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CHAPTER I

INTRODUCTION

1.1 Introduction to the Research Topic and Results

Lyapunov's Second Method provides a most general and powerful approach for the study of stability of nonlinear systems.

Consider the system

\[ \dot{x} = f(x), \quad (\dot{} = d/dt), \]  

(1.1)

where \( x = (x_1, \ldots, x_n) \) and \( f(x) = (f_1(x), \ldots, f_n(x)) \) are elements of an Euclidean n-space \( \mathbb{R}^n \), and the function \( f(x) \) has continuous first partial derivatives with respect to the components of \( x \).

Suppose one wishes to study the stability of a solution \( \Gamma \) of this system using the above method. By appropriate coordinate transformation one reduces \( \Gamma \) to the origin of the new coordinate system. Then stability of \( \Gamma \) is assured if there exists a positive definite scalar function \( V(x) \) whose total time derivative \( \dot{V}(x) \) along the solutions of the system is non-positive. The advantage of this approach is that the stability properties are obtained without any knowledge of the solutions but directly from the differential equations describing the system. However, in many cases, it is difficult if not impossible to find a function which satisfies the restrictions on \( V(x) \) and \( \dot{V}(x) \) just stated.

It is the purpose of this dissertation to provide a method for the investigation of the stability of the solution of (1.1). We will still resort to a positive definite scalar function \( V(x) \) but the usual
restrictions will be replaced by conditions on the function \( f(x) \) and its partial derivatives which guarantee the existence of a unique globally stable limiting trajectory (limit set) for the system (1.1).

Sufficient conditions for the boundedness of solutions in the future for general classes of nonlinear \( n \)-dimensional autonomous systems are also given.

The results are illustrated by application to several special cases including the example due to Zubov, and a special instance of coupled Liénard's equations.

1.2 Background

1.2.1 Liapunov's Second Method

For reference we will recall the fundamental stability theorems that bear a relationship to the present work.

Consider the system (1.1) with \( f(0) = 0 \). In what follows \( V(x) \) is a mapping from \( X^n \rightarrow \mathbb{R} \) (where \( \mathbb{R} \) = nonnegative real half line), having continuous partial derivatives w.r.t. \( x_i \), \( i = 1, \ldots, n \).

**Theorem 1.1** (Liapunov's stability theorem [23]):

If there exists a \( V(x) \) such that

(i) \( V(x) > 0 \) for all \( x \neq 0 \), \( V(0) = 0 \),

(ii) \( \dot{V}(x) \leq 0 \) for all \( x \neq 0 \),

then the origin of (1.1) is **asymptotically stable**.

In this and the following theorems, \( \dot{V}(x) \) is defined as the total time derivative of \( V(x) \) with respect to the system (1.1), i.e.

\[
\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i = (\text{grad } V) \cdot f
\]  

(1.2)
Theorem 1.2 (Liapunov's asymptotic stability theorem [23]):

If there exists a $V(x)$ such that

(i) $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$,
(ii) $\dot{V}(x) < 0$ for all $x \neq 0$,

then the origin of (1.1) is asymptotically stable.

The theorems just mentioned are local theorems, in other words, they are concerned with the behavior of the system in the neighborhood of the origin. In 1954, Barbashin and Krasovskii [4] extended Liapunov's theorem on asymptotic stability so that the region of asymptotic stability includes the whole state space. The origin is then said to be asymptotically stable in the large, (or globally asymptotically stable).

Theorem 1.3 (Barbashin and Krasovskii's theorem on asymptotic stability in the large [4]):

If

(i) $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$,
(ii) $\dot{V}(x) < 0$ for all $x \neq 0$,
(iii) $V(x) \to \infty$ as $\|x\| \to \infty$.

Then the origin of (1.1) is asymptotically stable in the large.

LaSalle [17,18] has pointed out that although Barbashin and Krasovskii proved the inverse theorem of the preceding one, yet it can be difficult to construct such a Liapunov function. In his paper [17], LaSalle has established a theorem which replaces condition (ii) of Theorem 1.3 by $\dot{V}(x) \leq 0$ and (iii) by a Lagrange stability condition. Here Lagrange stability is defined as the boundedness in the future of all solutions. For convenience we will state the Lagrange stability
theorem before this result of LaSalle.

**Theorem 1.4** (LaSalle's Lagrange stability theorem [17]):

Let $\Omega$ be a bounded neighborhood of the origin and $\Omega^c$ be its complement. If there exists a scalar function $W(x)$ with continuous first partials in $\Omega^c$ and satisfying

(i) $W(x) > 0$ for all $x$ in $\Omega^c$,
(ii) $\dot{W}(x) \leq 0$ for all $x$ in $\Omega^c$,
(iii) $W(x) \to \infty$ as $\|x\| \to \infty$,

then all solutions of (1.1) are bounded for $t \geq 0$.

**Theorem 1.5** (LaSalle's theorem on the asymptotic stability in the large [18]):

If there exists a scalar function $V(x)$ with continuous first partial derivatives satisfying

(i) $V(x) > 0$ for all $x \neq 0$,
(ii) $\dot{V}(x) \leq 0$ for all $x$, $\dot{V}$ does not vanish identically along any trajectory other than the origin,
(iii) the system (1.1) is Lagrange stable,

then the origin of the system (1.1) is asymptotically stable in the large.

Although it is often possible to find a function $V(x)$ to satisfy the previous theorems as being demonstrated by LaSalle in [17, 18], the requirement that $\dot{V}(x) \leq 0$ is difficult to be met in many cases. Furthermore, if the system (1.1) exhibits a limit cycle, then in order to investigate its global stability by applying the previous theorems, it would be necessary, as stated earlier, to transform the limit cycle
to the origin of the state space by a suitable coordinate transformation. However, for this purpose, we must have a priori knowledge of the equation of the limit cycle, which is usually unavailable. So we cannot use to advantage the above theorems in these cases.

In the subsequent chapters, we develop a direct method, which alleviates the aforementioned difficulties, for the determination of the stability properties of general classes of nonlinear n-dimensional autonomous systems.

1.2.2 Zubov's Approach and its Generalizations

The first general formulation of the Liapunov Second Method as a stability criterion is due to Zubov [34,35], who established certain theorems which constitute a stability criterion based on the construction of a Liapunov function \( V(x) \) for nonlinear systems.

For the system (1.1), Zubov's procedure is to consider the partial differential equation for a positive definite function \( V(x)(V = (\partial / \partial x_i)) \),

\[
\dot{V} = (V \nabla) \cdot f(x) = -G(x)(1 - V)
\]  

(1.3)

where \( G(x) \) is a positive semidefinite function not identically equal to zero on any non-trivial solution of the system (1.1). The solution of the partial differential equation (1.3) with \( V(0) = 0 \) then exhibits the stability property of the system (1.1). \( V(x) < 1 \) defines exactly the domain of attraction (with the set \( \{x: V(x) = 1\} \) as its boundary). Furthermore, if \( V(x) \to 1 \) as \( \|x\| \to \infty \), then the origin is asymptotically stable in the large.

In [35], Zubov extended this procedure to establish necessary and
sufficient conditions for the asymptotic stability of a periodic solution of the system (1.1). He assumes that system (1.1) has a periodic solution

\[ x = \varphi(t) \]  \hspace{1cm} (1.4)

with period \( T \). For the presentation of this result, the following definition is necessary.

**Definition 1.1** The set \( \Omega \) of all points \( x_o \) in n-space is said to be the region of attraction of the self-oscillation \( x = \varphi(t) \), if for \( x_o \in \Omega, \rho(x(t)) \to 0 \) as \( t \to \infty \), where \( x(t) \) is the solution of (1.1), with \( x_o \) as its initial condition, and \( \rho \) denotes the distance, from \( x(t) \) to \( \varphi(t) \) given by

\[ \rho(x(t)) = \inf_{t' \in [0,T]} \sqrt{\sum_{i=1}^{n} [x_i(t) - \varphi_i(t')]^2} \]  \hspace{1cm} (1.5)

**Theorem 1.6 (Zubov [35]):**

The region \( \Omega \) is the region of attraction of self-oscillation (1.4), if and only if there exist two functions \( V(x) \) and \( W(x) \) satisfying

(i) \( V(x) \) is defined on \( \Omega \) and continuous there;

\( W(x) \) is defined on the entire state space \( \mathbb{R}^n \),

(ii) \( V(x) = W(x) = 0 \) for all the points of curve (1.4),

(iii) \( V(x) \) will take positive values from the interval \((0,1)\) outside the curve (1.4), and \( W(x) > 0 \) for \( \rho(x) > 0 \);

(iv) \( \dot{x} = (VV) \cdot f(x) = -W(x) \sqrt{1 + \sum_{i=1}^{n} \xi_i^2(x)} (1 - V) \); \hspace{1cm} (1.6)

(v) \( V(x) \to 1 \) as \( x \) tends the boundary of \( \Omega \).

Zubov's approach is based solely upon the construction of a
Lyapunov function to satisfy partial differential equations of either the form (1.3) or (1.6). The difficulties in this approach rest on the following two facts: First, the partial differential equations (1.3) and (1.6) usually can not be solved in closed form despite the freedom in choice of $G(x)$ and $W(x)$; second, the periodic solution $x = \varphi(t)$ of (1.1) is usually unknown and not easy to be determined.

Szegö [28,29,30,31] made several generalizations of Zubov's results. In [28,29], he modified Zubov's method, and replaced Zubov's equation (1.3) by the following two partial differential equations

$$\frac{dV}{dt} = (\text{grad } V) \cdot f(x) = \frac{\psi(x)}{\beta(V)}$$  \hspace{1cm} (1.7)

$$\frac{dV}{dt} = (\text{grad } V) \cdot f(x) = -g(V)$$  \hspace{1cm} (1.8)

where $\psi(x)$ is negative definite on the trajectories of (1.1), $V(0) = 0$, and $\beta(V)$ and $g(V)$ are scalar functions satisfying respectively

1° $\int_0^V \beta(s) \, ds < \infty$, \hspace{1cm} (1.9)

2° $g(0) = 0$, $g(V(x)) \neq 0$ for all $x \notin M$, $M$ is the manifold on which $V(x) = 0$, $V(x) g(V) \geq 0$ in the whole space $\mathbb{R}^n$.

