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HYPERSONIC MINIMUM DRAG BODIES OF GIVEN LIFT UNDER VARIOUS GEOMETRIC CONSTRAINTS

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1. INTRODUCTION

Recently, there has arisen considerable interest in transport vehicles designed for a quasi-steady cruise at moderately hypersonic speeds (Refs. 1 and 2). For a given speed and altitude, the lift of such a vehicle is a constant fraction of the weight at every point of the cruise trajectory (Ref. 3); since the fuel consumed per unit range is proportional to the drag(*), it is of interest to find the vehicle shape which minimizes the drag for a given lift. At this moment, no definitive conclusion has been reached as to whether a hypersonic vehicle should have wings or not. Here, the class of wingless configurations is considered, and the optimization of the body shape is discussed under several geometric constraints imposed on the wetted area, the volume, the length, the thickness, or the thickness ratio.

In hypersonic flight, a positive slope for the upper side of the body produces a positive contribution to the drag and a negative contribution to the lift; therefore, it is convenient to have a flat-top at zero angle of attack (Ref. 4)(**). Also, the body should not produce a side force; hence, it is required to have a plane of symmetry perpendicular to the flat top. Furthermore, large values of the slope lead to large local contributions to the pressure drag; thus, it is logical to assume that the body is slender in the longitudinal sense. Finally, since Miele's similarity law (Ref. 5) can be used

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(*) Here, it is assumed that the specific fuel consumption is approximately independent of the power setting.

(**) This conclusion is consistent with the assumption of a Newtonian pressure law.

to transform optimum longitudinal contours obtained for a particular transversal contour into those valid for any other transversal contour, it is assumed in this work that the lower surface is axially symmetric about the line of intersection of the plane of symmetry and the flat top. It is emphasized that this semicircular cross section is not necessarily the best, but just the most convenient from the author's point of view.

Due to the speed limitation imposed by cooling requirements, the small lift coefficients associated with the optimum bodies, and the practical range of values of the weight per unit volume, it is assumed that the aircraft must fly at relatively low altitudes so that the boundary layer will be predominantly turbulent. Hence, it is assumed that the friction coefficient is adequately represented by a surface-averaged value which is considered to be independent of the body shape during the optimization. Finally, in the interest of simplicity and in view of the recent experimental and analytical justifications (Ref. 6), the pressure coefficient is assumed to be given by the modified Newtonian theory.

Thus, the problem of minimizing the drag of a slender, symmetric, flat-topped body of semicircular cross section in a moderately hypersonic flow is considered under the assumptions that the pressure coefficient is modified Newtonian and the surface-averaged skin-friction coefficient is constant. The indirect methods of the calculus of variations are employed, and the necessary conditions to be satisfied by an optimum body are derived for arbitrary conditions imposed on the lift, the wetted area, the volume, the length, and the thickness. The particular cases treated are the following: (a) given lift, (b) given lift and thickness, (c) given lift and wetted area, (d) given lift and length, (e) given lift and thickness ratio, and (f) given lift and volume.

4. BASIC EQUATIONS

In order to relate the aerodynamic and geometric quantities of a symmetic flat-topped body of semicircular cross section to its geometry, it is necessary to define two coordinate systems: a Cartesian system Oxyz and a cylindrical system Oxrθ. For the Cartesian coordinate system, the origin O is the apex of the body; the x-axis is the intersection of the plane of symmetry and the flat top and is positive toward the base; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed. For the cylindrical coordinate system, r is the distance of any point from the x-axis, and θ measures the angular position of the vector r with respect to the xy-plane.

If the hypotheses of the previous section are employed, if it is assumed that the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces, and if the lower surface is represented by the relationship $r = r(x)$, the drag D, the lift L, the wetted area S, and the volume V are given by (Ref. 4)

$$
\begin{aligned}
D/2\pi\epsilon q &= \int_{x_i}^{x_f} r(\dot{r}^3 + nK_f)\,dx \\
L/4\epsilon q &= \int_{x_i}^{x_f} r\dot{r}^2\,dx \\
S/(2+\pi) &= \int_{x_i}^{x_f} r\,dx \\
2V/\pi &= \int_{x_i}^{x_f} r^2\,dx
\end{aligned}
\tag{1}
$$

where ϵ is the constant factor modifying the Newtonian pressure coefficient*, q is the free-stream dynamic pressure, $\dot{r}$ is the derivative dr/dx,

$$n = (2 + \pi)/2\pi \quad , \quad K_f = C_f/\epsilon \tag{2}$$

and C_f is the constant surface-averaged skin-friction coefficient. The end coordinates are represented by the relations

$$\begin{aligned} x_i &= 0 \quad , \quad r_i = 0 \\ x_f &= \ell \quad , \quad r_f = t \end{aligned} \tag{3}$$

where the length ℓ and the thickness t can be either arbitrarily prescribed, related, or free.

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* It is assumed that the pressure coefficient is given by $C_p = 2\epsilon\dot{r}^2$.

3. MINIMAL PROBLEM

The problem of minimizing the drag for arbitrary values imposed on the lift, the wetted area, the volume, the length, the thickness, and the thickness ratio is now formulated as follows: "In the class of functions r(x) which satisfy the integral constraints (1-2) through (1-4) and the prescribed boundary condition, find that particular function which minimizes the integral (1-1)." According to standard variational procedures (see, for instance, Chapter 1 of Ref. 7), this problem is equivalent to that of minimizing the functional

$$I = \int_{x_i}^{x_f} F(r, \dot{r}, \lambda_1, \lambda_2, \lambda_3)\, dx \tag{4}$$

subject to the constraints (1-2) through (1-4) and the prescribed boundary conditions with the understanding that the fundamental function F is defined as

$$F = r\dot{r}^3 + \lambda_1 r\dot{r}^2 + (\lambda_2 + nK_f)r + \lambda_3 r^2 \tag{5}$$

where λ_1, λ_2, λ_3 denote constant Lagrange multipliers.

4. NECESSARY CONDITIONS

The function r(x) which extremizes the functional (4) must be a solution of the Euler equation

$$2r(3\dot{r} + \lambda_1)\ddot{r} + 2\dot{r}^3 + \lambda_1\dot{r}^2 - (\lambda_2 + nK_f) - 2\lambda_3 r = 0 \tag{6}$$

which admits the first integral

$$-2r\dot{r}^3 - \lambda_1 r\dot{r}^2 + (\lambda_2 + nK_f)r + \lambda_3 r^2 = C \tag{7}$$

where C is a constant.

The general solution of the Euler equation involves two integration constants whose values are determined by applying the prescribed boundary conditions and the natural boundary conditions. The latter are obtained from the transversality condition

$$\Big[C\delta x + r\dot{r}(3\dot{r} + 2\lambda_1)\,\delta r\Big]_i^f = 0 \tag{8}$$

which must be satisfied for every system of variations consistent with the prescribed boundary conditions. Since $\delta x_i = \delta r_i = 0$, the natural boundary conditions take the form

$$C = 0 \tag{9}$$

if the length is free,

$$t\dot{r}_f(3\dot{r}_f + 2\lambda_1) = 0 \tag{10}$$

if the thickness is free, and

$$C + \tau t \dot{r}_f (3\dot{r}_f + 2\lambda_1) = 0 \tag{11}$$

if the thickness ratio $\tau = t/\ell$ is prescribed.

Once the solution of the Euler equation is obtained, it is necessary to verify that it actually minimizes the functional (4). In this connection, the Weierstrass necessary condition

$$2\dot{r} + \lambda_1 \geq 0 \tag{12}$$

must be satisfied at every point of the optimum shape.

5. NONDIMENSIONAL VARIABLES

In the following sections, several particular cases are analyzed with the aid of the previous necessary conditions. In order to present the results in the most compact way, it is convenient to introduce the nondimensional coordinates

$$\xi = x/\ell \quad , \quad \rho = r/t \tag{13}$$

and the nondimensional variables

$$\begin{aligned}
E_* &= (L/D)\, K_f^{1/3} \\
S_* &= S(\varepsilon q/L)\, K_f^{2/3} \\
V_* &= V(\varepsilon q/L)^{3/2}\, K_f^{5/6} \\
\ell_* &= \ell(\varepsilon q/L)^{1/2}\, K_f^{1/2} \\
t_* &= t(\varepsilon q/L)^{1/2}\, K_f^{1/6} \\
\tau_* &= (t/\ell)\, K_f^{-1/3}
\end{aligned} \tag{14}$$

6. GIVEN LIFT

For this case, the first integral (7) in conjunction with the natural boundary condition (9) and the conditions $\lambda_2 = \lambda_3 = 0$ leads to the following differential equation to be satisfied by the optimum shape:

$$2\dot{r}^3 + \lambda_1 \dot{r}^2 = nK_f \tag{15}$$

Since the multiplier λ_1 and the term nK_f are constant, this equation is equivalent to

$$\dot{r} = \text{Const} \tag{16}$$

Integration of this differential equation of the shape yields

$$r = C_1 x + C_2 \tag{17}$$

and subsequent use of the end conditions (3) results in

$$C_1 = \tau \ , \quad C_2 = 0 \tag{18}$$

Hence, the optimum flat-topped body is the semicone whose shape, in the nondimensional coordinates (13), is given by

$$\rho = \xi \tag{19}$$

If the integrals (1) are evaluated for this shape and the dimensionless variables (14) are employed, the following relations are obtained:

$$
\begin{aligned}
E_* &= (2/\pi)\,\tau_*^2\,(\tau_*^3 + n)^{-1} \\
S_* &= (2+\pi)(2\tau_*)^{-2} \\
V_* &= (\pi/3)(2\tau_*)^{-5/2} \\
\ell_* &= 2(2\tau_*)^{-3/2} \\
t_* &= (2\tau_*)^{-1/2}
\end{aligned}
\qquad (20)
$$

which connect the dimensionless values of the lift-to-drag ratio, the wetted area, the volume, the length, and the thickness to the thickness ratio.

Since the thickness is free in this case, the natural boundary condition (10) must be satisfied. After eliminating the trivial solution $\dot{r}_f = 0$ and utilizing Eqs. (16) through (18), Eqs. (10) and (15) can be rewritten as follows:

$$3\tau + 2\lambda_1 = 0 \quad , \quad 2\tau^3 + \lambda_1\tau^2 = nK_f \qquad (21)$$

If the multiplier λ_1 is eliminated between these two relations, one obtains the following optimum value of the thickness ratio:

$$\tau_* = (2n)^{1/3} \cong 1.18 \qquad (22)$$

which is such that the friction drag is one-third of the total drag. Therefore, the quantities (20) become

$$
\begin{aligned}
E_* &= (4/3\pi)(2n)^{-1/3} \cong 0.360 \\
S_* &= (\pi/4)(2n)^{1/3} \cong 0.926 \\
V_* &= (\pi/12\sqrt{2})(2n)^{-5/6} \cong 0.123 \\
\ell_* &= (1/2)\, n^{-1/2} \cong 0.553 \\
t_* &= (1/\sqrt{2})(2n)^{-1/6} \cong 0.651
\end{aligned}
\tag{23}
$$

Equation (23-1) represents the highest lift-to-drag ratio which can be obtained using a flat-topped body of semicircular cross section with the flow parallel to the top (Ref. 4). Should the body be required to satisfy a certain number of geometric constraints, a decrease in the lift-to-drag ratio must be expected with respect to that predicted by Eq. (23-1).

7. GIVEN LIFT AND THICKNESS

As in the previous case, the relationships $C = \lambda_2 = \lambda_3 = 0$ hold and, hence, Eqs. (15) through (20) are valid. Since the lift and the thickness are prescribed, the parameter t_* is known a priori. Hence, because of Eq. (20-5), the optimum thickness ratio is given by (Fig. 2)

$$\tau_* = (1/2)\, t_*^{-2} \tag{24}$$

The lift-to-drag ratio becomes (Fig. 3)

$$E_* = (4/\pi)\, t_*^2 (1 + 8nt_*^6)^{-1} \tag{25}$$

and achieves the maximum value (23-1) for the value of t_* defined by Eq. (23-5).

