TU, Chang-Char, 1932--
I. A BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION IN A LIP$_\alpha$ REGION. II. NON-TANGENTIAL LIMITS OF A SOLUTION OF A BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION.

Rice University, Ph.D., 1967
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
I. A BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION IN A LIP$_\alpha$ REGION

II. NON-TANGENTIAL LIMITS OF A SOLUTION OF A BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION

by

Chang-Char Tu

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's signature:

Houston, Texas

May, 1967
ACKNOWLEDGMENTS

The author wishes to thank Professor B. F. Jones, Jr., who posed the problems treated in this thesis and gave invaluable criticisms, suggestions and helps. The author is grateful to Mrs. Nancy Singleton who typed this thesis expertly and to Miss Janet Gordon for general help.

Financial support was provided by the Rice University and the Schlumberger Fellowship.
CONTENTS

Part I  A Boundary Value Problem for the Heat Equation in a Lip_α Region  .......... 1

Part II Non-Tangential Limits of a Solution of a Boundary Value Problem for the Heat Equation  .......... 18

(A) A Privaloff-Plesner-Calderón Theorem for Solution of a Boundary Value Problem for the Heat Equation  .......... 18

(B) Stein's Theorem  .......... 28

Reference  .......... 54
I. A Boundary Value Problem for the
Heat Equation in a Lip_α Region

NOTATIONS AND CONVENTIONS:

(a) \( \eta(t) \in \text{Lip}_\alpha (\frac{1}{2} < \alpha < 1) \) defined on \([0,T]\). More precisely,
\[
|\eta(t) - \eta(\tau)| \leq M|t - \tau|^\alpha \quad \forall \ t, \tau \in [0,T], \text{for some } M > 0;
\]

(b) \( \Gamma_{t_0} = \{(x,t): |t-t_0| < M(x-\eta(t_0))^2, 0 < t_0 < T\}; \)

(c) since we confine ourselves to \([0,T]\), whenever \( t \) or \( \tau \) appear they are in \([0,T]\);

(d) whenever an expression like
\[
\frac{x-\eta(t)}{(t-\tau)^{3/2}} e^{-\frac{(x-\eta(t))^2}{4(t-\tau)}}
\]
appears, it will be considered 0 when \( t \leq \tau \);

(e) \( \lim_{t_0 \to \infty} \) will stand for \( \lim_{(x,t) \to (\eta(t_0),t_0)} \);

(f) \( c \) will be a generic constant, not the same in various places.

The main theorem in this section is to establish
the existence of a solution of a certain boundary value
problem for the heat equation. The hardest part in the
proof is to show that the solution takes on the right
boundary value in the right fashion. In order to achieve
this, we have to break up the proof into several lemmas.

We begin by considering the following problem:
\begin{align*}
\{ \begin{array}{l}
\frac{u_{xx}}{u_t} = 0 \quad \text{in } D = \{(x,t): 0 < t < T, \eta(t) < x\}, \\
u(\eta(t), t) = f(t) \text{ a.e.}, \text{ where } f(t) \in L^1(0,T).
\end{array} \right.
\end{align*}

The boundary value is taken in the sense
\[
\lim_{t \to t_0^+} u(x,t) = f(t_0). \quad \text{(See fig. 1).}
\]

**Lemma 1.1:** If \( f \in L^1(0,\infty), |H(t,\tau)| \leq 1, \frac{1}{2} < \beta < 1, \) then there exists one and only one solution for the following integral equation:
\[
\varphi(t) = f(t) + \int_0^t \varphi(\tau) \frac{H(t,\tau)}{(t-\tau)^\beta} d\tau \quad \text{a.e.,}
\]
and the solution is in \( L^1_{\text{loc}}(0,\infty). \)

**Proof:** Let \( K(t,\tau) = H(t,\tau)/(t-\tau)^\beta, \) and let \( K_m(t,\tau) = \int_0^t K(t,\tau)K_{m-1}(s,\tau) d\tau \) \( (K_1 = K). \) Then it is known that
(see Mihlin: Integral equations and applications, p. 25)
\[
|K_m(t,\tau)| \leq \frac{(t-\tau)^{m-1} - m\beta (\Gamma(1-\beta))^m}{\Gamma(m(1-\beta))}.
\]

Consider the infinite series
\[ f(t) + \sum_{1}^{\infty} g_m(t), \text{ where } g_m(t) = \int_{0}^{t} K_m(t, \tau) f(\tau) d\tau. \]

Then, by using the estimate on \( K_m(t, \tau) \) and the Stirling's formula \( \Gamma(p) \sim p^{-\frac{1}{2}} e^{-p} \), we have, for \( m > (1-\beta)^{-1} \),

\[
|g_m(t)|^m < c t^{1-\beta - \frac{1}{m}} (1-\beta)^{2m} e^{m^2 (1-\beta)^m \theta - 1} \quad (0 < \theta < 1),
\]

which tends to 0 as \( m \) approaches \( \infty \). Thus, the infinite series \( \sum_{m>1}^{} g_m(t) \) converges uniformly on compact subsets of \((0, \infty)\). Since \( f \) is an \( L^1 \) function and each term of the finite sum \( \sum_{1 \leq m \leq \frac{1}{1-\beta}} g_m(t) \) is the convolution of an \( L^1 \) function (redefine \( K(t, \tau) \) to be 0 if \( t < \tau \)) with another \( L^1 \) function, therefore

\[
\phi(t) = f(t) + \sum_{m=1}^{\infty} g_m(t)
\]

is a well-defined function and obviously \( \phi(t) \sim_{1 \text{ loc}} (0, \infty) \). Moreover, \( \phi(t) \) satisfies the integral equation. The uniqueness of \( \phi \) also follows easily.

\[Q.E.D.\]

**Lemma 1.2:** Let

\[
K(x, t, t_0, \tau) = \frac{x-\eta(\tau)}{(t-\tau)^{3/2}} e^{-\frac{(x-\eta(\tau))^2}{4(t-\tau)}} - \frac{x-\eta(t_0)}{(t-\tau)^{3/2}} e^{-\frac{(x-\eta(t_0))^2}{4(t-\tau)}}
\]
Then

\[ |K(x, t, t_o, \tau)| \leq c \left( |t_o - \tau|^\alpha - \frac{2}{3} + |t - \tau|^\alpha - \frac{2}{3} \right) \]

for \((x, t) \in \Gamma_{t_o}\) and \((2M)^{1+\alpha} (x-\eta(t_o))^{2\alpha-1} \leq 1\).

**Proof:** The last term in the expression for \(K\) is dominated by \(c |t_o - \tau|^{-\frac{2}{3}}\) as \(\eta(t) \in \text{Lip}_\alpha\).

By using mean value theorem and the fact that \(ye^{-y} < c\) \((y > 0)\), we have

\[
\frac{3}{2} (t - \tau)^{\frac{3}{2}} \left( (x - \eta(t))e^{-\frac{(x-\eta(t))}{4(t - \tau)}} - (x - \eta(t_o))e^{-\frac{(x-\eta(t_o))}{4(t - \tau)}} \right) |x - \eta(t_o) + c(\eta(t_o) - \eta(\tau))|^{2c(t - \tau)} = \ast,
\]

where \(0 < \theta < 1\).

**Case 1:** \(|t_o - t| \leq \frac{1}{2} |t_o - \tau|\).

In this case \(|t - \tau| = |t - t_o + t_o - \tau| \geq |t_o - \tau| - |t_o - t| \geq \frac{1}{2} |t_o - \tau|\).

\[ \ast \leq c |t_o - \tau|^{-\frac{3}{2}}. \]

**Case 2:** \(|t_o - t| > \frac{1}{2} |t_o - \tau|\).

In this case \(|\eta(t_o) - \eta(\tau)| < M|t_o - \tau|^{\alpha} < 2^\alpha M|t_o - t|^{\alpha}\).
\[ < 2^{\alpha M(M(x-\eta(t_o))^{2\alpha}} = 2^{\alpha M^{1+\alpha}(x-\eta(t_o))^{2\alpha}} \]
\[ \leq \frac{1}{2}(x-\eta(t_o)), \text{ as } 2^{1+\alpha M^{1+\alpha}(x-\eta(t_o))^{2\alpha}} \leq 1. \]
\[ \therefore \quad * \leq c \frac{(x-\eta(t_o))^{2\alpha}}{(t-\tau)^{3/2}} e^{-\frac{(x-\eta(t_o))^2}{c(t-\tau)}} \]
\[ = c \left( \frac{(x-\eta(t_o))^2}{c(t-\tau)} \right)^{\alpha} e^{-\frac{(x-\eta(t_o))^2}{c(t-\tau)}} \frac{1}{(t-\tau)^{3/2}} \]
\[ \leq c \left| t-\tau \right|^{\alpha-\frac{3}{2}}. \quad \text{Q.E.D.} \]

**Lemma 1.3:** Let \( \mu \) be a measure. Let \( |\mu|(E) \) be the total variation of \( \mu \) over \( E \), where \( E \) is a \( \mu \)-measurable set. Define

\[ \bar{\mu}(t_o) = \sup \left\{ \frac{|\mu|(I)}{|I|} : I \text{ is an interval centered at } t_o, \ |I| \text{ is the Lebesgue measure of } I \right\}. \]

Then

\[ \left| \int_0^{t_o} \frac{\eta(t_o)-\eta(\tau)}{(t_o-\tau)^{3/2}} e^{-\frac{(\eta(t_o)-\eta(\tau))^2}{4(t_o-\tau)}} \ d\mu(\tau) \right| \]
\[ \leq c \bar{\mu}(t_o). \]

**Proof:** Let \( k_o \) be chosen so that \( t_o-2^{k_o+1} \leq 0, \ t_o-2^{k_o} > 0. \)
Then, the left hand side of the inequality can be written as:

\[
\begin{align*}
&\sum_{k=\infty}^{k_0} \int_{2^k < |t_o - \tau| < 2^{k+1}} \frac{\eta(t_o) - \eta(\tau)}{(t_o - \tau)^{3/2}} e^{-\frac{(\eta(t_o) - \eta(\tau))^2}{4(t_o - \tau)}}
\leq c \sum_{k=\infty}^{k_0} \mu |\{ \tau : 2^k < |t_o - \tau| < 2^{k+1}\}| 2^k \alpha^{-\frac{3}{2}}
\leq c \sum_{k=\infty}^{k_0} \bar{\mu}(t_o) 2^k \alpha^{-\frac{3}{2}} 2^{k_0} \alpha^{-\frac{3}{2}}
\leq c \bar{\mu}(t_o) 2^{\alpha-k_0} \alpha^{-\frac{3}{2}} \leq c \bar{\mu}(t_o) \sum_{k=\infty}^{k_0} 2^k \alpha^{-\frac{1}{2}} k
\leq c \bar{\mu}(t_o) 2^{\alpha-k_0} \leq c \bar{\mu}(t_o)
\end{align*}
\]

Q.E.D.

**Lemma 1.4:** Let \( f \) be an \( L^1 \) function defined in \( (-\infty, \infty) \).

Let \( \bar{f}(t_o) \) be defined as

\[
\sup_{0 < \varepsilon < \infty} \frac{1}{2\varepsilon} \int_{t_o - \varepsilon}^{t_o + \varepsilon} |f(t)| dt.
\]

Let \( K(x,t) = \frac{1}{2\sqrt{\pi}} \frac{x}{t^{3/2}} e^{-\frac{x^2}{4t}} \) for \( t > 0 \) and zero otherwise.
Let \( G = \{(x,t) : |t| < x^2\} \) and \( G_{t_o} = (0,t_o) + G \). Then

\[
\sup_{(x,t) \in G_{t_o}} \left| \int_{-\infty}^{\infty} K(x,t-\tau)f(\tau)d\tau \right| \leq c\overline{f}(t_o).
\]

**Proof:** Let \( u(x,t) = \int_{-\infty}^{\infty} K(x,t-\tau)f(\tau)d\tau \). For \((x,t) \in G_{t_o}\), we can rewrite \((x,t)\) as \((x,t+t_o)\), then \((x,t) \in G\).

Consider, for \((x,t) \in G\),

\[
|u(x,t+t_o)| = \left| \int_{-\infty}^{\infty} K(x,t_o+t-\tau)f(\tau)d\tau \right| = \left| \int_{-\infty}^{\infty} K(x,t+s)f(t_o-s)ds \right|
\]

\[
= \left| \left( \int_{I_0}^{I_0} + \int_{I_k-I_{k-1}} \right) K(x,t+s)f(t_o-s)ds \right| \leq c\overline{f}(t_o)(a_o x^2 + \sum_{k=1}^{\infty} a_k 2^k x^2)
\]

where \( I_k = \{s: |s| < 2^k x^2\} \), \( a_o = \sup_{(x,t) \in G} K(x,t+s) \) and \( a_k = \sup_{(x,t) \in G, s \in I_k-I_{k-1}} K(x,t+s) \) \( k=1,2,\ldots \).

