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THE SELF-INTERSECTIONS OF IMMERSED MANIFOLDS

by

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0. **Introduction.** I will be concerned here with the self-intersections of differentiable manifolds immersed in euclidean space. Specifically, if $N$ is a compact manifold of dimension $n$ and $f:N \to \mathbb{R}^m$ is a **completely regular** immersion with $3n < 2m$, then the set of points

$$\overline{I}_f = \{ x \in n | \exists y, f(x) = f(y) \}$$

is a compact embedded submanifold of dimension $2n-m$ of $N$. Then $f \overline{I}_f$ is an embedded $(2n-m)$-dimensional submanifold of $\mathbb{R}^m$, $I_f$, which is doubly covered by $\overline{I}_f$.

**Definitions.** An **immersion** is a differentiable map $f: X \to V$ between two manifolds whose differential $f_*$ is injective at each point of $X$. By the inverse function theorem this is equivalent to the condition that $f$ be locally a regular embedding. A **regular homotopy** $H: N \times I \to V$ between two immersions $f$ and $g$ is a differentiable map such that $H(x,0) = f(x)$, $H(x,1) = g(x)$ and $H(N,t)$ is an immersion for all $t \in I$. An immersion $f$ is **completely regular** if i.) $f$ has no triple points. ii.) If $f(x) = f(y)$ and $T_x, T_y$ are the tangent spaces to $N$ at $x$ and $y$, then $f_* T_x$ and $f_* T_y$ span the tangent space to $V$ at $f(x) = f(y)$.

Completely regular immersions possess two important properties. They are **generic** in the sense of Thom [20]: Any immersion of a compact manifold and its derivatives up to any order may be approximated arbitrarily closely by a regularly homotopic completely regular immersion, and any immersion close enough to a completely regular immersion $g$, whose first two derivatives approximate those of $g$, is
completely regular. Secondly, they are stable, in the sense that under a sufficiently small regular homotopy the diffeomorphism type of the immersed manifold, in a suitably defined sense, does not change. In particular the intersection manifolds are diffeomorphic [11].

The property of intersections to be investigated here is their cobordism type, a concept introduced by Thom [20]. In fact, in the course of the proofs a stronger notion is needed, that of cobordism of manifolds with fixed-point free involutions, due to Conner and Floyd [2]. The cobordism type of $\overline{I}_f$ in this sense, the involution being the covering transformation $T$ of $\pi_f: \overline{I}_f \to I_f$, is shown to depend only on the regular homotopy class of $f$.

The principal result of Chapter I is the determination of the equivariant cobordism type of $\overline{I}_f$ for an immersion $f: N \to \mathbb{R}^m$, where $3n < 2m$ and $N$ is a $m$-manifold; that is, a manifold with trivial stable normal bundle.

Chapter II is devoted to producing examples. It is shown that there are a number of immersions of spheres in euclidean space whose self-intersections are real projective spaces. These may be thought of as analogous to the point-intersections treated by Whitney in [22], although I give no explicit construction.

Other material on self-intersections may be found in Lashof and Smale [9], where the fundamental classes of $\overline{I}_f$, and its analogues for higher order intersections, are
determined in \( N \), and in Whitney [23]. The methods used here are closest to those of Novikov [17], but his results are largely false. The origin of the difficulty seems to be a lemma stating that self-intersections which are non-trivially covered circles appear only in pairs, which may be cancelled by a regular homotopy. This is in contradiction to the computation following our Theorem of II, 5.)

I wish to express my gratitude to Professor Eldon Dyer, for his advice and encouragement over a period of years.

Thanks are also due to the City University of New York, where I was a guest during the preparation of this Thesis.
Chapter I.

1. Preliminaries.

A. All maps and manifolds discussed will be compact and differentiable of class $C^{\infty}$. It is known ([14] and elsewhere) that the latter restriction is not an essential one. Manifolds will be without boundary unless otherwise stated. I give below a resumé of the basic results from differential topology which will be used. The references are to proofs in $C^{\infty}$ language rather than to the original sources.

1.) For $\varepsilon > 0$ there is a real valued function defined on the reals satisfying: i.) $\mu(x) = 0, x \leq 0$. ii.) $\mu(x) = 1, x \geq \varepsilon$. iii.) $\frac{d\mu}{dx} > 0$. [14].

2.) Given a map $f: N^N \rightarrow N^m$ of manifolds, a critical point of $N$ is one at which the rank of the differential $f_*$ is less than $m$. A critical value is the image of a critical point. A point in $M$ not a critical value is called a regular value.

Theorem: (Sard) The Lebesgue measure of the set of critical values of a $C^{\infty}$ map is 0. [13].

3.) Let $M^m$ and $N^n$ be manifolds, $V^V$ a submanifold of $M$. A map $f: N \rightarrow M$ is transverse regular over $V$ if for $f(x) \in V$, the composition $T_x f_* \rightarrow T_{fx} \rightarrow T_{fx}/T/V_{fx}$ is surjective.

Theorem: (Thom) i.) Under the above conditions $f^{-1}(V)$ is a submanifold of $N$ of dimension $n-m+v$. ii.) Let $A$ be a closed subset of $N$ such that the transverse regularity condition holds for each point in $A \cap f^{-1}(V)$. Then $f$ may be approximated arbitrarily closely by a map $g$ transverse
regular over \( V \) such that \( f|A = g|A \) [14].

4.) **Theorem:** Let \( M \) and \( N \) be manifolds and \( f_0, f_1 \), be two isotopic embeddings \( N \rightarrow M \). Then there is an isotopy \( F \) of \( M \) over itself such that for all \( x \in N \), \( F_0(f_0(x)) = f_0x \), \( F_1(f_0(x)) = f_1x \). [15].

5.) If \( N^n \) is embedded in \( M^m \) a tubular neighborhood of \( N \) in \( M \) is the total space of an \((m-n)\)-disc bundle embedded in \( M \) whose zero section corresponds to \( N \). It is shown in [15] that these always exist.

B. Two manifolds \( N_0^n, N_1^n \) are (unoriented) cobordant if there exists a manifold with boundary \( W^{n+1} \) such that \( \partial W = N_0 \cup N_1 \). This can be shown to be an equivalence relation. The class of a manifold \( M \) will be written \([M]\).

Define operations on manifolds by \( N_0 + N_1 = N_0 \cup N_1 \), \( N \cdot V = N \times V \). The class of all manifolds under the relation of cobordism then form a graded commutative \( Z_2 \)-algebra \( \mathfrak{h} \). The additive inverse of a cobordism class is itself, since \( \partial(M \times I) = M \cup M \). The class of the empty manifold is the zero element. Thom [20] showed that \( \mathfrak{h} \) is a polynomial algebra with one generator in each dimension not of the form \( 2^i - 1 \). The cobordism class of a manifold \( N \) in \( \mathfrak{h} \) is determined as follows:

Let \( w_i \) be the \( i \)-th Stiefel-Whitney class of the tangent bundle of \( N \), [16], and \([N]\) its fundamental class. (All homology will have \( Z_2 \) coefficients unless otherwise stated). Let \( \pi \) be a sequence \( (i_1, \ldots, i_n) \) such that \( \sum_{j} j i_j = n \). The \( \pi \)-th Stiefel-Whitney number of \( N \) is the value of the index
$\langle w_1, w_2, \ldots, w_n, [N] \rangle$. Then two manifolds are cobordant iff these numbers are the same for all $\pi$.

