equal to 2.973. We work in the Coulomb or transverse gauge and choose \( k \cdot \hat{\varepsilon} = 0 \) in the c.m. coordinate system. The amplitude for photon emission by a neutron is obtained by dropping the first term and setting \( \mu \) equal to -1.913, the neutron magnetic moment. A factor \( 1/\sqrt{2k} \) which occurs in the photon field operator \( A(x) \)

\[
A(\lambda) = \sum_{k, \varepsilon} \frac{1}{\sqrt{2k}} a_{k, \varepsilon} \hat{\varepsilon} e^{i(k \cdot x - \omega t)} + \text{Hermitian conjugate}
\]

(where \( a_{k, \varepsilon} \hat{\varepsilon} \) is the usual photon annihilation operator) has been left out of \( V_{\text{em}} \) but will be included in the cross section formulas. The amplitude eq. (10) is the amplitude for photon emission by a proton due to an interaction Hamiltonian

\[
\mathcal{H}_{\text{em}} = \int j(x) \cdot A(x) \, d^3x,
\]

where \( j(x) \) is the proton current [28].

With the photon emission operator eq. (10), the proton-proton Bremsstrahlung amplitudes of diagrams (2c) and (2d) can be written explicitly, and we will do this presently. The rescattering amplitude of diagram (2d) is straightforward to write down but is too difficult to evaluate with the full complexity of the nucleon-nucleon \( t \)-matrix. We have left this term out of our detailed calculations, as have other authors. Estimates of the rescattering amplitude are made in sect. 6 on the basis of a simplified model and indicate that these terms contribute less than 5% to the Bremsstrahlung cross section. We have also left out any terms arising from modifications of the nucleon electromagnetic interaction by the nucleon-nucleon interaction. Estimates of one type of such term which arises in our calculation are contained in sect. 6 and show that this term is extremely small.

Consider a proton (numbered 1) of momentum \( r \) incident on a target proton (numbered 2) of momentum \( s \) scattering to a final state with proton \( (1) \) having momentum \( p \) and proton \( (2) \) having momentum \( q \), with the emission of a photon of momentum \( k \) and polarization \( \varepsilon \). For the moment we are considering the two protons as distinguishable, and we shall take account of particle identity later. The Bremsstrahlung amplitude \( T \) is conveniently written as

\[
T = -e \hat{\varepsilon} \cdot T,
\]

(11)

where we have factored out the proton charge and the photon polarization vector and defined a vector amplitude \( T \). Diagrams (2b) and (2c), which are all we consider in detail, give
Fig. 2. Diagrams corresponding to the Bremsstrahlung amplitude treating the nucleon-nucleon interaction exactly and the electromagnetic interaction to first order.

\[
T = \langle \frac{1}{2}(p - q) | t(E_{\rho q}) | \frac{1}{3}(r + k - s) \rangle [E - E(r) - E(s - k) - k]^{-1} (s - \frac{1}{2}i\mu \sigma_2 \times k) \\
+ \langle \frac{1}{3}(p - q) | t(E_{\rho q}) | \frac{1}{3}(r - k - s) \rangle [E - E(r - k) - E(s) - k]^{-1} (r - \frac{1}{2}i\mu \sigma_1 \times k) \\
+ (q - \frac{1}{2}i\mu \sigma_2 \times k)[E - E(p) - E(q + k)]^{-1} \langle \frac{1}{3}(p - q - k) | t(E_{\gamma S}) | \frac{1}{3}(r - s) \rangle \\
+ (p - \frac{1}{2}i\mu \sigma_1 \times k)[E - E(p + k) - E(q)]^{-1} \langle \frac{1}{3}(p + k - q) | t(E_{\gamma S}) | \frac{1}{3}(r - s) \rangle. \tag{12}
\]

The variables in eq. (12) have the following significance. The energy of a proton of momentum \( r \) is \( E(r) \). The total energy of the system is \( E = E(r) + E(s) = E(p) + E(q) + k \). The energy \( E_{\gamma S} \) is the energy of relative motion of two protons with relative momentum \( r - s \). Thus the \( t \)-matrix element

\[
\langle \frac{1}{3}(p + k - q) | t(E_{\gamma S}) | \frac{1}{3}(r - s) \rangle \tag{12'}
\]

is on-shell on the right but off-shell on the left. As it is written in eq. (12), \( T \) is a matrix in the spin spaces of protons 1 and 2 and a matrix element between Pauli spinors for each proton is understood.

So far the whole discussion has been non-relativistic but it actually simplifies the problem, and is certainly desirable at the highest energies for which experiments have been done, to treat the kinematics relativistically.
even though we still treat the dynamics of the proton-proton scattering non-relativistically. Accordingly, we use \( E(r) = (1 + r^2)^{1/2} \) in the kinematics and in the energy denominators in eq. (12). The energy \( E_{r,s} \) in the \( t \)-matrix is given its non-relativistic form, \( E_{r,s} = (r - s)^2 \).

The absolute square of the Bremsstrahlung amplitude eq. (12) averaged over initial proton spin orientations, summed over final proton spin orientations and summed over photon polarization directions is required. Thus

\[
|T|^2 = \frac{1}{4} e^2 \text{Tr}_1 \text{Tr}_2 \sum \hat{\hat{S}}(\hat{\hat{S}} \cdot T)\dagger(\hat{\hat{S}} \cdot T),
\]

where the traces are performed in the spin spaces of protons 1 and 2; and the polarization sum is over two polarization directions mutually perpendicular and perpendicular to \( \hat{k} \). The polarization sum reduces eq. (13) to

\[
|T|^2 = e^2 \{ \frac{1}{4} \text{Tr} \quad T^\dagger \cdot T \} = e^2 \ |T_\perp|^2,
\]

where

\[
T_\perp = T - \hat{k}(T \cdot \hat{k})
\]

is the component of \( T \) perpendicular to the photon propagation direction \( \hat{k} \). The calculation of the trace in eq. (14) is extremely tedious when the full complexity of the nucleon-nucleon \( t \)-matrix is included. The full results are too long to be included here but are available elsewhere [29].

The total Bremsstrahlung cross section \( \sigma \), from which various differential cross sections can be extracted, is written

\[
\sigma F = \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{(2\pi)^4 \delta^3(p + q + k - r - s)}{2k E(r)E(s)E(p)E(q)} \times \delta(E(p)) \delta(E(q) + k - E(r) - E(s)) e^2 |T_\perp|^2,
\]

where \( r \) and \( s \) are the momenta of the two protons in the initial state, \( p \) and \( q \) in the final state, and \( k \) is the photon momentum.

The factor \( 1/2k \) from the photon normalization has been included in the photon phase space \( d^3k/[2(2\pi)^3(2k)] \). The incident flux \( F \) would be simply \( |r - s| \), the relative velocity of the two protons in the initial state, for non-relativistic kinematics and incident plane waves of unit magnitude but, in accordance with our decision to treat the kinematics relativistically, we have modified \( F \) to be a relativistically invariant flux satisfying [30]

\[
E(r)E(s)F = \left[ (r \cdot s - E(r)E(s))^2 - 1 \right]^{1/2}.
\]

At the same time we have made a covariant normalization of the incident and outgoing proton plane waves so that they have the magnitude \( 1/\sqrt{E} \) rather than 1, which accounts for the energy factors in the denominator of eq. (16). The form eq. (16), familiar from a fully covariant theory, makes manifest the translational invariance of the Bremsstrahlung amplitude \( T \). We will use
this fact to calculate $T$ in the initial c.m. system for the sake of convenience, rather than in the laboratory system of coordinates where we evaluate eq. (16) for comparison to experiment.

Kinematical details of the evaluation of differential Bremsstrahlung cross sections from eq. (16) for the Harvard geometry and for the Thorndike geometry are contained in appendix A.