Now, \( s+t > |s| - |t| > 2^{k-1} x^2 - x^2 > 2^{k-2} x^2 \) for \( k \) large, as \( |s| \geq 2^{k-1} x^2 \) and \( |t| < x^2 \). Thus \( a_k < c_2 \frac{3k}{2} x^{-2} \) for \( k \) large. For the rest of \( k \)'s, which is finite in number, we use the fact that \( K(x,t+s) \) takes on its maximum value when \( t+s = cx^2 \) where \( c \) is a certain positive constant. Thus,
the infinite series \( \sum_{k=0}^{\infty} a_k 2^k x^2 \leq c \), and the desired result follows. \( \text{Q.E.D.} \)

From this lemma, it is easy to show that

\[
\lim_{t \to t_0} u(x,t) = f(t_0) \text{ a.e.} \quad \text{for} \quad \limsup_{t \to t_0} |u(x,t)-f(t_0)| < c \]

\[
\leq c(\bar{f}(t_0) + |f(t_0)|) \text{ a.e., and thus,} \quad \mu\{t_0: \limsup_{t \to t_0} |u(x,t)-f(t_0)| > \}
\]

\[
> \lambda \} \leq \mu\{t_0: \bar{f}(t_0) > \frac{\lambda}{2c} \}
\]

\[
+ \mu\{t_0: |f(t_0)| > \frac{\lambda}{2c} \} \leq \frac{c}{\lambda} \|f\|_1 \text{ where} \quad \mu\{\cdot\} \text{ denotes the}
\]

Lebesgue measure of the set \{\cdot\}. Let \( g \in C_0^\infty(-\infty, \infty) \) such that

\[
\|f-g\|_1 < \varepsilon.
\]

Then \( \mu\{t_0: \limsup_{t \to t_0} |u(x,t)-g(x,t)-f(t_0)+g(t_0)| > \}
\]

\[
> \lambda \} \leq \frac{c}{\lambda} \|f-g\|_1 (-\infty, \infty). \quad \text{Hence} \quad \lim_{t \to t_0} u(x,t) = f(t_0) \text{ a.e. as} \quad g(x,t) \equiv \int_{-\infty}^{\infty} K(x,t-\tau)g(\tau)d\tau \text{ tends to} \quad g(t_0)
\]

everywhere when limit is taken inside \( t_0 \). Therefore, we can proceed to prove

**Lemma 1.5**: \( \lim_{t \to t_0} \left| \int_{0}^{t} \frac{x-\eta(\tau)}{(t-\tau)^{3/2}} e^{-\frac{(x-\eta(\tau))^2}{4(t-\tau)}} d\tau \right| = 0 \)

\[
\int_{0}^{t_0} \frac{\eta(t_0)-\eta(\tau)}{(t_0-\tau)^{3/2}} e^{-\frac{(\eta(t_0)-\eta(\tau))^2}{4(t_0-\tau)}} d\tau = 0.
\]
Proof: We know from Lemma 1.4 and what follows it that

\[
\lim_{t \to \infty} \int_{0}^{t} \frac{x - \eta(t)}{(t - \tau)^{3/2}} e^{-\frac{(x - \eta(t))^2}{4(t - \tau)}} d\tau = 2\sqrt{\pi}.
\]

Thus, it is sufficient to show that

\[
\lim_{t \to \infty} \left[ \int_{0}^{t} \frac{x - \eta(t)}{(t - \tau)^{3/2}} e^{-\frac{(x - \eta(t))^2}{4(t - \tau)}} d\tau - \int_{0}^{t} \frac{x - \eta(t_0)}{(t - \tau)^{3/2}} e^{-\frac{(x - \eta(t))^2}{4(t - \tau)}} d\tau - \int_{0}^{t_0} \frac{\eta(t_0) - \eta(t)}{(t_0 - \tau)^{3/2}} e^{-\frac{(\eta(t_0) - \eta(t))^2}{4(t_0 - \tau)}} d\tau \right] = 0.
\]

Let \(K(x,t,t_0,\tau)\) be as in Lemma 1.2. Let \(\delta\) be any small positive number. Write the integral of \(K\) as:

\[
\int_{0}^{\infty} K(x,t,t_0,\tau) d\tau = \int_{0}^{t_0 - \delta} K(x,t,t_0,\tau) d\tau + \int_{t_0 - \delta}^{\infty} K(x,t,t_0,\tau) d\tau = I + II.
\]

I: By Lemma 1.2, \(K\) is bounded by a constant for \(\tau \in (0,t_0 - \delta)\). Therefore, by Lebesgue's theorem, we can pass the limit inside the integral sign and obtain \(\lim_{t \to \infty} I = 0\) since \(K(\eta(t_0),t_0,t_0,\tau) = 0\).

II: Since

\[
\left| \int_{t_0 - \delta}^{\infty} K(x,t,t_0,\tau) d\tau \right| = \left| \int_{t_0 - \delta}^{\max(t_0,t)} K(x,t,t_0,\tau) d\tau \right| \leq c \int_{t_0 - \delta}^{\max(t_0,t)} (|t_0 - \tau|^\alpha + |t - \tau|^\alpha) d\tau
\]

\(\alpha = \frac{3}{2}\).
\[
\frac{\max(t_o, t)}{t_o^{-\frac{\alpha}{2}}}
= c \left( |t_o - \tau|^{\alpha\cdot \frac{1}{2}} + |t - \tau|^{\alpha\cdot \frac{1}{2}} \right) \text{ for } t_o^{-\delta}.
\]

Thus, \(\limsup_{t_o} II \leq c\delta^{\alpha\cdot \frac{1}{2}}\).

**Q.E.D.**

**LEMMA 1.6:** Let \(\mu\) be a measure. Then,

\[
\sup_{t_o} \left| \int_{t_o}^{t} \frac{x - \eta(\tau)}{(t - \tau)^{3/2}} \frac{-(x - \eta(\tau))^2}{4(t - \tau)} \, d\mu(\tau) \right| \leq c\mu(t_o).
\]

**Proof:** We recall that the following inequalities are satisfied:

1. \(|t - t_o| < M(x - \eta(t_o))^2 \equiv Ma^2\);

2. \(|\eta(t) - \eta(t_o)| \leq M|t - t_o|^\alpha, \frac{1}{2} < \alpha < 1\);

3. \(|x - \eta(\tau) - a| = |\eta(t_o) - \eta(\tau)| \leq M|\tau - t_o|^\alpha \leq M|t - \tau|^\alpha + M|t - t_o|^\alpha \leq M^{1+\alpha} a^{2\alpha} + M|t - \tau|^\alpha\).

Decompose the integral as:

\[
\int_{0}^{t} \frac{x - \eta(\tau)}{(t - \tau)^{3/2}} \frac{-(x - \eta(\tau))^2}{4(t - \tau)} \, d\mu(\tau)
= \left( \int_{|t - t_o| < Ba^2} + \int_{Ba^2 < |\tau - t_o|} \right) \frac{x - \eta(\tau)}{(t - \tau)^{3/2}} \frac{-(x - \eta(\tau))^2}{4(t - \tau)} \, d\mu(\tau)
= I + II.
\]

B will be chosen in estimating II.
I: In this integral

\[ |x-\eta(\tau)-a| \leq M|t_0-\tau|^\alpha < MB^\alpha a^{2\alpha}. \]

Once \( B \) is chosen, no matter how big it is, there exists \( \epsilon > 0 \) such that \( 0 < \epsilon \leq \epsilon = MB^\alpha a^{2\alpha-1} < \frac{1}{2}. \) Thus \( a/2 < |x-\eta(\tau)| \)

\[ < \frac{3a}{2}, \text{ and } |I| \leq \int_{|\tau-t_0|<Ba^2} \frac{3a/2}{(t-\tau)^{3/2}} e^{-\frac{a^2}{16(t-\tau)}} d|\mu|(\tau). \]

Estimating the integrand by its maximum value, which occurs when \( t-\tau = ca^2 \) for some \( c \), we have

\[ |I| \leq c|\mu|(t_0-Ba^2,t_0+Ba^2)a^{-2} \leq c\bar{\mu}(t_0)2Ba^2a^{-2} = \bar{\mu}(t_0). \]

II: Notice that \( |t-\tau| \geq |t_0-\tau| - |t-t_0| > (B-M)a^2. \)

If \( B \) is chosen to be bigger than \( 3M \), then, \( |t_0-\tau-(t-\tau)| \)

\[ = |t-t_0| < Ma^2 < \frac{M}{B-M}|t-\tau| < \frac{1}{2}|t-\tau|. \] Thus, we may replace \( t-\tau \) by \( t_0-\tau \) in II. Using this and the fact that \( t_0+Ma^2 \)

\(< t_0+Ba^2 < \tau \Rightarrow t<\tau \) (by (1)), we have

\[ |I| \leq c \int_{\tau_0-Ba^2}^{\tau_0} \frac{|x-\eta(\tau)|}{(t_0-\tau)^{3/2}} e^{-\frac{(x-\eta(\tau))^2}{c(t_0-\tau)^2}} d|\mu|(\tau). \]

From previous discussion, we know that \( 0 < \alpha \leq \epsilon \)

\[ MB^\alpha a^{2\alpha-1} < \frac{1}{2} \) (\( \epsilon \) depends on \( M \) only). Therefore \( \exists \rho \) such

that \( B^\alpha \epsilon^{2\alpha-1} < \rho^\alpha < \frac{1}{2M} \), and hence \( \rho^\alpha > B^\alpha a^{2\alpha-1}a = Ba^2a^\alpha. \)

Now
\[ t_o^{-Ba^2} \int_0^{\frac{|x-\eta(\tau)|}{(t_o-\tau)^{3/2}}} e^{-\frac{(x-\eta(\tau))^2}{c(t_o-\tau)}} d|\mu|(\tau) \]

\[ = \left( t_o^{-pa^{1/\alpha}} + \int_{t_o^{-pa^{1/\alpha}}}^{t_o^{-Ba^2}} \right) \frac{|x-\eta(\tau)|}{(t_o-\tau)^{3/2}} e^{-\frac{(x-\eta(\tau))^2}{c(t_o-\tau)}} d|\mu|(\tau). \]

Consider the 2nd integral first. In this integral \(|x-\eta(\tau)-a| \leq M(t_o-\tau)^{\alpha} < M\rho^\alpha a < \frac{1}{2}a\), and this enables us to replace \(x-\eta(\tau)\) by \(a\).

Let \(k_o\) be a positive integer such that

\[ 2^{-k_o}pa^{1/\alpha} \leq Ba^2 \Rightarrow 2^{k_o} \geq \rho B^{-\frac{1}{\alpha}} \geq 1 \]

and

\[ 2^{-k_o+1}pa^{1/\alpha} > Ba^2 \Rightarrow 2^{k_o} < 2\rho B^{-\frac{1}{\alpha}} \].

Then,

\[ t_o^{-Ba^2} \int_{t_o^{-pa^{1/\alpha}}}^{a} \frac{a}{(t_o-\tau)^{3/2}} e^{-\frac{a^2}{c(t_o-\tau)}} d|\mu|(\tau) \]

\[ \leq c \sum_{k=1}^{k_o} \int_{t_o^{-2^{-k+1}pa^{1/\alpha}}}^{t_o^{-2^{k}pa^{1/\alpha}}} \frac{a}{(t_o-\tau)^{3/2}} e^{-\frac{a^2}{c(t_o-\tau)}} d|\mu|(\tau) \]

\[ \leq c\bar{\mu}(t_o) \sum_{k=1}^{k_o} 2^{-k+1}pa^{1/\alpha}[t_o-(t_o-2^{-k}pa^{1/\alpha})]^{-3/2} \]

\[ = 2c\bar{\mu}(t_o)\rho^{-\frac{1}{2}}a^{-\frac{1}{2}} \sum_{k=1}^{k_o} 2k/2 \leq c\rho^{-\frac{1}{2}}\mu(t_o)a^{-\frac{1}{2}} \frac{k_o}{2} \]
\[ c \rho^{\frac{1}{2}} \overline{\mu}(t_0) \frac{1 - \frac{1}{2\alpha}}{a^{\frac{1}{2\alpha} - 1}} \frac{1}{\rho^{\frac{1}{\alpha} - \frac{1}{2}}} \leq c \overline{\mu}(t_0). \]

As for the other integral, we choose \( k_1 \), an integer, such that \( 2^{k_1 + 1} \rho a^{1/\alpha} \geq t_o \Rightarrow 2^{k_1} \geq t_o (2\rho)^{-1} a^{-1/\alpha} \) and \( 2^{k_1} \rho a^{1/\alpha} < t_o \Rightarrow 2^{k_1} < t_o (2\rho)^{-1} a^{-1/\alpha} \). If the only integer satisfies these two conditions is a negative integer, then \( t_o < \rho a^{1/\alpha} \) and there is no such integral. So, we may assume \( k_1 \geq 0 \), then

\[
\int_{t_o - \rho a^{1/\alpha}}^{t_o} \frac{|x - \eta(\tau)|}{(t_o - \tau)^{3/2}} e^{-\frac{(x - \eta(\tau))^2}{c(t_o - \tau)}} d|\mu|(\tau)
\]

\[
= \sum_{k=0}^{k_1} \int_{t_o - 2^{k+1} \rho a^{1/\alpha}}^{t_o - 2^{k+1} \rho a^{1/\alpha}} \frac{|x - \eta(\tau)|}{(t_o - \tau)^{3/2}} e^{-\frac{(x - \eta(\tau))^2}{c(t_o - \tau)}} d|\mu|(\tau)
\]

\[ \leq \overline{\mu}(t_o) \rho a^{1/\alpha} \sum_{k=0}^{k_1} 2^{k+1} [t_o - (t_o - 2^{k+1} \rho a^{1/\alpha})^{-1}]^{-1} \max \frac{|x - \eta(\tau)|}{(t_o - \tau)^{3/2}} e^{-\frac{(x - \eta(\tau))^2}{c(t_o - \tau)}} \]

\[ = 2\overline{\mu}(t_o) \rho a^{1/\alpha} \sum_{k=0}^{k_1} \rho^{-1} a^{-1/\alpha} \max \frac{|x - \eta(\tau)|}{(t_o - \tau)^{3/2}} e^{-\frac{(x - \eta(\tau))^2}{c(t_o - \tau)}} \]

\[ = \ast. \]

