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TWO-DIMENSIONAL, HYPersonic Wings
OF MAXIMUM LIFT-TO-DRAG RATIO

by

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1. INTRODUCTION

At the present time, the problem of designing a reentry vehicle which is capable of being controlled during flight is of great interest. Since the range of such a vehicle is proportional to its lift-to-drag ratio, the design problem consists of finding the configuration which has the highest possible lift-to-drag ratio. A preliminary investigation (Ref. 1) shows that wing-like configurations produce higher lift-to-drag ratios than body-like configurations. Therefore, there are two reasons for analyzing two-dimensional wings. First, optimum two-dimensional wings can be transformed into optimum three-dimensional wings by employing the similarity law derived in Ref. 2. Second, wings may be used to augment the lift-to-drag ratio of body-like configurations, and at hypersonic speeds, the flow field over most of a wing is two-dimensional. In this paper, the problem of maximizing the lift-to-drag ratio of a two-dimensional wing in hypersonic flow is considered under the following hypotheses: (a) the upper surface is a plane parallel to the undisturbed flow direction, (b) the wing is slender in the chordwise sense, (c) the pressure coefficient is modified Newtonian, (d) the average skin-friction coefficient is constant, (e) the base drag is neglected, and (f) the effect of skin-friction forces on the lift is neglected.
2. **LIFT-TO-DRAG RATIO AND PROFILE AREA**

In order to relate the drag and the lift of a two-dimensional, flat-top wing to its geometry, the following Cartesian coordinate system $O_{xy}$ is introduced (Fig. 1): the $x$-axis (unit vector $\mathbf{u}_x$) is in the direction of the undisturbed flow and coincides with the flat top; the $y$-axis (unit vector $\mathbf{u}_y$) is perpendicular to the $x$-axis and is positive downward. Next, let $\mathbf{n}$ denote the unit vector which is inwardly normal to the infinitesimal element of wetted area, and let $\mathbf{t}$ denote the unit tangent vector in the flow direction. Hence, if the base drag is neglected, the drag per unit span $D$ and the lift per unit span $L$ are given by

$$D/q = \int_{s_u + s_L} \left[ C_p (\mathbf{n} \cdot \mathbf{u}_x) + C_f (\mathbf{t} \cdot \mathbf{u}_x) \right] ds$$

$$L/q = -\int_{s_u + s_L} \left[ C_p (\mathbf{n} \cdot \mathbf{u}_y) + C_f (\mathbf{t} \cdot \mathbf{u}_y) \right] ds$$

(1)

where $q$ is the free-stream dynamic pressure, $C_p$ is the pressure coefficient, $C_f$ is the skin-friction coefficient, $s$ is a curvilinear coordinate measured from the leading edge, and the subscripts $u$, $t$ refer to the upper and lower surfaces, respectively.

Since the upper surface is flat and parallel to the undisturbed flow direction, the following relations are valid:

$$\mathbf{n} \cdot \mathbf{u}_x = 0 , \quad \mathbf{t} \cdot \mathbf{u}_x = 1$$

$$\mathbf{n} \cdot \mathbf{u}_y = 1 , \quad \mathbf{t} \cdot \mathbf{u}_y = 0$$

(2)

$$ds = dx$$
Furthermore, if the lower surface is represented by the function \( y(x) \) and if the symbol \( \dot{y} \) denotes the derivative \( dy/dx \), simple geometric manipulations lead to the relations

\[
\vec{n} \cdot \vec{u}_x = \dot{y} / (1 + \dot{y}^2)^{1/2}, \quad \vec{t} \cdot \vec{u}_x = 1 / (1 + \dot{y}^2)^{1/2}
\]

\[
\vec{n} \cdot \vec{u}_y = -1 / (1 + \dot{y}^2)^{1/2}, \quad \vec{t} \cdot \vec{u}_y = \dot{y} / (1 + \dot{y}^2)^{1/2}
\]

\[
ds = (1 + \dot{y}^2)^{1/2} \, dx
\]

Therefore, assuming that the effect of the skin-friction forces on the lift is negligible with respect to the pressure forces, one can rewrite Eqs. (1) as

\[
D/q = \int_0^c \left( C_{p\ell} \dot{y} + C_{fu} \right) \, dx
\]

\[
L/q = \int_0^c \left( - C_{pu} + C_{p\ell} \right) \, dx
\]

where \( c \) is the chord.