3. THE NUCLEON-NUCLEON $t$-MATRIX

In order to evaluate the nucleon spin sums in eq. (14) it is necessary to make explicit the spin dependence of the nucleon-nucleon $t$-matrix elements. The $t$-matrix element

$$\langle p_f^1 | t(E) | p_i^1 \rangle$$

is understood as a matrix element between Pauli spinors $u$ for the two nucleons, of a spin operator $\mathcal{S}$ such that

$$\langle p_f^1 | t(E) | p_i^1 \rangle = \Pi (1) \Pi (2) \mathcal{S}(p_{i1}^1, p_{i2}^1, E, \sigma_1, \sigma_2) u_1(1) u_2(2).$$

(18)

The most general expansion of $\mathcal{S}$ for on-shell values of the relative momenta $p_i^1$ and $p_f^1$ is restricted by requirements of translational and rotational invariance, space reflection invariance, time reversal invariance, and the charge independence of nuclear forces [26]. In this expansion, $\mathcal{S}$ is expressed in terms of five Wolfenstein parameters $\alpha$, $\beta$, $\gamma$, $\delta$, and $\epsilon$ which are scalar functions of $E = p_f^2 - p_i^2$ and $\hat{p}_f \cdot \hat{p}_i$, and three mutually perpendicular unit vectors which are chosen to be

$$\hat{n} = \frac{\hat{p}_i \times \hat{p}_f}{|\hat{p}_i \times \hat{p}_f|}, \quad \hat{m} = \frac{\hat{p}_i \cdot \hat{p}_f - \hat{p}_f \cdot \hat{p}_i}{|\hat{p}_i \cdot \hat{p}_f - \hat{p}_f \cdot \hat{p}_i|} \quad \text{and} \quad \hat{l} = \frac{\hat{p}_i + \hat{p}_f}{|\hat{p}_i + \hat{p}_f|}.$$

Then the explicit spin dependence of the nucleon-nucleon on-shell $t$-matrix is

$$\mathcal{S} = \alpha + \beta \sigma_1 \cdot \hat{n} \sigma_2 \cdot \hat{n} + 2i\gamma (\sigma_1 + \sigma_2) \cdot \hat{n} + \delta \sigma_1 \cdot \hat{m} \sigma_2 \cdot \hat{m} + \epsilon \sigma_1 \cdot \hat{l} \sigma_2 \cdot \hat{l}. \quad (20)$$

The expansion eq. (20) must be modified in just two ways to hold also for off-shell matrix elements [31]. The Wolfenstein parameters must be understood now to be scalar functions of all the kinematic invariants $E = p_f^2$, $p_i^2$, $p_i^1$, and $\hat{p}_f \cdot \hat{p}_i$. Also, since off-shell matrix elements are not time reversal invariant, another term, which changes sign under time reversal but which vanishes on the energy shell, can enter eq. (20). The modified expansion, valid off the energy shell, has the form

$$\mathcal{S} = \alpha + \beta \sigma_1 \cdot \hat{n} \sigma_2 \cdot \hat{n} + 2i\gamma (\sigma_1 + \sigma_2) \cdot \hat{n} + \delta \sigma_1 \cdot \hat{m} \sigma_2 \cdot \hat{m} + \epsilon \sigma_1 \cdot \hat{l} \sigma_2 \cdot \hat{l} + G \sigma_1 \cdot \hat{m} \sigma_2 \cdot \hat{m} + \sigma_1 \cdot \hat{l} \sigma_2 \cdot \hat{l}$$

$$+ \sigma_1 \cdot \hat{l} \sigma_2 \cdot \hat{l} + C \sigma_1 \cdot \hat{m} \sigma_2 \cdot \hat{m} + \sigma_1 \cdot \hat{l} \sigma_2 \cdot \hat{l}$$

(21)

where the Wolfenstein parameter $G$ vanishes on the energy shell.

Following Goldberger and Watson [26], we relate the Wolfenstein param-
eters \(\alpha, \beta, \gamma, \delta, \epsilon\) and \(G\) first to a set of channel-spin amplitudes and through these to individual partial wave amplitudes of the nucleon-nucleon \(t\)-matrix. The channel-spin amplitude \(T_{m'\,m}\) for channel-spin one is defined as the matrix element between an initial state with spin one, \(z\)-projection \(m\), and a final state with spin one, \(z\)-projection \(m'\) of the operator \(\mathcal{D}\) of eq. (18). If the \(z\)-axis is chosen parallel to \(\bm{p}_1\) then

\[
T_{m'\,m} = e^{i(m' - m)\phi} \langle 1, m' | \mathcal{D} (\bm{p}_1, \mu_1, E, \sigma_1, \sigma_2) | 1, m \rangle ,
\]

where \(T_{m'\,m}\) depends only on \(\mu_1^2, \mu_1^2, E\) and \(\hat{\mu}_1 \cdot \hat{\mu}_1 = \cos \theta\), but not on \(\phi\), the azimuthal angle of \(\bm{p}_1\). The channel-spin zero amplitude \(T_{ss}\) is similarly defined as

\[
T_{ss} = \langle 0, 0 | \mathcal{D} (\mu_1, \mu_1, E, \sigma_1, \sigma_2) | 0, 0 \rangle
\]

and is independent of \(\phi\). The requirements of space symmetry and rotational invariance reduce the total number of channel-spin amplitudes to six independent ones,

\[
T_{ss}, \quad T_{00}, \quad T_{11} = T_{-11}, \quad T_{1-1} = T_{-11}, \quad T_{10} = -T_{-10}, \quad T_{01} = -T_{0-1}.
\]

In terms of these channel-spin amplitudes, the Wolfenstein parameters are

\[
\begin{align*}
\alpha &= \frac{1}{2} (2T_{11} + T_{00} + T_{ss}), \\
\beta &= \frac{1}{2} (-2T_{1-1} + T_{00} - T_{ss}), \\
\gamma &= \frac{1}{2} \sqrt{2} (T_{10} - T_{01}), \\
\delta &= \frac{1}{4} \left( [T_{11} + T_{1-1} - T_{ss}] - [T_{11} - T_{1-1} - T_{00}] \cos \theta - \sqrt{2} \sin \theta [T_{10} + T_{01}] \right), \\
\epsilon &= \frac{1}{4} \left( [T_{11} + T_{1-1} - T_{ss}] + [T_{11} - T_{1-1} - T_{00}] \cos \theta + \sqrt{2} \sin \theta [T_{10} + T_{01}] \right), \\
G &= \frac{1}{4} \left( [T_{11} - T_{1-1} - T_{00}] \sin \theta - \sqrt{2} \cos \theta [T_{10} + T_{01}] \right).
\end{align*}
\]

The channel-spin amplitudes can next be expressed in terms of partial wave amplitudes. We follow closely the exposition of Goldberger and Watson for on-shell amplitudes and use their notation, except for a common normalization factor which distinguishes our channel-spin amplitudes eq. (24), Wolfenstein parameters eq. (21), and \(t\)-matrix elements eq. (18) from theirs.

With our normalization, we have

\[
\begin{align*}
T_{11} &= -\frac{4\pi}{\mu} \sum_{L=0}^{\infty} \frac{1}{2} [(L+2)\alpha_L^{L+1} + (2L+1)\alpha_L^L + (L-1)\alpha_{L-1}^L] P_L(\cos \theta), \\
T_{00} &= -\frac{4\pi}{\mu} \sum_{L=0}^{\infty} [(L+1)\beta_L^{L+1} + L\beta_L^L - 1] P_L(\cos \theta),
\end{align*}
\]
\[ T_{1-1} = - \frac{4\pi}{p} \sum_{L=2}^{\infty} \left[ \frac{1}{L+1} a_{L}^{L+1} - \frac{2L+1}{L(L+1)} a_{L}^{L} + \frac{1}{L} a_{L}^{L-1} \right] P_{L}^{2}(\cos \theta), \]

\[ T_{10} = - \frac{4\pi}{p} \frac{1}{\sqrt{2}} \sum_{L=1}^{\infty} \left[ b_{L}^{L+1} - b_{L}^{L-1} \right] P_{L}^{1}(\cos \theta), \]

\[ T_{01} = - \frac{4\pi}{p} \frac{1}{\sqrt{2}} \sum_{L=1}^{\infty} \left[ \frac{L+2}{L+1} a_{L}^{L+1} + \frac{2L+1}{L(L+1)} a_{L}^{L} + \frac{L-1}{L} a_{L}^{L-1} \right] P_{L}^{1}(\cos \theta), \]

\[ T_{ss} = - \frac{4\pi}{p} \sum_{L=0}^{\infty} (2L+1)c_{L} P_{L}^{2}(\cos \theta). \] (28)

The entire \( \cos \theta \) dependence of the channel-spin amplitudes is contained in the Legendre functions of eq. (26). The partial wave amplitudes \( a, b \) and \( c \) are functions only of the initial and final momenta \( p_{i} \) and \( p_{f} \), and of the on-shell momentum \( p \). The partial wave amplitudes, on the energy shell, are expressed in terms of the individual phase shifts and mixing parameters as follows:

(i) the singlet-spin amplitude

\[ c_{J} = e^{i\delta_{0}(J)} \sin \delta_{0}(J), \] (27)

with \( \delta_{1}(J) \) the phase shift for the \( 3J_{f} \) partial wave, and

(ii) the uncoupled triplet-spin amplitude

\[ a_{J}^{J} = e^{i\delta_{1}(J)} \sin \delta_{1}(J), \] (28)

where \( \delta_{1}(J) \) the phase shift for the \( 3J_{f} \) partial wave and

(iii) the coupled triplet-spin amplitudes

\[ a_{J-1}^{J} = a_{J}^{J} \cos^{2} \epsilon_{J} + b_{J}^{J} \sin^{2} \epsilon_{J} - \sqrt{J/(J+1)} \frac{1}{2}(a_{J}^{J} - b_{J}^{J}) \sin^{2} \epsilon_{J}, \]

\[ a_{J+1}^{J} = a_{J}^{J} \sin^{2} \epsilon_{J} + b_{J}^{J} \cos^{2} \epsilon_{J} - \sqrt{J/(J+1)} \frac{1}{2}(a_{J}^{J} - b_{J}^{J}) \sin^{2} \epsilon_{J}, \]

\[ b_{J-1}^{J} = a_{J}^{J} \cos^{2} \epsilon_{J} + b_{J}^{J} \sin^{2} \epsilon_{J} + \sqrt{J/(J+1)} \frac{1}{2}(a_{J}^{J} - b_{J}^{J}) \sin^{2} \epsilon_{J}, \]

\[ b_{J+1}^{J} = a_{J}^{J} \sin^{2} \epsilon_{J} + b_{J}^{J} \cos^{2} \epsilon_{J} + \sqrt{J/(J+1)} \frac{1}{2}(a_{J}^{J} - b_{J}^{J}) \sin^{2} \epsilon_{J}, \] (29)

where \( a_{J}^{J} \) and \( b_{J}^{J} \) are amplitudes related to the eigenphase shifts \( \delta_{0}(J) \) and \( \delta_{1}(J) \) for the triplet-spin channels with \( L = J - 1 \) and \( L = J + 1 \) which are defined along with the mixing parameters \( \epsilon_{J} \) according to the convention of Blatt and Biedenharn [32].

For the proton-proton \( t \)-matrix, we have used the Tabakin potentials [23] to generate off-shell amplitudes for the \( 1S_{0}, 3P_{0,1,2} \) and \( 1D_{2} \) partial waves and off-shell one-pion-exchange amplitudes for the higher partial waves. All
the mixing parameters connecting $P$- and $F$-waves are zero on this model, and the required partial wave amplitudes on the energy shell are

\[ c_0 = e^{i\delta(1S_0)} \sin \delta(1S_0), \]

\[ c_2 = e^{i\delta(1D_2)} \sin \delta(1D_2), \]

\[ a_1^1 = e^{i\delta(3P_1)} \sin \delta(3P_1), \]

\[ a_2^2 = b_2^2 = e^{i\delta(3P_2)} \sin \delta(3P_2), \]

\[ a_1^0 = b_1^0 = e^{i\delta(3P_0)} \sin \delta(3P_0). \] (30)

A discussion of the Tabakin potentials and solutions of the Lippmann-Schwinger equation for the amplitudes eq. (30), off-shell, are presented in appendix B. Explicit expressions for these amplitudes are contained in ref. [29]. The one-pion-exchange amplitudes for higher partial waves are discussed in appendix C.

An explicit evaluation of the Wolfenstein parameters throughout the $p-p-\gamma$ calculations has shown that $G$, the Wolfenstein parameter for the non-time-reversal-invariant amplitude in the $p-p$ off-shell $l$-matrix element, is always negligible. In our calculations, $G$, which vanishes on the energy shell, varied from $10^{-7}$ to $10^{-9}$ times the other Wolfenstein parameters which are non-zero on the energy shell. Accordingly, we have set $G$ to zero throughout the calculations and have not included the non-time-reversal-invariant amplitudes in eq. (21) in the calculation of the trace eq. (14). To do so would greatly increase the amount of work required to evaluate eq. (14).

4. COMPARISON WITH EXPERIMENT

Our calculated proton-proton Bremsstrahlung cross sections are compared with the existing experimental results in figs. 3-7.

In fig. 3 the $p-p-\gamma$ cross section for the coplanar Harvard geometry with proton counters at equal angles $\theta = 30^\circ$ to the incident beam and coplanar with it, is plotted versus the incident proton laboratory kinetic energy $E_{\text{lab}}$. The smooth curve is the theoretical result obtained from eq. (A.6) with the electromagnetic interaction eq. (10) and with the $p-p$ interaction described by the Tabakin potentials (B.10,12) in $S$, $P$- and $D$-waves and off-shell one-pion-exchange (OPE) amplitudes (C.1) in the higher partial waves. The calculations at 158 MeV and 204 MeV agree with the experimental results of Gottschalk [6] and of Thorndike [8] within experimental errors, although they are below the central experimental values. At 62 MeV the calculated result coincides with the result reported by Halbert [9] of 2.4 $\mu$b/sr$^2$. The situation is less clear in comparison to the experiments of Slaus [7] and of Warner [5] at 46 MeV and 48 MeV. Our results are at the lower limit of
Fig. 3. The p-p-γ cross section \( \frac{d^2\sigma}{d\Omega_1 d\Omega_2} \) in the Harvard geometry with \( \theta = 30^\circ, \phi = 180^\circ \) as a function of the incident proton kinetic energy, \( E_{\text{lab}} \). The experimental points are those of Slaus (48 MeV), Warner (48 MeV), Gottschalk (158 MeV), and Thorndike (204 MeV). The smooth curve is calculated with Tabakin potentials for S, P and D waves and off-shell OPE for higher partial waves. The cross section calculated with magnetic moments set to zero is designated by an upright triangle.

Warner's measured value and below that of Slaus. At still lower energies there is a possible discrepancy between our calculations and the recently reported result [10] at 30 MeV of 1.85 \( \mu b/\text{sr}^2 \) (measured at \( \theta = 350^\circ \), however) and at 33.5 MeV of 3.0 \( \pm 0.8 \) \( \mu b/\text{sr}^2 \) (ref. [7]). Our calculated p-p-γ cross sections in the Harvard geometry decrease monotonically and rapidly to zero as discussed in more detail in sect. 5, and would disagree with any result which showed the cross section passing through a minimum. If a discrepancy at low energies persists, it would appear necessary to include the Coulomb interaction between the protons. This might be done within the context of our work using the Coulomb modified separable potentials of Harrington [33]. (We will return to the question of such possible refinements to the calculation in sect. 8.) We also show in fig. 3 p-p-γ cross sections (indicated by upright triangles) calculated with the proton magnetic moments set to zero.

The full cross section is dominated by the magnetic dipole term in eq. (10) due to the close cancellation of the electric dipole matrix element for the equal-mass, equal-charge proton pair [34, 35]. The experimental data and corresponding calculated values shown in fig. 3 are presented also in table 1.

Fig. 4 shows the comparison of our calculations with the experimental results of Gottschalk [6] at a bombarding energy \( E_{\text{lab}} = 158 \) MeV for the coplanar Harvard geometry with proton counters at variable angles \( \theta \) to the incident beam. The agreement between the smooth curve (Tabakin potentials and off-shell OPE) is seen to be within the experimental uncertainties. The inverted triangles are the cross sections calculated with magnetic moments.
set to zero. As $\theta$ approaches $43.7^0$ and the angle between the final state protons approaches $87.5^0$ ($180^0$ in the c.m. system) we get close to the elastically scattered beam and the momentum $k$ carried off by the photon goes to zero.

Fig. 4 shows the characteristic $(1/k)$ infrared divergence of the Bremsstrahlung cross section [36] in this limit. Since the magnetic interaction terms in eq. (10) do not contribute to the infrared divergence, the cross section is increasingly dominated by the electric interaction as $\theta \rightarrow 43.7^0$, in spite of the cancellations which reduce the electric dipole matrix element. The cancellations are not exact for finite proton momentum.