These ideas received the following extension from Conner and Floyd [2]: The objects considered are manifolds with fixed-point free involutions, that is, diffeomorphisms $T$ such that $T^2 = I$ and $Tx \neq x$ for all $x$. These will be written $(M, T)$. Two such, $(M_0, T_0)$, $(M_1, T_1)$, are cobordant if there exists $(W^{m+1}, T)$ such that $\partial(W, T) = (M_0, T_0) \cup (M_1, T_1)$, the union of the given manifolds with their given involutions. Note that the quotient manifolds $M_0/T_0$ and $M_1/T_1$ are then cobordant in the original sense. Disjoint union, as above, equips the objects under this relation with the structure of a graded vector space over $\mathbb{Z}_2$, denoted $\mathfrak{h}(B_{\mathbb{Z}_2})$. Further, $\mathfrak{h}(B_{\mathbb{Z}_2})$ is a module over $\mathfrak{h}$. If $[N] \in \mathfrak{h}$ and $[(M, T)] \in \mathfrak{h}(B_{\mathbb{Z}_2})$, then $(N \times M, 1 \times T)$ represents $[N][(M, T)]$, the involution $1 \times T$ acting in co-ordinate-wise fashion.

Conner and Floyd have shown that $\mathfrak{h}(B_{\mathbb{Z}_2})$ is a free module over $\mathfrak{h}$ with one generator in each dimension $\geq 0$. These may be taken to be $(S^n, A^n)$, the standard sphere furnished with the antipodal map. In other words any element may be expressed as a sum $[(M, T)] = \sum_{i=0}^{m} [V_i][(S^i, A^i)]$, where $V_i$ is of dimension $m-i$.

These equivariant cobordism classes are determined by involution numbers analogous to the Stiefel-Whitney numbers described above. Given $(M^m, T)$ there is in $H^1(M/T)$ an element $c$, the characteristic class of the double covering $M \longrightarrow M/T$. 
An involution number of \((M,T)\) is an integer \((\mod 2)\)
\[
\langle w_1 \ldots w_m \rangle = c^{r \cdot [M/T]}, \quad \text{where } \sum_j i_j + r = m, \text{ and the } w_j \text{ are the tangent Stiefel-Whitney classes of } M/T. \quad \text{As above, two manifolds with involution are equivariantly cobordant iff all their involution numbers agree.}
\]

As an illustration of the way that this concept will be used, consider manifolds of dimension one, that is, finite collections of circles. It is clear that all of these bound in \(\mathbb{R}^1\). However the covering induced by \((S^1, A^1)\) is just the usual double covering of the circle, whose characteristic class is the generator of \(H^1(S^1)\). Thus there is the nonzero involution number \(\langle c, [S^1] \rangle\), so \((S^1, A^1)\) does not bound in \(\mathbb{R}^1 (B_{\mathbb{Z}_2})\).

2.) **Invariance of cobordism type**

Consider an immersion \(f: N^n \longrightarrow \mathbb{R}^m, 3n < 2m\). Form the open manifold \(N \times N - \Delta_N\), where \(\Delta\) denotes the diagonal. Define \(f^2: N \times N - \Delta_N \longrightarrow \mathbb{R}^m \times \mathbb{R}^m\) by \(f^2(x,y) = (f(x), f(y))\). Since \(f^2_* = f_* \oplus f_*\), \(f^2\) is an immersion.

**Lemma 1.** If \(f\) is a completely regular immersion then \((f^2)^{-1} \Delta_{\mathbb{R}^m}\) is a compact submanifold of \(N \times N - \Delta_N\).

To show that \((f^2)^{-1} \Delta_{\mathbb{R}^m}\) is a manifold, we must show that \(f^2\) is transverse regular over \(\Delta_{\mathbb{R}^m}\). If for \((x,y) \in N \times N - \Delta_N\), \(f(x) \neq f(y)\), \(f^2\) is transverse regular at \((x,y)\) by definition. If \(f(x) = f(y)\), since \(f^2_*\) is injective it will suffice to show that there exist \(m\) linearly independent vectors \(\{r_i\} \in T(x,y)\) \(f^2_* r_i \in T/\Delta f^2(x,y)\). Now \(T/\Delta f^2(x,y)\) is just the subspace of vectors \((\alpha, \beta)\) such that if either \(\alpha\) or \(\beta\) is zero then both
are, and $a$ and $b$ are linearly dependent considered as elements
of $T_{fx} = T_{fy} = \mathbb{R}^m$. Thus we need to find $m$ independent
vectors $\{r_i\} = (S_1^1, S_1^2) \in T(x, y) = \mathbb{R}^n \oplus \mathbb{R}^n$ such that $f_* S_1^1$ and
$f_* S_1^2$ are independent. But we know that $f_* T_x$ and $f_* T_y$
span $T_{fx}$, so these may be found.

Equip $N$ with a Riemannian metric. Let $q(x) =
\inf\{d(x, y), y | f(x) = f(y)\}$. I claim that $\inf\_{x \in N} q(x) = b$,
$0 < b \leq \infty$. If not there is a sequence of points $\{y_i\}$ such
that $\lim\_{i \to \infty} q(y_i) = 0$. Since $N$ is compact $\{y_i\}$ has a limit
point $y$. For $\varepsilon > 0$ let $U_{\varepsilon} = \{x \in m | d(x, y) < \varepsilon\}$. We may
select $y_j \in U_{\varepsilon}$ so that $q(y_j) < \varepsilon$, i.e. There exists $w$,
$f(w) = f(y_j)$ and $d(y_j, w) < \frac{\varepsilon}{2}$. Then $d(y, w) \leq d(y_j, y) +
\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $m \in U_{\varepsilon}$ and no neighborhood of $y$ is
embedded, a contradiction.

Therefore if we choose a tubular neighborhood $V$ of $\Delta$
in $N \times N$ such that if $(x, y) \in V$, $d(x, y) < b$, then $f(x) \neq f(y)$
or equivalently $f^2(x, y) \notin \Delta_{Rm}$. Therefore $(f^2)^{-1} \Delta_{Rm}$ is
contained in the interior of $N \times N - V$ and is a compact
submanifold of $N \times N - \Delta_N$.

**Lemma 1:** If $f$ is completely regular, $3n < 2m$, $(f^2)^{-1} \Delta_{Rm}$ is
equivariantly diffeomorphic to $\mathcal{I}_f$, the involution on the
former being given by restriction from $T$: $(x, y) \rightarrow (y, x)$.