Thus for the first partial differential equation (1.7), the trivial solution $x = 0$ of (1.1) is asymptotically stable in a closed, bounded region $\Omega$ in which $\int_0^V \beta(s) \, ds > 0$ for all $x \neq 0$. If, in addition, $\int_0^V \beta(s) \, ds \to \infty$ as $\|x\| \to \infty$, then the solution $x = 0$ is asymptotically stable in the large.

For the second partial differential equation (1.8), the integral
manifold $M$ on which $V(x) = 0$ is asymptotically stable in the large. Szegő's approach is interesting because it brings in the idea of integrating Pfaffian forms to generate Liapunov functions.

In [30,31], Szegő further improved the method by imposing $\frac{dV}{dt}$ to be indefinite on a closed surface. He introduced the following concept.

**Definition 1.2 [30]** A scalar function $g(x) = \xi(x)$ is **indefinite on a closed surface**, if $\xi(x) = 0$ represents a closed bounded surface, and if the sign of $g$ inside $\xi(x) = 0$ is different from the sign of $g$ outside $\xi(x) = 0$.

In connection with this idea, Szegő stated the following theorem:

**Theorem 1.7 (Szegő [30]):** A sufficient condition for the local stability of the origin of (1.1) is that there exist a scalar function $V(x)$ with continuous first partial derivatives satisfying

(i) $V(x) > 0$ for all $x \neq 0$,

(ii) $\dot{V} = (\text{grad } V) \cdot f(x) = \psi(x) g[\xi(x)]$,

where $\psi(x)$ is a semidefinite function not identically equal to zero on any non-trivial solution of (1.1), and $g(\xi(x))$ is indefinite on a closed surface. Here $\xi(x) = 0$ is a closed bounded surface, or a family of closed surfaces, and $g(u)$ is such that $g(0) = 0$, and $g(u)/u > 0$.

It may well be profitable to present here a paraphrase of the preceding theorem.

Let $V_C$ denote the surface (belonging to the family $\mathcal{F}: V(x) = \text{constant}$) which is circumscribed to the surface $\xi(x) = 0$, and let $V_i$ be the surface in $\mathcal{F}$ which is inscribed to $\xi(x) = 0$. Then if we indicate by $V_C \geq V_C$ any surface of $\mathcal{F}$ outside $V_C$, and by
\[ 
\tilde{V}_i \leq V_i \text{ inside } V_i, \text{ then } \frac{d\tilde{V}}{dt} < 0 \text{ and } \frac{dV_i}{dt} < 0 \text{ will be semi-definite of different signs.}

In particular, if \( \frac{d\tilde{V}}{dt} \leq 0 \), and \( \psi(x) \to \infty \text{ as } \|x\| \to \infty \), then by the Lagrange stability theorem, all solutions of (1.1) are bounded in the future. If, in addition \( V_c = V_i \), then the surface \( V_c \) (or \( V_i \)) identifies exactly the globally stable limit set of the system (1.1).

This approach, however, still indicates the difficulties for high order systems. As mentioned by Szegö, in the case of systems of order higher than two with limit cycles, it is difficult to find a \( V(x) \) such that \( \dot{V}(x) \) is indefinite on a closed surface. Thus it is unlikely that the stability property of such systems may be determined by the preceding theorem.

1.2.3 The Moving Orthonormal System and Nonlocal Problems---

(The results of Diliberto, Hufford, Urabe, Pliss and de Figueiredo)

In order to investigate the behavior of a differential equation in the neighborhood of a given closed curve, it has been found convenient to use, instead of the original x-coordinate, the so-called moving orthonormal system which consists of one "parametric coordinate" measured along the closed curve, and n-1 "normal coordinates" measured in hyperplane perpendicular to the curve. Such a moving orthonormal system was first used by Poincaré. Among the applications of this system to nonlinear autonomous oscillations found in the literature hereafter, notable are those of Levinson [22], Diliberto and Hufford [12], Urabe [32] and Pliss [26].

In [22], Levinson required only the uniqueness of the transforma-
tion, and the \((n-1)\) normal coordinates are not necessarily measured in hyperplanes perpendicular to the curve. It seems that the existence of a moving orthonormal system was not proved for a general autonomous system until Diliberto and Hufford [12] established conditions for its existence and uniqueness for a continuous autonomous system satisfying a Lipschitz condition. Urabe [32] further derived such conditions for a continuous autonomous system without using the Lipschitz condition. Probably, the most important application of such a moving orthonormal system to the theory of nonlinear autonomous oscillations is found in the work of Pliss [26], who developed a test for the existence of a unique periodic solution just by means of this orthonormal system.

Consider the autonomous system \((1.1)\). Let \(x = \varphi(\theta)\) be a parametrically represented solution of the system \((1.1)\) that is located in a bounded region \(\Omega\) in which no equilibrium state exists. This implies that there is a constant \(\nu > 0\) such that

\[
\sum_{i=1}^{n} [\varphi'_i(\theta)]^2 > \nu \tag{1.10}
\]

holds in \(\Omega\). Since \(f(x)\) has continuous first partial derivatives in \(\Omega\), \(\varphi''(\theta)\) is bounded there. Hence the trajectory \(x = \varphi(\theta)\) has bounded curvature in \(\Omega\).

Then any point \(x\) of a trajectory \(x(t)\) lying near \(\varphi(\theta)\) can be represented by

\[
x_i = \varphi_i(\theta) + \sum_{j \neq 1}^{n-1} \xi_{ij}(\theta) y_j, \tag{1.11}
\]

where \(y_j, j = 1, ..., n-1\), are new orthogonal coordinates measured in the hyperplane normal to the trajectory \(x = \varphi(t)\). Since \(\varphi(\theta)\) has
bounded curvature, the hyperplanes corresponding to neighboring points on the trajectory will not intersect in a sufficiently small neighborhood of the trajectory. \( \zeta_{ij}(\theta) \) are functions of \( \varphi'_i(\theta) \) and \( \varphi_i(\theta) \), hence \( \zeta'_{ij}(\theta) \) are bounded for all \( \theta \geq 0 \).

The coordinates \( y_j, j = 1, \ldots, n-1 \), are selected in such a way that if

\[
(x - \varphi(\theta)) \cdot f(\varphi(\theta)) = 0,
\]

then

\[
\sum_{i=1}^{n} (x_i - \varphi_i(\theta))^2 = \sum_{j=1}^{n-1} y_j^2.
\]

(1.12)

(1.13)

The transformation (1.11) has replaced the variables \( (x_1, \ldots, x_n) \) by a set of new variables \( (\theta, y_1, \ldots, y_{n-1}) \). Differentiating (1.11) with respect to \( t \), and using (1.1) to eliminate \( x \) and \( \dot{x} \), one arrives at the following transformed equations:

\[
\frac{d\theta}{dt} = t(y, \theta)
\]

(1.14a)

\[
\frac{dy}{dt} = \tilde{g}(y, \theta)
\]

(1.14b)

It follows from (1.11) that the solution \( x = \varphi(t) \) of the system (1.1) is transformed into the solution \( \theta = t, y = 0 \) of system (1.14), so that on the trajectory \( x = \varphi(t) \), one has

\[
\tilde{\varphi}(0, \theta) \equiv 1, \quad \tilde{g}(0, \theta) \equiv 0.
\]

(1.15)

Next we would like to point out that the existence of periodic solutions and their stability in the large play an important role in the so-called nonlocal problems. Pliss has devoted extensively to this subject in [26].

In order to refer to these results, consider the fundamental system (1.
(1.1). Denote by $J_f(x)$ the Jacobian matrix of $f(x)$:

$$J_f(x) = \left( \begin{array}{c}
\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_2}{\partial x_n} \\
\vdots \\
\frac{\partial f_n}{\partial x_1}, \ldots, \frac{\partial f_n}{\partial x_n}
\end{array} \right), \quad (1.16)$$

and by $\|x\|$ the Euclidean norm of the vector $x$. If $p$ and $q$ are two elements of $X^n$, we will denote their distance $\|p-q\|$ by $\rho(p,q)$.

The results of Fliss are as follows:

**Theorem 1.8** [26, pp. 230]

Let $\Omega$ be a bounded region of $X^n$. If

(i) there exists a constant $\sigma > 0$ such that the inequality

$$(x-y)^T J_f(y) (x-y) \leq -\sigma \|x-y\|^2 \quad (1.17)$$

is satisfied, where $x \in \Omega$ and $y \in \Omega$ are two vectors such that

$$(x-y) \cdot f(y) = 0, \quad (1.18)$$

i.e. the difference $(x-y)$ is orthogonal to the direction field of (1.1) at the point $y$;

(ii) there is no critical point in $\bar{\Omega}$ (the closure of $\Omega$), then then

1) any solution $x = \varphi(p,t)$ in $\Omega$ is stable,

2) there exists a $\lambda > 0$ such that for every $\epsilon > 0$, one can find a $\delta > 0$ such that if $\rho(p,q) < \delta$, then there exists a $\eta(q)$, with $\|\eta(q)\| < \epsilon$, such that

$$\rho(\varphi(p,t + \eta(q)), \varphi(q,t)) < e^{-\lambda t} \quad (1.19)$$

holds.
Theorem 1.9 [26, pp. 238] If the conditions of Theorem 1.8 are satisfied, and there exists a solution $x = \varphi(q,t)$ of (1.1) in $\Omega$ for $t \geq 0$, with $\varphi(q,0) = q$, then $\Omega$ contains at least one periodic solution of (1.1).

Theorem 1.10 [26, pp. 238] If all the conditions of Theorem 1.8 are satisfied, assume that for any $p \in \hat{\Omega}$, the solution $\varphi(p,t)$ lies in $\Omega$ for $t > 0$. Then there exists a unique periodic solution of (1.1) which is asymptotically orbitally stable, and every solution approaches it as $t \to \infty$.

The preceding theorems (Theorems 1.8 - 1.10) have been developed altogether by means of a moving orthonormal system. In the subsequent chapter, new results are established via the same moving orthonormal system. These results are applicable to a wider class of nonlinear autonomous systems than those covered by Pliss's conditions. The approach followed represents an extension and generalization to high order systems of the method used by de Figueiredo in proving some of the results in [9,10,11].

1.2.4 Boundedness of Solutions by Liapunov's Method

The application of Liapunov's Direct Method to the problems of boundedness of solutions of a system has been extensively investigated by Yoshizawa [33]. In [17], LaSalle has also established independently a Lagrange stability theorem via a positive definite scalar function as mentioned in section 1.2.1. de Figueiredo [9,10,11] used an interesting Liapunov-like function to show the boundedness of all solutions of Liénard's equation while proving his existence theorems of a
periodic solution for Liénard's equation.