Fig. 5 compares our full calculation with Gottschalk's measured cross
Table 1
Proton-proton Bremsstrahlung cross sections (coplanar Harvard geometry)

<table>
<thead>
<tr>
<th>$E_{lab}$ (MeV)</th>
<th>$\theta$</th>
<th>$\frac{d^3\sigma}{d\Omega_1 d\Omega_2}$ ($\mu$b/ster$^2$)</th>
<th>calc.</th>
<th>exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>35°</td>
<td>0.88</td>
<td>1.85 ± 0.25</td>
<td>Warner</td>
</tr>
<tr>
<td>33.5</td>
<td>30</td>
<td>3.0 ± 0.8</td>
<td>$\text{slau's}$</td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>30</td>
<td>1.47</td>
<td>3.46 ± 0.6</td>
<td>$\text{slau's}$</td>
</tr>
<tr>
<td>48</td>
<td>30</td>
<td>2.12 ± 0.36</td>
<td>Warner</td>
<td></td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>3.04 ± 0.44</td>
<td></td>
<td></td>
</tr>
<tr>
<td>61.8</td>
<td>30</td>
<td>2.4 a)</td>
<td>1.32</td>
<td>Halbert</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.4 a)</td>
<td></td>
</tr>
<tr>
<td>158</td>
<td>30°</td>
<td>8.47</td>
<td>10.6 a)</td>
<td>Gottschalk</td>
</tr>
<tr>
<td>32.5</td>
<td></td>
<td>10.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
<td>14.0 a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>37.5</td>
<td></td>
<td>13.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>23.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>204</td>
<td>30</td>
<td>10.9</td>
<td>13 ± 2.2</td>
<td>Thorndike</td>
</tr>
</tbody>
</table>

a) Narrow azimuthal acceptance was used in these experiments.

sections [37] for a non-coplanar Harvard geometry at $E_{lab} = 158$ MeV. In the notation of fig. 1, $\theta$ is the angle between $\mathbf{p}$ and $\mathbf{r}$ and also between $\mathbf{q}$ and $\mathbf{r}$. The angle $\phi$ between the plane $(\mathbf{p}, \mathbf{r})$ and the plane $(\mathbf{q}, \mathbf{r})$ measures the non-coplanarity. For $\phi = 180^0$, the two proton counters are coplanar with the beam and have an angle $2\theta$ between them. As $\phi$ is decreased from $180^0$, the cross section decreases to zero at $\phi = 162^0$, beyond which point the Bremsstrahlung process is kinematically forbidden. The calculated results are within the limits of experimental error with, however, a tendency to be somewhat lower than the coplanar results. It is possible that this agreement could be improved. Gottschalk corrected his coplanar results for finite resolution by assuming a cross section decreasing linearly to zero at $\phi = 162^0$ and increased his coplanar cross sections, measured with a resolution $\phi = \pm 40^0$, by 2 $\mu$b/ster$^2$ to get a theoretical coplanar cross section. Our calculations indicate that near $\phi = 180^0$, the cross section falls off more slowly than linearly, and that it would be more appropriate to add a correction of only 0.5 $\mu$b/ster$^2$. Such over-correction of the coplanar cross sections for finite resolution would contribute to the fact that our calculations in fig. 3 are generally low.

Fig. 6 shows our calculations for the Thorndike geometry (eq. (A.8)) compared to the experimental results for an incident energy $E_{lab} = 204$ MeV. The cross sections are referred to the c.m. system. The photon counter is at 90° to the beam in the lab system (108° in the c.m. system). The full calculation with Tabakin potentials and off-shell OPE are in excellent agree-
Fig. 5. The p-p-γ cross section $d^2\sigma/d\Omega_1 d\Omega_2$ in the Harvard geometry with $\theta = 30^\circ$, $E_{\text{lab}} = 158$ MeV as a function of $\phi$. The experimental points are those of Gottschalk, the curve (T) is with Tabakin potentials and off-shell OPE.

Fig. 6. The p-p-γ cross section $d^2\sigma/d\Omega_1 d\Omega_2$ in the Thordike geometry with $E_{\text{lab}} = 204$ MeV, $\theta = 90^\circ$ in the lab as a function of photon energy. The experimental points are those of Thordike and the curve is with Tabakin potentials and off-shell OPE. We also reproduce Ueda's results and our own calculations for OPE only.
ment with Thorndike’s results. Thorndike [8] quotes an integrated cross section $\frac{d\sigma}{d\Omega_\gamma}$ for photon energies above 35 MeV

$$\frac{d\sigma}{d\Omega_\gamma} / k > 35 \text{ MeV} = 48 \pm 3.5 \text{ nb/sr}.$$  

Our theoretical result is 51 nb/sr.

In fig. 7 we present a number of calculations designed to test the sensitivity of the p-p-\(\gamma\) cross sections to the details of the p-p interaction. The calculations are for the coplanar Harvard geometry with \(\theta = 30^\circ\). The solid line (T) is a repetition of fig. 3, calculated with the Tabakin soft-core potentials (B.10, 12) and off-shell OPE amplitudes. Indistinguishable from these results are calculations using the Tabakin hard-shell potential (B.11). These two forms of the \(1S_0\) potential predict p-p-\(\gamma\) cross sections within 5 percent of each other. The curve (TL) is the result for Tabakin soft-core potentials in S-, P- and D-waves and no higher partial waves. Curve (OSL) is the same as (TL) except that off-shell Tabakin S-, P- and D-wave amplitudes are replaced by their on-shell values. Curve (PT) is the same as (T) except

![Graph](image-url)

**Fig. 7.** The p-p-\(\gamma\) cross section $d^2\sigma/d\Omega_1 d\Omega_\gamma$ in the Harvard geometry with \(\theta = 30^\circ\), \(\phi = 180^\circ\) as a function of the incident proton kinetic energy, $E_{lab}$, calculated with various models of the p-p interaction: (T) Tabakin potentials and off-shell OPE for \(L < 2\), (TL) Tabakin potentials and no partial waves with \(L > 2\), (OSL) on-shell Tabakin l-matrix elements and no partial waves with \(L > 2\), (PT) Tabakin potentials with P wave potential strengths increased 10%, and off-shell OPE for \(L > 2\), (CF) Tabakin potentials and on-shell OPE for \(L > 2\), (OPE) Off-shell OPE for all partial waves.
that the strengths, \( \lambda \), of the \( P \)-wave potentials (B.12) have been increased by 10\% to test the sensitivity of our results to Tabakin's fit to the \( p-p \) \( P \)-wave phase shifts. Tabakin's \( P \)-wave phase shifts are only a qualitative fit to the data but our Bremsstrahlung cross sections do not appear to be sensitive to small changes in the \( P \)-wave parameters. Curve (CF) is the same as (T) except that the off-shell OPE amplitudes in the higher partial waves are replaced by their on-shell values with the form factor \( F(\cdot) \) replaced by unity. Curve OPE is calculated with off-shell OPE amplitudes in all partial waves and reflects the limitation of OPE as a representation of the low partial waves.

Returning to fig. 6, we show the cross section in the Thorndike geometry calculated with off-shell OPE in all partial waves for comparison to Ueda's result [13]. We obtain an integrated cross section for \( k > 35 \) MeV,

\[
\frac{d\sigma}{d\Omega}^{OPE} = 40 \text{ nb/sr},
\]

compared to Ueda's result

\[
\frac{d\sigma}{d\Omega}^{UEDA} = 60 \text{ nb/sr},
\]

reasonable agreement considering the different approaches used.

In fig. 8 we show the \( p-p-\gamma \) cross section for the coplanar Harvard geometry with \( \theta = 30^\circ \) for a wide range of energies. These results are calculated with Tabakin potentials and off-shell OPE. The cross section increases rapidly from zero at zero bombarding energy and increases monotonically as a function of energy with no sign of any structure until it reaches a maximum of 30 \( \mu \text{b/sr}^2 \) at 700 MeV and then decreases slightly at 1000 MeV. At these energies our representation of the proton-proton interaction is expected to be inadequate and these can only be qualitative predictions.

5. LOW-ENERGY BEHAVIOUR

There has been some disagreement amongst the various calculations of the \( p-p-\gamma \) cross sections especially in the limit of zero proton bombarding energies where some results [12] indicate that the cross section increases after a minimum at about 50 MeV to a non-zero limit which has even been thought to be infinite and a manifestation of an infrared divergence [36]. Our own calculated cross sections for the Harvard geometry (fig. 8) fall rapidly and monotonically to zero for zero bombarding energy, \( E_{\text{lab}} \). It should be emphasized that all these extrapolations to zero bombarding energy are made neglecting the Coulomb interaction of the protons and are not intended to be a physically realistic description of the \( p-p-\gamma \) process below about \( E_{\text{lab}} = 10 \) MeV. The zero energy limit with the neglect of the Coulomb interaction does, however, provide a sensitive point of comparison between various calculations, and one which can be verified by comparison with a calculation using a simple nuclear force.

An explicit numerical evaluation of the factors entering the Bremsstrahl-
lung cross section eq. (A.6) shows the following dependence upon the incident proton kinetic energy $E_{\text{lab}}$ at energies below $\sim 1$ MeV:

$$p, q, r, s \approx (E_{\text{lab}})^{\frac{1}{2}}, \quad k \approx E_{\text{lab}}, \quad |T_\perp|^2 \approx 1, \quad J \approx (E_{\text{lab}})^{-\frac{1}{2}}$$  \hspace{1cm} (31)

and the interval of integration on $E_p$ varies as $(E_{\text{lab}})^{\frac{1}{2}}$. The differential cross section $d^2\sigma/d\Omega_1 d\Omega_2$ goes to zero like $(E_{\text{lab}})^{\frac{1}{2}}$ as shown in fig. 8.