At this point, it is necessary to specify the distribution of pressure coefficients and skin-friction coefficients (see Appendices A and B). In this connection, the pressure coefficient is approximated by the modified Newtonian law, which for a slender wing \( \dot{y}^2 < < 1 \) implies that

\[
C_{pu} = 0, \quad C_{p\ell} = 2n\dot{y}^2
\]

where \( n \) is a constant. Rather than specifying the individual skin-friction coefficients, it is assumed that the average skin-friction coefficient
\[ C_{fa} = \frac{1}{2c} \int_0^c (C_{fu} + C_{ft}) \, dx \]  

is constant. In view of these hypotheses, the expressions (4) for the drag and the lift can be rewritten as

\[ \frac{D}{2nq} = \int_0^c (\dot{y}^3 + K) \, dx \]  

\[ \frac{L}{2nq} = \int_0^c \dot{y} \, dx \]

where the constant \( K \) is defined as

\[ K = \frac{C_{fa}}{n} \]

In conclusion, the lift-to-drag ratio \( E = L/D \) of a slender, two-dimensional, flat-top wing in hypersonic flow is given by

\[ E = \frac{\int_{x_i}^{x_f} \dot{y}^2 \, dx}{\int_{x_i}^{x_f} (\dot{y}^3 + K) \, dx} \]

while the profile area \( A \) can be expressed as

\[ A = \int_{x_i}^{x_f} y \, dx \]

In these relations, the end coordinates are represented by the relations

\[ x_i = 0 \quad , \quad y_i = 0 \quad , \quad x_f = c \quad , \quad y_f = t \]
where, if $c_0$ denotes the maximum allowable chord, the chord is required to satisfy the inequality

$$c \leq c_0$$  \hspace{1cm} (12)

and where the thickness $t$ may be prescribed or free.
3. **FORMULATION OF THE MAXIMUM LIFT-TO-DRAG RATIO PROBLEM**

It is desired to find the airfoil shape which maximizes the lift-to-drag ratio subject to an upper limit on the chord and either of the following sets of geometric constraints: (a) free thickness and free profile area, (b) given thickness and free profile area, or (c) free thickness and given profile area. While each of these problems can be solved individually, it is more convenient to formulate the variational problem in general, derive the necessary conditions to be satisfied by an optimum airfoil, and then solve the above problems as particular cases. In this connection, the general statement of the maximum lift-to-drag ratio problem is as follows: In the class of functions \( y(x) \) which satisfy the integral constraint (10), the inequality constraint (12), and the prescribed boundary conditions, find that particular function which maximizes the functional (9).

According to Ref. 3, the proposed problem is equivalent to that of maximizing the functional

\[
I = \int_{x_i}^{x_f} F(y, \dot{y}, E, \lambda) \, dx
\]  

(13)

with respect to the functions \( y(x) \) satisfying the integral constraint (10), the inequality constraint (12), and the prescribed boundary conditions. In Eq. (13) the fundamental function \( F \) is defined as

\[
F = y^2 - E(\dot{y}^3 + K) - \lambda y
\]  

(14)

where the constant \( E \) is the unknown maximum lift-to-drag ratio and \( \lambda \) is an undetermined, constant Lagrange multiplier.
4. NECESSARY CONDITIONS

Accounting for the relationships (11), one can write the first variation of the functional (13) as (see Chapter 1 of Ref. 4)

\[
\delta I = \int_{x_i}^{x_f} (F_y - dF_y/dx) \delta y \, dx + (F - \dot{y} F_{yy}) \dot{\delta c} + (F_{yy}) \dot{\delta t}
\]  

(15)

where the symbol \( \delta \) denotes a variation taken at a constant station \( x \) and the symbol \( \tilde{\delta} \) denotes a variation consistent with the prescribed boundary conditions. Furthermore, the second variation of the functional (13) for fixed end points is given by

\[
\delta^2 I = \int_{x_i}^{x_f} [F_{yy} (\tilde{\delta y})^2 + 2 F_{yy} \tilde{\delta y} \dot{\delta y} + F_{yy} (\dot{\delta y})^2] \, dx
\]  

(16)

The first and second variations lead to the Euler equation, the natural boundary conditions, and the Legendre condition which are now derived.