8. GIVEN LIFT AND WETTED AREA

For this case, the relationships $C = \lambda_3 = 0$ are valid and, as a consequence, the first integral (7) reduces to

$$2\dot{r}^3 + \lambda_1 \dot{r}^2 = \lambda_2 + nK_f \tag{26}$$

Since λ_1, λ_2, and nK_f are constant, this equation has the solution (16) so that Eqs. (17) through (20) are valid. Since the lift and the wetted area are prescribed, the parameter S_* is known a priori. Hence, because of Eq. (20-2), the optimum thickness ratio is given by (Fig. 4)

$$\tau_* = (1/2)(2+\pi)^{1/2} S_*^{-1/2} \tag{27}$$

The corresponding length can be written as (Fig. 5)

$$\ell_* = 2(2+\pi)^{-3/4} S_*^{3/4} \tag{28}$$

Finally, the lift-to-drag ratio becomes (Fig. 6)

$$E_* = \frac{4S_*^{1/2}}{\pi(2+\pi)^{1/2} + 4S_*^{3/2}} \tag{29}$$

and achieves the maximum value (23-1) for the value of S_* defined by Eq. (23-2).

9. GIVEN LIFT AND LENGTH

For this case, the first integral (7) in conjunction with the conditions $\lambda_2 = \lambda_3 = 0$ leads to the following differential equation of the optimum shape:

$$- 2r\dot{r}^3 - \lambda_1 r\dot{r}^2 + nK_f r = C \tag{30}$$

By evaluating this relationship at the initial point, we see that two classes of solutions are possible

$$\text{Class I} \qquad \dot{r}_i = \infty \quad , \quad C < 0$$
$$\text{Class II} \qquad \dot{r}_i = \text{finite} \; , \quad C = 0 \tag{31}$$

For both classes, the natural boundary condition (10) holds and implies that

$$\dot{r}_f = - (2/3)\,\lambda_1 \tag{32}$$

Solutions of Class I. For these solutions, the first integral (30) leads to the following parametric representation of the optimum shape:

$$x = \int_0^r (1/\dot{r})\,dr = 2C \int_\infty^{\dot{r}} (3\dot{r} + \lambda_1)(2\dot{r}^3 + \lambda_1\dot{r}^2 - nK_f)^{-2} d\dot{r}$$
$$r = - C(2\dot{r}^3 + \lambda_1\dot{r}^2 - nK_f)^{-1} \tag{33}$$

After the nondimensional coordinates (13) are employed in combination with the definitions

$$\alpha = (1/\dot{r})K_f^{1/3} \quad , \quad \lambda = \lambda_1 K_f^{-1/3} \tag{34}$$

Eqs. (33) can be rewritten as

$$\xi = A(\alpha, \lambda)/A(\alpha_f, \lambda) \quad , \quad \rho = B(\alpha, \lambda)/B(\alpha_f, \lambda) \tag{35}$$

where

$$\begin{aligned} A(\alpha, \lambda) &= \int_0^{\alpha} u^3(3 + \lambda u)(2 + \lambda u - nu^3)^{-2} du \\ B(\alpha, \lambda) &= \alpha^3(2 + \lambda\alpha - n\alpha^3)^{-1} \end{aligned} \tag{36}$$

and where, because of Eq. (32),

$$\alpha_f = - 3/2\lambda \tag{37}$$

The next stop consists of relating the quantity λ to the prescribed values of the lift and the length as well as calculating the maximum lift-to-drag ratio. By combining Eqs. (35) with the integrals (1) and the definitions (14) and (34), we obtain the relationships

$$\begin{aligned} \ell_* &= (1/\sqrt{2})A(\alpha_f, \lambda)[G(\alpha_f, \lambda)]^{-1/2} \\ \tau_* &= (1/2)\,[B(\alpha_f, \lambda)/A(\alpha_f, \lambda)] \\ E_* &= (2/\pi)\,[G(\alpha_f, \lambda)/H(\alpha_f, \lambda)] \end{aligned} \tag{38}$$

where

$$\begin{aligned} G(\alpha, \lambda) &= \int_0^{\alpha} u^4(3 + \lambda u)(2 + \lambda u - nu^3)^{-3} du \\ H(\alpha, \lambda) &= \int_0^{\alpha} u^3(1 + nu^3)(3 + \lambda u)(2 + \lambda u - nu^3)^{-3} du \end{aligned} \tag{39}$$

The final step consists of eliminating the quantities α_f and λ from Eqs. (35) through (38). If this is done, one obtains the functional relationships

$$\rho = f_1(\xi, \ell_*) \tag{40}$$

and

$$\tau_* = f_2(\ell_*) \quad , \quad E_* = f_3(\ell_*) \tag{41}$$

which are plotted in Figs. 7 through 9 and are valid in the range

$$0 \leq \ell_* \leq (1/2)n^{-1/2} \tag{42}$$

At the upper limit of this range, the lift-to-drag ratio (41-2) achieves the maximum value (23-1).

<u>Solutions of Class II</u>. For these solutions, the first integral (30) is solved by

$$r = 0 \quad \text{or} \quad 2\dot{r}^3 + \lambda_1 \dot{r}^2 = nK_f \tag{43}$$

If the corner conditions (see Chapter 1 of Ref. 7) and the natural boundary condition (10) are combined with Eqs. (43), we see that the optimum body is a spike of zero thickness followed by a semicone of apex angle identical with that found in Section 6. If the nondimensional coordinates (12) are used, the shape is given by (Fig. 7)

$$\rho = 0 \quad , \quad 0 \le \xi \le \xi_o$$
$$\rho = (\xi - \xi_o)/(1 - \xi_o) \quad , \quad \xi_o \le \xi \le 1 \tag{44}$$

where the dimensionless abscissa of the transition point satisfies the relationship

$$\xi_o = 1 - (1/2)n^{-1/2} \ell_*^{-1} \tag{45}$$

The thickness ratio and the lift-to-drag ratio of this body are given by (Figs. 8 and 9)

$$\tau_* = (1/\sqrt{2})(2n)^{-1/6} \ell_*^{-1} \quad , \quad E_* = (4/3\pi)(2n)^{-1/3} \tag{46}$$

These solutions occur in the range

$$(1/2)n^{-1/2} \le \ell_* \le \infty \tag{47}$$

and are characterized by a lift-to-drag ratio identical with that obtained in the case where the lift is the only prescribed quantity. The reason is that, if a spike of zero thickness is added to the semicone of optimum apex angle, the drag and the lift are unaffected and, consequently, the lift-to-drag ratio does not change.

10. GIVEN LIFT AND THICKNESS RATIO

This case is very similar to that of the previous section in that the differential equation (30) for the optimum shape is still valid and leads to the two classes of solutions (31). However, the natural boundary condition of the previous section does not apply to this problem and must be replaced by Eq. (11). The latter, in combination with Eq. (30) evaluated at the final point, yields the following alternate form of the natural boundary condition

$$2\dot{r}_f^3 + \lambda_1 \dot{r}_f^2 - nK_f - \tau \dot{r}_f (3\dot{r}_f + 2\lambda_1) = 0 \tag{48}$$

Solutions of Class I. For these solutions, the differential equation (30) leads once more to the parametric equations (33) for the optimum shape. Furthermore, the nondimensional coordinates (13) and the definitions (34) lead to Eqs. (35) and (36). However, at this point, it must be noted that Eq. (37) is no longer valid and must be replaced by

$$\tau_* = K(\alpha_f, \lambda)/B(\alpha_f, \lambda) \tag{49}$$

where

$$K(\alpha, \lambda) = \alpha^2/(3 + 2\lambda\alpha) \tag{50}$$

This relationship was obtained by combining the definitions (14-6) and (34) with the natural boundary condition (48).

The next stop consists of relating the quantities λ and α_f to the prescribed values of the lift and thickness ratio. As in the previous section, the combination of Eqs. (35) with the integrals (1) and the definitions (14) and (34) once again yields Eqs. (38) and (39). Hence, the final step consists of eliminating the quantities λ and α_f from Eqs. (35), (38), and (49) to obtain the functional relationships

$$\rho = f_1(\xi, \tau_*) \tag{51}$$

and

$$\ell_* = f_2(\tau_*) \quad , \quad E_* = f_3(\tau_*) \tag{52}$$

which are plotted in Figs. 10 through 12 and are valid in the range

$$(2n)^{1/3} \leq \tau_* \leq \infty \tag{53}$$

At the lower limit of this range, the lift-to-drag ratio (52-2) achieves the maximum value (23-1).

Solutions of Class II. For these solutions, Eqs. (43) are still valid and the natural boundary condition (49) reduces to Eq. (37). Hence, as in the previous section, the optimum shape is composed of a spike of zero thickness followed by a semicone whose apex angle is identical with that found in Section 6. Analytically, the shape is given by Eqs. (44) where the dimensionless abscissa satisfies the relationship

$$\xi_o = 1 - (2n)^{-1/3} \tau_* \tag{54}$$

The length and the lift-to-drag ratio of this body are given by

$$\ell_* = (1/\sqrt{2})(2n)^{-1/6}\,\tau_*^{-1} \quad , \quad E_* = (4/3\pi)(2n)^{-1/3} \tag{55}$$

These solutions occur in the range (Figs. 10 through 12)

$$0 \leq \tau_* \leq (2n)^{1/3} \tag{56}$$

and are characterized by a lift-to-drag ratio identical with that obtained in the case where the lift is the only prescribed quantity.

11. GIVEN LIFT AND VOLUME

For this case, the first integral (7) in conjunction with the conditions $C = \lambda_2 = 0$ leads to the following differential equation of the optimum shape:

$$- 2\dot{r}^3 - \lambda_1 \dot{r}^2 + nK_f + \lambda_3 r = 0 \tag{57}$$

which, at the initial point, becomes

$$2\dot{r}_i^3 + \lambda_1 \dot{r}_i^2 - nK_f = 0 \tag{58}$$

Furthermore, by applying the natural boundary condition (10), we see that two classes of solutions are possible

$$\text{Class I} \qquad \dot{r}_f = -(2/3)\,\lambda_1$$
$$\text{Class II} \qquad \dot{r}_f = 0 \tag{59}$$

As a first step, we solve Eq. (57) in parametric form as follows:

$$x = \int_0^r (1/\dot{r})dr = (2/\lambda_3)\int_{\dot{r}_i}^{\dot{r}_f} (3\dot{r} + \lambda_1)\, d\dot{r}$$
$$r = (1/\lambda_3)(2\dot{r}^3 + \lambda_1 \dot{r}^2 - nK_f) \tag{60}$$

Next, we introduce the nondimensional coordinates (12) in conjunction with the definitions

$$\beta = \dot{r}K_f^{-1/3} \quad , \quad \lambda = \lambda_1 K_f^{-1/3} \tag{61}$$

and rewrite Eqs. (60) in the form

$$\xi = A(\beta, \beta_i)/(A(\beta_f, \beta_i) \quad , \quad \rho = B(\beta, \beta_i)/B(\beta_f, \beta_i) \tag{62}$$

where

$$\begin{aligned} A(\beta, \beta_i) &= \pm \left[3u^2 + 2\lambda(\beta_i)u\right]_{\beta_i}^{\beta} \\ B(\beta, \beta_i) &= \pm \left[2\beta^3 + \lambda(\beta_i)\beta^2 - n\right] \end{aligned} \tag{63}$$

and where the upper and lower signs hold for $\beta_i \gtrless (2n)^{1/3}$, respectively. In the above relationships, the function $\lambda(\beta_i)$ is implicitly defined by Eq. (58), which can be rewritten as

$$\lambda(\beta_i) = -2\beta_i + n/\beta_i^2 \tag{64}$$

Furthermore, the function $\beta_f(\beta_i)$ is defined by either of Eqs. (59), that is, by

$$\begin{aligned} &\underline{\text{Class I}} \qquad && \beta_f = -(2/3)\lambda(\beta_i) \\ &\underline{\text{Class II}} \qquad && \beta_f = 0 \end{aligned} \tag{65}$$