The low-energy behaviour of the Bremsstrahlung amplitude can be substantiated from eq. (12) by neglecting the magnetic moment terms and by
replacing the off-shell $t$-matrix elements by a constant zero-range ap­proximation, $\langle t_0 \rangle$. A straightforward calculation yields (in the c.m.)

$$T \rightarrow \langle t_0 \rangle \left( -r \frac{2r \cdot k}{(k + \frac{1}{2}k^2)^2 - (r \cdot k)^2} + p \frac{2(p \cdot k + \frac{1}{2}k^2)}{k^2 - (p \cdot k + \frac{1}{2}k^2)^2} \right).$$

(32)

Neglecting $k^2$, $p \cdot k$ and $r \cdot k$ compared to $k$, we obtain

$$T \rightarrow \langle t_0 \rangle \left\{ p + \frac{2p \cdot k}{k^2} - 2 \frac{r \cdot k}{k^2} r \right\}. \quad (33)$$

The second and third terms in eq. (33) are of zeroth order in the incident energy, confirming the numerical result in eq. (31), that the Bremsstrahlung amplitudes go to a constant at zero energies.

The well-known low-energy theorem [36] relating the Bremsstrahlung cross section to the elastic cross section is not directly applicable here. In an expansion of the Bremsstrahlung cross section in powers of $(k/E_{\text{lab}})^{-1}$, the coefficient of the term $(k/E_{\text{lab}})^{-1}$ which dominates at low-photon energy, is proportional to the elastic cross section. However, in the Harvard geometry where the protons are detected at inelastic angles, the photon momentum is not small compared to the incident energy.

6. ESTIMATE OF RESCATTERING AND NON-LOCAL CORRECTIONS

We have used two simple models to estimate the contributions of diagrams (d) and (e) of fig. 2, which we term the rescattering and non-local corrections, respectively. To make these calculations tractable, we consider non-identical spinless nucleons interacting through one-component separable potentials (B.1) acting in $S$-, $P$- and $D$-waves only. In the notation of appendix B, model "1" has $S$-, $P$- and $D$-separable potentials with the form factor

$$v_L(p) = \frac{p^L}{\langle p^2 + a_L^2 \rangle^{\frac{3}{2}(L+2)}}, \quad (34)$$

the same as the first term in the Tabakin potentials. The strengths $\lambda_L$ and ranges $a_L^1$ of the model potentials are chosen equal to the corresponding Tabakin parameters. We have further simplified the calculations in model "2" by assuming a $P$-wave potential

$$v_1(p) = \frac{p}{\langle p^2 + a_1^2 \rangle} \quad (35)$$

with the same range $a_1^2$ as Tabakin but with a strength twice Tabakin's value to give a representative phase shift.

With the simple model we can easily evaluate $T$ in eq. (12) and the $p-p-\gamma$ cross section without rescattering or non-local effects. Since the model does not include spins, we have no magnetic moment interactions. The magnetic moment interaction is actually increasingly important at high energies for $p-p-\gamma$, so the simple model does not correspond very closely to the full calculation. The full calculation with magnetic moments turned off is compared to the $p-p-\gamma$ cross section predicted by models "1" and "2" for the
coplanar Harvard geometry with $\theta = 30^\circ$ in fig. 9. The results coincide when the nucleon-nucleon interaction is the same $S$-wave potential in the two calculations. The spinless nucleons do not take proper account of the higher partial waves but, in spite of the crudity of the models, we hope our conclusions about rescattering and non-local corrections will have some qualitative validity.

The rescattering correction to $T$ from diagram (d) with initial and final interactions in angular momentum states $L_1$ and $L_2$, respectively, is, in the c.m. system,

$$T_R(L_1, L_2) = -\int \frac{d^3w}{(2\pi)^3} \left\langle \frac{1}{2}(p - q) \right| \ell(E_{pq}) \left| w + \frac{1}{2}k \right\rangle_{L_2}$$

$$\times [E - E(w + k) - E(w) - k + i\eta]^{-1} w$$

$$\times [E - 2E(w) + i\eta]^{-1} \langle w \mid \ell(E_{rs}) \mid \frac{1}{2}(r - s) \rangle_{L_1}$$

(36)

The other diagram (d) for p-p-$\gamma$ satisfies

$$T^\text{ex}_R(L_1, L_2) = (-)^{L_1+L_2} T_R(L_1, L_2)$$

(37)

so for p-p-$\gamma$ the rescattering corrections are

$$2(T_R(0, 2) + T_R(2, 0) + T_R(1, 1))$$

since $T_R(0, 0)$ vanishes and the (S, P) and (P, D) terms cancel. If we use non-relativistic kinematics and the P-wave potential of model "2", the integrations in eq. (36) can all be reduced to a one-dimensional integral over $w$ which is done numerically. This is not the case for $T_R(1, 1)$ if we use the P-wave potential of model "1", so we have not evaluated the p-p-$\gamma$ rescattering correction using this model. The integrals to be done have denominators with four quadratic factors, two of them linear in $w \cdot k$. The integrals over angles are done by partial fractions leaving a one-dimensional integral with a singularity structure consisting of eight logarithmic branch points, two algebraic branch points, and two simple poles. We integrate numerically along a straight path in the fourth quadrant of the complex $w$-plane, chosen to pass equally distant from the two nearest singularities. A predictor-corrector method was used with a step size half the distance to the nearest singularity.

The rescattering corrections calculated with model "2" for p-p-$\gamma$ cross sections in the coplanar Harvard geometry at $\theta = 30^\circ$ are shown in fig. 10, and are less than 3% of the model cross sections. The corrections are small because of the cancellation of the electric dipole (S, P) terms, $T_R(0, 1)$ and $T_R(1, 0)$, for the p-p system. These terms are non-zero in n-p-$\gamma$ and, as we discuss in the next section, can make the rescattering corrections to n-p-$\gamma$ very important. The P-wave potential of model "1" is weaker than that of model "2" and we anticipate that model "1" would have even smaller rescattering corrections.
Fig. 9. The $p-p-\gamma$ cross section $d^2\sigma/d\Omega_1 d\Omega_2$ in the Harvard geometry with $\theta = 30^\circ$, $\phi = 180^\circ$ calculated with the spinless nucleon models of sect. 6, labeled P1 and P2 compared to the full calculation C with Tabakin potentials and off-shell OPE but with the magnetic moments set to zero. Below are the corresponding $n-p-\gamma$ results.
Fig. 10. The percent rescattering corrections to the model p-p-γ and n-p-γ cross sections of fig. 9.

Sobel [28] has estimated the importance of rescattering effects in p-p-γ by an approximate integration of eq. (36) using on-shell \( t \)-matrix elements of local potentials, and reaches a similar conclusion to ours, that rescattering terms contribute less than five percent to the p-p-γ cross section for the Gottschalk experiment.

We have also estimated the effect of one type of non-local correction, that due to an electromagnetic coupling induced by the velocity dependent non-local separable potential between the two nucleons. Yamaguchi [39] points out that in order to leave the Schrödinger equation with a non-local potential gauge invariant, it is necessary to modify the potential according to

\[
V(r, r') \to V(r, r') \exp \left\{ ie \int_{r'}^{r} A \cdot ds \right\}
\]

(38)

for a single particle of charge \( e \). For two protons 1 and 2 interacting through a non-local potential, the generalization of eq. (38) is

\[
V(r_1, r_2; r'_1, r'_2) \to V(r_1, r_2; r'_1, r'_2) \exp \left\{ ie \int_{r'_1}^{r_1} A \cdot ds + ie \int_{r'_2}^{r_2} A \cdot ds \right\}.
\]

(39)

The integrals are path dependent, but, following Yamaguchi, and Osborn and Foldy [40], we assume a straight-line path for simplicity. Then, keeping just first order in \( e \) in the expansion of the exponential, we obtain a modification to the minimal electromagnetic interaction due to the non-local poten-
tial. An S-wave separable potential of the form (B.1) is modified, for emission of a photon of momentum \( k \), to

\[
-\nu(p)\lambda\nu(p') \rightarrow -\nu(p)\lambda\nu(p') \\
+ e \varepsilon \cdot p' \int_0^1 dx \, \nu(p) \lambda [\nu'(p' + \frac{1}{2}kx) - \nu'(p' - \frac{1}{2}kx)] \\
+ e \varepsilon \cdot p \int_0^1 dx \, [\nu'(p - \frac{1}{2}kx) - \nu'(p + \frac{1}{2}kx)] \lambda \nu(p'),
\]

where \( \nu'(p) = \partial \nu/\partial p^2 = -(\sum_{i} k_i^2)^{-2} \) for our models eq. (33). We have kept non-local terms arising from the S-wave interaction only.

The non-local contribution eq. (40) gives consistently small corrections. The p-p-\( \gamma \) cross section for models "1" and "2" is increased about \( \frac{1}{7} \mu b/sr^2 \) in the coplanar Harvard geometry at \( \theta = 300^\circ \), as shown in fig. 11. We might expect the non-local correction to be relatively less important for the larger cross section of the full calculation if we are correctly estimating its amplitude.