When \( \tau \in (t_o - 2^{k+1} \rho a^{1/\alpha}, t_o - 2^{k} \rho a^{1/\alpha}) \),

\[ |x - \eta(\tau) - a| \leq M(t_o - \tau)^{\alpha} \leq c 2^k a. \] Thus
\[ \frac{|x-\eta(\tau)|}{(t_o-\tau)\frac{3}{2}} \leq (c2^{k\alpha}a)(2^{k\rho}a^{1/\alpha})^{-\frac{1}{2}}, \text{ as } 2^{k\rho}a^{1/\alpha} < t_o-\tau. \text{ Thus,} \]

\[ \star \leq c\mu(t_o) \sum_{k=0}^{k_1} \frac{k\alpha-k}{2a} \frac{1-\frac{1}{2\alpha}}{a} \]

\[ \leq c\mu(t_o) a \frac{1-\frac{1}{2\alpha}}{2a} k_o(\alpha^{-\frac{1}{2}}) \]

\[ \leq c\mu(t_o) a \frac{1-\frac{1}{2\alpha}}{2a} (t_o/\rho)^{\alpha^{-\frac{1}{2}}} \frac{1}{a^{\alpha^{-\frac{1}{2}}}} \]

\[ \leq c\mu(t_o). \quad \text{Q.E.D.} \]

**Lemma 1.7:** Let \( \mu \) be a measure, and assume that \( \mu'(t_o) \)

\[ = \lim_{\varepsilon \to 0} (1/2\varepsilon) \int_{t_o-\varepsilon}^{t_o+\varepsilon} d\mu \text{ exists. Let} \]

\[ T\mu(x,t) = \int_{0}^{t} \frac{x-\eta(\tau)}{(t-\tau)^{3/2}} e^{-\frac{(x-\eta(\tau))^2}{4(t-\tau)}} d\mu(\tau) \]

\[ = \int_{0}^{t_o} \frac{\eta(t_o)-\eta(\tau)}{(t_o-\tau)^{3/2}} e^{-\frac{(\eta(t_o)-\eta(\tau))^2}{4(t_o-\tau)}} d\mu(\tau) - 2\sqrt{\pi} \mu'(t_o). \]

Then, \( \lim \sup_{\Gamma_{t_o}} |T\mu(x,t)| \leq c |\mu|^* (t_o), \text{ where } |\mu|^* (t_o) \)

\[ = \lim_{\varepsilon \to 0} (1/2\varepsilon) \int_{t_o-\varepsilon}^{t_o+\varepsilon} d|\mu|. \]
**Proof:** It is clear that \(|2\sqrt{\pi} \mu'(t_o)| \leq c\mu(t_o)|. Therefore, by Lemma 1.3 and Lemma 1.6, \(\sup_{\Gamma t_o} |T\mu(x,t)| \leq c\mu(t_o).\)

Let \(\varphi_\varepsilon(t)\) be a \(C^\infty\) function such that \(\varphi_\varepsilon(t) = 1\) if \(|t-t_o| \geq \varepsilon\), \(\varphi_\varepsilon(t) = 0\) if \(|t-t_o| \leq \varepsilon/2\) and \(0 < \varphi_\varepsilon(t) < 1\) elsewhere.

Let \(\mu_\varepsilon = \varphi_\varepsilon \mu, (i.e., \int f(\tau) d\mu_\varepsilon(\tau) = \int f(\tau) \varphi_\varepsilon(\tau) d\mu(\tau)), \varphi_\varepsilon = (1-\varphi_\varepsilon)\mu\). Thus, \(\mu = \mu_\varepsilon + \nu_\varepsilon\).

Since \(\varphi_\varepsilon(t) = 0\) in \([t_o - \frac{\varepsilon}{2}, t_o + \frac{\varepsilon}{2}]\), it follows that \(\mu_\varepsilon'(t_o) = 0\), and

\[
\lim_{\Gamma t_o} \left[ \int_0^t \frac{x - \eta(\tau)}{(t - \tau)^{3/2}} e^{-\frac{(x - \eta(\tau))^2}{4(t - \tau)}} \varphi_\varepsilon(\tau) d\mu(\tau) \right.
\]

\[
- \int_0^{t_o} \frac{\eta(t_o) - \eta(\tau)}{(t_o - \tau)^{3/2}} e^{-\frac{(\eta(t_o) - \eta(\tau))^2}{4(t_o - \tau)}} \varphi_\varepsilon(\tau) d\mu(\tau) \]

\[
= 0.
\]

\[
\lim_{\Gamma t_o} \sup |T\mu(x,t)| = \lim_{\Gamma t_o} \sup |T\nu_\varepsilon(x,t)|
\]

\[
\leq \sup_{\Gamma t_o} |T\nu_\varepsilon(x,t)| \leq c\nu_\varepsilon(t_o)
\]

\[
= \sup_{0<\delta<\infty} \frac{1}{2\delta} \int_{t_o-\delta}^{t_o+\delta} (1-\varphi_\varepsilon(\tau)) d|\mu|(|\tau|)
\]
\[
= \sup_{0 < \delta < \epsilon} \frac{1}{2\delta} \int_{t_0 - \delta}^{t_0 + \delta} (1 - \nu_\epsilon(\tau)) d|\mu| (\tau)
\]
\[
\leq \sup_{0 < \delta < \epsilon} \frac{1}{2\delta} \int_{t_0 - \delta}^{t_0 + \delta} d|\mu| (\tau).
\]
\[
\therefore \limsup_{\Gamma_{t_0}} |T\mu(x,t)| \leq \limsup_{\epsilon \to 0} \sup_{0 < \delta < \epsilon} \frac{1}{2\delta} \int_{t_0 - \delta}^{t_0 + \delta} d|\mu| (\tau)
\]
\[
= |\mu|^*(\tau). \quad \text{Q.E.D.}
\]

**Lemma 1.8:** Let \( \mu \) be a measure absolutely continuous with respect to Lebesgue measure such that \( \mu'(t_0) \) exists. Let \( T\mu(x,t) \) be as in the previous lemma. Then \( \limsup_{\Gamma_{t_0}} |T\mu(x,t)| = 0. \)

**Proof:** Let the expression for \( T\mu(x,t) \) in the previous lemma, with \( d\mu(\tau) \) replaced by \( d\tau \), be \( T_1(x,t) \). Then,
\[
\limsup_{\Gamma_{t_0}} |T\mu(x,t)| = \limsup_{\Gamma_{t_0}} |T(\mu - \mu'(t_0)1)(x,t) |
\]
\[
+ \mu'(t_0)T_1(x,t) | \leq \limsup_{\Gamma_{t_0}} |T(\mu - \mu'(t_0)1)(x,t) | \quad \text{(by Lemma 1.5)}
\]
\[
\leq |\mu - \mu'(t_0)1|^*(t_0) = 0. \quad \text{Q.E.D.}
\]

**Theorem 1:** The function
\[
u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{0}^{t} \frac{x - \eta(\tau)}{(t - \tau)^{3/2}} e^{-\frac{(x - \eta(\tau))^2}{4(t - \tau)}} \phi(\tau) d\tau
\]
is a solution of (P), where \( \varphi \) is the solution of the following integral equation:

\[
\varphi(t) = f(t) - \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\eta(t) - \eta(\tau)}{(t-\tau)^{3/2}} \frac{-(\eta(t) - \eta(\tau))^2}{4(t-\tau)} \varphi(\tau) d\tau,
\]

and \( f(t) \) is the given boundary value.

**Proof:** The existence and uniqueness of \( \varphi \) which is in \( L^1(0,T) \) have been established in Lemma 1.1. It is easily checked that \( u(x,t) \) satisfies the heat equation in the domain indicated in (P). As to the boundary value, we have shown, in Lemma 1.8., that

\[
\lim_{\Gamma_{t_0}} u(x,t) = \lim_{\Gamma_{t_0}} \int_{t_0}^t \frac{1}{2\sqrt{\pi}} \int_0^t \frac{x-\eta(\tau)}{(t-\tau)^{3/2}} \frac{-(x-\eta(\tau))^2}{4(t-\tau)} \varphi(\tau) d\tau
\]

\[
= \frac{1}{2\sqrt{\pi}} \int_0^{t_0} \frac{\eta(t_0) - \eta(\tau)}{(t_0-\tau)^{3/2}} \frac{-(\eta(t_0) - \eta(\tau))^2}{4(t_0-\tau)} \varphi(\tau) d\tau + \varphi(t_0),
\]

a.e., (here we replace \( \mu(\tau) \) by the indefinite integral of \( \varphi \)). Thus, \( u(x,t) \) will take on the right boundary value in the right manner if and only if \( \varphi \) satisfies the mentioned integral equation. This derivation is standard, for reference, see E. Goursat: *A course in Mathematical Analysis*, Vol. III, part one, p. 308-319. Q.E.D.

Note: All measures are Borel measures.
II. Non-tangential Limits of a Solution of a Boundary Value Problem for the Heat Equation.

(A) A Privaloff-Plesner-Calderón Theorem for a solution of a boundary value problem for the heat equation.

As the title of this section indicates, we shall establish the fact that a solution of such a problem is "parabolically" bounded if and only if it has "parabolic" limit a.e., the precise statement of this theorem and the definitions of the terms will appear subsequently.

A theorem of this type for functions harmonic in a half space was proved first by A. P. Calderón [1] and then it was generalized to 3-dim Liapunor domain by Widman [5]. Recently Hunt and Wheeden used a technique similar to Carleson [2] and were able to prove this theorem for harmonic function in a more general setting. In 1964, Hattemer [4] proved an analogous theorem for a solution of an initial value problem for the heat equation. Here we deal with a solution of a boundary value problem for the heat equation. In order to prove this theorem, it is essential to prove Lemma 2.1 (Calderón's Lemma, [1]) which allows us to disregard the various "sizes" of the "parabolic cones" and look at those of a fixed size only. Before we start to prove Lemma 2.1, we need a density theorem. First of all, with a slightly modified argument, we can draw the following conclusion from a theorem of
Edwards and Hewitt ([3] Theorem 2.):

Let \( B^\gamma_a = \{(x_1, \ldots, x_n) \in \mathbb{R}^n: \ |x_i - a_i| \leq \frac{1}{2}\gamma, \quad i = 1, \ldots, n-1, \ |x_n - a_n| \leq \frac{1}{2}\gamma^2 \} \), where \((a_1, \ldots, a_n) = a\) is a point in \(\mathbb{R}^n\), \(0<\gamma<\infty\). Let \(E\) be a Lebesgue measurable set in \(\mathbb{R}^n\). Then

\[
\lim_{\gamma \to 0} \frac{1}{m(B^\gamma_a)} \int_{B^\gamma_a} |f(x) - f(a)| \, dx = 0
\]

\(\gamma \in E - E'\), where \(E' \subset E\), \(m(E') = 0\) (\(m(A)\) denotes the Lebesgue measure of the set \(A\)), and \(E'\) depends on \(f\) and \(f\) is an \(L^1\) function.

From this assertion, it is trivial to obtain the density theorem we need:

Let \(B^\gamma_a\) be defined as above. Let \(\{S^a_r\}_{r=1}^\infty\) be a sequence of Lebesgue measurable sets such that

(i) \(\{S^a_r\}\) shrinks to the point \(a\) as \(r \to 0\),

by this we mean \(d(r)\) is a monotone decreasing function of \(r\) tending to zero, as \(r \to 0\), where \(d(r) = \max\{|a-x|: x \in S^a_r\}\);

(ii) to each \(r \geq \gamma\) such that \(S^a_r \subset B^\gamma_a\) and \(m(S^a_r)/m(B^\gamma_a) \geq \beta > 0\). If \(f(x)\) is an \(L^1\) function, and \(E\) is a Lebesgue measurable set in \(\mathbb{R}^n\), then

\[
\lim_{S^a_r \to a} \frac{1}{m(S^a_r)} \int_{S^a_r} |f(x) - f(a)| \, dx = 0
\]
\[ V \alpha \in E - E', \text{ where } E' \subset E, \ m(E') = 0 \text{ and } E' \text{ depends on } f. \]

**NOTATIONS AND CONVENTIONS**

\[ \Gamma(x, t, y) = \left( \frac{4\pi}{2} \right) \frac{n+3}{t} e^{\frac{|x|^2 + y^2}{4t}}, \quad t > 0, \]

\[ = 0, \quad t \leq 0, \]

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \ |x|^2 = x_1^2 + \ldots + x_n^2; \)

\( y \in \mathbb{R}_+ = (0, \infty). \)

Let \( P(x, t; \alpha) = \{(z, \tau, y) : (z, \tau) \in \mathbb{R}^{n+1}, y \in \mathbb{R}_+, \alpha > 0, |z - x|^2 + |\tau - t| < \alpha y^2\}, \)

\[ P_h(x, t; \alpha) = P(x, t; \alpha) \cap \{(z, \tau, y) : (z, \tau) \in \mathbb{R}^{n+1}, 0 < y < h\}. \]

The sets \( P(x, t; \alpha) \) and \( P_h(x, t; \alpha) \) are called a **parabolic cones**.

**LEMMA 2.1.** Suppose that

1. \( E \) is a Lebesgue measurable set in \( \mathbb{R}^{n+1} \), with positive finite Lebesgue measure \( m(E) \);
2. \( \alpha(x, t), h(x, t) \) are positive numbers defined for each \( (x, t) \in E; \)
3. \( f \) is a non-negative continuous function on \( \mathbb{R}_+^{n+2} = \{(x, t, y) : x \in \mathbb{R}^{n}, t \in \mathbb{R}, y \in \mathbb{R}_+\}; \)
4. \( \sup_{f < \infty} V(x, t) \in E; \)
\[ P_h(x, t)(x, t; \alpha(x, t)) \]
5. \( \alpha, \varepsilon \) are two positive numbers.

Then \( \exists h \in (0, \infty) \) and a compact set \( E' \subset E \) such that \( m(E') > m(E) - \varepsilon \), and
\[ \sup \bigcup_{(x,t) \in E'} P_h(x,t; \alpha) \quad f < \infty. \]

**Proof:** Let \( E_{k,N} = \{ (x,t) \in E : f < N \text{ in } P_{1/k}(x,t; 1/k) \} \), it is a measurable set since it is open in \( E \). Moreover, \( E = \bigcup E_{k,N} \).