4.1. Euler Equation. If a system of variations is chosen such that the chord and the thickness are unchanged, that is, such that \( \delta c = \delta t = 0 \), the first variation (15) reduces to

\[
\delta I = \int_{x_i}^{x_f} (F_y - dF_y/dx) \tilde{\delta y} \, dx
\]  

(17)

and must vanish for every admissible distribution of variations \( \tilde{\delta y}(x) \). Hence, one obtains the Euler equation

\[
F_y - dF_y/dx = 0
\]  

(18)
which can be written explicitly as

\[ \lambda + d(2\dot{y} - 3E\dot{y}^2)/dx = 0 \]  

(19)

This equation admits the first integral

\[ 3E\dot{y}^2 - 2\dot{y} - \lambda x = C \]  

(20)

where C is a constant; a second integral can be obtained, but it is more convenient to derive it when analyzing the particular cases.

4.2. **Natural Boundary Conditions.** The integration constants which appear in the general solution of the Euler equation can be determined by applying the prescribed boundary conditions and the natural boundary conditions. The latter are obtained from the remainder of the first variation which, through the combination of Eqs. (15) and (18), is given by

\[ \delta I = (F - \dot{y}F_y) \delta c + (F_y \dot{y}) \delta t \]  

(21)

If we choose a system of variations such that the chord does not change \((\delta c = 0)\) but the thickness is free \((\delta t \neq 0)\), Eq. (21) must vanish regardless of the value of \(\delta t\). Hence, we obtain the following natural boundary condition

\[ (F_y \dot{y}) = 0 \]  

(22)

which can be written explicitly as

\[ t = \text{free:} \quad (3E\dot{y}^2 - 2\dot{y}) = 0 \]  

(23)
The inequality constraint (12) implies that we may have solutions where 
\( c < c_0 \) and solutions where \( c = c_0 \). For the former, the first variation (21) 
must vanish regardless of the choice of \( \delta c \). Hence, one obtains the natural 
boundary condition

\[
(F - \dot{y}F_y)_f = 0
\]  
(24)

which can be written explicitly as

\[
c < c_0; \quad (2E \dot{y}^3 - \dot{y}^2 - \lambda y - EK)_f = 0
\]  
(25)

On the other hand, if we are at the boundary, only one-sided variations of the 
form \( \delta c < 0 \) are admissible. Therefore, since the first variation must be 
negative for a maximum, we see that the inequality

\[
c = c_0; \quad (F - \dot{y}F_y)_f > 0
\]  
(26)

must be satisfied for these solutions.

4.3. **Legendre Condition.** Owing to the fact that \( F_{yy} = F_{yy} = 0 \), the second 
variation (16) reduces to

\[
\delta^2 I = \int_{x_1}^{x_f} F_{..} (\dot{y})^2 \, dx
\]  
(27)

and must be negative for a maximum. Hence, one obtains the Legendre condition

\[
F_{..} < 0
\]  
(28)
which, for the problems considered here, represents a sufficient condition for a weak maximum. The explicit form of this inequality requires that

\[ \dot{y} > \frac{1}{3}E \] (29)
5. NONDIMENSIONAL EQUATIONS

Before analyzing the particular cases, it is convenient to nondimensionalize the previous relations. In this connection, we introduce the dimensionless coordinates

\[ \xi = \frac{x}{c} \quad , \quad \eta = \frac{y}{t} \tag{30} \]

and the thickness ratio

\[ \tau = \frac{t}{c} \tag{31} \]

Furthermore, the following nondimensional parameters are defined:

\[ E_* = E K^{1/3} \]
\[ \tau_* = \tau K^{-1/3} \]
\[ A_* = A c^{-2} K^{-1/3} \]
\[ \lambda_* = \lambda c K^{-1/3} \]
\[ C_* = C K^{-1/3} \tag{32} \]