This discussion of the effect of induced electromagnetic interactions is obviously model dependent and leaves the question of exchange currents or velocity dependent effects of local potentials completely untouched. However, we have shown that it is possible to neglect these non-local terms in the Yamaguchi and, almost certainly, also in the Tabakin potentials.

![Graph](image)  

Fig. 11. The non-local corrections to the p-p-\( \gamma \) and n-p-\( \gamma \) cross sections of fig. 9. The p-p-\( \gamma \) corrections are multiplied by four.
7. NEUTRON-PROTON BREMSSTRAHLUNG

The results of our calculations for neutron-proton Bremsstrahlung cross sections for the coplanar Harvard geometry with particle detectors at $30^\circ$ to the incident proton beam are shown in fig. 12 as a function of incident proton energy. These cross sections have been calculated using just the Tabakin S-, P- and D-wave interactions and taking no account of higher partial waves, so they are expected to have only a qualitative significance at higher energies judging from fig. 7. Furthermore, the neglected rescattering corrections, estimated according to models "1" and "2" of sect. 6, are significantly large for n-p-$\gamma$ since the electric dipole (S, P) rescattering terms, $T_R(1,0)$ and $T_R(0,1)$ of eq. (36), do not vanish as in p-p-$\gamma$.

The rescattering corrections are sensitive to the model of the P-wave interaction, but as we see in fig. 10 both models give rescattering corrections greater than ten percent, and sometimes as large as fifty percent, of the first order cross section. The effect of the non-local terms has been estimated for n-p-$\gamma$ as for p-p-$\gamma$ in sect. 6, and appears to be small and less than one $\mu$b/sr$^2$ at all energies for this geometry.

The n-p-$\gamma$ cross section for the Thorndike geometry at an incident proton energy of 197 MeV is shown in fig. 13. The integrated cross section for photon c.m. energies between 40 and 90 MeV at $\theta = 90^\circ$ in the lab is

![Graph showing n-p-$\gamma$ cross section as a function of incident proton energy.](image)

Fig. 12. The n-p-$\gamma$ cross section $d^2\sigma/d\Omega_1 d\Omega_2$ in the Harvard geometry with $\theta = 30^\circ$, $\phi = 180^\circ$ as a function of incident proton kinetic energy, $E_{\text{lab}}$, calculated with Tabakin potentials in S-, P- and D-waves and no higher partial waves.
\( \frac{d\sigma}{d\Omega_\gamma} \) _calc \( = \, 0.36 \, \mu b/\text{sr}. \)

Rothe et al. [41] have extracted an n-p-\( \gamma \) cross section from their measurements with deuterium targets and obtain

\( \frac{d\sigma}{d\Omega_\gamma} \) _exp \( = \, 2.5 \pm 0.8 \, \mu b/\text{sr}, \)

in considerable disagreement with our calculations. We feel that it would be reasonable to increase our calculated results by as much as a factor of 4 to take account of rescattering effects and the effects of higher partial waves but we are unable to account for the large discrepancy that exists at present.

8. CONCLUSIONS AND DISCUSSION

We conclude that our results are in good agreement with the p-p-\( \gamma \) experiments at \( E_{\text{lab}} = 48, 62, 158 \) and 204 MeV. Our calculations favor the lower results of Warner at 48 MeV over the result of Slaus at 46 MeV and are lower than the recent results at 30 MeV, but still within the experimental limits at this energy.

The results of fig. 7 demonstrate that it will not be easy to differentiate potentials by their predictions of p-p Bremsstrahlung cross sections. We are only equipped to calculate with separable potentials so we cannot compare the predictions of the various local potentials, but from fig. 7 we see that:

(i) The Tabakin soft-core and Tabakin hard-shell \( ^1S_0 \) potentials make virtually identical predictions (curve T) for the p-p-\( \gamma \) cross sections, in spite of the fact that the short-range behaviour of their wave functions is

![Graph](image-url) - Fig. 13. The n-p-\( \gamma \) cross section \( d^2\sigma/d\Omega_\gamma \, dk \) in the Thordike geometry with \( \theta = 90^\circ \) in the lab system and \( E_{\text{lab}} = 197 \, \text{MeV} \). The calculation employs Tabakin S-, P- and D-wave potentials and no higher partial-wave interactions.
distinctly different [43]. Signell and Marker (ref. [44])* have compared various local potentials [2] and have reached similar conclusions.

(ii) The effect of changes in the Tabakin P-wave potentials (the difference PT-T) is as large as the effect of the off-shell corrections (the difference OSL-TL) and much larger than the difference between the soft-core and hard-shell models of the off-shell corrections. Local potentials which are much more carefully fit to the experimental data do not have as large possible errors as the Tabakin potentials, but we can still anticipate that the uncertainties in the predictions of equivalent potentials will be comparable to their differences.

(iii) The difference between on-shell and off-shell one-pion-exchange amplitudes for the higher partial waves (CF and T) is about 25% of the cross section. It is possible that a more refined treatment of the higher waves would bring our results into precise agreement with the central values of Gottschalk and Thordanike (at the same time probably disrupting the agreement with Thordanike's photon counter experiment) but it is likely that there would be enough ambiguities in such a treatment to bring the prediction of all potentials (with the same on-shell predictions OSL) into as good agreement.

On the basis of our calculations there is a mild discrepancy between experiments in the Harvard geometry and in the Thordanike geometry which could be reduced by lowering the cross sections in the coplanar Harvard geometry at 158 MeV and 204 MeV by 1 μb/sr². We suggest from our calculations (fig. 5) for non-coplanar Harvard geometry, that these results have been over-corrected for finite resolution and should, in fact, be reduced.

There remains a possible discrepancy with the recent results at 30 MeV which indicate that the cross section is increasing at low energies after a minimum near 50 MeV. If this is true, there are a number of possible resolutions of discrepancy:

(i) There is an error in our work. We feel that this is not a likely possibility in view of the detailed agreement with experiments we have obtained, and the zero-energy behaviour of our cross sections which is the result of anticipated cancellations that would not occur with errors.

(ii) The low-energy results are more sensitive to the details of the potential and would increase at 30 MeV for the correct local potential. This is not a likely possibility in view of the results of fig. 7 which show that calculations with on-shell matrix elements (presumably common to all acceptable potentials) approach the off-shell calculations at low energies. We cannot absolutely rule out the possibility that the low-energy results might be particularly sensitive to the choice of potential, but our conclusions are substantiated by the calculations of Signell and Marker [44].

(iii) The Coulomb interaction between the protons is important at 30 MeV. This is the most reasonable possibility and must be investigated. A first attempt by us to incorporate the Coulomb repulsion by using Harrington's

* Their calculations do not agree with ours, especially at low energies, but presumably we can use their results to compare potentials.
modified separable potentials [33] for the p-p interaction, leads to the conclusion that the Coulomb effect is small above about 10 MeV [45], and furthermore, that the Coulomb repulsion reduces the attractive nuclear $1S_0$ interaction and first reduces rather than increases the cross section. However, if the discrepancy persists, it will be necessary to incorporate the Coulomb interaction in a detailed way. There is considerable interest in doing so at even lower energies and we understand that calculations in progress [20,21] do take it into account.

(iv) The estimates which show rescattering effects to be small are incorrect at low energies because of the importance of the Coulomb interaction in intermediate states.

If it is necessary to incorporate (iii) and/or (iv), above, then it is probably best to abandon the multiple-scattering approach and return to a distorted wave calculation (eq. 1) and calculate individual multipole amplitudes between individual distorted partial waves as is being done by Brolley and Morrison [21], and by Heller and Rich [20].

The final question to which we turn is the general problem of refining the calculations. There are a number of obvious and desirable improvements that can be made, but usually only at the expense of a vast increase in the complexity of the calculation or even of a complete change in its format. Among these improvements we list the following:

(i) Extend the Tabakin potentials to higher partial waves so that we depend on off-shell OPE amplitudes only for $L > 5$. Strobel and Ericson [19] have such calculations in progress.

(ii) Repeat our calculations with $t$-matrix elements generated numerically from local potentials. This would require only minor changes in the computer programs but would necessitate a faster computer than the IBM 7040 used so far.

(iii) Include the electromagnetic form factor of the protons. This is certainly a small effect even at the highest photon energies, $k \approx 90$ MeV in the Thordike experiment.

(iv) Include the rescattering effects exactly. This is too difficult to do in the rescattering formalism employed here and we would have to redo the problem using the full DWBA (eq. 1) directly. This approach would make it possible to incorporate Coulomb effects and do a valid low-energy calculation as discussed above.