Let \((x_o, t_o) \in E_{k,N}\) be a point of density \(1\) with respect to \(E_{k,N}\) and \( \{ B_{(x_o, t_o)}^\gamma \} \). (A point \( a \) is said to have density \(1\) with respect to a measurable set \( E \) and \( \{ B_a^\gamma \} \) if \( \lim \frac{1}{m(S_r^a)} \int_{S_r^a} \chi_E \, d\mu = 1 \), where \( \mu \) is the Lebesgue measure and \( \{ S_r^a \} \) and \( \{ B_a^\gamma \} \) are as before.) Then \( a \) \( h = h(E, x_o, t_o, k, N) \) such that \( P_h(x_o, t_o; \alpha) \subset \bigcup_{(x,t) \in E_{k,N}} P_{1/k}(x,t; 1/k) \).

For, let \((z, \tau, y) \in P_h(x_o, t_o; \alpha)\), and \( \Sigma = \{(x,t) : |x-z|^2 + |t-\tau| < y^2/k \} \) where \(0 < y < h\) and \(|z-x_o|^2 + |\tau-t_o| < \alpha y^2\).

Then, \((x,t) \in \Sigma\) implies that \(|x-x_o| \leq |x-z| + |z-x_o| < \left( \frac{1}{\sqrt{k}} + \sqrt{\alpha} \right) y\) and \(|t-t_o| \leq |t-\tau| + |\tau-t_o| < \left( \frac{1}{k} + \alpha \right)y^2\).

Thus \( \Sigma \) is contained in an \( n+1 \)-dimension box \( B \) centered at \((x_o, t_o)\) with length \(2\left( \frac{1}{k} + \alpha \right)y^2\) in the \( t \) direction and length \(2\left( \frac{1}{\sqrt{k}} + \sqrt{\alpha} \right)y\) in each \( x_i \) direction. Moreover,
m(Σ)/m(B) is a positive constant independent of y. Thus, by the definition of density, m(E_{k,N} \cap Σ)/m(Σ) \to 1 as y \to 0. Choose h < \frac{1}{k} such that m(E_{k,N} \cap Σ) > 0 for 0 < y < h, (this h depends on E, x_0, t_0, k, N). There exists (x, t) \in E_{k,N} \cap Σ; for this (x, t), |x - z|^2 + |t - τ| < y^2/k and hence (z, τ, y) ∈ P_1/k(x, t; 1/k).

Now we define E_N = \{(x, t) ∈ E: f < N in P_1/N(x, t; α)\}. Since for almost all (x_0, t_0) of E_{k,N} are of density 1 with respect to E_{k,N} by the density theorem, for these (x_0, t_0), P_h(x_0, t_0; α) ⊆ \bigcup_{(x, t) \in E_{k,N}} P_1/k(x, t; 1/k) and so for almost all (x_0, t_0) ∈ E_{k,N}, (x, t, y) ∈ P_h(x_0, t_0; α) implies f < N.

Hence, E_{k,N} ⊆ \bigcup_{M=1}^{∞} E_M a.e. and thus E ⊆ \bigcup_{M=1}^{∞} E_M a.e. as E = \bigcup_{E_{k,N}}. Finally, \{E_M\} is a monotone increasing sequence of sets, so ∃ M such that m(E_M) > m(E) - ε. The lemma now follows from the fact that we can approximate E_M from inside by compact sets. Q.E.D.

**Definition:** If u(x, t, y) is defined for (x, t) ∈ R^{n+1}, y ∈ R_+, then u is said to have a parabolic limit u(x_0, t_0, 0) at (x_0, t_0) ∈ R^{n+1} if u(x, t, y) tends to u(x_0, t_0, 0) as (x, t, y) tends to (x_0, t_0, 0) in parabolic cone P(x_0, t_0; α). We shall denote this by

$$\lim_{P(x_0, t_0, α)} u(x, t, y) = u(x_0, t_0, 0).$$
A function $u(x,t,y)$ is said to be \textbf{parabolically bounded} at $(x_0,t_0) \in \mathbb{R}^{n+1}$ if $\exists M > 0$, $h > 0$ and $\alpha > 0$ such that $|u(x,t,y)| \leq M$ for $(x,t,y) \in P_h(x_0,t_0;\alpha)$.

**Theorem 2:** If $u(x,t,y)$ satisfies $u_t = u_{x_1}x_1 + \cdots + u_{x_n}x_n + u_{yy}$ in $\mathbb{R}^{n+2}$, and if $u$ is parabolically bounded on a measurable set $S \subset \mathbb{R}^{n+1}$, then $u$ has parabolic limits a.e. in $S$.

**Proof:** First of all, we outline our proof.

For each $(x,t) \in S$ there exist positive numbers $M, h, \alpha$, all depending on $(x,t)$, such that $|u(z,s,y)| \leq M$ for $(z,s,y) \in P_h(x,t;\alpha)$. By Lemma 2.1, for any fixed $r(>0)$, $\varepsilon > 0$, $\exists k > 0$ such that $\sup_{S_r} |u(z,s,y)| < \infty$, where $S_r = \bigcup_{(x,t) \in S_r} P_{2k}(x,t;r)$ and $S_r$ is a subset of $S$ such that $m(S-S_r) < \frac{\varepsilon}{2r}$. If we can prove that $\lim_{(z,s,y) \in P_{2k}(x,t;r)} u(z,s,y)$ exists for almost all $(x,t) \in S_r$, then we are done.

For, if we let $S_r'$ be the subset of $S_r$ such that the limit does not exist, then $m(S_r') = 0$. The set of points in $S$ at which $u$ fails to have a parabolic limit is $E = \bigcup_{r=1}^{\infty} \{ (x,t) \in S: r=1 \}

\lim_{(z,s,y) \in P_{2k}(x,t;r)} u(z,s,y)$ does not exist}. Then $E \subset \bigcup_{r=1}^{\infty} (S-S_r) \cup S_r'$ and $m(E) \leq \sum_{r=1}^{\infty} \frac{\varepsilon}{2r} = \varepsilon$. Since $\varepsilon$ is arbitrary,
the theorem follows.

Now, we may fix out attention on $S_r$ ($r$ is fixed) and try to move $m(S_r') = 0$. We can assume $M=1$. Since $S_r$ can be covered by a countable number of unit hypercubes, we may as well assume that $S_r$ itself is contained in a unit hypercube. Finally, we enlarge $S_r$ to the set $\{(x,t) \in Q: Q$ is a closed unit hypercube containing $S_r$, $|u(z,s,y)| \leq 1$ for $(z,s,y) \in P_{2k}(x,t;r)\}$; rename this set $S$. $S$ is obviously closed.

Let

$$\mathcal{S} = \bigcup_{(x,t) \in S} P_k(x,t;r); \mathcal{S} = \partial S;$$

$$\mathcal{S}_k = \{(x,t,y): (x,t,y + \frac{1}{k}) \in \mathcal{S}\};$$

$$\mathcal{U}_k = \{x,t,0): (x,t,1/k) \in \mathcal{S}\} = \mathcal{S}_k \cap (\mathbb{R}^{n+1} \times \{0\});$$

$\chi_k$ the characteristic function of $\mathcal{U}_k$;

$$u_k(x,t,y) = u(x,t,y + 1/k);$$

$$f_k(x,t) = \chi_k(x,t)u_k(x,t,0) = \chi_k(x,t)u(x,t,1/k);$$

$$\varphi_k(x,t) = \int_{\mathbb{R}^{n+1}} \Gamma(x-z,t-s,y)f_k(z,s)dzds;$$

$$\psi_k(x,t,y) = u_k(x,t,y) - \varphi_k(x,t,y).$$

Since $\|f_k\|_2 \leq \|\chi_k\|_2 \leq c_r$, $\exists$ a subsequence $\{f_{k'}\}$ of $\{f_k\}$ such that $f_{k'}$ converges weakly to an $L^2$ function $f$.

Thus, $\lim_{k' \to \infty} \varphi_{k'}(x,t,y) = \int_{\mathbb{R}^{n+1}} \Gamma(x-z,t-s,y)f_{k'}(z,s)dzds$.
\[
\Gamma(x-z,t-s,y)f(z,s)dzds = \varphi(x,t,y), \text{ as } \Gamma(x-z,t-s,y)
\]
considered as a function of \((z,s)\) is in \(L^2\);
and
\[
\lim_{k' \to \infty} \psi_{k'}(x,t,y) = \lim_{k' \to \infty} (u_{k'}(x,t,y) - \varphi_{k'}(x,t,y)) = u(x,t,y)
\]

- \(\varphi_{k}(x,t,y)\) exists for every \((x,t,y) \in \mathbb{R}^{n+2}_+\). But, \(\varphi(x,t,y)\)

has parabolic limits a.e. in \(S\), (this follows from Lemma 1.4 by replacing \(\eta(t)\) by 0 and generalizing to \(n\) spatial variables).

Therefore, it remains to show that \(\psi(x,t,y) (= u(x,t,y) - \varphi(x,t,y))\) has parabolic limit a.e. in \(S\); as a matter of fact, we shall show that \(\lim_{P(x,t;r)} \psi(a,s,y) = 0\) for almost all \((x,t) \in S\).

Notice that \(|\psi_{k}| \leq 2\) in \(\varnothing\) and for a point \((x,t) \in S\), \(\chi_{k}\) is identically 1 in a neighborhood of \((x,t)\), so \(f_{k}(z,s) = \chi_{k}(z,s)u_{k}(z,s)\) is continuous in this neighborhood of \((x,t)\), hence \(\varphi_{k}(z,s,y)\) converges to \(\chi_{k}(x,t)u(x,t,1/k)\) as \((z,s,y) \in \varnothing\) tends to \((x,t) \in S\). Thus \(\psi_{k}(z,s,y)\) tends to 0 as \((z,s,y) \in \varnothing\) tends to \((x,t) \in S\).

If we can construct a function \(w(x,t,y)\) in \(\mathbb{R}^{n+2}_+\) such that \(w_t = w_{x_1x_1} + \cdots + w_{x_nx_n} + w_{yy}\) and \(w\) satisfies (a) \(w(x,t,y) \geq 0, (x,t,y) \in \mathbb{R}^{n+2}_+\); (b) \(w(x,t,y) \geq 2\) for \((x,t,y) \in \varnothing-S\); and (c) for almost all \((x,t) \in S\) \(\lim_{P(x,t;r)} w(z,s,y) = 0\).

Then the theorem follows. For, \(w \pm \psi_{k} \geq 0\) in \(\varnothing-S\), and the \(\lim \inf\) of

\([w(z,s,y) \pm \psi_{k}(z,s,y)] \geq 0\) as \((z,s,y) \in \varnothing\) tends to any \((x,t) \in S\).
implies \( w(x,t,y) \pm \psi_k(x,t,y) \geq 0 \) \( \forall (x,t,y) \in \mathbb{S} \). (If not, then \\
\( \exists (x^*_0,t^*_0,y^*_0) \in \mathbb{S}, \ w(x^*_0,t^*_0,y^*_0) \pm \psi_k(x^*_0,t^*_0,y^*_0) \leq c < 0 \). Let \\
\( \mathbb{S}^n = \mathbb{S} \cap \{(x,t,y) : \frac{1}{n} < y < 1\} \), then \( \exists N \) such that \( (x^*_0,t^*_0,y^*_0) \in \mathbb{S}^n \) for \( n \geq N \). Let \( n_0 > N \), then \( \exists \) a point on \( \partial \mathbb{S}^{n_0} \) with \( t \) coordinate less than or equal to \( t^*_0 \) such that the value of \( w \pm \psi_k \) is less than or equal to \( c \) at this point. Since \( w \pm \psi_k \geq 0 \) on \( \mathbb{S} - S \), so this point must have coordinates like \( (x_m,t_m,1/m) \). Since there exists at least one such point, thus, this sequence of points will tend to a point in \( S \) as \( m \to \infty \). But the \( \lim \inf \) of \\
\( (w \pm \psi_n) \geq 0 \) when points in \( \mathbb{S} \) tend to a point in \( S \), this is a contradiction. So, \( |\psi_k| \leq w \) in \( \mathbb{S} \). Let \( k \) tend to infinity first and then let \( (z,s,y) \) tend to \( (x,t) \in S \), we have \\
\( \lim \psi(z,s,y) = 0 \) a.e. in \( S \), by (c). Hence the theorem \\
P(x,t;r) \\
is proved.

Construction of \( w \):

Let \( w(x,t,y) = 2y + c \int_{\mathbb{R}^{n+1}} \Gamma(x-z,t-s,y)\chi(z,s)dzds \), \\
where \( c \) is a positive constant to be determined later and \( \chi \) is the characteristic function of \( \mathbb{R}^{n+1} - S \).