If \( \hat{\eta} \) denotes the derivative \( d\eta/d\xi \), the first integral (20) can be rewritten as

\[ 3E_* \tau_*^2 \eta_*^2 - 2\tau_* \dot{\eta}_* - \lambda_* \xi = C_* \tag{33} \]

and must be solved in conjunction with the end conditions

\[ \xi_i = 0 \quad , \quad \eta_i = 0 \quad , \quad \xi_f = 1 \quad , \quad \eta_f = 1 \tag{34} \]
Next, the natural boundary conditions (23), (25), and (26) become

\[
\begin{align*}
\text{free:} & \quad (3E_* \tau_*^2 \eta^2 - 2\tau_* \dot{\eta})_f = 0 \\
\leq c_o : & \quad (2E_* \tau_*^3 \eta^3 - \tau_*^2 \eta^2 - \lambda_* \tau_* \eta - E_*)_f = 0 \\
\geq c_o : & \quad (2E_* \tau_*^3 \eta^3 - \tau_*^2 \eta^2 - \lambda_* \tau_* \eta - E_*)_f > 0
\end{align*}
\]

(35)

while the Legendre condition (29) can be expressed as

\[
\dot{\eta} > 1/3E_* \tau_*
\]

(36)

Finally, the lift-to-drag ratio (9) and the profile area (10) supply the relations

\[
E_* = \frac{\tau_*^2 \int_0^1 \eta^2 d\xi}{\tau_*^3 \int_0^1 \eta^3 d\xi + 1}, \quad A_* = \tau_* \int_0^1 \eta d\xi
\]

(37)

In the following sections, several particular cases are analyzed. With regard to the solution process, we first determine those solutions for which the chord has the maximum allowable value \(c = c_o\). Then, the left-hand side of Ineq. (35-3) is calculated, and when it vanishes, we switch to the solutions where the chord is free \(c < c_o\).
6. **ABSOLUTE MAXIMUM LIFT-TO-Drag RATIO**

If the thickness and the profile area are free, the first integral (33) must be solved in conjunction with the natural boundary condition (35-1) and the relation \( \lambda_\ast = 0 \). Doing so, one sees that \( C_\ast = 0 \) and that the optimum shape satisfies the differential equation

\[
\dot{\eta} = \text{Const} \tag{38}
\]

Integration of this equation subject to the end conditions (34) implies that the optimum airfoil shape is the wedge

\[
\eta = \xi \tag{39}
\]

Hence, by combining Eqs. (33), (37-1), and (39), it is found that the thickness ratio parameter and the lift-to-drag ratio parameter are given by

\[
\tau_\ast = \sqrt[3]{2} \approx 1.26 \quad , \quad E_\ast = \frac{3\sqrt[4]{4}}{3} \approx 0.529 \tag{40}
\]

Incidentally, these solutions are characterized by the following value of the area parameter:

\[
A_\ast = 1/ \frac{3\sqrt[4]{4}}{3} \approx 0.630 \tag{41}
\]

If Eqs. (39) and (40) are substituted into the left-hand side of Ineq. (35-3), one sees that it vanishes identically. Therefore, since the above results are valid regardless of whether the chord is at its maximum allowable value or free, Eq. (40-2) represents the absolute maximum lift-to-drag ratio for a two-dimensional, flat-top wing.
In order to find the actual thickness ratio and lift-to-drag ratio, it is necessary to employ an iteration procedure. In this connection, one guesses a value for the thickness ratio and calculates the quantity $K = C_{\tau}/n$ (see Appendix C). Then, Eq. (40-1) yields a new starting value for $\tau$, and the procedure is repeated until the starting value and the calculated value agree to some predetermined degree of accuracy. Finally, the lift-to-drag ratio is determined from Eq. (40-2). A calculation has been carried out for the free-stream Mach number $M = 10$ and the free-stream Reynolds number based on the chord $Re = 5 \times 10^6$, and it is found that $\tau = 0.0872$ and $E = 7.64$ for a laminar boundary layer.
7. **GIVEN THICKNESS**