The next step consists of relating the quantity β_i to the prescribed values of the lift and the volume as well as calculating the unknown values of the length, the thickness ratio, and the lift-to-drag ratio. By combining Eqs. (62) with the integrals (1) and the definitions (13), (14), and (61), we obtain the relationships

$$\begin{aligned} \ell_* &= (1/2\sqrt{2})A(\beta_f, \beta_i)[G(\beta_f, \beta_i)]^{-1/2} \\ \tau_* &= B(\beta_f, \beta_i)/A(\beta_f, \beta_i) \\ E_* &= (2/\pi)[G(\beta_f, \beta_i)/H(\beta_f, \beta_i)] \\ V_* &= (\pi/16\sqrt{2})K(\beta_f, \beta_i)[G(\beta_f, \beta_i)]^{-3/2} \end{aligned} \tag{66}$$

where

$$\begin{aligned} G(\beta, \beta_i) &= \left[\frac{6}{7}u^7 + \frac{5}{6}\lambda u^6 + \frac{1}{5}\lambda^2 u^5 - \frac{3}{4}nu^4 - \frac{1}{3}\lambda nu^3\right]_{\beta_i}^{\beta} \\ H(\beta, \beta_i) &= \left[\frac{3}{4}u^8 + \frac{5}{7}\lambda u^7 + \frac{1}{6}\lambda^2 u^6 + \frac{3}{5}nu^5 + \lambda nu^4 + \frac{1}{3}\lambda^2 nu^3 - \frac{3}{2}n^2u^2 - \lambda n^2 u\right]_{\beta_i}^{\beta} \\ K(\beta, \beta_i) &= \pm\left[\frac{3}{2}u^8 + \frac{16}{7}\lambda u^7 + \frac{7}{6}\lambda^2 u^6 + \frac{1}{5}(\lambda^3 - 12n)u^5 - \frac{5}{2}\lambda nu^4 - \frac{2}{3}\lambda^2 nu^3 + \frac{3}{2}n^2u^2 + \lambda n^2 u\right]_{\beta_i}^{\beta} \end{aligned} \tag{67}$$

The final step consists of eliminating the quantities β_i and β_f from Eqs. (62) through (66). If this is done, one obtains the functional relationships

$$\rho = f_1(\xi, V_*) \tag{68}$$

and

$$\ell_* = f_2(V_*) \quad , \quad \tau_* = f_3(V_*) \quad , \quad E_* = f_4(V_*) \tag{69}$$

which are valid in the range

$$\underline{\text{Class I}} \qquad 0 \le V_* \le (\pi/40)\, 7^{3/2}\, 2^{-1/6}\, n^{-5/6}$$

$$\underline{\text{Class II}} \qquad (\pi/40)\, 7^{3/2}\, 2^{-1/6}\, n^{-5/6} \le V_* \le \infty \tag{70}$$

Because of the practical values of the parameter V_* , only solutions of Class I are significant from an engineering point of view (see Figs. 13 through 16). For these solutions, the lift-to-drag ratio E_* achieves the maximum value (23-1) for the value of V_* defined by Eq. (23-3).

12. DISCUSSION OF RESULTS

In the previous sections, the problem of minimizing the drag of a slender, symmetric, flat-topped body of semicircular cross section in a moderately hypersonic flow is investigated under the assumptions that the pressure coefficient is modified Newtonian and the surface-averaged skin-friction coefficient is constant. The indirect methods of the calculus of variations are employed, and the necessary conditions to be satisfied by an optimum body are derived for arbitrary conditions imposed on the lift, the wetted area, the volume, the length, the thickness, and the thickness ratio. The particular problems treated are the following: (a) given lift, (b) given lift and thickness, (c) given lift and wetted area, (d) given lift and length, (e) given lift and thickness ratio, and (f) given lift and volume.

For case (a), the optimum body is a semicone of thickness ratio $\tau = 1.18\,(C_f/\epsilon)^{1/3}$ and lift-to-drag ratio $E = 0.360\,(C_f/\epsilon)^{-1/3}$. For each of the cases (b) and (c), the optimum body is a semicone but the thickness ratio and lift-to-drag ratio are generally different from those pertaining to case (a). For case (d), two solutions are possible depending on whether the length is smaller or larger than that associated with the optimum semicone of case (a). If the length is smaller, a blunt-nosed body is obtained while, if the length is larger, the optimum body is composed of the semicone of case (a) preceded by a spike of zero thickness. For case (e), two solutions are possible depending on whether the thickness ratio is larger or smaller than that associated with the optimum semicone of case (a). If the thickness ratio is larger, a blunt-nosed body is obtained while, if the thickness ratio is smaller, the optimum body is composed of the semicone of case (a) preceded by a spike of zero thickness. Finally, for case (f), the solution is generally a sharp-nosed body. However, depending on whether the volume is smaller or larger than that associated with the semicone of case (a), the optimum body is concave or convex, respectively.

Among the preceding solutions, those leading to spiked and concave bodies are certainly suspect from the point of view of the applicability of the modified Newtonian pressure distribution (Ref. 8); hence, one must investigate these cases more thoroughly. First, consider the spike-semicone solutions. Clearly, the spike of zero thickness is a physical imposibility, yet the constraints which force the spike to be a part of the solution must be satisfied; hence, in practice, the spike of zero thickness must be replaced by one having the smallest thickness possible from the structural and heat transfer points of view. According to the theory of this work, the spike of zero thickness does not influence the pressure distribution on the semicone which follows it. Now, the question arises whether or not the pressure distribution is essentially unchanged when the semicone is preceded by a very slender spike. The answer to this question depends on whether or not the adverse pressure gradient, induced by the body-spike junction, is sufficient to cause flow separation. If a significant region of separated flow is present, the Newtonian pressure distribution leads to considerable error; however, if the flow does not separate appreciably, the Newtonian pressure distribution is valid. It can be seen from Ref. 9 that, for slender cones, no separation occurs for a large range of Mach and Reynolds numbers. Since this includes the range of interest of this paper, we conclude that the use of the modified Newtonian pressure distribution for the spike-semicone bodies is justified.

Now, consider the use of the Newtonian pressure distribution in conjunction with the concave bodies obtained in Section 11. Clearly, the concavity of the body tends to invalidate the Newtonian pressure distribution (Ref. 8). The actual pressure distribution is more accurately represented by the Newton-Busemann law, which includes the effect of the longitudinal curvature of the body. However, due to the small values of the curvature associated with the practical values of the dimensionless volume, it is concluded that the Newtonian pressure distribution is adequate for preliminary considerations. If greater precision is required, the optimum problem must be restudied using a more complex physical model.

Many of the optimum bodies of the previous sections are sharp-nosed and, hence, impractical from the heat transfer point of view. In order to contain the nose heating rate within acceptable limits, it is necessary to blunt the nose somewhat. For a two-dimensional wing, a slight blunting of the leading edge leads to a substantial decrease in the maximum lift-to-drag ratio (Ref. 10). However, both theory and experiments show that this is not the case with a bodylike configuration (Ref. 8).

Finally, the question arises as to just how the equations of the preceding sections are to be employed in order to determine an optimum body. In this connection, it is convenient to utilize a particular example, that of the minimum drag body in quasi-steady, level cruise at a given altitude h, Mach number M, total weight W, and constant average wall temperature T_W. For these conditions, the lift is determined by the equation of motion on the normal to the flight path (Ref. 3)

$$L \cong W\left(1 - \frac{M^2 a^2}{r_o g_o}\right) \tag{71}$$

in which r_o is the radius of the Earth, g_o the acceleration of gravity at sea level, and a = a(h) the speed of sound. If no geometric constraint is imposed, the results of Section 6 apply, and the optimum body is the semicone of thickness ratio and length

$$\tau = 1.18(C_f/\varepsilon)^{1/3} \quad , \quad \ell = 0.553(L/qC_f)^{1/2} \tag{72}$$

In these relations, the dynamic pressure q is defined by

$$q = \gamma p M^2/2 \tag{73}$$

where γ is the ratio of specific heats and p = p(h) is the free-stream static pressure. Equations (72) and (73) would completely determine the optimum body, if C_f and ε were known numbers. In practice, this is not the case, since these quantities depend on the geometry of the body. Therefore, an iteration procedure is necessary. For a conical body, the constant modifying the Newtonian pressure distribution is given by (Ref. 11)

$$\epsilon = (\gamma + 7)/8 + 3/20M^2\tau^2 \tag{74}$$

and, therefore, depends on the thickness ratio. Neglecting dissociation, ionization, and mass addition, the surface-averaged skin-friction coefficient is a function of the form

$$C_f = C_f(h, M, T_W, \tau, \ell) \tag{75}$$

to be determined through the solution of (a) the conical flow equations for the inviscid flow over the bottom, (b) the boundary layer equations over the bottom, and (c) the boundary layer equations over the top (Refs. 12 and 13). With these considerations in mind, the following procedure must be employed: (a) assume a pair of reasonable values for ϵ and C_f; (b) determine τ and ℓ using Eqs. (72); (c) calculate a new pair of values for ϵ and C_f using Eqs. (74) and (75); and (d) proceed iteratively until there is no significant difference between the assumed values and those calculated.

Notice, that in this example, the optimum body is conical; hence, the solution of the boundary layer equations is not complicated by the presence of a pressure gradient, and the surface-averaged skin-friction coefficient is accurately represented by the relationship (75). Should the problem be such that the optimum body is not conical, the relationship (75) would only be an approximation to the exact relation, since the actual value of C_f depends on the pressure distribution over the body and, hence, its shape. It should be noted that (a) for a relatively cold wall, the dependence of the local skin-friction coefficient on the pressure gradient is small, (b) if the body is sharp-nosed, the pressure gradient is small, and (c) if the body is blunt-nosed, the region of significant pressure gradient is a relatively small area at the nose where the body behaves as a 3/4-power law body. For these reasons, the author feels that the relationship (75) adequately represents the surface-averaged skin-friction coefficient even for the case where the pressure varies along the surface.