The approach they developed may, in principle, be used for any high order system, but in their illustrative examples, only second order systems can be found.

In chapter III of this dissertation, we present a new set of sufficient conditions for the boundedness of solutions in the future of a class of n coupled Liénard's equations (in three general forms). These results constitute an extension of Yoshizawa, LaSalle and de Figueiredo's work. The coupled systems represent the generalized equations for anode currents of n capacitively and/or resistively coupled anode oscillators. We give also an n-dimensional generalization of Zubov's third order example, and by using the above method, show that all solutions of this class of systems are bounded in the future.
CHAPTER II

CONDITIONS FOR THE EXISTENCE OF A UNIQUE
GLOBALLY STABLE LIMIT SET

2.1 Introduction

In this chapter, we consider the system

\[ \dot{x} = f(x) , \]  

(2.1)

where \( x \) and \( f \) are elements of the real n-space \( \mathbb{R}^n \), and \( f(x) \) has continuous first partial derivatives on \( \mathbb{R}^n \) with respect to the components of \( x \). We will establish sufficient conditions for the existence of a unique globally asymptotically orbitally stable limit set of the system (2.1). For this purpose, we will rely on a positive definite Liapunov-like function \( V(x) \). But we impose a set of conditions on function \( f(x) \) and its first partial derivatives which remove the requirement that \( \dot{V}(x) \leq 0 \).

2.2 On a Globally Asymptotically Orbitally Stable Limit Set

In what follows, \( J_f(x) \) will still denote the Jacobian matrix of the function \( f(x) \), \( E \) is the unit matrix, and \( V = V(x) > 0 \forall x \neq 0, V(0) = 0 \) is a Liapunov-like function.

Our main result is

Theorem 2.1 If

(i) all solutions of (2.1) tend to a bounded region \( \Omega \subset \mathbb{R}^n \) as \( t \to \infty \),

(ii) there is no critical point in the region \( \Omega \),
(iii) there exists a constant \( \sigma > 0 \) such that the inequality
\[
(x-y)^T \frac{\dot{V}(x)}{V(x)} E - 2f(x) \cdot (x-y) \geq \sigma \|x-y\|^2
\]
holds in \( \Omega \), where \( x \in \Omega \) and \( y \in \Omega \) are two vectors such that
\[
(x-y) \cdot f(y) = 0,
\]
then there exists a unique globally asymptotically orbitally stable limiting trajectory of (2.1).

**Proof:** In what follows, the statements up to (2.15)b are taken from Pliss [26], and equations included here are used merely as an introduction to the rest of the proof. Consider a trajectory \( x = \varphi(p,t) \), with \( \varphi(p,t) = p \). By hypothesis, the trajectory \( x = \varphi(p,t) \) is confined in the bounded region \( \Omega \) as \( t \to \infty \) which is free of critical points. This together with the continuously differentiability \( p \) property of \( f(x) \) implies that the transformation (1.11), i.e.
\[
x_i = \varphi_1(\theta) + \sum_{j=1}^{n-1} \zeta_{ij}(\theta) y_j,
\]
is valid for the trajectory \( x = \varphi(p,t) \), where the \( (n-1) \) tuple of real numbers \( (y_1, \ldots, y_{n-1}) \) are measured in the hyperplane perpendicular to the trajectory \( x = \varphi(p,t) \) at the point \( x = \varphi(\theta) \), and they are, as in Section 1.2.3, selected so that if
\[
(x - \varphi(\theta)) \cdot f(\varphi(\theta)) = 0,
\]
then
\[
\sum_{i=1}^{n} (x_i - \varphi_i(\theta))^2 = \sum_{j=1}^{n-1} y_j^2.
\]
According to the transformed equations (1.14), if we set
\[
\theta = t + \tau,
\]
then we have
\[
\frac{dt}{dt} = \Theta(y, \tau, t)
\]  \hspace{1cm} (2.8a)
\[
\frac{dy}{dt} = g(y, \tau, t),
\]  \hspace{1cm} (2.8b)
and (1.15) becomes
\[
\Theta(0, \tau, t) = 0, \quad g(0, \tau, t) = 0.
\]  \hspace{1cm} (2.9)
The first identity of (2.9) implies that for a \( \xi > 0 \), there is a \( \delta > 0 \) such that
\[
|\Theta(y, \tau, t)| < \xi \|y\|
\]  \hspace{1cm} (2.10)
is satisfied when \( \|y\| < \delta \). It implies from the second identity of (2.9) that the function \( g_i(y, \tau, t) \) can be written as
\[
g_i(y, \tau, t) = a_{i1}(t) y_1 + \cdots + a_{in-1}(t)y_{n-1} + A_i(y, \tau, t),
\]  \hspace{1cm} (2.11)
where \( a_{ij}(t); i, j = 1, \ldots, n-1 \), are continuous and bounded for all \( t \geq t_0 \), and
\[
|A_i(y, \tau, t)| < \gamma \|y\| \quad i = 1, \ldots, n-1
\]  \hspace{1cm} (2.12)
holds for sufficiently small \( |\tau| \) and \( \|y\| \), and \( \gamma \) is a constant which can be made arbitrarily small by selecting \( |\tau| \) and \( \|y\| \) sufficiently small. Hence there is a \( \delta > 0 \) such that
\[
|\tau| < \delta, \quad \|y\| < \delta
\]  \hspace{1cm} (2.13)
implies both (2.12) and
\[
\gamma < \frac{\sigma}{16(n-1)}.
\]  \hspace{1cm} (2.14)
Thus the system (2.8) is rewritten as
\[
\frac{dt}{dt} = \Theta(y, \tau, t),
\]  \hspace{1cm} (2.15a)
\[
\frac{dy_i}{dt} = a_{i1} y_1 + \ldots + a_{in-1} y_{n-1} + A_i(y, \tau, t). \tag{2.15b}
\]

Let us consider a function of the form
\[
W = \sum_{i=1}^{n-1} \frac{y_i^2}{V(x)} \tag{2.16a}
\]
\[
= \sum_{i=1}^n \left( x_i - \varphi_i(t+\tau) \right)^2 / V(x), \tag{2.16b}
\]

with which, up to equation (2.33), we will follow a procedure similar to the one used by Pliss.

Upon using system (2.15), we obtain the time derivative of (2.16a) as:
\[
\dot{W} = \frac{1}{V^2(x)} \left(2V(x) \sum_{i=1}^{n-1} y_i \left( a_{i1} y_1 + \ldots + a_{in-1} y_{n-1} \right) + 2V \sum_{i=1}^{n-1} y_i A_i \right)
\]
\[
- \dot{V}(x) \sum_{i=1}^{n-1} y_i^2 \right). \tag{2.17}
\]

On the other hand, it follows from (2.16b) that
\[
\dot{W} = \frac{1}{V^2(x)} \left(2V(x) \sum_{i=1}^{n-1} \left( x_i - \varphi_i(t+\tau) \right) \left( f_i(x) - \dot{f}_i(\varphi(t+\tau)) \right) \left(1 + \frac{dt}{dt-1} \right) \right)
\]
\[
- \dot{V}(x) \sum_{i=1}^n \left( x_i - \varphi_i(t+\tau) \right)^2 \right). \tag{2.18}
\]

Since the vector \( x - \varphi(t+\tau) \) is orthogonal to \( f(\varphi(t+\tau)) \), (2.18) can also be written as
\[
\dot{W} = \frac{1}{V^2(x)} \left(2V(x) \sum_{i=1}^n \left( x_i - \varphi_i(t+\tau) \right) \left( f_i(x) - \dot{f}_i(\varphi(t+\tau)) \right) \right)
\]
\[
- \dot{V}(x) \sum_{i=1}^n \left( x_i - \varphi_i(t+\tau) \right)^2 \right). \tag{2.19}
\]

Letting
\[
x^* = \varphi(t+\tau) + \mu(x - \varphi(t+\tau)), \quad 0 \leq \mu \leq 1, \tag{2.20}
\]
then we have
\[ f_i(x) - f_i(\varphi(t+\tau)) = \sum_{j=1}^{n} \int_{0}^{1} \frac{\partial f_i(x^*)}{\partial x_j} (x_j - \varphi_j(t+\tau)) \, du. \tag{2.21} \]

It then yields from (2.19) that
\[
\dot{W} = \frac{1}{\sqrt{V(x)}} \left( 2V(x) \sum_{i=1}^{n} (x_i - \varphi_i(t+\tau)) \right) \sum_{j=1}^{n} \int_{0}^{1} \frac{\partial f_i(x^*)}{\partial x_j} (x_j - \varphi_j(t+\tau)) \, du \\
- \dot{V}(x) \sum_{i=1}^{n} (x_i - \varphi_i(t+\tau))^2 \right)
\]
\[ = - \frac{1}{\sqrt{V(x)}} (x - \varphi(t+\tau))^T \left[ \dot{V}(x) E - 2V(x) \sum_{i=1}^{n} J_i(x^*) \, du \right] (x - \varphi(t+\tau)) \]
\[ = - \frac{1}{V(x)} (x - \varphi(t+\tau))^T \left[ \dot{V}(x) E - 2J_e(x) + 2 \sum_{i=1}^{n} \left[ J_i(x) - J_i(x^*) \right] \, du \right] (x - \varphi(t+\tau)). \tag{2.22} \]

Since \( f(x) \) is continuously differentiable, the matrix \( J_e(x) \)
is continuous in \( \Omega \); moreover, \( J_e(x) - J_e(x^*) \) vanishes if \( x - \varphi(t+\tau) = 0 \),there exists a constant \( \delta > 0 \) such that
\[ \|x - \varphi(t+\tau)\| < \delta \tag{2.23} \]
implies
\[ \|2(x - \varphi(t+\tau))^T \left[ \sum_{i=1}^{n} \left[ J_i(x) - J_i(x^*) \right] \, du \right] (x - \varphi(t+\tau))\| < \frac{\sigma}{2} \|x - \varphi(t+\tau)\|^2 \tag{2.24} \]
Hence by the condition (iii) of the theorem, we have the inequality
\[ \dot{W} \leq - \frac{\sigma}{2V(x)} \|x - \varphi(t+\tau)\|^2 = - \frac{\sigma}{2V(x)} \|y\|^2 < 0 \tag{2.25} \]
provided \( \|x - \varphi(t+\tau)\| = \|y\| < \delta \).