Consider the projection on the first factor $p: N \times N - \Delta \rightarrow N$.
The restriction $\overline{p}$ to $(f^2)^{-1} \Delta_{Rm}$ is then differentiable.
$\overline{p}$ is one-to-one, for if we had $(z, x), (z, y) \in (f^2)^{-1} \Delta_{Rm}$,
then \( f(x) = f(y) = f(z) \), a triple point. Since \( (f^2)^{-1} \Delta_R^m \) is compact, if \( \overline{p} \) is shown to be regular, it will follow that it is a n embedding. There is the commutative diagram

\[
\begin{array}{ccc}
N \times N - \Delta_n & \xrightarrow{\pi^2} & \mathbb{R}^m \times \mathbb{R}^m \\
\downarrow{p} & & \downarrow{\pi} \\
N & \xrightarrow{f} & \mathbb{R}^m \\
\end{array}
\]

Let \( x \in (f^2)^{-1} \Delta_R^m \) and \( v \in T_x \). \( f^2 \) is an immersion so \( f^2_\ast v \neq 0 \) if \( v \neq 0 \). \( f^2 (f^2)^{-1} \Delta_R^m \subset \Delta_R^m \), so \( f^2_\ast v \in T/\Delta f^2_\ast \). But if \( w \in T/\Delta f^2_\ast \pi_\ast w = 0 \) iff \( w = 0 \). Therefore \( p_\ast v \neq 0 \) unless \( v = 0 \).

Now \( \overline{p} (f^2)^{-1} \Delta_R^m \) and \( \overline{T}_f \) coincide as sets. From the above \( (f^2)^{-1} \Delta_R^m \) and \( \overline{T}_f \) are diffeomorphic. Further \( \overline{p} (x,y) = x \), \( \overline{p} T(x,y) = y \), so \( \overline{p} \) commutes with the covering transformation of \( \overline{T}_f \rightarrow I_f \).

**Proposition:** If \( f \) and \( g \) are regularly homotopic completely regular immersions \( N \rightarrow \mathbb{R}^m \), and \( 3n < 2m \), then \( \overline{T}_f \) and \( \overline{T}_g \) are equivariantly cobordant.

Let \( F : N \times I \rightarrow \mathbb{R}^m \) be a regular homotopy between \( f \) and \( g \) which is constant for \( t \leq \frac{1}{3} \), \( t \geq \frac{2}{3} \), \( t \in I \). Define \( F^2 : N \times N \times I \rightarrow \mathbb{R}^m \times \mathbb{R}^m \times I \) by \( F^2(x,y,t) = [F(x,t), F(y,t), t] \). Then \( F^2 \) is an immersion, which by the proof of lemma 1 is transverse regular to \( \Delta_R^m \times I \) for \( t \leq \frac{1}{3} \) and \( t \geq \frac{2}{3} \). Replace \( F^2 \) by a map \( H \) transverse regular to \( \Delta_R^m \times I \) and \( = F^2 \) in the above intervals. Then \( H^{-1}(\Delta_R^m) \) is a cobordism between \( (f^2)^{-1}(\Delta_R^m \times 0) \) and \( (f^2)^{-1}(\Delta_R^m \times 1) \). There is an involution on
$H^{-1}(\Delta_{\mathbb{R}}^m) \times I$ obtained by restriction from $(x,y,t) \rightarrow (y,x,t)$.
The proposition now follows from lemma 2.

3.) The Main Theorem.

The intersection number of a completely regular immersion $f: N \rightarrow \mathbb{R}^{2n}$ is the number (mod 2) of point self-intersections. Whitney [22] showed that for all $n$ there exist immersions with intersection number taking either value.

Given a formal power series $a = \sum \alpha_i x^i$ over $\mathbb{Z}_2$. It is known that $a$ is a unit iff $\alpha_0 = 1$, and in that case its inverse may be formed by rational operations. Let $w_N$ denote the total tangent Stiefel-Whitney class of a manifold $N$.

Theorem: Let $N^n$ be a $\pi$-manifold, $f: N \rightarrow \mathbb{R}^m$ be a completely regular immersion with $3n < 2m$. Let $j_f$ be the composition $N \xrightarrow{f} \mathbb{R}^m \xrightarrow{i} \mathbb{R}^{2n}$, and $c$ be the characteristic class of the covering $\overline{I}_f \rightarrow I_f$. Then $w_{I_f} = (1+c)^{n-m} = \frac{1}{(1+c)^{m-n}}$. If the intersection number of $j_f$ is 0, then $c^{2n-m} = 0$ and $\overline{I}_f$ equivariantly bounds. If the intersection number of $j_f$ is 1, then $c^{2n-m} \neq 0$ and $\overline{I}_f$ is equivariantly nonbounding.

Thus the cobordism type of $I_f$ and the equivariant cobordism type of $\overline{I}_f$ may be computed, as explained in 1B, and depend only on $m,n$, and the intersection number of $j_f$.

Remarks: Note that in the expansion $[\overline{I}_f] = \sum_{i=0}^{2n-m} [M_i][S^i, A^i]$, the only term which can have nonzero involution number $\langle c^{2n-m}, [M_i \times S^i / A, i] \rangle$ is $[S^{2n-m}, A^{2n-m}]$, since $M_i \times S^i / A, i = M_i \times P^i$. 
where \( \mathbb{P}^i \) is real projective \( i \)-space. Thus if \( \overline{I}_f \) does not equivariantly bound, the coefficient of \([S^{2n-m}, A^{2n-m}]\) is 1. I have no examples of immersions for which any other terms are not zero, but nothing seems to preclude the possibility of their existence.

The theorem is almost vacuous when \( n \) is even. Smale and Lashof [8] have shown that if in this case the intersection number is not zero, then the Euler class \( \chi \in H^2(N; \mathbb{Z}) \) of the normal bundle of \( N \) in \( \mathbb{R}^{2n} \) is not zero. But Hirsch [5] shows that \( \chi = 0 \) is a necessary and sufficient condition for an immersion of \( N \) in \( \mathbb{R}^{2n} \) to be regularly homotopic to one in \( \mathbb{R}^{2n-1} \).

The proof of the theorem will be preceded by several lemmas. Let \( k: \mathbb{R}^m \to \mathbb{R}^{m+1} \) be the standard inclusion.

**Lemma 3:** Let \( f: N^n \to \mathbb{R}^m \) be a completely regular immersion, \( 3n < 2m \), with \( \overline{I}_f \) representing \( \sum_{i=0}^{2n-m} [M_i][S^i, A^i] \). Then for any completely regular immersion \( g \) regularly homotopic to \( kf \), \( \overline{I}_g \) represents \( \sum_{i=1}^{2n-m} [M_i][S^{i-1}, A^{i-1}] \).

(The restriction \( 3n < 2m \) is actually not essential.) By the proposition it will suffice to find one such \( g \). The proof will be divided into two parts.