(v) Make detailed estimates of the induced interactions occurring through the velocity dependent p-p interactions. Our results based on a simple model are reassuring but it would be possible to include these terms in the full Tabakin calculation.

(vi) Make a realistic calculation of meson-exchange current effects. This would be a major undertaking and could be based on the Bryan-Scott [46] one-boson exchange potentials within the context of a potential model of p-p interaction, or on the Scotti-Wong [47] one-boson exchange amplitudes with (inelastic) unitarity imposed through the $N/D$ dispersion relations.

It is clear that p-p Bremsstrahlung is an intricate problem which will continue to excite theoretical and experimental interest until it is understood in complete detail. We believe that our calculations at intermediate energies
(50-200 MeV) show that the process can be understood quantitatively using a potential model of the p-p interaction, but that uncertainties in the calculation mask the differences amongst potentials reproducing the experimental phase shifts.

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APPENDIX A. KINEMATICS

Let \( r \) be the momentum in the laboratory system of coordinates of a proton incident on a target proton of momentum \( s \) (= 0), and \( p \) and \( q \) be the momenta of the two protons after scattering with the production of a photon of momentum \( k \) (see fig. 2). Momentum conservation requires (with \( \hbar = c = M_p = 1 \))

\[
p + q + k = r + s,
\]

and energy conservation requires

\[
(1 + p^2)^{1/2} + (1 + q^2)^{1/2} + k = (1 + r^2)^{1/2} + (1 + s^2)^{1/2}.
\]

Eliminating the photon momentum between (A.1) and (A.2), we obtain

\[
q = \gamma \beta (\pm \sqrt{\beta^2 - 4\gamma^2})
\]

where

\[
\beta = 2ab / (b^2 - a^2), \quad \gamma = (a^2 - c^2) / (b^2 - c^2),
\]

with

\[
a = -2\gamma^2 x_1 - [4 + 2\sqrt{1 + \gamma^2} - 2\sqrt{1 + \gamma^2} - 2\sqrt{1 + \gamma^2} \sqrt{1 + p^2}],
\]

\[
b = -2\gamma^2 x_2 - 2\gamma^2 x_{12},
\]

\[
c = -2 - 2\sqrt{1 + \gamma^2} + 2\sqrt{1 + p^2},
\]

\[
x_1 = \hat{p} \cdot \hat{\rho}, \quad x_2 = \hat{p} \cdot \hat{q}, \quad x_{12} = \hat{p} \cdot \hat{q}.
\]

Both solutions \( q \) to (A.3) are positive, physically allowed proton momenta for the second proton momentum \( p \) between those extreme values where the radical \( (\beta^2 - 4\gamma^2) \) vanishes.

The cross section eq. (16) can be integrated analytically over photon momentum \( k \) and over the magnitude of one final-state proton momentum \( q \) and then numerically over the other final-state proton energy \( E_p \) to give the double-differential cross section appropriate for the Harvard geometry. We obtain the cross section in \( \mu b/sr^2 \).
\[
\frac{d^2 \sigma}{d \Omega_1 d \Omega_2} = 4 \cdot \frac{4\pi}{137} \left( \frac{197}{938} \right)^2 \times 10^4 \left\{ \frac{\sum_{q} \int \frac{dE_p}{q} \cdot b_q}{(2\pi)^5} \right\} |T_\perp|^2
\]
(A.6)

where \( J = \left| 1 - \sqrt{1 + q^2 \cdot \hat{q} \cdot \hat{k}} / q \right| \). The summation in eq. (A.6) is over the two allowed value of \( q \), and the integration is over proton energies \( E_p \) for which the radical \((\beta^2 - 4\gamma^2)\) in eq. (A.3) is non-negative. The photon momentum \( k \) is determined from eq. (A.2). The numerical factor \((\frac{197}{938})^2 \times 10^4\) converts the cross section from nucleon units \((\hbar = c = M_\beta = 1)\) to \( \mu \text{b} \). All kinematic variables appearing explicitly in eq. (A.6) are in the laboratory system of coordinates as defined by eqs. (A.1) and (A.2). However, since \( |T_\perp|^2 \) in eqs. (16) and (A.6) is translationally invariant, we have evaluated this Bremsstrahlung matrix element in the c.m. system where, also, we impose the gauge condition

\[ \hat{e} \cdot \hat{k} = 0. \]  
(A.7)

The factor \( 4 \) in eq. (A.6) arises from the identity of the two protons in the following way: the amplitudes already possess the proper antisymmetry to describe identical particles but they must be normalized properly by multiplying our amplitudes normalized according to eq. (30) by \( \sqrt{2} \). In addition, the physical counters do not distinguish protons (1) and (2), so an extra factor of \( 2 \) is required, giving rise to the overall factor \( 4 \). For \( (n-p-\gamma) \) cross sections this factor \( 4 \) is replaced by unity.

The other geometry for which experimental results are available is that of Thorndike who observes the Bremsstrahlung photon. The appropriate differential cross section is obtained from eq. (16), which can be integrated analytically over two proton energies and one proton solid angle using delta-functions of energy and momentum, and then numerically over the solid angle of the second proton of momentum \( p \). The numerical integration is done over the angle \( \theta \) between \( p \) and \( r \) and over the angle \( \phi' \) between the plane \((K, r)\) and the plane \((p, r)\). The differential cross section in \( \mu \text{b}/(\text{sr} \cdot \text{MeV}) \), measured in the c.m. coordinate system, is

\[
\frac{d^2 \sigma}{d \Omega_\gamma d k} \text{c.m.} = \frac{\hbar}{2F} \frac{4\pi}{137} \left( \frac{197}{938} \right)^2 \times 10^4 \times \left( \frac{2}{2} \right) \times \frac{1}{(2\pi)^5}
\times \int_{0}^{1} \cos (\theta') \int_{0}^{2\pi} d \phi' \frac{|T_\perp|^2}{J_T},
\]  
(A.8)

with \( J_T = |E - k + \sqrt{1 + p^2 \cdot \hat{p} \cdot \hat{k}} / p^2 \). \( E \) is the total energy in the c.m. and all variables in eq. (A.8) refer to the c.m. system. The invariant flux \( F \) is defined in eq. (17). The factor \( 2/2 \) accounts for the identity of the two protons. A factor \( (\sqrt{2})^2 \) arises from the proper normalization of the amplitudes eq. (30), as in eq. (A.6). The factor of 2 which entered eq. (A.6) because the proton counters do not distinguish protons (1) and (2), enters eq. (A.8) implicitly through the unrestricted integration of both momenta \( p \) and \( q \) over all solid angles and means that the pair of protons is being counted twice. A factor \( (\frac{1}{2}) \) reduces this double counting of the pair of protons to the correct single counting. This factor is one for \( n-p-\gamma \).
APPENDIX B. THE TABAKIN POTENTIALS

The tabakin non-local separable potentials have the form in momentum space

\[ (p_i | V | p_1) = \sum_L \{ -(2L + 1) P_L (\hat{p}_i \cdot \hat{p}_1) v_L (p_i) \lambda v_L (p_1) \} \],

(\ref{tabakin_potential})

where, for simplicity in this appendix, we are ignoring the spin-isospin complexity of the scattering. Momentum eigenstates are normalized

\[ (p | p') = (2\pi)^3 \delta^3 (p - p') \].

(\ref{momentum_normalization})

The separable potential in each partial wave can have a number of components and can be written in matrix notation with \( \nu \) a vector and \( \lambda \) a (diagonal) matrix, so

\[ v_L (\hat{p}_i) \lambda v_L (p_1) = \sum_j v_{Lj} (\hat{p}_i) (\lambda_{Lj}) v_{Lj} (p_1) \].

(\ref{matrix_notation})

The \( t \)-matrix element has the partial wave expansion

\[ (p_i | t^0 | p_1) = \sum_L \{ t_L (p; p_i, p_1) (2L + 1) P_L (\hat{p}_i \cdot \hat{p}_1) \} \],

(\ref{t_matrix_expansion})

and satisfies the Lippmann-Schwinger equation

\[ t_L (p; p_i, p_1) = -i L (p_1) v_L (p_1) - \int \frac{d^3q}{(2\pi)^3} \frac{v_L (p_i) \lambda L v_L (q) t_L (p; q, p_1)}{p^2 - q^2 + i\eta} \].