Let
\[ \delta = \min \{ \delta, \, \delta, \, \delta \} \tag{2.26} \]
By (2.12), (2.14), (2.17) and (2.25), we find that
\[ \frac{1}{V(x)} \left[ 2 \sum_{i=1}^{n} \int_{0}^{1} \left( a_{i1}y_1 + \cdots + a_{in-1}y_{n-1} \right) - \frac{\dot{V}(x)}{V(x)} \sum_{i=1}^{n} y_i^2 \right] < - \frac{3\sigma}{8V(x)} \|y\|^2 \tag{2.27} \]
At this stage, it is convenient to introduce a new variable $z$ as follows:

$$z = e^{\lambda t} y, \quad \lambda > 0,$$

(2.28)

then according to (2.15)b, we have

$$\frac{dz_i}{dt} = a_{ii} z_i + \ldots + \left(a_{ii} \Delta \lambda\right) z_i + \ldots + a_{in-1} z_{n-1} + e^{\lambda t} A_i (e^{-\lambda t} z_i, \tau, t)$$

(2.29)

and for $|\tau| < \delta$, $||z|| < \delta$,

$$|A_i (e^{-\lambda t} z_i, \tau, t)| < \frac{\sigma}{16(n-1)} ||z|| e^{-\lambda t}.$$  

(2.30)

By (2.29), it follows

$$\frac{d}{dt} \left( \sum_{i=1}^{n-1} \frac{z_i^2}{V(x)} \right) = \frac{1}{V(x)} \left( 2V(x) \sum_{i=1}^{n-1} \frac{dz_i}{dt} - V(x) \sum_{i=1}^{n-1} \frac{z_i^2}{V(x)} \right)$$

$$= \frac{1}{V(x)} \left( 2V(x) \sum_{i=1}^{n-1} \left[ a_{ii} z_i + \ldots + \left( a_{ii} \Delta \lambda\right) z_i + \ldots + a_{in-1} z_{n-1} \right.$$

$$\left. + e^{\lambda t} A_i \right] - V \sum_{i=1}^{n-1} \frac{z_i^2}{V} \right)$$

$$= \frac{1}{V(x)} \left( 2 \sum_{i=1}^{n-1} \left[ a_{ii} z_i + \ldots + a_{in-1} z_{n-1} \right] + 2 \lambda \sum_{i=1}^{n-1} \frac{z_i^2}{V} + 2 \sum_{i=1}^{n-1} \frac{z_i e^{\lambda t} A_i}{V} \right.$$ 

$$\left. - \frac{V}{n-1} \sum_{i=1}^{n-1} \frac{z_i^2}{V} \right).$$

(2.31)

If $|\tau| < \delta$, $||z|| < \delta$, then from (2.27), (2.29) and (2.30), we obtain

$$\frac{d}{dt} \left( \sum_{i=1}^{n-1} \frac{z_i^2}{V(x)} \right) < - \frac{3\sigma}{8V} ||z||^2 + \frac{\sigma}{8V} ||z||^2 + \frac{2\lambda}{V} ||z||^2$$

$$= - \frac{\sigma}{4V} ||z||^2 + \frac{2\lambda}{V} ||z||^2.$$  

(2.32)

Hence if $\lambda$ is chosen to be $0 < \lambda < \frac{\sigma}{16}$, then we have
\[
\frac{d}{dt} \left( \sum_{i=1}^{n-1} \frac{z_i^2}{V(x)} \right) < \frac{\sigma}{8V} \|z\|^2 < 0
\]  \hspace{1cm} (2.33)

Provided \(|\tau| < \delta, \|z\| < \delta\).

Consider an arbitrary solution of (2.1) with initial conditions
\[
|\tau(t_o)| < \epsilon_0, \|z(t_o)\| < \epsilon_0,
\]  \hspace{1cm} (2.34)

where \(\epsilon_0 < \delta\) will be specified later.

Due to continuity of \(x(t)\) and \(\varphi(t)\), there exists a positive interval \(I = (t_1, t^*)\) such that if (2.34) is given, then
\[
|\tau(t)| < \delta, \|z(t)\| < \delta
\]  \hspace{1cm} (2.35)

hold for \(t \in I\). Hence in this interval, the inequalities (2.25) and (2.33) also hold.

Let \(\Omega\) be the bounded region defined in the hypothesis. If we let
\[
N = \sup_{x \in \Omega} V(x)
\]  \hspace{1cm} (2.36a)

and
\[
\epsilon = \frac{\frac{1}{2} \epsilon_0}{V(x(t_o))},
\]  \hspace{1cm} (2.36b)

then from (2.33), one deduces the inequality
\[
\|z\| < \epsilon, \quad \text{for } t \in I.
\]  \hspace{1cm} (2.37)

And by (2.28), we have
\[
\|y\| < \epsilon e^{-\lambda t}, \quad \text{for } t \in I.
\]  \hspace{1cm} (2.38)

According to (2.10),
\[
|\Theta(y, \tau, t)| < \xi \epsilon e^{-\lambda t}, \quad \text{for } t \in I,
\]  \hspace{1cm} (2.39)

by the first equation of (2.8), we obtain the following integral
equation:
\[
\begin{align*}
\tau(t) & \neq \tau(t_o) + \int_{t_o}^{t} \Theta(y, \tau, t) dt \\
& < |\tau(t_o)| + \frac{\delta \epsilon}{\lambda} (e^{-\lambda t_o} - e^{-\lambda t}) \\
& < \epsilon_o + \frac{\delta \epsilon}{\lambda} e^{-\lambda t_o} = \epsilon_o [1 + \frac{\delta \epsilon}{\lambda} \frac{N^\frac{1}{2}}{V^2(t_o)} e^{-\lambda t_o}] \equiv \bar{\eta}.
\end{align*}
\]

(2.40)

Hence if we select \( \epsilon_o \) so small that
\[
\bar{\eta} = \epsilon_o [1 + \frac{\delta \epsilon}{\lambda} \frac{N^\frac{1}{2}}{V^2(t_o)} e^{-\lambda t_o}] < \delta
\]

\[
\epsilon = \frac{\delta \epsilon}{\lambda} \frac{N^\frac{1}{2}}{V^2(t_o)} < \delta,
\]

(2.41)

then (2.35) is satisfied for \( t \in I \). On the other hand, in view of (2.36) and (2.40), \( \tau(t) \) and \( \|y(t)\| \) are bounded for all \( t \geq t_o \).

Hence by choosing \( \epsilon_o \) sufficiently small to guarantee the inequality (2.41), then (2.35) (which is required for our approximation in (2.25) and (2.33) to be valid) will hold for all \( t \geq t_o \). Therefore the inequalities (2.25) and (2.33) are satisfied for all \( t \geq t_o \) provided such a \( \epsilon_o \) is to be selected as the initial condition in (2.34).

Returning to (2.25), let
\[
\frac{\|y\|^2}{V} = \psi.
\]

(2.42)

Since (2.25) holds for all \( t \geq t_o \), \( \psi(t) \) decreases monotonically with \( t \), and
\[
\lim_{t \to \infty} \psi(t) = \psi_o \geq 0.
\]

(2.43)

Suppose \( \psi_o > 0 \), then according to (2.42), \( \frac{\|y\|^2}{V} \neq 0 \) at \( \psi_o \).
While by (2.25), $\frac{d}{dt} \psi(t) < 0$ for $\|y\|^2/V \neq 0$. So $\psi(t)$ can not remain stationary w.r.t. time at $\psi_o$. This contradicts our supposition that $\psi_o > 0$. From this and (2.43), we conclude that $\psi_o = 0$.

By the first condition of the theorem, it follows that $V(x)$ is bounded for all $t \geq t_o$. Hence $\psi_o = 0$ implies that

$$\lim_{t \to \infty} \|y\|^2 = 0,$$  \hspace{1cm} (2.44)

or what is the same (see (2.6))

$$\lim_{t \to \infty} \|x - \varphi(t + T)\| = 0$$  \hspace{1cm} (2.45)

Hence we conclude that the trajectory $\varphi(t)$ is approached asymptotically orbitally by every other trajectory sufficiently near to $\varphi(t)$.

We note that this applies to any other trajectory in $\Omega$, by the continuity of the solution of (2.1) w.r.t. the initial conditions, it implies that all trajectories approach one another as $t \to \infty$, and hence they can only tend to a single trajectory.

This proves that, under the conditions of the theorem, there exists a unique globally asymptotically orbitally stable limiting trajectory of (2.1) which is approached by every other trajectory as $t \to \infty$.

Q.E.D.

Remark 1. In the formulation of the preceding theorem, a positive definite Liapunov-like function $V(x)$ has been used. Since we made no restrictions on the form of $V(x)$, any suitable positive definite function can be used as a realization of the function $V(x)$. In par-
ticular, a positive definite function \( V(x^{(l)}) \) w.r.t. \( x^{(l)} = (x_1, \ldots, x_l) \) defined as

\[
V(x^{(l)}) = \sum_{i=1}^{l} x_i^2 \quad 2 \leq l \leq n
\]

(2.46)
can be adopted as a candidate for the Liapunov-like function in the preceding theorem. Thus we have

**Corollary 2.1** The result of Theorem 2.1 holds if all the conditions of that theorem are satisfied except that \( V(x) \) is replaced by \( V(x^{(l)}) \). Or equivalently, (2.2) is replaced by (2.47) below:

\[
(x-y)^T \left[ \sum_{i=1}^{l} \frac{i-1}{l} E - J_f(x) \right] (x-y) \geq \frac{\sigma}{2} \|x-y\|^2, \quad 2 \leq l \leq n.
\]

(2.47)

**Remark 2.** We note from the preceding corollary that, if we let \( V(x) = 1 \), then Theorem 2.1 reduces to Pliss's Theorems 1.8 - 1.10.
CHAPTER III

ON THE BOUNDEDNESS OF SOLUTIONS OF TWO GENERAL
CLASSES OF MULTIDIMENSIONAL NONLINEAR AUTONOMOUS SYSTEMS

3.1 Introduction

In the present chapter, we study the boundedness of solutions of
two classes of n-dimensional nonlinear autonomous systems, namely, (a)
a system of n coupled Liénard's equations (in three general forms),
and (b) a n-dimensional generalization of Zubov's third order example.
As pointed out in chapter I, equations of this type have physical
relevance, and the methods of our theorems represent an extensions of
those used with two-dimensional systems by Yoshizawa [33], LaSalle [17],
and de Figueiredo [9, 10, 11].