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LIST OF CAPTIONS

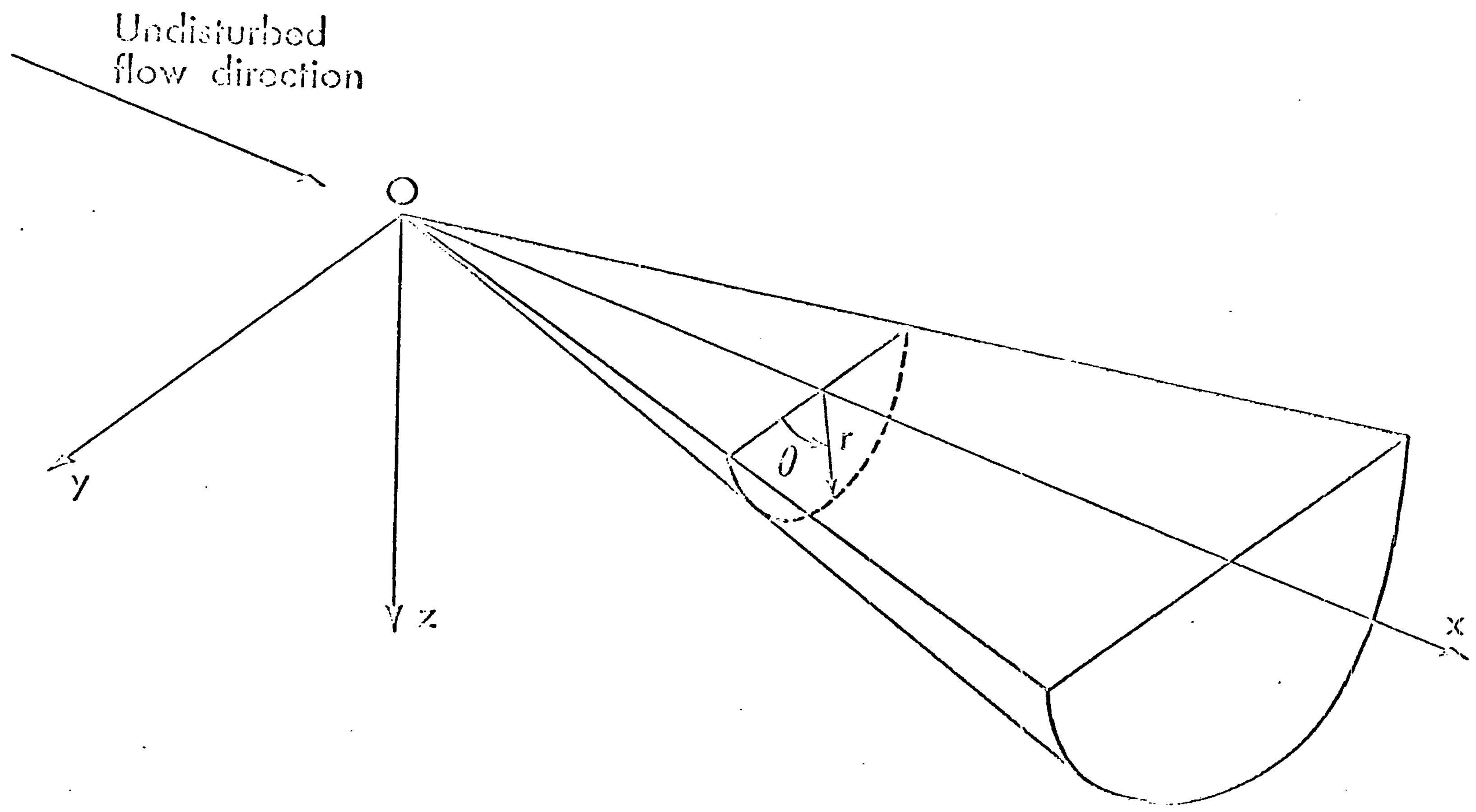

Fig. 1 Coordinate system.

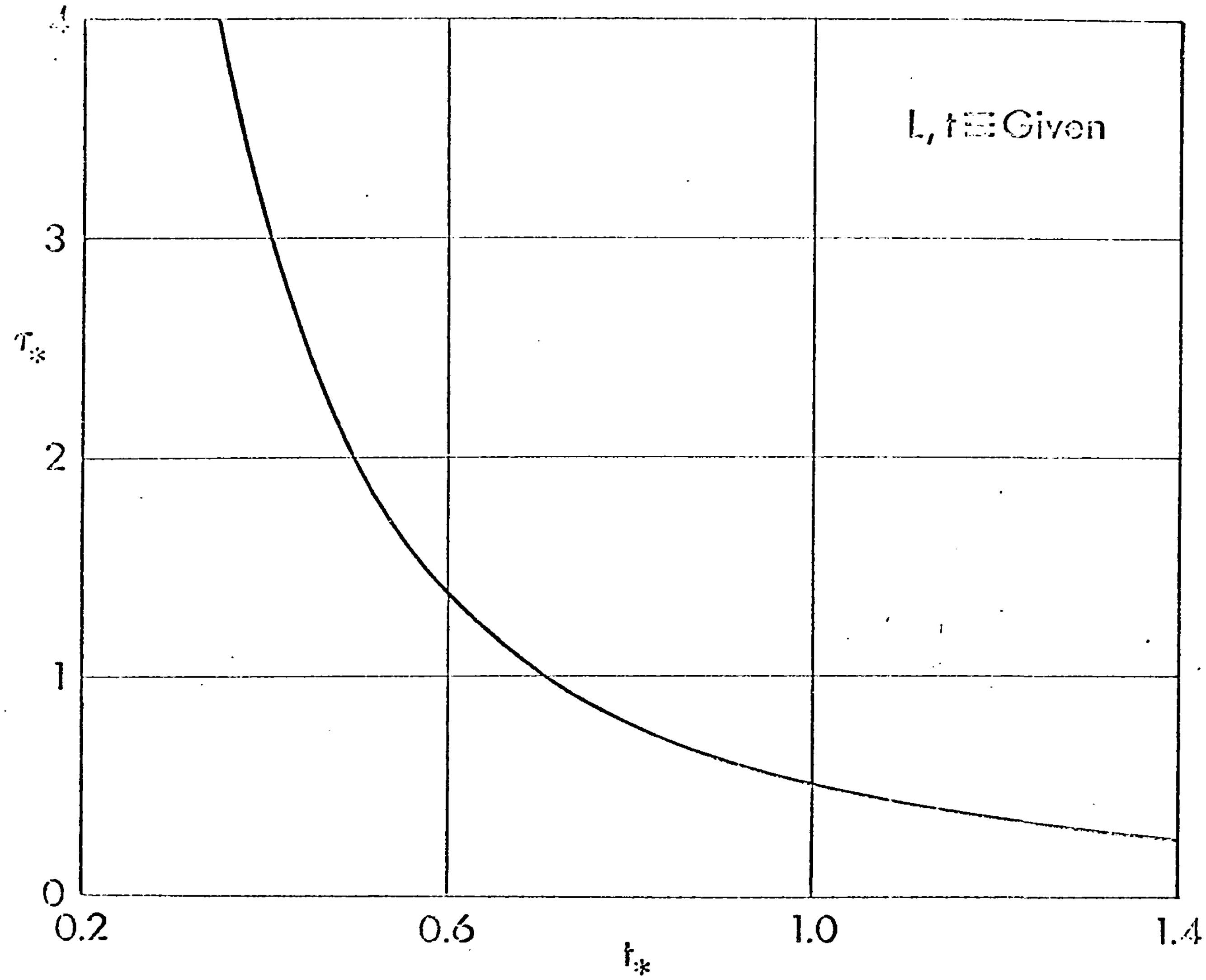

Fig. 2 Optimum thickness ratio.

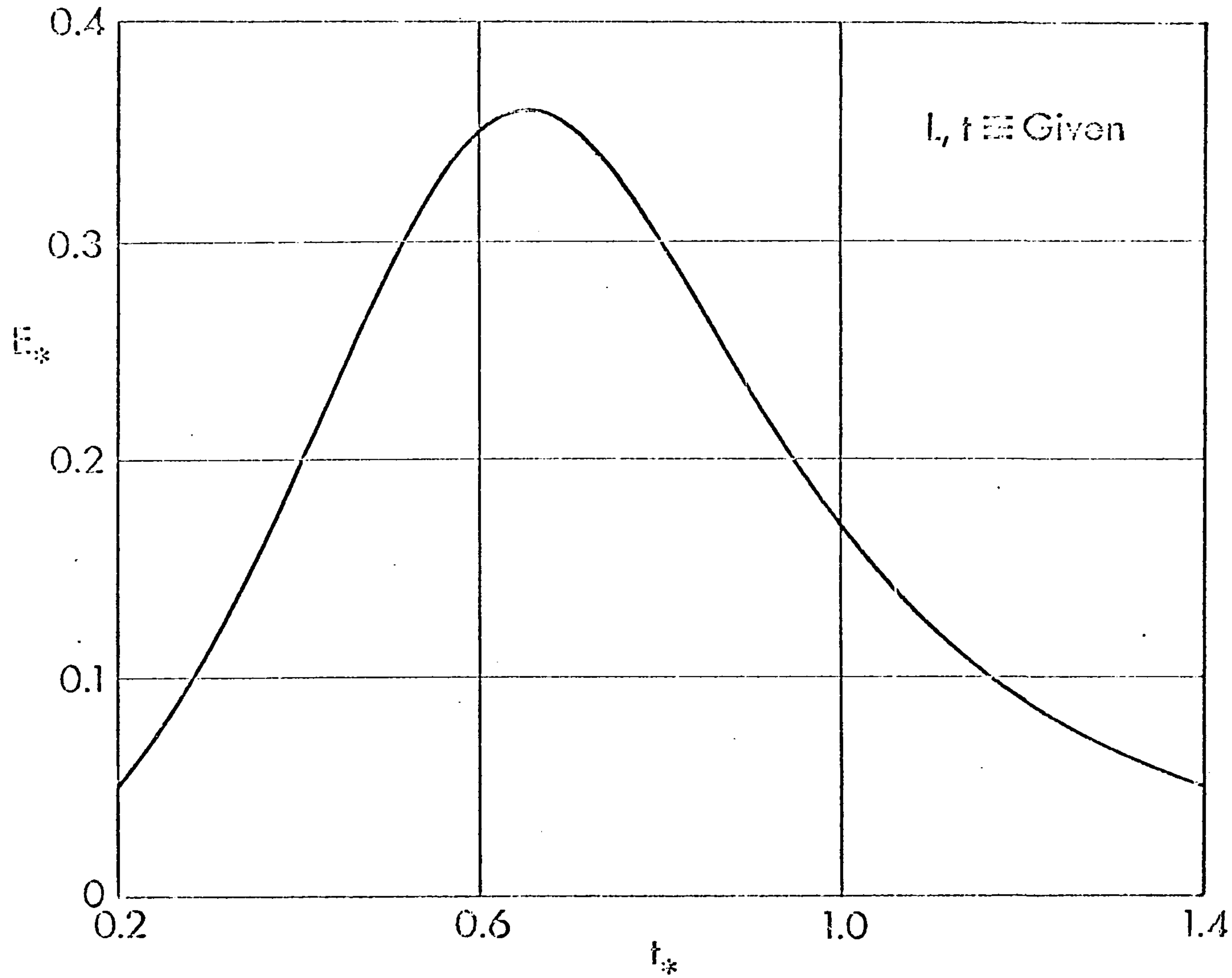

Fig. 3 Maximum lift-to-drag ratio.

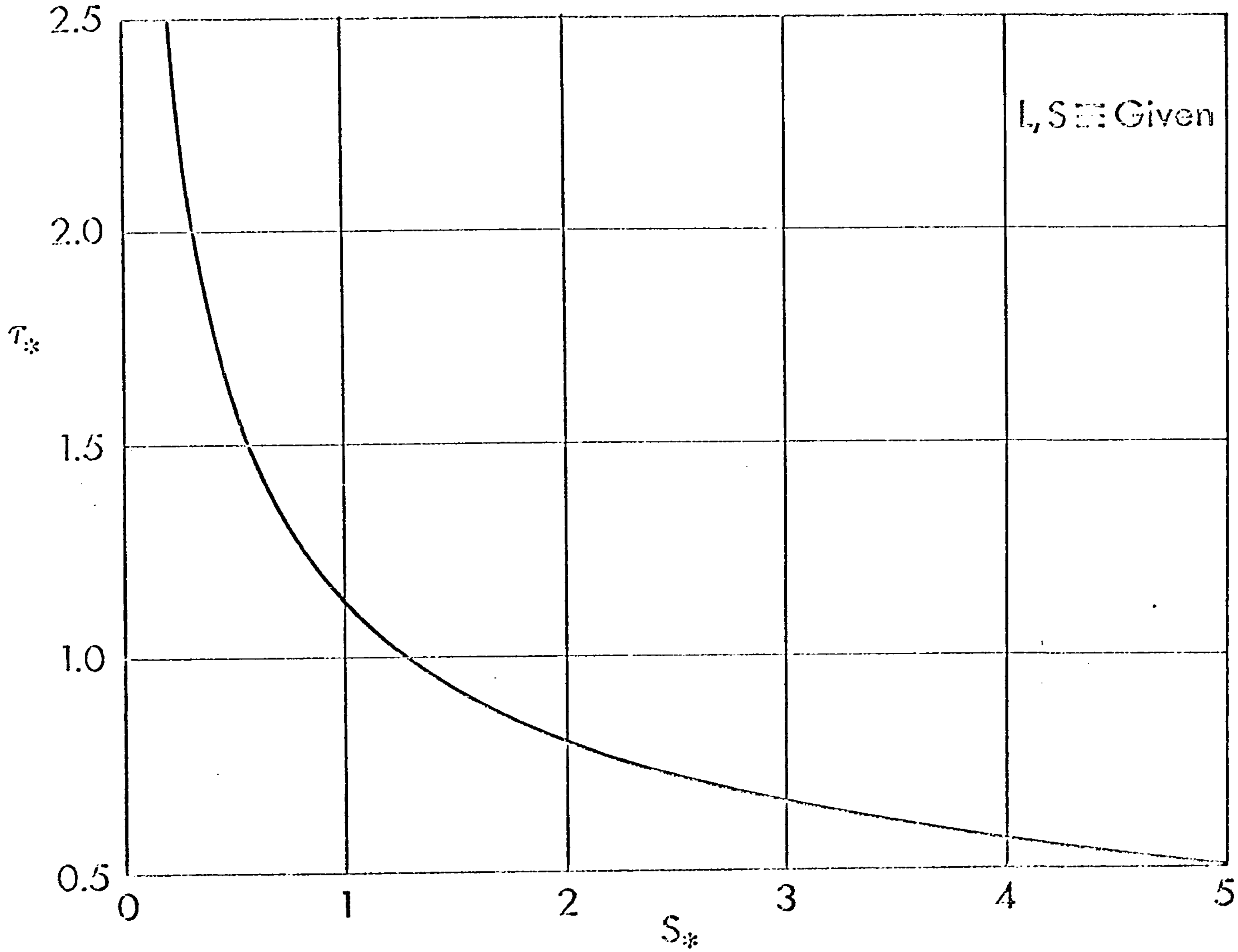

Fig. 4 Optimum thickness ratio.