Given a manifold with involution \((V^2, T)\), the line bundle \( E_V \) associated to \( I \) has total space the quotient of \( V \times \mathbb{R} \) by the relation \((x, s) \sim (Tx, -s)\), base space \( V/T \), and projection \( p_V: (x, s) \to [x] \), where \([x]\) is the class of \( x \) in \( V/T \).

1.) Let \( V' \) be a manifold with involution \( T \), representing
\[ \sum_{i=0}^{v} [M_i][S_i^1, A_i^1] \in n \left( B_{Z_2} \right). \] Then if \( r: V/T \to E_v \) is a section which is transverse to the zero section and \( O_r \) is the set of zeroes, then \( (P_{V^{-1}}O_r, T/P_{V^{-1}}O_r) \) represents \( \sum_{i=1}^{v} [M_i][S_i^{i-1}, A_i^{i-1}] \).

By the hypotheses there exists an equivariant cobordism \( (W, R) \) between \( (V, T) \) and \( \bigcup_{i=0}^{v} (M_i \times S_i^1, 1 \times A_i^1) = (X, A) \). I claim that there is a section \( q_i \) of \( E_{M_i} \times S_i^1 \), transverse to the zero section and such that \( O_{q_i} \) is \( M_i \times S_i^{i-1}/T = M_i \times P_i^{i-1} \).

Consider \( S_i^1 \) embedded as the unit sphere in \( R^{i+1} \) and define \( h(x), x \in S_i^1 \), to be the \((i+1)-st\) co-ordinate \( x_{i+1} \) of \( x \).

Then define \( \overline{q_i}(y, x) = (y, x, h(x)) \in M_i \times S_i^1 \times R \). It is evident that \( h(A_i^1x) = h(-x) = -h(x) \). Thus \( \overline{q_i}(x) \) respects the identifications made in defining \( E_{M_i} \times S_i^1 \) and passes to a section \( q_i: M_i \times S_i^1/A_i = M_i \times P_i^1 \to E_{M_i} \times S_i^1 \). Since \( \frac{\partial h}{\partial x_{i+1}} \neq 0 \) on \( S_i^{i-1}, \overline{q_i} \) is transverse to the zero section and thus so is \( q_i \).

Define \( q: X \to E_X \) to be the union of the \( q_i \). Then \( O_q = \bigcup_{i=1}^{v} M_i \times P_i^{i-1} \).

\( X/A \) and \( V/T \) have neighborhoods diffeomorphic to the products \( X/A \times I \) and \( V/T \times I \) in \( W/R \). Define a section \( v: W/R \to E_w \) to be the products \( q \times I \) and \( r \times I \) in these neighborhoods and transverse to the zero section elsewhere. Then \( O_v \) is a manifold with boundary \( O_r \cup O_q \). \( (P_{W^{-1}}O_v, R/P_{W^{-1}}O_v) \) is a manifold with involution, restricting on its boundary to \( (P_{V^{-1}}O_r, T/P_{V^{-1}}O_r) \cup (P_{X^{-1}}O_q, A/P_{X^{-1}}O_q). \) But \( (P_{X^{-1}}O_q, A/P_{X^{-1}}O_q) = \bigcup_{i=0}^{v} (M_i \times S_i^{i-1}, 1 \times A_i^{i-1}) \), which
establishes 1.)

In place of the complicated symbols \( E_{\overline{I}_f}, p_{\overline{I}_f} \), I will write \( E_f, p_f \).

2.) Let \( f \) be an immersion as above. Then if \( r \) is any section of \( E_f \) tranverse to the zero section, there is an immersion \( g \) regularly homotopic to \( kf \) such that \( \overline{I}_g = (p_f^{-1}0_r, T_f|p_f^{-1}0_r) \).

\( g \) will be constructed by first defining a map \( m: \overline{I}_f \rightarrow I_f \times R \subset R^{m+1} \) and then extending \( m \) to an immersion of \( N \).

By the definition of \( E_f \) there is a bundle map \( s: \overline{I}_f \times R \rightarrow E_f \). Then there is the diagram

\[
\begin{array}{ccc}
\overline{I}_f \times R & \xrightarrow{s} & E_f \\
\downarrow & & \downarrow p_f \\
\overline{I}_f & \xrightarrow{\pi} & I_f
\end{array}
\]

Let \( \overline{r} \) be the section of \( \overline{I}_f \) induced from \( r \) by \( \pi \). That is, \( \overline{r}(x) = s^{-1}r \pi(x) \). Define \( m(x) = (f(x), \ell \overline{r} (x)) \). Then since \( \ell \overline{r} x = - \ell \overline{r} T x \), by the identifications made in defining \( E_f \), there exists \( y \) such that \( m(x) = m(y) \) iff \( x \in -10_r \).

Further \( m \) \( \overline{I}_f \) meets \( I_f \) tranversely in \( I_f \times R \) since \( r \) meets the zero section in \( E_f \) transversely and the bundle map \( s \) is locally an isomorphism.

Let \( U \) be a tubular \( \varepsilon \)-neighborhood of \( \overline{I}_f \) in \( N \) for small \( \varepsilon > 0 \).
Let $d(x)$ be the distance of $x$ from $\overline{I_f}$ in $N$. Recall the function $\mu$ defined in 1.). For $y \in R$, $y > 0$ define $\rho(y) = 1 - \mu(y)$. Then $\rho(0) = 1$, $\rho(y) = 0$ if $y \geq \epsilon$.

Define $g$ as follows: If $x \notin u$, $g(x) = kf(x)$. If $w \in \overline{I_f}$ and $x$ is in the fibre of $w$ in $U$, $U_w$, $g(x) = [f(x), \rho(d(x)) \cdot t\overline{w}(w)] \in R^{m+1} = R^{m+1}$. When $g$ extends $m$ and its self-intersections are precisely those of $m$. Define a homotopy $H$ between $kf$ and $g$ by $H(x,t) = [f(x),0]$ if $x \notin u$, and if $x \in U_w$, $H(x,t) = [f(x), \rho(d(x))) \cdot t\overline{w}(w) \cdot t]$.

Then $g$ is an immersion. This is clear if $x \notin u$. If $x \in U_w$, we may select a neighborhood of $w$ in $\overline{I_f}$ on which $m$ is an embedding. Then it is clear that $g$ embeds the restriction of $N$ to this neighborhood. Similarly $H(N,t)$ is an immersion for all $t$, so $H$ is a regular homotopy. Finally we must show that at any intersection point $z$ of $g$ the intersection is transverse. $z$ is also an intersection point of $f$. The projection $R^{m+1} = R \times R^m \rightarrow R^m$ carries $g$ into $f$, which was assumed to be completely regular, so the tangent spaces of the two leaves at $z$ span the subspace of the tangent space of $R^{m+1}$ orthogonal to the last co-ordinate. Since $g$ extends $m$ and the two leaves of $m(\overline{I_f})$ at each intersection were shown to span the tangent space to $L_f \times R$, where $R$ is the last co-ordinate, the two tangent spaces at $z$ span the tangent space to $R^{m+1}$.