3.2 n Coupled Liénard's Equations

3.2.1 A Class of Coupled Liénard's Equations (Coupling Through
the Spring Term)

Let us now first consider a system of n coupled Lienard's equa-
tions of the form

\[ \ddot{x}_i + f_i(x_i) \dot{x}_i + g_i(x) = 0 \quad (3.1) \]
\[ x = (x_1, \ldots, x_n); \ i = 1, \ldots, n \]

where \( g_i(x) = \frac{\partial U(x)}{\partial x_i} \) and \( U(x) \) is a positive definite potential
function, and both \( f_i \) and \( g_i \) are continuous and satisfy a condition
(e.g. a Lipschitz condition) to assure local uniqueness of the solu-
tions of (3.1) throughout the \((x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)\) space.
System (3.1) is equivalent to

\[
\dot{x}_i = y_i \quad i = 1, \ldots, n. \quad (3.2)
\]

\[
\dot{y}_i = -g_i(x) - f_i(x_i) y_i
\]

For arbitrary positive constants \(\alpha, \gamma\) and \(\sigma\), define

\[
M_i(\alpha) = \sup_{x} \left| \int_0^{x_i} f_i(s) \, ds \right| = \sup_{x} |F_i(x_i)|, \quad \text{for} \ |x_i| < \alpha, \quad (3.3)
\]

\[
\gamma_j = \sup_{\nabla x} \left[ - \sum_{i=1}^{n} g_i(F_i - \frac{\sigma}{\alpha} x_i) \right] \quad (3.4a)
\]

\[
\gamma = \sup_{\nabla x} \left[ - \sum_{i=1}^{n} g_i(F_i - \frac{\sigma}{\alpha} x_i) \right] \quad (3.4b)
\]

\[
N_j = \sum_{i=1}^{n} \left[ \frac{\sigma}{4\alpha} [M_i(\alpha) + \sigma]^2 \right] + \gamma_j \quad (3.5a)
\]

\[
\hat{N}_j = \sum_{i=1}^{n} \frac{\sigma}{4\alpha} [M_i(\alpha) + \sigma]^2 + \gamma \quad (3.5b)
\]

\[
K = \max_{1 \leq i \leq n} \left( \frac{1}{2} \left[ (M_j(\alpha) + \sigma)^2 + \sqrt{(M_j(\alpha) + \sigma)^2 + 4\alpha \hat{N}_j / \sigma} \right] \right) \quad (3.6)
\]

We assert:

**Theorem 3.1** All solutions of (3.2) (hence, of (3.1)) are bounded for \(t \geq 0\) provided the following conditions are satisfied:

(i) \(U(x) \to \infty\) as \(\|x\| \to \infty\);

(ii) for a fixed constant \(\sigma > 0\), there is an \(\hat{\alpha} > 0\) such that

\[
F_i(x_i)/x_i > \sigma/|x_i| \quad \text{for} \ |x_i| > \hat{\alpha} \quad i = 1, \ldots, n,
\]

(iii) \(x_i g_i(x) > 0\) for \(|x_i| > \hat{\alpha}\) and \(\nabla x_j, j \neq i, i = 1, \ldots, n,
\]

(iv) there is an \(\alpha > \hat{\alpha}\) such that

\[
g_j(x) [F_j(x_j) - \sigma] > N_j, \quad j = 1, \ldots, n,
\]

"-" for \(x_j \geq \alpha\)

"+" for \(x_j \leq -\alpha\), and \(\nabla x_i, i \neq j\).
Proof. Under these conditions, the boundedness of all solutions of (3.2) in the future will be established by selecting a suitable scalar function $W(x,y)$ which satisfies the requirements of the Lagrange stability theorem (this theorem is quoted in chapter I).

Let

$$\Omega = \{(x,y) : |x_i| \leq \alpha, |y_i| \leq \beta, \ i = 1, \ldots, n\},$$

where $\alpha$ is defined in condition (iv) and $\beta$ is a positive constant such that

$$\beta > K.$$  \hspace{1cm}(3.8)

In $\Omega^c$, the complement of $\Omega$, choose

$$W(x,y) = U(x) + \frac{1}{2} \sum_{i=1}^{n} [y_i + F_i(x_i) - h_i(x_i)]^2$$  \hspace{1cm}(3.9)

where

$$h_i(x_i) = \begin{cases} 
\sigma & x_i \geq \alpha \\
\sigma x_i / \alpha & |x_i| \leq \alpha \\
-\sigma & x_i \leq -\alpha 
\end{cases} \quad i = 1, \ldots, n$$ \hspace{1cm}(3.10)

According to (3.2), the total time derivative of $W(x,y)$ is

$$\dot{W}(x,y) = -\sum_{i=1}^{n} h_i'(x_i) y_i [y_i + F_i(x_i) - h_i(x_i)]$$

$$- \sum_{i=1}^{n} g_i(x)[F_i(x_i) - h_i(x_i)].$$  \hspace{1cm}(3.11)

We denote by $A_j$, $j = 1, \ldots, n$, the regions

$$|x_j| \geq \alpha; \ |x_i| \leq \alpha, \ i = 1, \ldots, n, \ i \neq j$$

$$|y_i| \leq \beta, \ i = 1, \ldots, n$$  \hspace{1cm}(3.12)

and by $B_j$, $j = 1, \ldots, n$, the regions

$$|y_j| \geq \beta; \ |x_i| \leq \alpha, \ i = 1, \ldots, n;$$

$$|y_i| \leq \beta, \ i = 1, \ldots, n, \ i \neq j.$$  \hspace{1cm}(3.13)
It is not difficult to see, from (3.11), that if $\dot{W} < 0$ in all of the regions $A_j$ and $B_j$, $j = 1, \ldots, n$, then $\dot{W} < 0$ in $\Omega^c$. [Note that

$$\left( \bigcup_{j=1}^{n} A_j \right) \cup \left( \bigcup_{j=1}^{n} B_j \right) \subset \Omega^c.$$  

(3.14)

Let us now consider the regions $A_j$, $j = 1, \ldots, n$. $\dot{W}(x,y)$ in $A_j$ takes the form

$$\dot{W} = -\sum_{i=1}^{n} \frac{\sigma}{\alpha} y_i (y_i + F_i - \frac{\sigma}{\alpha} x_i) - \sum_{i=1}^{n} g_i (F_i - \frac{\sigma}{\alpha} x_i) - g_j (F_j - h_j).$$  

(3.15)

Conditions (ii) - (iv) then guarantee that $\dot{W} < 0$ in $A_j$.

Indeed

$$\dot{W} \leq \sum_{i=1}^{n} \frac{\sigma}{\alpha x_i} [M_i(\alpha) + \sigma]^2 + \gamma_j - g_j (F_j + \sigma) < 0$$  

in $A_j$. Where "-" applies for $x_j \geq \alpha$ and "+" for $x_j \leq -\alpha$.

Let us consider next the regions $B_j$, $j = 1, \ldots, n$. In these regions,

$$\dot{W} = -\sum_{i=1}^{n} \frac{\sigma}{\alpha} y_i (y_i + F_i - \frac{\sigma}{\alpha} x_i) - g_i (F_i - \frac{\sigma}{\alpha} x_i) - \frac{\sigma}{\alpha} \gamma_j (y_j + F_j - \frac{\sigma}{\alpha} x_i)$$  

(3.16)

According to (3.8) and (3.13), $\dot{W} < 0$ in $B_j$, $j = 1, \ldots, n$.

Hence $\dot{W} < 0$ in $\Omega^c$. By the Lagrange stability theorem, we then conclude that all solutions of (3.2) (or equivalently, of (3.1)) are bounded for $t \geq 0$.

Q.E.D.

The preceding condition on $F_i(x_i)$ may be relaxed to include cases where the system remains negatively damped for $x_i$ of one sign of
large magnitude, such as in the case of two stroke oscillations [9, 10, 11, 20]. Thus:

**Theorem 3.2**: The result of Theorem 3.1 holds if conditions (ii) and (iv) are replaced by (ii)' and (iv)' below:

(ii)' for arbitrary constants $\sigma$ and $\hat{\xi} > 0$, there is an $\hat{\alpha} > 0$

such that

\[
    F_i(x_i) > \sigma \quad \text{for } x_i > \hat{\alpha} \\
    F_i(x_i) < \hat{\xi} \quad \text{for } x_i < -\hat{\alpha}
\]

(iv)'

there is an $\alpha \geq \hat{\alpha}$ and $\hat{\xi} \geq \hat{\xi}$ such that

\[
    g_j(x) [F_j(x_j) - \varphi] > N_j(\varepsilon, \gamma, \kappa)
\]

where $\varphi = \sigma$ if $x_j \geq \alpha$, $\varphi = \hat{\xi}$ if $x_j \leq -\alpha$; and $x_i, i \neq j$.

**Proof.** The above result is proved in much the same way as Theorem 3.1 except that the functions $h_i(x_i)$ are now replaced by

\[
h_i(x_i) = \begin{cases} 
    \sigma & x_i \geq \alpha \\
    \sigma x_i / \alpha & |x_i| \leq \alpha \\
    \xi & x_i \leq -\alpha
\end{cases}
\]

(3.17)

3.2.2 Another Class of Coupled Liénard's Equations (Coupling Through Both the Spring and Damping Terms)

Another interesting general class of coupled Liénard's equations is characterized by a system of the form

\[
    \ddot{x}_i + (\text{grad } F_i(x)) \cdot \dot{x} + g_i(x) = 0, \quad i = 1, \ldots, n
\]

(3.18)

where $g_i(x) = \partial U(x)/\partial x_i$ and $U(x)$ is a positive definite function.

This system differs from (3.1) only as regards the damping term which, in the present case, provides additional coupling.

We now state without proof:
Theorem 3.3 If the conditions (ii) and (iv) of Theorem 3.1 are replaced by the conditions (ii)' and (iv)' below, then the result of that theorem is valid:

(ii)' for a fixed constant $\sigma > 0$, there is an $\hat{\alpha} > 0$ such that

$$F_i(x)/x_i > \sigma/|x_i|\quad \text{for}\quad |x_i| > \hat{\alpha}, \ i = 1, \ldots, \ n,$$

(iv)' there is an $\alpha \geq \hat{\alpha}$ such that

$$g_j(x) [F_j(x) \mp \sigma] > N_j(\alpha, \gamma, \omega) \quad j = 1, \ldots, \ n$$

"-" for $x_j \geq \alpha$

"+" for $x_j \leq -\alpha$, and $\forall \ x_i, \ i \neq j$.