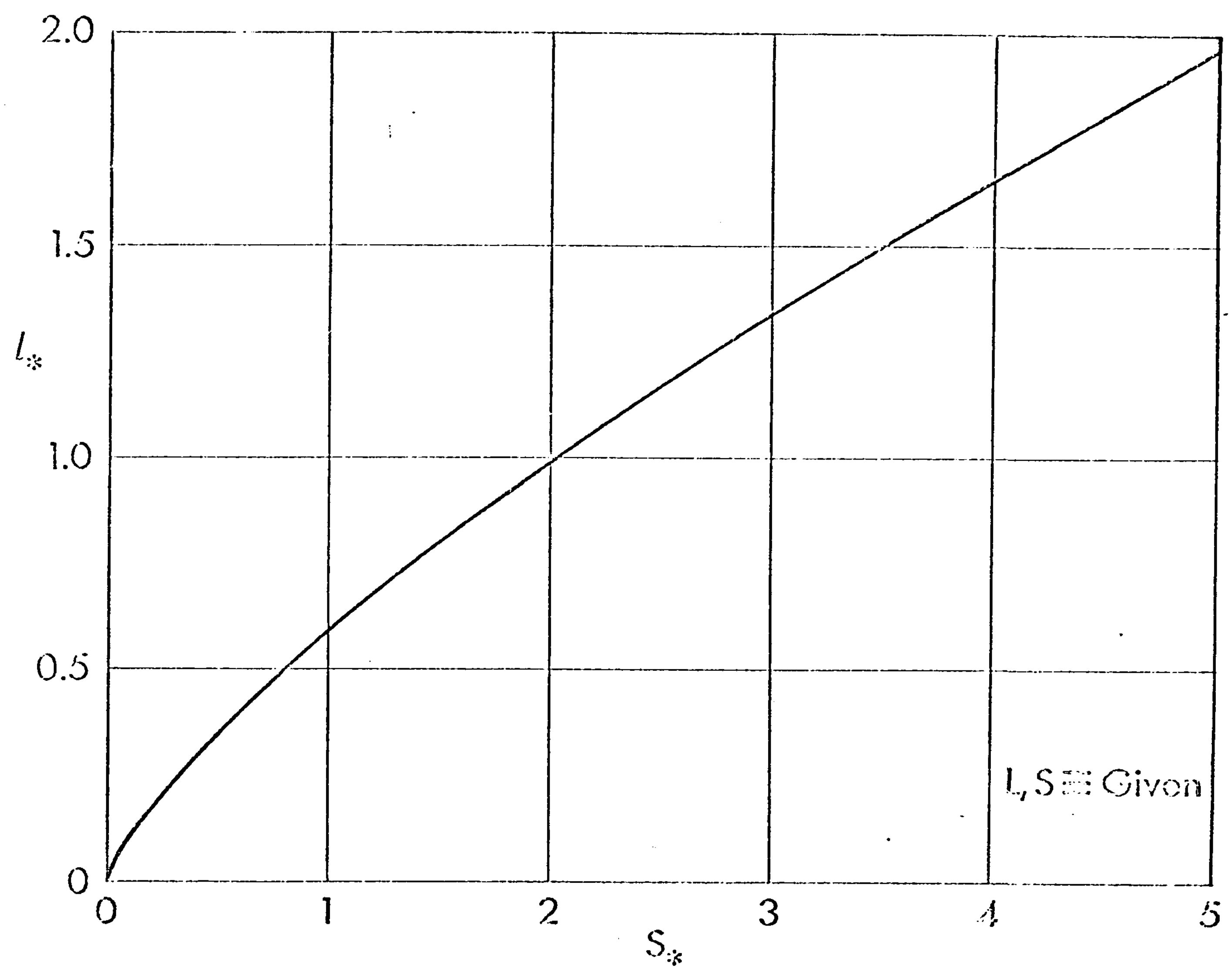

Fig. 5 Optimum length.

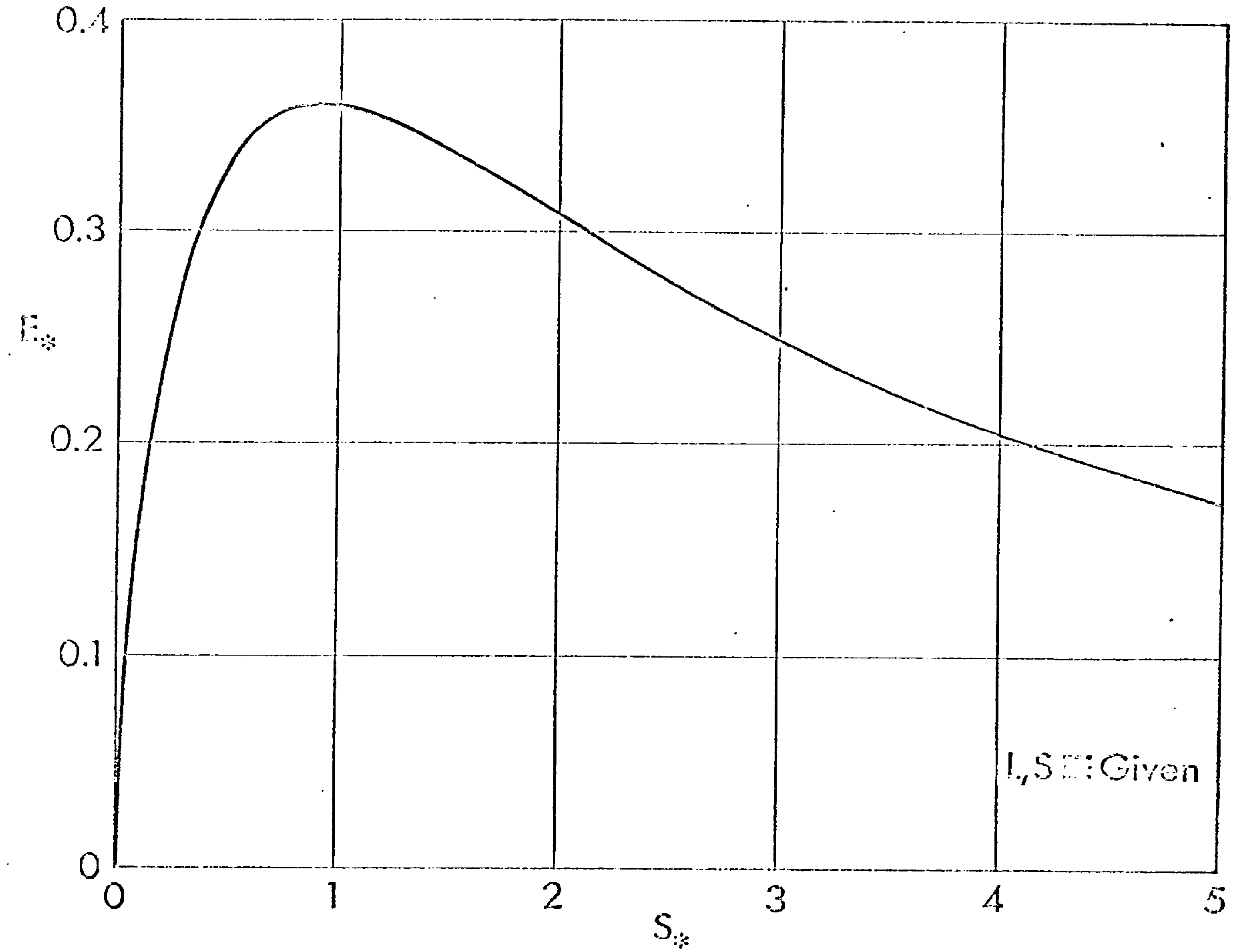

Fig. 6   Maximum lift-to-drag ratio.

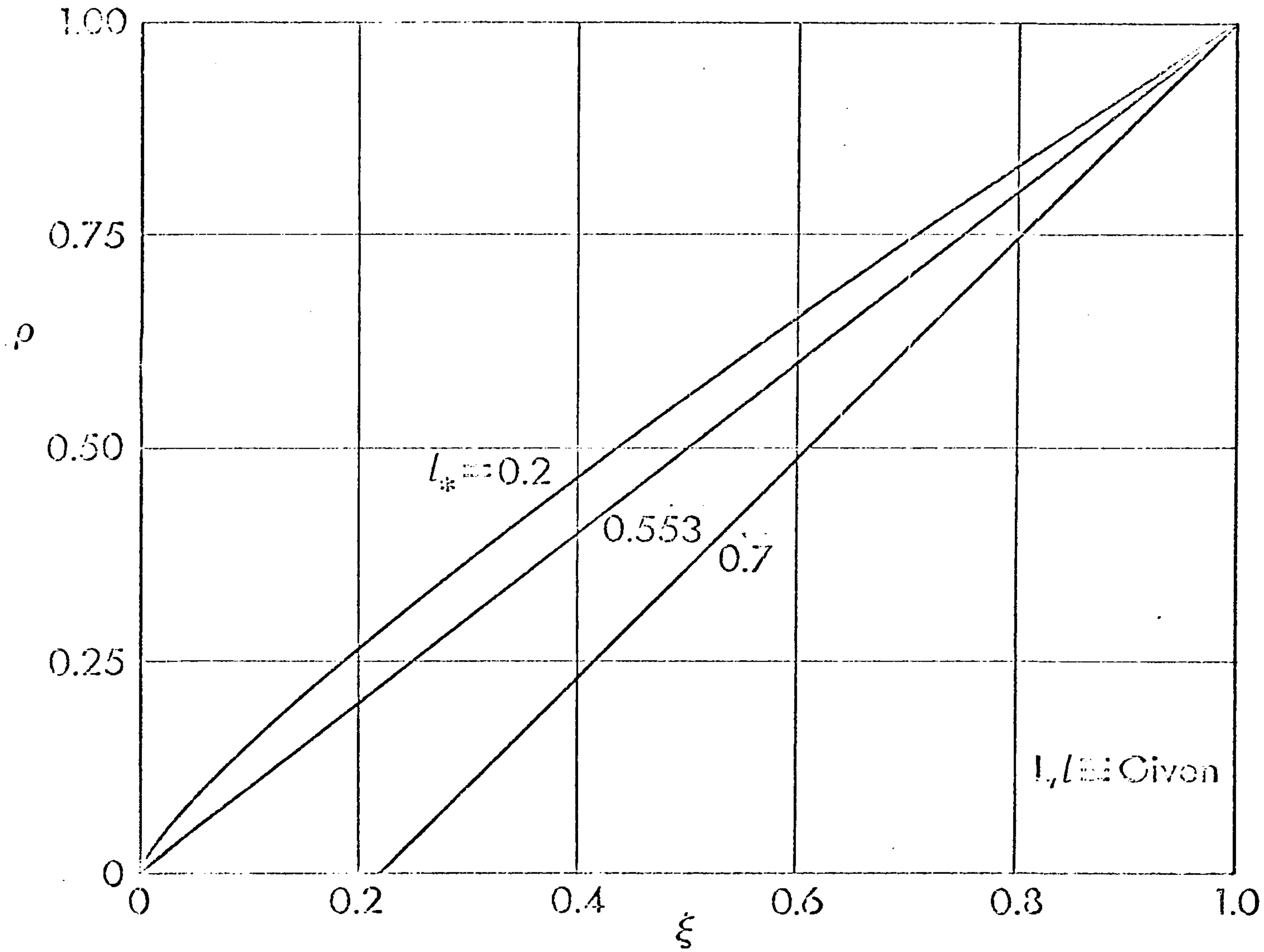

Fig. 7 Optimum shape.