This \( w(x,t,y) \) satisfies conditions (a) and (c) obviously. Since the integral is non-negative, \( w \geq 2 \) on \( y = 1 \). For the other parts of \( \mathbb{S} - S \), we notice first that if \( (x,t,y) \in \mathbb{S} - S \), then \( y > 0 \), as \( S \) is closed. But, if \( (x,t,y) \in \mathbb{S} \) and \( 0 < y < 1 \) then \( \mathcal{E} = \{(z,s) \in \mathbb{R}^{n+1} : |x-z|^2 + |s-t| < ry^2\} \)
does not intersect $S$; otherwise there will be a point $(x', t')$
$s.t. \quad P_k(x', t'; r) \subseteq \emptyset$, thus $(x, t, y) \notin \emptyset - \emptyset$. So,
we have, $w(x, t, y) \geq c \int_{\emptyset} \Gamma(x-z, t-s, y) dz ds$

$$w(x, t, y) \geq c \int_{\emptyset} \Gamma(x-z, t-s, y) dz ds$$

$$= c \left( \frac{n+1}{2} \int_{\left| z-x \right|^2 + |s-t| < k} \right) e^{-\frac{|x-z|^2 + y^2}{4(t-s)}} dz ds$$

$$= c \left( \frac{n+1}{2} \int_{\left| z-x \right|^2 + |s-t| < k} \right) e^{-\frac{|x-z|^2 + y^2}{4(t-s)}} dz ds$$

$$= c \left( \frac{n+1}{2} \int_{\left| z-x \right|^2 + |s-t| < k} \right) e^{-\frac{|x-z|^2 + y^2}{4(t-s)}} dz ds$$

This proves the theorem if we choose $c$ so that the last
quantity is $\geq 2$. \hfill \text{Q.E.D.}
(B) Stein's Theorem

In this part of section II we shall prove Stein's theorem ([5 ], theorem 1) for the functions we have been discussing; this theorem changes the qualitative character of a function -- having parabolic limits a.e. -- to a quantitative one -- the finiteness of the "generalized area function". But in the original proof, Stein established a regularization theorem for the regions he considered (lemma 3, [5]), and we fail to have an equivalent theorem here because of the geometry of the parabolic cones; thus, a slightly different argument is needed, and this argument is provided by a certain form of Green's theorem (lemma 2.2).

We shall use $|\nabla u|^2$ to denote $\sum_{i=1}^{n} |u_{x_i}|^2 + |u_y|^2,$

and $c$ to denote a generic constant independent of essential factors.

**Lemma 2.2.** Let

$D_N = \bigcup_{k=1,\ldots,N} \{(x_k, t_k, y_o) + P_\alpha(0,0;\alpha) : (x_k, t_k, y_o) \in \mathbb{R}^{n+2}, y_o < 0\},$

$B_o = D_N \cap \{(x, t, 0) : (x, t) \in \mathbb{R}^{n+1}\},$

$B_1 = D_N \cap \{(x, t, 1) : (x, t) \in \mathbb{R}^{n+1}\},$

$R_N = \text{the part of } D_N \text{ between } B_o \text{ and } B_1, S = \partial R_N - (B_o \cup B_1).$

Let $\Pi A$ denote the orthogonal projection of a set $A \subset \mathbb{R}^{n+2}$ onto $\mathbb{R}^{n+1}$ space (i.e., $(x, t)$ space).
If \( u(x,t,y) \in C^1(\overline{\mathbb{R}}_N) \), then

\begin{align*}
(1) & \quad \int_{\mathbb{R}^N} u_{x_i}(x,t,y) dv = \int_{\mathbb{P}^k} u(x,t,\delta(x,t)) \delta_{x_i}(x,t) dx dt \ (i=1,2,\ldots,n); \\
(2) & \quad \int_{\mathbb{R}^N} u_t(x,t,y) dv = \int_{\mathbb{P}^k} u(x,t,\delta(x,t)) \delta_t(x,t) dx dt; \\
(3) & \quad \int_{\mathbb{R}^N} u_y(x,t,y) dv = -\int_{\mathbb{B}_0} u(x,t,0) dv + \int_{\mathbb{P} \mathbb{B}_1} u(x,t,1) dv dt \\
& \quad - \int_{\mathbb{P}^k} u(x,t,\delta(x,t)) dx dt
\end{align*}

where \( y = \delta(x,t) \) is the equation of \( S \).

**Proof:** Let \( \varphi \in C_0^\infty(\mathbb{R}^{n+1}) \) such that \( \int \varphi = 1, \varphi \geq 0 \). Let

\[
f_\eta(x,t) = \eta^{-n-1} \int_{\mathbb{R}^{n+1}} \varphi(z/\eta,s/\eta) \delta(x-z,t-s) dz ds;
\]

then \( f_\eta(x,t) \) tends to \( \delta(x,t) \) uniformly as \( \eta \to 0 \). For a given positive \( \varepsilon \), choose \( \eta = \eta(\varepsilon) \) such that \( |f_\eta(x,t) - \delta(x,t)| < \varepsilon \) and set

\[
\delta_\varepsilon(x,t) = f_\eta(\varepsilon)(x,t) + 2\varepsilon.
\]

Taking a subset of this collection \( \{ \delta_\varepsilon(x,t) \} \), reindex \( \varepsilon \) if necessary, we have:

(a) \( \delta_\varepsilon(x,t) \geq \delta(x,t) \);

(b) \( \delta_\varepsilon_1(x,t) \geq \delta_\varepsilon_2(x,t) \), if \( \varepsilon_1 > \varepsilon_2 \);

(c) \( \delta_\varepsilon(x,t) \to \delta(x,t) \) as \( \varepsilon \to 0 \). Let the region (in \( \mathbb{R}^{n+2} \)) bounded by the surface \( y = \delta_\varepsilon(x,t), \mathbb{B}_1 \), and the hyperplane \( y = k(\varepsilon) \) (\( k(\varepsilon) \) is a positive constant less than 1, \( k(\varepsilon) = \delta_\varepsilon(x,t) \), \( (x,t) \in \mathbb{B}_0 \).
by (c), \( k(\epsilon) \) can be chosen so that \( k(\epsilon) \to 0 \) as \( \epsilon \to 0 \) be \( R_{N\epsilon} \).

Let \( B_{1\epsilon} = B_1 \cap \partial R_{N\epsilon} \), \( B_{0\epsilon} = \{ y = k(\epsilon) \} \cap \partial R_{N\epsilon} \).

It is obvious that when \( \epsilon \) tends to zero,
\[
\int_{\partial R_N} u(x_1) \, dv \rightarrow \int_{\partial R_N} u(x_1) \, dv.
\]
Let \( S_\epsilon \) be the lateral surface of \( R_{N\epsilon} \);
\[
\int_{\partial R_{N\epsilon}} u(x_1) \, dv \rightarrow \int_{S_\epsilon} u(\delta \epsilon) x_1 \, dxdt,
\]
then by Green's theorem, as the \( x_1 \)-component of the outer unit normal vector on \( B_{0\epsilon} \) and \( B_{1\epsilon} \) is zero.

Notice that the equation of \( \partial R_N \) is of one of the three forms: (a) \( y = 0 \), \( (B_0) \); (b) \( y = 1 \), \( (B_1) \); (c) \( y = a^{-\frac{1}{2}} (|x-x_k|^2 + |t-t_k|^2)^{\frac{1}{2}} - y_o \) for some \( 1 \leq k \leq N \), \( (S) \); since the orthogonal projection on \( \mathbb{R}^{n+1} \) of the set of all points in \( S \) which are common to at least two parabolic cones is a subset of
\[
\mathcal{U}(x,t) = \{ |x-x_i|^2 + |t-t_i| - y_o \in \mathbb{R}^{n+1}, i,j = 1,\ldots,N, i \neq j \},
\]
but \( \{(x,t) : \{ |x-x_j|^2 + |t-t_j| - y_o = |x-x_j|^2 + |t-t_j| - y_o \in \mathbb{R}^{n+1}, i \neq j \} \), as a subset in \( \mathbb{R}^{n+1} \), has measure zero. Since \( t \) can be solved as a continuous function of \( x \) provided \( t-t_i \) and \( t-t_j \) have different signs; if \( t-t_i \) and \( t-t_j \) have the same sign then this set forms an \( n \)-dimensional
hyperplane $|x-x_i|^2 - |x-x_j|^2 = \pm(t_i-t_j)$ again it is measure 
zero as $x_i \neq x_j$ or $t_i = \pm t_j$. Therefore finite union of such 
sets is of measure zero in $\mathbb{R}^{n+1}$. Thus $\delta_{x_i}^\epsilon \ i = 1, \ldots, n$
and $\delta_t^\epsilon$ exist for almost all $(x,t) \in \mathbb{P}S$; moreover, outside 
of the set they don't exist, they are continuous and bounded 
uniformly. Since $\delta$ is absolutely continuous in each variable, 
therefore $(\delta^\epsilon)_{x_i} = (\delta_{x_i})^\epsilon$, $(\delta^\epsilon)_t = (\delta_t)^\epsilon$ where $(\delta_{x_i})^\epsilon$ and 
$(\delta_t)^\epsilon$ denote the mollified $\delta_{x_i}$, $\delta_t$ respectively. So, $(\delta^\epsilon)_{x_i}$ converges to $\delta_{x_i}$ in $L^1$ and we have,

$$
\left| \int_{\mathbb{P}S} u\delta_{x_i} \, dx \, dt - \int_{\mathbb{P}S^\epsilon} u(\delta^\epsilon)_{x_i} \, dx \, dt \right| \leq \int_{\mathbb{P}S} u(\delta_{x_i} - (\delta^\epsilon)_{x_i}) \, dx \, dt
$$

$$
+ \int_{\mathbb{P}S \setminus \mathbb{P}S^\epsilon} u\delta_{x_i} \, dx \, dt.
$$

The first term of the right-hand side tends to 0 as $\epsilon \to 0$
because of the $L^1$ convergence, the second term tends to 0 
as $\epsilon \to 0$ by trivial estimates. This proves (1), and (2) also holds by similar argument.

For (3), again we have $\int_{R_{N\epsilon}} u_y \, dv \longrightarrow \int_{R_N} u_y \, dv$ as $\epsilon \to 0$. By Green's theorem

$$
\int_{R_{N\epsilon}} u_y \, dv = -\int_{\mathbb{P}B_{0\epsilon}} u(x,t,k(\epsilon)) \, dx \, dt + \int_{\mathbb{P}B_{1\epsilon}} u(x,t,1) \, dx \, dt
$$

$\int_{\mathbb{P}S} u(x,t,\delta^\epsilon(x,t)) \, dx \, dt$;

the right-hand side tends to
\[ - \int u(x,t,0)dxdt + \int u(x,t,1)dxdt = \int u(x,t,\delta(x,t))dxdt \text{ as } \varepsilon \to 0, \]

by the uniform continuity of \( u \) and trivial estimates. Q.E.D.

**Remark:** Similar proof for Lemma 2.2 shows that if \( u \in C^1(\Omega) \) where \( \Omega = P_h(x_o, t_c; \alpha) \) then

\[ \int_{\partial \Omega} u_{x_i} (x,t,y) dx dy = \int_{\partial \Omega} u(x,t,\delta(x,t)) \delta_{x_i} dx dt \quad i = 1, 2, \ldots, n; \]

\[ \int_{\partial \Omega} u_{t} (x,t,y) dx dy = \int_{\partial \Omega} u(x,t,\delta(x,t)) \delta_{t} dx dt; \]

\[ \int_{\partial \Omega} u_{y} (x,t,y) dx dy = -\int_{\partial \Omega} u(x,t,\delta(x,t)) dx dt + \int_{\partial \Omega} u(x,t,h) dx dt, \]

where \( s \) is the lateral surface of \( \partial \Omega \), \( B \) is the rest of \( \partial \Omega \).

**Lemma 2.3:** If \( u(x,t,y) \) satisfies the heat equation

\[ u_{x_1 x_1} + \cdots + u_{x_n x_n} + u_{yy} = u_t \text{ in } P_k(x_o, t_o; \beta) \text{ and } |u| \leq 1 \]

there, then

\[ |y_{x_1}|, |y_{x_2}|, |y_2 x_1 x_1|, |y_2 u_{x_1 y}|, |y_2 u_{yy}|, |y_2 y_{x_1}|, |y_3 u_{ty}| - |y_3 u_{tx_1}| \]

are all bounded by \( c \) in \( P_h(x_o, t_o; \alpha) \) \( i = 1, 2, \ldots, n, \)

where \( \alpha < \beta, \ h < h, \) and \( c \) depends only on \( \alpha, \beta, h, k, n. \)

**Proof:** We observe that if \( D \) is a domain in \( \mathbb{R}_+^{n+2}, (x_o, t_o, y_o) \)

is a point in \( D. \) Let \( K_\rho(x_o, t_o, y_o) = \{(x,t,y) : |x_i - x_{o_i}| < \frac{1}{2} \rho, \]

\[ |y-y_o| < \frac{1}{2} \rho, 0 < t_o - t < \rho^2 \}. \) Suppose \( D \supset K_1 \) and \( u(x,t,y) \) satisfies the heat equation in \( D \) and \( |u| \leq 1 \) in \( D \) then by the integral representation of \( u \) (see [4] section 2) we have

\[ |u_{x_1}(x_o, t_o, y_o)| \leq c \text{ and } |u_y(x_o, t_o, y_o)| \leq c \text{ and } |u_t(x_o, t_o, y_o)| \leq c. \]

If \( K_\rho \subset D, \)
then by changing \((x, t, y)\) to \((\bar{x}, \bar{t}, \bar{y})\) where \(\rho \bar{x} = x, \rho^2 \bar{t} = t, \rho \bar{y} = y\) we have
\[ |\rho \ u_{x_1}(x_o, t_o, y_o)| \leq c, \ |\rho^2 \ u_t(x_o, t_o, y_o)| \leq c, \]
\[ |\rho \ u_y(x_o, t_o, y_o)| \leq c. \]

Now, let \((x, t, y)\) be any point in \(P_h(x_o, t_o; \alpha)\). Since \(h < k, \alpha < \beta\), there exists fixed constant \(c_o > 0\) such that \(K_{c_o y}(x, t, y)\) lies in \(P_k(x_o, t_o; \beta)\). By the observation we just made
\[ |y \ u_{x_1}(x, t, y)| \leq c, \ |y^2 \ u_t(x, t, y)| \leq c \] and \( |y \ u_y(x, t, y)| \leq c. \)
The others can be proved similarly. Q.E.D.

**Lemma 2.4:** If \(u(x, t, y)\) satisfies the heat equation in \(P_k(x_o, t_o; \beta)\) and
\[ \iint_{P_k(x_o, t_o; \beta)} y^{-n-1}(|\nabla u|^2 + y^2 |u_t|^2)\,dx\,dt\,dy \leq 1. \]
Then, \(y |\nabla u| \leq c, \ y^2 |u_t| \leq c\) in \(P_h(x_o, t_o; \alpha)\) where \(\alpha < \beta, h < k, c\)
depends only on \(\alpha, \beta, h, k, n\).