3.2.3 A Class of n Coupled Rayleigh Equations

Let us turn our attention next to the $n$ coupled equations of the following forms

$$\ddot{x}_i + F_i(\dot{x}_i) + g_i(x) = 0, \ i = 1, \ldots, \ n \quad (3.19)$$

where $g_i$ is defined in the same way as before.

It may be noted that if the coupling term is linear, then it is possible to transform (3.1) to (3.19). We will now establish a variant of Theorem 3.1 applicable to the equation (3.19).

In the present case, for some positive constants $\beta$, $\hat{\beta}$ and $\sigma$, let

$$M_i(\beta) = \sup |F_i(\dot{x}_i)| \quad \text{for} \quad |\dot{x}_i| \leq \beta, \ i = 1, \ldots, \ n \quad (3.20)$$

$$K_i(\beta, \sigma) = \max [-\frac{\sigma}{\beta} x_i [g_i(x) \mp M_i(\beta)]], \ \forall \ x_i, \ \ldots, \ x_n \quad (3.21)$$

$$N_j(\beta, \sigma) = \sum_{i=1}^{n} K_i + 2 \beta [M_i(\hat{\beta}) + \sigma], \ j = 1, \ldots, \ n \quad (3.22)$$

$$\hat{N}_j(\beta, \sigma) = \sum_{i=1}^{n} K_i + \sum_{i=1}^{n} \beta [M_i(\hat{\beta}) + \sigma], \ j = 1, \ldots, \ n \quad (3.23)$$
We then have:

**Theorem 3.4** Every solution of (3.19) is bounded in the future provided

(i) **for a given constant** $\sigma > 0$,

$$U(x) + \sigma \sum_{i=1}^{n} x_i \to +\infty \text{ as } \|x\| \to \infty,$$

(ii) there is an $\hat{\beta} > 0$ such that

$$\frac{F_i(\dot{x}_i)}{\dot{x}_i} > \frac{\sigma}{|\dot{x}_i|} \quad \text{for} \quad |\dot{x}_i| > \hat{\beta}, \quad i = 1, \ldots, n$$

and $\forall \dot{x}_j, \ j \neq i$,

(iii) there is $\beta \geq \hat{\beta}$ such that

$$\ddot{x}_j [F_j(\dot{x}_j) \pm \sigma] > N_j(\hat{\beta}, \sigma), \quad "-" \text{ for } \dot{x}_j \geq \beta$$

$$"+" \text{ for } \dot{x}_j \leq \beta \beta$$

and $\forall \dot{x}_k, \ k \neq j$,

(iv) $x_i g_i(x) > 0 \quad \forall x_1, \ldots, x_n$,

(v) there is an $\hat{\alpha} > 0$ such that

$$|g_i(x)| > N_i(\beta) \quad \text{for} \quad |x_i| > \hat{\alpha}, \quad i = 1, \ldots, n,$$

and $\forall x_j, \ j \neq i$.

**Proof.** The proof of this theorem is similar to that of Theorem 3.1. We will simply point out the changes that need be made in some of the equations and inequalities.

Instead of (3.2), we have

$$\dot{x}_i = y_i$$

$$\dot{y}_i = -g_i(x) - F_i(y_i), \quad i = 1, \ldots, n.$$

(3.24)

The region $\Omega$ is defined as in (3.7). But now $\beta$ is defined in the condition (iii) of the present theorem, and $\alpha > \hat{\alpha}$ is such that
the following inequalities
\[
\frac{\sigma}{\beta} x_j (g_j + F_j) > \hat{N}_j (\hat{\beta}, \sigma), \quad j = 1, \ldots, n
\] (3.25)
are satisfied for the regions \( A_j, j = 1, \ldots, n \).

The scalar function \( W(x, y) \) assumes the form
\[
W(x, y) = U(x) + \frac{\lambda}{2} \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} x_i h_i (y_i)
\] (3.26)
where
\[
h_i (y_i) = \begin{cases} 
\sigma & y_i \geq \beta \\
\frac{\sigma y_i}{\beta} & |y_i| \leq \beta \\
\sigma & y_i \leq -\beta
\end{cases} \quad i = 1, \ldots, n,
\] (3.27)
and the corresponding \( \dot{W}(x, y) \) is
\[
\dot{W}(x, y) = -\sum_{i=1}^{n} h_i (y_i) x_i [g_i (x) + F_i (y_i)] - \sum_{i=1}^{n} y_i [F_i (y_i) - h_i (y_i)]
\] (3.28)
Q.E.D.

An extension of this theorem similar to that embodied by Theorem

3.3 can readily be made.

3.3 A n-dimensional Generalization of Zubov's Example

Zubov's example of a third order system [35, pp. 203] may be

generalized to a class of n-dimensional systems defined by the set of

equations:
\[
\begin{aligned}
\dot{x}_1 &= \alpha x_1 + x_2 + \ldots + x_{n-1} - x_1 \sum_{i=1}^{n-1} x_i^2 \\
\dot{x}_2 &= -x_1 + \alpha x_2 + \ldots + x_{n-1} - x_2 \sum_{i=1}^{n-1} x_i^2 \\
&\ldots \ldots \ldots \\
\dot{x}_{n-1} &= -x_1 - x_2 - \ldots + \alpha x_{n-1} - x_{n-1} \sum_{i=1}^{n-1} x_i^2 \\
\dot{x}_n &= -x_n
\end{aligned}
\] (3.29)
where \( \alpha > 0 \).
For this system, we try a scalar function

\[ \dot{W}(x) = \frac{1}{\epsilon} \sum_{i=1}^{n} x_i^2. \]  

(3.30)

From (3.29), we have

\[ \dot{W}(x) = \left( \sum_{i=1}^{n-1} x_i^2 \right) \left[ \alpha - \left( \sum_{i=1}^{n-1} x_i^2 \right) \right] - x_n^2. \]  

(3.31)

It is then not difficult to see that \( W > 0 \) and \( \dot{W} < 0 \) in the region \( \Omega^c \), where \( \Omega \) is defined as

\[ \Omega^c : \quad \begin{align*}
  x_1^2 + \ldots + x_{n-1}^2 &< \alpha + \epsilon \\
  x_n^2 &< \frac{\alpha^2}{\epsilon} + \epsilon
\end{align*} \]  

(3.32)

\( \epsilon \) being a sufficiently small positive constant. Hence the system (3.29) is Lagrange stable.

An example of this class will be discussed in the following chapter.
CHAPTER IV

SPECIAL CASES AND EXAMPLES OF BOUNDEDNESS
OF SOLUTIONS AND UNIQUENESS OF A STABLE LIMIT SET

4.1 Introduction

The examples that follow are used to illustrate the applications of the stability and boundedness criteria developed in the previous chapters. A second order system is considered firstly to provide us easier visualization of, and justification for our criteria. Some generalizations of Zubov's classical example are studied subsequently. A special case of coupled Lienard's equations is illustrated in the last section of this chapter by applying our boundedness criteria developed in Chapter III.

The generality and usefulness of our results are apparent from the fact that in all these examples, Pliss's [26] conditions are violated.

4.2 Generalizations of Zubov's Example

Let us first take a simplest second order nonlinear system with limit cycle as our example 1.

Example 1 Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= -x_1 + x_2 - x_2(x_1^2 + x_2^2)
\end{align*}
\]  

(4.1)

for which we assume a Liapunov-like function of the form

\[
V(x) = x_1^2 + x_2^2.
\]  

(4.2)

A simple computation yields
\[
\dot{V}(x) = 2(x_1^2 + x_2^2)(1 - (x_1^2 + x_2^2)) \tag{4.3}
\]

If we let \( \Omega \) be the region defined as

\[
\Omega = \{(x_1, x_2): x_1^2 + x_2^2 \leq 1 + \varepsilon \} \tag{4.4}
\]

where \( \varepsilon \) is any small positive constant. Then it is readily seen that \( V(x) \) plays the same role as \( W(x) \) in the Lagrange Stability Theorem, i.e. \( V(x) > 0 \) and \( \dot{V}(x) < 0 \) in \( \Omega^c \). Hence every solution of (4.1) is bounded in the future.

Next we find that the Jacobian matrix of the right hand side functions of (4.1) is

\[
J_F(x) = \begin{pmatrix}
1-(x_1^2 + x_2^2)-2x_1^2 & 1-2x_1x_2 \\
-1-2x_1x_2 & 1-(x_1^2 + x_2^2)-2x_2^2
\end{pmatrix} \tag{4.5}
\]

Since the matrix

\[
\frac{\dot{V}(x)}{V(x)} - 2J_F(x) = 2 \begin{pmatrix}
2x_1^2 & -1+2x_1x_2 \\
1+2x_1x_2 & 2x_2^2
\end{pmatrix} \tag{4.6}
\]

is positive definite w.r.t. the origin of \( x_1-x_2 \) phase plane, it is not difficult to see that condition (iii) of Theorem 2.1 is satisfied in the region \( \{(x_1,x_2): x_1^2 + x_2^2 \geq \varepsilon \} \). Hence the system (4.1) satisfies all the conditions of that theorem. This implies that there exists a globally stable limiting trajectory of (4.1). Indeed, this trajectory is a limit cycle defined by \( x_1^2 + x_2^2 = 1 \) on \( x_1-x_2 \) phase plane.

We are most interested in high order nonlinear autonomous systems. The following are typical examples for the existence of a unique limiting trajectory of high order systems which constitute a generalization of Zubov's third order example.
Example 2 Consider the system
\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= -x_1 + x_2 - x_2(x_1^2 + x_2^2) \\
\dot{x}_3 &= -x_3(x_1^2 + x_2^2).
\end{align*}
\tag{4.7}
\]

The boundedness of solutions of (4.7) in the future is assured if we select \( W(x) \) as
\[
W(x) = \sum_{i=1}^{3} x_i^2.
\tag{4.8}
\]
Indeed, \( \dot{W}(x) = 2(x_1^2 + x_2^2)(1-(x_1^2 + x_2^2 + x_3^2)) < 0 \) in \( \Omega^c \), where
\[
\Omega^c = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 > 1 + \epsilon\},
\tag{4.9}
\]
and \( \epsilon > 0 \), hence the system (4.7) satisfies Lagrange stability theorem.