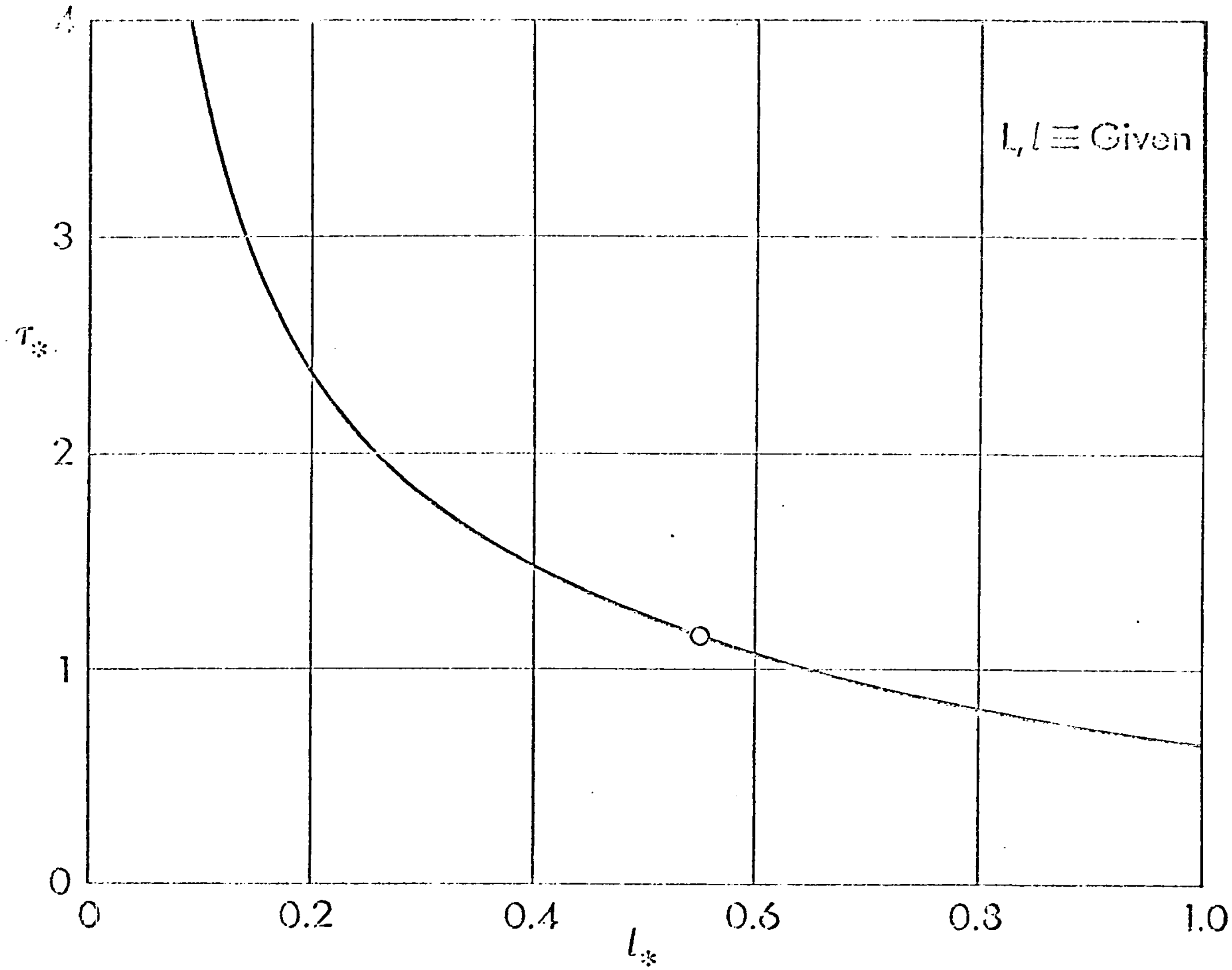

Fig. 8 Optimum thickness ratio.

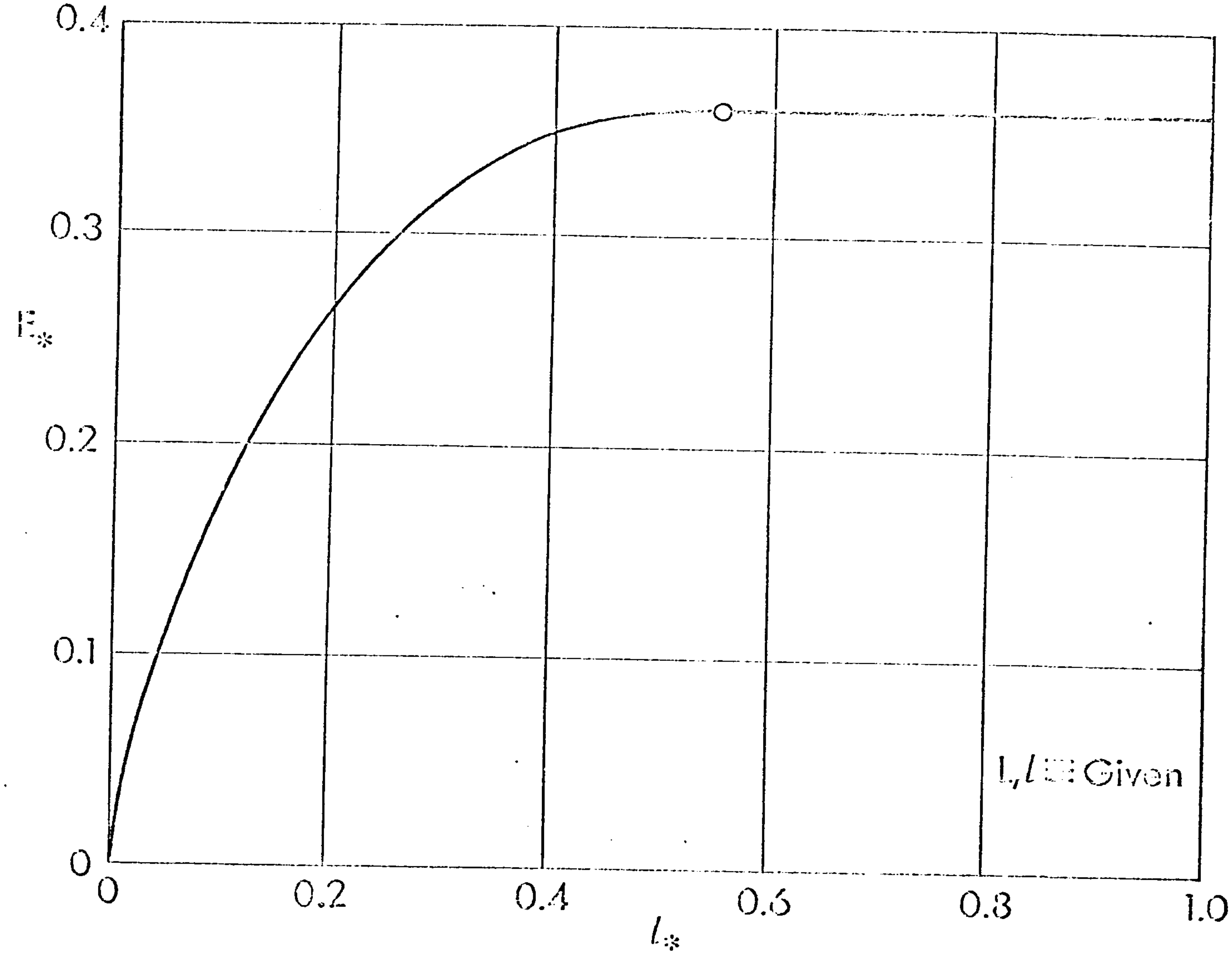

Fig. 9   Maximum lift-to-drag ratio.

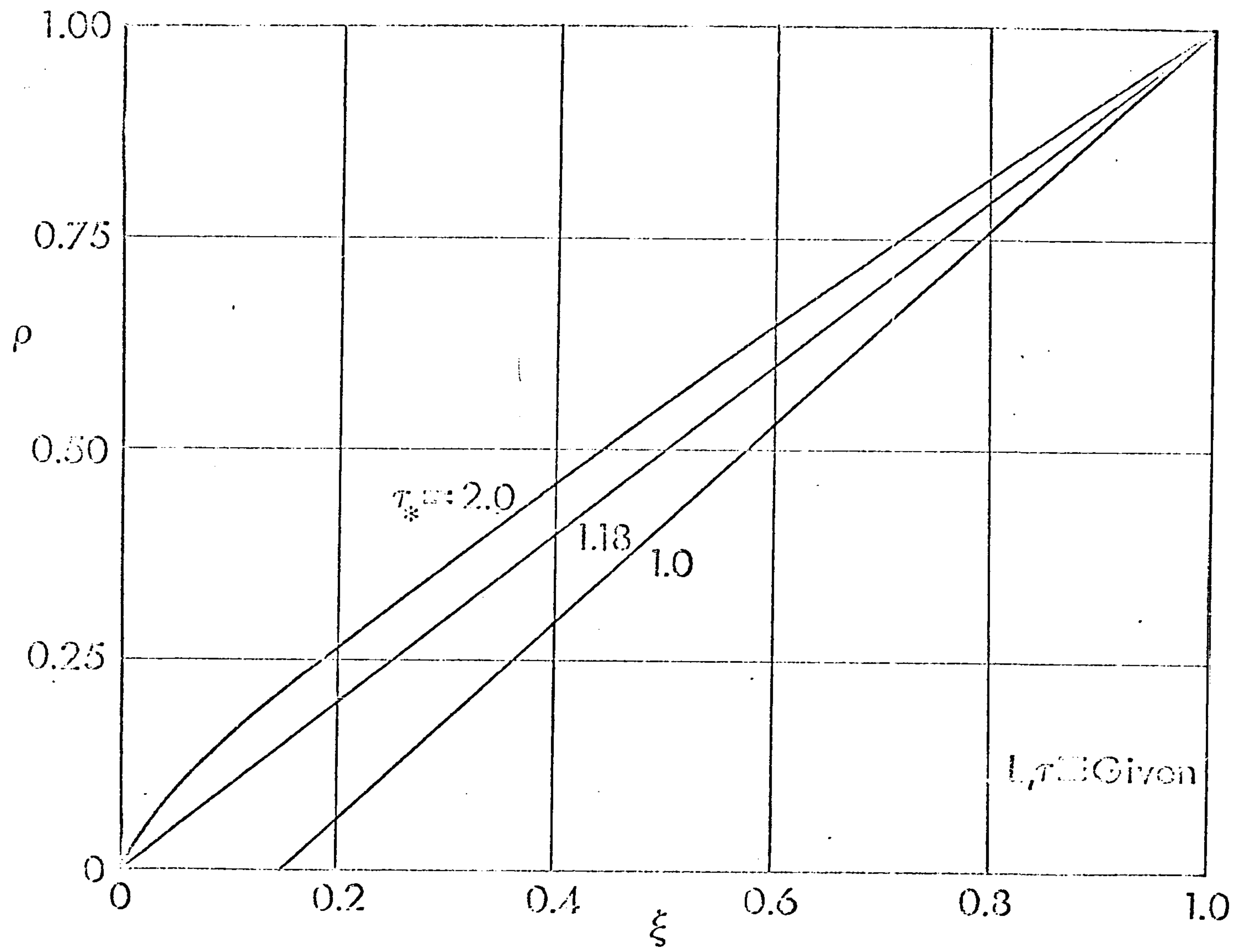

Fig. 10 Optimum shape.