**Proof:** By a known estimate (Lemma 5, [4]) we have
\[ |u_{x_1}(x_o, t_o, y_o)|^2 \leq c \rho^{-n-3} \int_{S_\rho(x_o, t_o, y_o)} |u_{x_1}(x, t, y)|^2\,dx\,dt\,dy \]
if \(u(x, t, y)\) satisfies the heat equation in \(D, (x_o, t_o, y_o) \in D\)
and \(K_\rho(x_o, t_o, y_o) \subset D, S_\rho(x_o, t_o, y_o) = K_\rho(x_o, t_o, y_o) - K_{2\rho}(x_o, t_o, y_o),\)
where the notation is the same as in Lemma 2.3. Again, for
any \((x, t, y) \in P_h(x_o, t_o; \alpha)\), there exists \(c_o(>0)\) such that
\(K_{c_o y}(x, t, y)\) lies in \(P_k(x_o, t_o; \beta)\).
Thus, in our case we have

\[ |u_{x_1}(x, t, y)|^2 \leq cy^{-n-3} \int_{S_\rho(x, t, y)} |u_{x_1}(z, \tau, \eta)|^2 dz d\tau d\eta \]

\[ \leq cy^{-2} \int_{S_\rho(x, t, y)} |u_{x_1}(z, \tau, \eta)|^2 dz d\tau d\eta, \]

the last inequality is due to the fact that \( \eta < 2y \) in \( S_\rho(x, t, y) \).

Adding, we have

\[ y^2 |v_g u|^2 \leq \int_{P_k(x_0, t_0; \beta)} \eta^{-n-1} |v_g u|^2 dz d\tau d\eta \leq 1. \]

A similar argument shows that

\[ y^2 |u_t| \leq 1 \text{ in } P_h(x_0, t_0; \alpha). \quad \text{Q.E.D.} \]

**THEOREM 3:** Let \( u(x, t, y) \) satisfy \( u_{x_1 x_1} + \cdots + u_{x_n x_n} + u_{yy} = u_t \)
in \( \mathbb{R}^{n+2}_+ \). Let \( E \) be a measurable set in \( \mathbb{R}^{n+1}_+ \) ((x,t)space).

(a) If for each \( (x_0, t_0) \in E \), \( u(x, t, y) \) is bounded in
\[ P(x_0, t_0) = P_h(x_0, t_0) (x_0, t_0; a(x_0, t_0)) \], then the \textit{generalized area function}

\[ A(x_0, t_0) = \int_{P(x_0, t_0)} y^{-n-1} \left( |v_s u(x, t, y)|^2 + y^2 |u_t(x, t, y)|^2 \right) dx dt dy \]

is finite for almost all \((x_0, t_0) \in E\).

(b) Conversely, if \(A(x_0, t_0)\) is finite for every \((x_0, t_0) \in E\), then \(u(x, t, y)\) has a parabolic limit for almost all \((x_0, t_0) \in E\).

\textbf{Proof:} \quad (a) We can assume that \(E\) is compact; by an argument similar to the one in Lemma 2.1 we can assume that \(u(x, t, y)\) is uniformly bounded in the region

\[ \tilde{R} = \bigcup_{(x_0, t_0) \in E} P_k(x_0, t_0; \beta) \], where \(\beta\) is any positive number and \(k\) is sufficiently small. Since \(E\) is compact and \(u\) is continuous we can even allow \(k\) to be large.

We claim that if we can prove

\[ A(x_0, t_0) = \int_{P_h(x_0, t_0; a)} y^{-n-1} \left( |v_s u|^2 + y^2 |u_t|^2 \right) dx dt dy \]

is finite for almost all \((x_0, t_0) \in E\), where \(h, a\) are two fixed positive numbers such that \(k > h, \beta > a\), then (a) part is proved.

For, we can choose sequences \(\beta_m\) and \(\alpha_m\) such that \(\beta_m > \alpha_m \to \infty\) and have

\[ A(x_0, t_0) = \int_{P_h(x_0, t_0; \alpha_m)} y^{-n-1} \left( |v_s u|^2 + y^2 |u_t|^2 \right) dx dt dy \]

finite for almost all \((x_0, t_0) \in E\) for each \(m\). For each \(m\) we discard a set of measure zero and finally have the desired
result as $P_h(x_o, t_o; \alpha_m)$ contains $P(x_o, t_o)$ eventually.

Let $R = \bigcup_{(x_o, t_o) \in E} P_h(x_o, t_o; \alpha)$; we want to show that

$$\int_{E} A(x_o, t_o) dx_o dt_o < \infty \text{ (this } A(x_o, t_o) \text{ is the generalized area)}$$

function over $P_h(x_o, t_o; \alpha))$. By Fubini's theorem, it is the same to show

$$\int_{R} \{ \int_{E} \psi(x_o, t_o, x, t, y) dx_o dt_o \} y^{-n-1} (|\nabla u|^2 + y^2 |u_t|^2) dx dt dy < \infty,$$

where $\psi(x_o, t_o, x, t, y)$ is the characteristic function of $P_h(x_o, t_o; \alpha)$. Since

$$\int_{E} \psi(x_o, t_o, x, t, y) dx_o dt_o \leq \int_{|x-x_o|^2 + |t-t_o| < ay^2} dx_o dt_o \leq cy^{n+2},$$

we finally reduce the problem to showing

$$\int_{R} y(|\nabla u|^2 + y^2 |u_t|^2) dx dt dy < \infty.$$

Let $\{S_N\}$ be an increasing sequence of compact subsets of $R$ such that $S_N \uparrow R$. Then each $S_N$ is contained in a finite union of parabolic cones with vertices in $E$. Let this finite union be $R_N$. If we can show that

$$\int_{R_N} y(|\nabla u|^2 + y^2 |u_t|^2) dx dt dy \leq c,$$

where $c$ is independent of the number of cones $N$, then by the previous remarks, (a) will be proved. We proceed to show this in two steps:
(A) \[ \int_{R_N} y |v du|^2 \, dx \, dt \, dy: \]

Notice that the function \( u(x,t,y) \) as given in the hypothesis does not satisfy the condition required in the remark following Lemma 2.2, which is needed in the argument. Therefore, we consider a function \( u(\eta)(x,t,y) = u(x,t,y+\eta), \eta > 0 \), and proceed to show \( \int_{R_N} y |v du(\eta)|^2 \, dx \, dt \, dy \leq c \), where \( c \) is independent of \( N \) and \( \eta \). Then we let \( \eta \to 0 \) and get what we want to establish. For simplicity's sake we still use the notation \( u \) instead of \( u(\eta) \), and note that \( c \) will not depend on \( \eta \).

Let \( B^2_N \) be the part of \( \partial R_N \) which has \( y \)-coordinate \( h \) and let \( B^1_N \) be the rest of \( \partial R_N \). By Lemma 2.2 and the remark following it, for smooth \( F(x,t,y) \) and \( G(x,t,y) \) we have

\[
\int_{R_N} \left[ F(G_{x_1x_1} + \cdots + G_{x_nx_n} + G_{y} + G_{\tau}) - G(F_{x_1x_1} + \cdots + F_{x_nx_n} + F_{y} - F_{\tau}) \right] \, dx \, dt \, dy
\]

\[
= \sum_{i=1}^{n} \int_{B^1_N} \left( FG_{x_i} - GF_{x_i} \right) \delta_{x_i} \, dx \, dt - \int_{B^1_N} \left( FG_{y} - GF_{y} \right) \, dx \, dt + \int_{B^2_N} \left( FG_{y} - GF_{y} \right) \, dx \, dt
\]

\[
+ \int_{B^1_N} FG_{\tau} \, dx \, dt,
\]

where \( y = \delta(x,t) \) is the equation of \( B^1_N \). Let \( F = u^2 \), \( G = y \) (these two functions obviously satisfy the conditions for Lemma 2.2 and the remark following it). Then we have

\[
2 \int_{R_N} y |v du|^2 \, dx \, dt \, dy = \sum_{i=1}^{n} \int_{B^1_N} y(u^2)_{x_i} \delta_{x_i} \, dx \, dt + \int_{B^1_N} (u^2 - y(u^2)) \, dx \, dt
\]
\[- \int_{\Pi B^2_N} (u^2 - y(u^2)_y) \, dx \, dt - \int_{\Pi B^1_N} yu^2 \delta_t \, dx \, dt.\]

Since \(|u|\) is bounded, Lemma 2.3 shows \(|yu_y|\) is bounded. Since \(\Pi B^1_N\) and \(\Pi B^2_N\) are of finite measure, the second and third expressions on the right side of the above equation are bounded. Therefore, we need only treat the first and fourth expressions.

(i) \[
\sum_{i=1}^{n} \int_{\Pi B^1_N} uy \, u_{x_i} \delta_{x_i} \, dx \, dt \leq \sum_{i=1}^{n} \int_{\Pi B^1_N} |u|(|yu_{x_i}| |\delta_{x_i}|) \, dx \, dt
\]
\[
\leq c \sum_{i=1}^{n} \int_{\Pi B^1_N} |\delta_{x_i}| \, dx \, dt \leq c,
\]

as \(|\delta_{x_i}| = c|x_i - x_0|/(|x - x_0|^2 + |t - t_0|)^{\frac{1}{2}}\) for some \((x_0, t_0) \in E\).

(ii) \[
\int_{\Pi B^1_N} u^2 y \delta_t \, dx \, dt \leq c \int_{\Pi B^1_N} y |\delta_t| \, dx \, dt
\]
\[
= c \int_{\Pi B^1_N} |\delta_t| \, dx \, dt \quad \text{(as} \ y = \delta(x, t) \text{ on} \ B^1_N) \]
\[
= c \int_{\Pi B^1_N} |(\delta^2)_t| \, dx \, dt \leq c.
\]

(B) \[
\iint_{R_N} y^3 |u_t|^2 \, dx \, dt \, dy:
\]

Since
\[
\int_{R_N} y^3 |u_t|^2 \, dx \, dt \, dy = \int_{R_N} (y^3 u \, u_t)_t \, dx \, dt \, dy - \int_{R_N} y^3 u \, u_{tt} \, dx \, dt \, dy,
\]
we consider the two integrals separately. The boundedness of the first is easily obtained, as
\[
\left| \int_{R_N} (y^3 u_{tt}) \, dx \, dt \, dy \right| = \left| \int_{\Pi B_N^1} y^3 u_{tt} \, dx \, dt \right|
\]
\[
\leq \int_{\Pi B_N^1} |u| \, |y^2 u_{t}| \, |y^5_{tt}| \, dx \, dt \, dy \leq c, \text{ as in (ii) above.}
\]
The second term is slightly more complicated. Since
\[
\int_{R_N} y^3 u_{tt} \, dx \, dt \, dy = \int_{R_N} y^3 u(x_1 x_{11} + \cdots + u_{x_1} x_{11} + u_{yy}) \, dx \, dt \, dy
\]
\[
= \sum_{i=1}^{n} \int_{R_N} y^3 u(u_{tt})_{x_1 x_{1}} \, dx \, dt \, dy + \int_{R_N} y^3 u(u_{tt})_{yy} \, dx \, dt \, dy
\]
\[
= \sum_{i=1}^{n} \int_{\Pi B_N^1} y^3 u(u_{tt})_{x_1 x_{1}} \, dx \, dt - \int_{\Pi B_N^1} y^3 u(u_{tt})_{yy} \, dx \, dt
\]
\[
+ \int_{\Pi B_N^2} y^3 u(u_{tt})_{y} \, dx \, dt - \sum_{i=1}^{n} \int_{R_N} (y^3 u)_{x_1} (u_{tt})_{x_{1}} \, dx \, dt \, dy
\]
\[
- \int_{R_N} (y^3 u)_{y} (u_{tt})_{y} \, dx \, dt \, dy,
\]
we have to consider
\[
\star = \sum_{i=1}^{n} \int_{R_N} (y^3 u)_{x_1} (u_{tt})_{x_{1}} \, dx \, dt \, dy + \int_{R_N} (y^3 u)_{y} (u_{tt})_{y} \, dx \, dt \, dy
\]
only, as the rest are bounded.

Now,
\[ * = \sum_{i=1}^{n} \int_{R_N} y^3 u_{x_1}^2 (u_{x_1})_{x_1} dx dt dy + \int_{R_N} y^3 u_y (u_{x_1})_{y} dx dt dy + 3 \int_{R_N} y^2 u (u_{x_1})_{y} dx dt dy \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \int_{R_N} (y^3 u_{x_1}^2)_{x_1} dx dt dy + \frac{1}{2} \int_{R_N} (y^3 u_{x_1}^2)_{y} dx dt dy + 3 \int_{R_N} y^2 u (u_{x_1})_{y} dx dt dy \]

\[ = \frac{1}{2} \int_{\Pi B_{1}} y^3 (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_{y}^2)_{x_1} dx dt + 3 \int_{R_N} y^2 u (u_{x_1})_{y} dx dt dy. \]

The integral over \( \Pi B_{1} \) is bounded as \( |yu_{x_1}|, |y_{x_1}| \) are bounded.