Since the profile area is free, the relationship \( \lambda_\ast = 0 \) holds, and the first integral implies that Eq. (38), and hence Eq. (39), is valid here. Consequently, the optimum airfoil shape is a wedge whose lift-to-drag ratio satisfies the relation (Fig. 2)

\[
E_\ast = \frac{\tau_\ast^2}{(\tau_\ast^3 + 1)}
\]  

Substitution of Eqs. (39) and (42) into Ineq. (35-3) shows that these solutions are valid providing

\[
\tau_\ast > \frac{3}{\sqrt{2}}
\]  

that is, providing

\[
t > c_0 \frac{3}{\sqrt{2K}}
\]  

The remaining solutions can be obtained by combining Eqs. (39) and (42) with the natural boundary condition (35-2). If this is done, one finds that the solutions for

\[
t \leq c_0 \frac{3}{\sqrt{2K}}
\]  

are identical with those presented in Section 6. Hence, Eq. (40-1) implies that the optimum chord is given by

\[
c = t/ \frac{3}{\sqrt{2K}}
\]
The calculation of the lift-to-drag ratio for the solutions where $c = c_0$ is straightforward; since the thickness ratio is known a priori, the quantity $K$ is a constant as shown in Appendix C. On the other hand, the iteration procedure for the solutions where $c < c_0$ has been discussed in Section 6.
8. **GIVEN PROFILE AREA**

If the profile area is prescribed, the geometry of the optimum airfoil is governed by the differential equation (33). Since the thickness is free, the natural boundary condition (35-1) applies and, when combined with Eq. (33), leads to the relation

\[ C_\ast = - \lambda_\ast \]  

(47)

Hence, in the light of the Legendre condition (36), Eq. (33) can be rewritten as

\[ \dot{\eta} = \left(1/3E_\ast \tau_\ast\right) \left\{ 1 + \left[1 - \alpha(1 - \xi)\right]^{1/2} \right\} \]  

(48)

where the quantity \( \alpha \) is defined by

\[ \alpha = 3E_\ast \lambda_\ast \]  

(49)

If one introduces the function

\[ G(\xi, \alpha) = \xi + (2/3\alpha) \left\{ [1 - \alpha(1 - \xi)]^{3/2} - [1 - \alpha]^{3/2} \right\} \]  

(50)

the integration of the differential equation (48) subject to the end conditions (34) leads to the expression for the optimum shape

\[ \eta = G(\xi, \alpha)/G(1, \alpha) \]  

(51)

as well as the relation

\[ 3E_\ast \tau_\ast = G(1, \alpha) \]  

(52)
The next step is to combine Eqs. (37), (51), and (52) to obtain the results

\[ \tau_* = G(1, \alpha)/M(\alpha) \quad , \quad E_* = M(\alpha)/3 \quad , \quad A_* = N(\alpha)/M(\alpha) \]  \hspace{1cm} (53)

where

\[ M(\alpha) = \left\{ 2 - (2/5\alpha) \left[ (4 + \alpha)(1 - \alpha)^{3/2} - 4 \right] \right\}^{1/3} \]  \hspace{1cm} (54)

\[ N(\alpha) = 1/2 - (2/15\alpha^2)\left[ (2 + 3\alpha)(1 - \alpha)^{3/2} - 2 \right] \]

Consequently, eliminating the quantity \( \alpha \) between Eqs. (51) and (53), one obtains the following functional relations

\[ \eta = \eta(5, A_*) \]  \hspace{1cm} (55)

\[ \tau_* = \tau_*(A_*) \quad , \quad E_* = E_*(A_*) \]

which are plotted in Figs. 3 through 5. Finally, the inequality (35-3) is satisfied providing

\[ \alpha < 0 \]  \hspace{1cm} (56)

that is, providing

\[ A_* > 1/\sqrt[3]{4} \]  \hspace{1cm} (57)