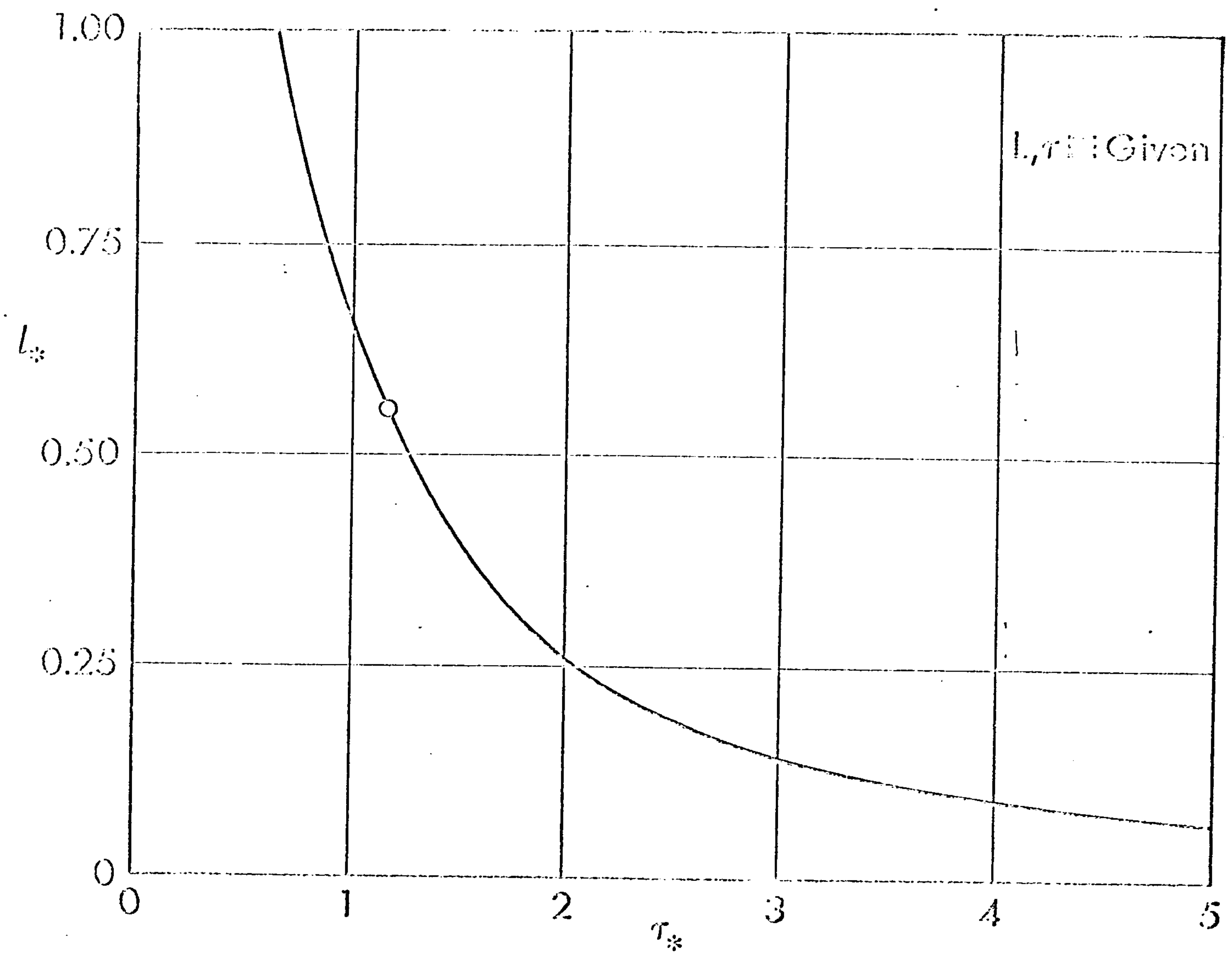

Fig. 11 Optimum length.

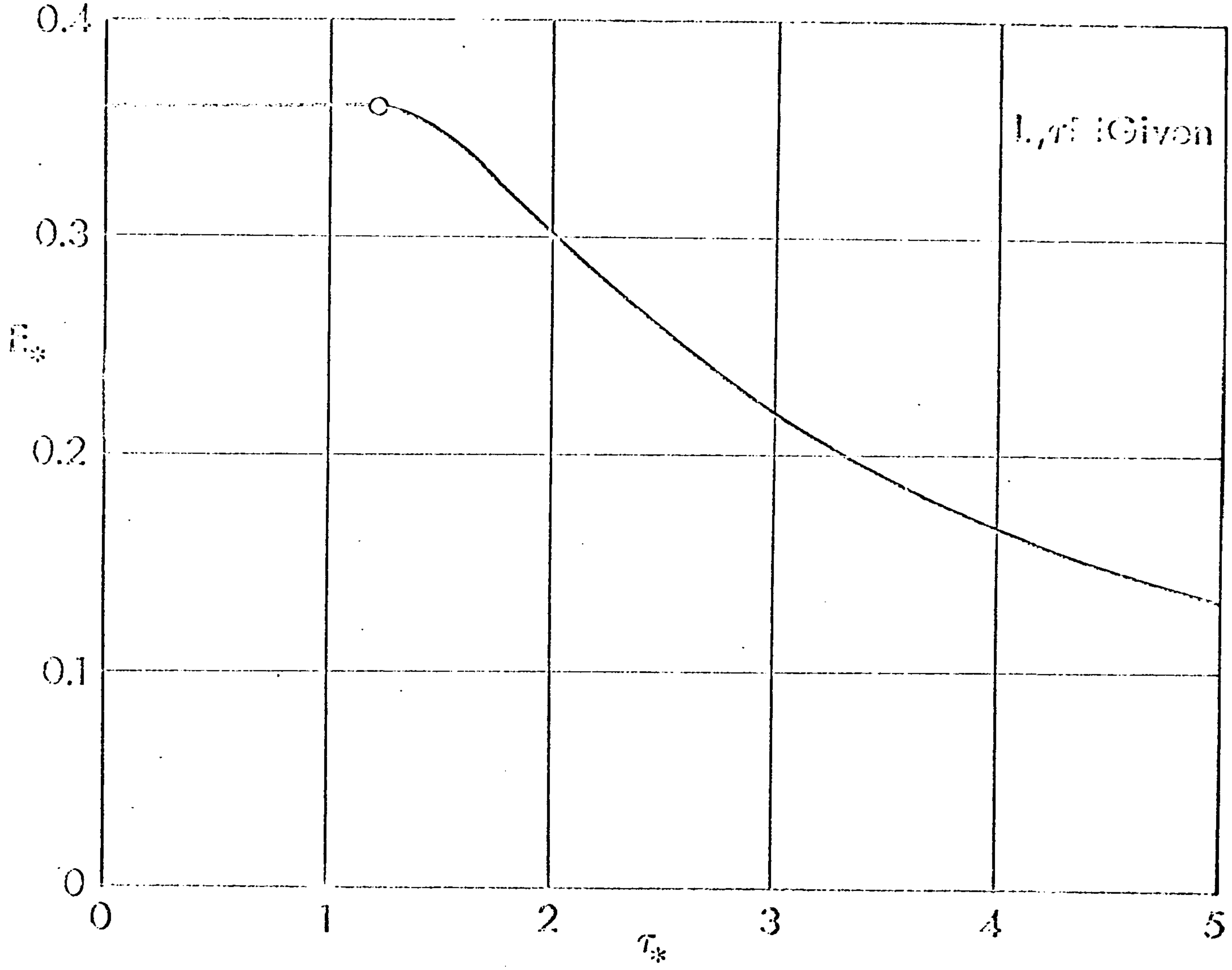

Fig. 12 Maximum lift-to-drag ratio.

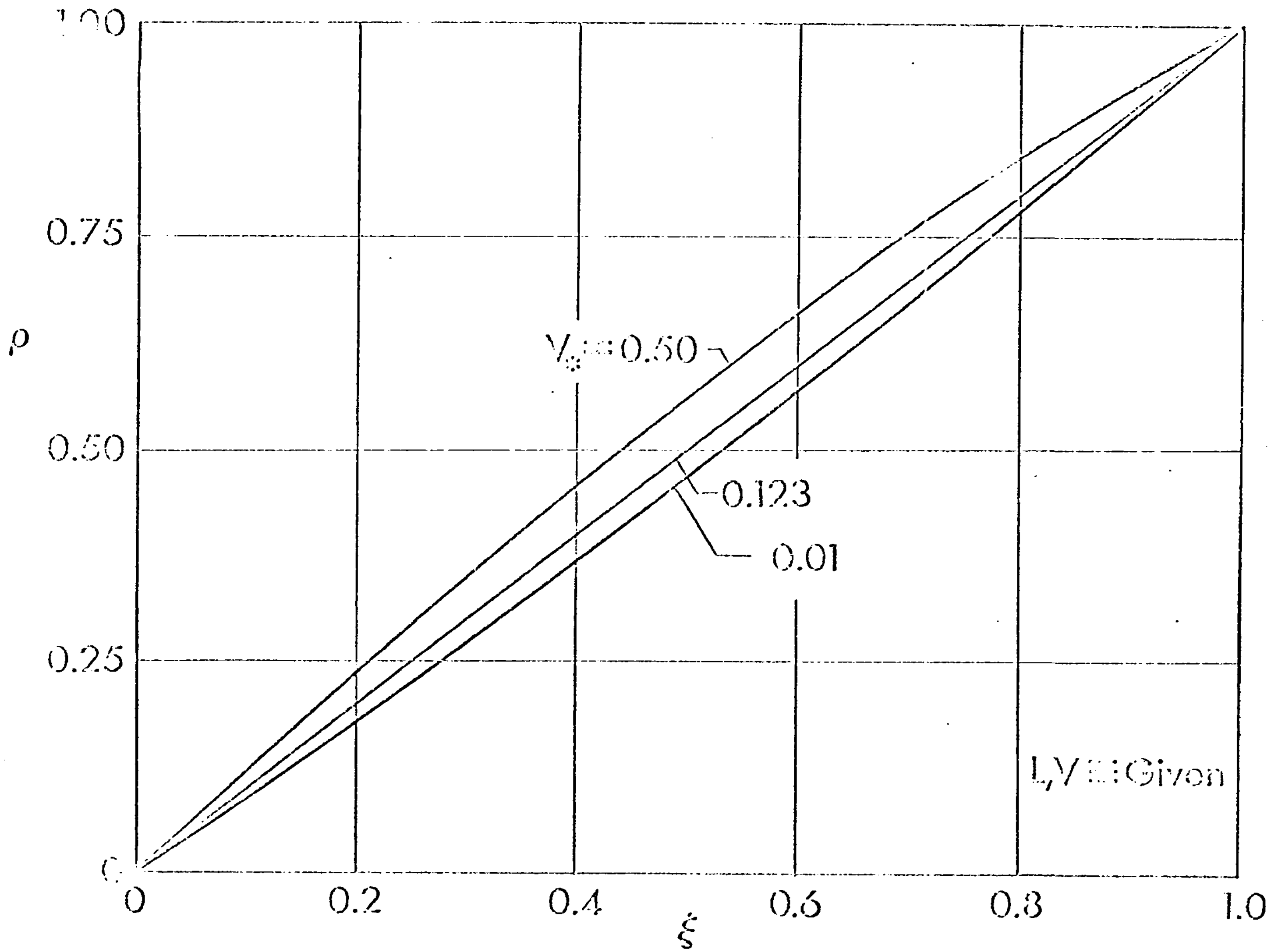

Fig. 13 Optimum shape.