In this system, a Liapunov-like function of the form
\[
V(x^{(2)}) = x_1^2 + x_2^2
\tag{4.10}
\]
is chosen. Then we find that
\[
\dot{V}(x)/V(x) - 2J(x) = 2 \begin{pmatrix} 2x_1^2 & -1+2x_1x_2 & 0 \\
1+2x_1x_2 & 2x_2^2 & 0 \\
2x_1x_3 & 2x_2x_3 & 1 \end{pmatrix}
\tag{4.11}
\]
which is positive definite matrix. Hence conditions of Corollary 2.1 are satisfied for the system (4.7). This implies that there exists a globally stable limit cycle in the \( X^3 \) space.

We want to give a special version of the system (3.29) as our last example of this section:

Example 3 Consider
\[ \dot{x}_1 = x_1 + x_2 - x_1 \left( \sum_{i=1}^{n} x_i^2 \right) \]
\[ \dot{x}_2 = -x_1 + x_2 - x_2 \left( \sum_{i=1}^{n} x_i^2 \right) \]  
\[ \dot{x}_j = x_j - x_j \left( \sum_{i=1}^{n} x_i^2 \right) \quad j = 3, \ldots, n \quad (4.12) \]

Let
\[ V(x) = \sum_{i=1}^{n} x_i^2 \]  
\[ (4.13) \]

Since
\[ \dot{V}(x) = 2 \left( \sum_{i=1}^{n} x_i^2 \right)(1- \sum_{i=1}^{n} x_i^2) \]
\[ < 0 \quad \text{in} \quad \Omega^c, \]

where \( \Omega \) is defined in the same way as before:
\[ \Omega = \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i^2 \leq 1 + \varepsilon \} \quad (4.15) \]

So that all solutions of (4.12) are bounded in the future.

Now the matrix
\[ \frac{\dot{V}(x)}{V(x)} - 2J_f(x) = 2 \begin{pmatrix} 2x_1^2 & -1+2x_1x_2 & 2x_1x_3 & \cdots & 2x_1x_n \\ 1+2x_1x_2 & 2x_2^2 & 2x_2x_3 & \cdots & 2x_2x_n \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 2x_1x_n & 2x_2x_n & \cdots & 2x_n^2 \end{pmatrix} \]
\[ (4.16) \]

is a positive definite one, hence we conclude that all the conditions of Theorem 2.1 are satisfied for (4.12). Therefore there exists a unique globally asymptotically orbitally stable limiting trajectory of (4.12).

4.3 Special Case of Coupled Liénard's Equations

The following two-degree-of-freedom system, examined previously by Aggarwal [1] will be chosen for our example on coupled systems.
Example 4 Consider
\begin{align}
\ddot{x}_1 + A_1 \dot{x}_1^3 - B_1 \dot{x}_1 + U_1(x_1, x_2) &= 0 \\
\ddot{x}_2 + A_2 \dot{x}_2^3 - B_2 \dot{x}_2 + U_2(x_1, x_2) &= 0.
\end{align}
(4.17)

The equivalent system is
\begin{align}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \theta U_1 - A_1 x_3^3 + B_1 x_3 \\
\dot{x}_4 &= -U_2 - A_2 x_4^3 + B_2 x_4.
\end{align}
(4.18)

Suppose this system is subject to the following conditions:
\begin{enumerate}
\item $U_1 \equiv \frac{\partial U}{\partial x_1}$, $U_2 \equiv \frac{\partial U}{\partial x_2}$; and $U(x_1, x_2)$ is a positive definite potential function.
\item $x_1 U_1 > 0$, $x_2 U_2 > 0$ for all $x_1$ and $x_2$, moreover, $x_1 U_1$ and $x_2 U_2$ increase monotonically.
\item $A_1$, $A_2$, $B_1$ and $B_2$ are all positive constants such that, for a sufficiently small positive constant $\sigma$, there exists an $\beta > \max \left\{ \sqrt{\frac{B_1}{A_1}}, \sqrt{\frac{B_2}{A_2}} \right\}$ satisfying
\begin{align}
\beta^2 (A_1 \beta^2 - B_1) - \sigma \beta > K_2(\beta, \sigma) + \frac{1}{4A_1} (B_1 + \frac{\sigma}{\beta})^2 \\
\text{and}
\beta^2 (A_2 \beta^2 - B_2) - \sigma \beta > K_1(\beta, \sigma) + \frac{1}{4A_2} (B_2 + \frac{\sigma}{\beta})^2,
\end{align}
(4.19a)
\[ (4.19b) \]

where
\begin{align}
K_1(\beta, \sigma) &= \max \left\{-\frac{\sigma}{\beta} x_i(U_i + A_i \dot{x}_i^3 - B_i \dot{x}_i) \right\} \text{ i = 1, 2 for } |\dot{x}_i| \leq \beta.
\end{align}
(4.20b)
There is a \( \alpha > 0 \) such that the inequalities

\[
\frac{\sigma}{\beta} x_1 (U_1 + \frac{2B_1}{3} \sqrt{\frac{B_1}{3A_1}}) > \frac{2}{4A_1} (B_1 + \frac{\sigma}{\beta})^2 + K_2 (\beta, \sigma) \quad "\mu" \quad \text{for} \quad x_1 > \alpha
\]

\[
\"\mu" \quad \text{for} \quad x_1 < -\alpha
\]

\[
\frac{\sigma}{\beta} x_2 (U_2 + \frac{2B_2}{3} \sqrt{\frac{B_2}{3A_2}}) > \frac{2}{4A_2} (B_2 + \frac{\sigma}{\beta})^2 + K_1 (\beta, \sigma) \quad "\mu" \quad \text{for} \quad x_2 > \alpha
\]

\[
\"\mu" \quad \text{for} \quad x_2 < -\alpha
\]

are satisfied.

Then all the conditions in Theorem 3.4 are clearly fulfilled.

\( \Omega \) is now defined as

\[
\Omega = \{ (x_1, \ldots, x_4): |x_1| \leq \alpha, \ |x_2| \leq \alpha, \ |x_3| \leq \beta, \ |x_4| \leq \beta \}
\]

and (3.26) and (3.28) take the form:

\[
W(x) = U_1 x_1 + \frac{1}{2} (x_3^2 + x_4^2) + x_1 h_1(x_3) + x_2 h_2(x_4) \quad (4.24)
\]

\[
\dot{W}(x) = -h_1^i x_1 (U_1 + A_1 x_3^3 - B_1 x_3) - h_2^i x_2 (U_2 + A_2 x_4^3 - B_2 x_4)
\]

\[
- x_3 (A_1 x_3^3 - B_1 x_3 - h_1(x_3)) - x_4 (A_2 x_4^3 - B_2 x_4 - h_2(x_4)) \quad (4.25)
\]

where \( h_i, \ i = 1, 2, \) are defined in (3.27).

It follows then \( W > 0 \) and \( \dot{W} < 0 \) in \( \Omega^c \). Thus by Theorem 3.4, we conclude that all solutions of (4.17) are bounded in the future.

Now if we select

\[
V(x) = U(x_1, x_2) + \frac{1}{2} (x_3^2 + x_4^2), \quad (4.26)
\]

we find that the matrix

\[
\frac{\dot{V}(x)}{V(x)} - 2J_f(x) \quad (4.27)
\]
is no longer a positive definite one. Therefore no conclusion about the existence of a unique globally stable limiting trajectory of (4.17) (or equivalently (4.18)) can be drawn from our criteria.
CHAPTER V

CONCLUSIONS AND COMPARISONS

Previous studies on the global asymptotic stability of n-dimensional autonomous systems of the form

\[ \dot{x} = f(x) \quad , \quad (*) = d/dt \]  \hspace{1cm} (5.1)

center around the results of Liapunov, complemented by Barbasin, Krasovskii and LaSalle. Invariably the limit set in question is chosen as the trivial solution (the origin of the state space), and the global asymptotic stability is established by invoking a positive definite scalar function \( V(x) \) such that \( \dot{V}(x) \) is nonpositive for \( x \neq 0 \).

Although attempts have been made by Zubov, Szegö, Pliss and others to extend this method to cases where the limit set is not the trivial solution, the criteria developed have proved to be too restrictive to be applied to most higher order nonlinear systems.

In the present work we have given sufficient conditions to guarantee simultaneously both the existence and the global stability of a limit set \( \Gamma \) of (5.1). Thus to verify our criteria, previous knowledge of the explicit form of \( \Gamma \) is unnecessary. \( \Gamma \) need not be a trivial solution. It can in fact be a limit cycle for which an analytical expression is unavailable (such as in the case of the van der Pol equation). Our results are proved via a positive definite function \( V(x) \) and a set of conditions expressed in terms of \( V(x) \), the components of \( f \) and their partial derivatives is presented.

We have also developed conditions for the boundedness in the
future of the solutions for general classes of systems of the form (4.1) (such as the class of n coupled Lienard's equation and a n-dimensional generalization of Zubov's example).

By means of various examples we have shown how these results can be verified in particular instances.

The following may be recommended as suggestions for further work: (a) to extend the present method to the investigation of forced non-linear systems, (b) to identify the unique globally stable limiting trajectory under the present conditions, and (c) in some high order cases of interest to develop methods to approximate the region of attraction Ω of a limiting trajectory Γ, in those instances in which Γ is stable only in Ω (and not globally stable).
APPENDIX A

ON THE EXISTENCE OF GLOBALLY STABLE LIMIT SET OF
SECOND ORDER NONLINEAR AUTONOMOUS SYSTEMS

In this appendix, we will present a new set of sufficient conditions for the existence of a unique globally stable limit set of the second order nonlinear autonomous system

\[ \dot{x} = f(x) , \quad (\dot{} = d/dt) \]  

(A.1)

where \( x = (x_1, x_2) \) and \( f(x) = (f_1(x), f_2(x)) \) are elements of the real 2-space \( \mathbb{R}^2 \), and \( f(x) \) has continuous first partial derivatives on \( \mathbb{R}^2 \) with respect to the components of \( x \). The approach followed represents a modification of the one used in Theorem 2.1 of Chapter II. We will still resort to a transformation similar to (2.4), but in the present case, it is not necessary to be confined in the orthonormal hyperplanes. In fact, the new set of variables are measured in an oblique hyperplane (a line for the case of second order system) on which the "ray" connecting a point on the given curve and the origin may lie. Similar results have been derived by R. J. P. de Figueiredo.**

Our result may be stated as the following

Theorem A.1 If

(i) all solutions of (A.1) tend to a bounded region \( \Omega \subset \mathbb{R}^2 \) as \( t \to \infty \),

**Private communication.
(ii) \[ \sum_{i=1}^{2} \sum_{j=1}^{2} x_i x_j \frac{\partial f_i}{\partial x_j} - \sum_{i=1}^{2} x_i f_i \leq \varepsilon \|x\|^2, \quad \varepsilon > 0, \]

(iii) \[ x_2 f_1 - x_1 f_2 = (\frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2)x_2 - (\frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2)x_1 \neq 0 \]

\[ \forall x_1, x_2 \neq 0, \]

then there exists a unique globally asymptotically orbitally stable limiting trajectory of (A.1).