Combining 1.) and 2.) yields the lemma.

Corollary: If $f$ is an immersion as above and $\overline{I_f} = (S^{2n-m},A^{2n-m})$, 
then we may choose $g$ so that $\mathcal{I}_g = (S^{2n-m-1}, A^{2n-m-1})$.

In this case the construction of a section $q$ over $X/A$ made in 1.) may be applied directly on $\mathcal{I}_f$. Section 2.) needs no modification.

**Lemma 4:** Let $N^n$ be a $\pi$-manifold, $f: N \to R^m$ an immersion with $3n < 2m$. Let $\alpha$ be the line bundle over $\mathcal{I}_f$ associated to $\pi$: $\overline{I}_f - I_f$, $e$ the trivial bundle, and $c$ the characteristic class of $\alpha$. Then the normal bundle $\nu$ of $I_f$ in $R^m$ is $\alpha^{m-n} \oplus e^{m-n}$ and thus its total Stiefel-Whitney class is $(1+c)^{m-n}$.

I claim that $\overline{I}_f$ is a $\pi$-manifold: By lemma 2 it is diffeomorphic to $(f^2)^{-1} \Delta^m_{R^m} \subset N \times N$. The normal bundle of $(f^2)^{-1} \Delta^m_{R^m}$ in $N \times N$ is trivial, since it is induced from that of $\Delta^m_{R^m}$ in $R^m \times R^m$. Since $N$ is a $\pi$-manifold its stable normal bundle is trivial, and thus so is that of $N \times N$. Therefore the stable normal bundle of $\overline{I}_f$ is trivial.

$\overline{I}_f$ is contained in $N$ with normal bundle $\xi$ of dimension $n-(2n-m) = m-n$. Since $N$ is a $\pi$-manifold $\xi$ is stably trivial. Since $3n < 2m$, $m-n > 2n-m$ and $\xi$ is trivial [7]. Let $\{r_i\}$, $i = 1, \ldots, m-n$, be a framing of $\xi$. If $f(x) = f(y) = z$, the normal space to $I_f$ at $z$ is the sum of the normal spaces to $\overline{I}_f$ in $N$ at $x$ and $y$, since the intersection is transverse. Each $r_i$ thus determines a two-dimensional subspace of the normal space at each point of $I_f$, and so a two dimensional subbundle $\nu_i$ of $\nu$. The $\nu_i$ are mutually isomorphic, for an automorphism of $\xi$ taking $r_i$ to $r_j$ induces an automorphism of $\nu$ taking $\nu_i$ to $\nu_j$. 
Consider the trivial 2-plane bundle $\theta$ over $\overline{\Gamma}_f$ spanned by $r_1$, and a second vector field $s_1$. Define a bundle map $\iota: \theta \to \nu_1$ by $\iota r_1(x) = f_\ast r_1(x)$, $\iota s_1(x) = f_\ast r_1(Tx)$. The identifications made under this map are $ar_1(x) + bs_1(x) \sim as_1(Tx) + br_1(Tx)$, which is to say that the map from the fibre at $x$ to that at $Tx$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This matrix is similar to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore the group $\mathbb{Z}_2$ acts on the fiber of $\nu_1$ with 2 invariant subspaces and $\nu_1$ splits as the sum of a trivial line bundle and the line bundle defined from $\overline{\Gamma}_f \times \mathbb{R}$ by the identifications $(x,t) \sim (Tx,-t)$, which is the line bundle of the covering.

**Proof of the Theorem:** It follows from lemma 4 that the normal Stiefel-Whitney class of $I_f$ is $(1+c)^{m-n}$, where $c$ is truncated to some height $\leq 2n-m$. Thus by Whitney duality the tangent class is $\frac{1}{(1+c)^{m-n}}$. Suppose that the intersection number of $jf$ is 0. Then $c^{2m-n} = 0$, for if not $I_f$ has the nonzero involution number $\langle c^{2m-n}, [I_f] \rangle$, and since $[S^{2n-m}, A^{2n-m}]$ is the only term with this involution number, it must have coefficient 1 in $\overline{\Gamma}_f$. By descending induction using lemma 3 the intersection number of $jf$ is 1, a contradiction.

Then $\overline{\Gamma}_f$ has no nonzero involution numbers and is an equivariant boundary.

If the intersection number of $jf$ is 1, we see using lemma 3 that the coefficient at $[S^{2n-m}, A^{2n-m}]$ is 1, and so $\langle c^{2n-m}, [I_f] \rangle \neq 0$, and $\overline{\Gamma}_f$ does not equivariantly bound.
4.) **Proposition:** Suppose $N$ is a $\pi$-manifold and $f: N \to \mathbb{R}^m$ is an immersion with $3n + 2 \leq 2m$ and $n > 4$. Then if the Arf invariant of manifolds of dimension $n$ is zero $\overline{I}_f$ may be represented as the self-intersection of an immersed $n$-sphere.

Haefliger [3] and Levine [11] have developed a method which permits an embedded $n$-manifold in $\mathbb{R}^m$, $3n < 2m$, with trivial normal bundle, to be transformed by sequence of spherical modifications into an embedded sphere, except possibly when $n = 4k + 2$. In this case the only obstruction is the Arf invariant, which has been shown to vanish when $n \neq 2^k - 2$, $k \geq 5$ (W. Browder).

I will indicate the modifications necessary when $N$ is an immersed $\pi$-manifold. The surgeries are performed on embedded spheres in $N$ of dimension $\leq \lfloor \frac{n}{2} \rfloor$, $\lfloor \rfloor$ denoting the integral part of a fraction. We must be able to choose homotopic embedded spheres disjoint from $\overline{I}_f$. This will be possible, by general position, if $\lfloor \frac{n}{2} \rfloor + 2n - m < n$. Since $2(\lfloor \frac{n}{2} \rfloor + n - m) \leq n + 2n - 2m = 3n - 2m < 0$, that is always the case.

Secondly, the normal bundle of $N$ in $\mathbb{R}^m$ is not necessarily trivial. However an examination of the proofs shows that it is sufficient that the normal bundle over the $\lfloor \frac{n}{2} \rfloor$-skeleton of $N$ be trivial. The obstructions lie in $H^r(N; \pi_{r-1}(SO(m-n)))$, $r \leq \lfloor \frac{n}{2} \rfloor$. Since $N$ is a $\pi$-manifold their stable suspensions $H^r(N; \pi_{r-1}(SO(m-n))) \to H^r(N; \pi_{r-1}(SO))$ are 0. $\pi_i(SO(m-n)) \to \pi_i(SO)$ is an isomorphism if $i \leq m - n - 2$. We have $2(\lfloor \frac{n}{2} \rfloor - 1) \leq n - 2 \leq 2m - 2n - 4 = 2(m - n - 2)$ since $3n + 2 \leq 2m$. Thus the obstructions are 0.
Chapter II.