(\ref{lippmann_schwinger})

With our normalization, the on-shell \( t \)-matrix element is just

\[ t_L (p; p, p) = -(-4\pi/\rho) \cdot e^{i\delta_L} \sin \delta_L \]

(\ref{on_shell_t_matrix})

in terms of the appropriate phase shift \( \delta_L \). Off-shell \( t \)-matrix elements which satisfy eq. (\ref{lippmann_schwinger}) can be written, in matrix notation for a many component separable potential, as

\[ t_L (p; p_i, p_1) = -i L (p_1) \Omega_L (p) \lambda_L v_L (p_1) \],

(\ref{off_shell_t_matrix})

in terms of the matrix

\[ \Omega_L (p) = [1 + \Lambda_L (p)]^{-1} \],

(\ref{omega_matrix})

where

\[ [\Lambda_L (p)]_{ij} = [\lambda_L]_{ii} \int \frac{d^3q}{(2\pi)^3} \frac{v_{Li} (q) v_{Lj} (q)}{p^2 - q^2 + i\eta} \]

\[ = \frac{1}{2\pi} [\lambda_L]_{ii} R_{ij} \],

(\ref{lambda_matrix})

defining the matrix \( R_{ij} \).

Tabakin [23] has obtained a fit to the \( p-p \) phase shifts with \( ^1S_0, ^3P_{0,1,2} \)
and $^1D_2$ separable potentials of the following forms (in the notation of eq. (B.3).

(i) For $^1S_0$: \[
\begin{align*}
v_1(p) &= (p^2 + a^2)^{-1}, \\
v_2(p) &= p^2[ (p - d)^2 + b^2 ]^{-1} [ (p + d)^2 + b^2 ]^{-1},
\end{align*} \tag{B.10}
\]
just a two-component separable potential.

The first component is an attractive potential with $\lambda_1 = 4\pi a V_\gamma > 0$, the second component is repulsive (simulating a hard core) with $\lambda_2 = 4\pi b V_\beta < 0$, and dominates at high momentum. Tabakin chooses for this "soft-core" potential

\[
V_\gamma = 115.9 \text{ MeV}, \quad V_\beta = -235.6 \text{ MeV}, \quad a^{-1} = 0.834 \text{ fm},
\]
\[
b^{-1} = 0.801 \text{ fm} \quad \text{and} \quad d^{-1} = 0.694 \text{ fm}
\]
as a best fit the p-p $^1S_0$ phase shift. Tabakin offers an alternative form, termed a "hard-shell" potential, which differs from the above by the choice
\[
v_2(p) = \sin (p/b)/pb,
\]
and the parameters
\[
\begin{align*}
V_\gamma &= 200 \text{ MeV}, \quad V_\beta = -44 \times 10^3 \text{ MeV}, \\
a^{-1} &= 0.755 \text{ fm} \quad \text{and} \quad b^{-1} = 0.182 \text{ fm}.
\end{align*}
\]
The "hard-shell" potential is more effective in excluding the nucleon-nucleon wave function from the internal region and more nearly simulates a hard-core potential.

(ii) For $^3P$: \[
\begin{align*}
v_1(p) &= p(p^2 + a^2)^{-\frac{3}{2}}, \\
v_2(p) &= p^3(p^2 + a^2)^{-\frac{3}{2}},
\end{align*} \tag{B.12}
\]
and $\lambda_1 = 4\pi a V_\gamma$, $\lambda_2 = 4\pi a V_\beta$ with the choice of parameters,
\[
\begin{align*}
^3P_0 \quad V_\gamma &= 287.7 \text{ MeV}, \quad V_\beta = -1067.7 \text{ MeV}, \quad a^{-1} = 0.714 \text{ fm}, \\
^3P_1 \quad V_\gamma &= -107.6 \text{ MeV}, \quad V_\beta = -531.2 \text{ MeV}, \quad a^{-1} = 0.800 \text{ fm}, \\
^3P_2 \quad V_\gamma &= 103.7 \text{ MeV}, \quad V_\beta = 394.5 \text{ MeV}, \quad a^{-1} = 0.625 \text{ fm}.
\end{align*} \tag{B.13}
\]

(iii) For $^1D_2$: \[
v_1 = p^2(p^2 + a^2)^{-2}, \quad v_2 = p^2(p^2 + a^2)^{-\frac{3}{2}},
\]
a one component potential, and $\lambda_1 = 4\pi a V_\gamma$ with $V_\gamma = 297.1 \text{ MeV}$, $a^{-1} = 0.565 \text{ fm}$.

The phase shifts obtained with these potentials are compared to the experimental phase shifts in ref. [23].

The partial-wave amplitudes of eq. (30) are related to the off-shell solu-
tions of eq. (B.5) normalized according to eq. (B.6) by the factor $-\rho/4\pi$.
Thus, for example,

$$C_0(p; \rho_T, p_1) = \frac{-\rho}{4\pi} t_1 S_0 \langle p; \rho_T, p_1 \rangle.$$  \hspace{0.5cm} \text{(B.14)}$$

APPENDIX C. THE ONE-PION-EXCHANGE AMPLITUDES

The OPE contribution to the p-p $t$-matrix elements are given, for example, by Cziffra et al. [48]. With our normalization, the OPE channel spin amplitudes eq. (26) are

$$T_{11}^{\pi} = T_{-1-1}^{\pi} = (-2\pi) \left\{ -\frac{g^2}{8E} [A(1+x) - B(1-x)] \right\},$$

$$T_{10}^{\pi} = T_{01}^{\pi} = T_{-10}^{\pi} = T_{0-1}^{\pi} = (-2\pi) \left\{ -\frac{g^2}{8E} \sqrt{2} (A + B) \sin \theta \right\},$$

$$T_{1-1}^{\pi} = T_{-11}^{\pi} = (-2\pi) \left\{ -\frac{g^2}{8E} [A(1-x) - B(1+x)] \right\},$$

$$T_{00}^{\pi} = (-2\pi) \left\{ \frac{g^2}{4E} (A + B) x \right\},$$

$$T_{ss}^{\pi} = (-2\pi) \left\{ -\frac{g^2}{4E} (A + B) \right\}.$$  \hspace{0.5cm} \text{(C.1)}$$

In the standard notation of MacGregor et al. [49], $\theta$ is the scattering angle between initial and final c.m. momenta of the two protons, $p_i$ and $p_f$, $x = \cos \theta$, $g^2 = 14.0$ is the pion-nucleon coupling constant, $E$ is the total c.m. energy, $A = (1+x)/(x_0 + x)$ and $B = (1-x)/(x_0 - x)$ with $x_0 = 1 + \mu^2/2\rho^2$, where $\mu$ is the pion mass, and $\rho$ is the (on-shell) c.m. momentum of one of the protons.

Ferrari and Selleri [50] have multiplied the OPE amplitudes by a form factor $F(\Delta^2)$ to account for the off-shell propagation of the virtual pion, whose four-momentum squared, $\Delta^2 \neq -\mu^2$. Their phenomenological form factor,

$$F(\Delta^2) = 0.72 \left[ 1 + \frac{\Delta^2 + \mu^2}{4.73 \mu^2} \right]^{-1} + 0.28 \left[ 1 + \frac{\Delta^2 + \mu^2}{32 \mu^2} \right]^{-2},$$  \hspace{0.5cm} \text{(C.2)}$$

equal to one when $\Delta^2 = -\mu^2$, reduces the OPE amplitude away from the pion pole. Ueda [14] has used the Ferrari-Selleri form factor to reduce the OPE amplitudes in his calculation of p-p Bremsstrahlung, and to bring his results into agreement with Thorndike's experiments.

We must project out the S, P and D partial wave amplitudes from the amplitudes eq. (C.1) and replace them by Tabakin amplitudes. With the form factor $F(\Delta^2)$ included, the partial wave projection must be done numerically to give a set of simultaneous linear equations for the OPE partial wave amplitudes $a$, $b$ and $c$ of eq. (26).
PROTON–PROTON BREMSSTRAHLUNG

ADDED NOTE

We are grateful to Professor Z. Koba for pointing out an error in sect. 5 of our original manuscript.
Since this work was submitted for publication we have received preprints of related work from
(3) V. Brown, Proton–proton Bremsstrahlung including rescattering;
which contain results in close agreement with our calculations.

NOTE ADDED IN PROOF

We have also been informed by Professor Cromer that the calculations of reference [11] and of M. Sobel and A. Cromer, Phys. Rev. 158 (1967) 1157,
have now been brought into agreement with our work.

REFERENCES

A. Simon, Phys. Rev. 79 (1950) 573;
R. E. Cutkosky, Phys. Rev. 103 (1957) 505;
B. Kursunoglu, Phys. Rev. 105 (1957) 1846;
C. Dullemond and J. De Swart, Physica 26 (1960) 664;
J. Hamada and I. Johnston, Nucl. Phys. 34 (1962) 382;
R. Bryan and B. Scott, Phys. Rev. 135 (1964) 2434;
M. Sobel, Thesis, Harvard University (1963);

A. Cromer, ibid.

[39] Y. Yamaguchi, loc. cit.;
[40] L. Foldy and D. Osborne, Phys. Rev. 80 (1951) 100.