**Proof.** Since most parts of the proof of this theorem are similar to that of Theorem 2.1, only those arguments which are different from the previous ones will be elaborated here.

Let us consider a trajectory \( x(t,x_0) \) of (A.1) with \( x(t_0,x_0) = x_0 \), then a perturbed motion \( y(t) \) lying sufficiently near the trajectory \( x(t,x_0) \) can be represented by the following transformation:

\[ y_1(t+\tau) = x_1(t) + \frac{x_1(t)}{v(x(t))} \eta \]

\[ y_2(t+\tau) = x_2(t) + \frac{x_2(t)}{v(x(t))} \eta, \quad (A.2) \]

where \( v^2(x) = x_1^2 + x_2^2 \).

Since its Jacobian determinant does not vanish anywhere except at the origin provided condition (iii) of the theorem is satisfied, and \( \tau \) and \( \eta \) are sufficiently small, the transformation (A.2) is valid (see Appendix B).

From (A.2), it follows that

\[ \sum_{i=1}^{2} (y_i(t+\tau) - x_i(t))^2 = \eta^2 \]

(A.3)

As in (2.8), the transformed equations resulted from (A.2) are

\[ \frac{d\tau}{dt} = g_1(\eta,\tau,t) \quad (A.4) \]

\[ \frac{d\eta}{dt} = g_2(\eta,\tau,t). \quad (A.5) \]
On the trajectory $x(t, x_0)$, it is clear that

$$g_1(0, \tau, t) \equiv 0 \quad \text{(A.6)}$$
$$g_2(0, \tau, t) \equiv 0 . \quad \text{(A.7)}$$

If $g_1(\eta, \tau, t)$ is expanded in Taylor series about $\eta = 0$, it assumes the form

$$g_1(\eta, \tau, t) = g_1(0, \tau, t) + \frac{\partial g_1(0, \tau, t)}{\partial \eta} \eta + \frac{1}{2} \frac{\partial^2 g_1(0, \tau, t)}{\partial \eta^2} \eta^2 + \ldots \quad \text{(A.8)}$$

The condition (iii) of the theorem yields

$$\frac{\partial g_1(0, \tau, t)}{\partial \eta} = 0 . \quad \text{(A.9)}$$

Hence from (A.6), (A.8) and (A.9), one has the following inequality

$$\left| \frac{d\tau}{dt} \right| = \left| g_1(\eta, \tau, t) \right| < \xi \eta^2 , \quad \xi > 0 \text{ is a constant,} \quad \text{(A.10)}$$

provided that $|\eta| < \delta , \delta > 0$ is a sufficiently small constant.

We have, as in (2.11), (2.12) and (2.14), the following expressions respectively:

$$g_2(\eta, \tau, t) = \gamma(t) \eta + \Gamma(\eta, \tau, t) , \quad \text{(A.11)}$$

$$|\Gamma(\eta, \tau, t)| < \alpha |\eta| , \quad \text{(A.12)}$$

$$\alpha < \frac{\sigma}{\delta} , \quad \text{(A.13)}$$

provided that $|\eta| < \tilde{\delta} , \tilde{\delta} > 0$ a constant.

We now consider the evolution of the following function

$$\sum_{i=1}^{2} \left( y_i(t+\tau) - x_i(t) \right)^2 / v^2(x(t)) . \quad \text{(A.14)}$$

According to (A.2), (A.10), and applying mean-value theorem to the function $f_1(y(t+\tau)) - f_1(x(t))$, $i = 1, 2$, one deduces the time
derivative of the function (A.14) as

\[
\frac{d}{dt} \sum_{i=1}^{2} \left( y_i(t+\tau) - x_i(t) \right)^2 / v^2(x(t))
\]

\[
= \frac{2}{v^2} \left[ \sum_{i=1}^{2} (y_i(t+\tau) - x_i(t)) \left[ (1 + \frac{d\tau}{dt}) f_i(y(t+\tau)) - f_i(x(t)) \right] \right]
\]

\[
- \sum_{i=1}^{2} (y_i(t+\tau) - x_i(t))^2
\]

\[
= \frac{2\tau}{v^2} \left[ \sum_{i=1}^{2} \frac{\partial f_i}{\partial x_i} \frac{\partial f_i}{\partial x_j} \right] + \frac{2\tau}{v^2} \sum_{i=1}^{2} \int_{0}^{1} \frac{\partial f_i(\hat{x})}{\partial x_j} \frac{\partial f_i(x)}{\partial x_j} d\mu + \frac{2\tau}{v} \sum_{i=1}^{2} f_i(y(t+\tau))
\]

(A.13)

where

\[
\hat{x} = x + \mu(y(t+\tau) - x), \quad 0 \leq \mu \leq 1.
\]

(A.16)

By the condition (ii) of the theorem, we have

\[
\left| \frac{1}{v^2} \sum_{i=1}^{2} \frac{\partial f_i}{\partial x_i} \right| \geq \sigma.
\]

(A.17)

And by the continuous differentiability of the function \( f(x) \), and (A.16), there is a \( \hat{\delta} > 0 \), such that if

\[
|\tau| < \hat{\delta}, |\eta| < \hat{\delta}
\]

(A.18)

then

\[
\frac{1}{v^2} \left| \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{0}^{1} \frac{\partial f_i(\hat{x})}{\partial x_j} \frac{\partial f_i(x)}{\partial x_j} d\mu \right| < \frac{\sigma}{8}
\]

(A.19)

and

\[
\frac{1}{v} \left| \sum_{i=1}^{2} \int_{0}^{1} f_i(y(t+\tau)) \right| < \frac{\sigma}{8}
\]

(A.20)

Hence from (A.17), (A.19) and (A.20), we have
\[
\frac{d}{dt} \left( \sum_{i=1}^{2} (y_i(t+\tau) - x_i(t))^2/v^2(x) \right) < -\frac{\sigma \eta^2}{2v^2} < 0 \quad (A.21)
\]

provided that
\[
|\tau| < \delta \ , \ |\eta| < \delta \quad (A.22)
\]
is satisfied, where
\[
\delta = \min(\delta, \delta', \delta) \ . \quad (A.23)
\]

Introducing a variable
\[
\zeta = \eta e^{\lambda t} \ , \ \lambda > 0 \ , \quad (A.24)
\]
and proceeding in much the same way as before while choosing \( \lambda \) to be
\[0 < \lambda < \frac{\sigma}{4}\], we obtain that
\[
\frac{d}{dt} \left( \frac{\zeta^2}{v^2} \right) < -\frac{\sigma \zeta^2}{2v^2} < 0 \quad (A.25)
\]
provided \( |\tau| < \delta \) and \( |\zeta| < \delta \).

Suppose at initial time \( t = t_0 \), it is given that
\[
|\tau(t_0)| < \epsilon_o \ , \ |\zeta(t_0)| < \epsilon_o \ . \quad (A.26)
\]

where \( \epsilon_o < \delta \) will be specified later.

With the same reasoning as before, there is a positive interval
\[
I = (t_0, t^*) \quad (A.27)
\]
such that
\[
|\tau(t)| < \delta \ , \ |\zeta(t)| < \delta_o \quad (A.27)
\]
will be satisfied for \( t \in I \).

Let
\[
N = \text{Sup}_{x \in \Omega} v(x) \quad (A.28)
\]

\[
\epsilon = N^{\frac{1}{2}} \epsilon_o/v(t_0) \ , \quad (A.29)
\]

then from (A.25), it follows that
\[ |z| < \varepsilon \quad \text{for} \quad t \in I, \quad (A.30) \]

or by (A.24),
\[ |\eta| < \varepsilon e^{-\lambda t} \quad \text{for} \quad t \in I. \quad (A.31) \]

By (A.10) and (A.4), we have
\[ |\tau| < \varepsilon_0 [1 + \xi N^2 \varepsilon_0 e^{-2\lambda t_0} / 2\lambda v(t_0)] = \psi. \quad (A.32) \]

Hence if we select \( \varepsilon_0 \) so small that
\[ \psi < \delta \]
\[ \varepsilon < \delta, \quad (A.33) \]

then (A.27) is satisfied for \( t \in I \). By the same reasoning used after (2.41) in the proof of Theorem 2.1, we concluded that, under the conditions of the present theorem, there exists a unique globally asymptotically orbitally stable limiting trajectory of (A.1).

Q.E.D.
APPENDIX B

JACOBIAN DETERMINANT OF (A.2)

The Jacobian determinant for

\[ y_1(\theta) = x_1(t) + \frac{x_1(t)}{v} \eta \]

\[ y_2(\theta) = x_2(t) + \frac{x_2(t)}{v} \eta \]  \hspace{1cm} (B.1)

is

\[ J = \frac{\partial (y_1, y_2)}{\partial (\eta, t)} = \begin{vmatrix} x_1/v & x_2/v \\ f_1 + \frac{\eta x_2}{v^3} (x_2 f_1 - x_1 f_2) & f_2 + \frac{\eta x_1}{v^3} (x_2 f_1 - x_1 f_2) \end{vmatrix} \]

\[ = \frac{(x_1 f_2 - x_2 f_1)}{v^2} (v + \eta). \]  \hspace{1cm} (B.2)

By the condition (iii) of Theorem A.1, (B.2) does not vanish anywhere except at the origin, hence the transformation (B.1) (or (A.2)) is valid.
BIBLIOGRAPHY