5.) In the last proposition of Chapter I it was shown that in those dimensions where the Arf invariant of a closed manifold is always zero, a manifold with involution arising as the self-intersection of an immersion of a \( \pi \)-manifold \( \mathbb{N}^n \) with \( 3n + 2 \leq m \), may be realized as the self-intersection of an immersion of the sphere, \( S^n \rightarrow \mathbb{R}^m \). In this section I apply the theory of immersions of spheres to characterize those dimensions \( m \) and \( n \), \( 3n + 2 \leq 2m \), for which there exist a \( \pi \)-manifold \( \mathbb{N}^n \) and an immersion \( \mathbb{N} \rightarrow \mathbb{R}^m \) with self-intersection which is not an equivariant boundary.

The fundamental result of this theory is due to Smale [18].

**Theorem:** The group of immersions of \( n \)-spheres in \( \mathbb{R}^m \), \( m > n+1 \), is isomorphic to the homotopy group \( \pi_n(V_{m,n}) \).

\( V_{m,n} \) denotes the Stiefel manifold of orthogonal \( n \)-frames in \( m \)-space. The construction of the above isomorphism shows that in the case of \( n \) odd, \( n > 1 \), the immersion with intersection number 1 corresponds to the nonzero element of \( \pi_n(V_{2n,n}) \cong \mathbb{Z}_2([18], 327-328) \). It also shows that the inclusion \( \mathbb{R}^m \rightarrow \mathbb{R}^{m+1} \) of immersions induces the map \( \pi_n(V_{m,n}) \rightarrow \pi_n(V_{m+1,n}) \) given by the natural map \( V_{m,n} \rightarrow V_{m+1,n} \) ([18], 327).

As remarked in Chapter I, 3.), if \( n \) is even the only examples of \( \pi \)-manifolds of dimension \( n \) immersed in \( \mathbb{R}^m \) with self-intersections which do not bound equivariantly occur when \( m = 2n \).
We therefore restrict attention to odd \( n \).

**Theorem:** Consider \( n, m \) with \( 3n + 2 \leq 2m \) and \( n \) odd. Then if the Arf invariant of closed \( \pi \)-manifolds of dimension \( n \) is 0, there exists an \( n \)-dimensional \( \pi \)-manifold immersed in \( \mathbb{R}^m \) with equivariantly nonbounding self-intersection iff the natural map \( \iota_*: \pi_n(\mathbb{V}_{m,n}) \to \pi_n(\mathbb{V}_{2n,n}) \) is not zero.

By the proposition cited above it is sufficient to consider immersions of spheres. By the remarks above if \( \iota_* \) is nonzero there is an immersion \( f \) of \( S^n \) in \( \mathbb{R}^m \) which goes under the inclusion \( k: \mathbb{R}^m \to \mathbb{R}^{2n} \) into an immersion with intersection number 1. Applying the Theorem of I, 3.) shows that \( \overline{T}_f \) does not bound in \( \mathcal{H}(B_{2^2}) \).

On the other hand if \( \overline{T}_f \) does not bound equivariantly, by the same Theorem \( k_f \) has intersection number 1, and thus the element representing \( f \) in is carried by \( \iota_* \) to the nonzero element of \( \pi_n(\mathbb{V}_{2n,n}) \).

Since I will give superior examples of self-intersections later by another method, \( \iota_* \) will be shown here to be non-trivial only in the case \( n = 4k+3, m = 8k+4, k > 0 \). This will establish the existence of immersions which are non-orientable nonbounding surfaces. The values of homotopy groups are taken from [19] and [12].

\( V_{m,n} \) is defined as \( \frac{O(m)}{O(m-n)} \), where \( O(r) \) is an orthogonal group. The map \( V_{m,n} \to V_{m+1,n} \) may be factored by an inclusion and a fibration as
\[ v_{m,n} \overset{i}{\rightarrow} v_{m+1,n+1} \overset{j}{\rightarrow} v_{m+1,n} \]

\[ \frac{O(m)}{O(m-n)} \overset{i}{\rightarrow} \frac{O(m+1)}{O(m-n)} \overset{p}{\rightarrow} \frac{O(m+1)}{O(m)} = s^m, \]

from whence the exact sequence

\[ \pi_{i+1}(S^m) \overset{\alpha}{\rightarrow} \pi_i \left( \frac{O(m)}{O(m-n)} \right) \overset{i*}{\rightarrow} \pi_i \left( \frac{O(m+1)}{O(m-n)} \right) \overset{p*}{\rightarrow} \pi_i(S^m). \]

Thus \( i_* \) is an isomorphism if \( i+1 < m \). \( j \) is the projection of the fibration

\[ s^{m-n} = \frac{O(m-n+1)}{O(m-n)} \overset{k}{\rightarrow} \frac{O(m+1)}{O(m-n)} \overset{i}{\rightarrow} \frac{O(m+1)}{O(m-n+1)}, \]

so there is the sequence

\[ \pi_i(s^{m-n}) \overset{j}{\rightarrow} \pi_i(v_{m+1,n+1}) \overset{j*}{\rightarrow} \pi_i(v_{m+1,n}) \overset{\alpha}{\rightarrow} \pi_{i-1}(s^{m-n}). \]

The inclusion

\[ \pi_{4k+3}(v_{8k+4}, 4k+3) \overset{j}{\rightarrow} \pi_{4k+3}(v_{8k+6}, 4k+3) \]

factors as

\[ \pi_{4k+3}(v_{8k+4}, 4k+3) \overset{i_*}{\approx} \pi_{4k+3}(v_{8k+5}, 4k+4) \overset{j}{\rightarrow} \pi_{4k+3}(v_{8k+5}, 4k+3) \overset{i_*}{\approx} \pi_{4k+3}(v_{8k+6}, 4k+4) \overset{j}{\rightarrow} \pi_{4k+3}(v_{8k+6}, 4k+3), \]

where the \( i_* \)'s are isomorphisms from the above remarks.

From the sequences

\[ \pi_{4k+3}(s^{4k+1}) \overset{j}{\rightarrow} \pi_{4k+3}(v_{8k+5}, 4k+4) \overset{j}{\rightarrow} \pi_{4k+3}(v_{8k+5}, 4k+3) \]

\[ \overset{i_1}{\rightarrow} \overset{i_2}{\rightarrow} \overset{i_3}{\rightarrow} \]

\[ \overset{j_1}{\rightarrow} \overset{j_2}{\rightarrow} \overset{j_3}{\rightarrow} \]

\[ \overset{Z_2}{\rightarrow} \overset{Z_8}{\rightarrow} \overset{Z_4}{\rightarrow} \]
and
\[ \pi_{4k+3}(V_{8k+6}, 4k+4) \xrightarrow{j_1^2} \pi_{4k+3}(V_{8k+6}, 4k+3) \xrightarrow{j_2^2} \pi_{4k+2}(S^{4k+2}) \]
\[ \mathbb{Z}_4 \xrightarrow{\mathbb{Z}_2} \mathbb{Z} \]

we see that \( j_1^2 \) and \( j_2^2 \) are epimorphisms, so \( \mu_\star \) also is.