As to the volume integral, we proceed as follows:

\[ \int_{R_N} y^2 u (u_{x_1})_{y} dx dt dy = \sum_{i=1}^{n} \int_{R_N} y^2 u (u_{y})_{x_1} x_1 dx dt dy + \int_{R_N} y^2 u (u_{y})_{y y} dx dt dy \]

\[ = \sum_{i=1}^{n} \int_{\Pi B_{1}} y^2 u u_{y x_1} x_1 dx dt - \int_{\Pi B_{1}} y^2 u u_{y y} dx dt \]

\[ + \int_{\Pi B_{2}} y^2 u u_{y y} dx dt - \sum_{i=1}^{n} \int_{R_N} (y^2 u)_{x_1} u_{y x_1} dx dt dy \]

\[ - \int_{R_N} (y^2 u)_{y} u_{y y} dx dt dy. \]

Once again because of Lemma 2.3, it is sufficient for us to consider the volume integrals only:

\[ \sum_{i=1}^{n} \int_{R_N} (y^2 u)_{x_1} u_{y x_1} dx dt dy + \int_{R_N} (y^2 u)_{y} u_{y y} dx dt dy \]

\[ = \sum_{i=1}^{n} \int_{R_N} y^2 u_{x_1} u_{y y} dx dt dy + \int_{R_N} y^2 u_{y y} dx dt dy + \int_{R_N} 2y u_{y y} dx dt dy. \]
Finally we observe that

\[ \sum_{i=1}^{n} \int_{R_N} y^2 u_{x_i} u_{x_i y} \, dx \, dt \, dy + \int_{R_N} y^2 u_y u_{yy} \, dx \, dt \, dy \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \int_{R_N} y^2 (u_{x_i})_y \, dx \, dt \, dy + \frac{1}{2} \int_{R_N} y^2 (u_y)_y \, dx \, dt \, dy \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \int_{R_N} (y^2 u_{x_i})_y \, dx \, dt \, dy + \frac{1}{2} \int_{R_N} (y^2 u_y)_y \, dx \, dt \, dy \]

\[ - \frac{1}{2} \sum_{i=1}^{n} \int_{R_N} 2y u_{x_i}^2 \, dx \, dt \, dy - \frac{1}{2} \int_{R_N} 2y u_y^2 \, dx \, dt \, dy \]

\[ = -\frac{1}{2} \sum_{i=1}^{n} \int_{\Pi B^1_N} y^2 u_{x_i}^2 \, dx \, dt - \frac{1}{2} \int_{\Pi B^1_N} y^2 u_y^2 \, dx \, dt \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \int_{\Pi B^2_N} y^2 u_{x_i}^2 \, dx \, dt + \frac{1}{2} \int_{\Pi B^2_N} y^2 u_y^2 \, dx \, dt - \int_{R_N} y|v_s u|^2 \, dx \, dt \, dy. \]

The volume integral is the same as the integral estimated in (A), the rest of the terms are bounded by Lemma 2.3.

Finally,

\[ 2 \int_{R_N} y u_y u_{yy} \, dx \, dt \, dy = 2 \int_{R_N} (uy u_y)_y \, dx \, dt \, dy - 2 \int_{R_N} (uy)_y u_y \, dx \, dt \, dy \]

\[ = -2 \int_{\Pi B^1_N} uy u_y \, dx \, dt + 2 \int_{\Pi B^2_N} uy u_y \, dx \, dt - 2 \int_{R_N} y(u_y)_y \, dx \, dt \, dy \]

\[ - 2 \int_{R_N} u_y u_{yy} \, dx \, dt \, dy. \]
The first two terms are bounded by Lemma 2.3, the third term is bounded, by (A), the last term is \( \int_{R_N} (u^2)_y \, dx \, dt \, dy \), which is bounded.

(b) Let \( E_0 \) be a subset in \( R^{n+1} \) such that

\[
\int_{P(x_0, t_0)} y^{-n-1}(|\nabla u|^2 + y^2 |u_t|^2) \, dx \, dt \, dy < \infty \quad \forall (x_0, t_0) \in E_0,
\]

where \( P(x_0, t_0) = P_h(x_0, t_0)(x_0, t_0; \alpha(x_0, t_0)) \). In this part we want to show that \( u(x, t, y) \) has parabolic limit for almost every \( (x_0, t_0) \in E_0 \). Using the same arguments as in proving Lemma 2.1, we can assume that:

1. \( \int_{P_k(x_0, t_0; \beta)} y^{-n-1}(|\nabla u|^2 + y^2 |u_t|^2) \, dx \, dt \, dy \) is uniformly bounded as \( (x_0, t_0) \) ranges over \( E_0 \), where \( \beta, k \) are some fixed positive numbers, \( \beta > \frac{1}{2} \).

2. \( E_0 \) is bounded.

Moreover, for any given \( \delta > 0 \), we can pick a closed set \( E, E \subseteq E_0 \) which satisfies the following two additional properties:

3. \( m(E_0 - E) < \delta \),

4. there exists a fixed \( \rho_\delta \), so that
\[
\frac{m(\{(z, \tau) : |x-z|^2 + |t-\tau| \leq \rho\} \cap E_\rho)}{m(\{(z, \tau) : |x-z|^2 + |t-\tau| \leq \rho\})} \geq \frac{1}{2}
\]

if \((x,t) \in E, 0<\rho<\rho_0\).

The existence of such an \(E\) can be shown as follows:

Let \((x,t) \in E_\rho\) be a point of density of \(E_\rho\). Then

\[
\lim_{\rho \to 0} \frac{m(\Sigma_\rho (x,t) \cap E_\rho)}{m(\Sigma_\rho (x,t))} = 1,
\]

where \(\Sigma_\rho (x,t) = \{(z, \tau) : |x-z|^2 + |t-\tau| \leq \rho\}\).

Hence there exists \(\rho(x,t) \leq c\), independent of \((x,t)\) as \(E_\rho\) is bounded) such that

\[
\frac{m(\Sigma_\rho (x,t) \cap E_\rho)}{m(\Sigma_\rho (x,t))} \geq \frac{1}{2}
\]

for all \(0<\rho \leq \rho(x,t)\).

Let \(\bar{\rho}(x,t) = \sup \{\rho(x,t) > 0 : \frac{m(\Sigma_\rho (x,t) \cap E_\rho)}{m(\Sigma_\rho (x,t))} \geq \frac{1}{2}\}
\]

for \(0<\rho \leq \rho(x,t)\). As \((x,t)\) is a point of density of \(E_\rho\) and \(E_\rho\) is bounded, \(0<\bar{\rho}(x,t)\) and \(\bar{\rho}(x,t) \leq c\) independent of \((x,t)\).

If \(E^*\) is the set of all points of density of \(E_\rho\), then

\[m(E_\rho - E^*) = 0\] (see page 19). The function \(x,t \mapsto \bar{\rho}(x,t)\)

is a measurable function on \(E^*\), since

\[
\frac{m(\Sigma_\rho (x,t) \cap E_\rho)}{m(\Sigma_\rho (x,t))}
\]

is a continuous function of \(x, t\) and \(\rho\). Let

\[M_1 = \{(x,t) \in E^* : 1 \leq \bar{\rho}(x,t)\}, M_k = \{(x,t) \in E^* : \frac{1}{k} \leq \bar{\rho}(x,t) < \frac{1}{k-1}\}\]

\[k = 2, 3, \ldots \] Then \(M_k \cap M_j = \emptyset\) for \(k \neq j\) and \(E^* = \bigcup_{k=1}^{\infty} M_k\).
Thus, \( m(E^*) = \sum_{k=1}^{\infty} m(M_k) < \infty \). Choose \( k_0 \) such that \( \sum_{k=k_0}^{\infty} m(M_k) < \delta \), let \( E = \bigcup_{k=1}^{k_0-1} M_k \). Then \( m(E_0^-E) = m(E^*-E) < \delta \) and, letting

\[
\rho_\delta = \frac{1}{k_0},
\]

we have

\[
\frac{m(\Sigma_{\rho}(x,t) \cap E_0)}{m(\Sigma_{\rho}(x,t))} \geq \frac{1}{2}
\]

for \((x,t) \in E, 0 < \rho < \rho_\delta\).

We now fix the set \( E \), and show that \( u(x,t,y) \) has parabolic limit for almost every \((x,t) \in E\). As \( m(E_0^-E) \) is arbitrarily small, this suffices to prove the theorem.

We prove this part in two steps.

(A) Let \( R = \bigcup_{(x_0,t_0) \in E} P_h(x_0,t_0;\alpha), 0 < \alpha < \beta, 0 < h < k, \)

\[
R_N = \bigcup_{i=1}^{N} P_h(x_i,t_i;\alpha) \quad (x_i,t_i) \in E, \text{ let } B_N^1 \text{ be the lateral boundary of } R_N \text{ given by the equation } y = \delta(x,t) \text{ and } B_N^2 \text{ be the rest of } \partial R_N. \]

We want to show that

\[
\int_{\Pi B_N^1} |u(x,t,\delta(x,t))|^2 \, dx \, dt \leq c,
\]

where \( c \) is independent of \( N \). In order to show this, we shall prove the following first:

\[
\int_{R_N} y |\nabla u|^2 \, dx \, dt \, dy \leq c,
\]

\( c \) is independent of \( N \).
By the reduced hypothesis (1), we know that
\[ \int_{P_k(x_0, t_0; \beta)} y^{-n-1} \vert \nabla u \vert^2 dx dt dy \leq c \quad \forall (x_0, t_0) \in E_0. \]
Integrating over \( E_0 \),
\[ \int_{E_0} (\int_{P_k(x_0, t_0; \beta)} y^{-n-1} \vert \nabla u \vert^2 dx dt dy) dx_0 dt_0 \leq c. \]
This is the same as the inequality
\[ \int_{\chi_{E_0}(x_0, t_0)} \psi(x_0, t_0; x, t, y) y^{-n-1} \vert \nabla u \vert^2 dx dt dy dx_0 dt_0 \leq c, \]
where \( \psi(x_0, t_0; x, t, y) \) is the characteristic function for \( P_k(x_0, t_0; \beta) \) and \( \chi_{E_0} \) is the characteristic function for \( E_0 \).

Since \((x, t, y) \in R = (x_1, t_1) \in E \) such that \((x, t, y) \in P_h(x_1, t_1; \alpha) \)
thus if \((x_0, t_0)\) is such that \(|x_0 - x_1|^2 + |t_0 - t_1| < (\sqrt{\beta} - \sqrt{\alpha})^2 y^2\)
then
\[ |x-x_0|^2 + |t-t_0| \leq |x-x_1|^2 + |x_1-x_0|^2 + 2|x_1-x_0| |x-x_1| + |t-t_1| + |t_1-t_0| < \alpha y^2 + 2\sqrt{\beta} y \sqrt{\alpha} y + (\sqrt{\beta} - \sqrt{\alpha})^2 y^2 = \beta y^2, \]

hence
\[ \int_{E_0} \psi(x_0, t_0; x, t, y) dx_0 dt_0 \]
\[ \geq \int_{E_0 \cap \{(x_0, t_0) : |x_0-x_1|^2 + |t_0-t_1| < (\sqrt{\beta} - \sqrt{\alpha})^2 y\}} dx_0 dt_0 \geq cy^{n+2} \]
\((c > 0)\), if we choose \( \alpha \) close enough to \( \beta \) so that the condition (4) for the set \( E \) holds. Hence we have
\[ 0 \leq \int_{R_N} y |v_s u|^2 \, dx \, dt \leq \int_{R} y |v_s u|^2 \, dx \, dt \]

\[ \leq c \int_{E_0} \int_{P_k(x_0,t_0;\beta)} y^{-n-1} |v_s u|^2 \, dx \, dy \, dx_0 \, dt_0 \leq c. \]

As before, we have to translate the function \( u(x,t,y) \) in order to be able to apply Lemma 2.2.

Let \( \eta \) be a sufficiently small positive number, let \( \tau = h - \eta > 0 \), and \( R' = \bigcup_{i=1}^{N} P_{t_i}(x_i,t_i;\alpha) \). Let \( v(x,t,y) = u(x,t,y+\eta); \) then

\[ 0 \leq \int_{R'_{N}} y |v_s v(x,t,y)|^2 \, dx \, dy \leq \int_{R'_{N}} y |v_s u(x,t,y+\eta)|^2 \, dx \, dy \]

\[ = \int_{R_N} (y-\eta) |v_s u(x,t,y)|^2 \, dx \, dy \leq \int_{R_N} y |v_s u(x,t,y)|^2 \, dx \, dy \]

\[ \eta < y \]

\[ \leq \int_{R_N} y |v_s u(x,t,y)|^2 \, dx \, dt \leq c. \]

From here on to the end of (A), we still use the notation \( u(x,t,y) \) instead of \( v(x,t,y) \), \( R_N \) instead of \( R'_N \), but we should note that \( u \) is translated and \( R_N \) is a smaller region. Once we get an \( \eta \)-independent estimate on this new function \( u \), we let \( \eta \to 0 \) and thus obtain the result we want.
Noticing that \( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y^2} u^2 \)

\[ = 2(u_{x_1}^2 + \cdots + u_{x_n}^2 + u_{y}^2) + 2u(u_{x_1 x_1} + \cdots + u_{x_n x_n} + u_{yy}), \]

we have

\[ 0 \leq \sum_{i=1}^{n} \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y\left( \frac{1}{2} u_{x_1 x_1}^2 - uu_{x_1 x_1} \right) dx dt dy + \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y\left( \frac{1}{2} u_{yy}^2 - uu_{yy} \right) dx dt dy \]

\[ \leq c. \]

Using the differential equation, we obtain

\[ (**0) \leq \sum_{i=1}^{n} \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y(u^2)_{x_1 x_1} dx dt dy + \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y(u^2)_{yy} dx dt dy \]

\[ - \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y u u_t dx dt dy \leq c. \]

Using Lemma 2.2, the integrals involving spatial derivatives reduce to the following:

\[ \sum_{i=1}^{n} \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y(u^2)_{x_1 x_1} dx dt dy + \frac{1}{2} \int_{\mathbb{R}_N} \int_{\mathbb{R}_N} y(u^2)_{yy} dx dt dy \]

\[ = \sum_{i=1}^{n} \int_{\mathbb{P}^1_{B_N}} y u u_{x_1} dx dt - \int_{\mathbb{P}^1_{B_N}} y u u_{x_1} dx dt + \int_{\mathbb{P}^2_{B_N}} y u u_{y} dx dt \]

\[ + \frac{1}{2} \int_{\mathbb{P}^1_{B_N}} u^2 dx dt - \frac{1}{2} \int_{\mathbb{P}^2_{B_N}} u^2 dx dt. \]
Thus, (**) becomes

\[ 0 \leq \sum_{i=1}^{n} \int_{\Pi B_{N}^{1}} yu_{x_{i} x_{i}} \delta_{x_{i}} dx_{i} dt - \int_{\Pi B_{N}^{1}} yu_{y} dx_{i} dt + \int_{\Pi B_{N}^{1}} yu_{y} dx_{i} dt \]
\[ - \frac{1}{2} \left[ \int_{\Pi B_{N}^{1}} yu^{2} \delta_{x} dx_{i} dt - \int_{\Pi B_{N}^{1}} u^{2} dx_{i} dt + \int_{\Pi B_{N}^{1}} u^{2} dx_{i} dt \right] \leq c. \]

Taking absolute values, we have

\[ \left| \sum_{i=1}^{n} \int_{\Pi B_{N}^{1}} yu_{x_{i} x_{i}} \delta_{x_{i}} dx_{i} dt - \int_{\Pi B_{N}^{1}} yu_{y} dx_{i} dt - \frac{1}{2} \int_{\Pi B_{N}^{1}} u^{2}(y \delta_{x} - 1) dx_{i} dt \right| \]
\[ \leq c + \left| \int_{\Pi B_{N}^{1}} (yu_{y} - \frac{1}{2} u^{2}) dx_{i} dt \right| \leq c, \]

as the measure of \( \Pi B_{N}^{2} \) is finite, \( u^{2} \) is bounded on \( B_{N}^{2} \) and \( yu_{y} \) is also bounded there by Lemma 2.4.