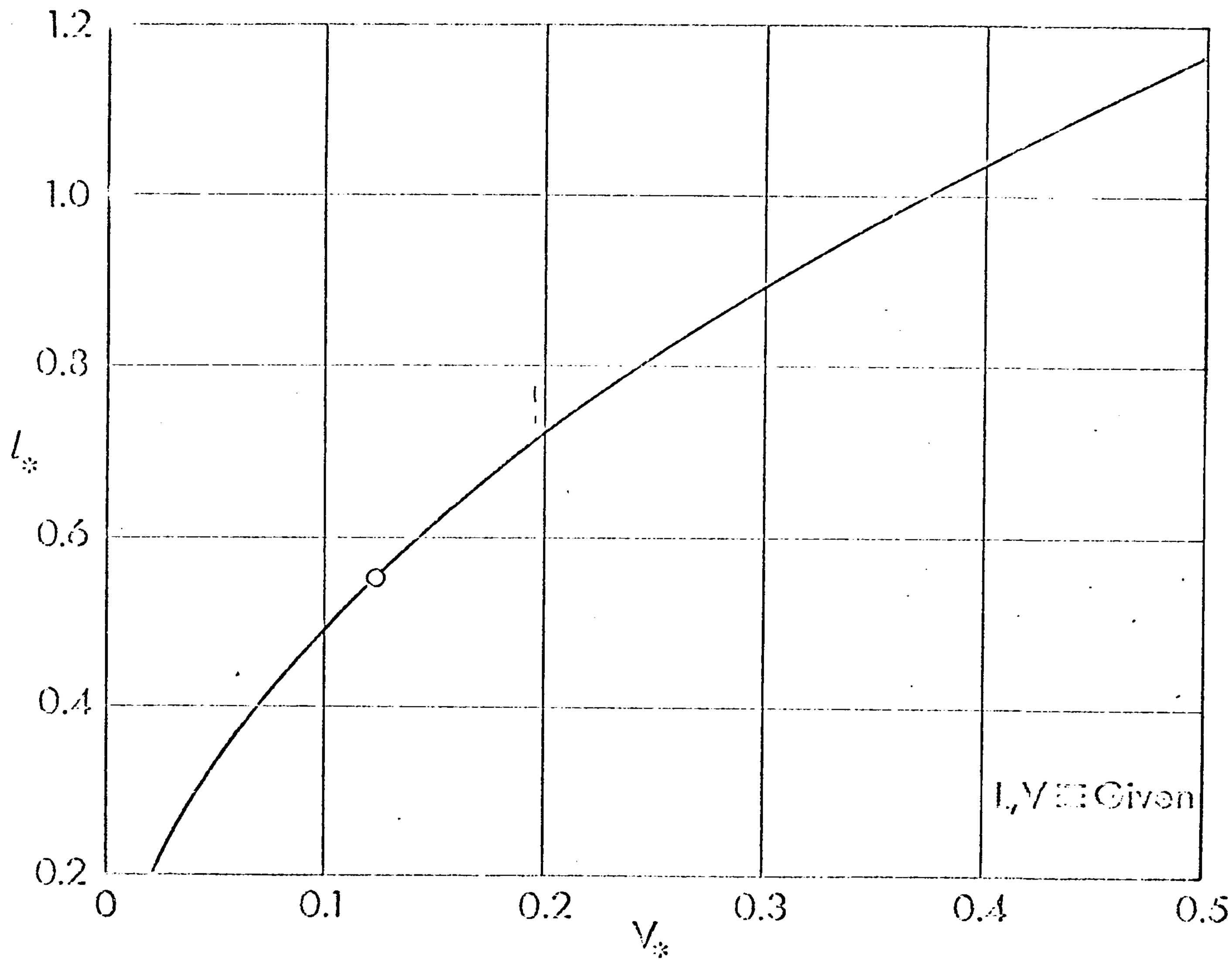

Fig. 14 Optimum length.

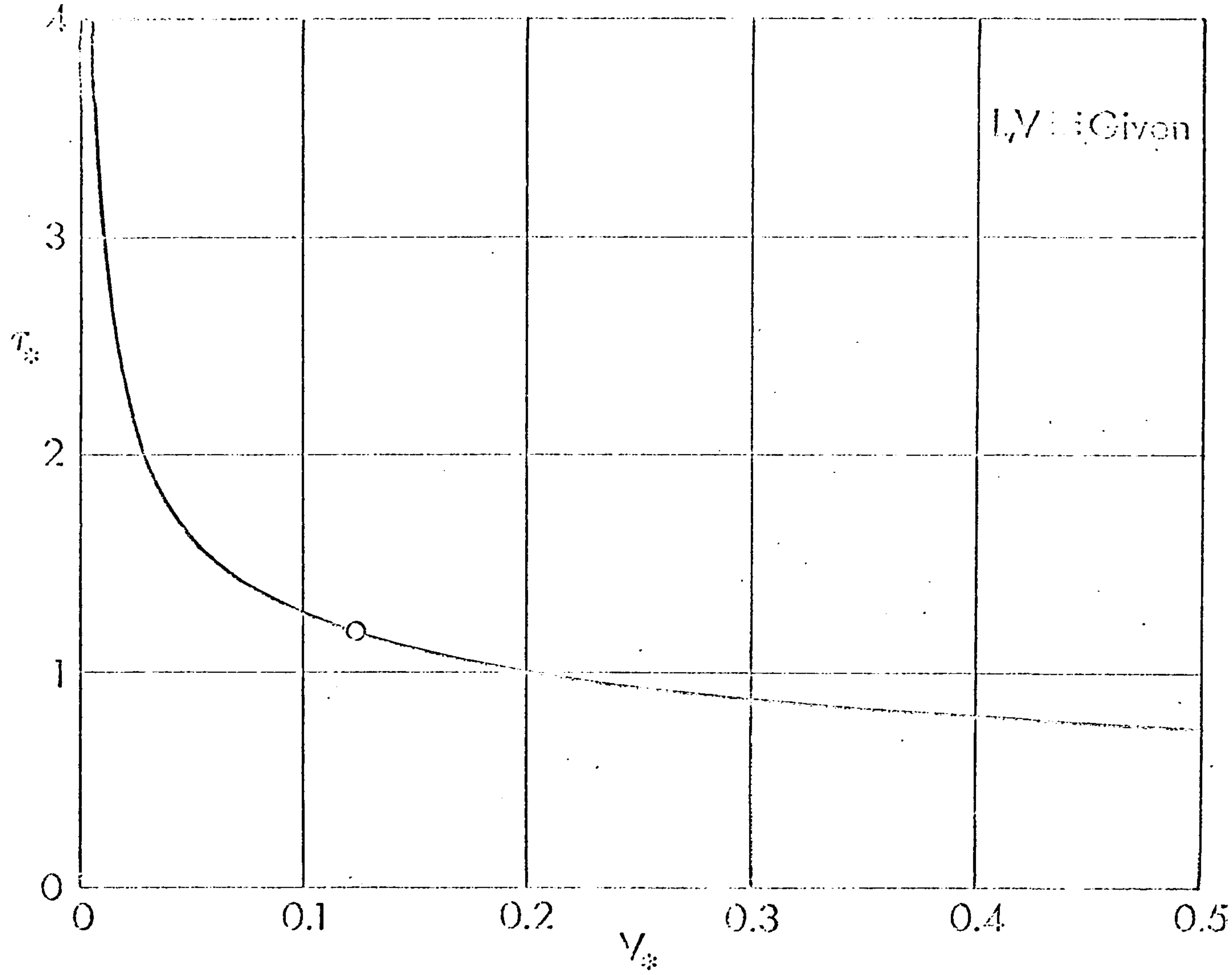

Fig. 15 Optimum thickness ratio.

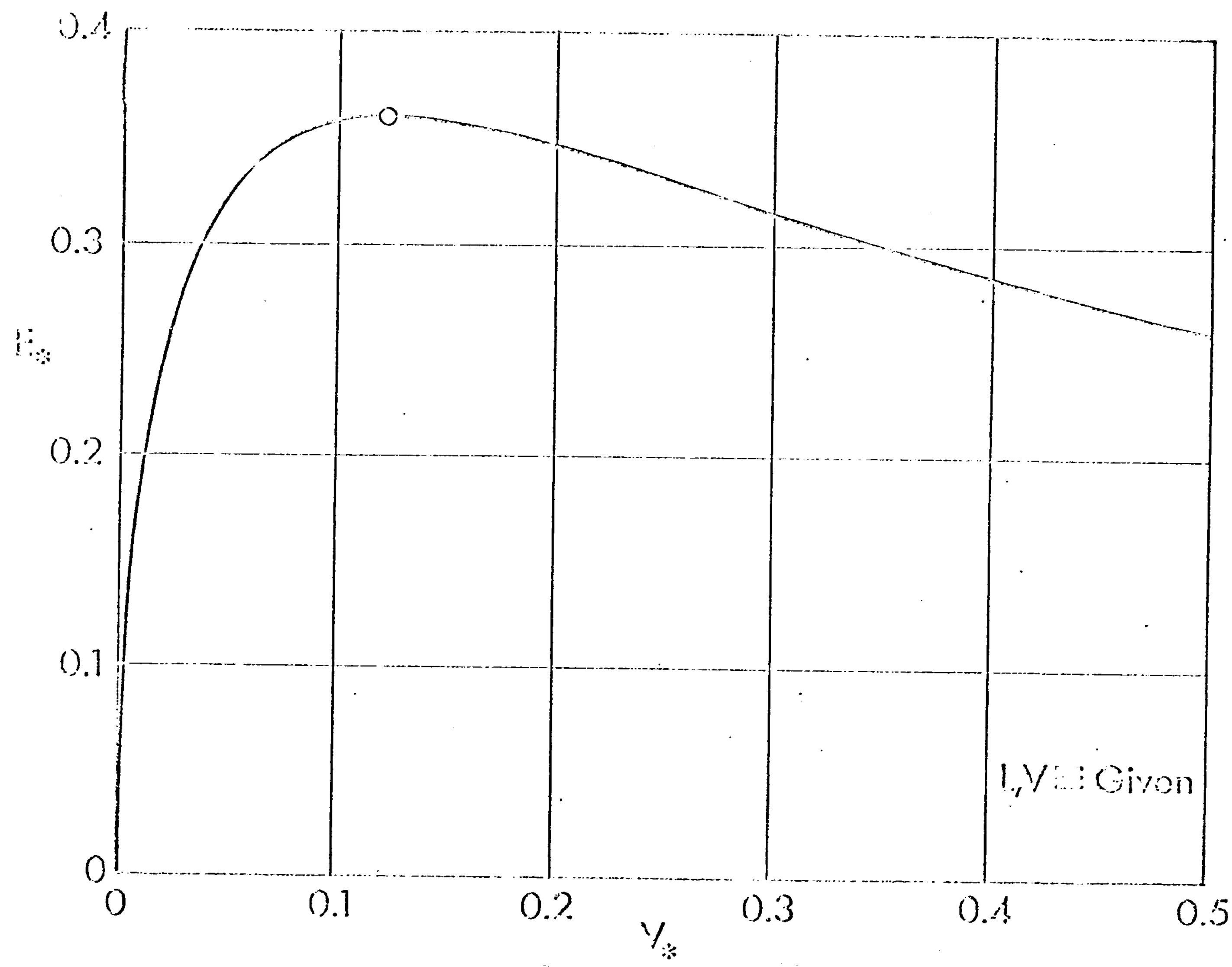

Fig. 16 Maximum lift-to-drag ratio.