Hence, these solutions are valid if

\[ \Lambda > \left( c_o^2 / 2 \right)^{3/\sqrt{2K}} \]  \hspace{1cm} (58)
The remaining solutions can be obtained by combining the above equations with the natural boundary condition (35-2). If this is done, one sees that the solutions for

\[ A \leq \left( \frac{c_0^2}{2} \right)^{\frac{3}{\sqrt{2K}}} \]  

(59)

are identical with those presented in Section 6. Hence, Eqs. (40-1) and (41) imply that the optimum chord and the optimum thickness are given by

\[ c = (2A)^{1/2} (2K)^{-1/6}, \quad t = (2A)^{1/2} (2K)^{1/6} \]  

(60)
9. **EFFECT OF BASE DRAG**

In the previous sections, the problem of maximizing the lift-to-drag ratio was considered under the assumption that the base drag is negligible. However, at moderate hypersonic speeds, the base drag can have the same order of magnitude as the drag of the remaining part of the wing. Therefore, the maximum lift-to-drag ratio problem is reformulated here with the base drag included, but only the case of given chord is considered.

If $\gamma$ denotes the ratio of specific heats and $M$ the free-stream Mach number, the base pressure coefficient can be approximated by the relation (see Appendix A)

$$C_p = -\frac{2}{\gamma M^2}$$  \hspace{1cm} (61)

Hence, introducing the definition

$$\beta = \frac{1}{\gamma n M^2}$$  \hspace{1cm} (62)

one can express the lift-to-drag ratio as

$$E = \frac{\int_{x_i}^{x_f} y^2 \, dx}{\int_{x_i}^{x_f} (y^3 + K + \beta y) \, dx}$$  \hspace{1cm} (63)

which leads to the following fundamental function:

$$F = y^2 - E(y^3 + K + \beta y)$$  \hspace{1cm} (64)
The combined use of the Euler equation (18) and the natural boundary condition (22) shows that the optimum shape must be a solution of the differential equation

\[ F_\gamma = 0 \]  

The explicit form of this equation is given by

\[ 2\dot{y} - 3E\dot{y}^2 - E\beta = 0 \]  

which implies that the slope is constant and that the optimum airfoil shape is a wedge. Hence, since \( \dot{y} = \tau \), Eqs. (63) and (66) can be used to obtain the relations

\[ \beta_* = \left(1/\tau_* \right)(\tau_*^3 - 2) \quad , \quad E_* = \tau_*^2/(2\tau_*^3 - 1) \]  

where the known constant \( \beta_* \) is defined as

\[ \beta_* = \beta_k^{-2/3} \]  

Consequently, the optimum thickness ratio and the maximum lift-to-drag ratio satisfy functional relations of the form

\[ \tau_* = \tau_*(\beta_*) \quad , \quad E_* = E_*(\beta_*) \]  

which are plotted in Figs. 6 and 7 and show that the base drag causes a decrease in the maximum lift-to-drag ratio. As a matter of fact, for \( M = 10 \) and \( Re = 5 \times 10^6 \) (see Appendix C), it is seen that \( \tau = 0.107 \) and \( E = 5.54 \) for a laminar boundary layer.
10. **EFFECT OF LEADING EDGE BLUNTING**

In all the previous problems, the optimum shapes are characterized by sharp leading edges and may be undesirable from a heat transfer point of view. As a consequence, the problem of maximizing the lift-to-drag ratio for a given chord is reconsidered here under the assumption that the leading edge is slightly blunt (Fig. 8).

Assuming that the leading edge thickness $\tilde{t}$ is small with respect to both the chord $c$ and the thickness $t$, one can approximate the drag of a blunted airfoil by the sum of the leading edge drag and the drag of the same airfoil with a sharp leading edge. Hence, if the leading edge drag is calculated without employing the slender-wing approximation (see Appendix A), the lift-to-drag ratio can be expressed as

$$
E = \frac{\int_{x_i}^{x_f} \frac{y^2}{\rho} \, dx}{\int_{x_i}^{x_f} (\dot{y}^3 + K + \rho) \, dx}
$$

where $\rho = (\tilde{C}_D/2)(\tilde{t}/c)$ is the nondimensional leading edge thickness and $\tilde{C}_D = D/\tilde{q}t$ is the leading edge drag coefficient. Since the fundamental function is given by

$$
F = \dot{y}^2 - E(\dot{y}^3 + K + \rho)
$$

the combined use of the Euler equation (18) and the natural boundary condition (22) leads to the differential equation (65) which can be written explicitly as

$$
3E\dot{y}^2 - 2\dot{y} = 0
$$
and which implies that the optimum shape is a wedge whose slope is given by

\[ \hat{y} = 2/3E = \tau \]  \hspace{1cm} (73)