6.) Throughout this section \( P^r \) will denote real projective space and \( S^r \) the sphere considered as a covering space of \( P^r \). I will consider immersions \( f: P^n \to R^m \) with \( 3n + 1 < 2m \). The method used here to find examples of self-intersections consists essentially of considering immersions of projective spaces, passing to the associated immersions of spheres given by the covering projection, and putting these into general position.

\( P^r \) is covered by \( r+1 \) co-ordinate neighborhoods \( \{V_i\} \), where \( V_i \) is the region where the \( i \)-th homogeneous co-ordinate is nonzero. \( V_i \) lifts to \( S^r \) as the union of two open hemispheres \( U_i^+ \) and \( U_i^- \).

**Lemma 5:** If \( f: P^n \to R^m \) is a completely regular immersion, \( 3n + 1 < 2m \), and the normal bundle of \( P^n \) restricted to \( P^{2n-m} \) is trivial, then there is a completely regular immersion \( g: S^n \to R^m \) whose self-intersection is the disjoint union of \( P^{2n-m} \) and another manifold, \( X^{2n-m} \).

Since \( 2n-m + 2n - m < n \) there is by general position an isotopy of \( P^{2n-m} \) in \( P^n \) separating \( P^{2n-m} \) from \( I_f \). By the isotopy extension Theorem of 1.), this isotopy extends to
an isotopy \( H: P^n \times I \to P^n \) such that for \( fH_1: P^n \to R^m \), 
\( fH_1(P^{2n-m}) \cap I_{fH_1} = \emptyset \). I will assume this to be the case, 
writing \( f \) for \( fH_1 \).

\( P^{2n-m} \subset P^n \) may be considered as the intersection of \( m-n \) 
copies of \( P^{n-1} \), \( \{P^{n-1}_i\} \), where \( P^{n-1}_i \) is the set of zeros of the 
i-th homogeneous co-ordinate. Recall the function \( \mu \) defined 
in 1.). Define \( \zeta = [0,\infty) \to R \) by \( \zeta(x) = x(1-\mu(x)) \). Then 
\( \frac{d\zeta}{dx}_{x=0} = 1 \) and \( \zeta(x) = 0, x > \epsilon \), for any chosen \( \epsilon > 0 \). Let \( P^n \) 
have a Riemannian metric \( d \) and define \( C^\infty \) functions \( h_1 = P^n \to R \) 
by \( h_1(x) = \zeta(d(x, P^{n-1})). \)

Let \( K \) be a tubular \( \epsilon \)-neighborhood of \( P^{2n-m} \) in \( P^n \), with 
\( \epsilon \) chosen so that \( K \cap \overline{I_f} = \emptyset \). Then since by hypothesis the 
normal bundle of \( P^n \) restricted to \( P^{2n-m} \) is trivial and \( K \) 
contracts onto \( P^{2n-m} \), we may choose a framing \( \{r_i\} \), \( i = 0, \ldots, 
m-n-1 \) of the normal bundle of \( P^n \) restricted to \( K \). Define 
\( v_i(x) = (1-\mu(d(x, P^{2n-m})))r_i(x) \). Then for \( x \) in the interior 
of \( K \), \( \{v_i(x)\} \) is an orthogonal normal \((m-n)\)-frame, and \( |v_i| \) 
goes smoothly to zero on the boundary of \( K \).

In order to follow the idea of the construction I am 
about to make, consider \( P^1 = S^1 \) embedded as the unit circle 
in the plane, doubly covered by the circle under the map 
\( p: \theta \to 2\theta \). Let \( r_1 \) be the outward normal near \( \theta = 0 \). Let 
\( \delta: P^1 \to R \) be defined by \( \delta(\theta) = 1, 0 < \theta < \pi, \delta(\theta) = -1 \), 
\( \pi < \theta < 2\pi, \delta(0) = \delta(\pi) = 0 \). Then the map 
\[ \theta \to p(\theta) + h_0(\theta)\delta(\theta)v_1(p\theta) \]
is an immersion of \( S^1 \) with a double point \( 0, \pi \) - 0, where the
intersection is transverse.

Let \( \delta_i(x) = 1 \) if \( x \in U^+_i \), \(-1 \) if \( x \in U^-_i \), \( = 0 \) if \( x \notin U^+_i \) or \( U^-_i \). Consider the normal disc bundle to \( P^n \) as a submanifold of \( R^m \). Then define a map \( g': S^n \to R^m \) as follows: Let \( p: S^n \to P^n \) be the projection. If \( x \notin p^{-1}K \) let \( g'(x) = fp(x) \). For \( y \in p^{-1}K \)

\[
g'(y) = \sum_{i=0}^{m-n-1} \delta_i(y)h_i(py) v_i(py) .
\]

We must show that \( g' \) is an immersion. For \( x \notin p^{-1}K \) this is clear since \( f \) is an immersion. For \( x \in p^{-1}K \) it will suffice to show that each \( \delta_i h_i \) is a \( C^\infty \) function of \( x \), for then \( g' \) will give a local section over \( K \), since \( p \) is a local diffeomorphism, and thus \( g' \) will be locally an embedding.

If \( x \in U^+_i \), \( \delta_i(x) h_i(px) = h_i(px) \). If \( x \in U^-_i \), \( \delta_i(x) h_i(px) = -h_i(px) \). Thus there remains only the case \( x \notin U^+_i \), \( U^-_i \). Let \( S^{n-1}_i = S^n - (U^+_i \cup U^-_i) \). Since \( p \) is locally a diffeomorphism, in a small neighborhood \( Z \) of \( x \) we may write

\[
\delta_i(x)h_i(px) = \delta_i(x)d(x,S^{n-1}_i)(1-\mu(d(x,S^{n-1}_i))),
\]

where \( d \) is the metric on \( Z \) induced by \( p \) from \( d \). Choose local co-ordinates in \( Z \) such that \( z_1, \ldots, z_{n-1} \) are local co-ordinates for \( S^{n-1}_i \) and \( |z_n| = d(z,S^{n-1}_i) \), the positive \( z_n \) direction being into \( U^+_i \). Then locally \( \delta_i(z) h_i(pz) = \pm z_n(1-\mu(|z_n|)) \), which is \( C^\infty \) since \( \mu(|z_n|) \) is.

If \( x_1, x_2 \in p^{-1}x, x \in K-p^{2n-m} \), then \( \delta_i(x_1) \neq \delta_i(x_2) \) for some \( i, 0 \leq i < m-n \). Thus the self-intersection set of \( g'|p^{-1}K \) is \( p^{2n-m} \).