Therefore, we have

\[ \frac{1}{2} \int_{\Pi B_{N}^{1}} (1 - y \delta_{x}) u^{2} dx_{i} dt \leq \sum_{i=1}^{n} \int_{\Pi B_{N}^{1}} yu_{x_{i} x_{i}} \delta_{x_{i}} dx_{i} dt + \int_{\Pi B_{N}^{1}} yu_{y} dx_{i} dt + c. \]

Now almost every point in \( \Pi B_{N}^{1} \) has a neighborhood in which \( y = \delta(x, t) \) is given by an equation \( \alpha y^{2} = |x - x_{o}|^{2} + |t - t_{o}| \) for some \( (x_{o}, t_{o}) \). Thus, \( 2\alpha \delta_{x} = \pm 1 \) a.e. As \( \alpha > \frac{1}{2} \), \( 1 - y \delta_{x} \)

\[ \geq 1 - \frac{1}{2\alpha} > 0 \text{ a.e.} \]

Thus,
\[
\int \frac{u^2}{dx} dt \leq c_1 \sum_{i=1}^{N} \int_{B_{1}^{\frac{1}{N}}} \left| yu_{x_i} \delta x_i \right| dx dt + \int_{B_{1}^{\frac{1}{N}}} \left| yu_y \right| dx dt + c
\]
\[
\leq c_1 \int_{B_{1}^{\frac{1}{N}}} \left| u \right| dx dt + c,
\]
where \(c_1, c\) are positive constants independent of \(N\) and the last inequality is obtained by Lemma 2.4 together with the fact that \(\left| \delta x_i \right|\) is uniformly bounded a.e. as in Lemma 2.2.

Finally by Hölder's inequality we have
\[
\int_{B_{1}^{\frac{1}{N}}} u^2 dx dt \leq c_2 \left( \int_{B_{1}^{\frac{1}{N}}} u^2 dx dt \right)^{\frac{1}{2}} + c, \quad c_2 > 0, \quad \text{independent of } N,
\]
and thus \(\int_{B_{1}^{\frac{1}{N}}} u^2 dx dt \leq c.\) This proves part (A).

(B) Let \(R_{N\epsilon} = \{(x, t, y) : (x, t, y-\epsilon) \in R^1_N\}\), and \(B_{1}^{\frac{1}{N}\epsilon}\) be the lateral boundary of \(R_{N\epsilon}\), \(B_{2}^{\frac{1}{N}\epsilon}\) be the rest of \(\partial R_{N\epsilon}\). Let \(f_{N\epsilon}(x, t) = u(x, t, \delta(x, t, \epsilon))\) for \((x, t) \in B_{1}^{\frac{1}{N}\epsilon}, f_{N\epsilon}(x, t) = 0\) otherwise, where \(y = \delta(x, t, \epsilon)\) is the equation for \(B_{1}^{\frac{1}{N}\epsilon}\). Then, we have by part (A) \[\|f_{N\epsilon}\|_{L^2(E)} \leq c \text{ independent of } N \text{ and } \epsilon.\]

Let
\[
v_{N\epsilon}(x, t, y) = \int_{\mathbb{R}^{n+1}} \Gamma(x-z, t-\tau, y) \left| f_{N\epsilon}(z, \tau) \right| dz d\tau,
\]
where \(\Gamma(x, t, y) = \frac{1}{(4\pi)^{\frac{n+1}{2}}} \frac{n+3}{2} \left( \frac{1}{y} \right)^{\frac{n+3}{2}} \exp(-\frac{|x|^2 + y^2}{4t}).\) We shall prove that there exist positive constants \(c_1, c_2\) independent
of $N$ and $\varepsilon$, such that $|u(x,t,y)| \leq c_1 v_{Ne}(x,t,y) + c_2$ for $(x,t,y) \in R_{Ne}$.

By the maximum principle for solutions of the heat equation, it suffices to show the above bound holds for $(x,t,y) \in \partial R_{Ne}$. Since $u(x,t,y)$ is continuous in $R^{n+2}_+$, $E$ is a bounded set, and $v_{Ne} \geq 0$, a sufficiently large $c_2$ can be chosen, independent of $N$ and $\varepsilon$, such that

$$|u(x,t,y)| \leq c_1 v_{Ne}(x,t,y) + c_2 \text{ for } (x,t,y) \in B^2_{Ne}.$$ 

Let $\beta^*$, $k^*$ be such that $\alpha < \beta^* < \beta$, $h < k^* < k$. Then there exists a constant $c$, depending on $\alpha, h, \beta^*, k^*$ only, such that for any $(x,t,y) \in B^1_{Ne}$, $K(x,t,y) = \{(z,\tau,\xi) : |z_i - x_i| < cy, i = 1, \ldots, n, |y - r| < cy, 0 < t - \tau < c^2 y^2 \} \subset \bigcup_{i=1}^{N} P^k_{k^*}(x_i, t_i; \beta^*)$.

Let $(x,t,y) \in B^1_{Ne}$ and $S = B^1_{Ne} \cap K(x,t,y)$. Then it is clear that $c$ can be chosen such that $m(\Pi S) \geq cy^{n+2}$ for some $c > 0$ independent of $N$ and $\varepsilon$. Let $(z,\tau,\xi) \in S$. Then $(z,t,y) \in K(x,t,y)$ and thus the mean value theorem implies

$$|u(x,t,y) - u(z,\tau,\xi)| \leq c \sup_t |v_{\xi} u| y + c \sup_t |u_{t}| y^2$$

where $t$ is a line segment joining $(x,t,y)$ and $(z,\tau,\xi)$.

Since $R_{Ne} \subset \bigcup_{i=1}^{N} P_k(x_i, t_i; \beta)$ if $\varepsilon$ is very small, and by hypothesis

$$\int_{\bigcup_{i=1}^{N} P_k(x_i, t_i; \beta)} y^{-n-1}(|v_{\xi} u|^2 + y^2 |u_{t}|^2) dx dt dy \leq c,$$
Lemma 2.4 implies that \(|u(x,t,y) - u(z,\tau,r)| \leq c\). Integrating this inequality gives

\[
|u(x,t,y)| \leq \frac{1}{m(\Pi S')} \int_{\Pi S'} |u(z,\tau,\delta(z,\tau,\varepsilon))| dz d\tau + c,
\]

where \(S' = \{(z,\tau,r) : (z,\tau,r) \in S \text{ and } t-\tau \geq \frac{1}{2}c^2y^2\}\).

Consider \(P = \{(x,t,y) : |x|^2 + |t| < a^2y^2\}\), let \((x_0,t_0,y_0)\) be a point on the surface of this cone, let

\(K = \{(x,t,y) : |x-x_0| < \varepsilon a y_0, |t-t_0| < \varepsilon^2 a^2 y_0^2, |y-y_0| < 2\varepsilon y_0\}\),

finally, let \(L' = \{(x,t,y) \in \partial P \cap K : t_0 - t > \frac{1}{2} \varepsilon^2 a^2 y_0^2\}\).

Now, \(\Pi L' = \{(x,t) : |x|^2 + |t| = a^2 y^2, |y-y_0| < 2\varepsilon y_0, |x-x_0| < \varepsilon a y_0, t_0 - \varepsilon^2 a^2 y_0^2 < t < t_0 - \frac{1}{2} \varepsilon^2 a^2 y_0^2\}; \text{ if } \delta < \varepsilon^2 / 4\)

then \(L = \{(x,t) : |x-x_0| < \delta a y_0, |t-t_0| + (3\varepsilon^2 a^2 y_0^2 / 4) < \delta a^2 y_0\}\)

\(\subset L'\). For, \(|x-x_0| < \delta a y_0 = |x-x_0| < \varepsilon a y_0\) as \(\varepsilon\) is small;

\(t-t_0 < (-3\varepsilon^2 a^2 y_0^2 / 4) + \delta a^2 y_0^2 < -\frac{1}{2} \varepsilon^2 a^2 y_0^2\) and \(t-t_0 > (-3\varepsilon^2 a^2 y_0^2 / 4)\)
- $\delta \sigma^2 y_o^2 > -\varepsilon^2 \sigma^2 y_o^2$; finally, since $|x_o|^2 + |t_o| = \alpha^2 y_o^2$ and

$\sqrt{|x|^2 + |t|}$ is a metric,

$$|\sqrt{|x|^2 + |t|} - \sqrt{y_o^2}| \leq \sqrt{|x-x_o|^2 + |t-t_o|}$$

$$\leq \sqrt{\delta \sigma^2 y_o^2 + (3\varepsilon^2 \sigma^2 y_o^2/4) + \delta \sigma^2 y_o^2}$$

$$\leq 2\varepsilon \sigma y_o.$$  

Thus, $|y-y_o| = |(\sqrt{|x|^2 + |t|}/\alpha) - y_o| < 2\varepsilon y_o$.

The above argument shows $m(\mathbb{S}^1) \geq cy^{n+2}$ also and when $t-\tau \geq \frac{1}{2}c^2 y^2$, $\Gamma(x-z,t-\tau,y) \geq cy^{-n-2}$. Thus,

$$|u(x,t,y)| \leq c_1 \int_{\mathbb{S}^1} \Gamma(x-z,t-\tau,y) |f_{N\varepsilon}(z,\tau)| dzd\tau + c_2$$

$$\leq c_1 \int_{\mathbb{R}^{n+1}} \Gamma(x-z,t-\tau,y) |f_{N\varepsilon}(z,\tau)| dzd\tau + c_2.$$  

Since $\|f_{N\varepsilon}\|_{L^2} \leq c$, independent of $N$ and $\varepsilon$ for each $N$ there exists a subsequence $\{|f_{N_{\varepsilon_k}}\}|$ of $\{|f_{N\varepsilon}\}|$ which converges to a function $f_N \in L^2(\mathbb{R}^{n+1})$ in the weak $L^2$ sense.

Let $v_N(x,t,y) = \int_{\mathbb{R}^{n+1}} \Gamma(x-z,t-\tau,y) f_N(z,\tau) dzd\tau$.

Then for each $(x,t,y)$, $y>0$, $v_{N\varepsilon}(x,t,y) \longrightarrow v_N(x,t,y)$ since $\Gamma(x-z,t-\tau,y) \in L^2$ as a function of $z,\tau$. Since $R_{N\varepsilon} \xrightarrow{\varepsilon \to 0} R_N$ as $\varepsilon \to 0$, we have
\[ |u(x,t,y)| \leq c \nu_N(x,t,y) + c_2 \quad \forall (x,t,y) \in R_N. \]

Finally, since \( \|f_N\|_{L^2} \leq c \), there exists a subsequence \( \{f_{N_k}\} \) of \( \{f_N\} \) which converges to \( f \in L^2 \) in the weak \( L^2 \) sense.

Let \( v(x,t,y) = \int_{R^{n+1}} \Gamma(x-z,t-\tau,y)f(z,\tau)dzd\tau. \)

Then, \( v_{N_k}(x,t,y) \rightarrow v(x,t,y) \) for each \( (x,t,y), \quad y > 0. \)

If \( A \) is any compact subset of \( R \), then there exists \( k \) such that \( R_{N_k} \supset A \). From what we have proved, if \( j \geq 0 \), then

\[ |u(x,t,y)| \leq c_1 v_{N_k+j}(x,t,y) + c_2 \quad \forall (x,t,y) \in A \subset R_{N_k}. \]

Letting \( j \rightarrow \infty \), we have

\[ |u(x,t,y)| \leq c_1 v(x,t,y) + c_2 \quad \forall (x,t,y) \in A. \]

Since \( A \) is any compact subset of \( R \),

\[ |u(x,t,y)| \leq c_1 v(x,t,y) + c_2 \quad \forall (x,t,y) \in R. \]

Since \( v(x,t,y) \) is parabolically bounded for almost all \( (x,t) \in R^{n+1} \) (Lemma 1.4), thus, \( u(x,t,y) \) is parabolically bounded for almost all \( (x,t) \in E \) and hence \( u(x,t,y) \) has parabolic limit for almost all \( (x,t) \in E \) (Theorem 2).

\[ Q.E.D. \]
References