Finally, if the following definition is introduced:

\[ \rho_\ast = \rho/K \]  \hspace{1cm} (74)

Eqs. (70) and (73) can be used to obtain the relations

\[ \tau_\ast = \sqrt{2} \frac{1}{\frac{1}{2} (1 + \rho_\ast)^{1/3}} \hspace{1cm} E_\ast = (\sqrt{4/3})(1 + \rho_\ast)^{-1/3} \]  \hspace{1cm} (75)

which are plotted in Figs. 9 and 10. As expected, leading edge blunting causes a decrease in the maximum lift-to-drag ratio.

To calculate the actual thickness ratio and lift-to-drag ratio, the leading edge is assumed to be a circular arc of thickness \( \bar{t}/c = 2 \times 10^{-3} \) whose drag coefficient is \( \bar{C}_D = 4/3 \). Hence, for \( M = 10 \) and \( Re = 5 \times 10^6 \) (see Appendix C), it is seen that \( \tau = 0.155 \) and \( E = 4.30 \) for a laminar boundary layer. The reason why this value of the lift-to-drag ratio is so much lower than that obtained in Section 6 is that the blunt leading edge contributes an amount to the drag which is the same order of magnitude as the pressure drag or the skin-friction drag of the remainder of the airfoil.
11. **EFFECT OF INCLINING THE UPPER SURFACE**

In the previous two sections, it was found that base drag and leading edge blunting cause a considerable decrease in the maximum lift-to-drag ratio predicted by Eq. (40-2). In this section, it is shown that, by inclining the upper surface to take advantage of the flow expansion (Fig. 11), some of this loss can be recovered.

If the geometry of the upper surface is represented by the function $z(x)$, the pressure coefficient is negative and can be written as (see Appendix A)

$$C_{pu} = - 2n P(\dot{z})$$  \hspace{1cm} (76)

where the function $P$ is defined as

$$P(\dot{z}) = \left(1/\gamma n M^2\right) \left\{1 - \left[1 - (\gamma - 1)M\dot{z}/2\right]^{\gamma/(\gamma - 1)}\right\}$$  \hspace{1cm} (77)

Consequently, representing the geometry of the lower surface by the function $y(x)$, one can express the lift-to-drag ratio in the form

$$E = \frac{\int_{x_i}^{x_f} [\dot{y}^2 + P(\dot{z})] \, dx}{\int_{x_i}^{x_f} [\dot{y}^3 + \dot{z}P(\dot{z}) + K] \, dx}$$  \hspace{1cm} (78)

which leads to the following fundamental function:

$$F = \dot{y}^2 + P(\dot{z}) - E[\dot{y}^3 + \dot{z}P(\dot{z}) + K]$$  \hspace{1cm} (79)
Owing to the fact that there are now two dependent variables \( \ddot{y} \) and \( z \), two Euler equations must be satisfied (see Chapter 2 of Ref. 4), that is,

\[
F_y - \frac{dF_y}{dx} = 0, \quad F_z - \frac{dF_z}{dx} = 0
\]  

(80)

which, since \( F_y = F_z = 0 \), admit the first integrals

\[
F_y = \text{Const}, \quad F_z = \text{Const}
\]  

(81)

Furthermore, the transversality condition for this problem is given by

\[
\left. \left[ (F - \dot{y}F_y - \dot{z}F_z) \delta x + F_y \delta y + F_z \delta z \right] \right|_1^f = 0
\]  

(82)

and, since the initial abscissa, the final abscissa, and the initial ordinates are prescribed, implies that

\[
(F_y)_1 = 0, \quad (F_z)_f = 0
\]  

(83)