We must show that the intersection is transverse.
To do this it will suffice to show that at an intersection point \(x\), the leaves containing \(x_1\) and \(x_2\) have distinct derivatives with respect to \(m-n\) independent directions normal to \(f^{2n-m}\) in \(f^n\). Let \(W\) be a small neighborhood of \(f(p_{x_1}) = f(p_{x_2})\) in \(f(P^n)\). Let \(W_1\) and \(W_2\) be the components of \(p^{-1}W\) which contain \(x_1\) and \(x_2\).

I will define a partial co-ordinate system in \(W\). Let \(w_i, i = 0, \ldots, m-n-1,\) be the normal co-ordinate to \(P_i^{n-1}\) in \(P^n\) with positive direction chosen so that \(w_i(py) > 0\) if \(y \in W_1 \cap U_i^+\), and thus \(w_i(py) < 0\) if \(y \in W_2 \cap U_i^-\). Then the \(w_i\) are independent functions on \(W\), since the \(P_i^{n-1}\) intersect transversely in \(P^n\), and they lift under \(p^{-1}\) to give partial co-ordinate systems \(\{w_1^1\}, \{w_1^2\}\) in \(W_1, W_2\). Let \(g'_1 = g'|_{W_1}, g'_2 = g'|_{W_2}\).

If \(W\) is chosen so that \(W \subseteq K\), the normal bundle of \(P^n\) restricted to \(W\) is trivial, and the vectors \(v_1\) form a partial co-ordinate system for the total space of this bundle. Since \(p\) is a diffeomorphism on each of \(W_1, W_2\), we may write \(g'_1\) and \(g'_2\) in terms of the co-ordinates \(\{w_i^1\}\) instead of \(\{w_1^1\}, \{w_2^2\}\).

Then

\[
\frac{\partial g'_1}{\partial w_i} x = \frac{\partial}{\partial w_i} \sum_{k=0}^{m-n-1} \delta_k (x_j) h_k(w)v_k(w).
\]

Rewriting as above this is equal to

\[
\frac{\partial}{\partial w_i} x (-1)^{j+1} \sum_{k=0}^{m-n-1} w_k(1-\mu(|w_k|))v_k(w) =
\]
\[ (-1)^{j+1} \frac{\partial}{\partial w_i} w_i (1 - \mu(|w_i|)) v_i(w) = (-1)^{j+1} v_i(x). \]

Since
\[ v_2(x) \neq 0, \quad \frac{\partial g_1'}{\partial w_i} \neq \frac{\partial g_2'}{\partial w_i} \text{ for } 0 \leq i < m-n-1 \]
and the intersection is transverse.

Define \( g = g' \) in \( p^{-1} K \), and to be a completely regular immersion close to \( f \) in \( S^n \cdot p^{-1} K \).

**Lemma 6:** Let \( A^a \) and \( B^b \) be disjoint submanifolds of \( S^n \) such that \( a + b + 2 \leq n \). Then \( A \) and \( B \) are separated by an \((n-1)\)-sphere embedded in \( S^n \).

By the isotopy extension theorem this will follow if there is an isotopy of \( A \cup B \) in \( S^n \) taking \( A \) into \( D^n_+ \) and \( B \) into \( D^n_- \). By expanding radially from points \( x_a \in D^n_- \), \( x_b \in D^n_+ \) in the complements of \( A \) and \( B \) respectively, there exist isotopies \( H_a : A \times I \to S^n \times I \) taking \( A \) into \( D^n_+ \) and \( H_b : B \times I \to S^n \times I \) taking \( B \) into \( D^n_- \). Since \((a+1)+(b+1) < n+1\), the tracks of these isotopies may be separated in \( S^n \times I \), by general position.

Thus since \( 2n-m+2n-m+1 < n \), \( p^{-1} p^{2n-m} \) and \( g^{-1} \chi \) may be separated by an \((n-1)\)-sphere in \( S^n \), which is then embedded by \( g \). Thus under the hypotheses of lemma 1 there exists a disc \( D^n \) immersed in \( \mathbb{R}^m \) with \( p^{2n-m} \) as self-intersection and which is embedded in a neighborhood \( V \) of \( \partial D^n \).

**Lemma 7:** Under the conditions of lemma 5 there exists an immersion \( h : S^n \to \mathbb{R}^{m+1} \) with self-intersection \( p^{2n-m-1} \).
Taking the composition $D^n \xrightarrow{\varphi} R^m \xrightarrow{g} R^{m+1}$ we may attach a cone $C$ over $\varphi D^n$, forming a sphere $S^n$ which is (topologically) embedded on $C \cup V$. Now by a theorem of Haefliger [3], since $3n < 2m$, this embedding may be approximated by a differentiable embedding of $C \cup V$, leaving $g$ unaltered outside of $V$.

Now we may apply the corollary to Lemma 3 of Chapter I to produce an immersion $h: S^n \to R^{m+1}$ with the desired self-intersection.

**Theorem:** The conditions of Lemma 5 can be satisfied if

- $m = 2n, n > 1$.
- $m = 2n, 2n-1, n > 1$ odd.
- $m = 2n, \ldots, 2n-3, n=4s+3, s > 1$.
- $m = 2n, \ldots, 2n-6, n=8s+7, s > 1$.
- $m = 2n, \ldots, 2n-8, n=16s+15, s \neq 2^q, s > 1$.

Thus there are immersed spheres whose self-intersections are projective spaces of dimension $< 8$.

It should be remarked that the restriction on dimension is due entirely to the limitations of the present knowledge of immersions of projective spaces.

The existence of the required immersions follows from Hirsch's table [6].

The normal bundle of $P^n$ restricted to $P^{2n-m}$ is of dimension $m-n$, and $m-n > 2n-m$ since $2m > 3n$. Thus if it is stably trivial it is trivial. The normal bundle of $P^{2n-m}$ splits as the sum of the normal bundle of $P^{2n-m}$ in $P^n$ and the normal bundle of $P^n$ restricted to $P^{2n-m}$.
The former is the \((m-n)\)-fold sum of the canonical line bundle \(\alpha\). The tangent bundle of \(P^{2n-m}\) is stably the \((2n-m+1)\)-fold sum of \(\alpha\) \([16]\). Thus to apply lemma 5 we must have that the stable normal bundle of \(P^{2n-m}\) is \((m-n)\) \(\alpha\). Let \(c_{2n-m}\) be the order of \(\alpha\) in \(\widetilde{KO}(P^{2n-m})\) (the reduced \(KO\) group). Since the stable normal bundle is uniquely characterized by the property that its sum with the stable tangent bundle is trivial, it will suffice that \((m-n)+(2n-m+1)=n+1\equiv 0\) (mod. \(c_{2n-m}\)). This may be verified in the above cases from the table below \([1]\).

\[
\begin{bmatrix}
  k = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  c_k = & 1 & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 16
\end{bmatrix}
\]
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