Consequently, the optimum lower and upper surfaces must be solutions of the differential equations

\[
F_y = 0, \quad F_z = 0
\]  

(84)

If \( P' \) denotes the derivative \( dP/d\dot{z} \), these equations can be written explicitly as

\[
2\dot{y} - 3E\dot{y}^2 = 0, \quad P' - E(P + \dot{z}P') = 0
\]  

(85)
and imply that the slopes of the lower and upper surfaces are constant. This being the case, Eq. (78) can be rewritten as

\[ E = \frac{(\dot{y}^2 + P)(\dot{y}^3 + \dot{z}P + K)}{} \quad (86) \]

so that Eqs. (85) and (86) form a system of three equations in the unknowns \( \dot{y}, \dot{z}, \) and \( E \).

Introducing the definitions

\[ \dot{y}_* = \dot{y}K^{-1/3}, \quad \dot{z}_* = \dot{z}K^{-1/3}, \quad M_* = MK^{1/3} \quad (87) \]

one can combine Eqs. (85) and (86) to obtain the following equation which determines the slope \( \dot{z}_* \) of the upper surface:

\[ 27P_*^3 - 27P_*^2P_*' - 4(P_* + \dot{z}_*P_*')^3 = 0 \quad (88) \]

Once \( \dot{z}_* \) is known, the slope \( \dot{y}_* \) of the lower surface can be obtained from the relation

\[ \dot{y}_* = 2(P_* + \dot{z}_*P_*')/3P_*' \quad (89) \]

so that the optimum thickness ratio and the maximum lift-to-drag ratio become

\[ \tau_* = \dot{y}_* - \dot{z}_*, \quad E_* = 2/3\dot{y}_* \quad (90) \]
The complete solution of this problem is represented by the functional relations

\[ \dot{y}_* = \dot{y}_* (M_*, n, \gamma), \quad \dot{z}_* = \dot{z}_* (M_*, n, \gamma) \]

\[ \tau_* = \tau_* (M_*, n, \gamma), \quad E_* = E_* (M_*, n, \gamma) \]  \hspace{1cm} (91)

which are plotted in Figs. 12 through 15 for \( \gamma = 1.4 \). From these diagrams, it is seen that the maximum lift-to-drag ratio can be increased substantially by inclining the upper surface. For \( M = 10 \) and \( \text{Re} = 5 \times 10^6 \) (see Appendix C), we have \( \tau = 0.0238 \) and \( E = 9.97 \) for a laminar boundary layer.
12. **DISCUSSION AND CONCLUSIONS**

The problem of maximizing the lift-to-drag ratio of a slender, two-dimensional, flat-top wing has been considered under the assumptions that the pressure coefficient is modified Newtonian, the average skin-friction coefficient is constant during the optimization process, the base drag is negligible, and the chord must be less than or equal to a certain value. Using the indirect methods of the calculus of variations, we have determined the airfoil which produces the absolute maximum lift-to-drag ratio as well as the optimum airfoil for a given thickness or a given profile area. While analytical expressions have been found for the optimum shape, it is necessary to employ an iteration procedure to determine the optimum dimensions and the maximum lift-to-drag ratio.

With regard to the hypotheses employed, several comments are in order: (a) the modified Newtonian pressure law is an empirically valid approximation of the actual pressure distribution on nonconcave airfoils; therefore, since the resulting optimum shapes are of this type, the use of the modified Newtonian pressure law is justified, (b) in general, the average skin-friction coefficient is not constant but depends on the wing shape and size; however, it has been shown in Ref. 5 that the optimum wedge calculated with a constant $C_{fa}$ has approximately the same lift-to-drag ratio as the optimum wedge calculated with a variable $C_{fa}$, and (c) nonslender wings have been considered in Ref. 6, and it was concluded that, for values of the skin-friction coefficient having engineering interest, the slender-wing approximation is completely justified.
In the final sections of this paper, the effects of base drag, leading edge blunting, and inclining the upper surface on the absolute maximum lift-to-drag ratio for a flat-top wing with a sharp leading edge and negligible base drag have been considered. It was found that base drag and leading edge blunting cause a decrease while inclining the upper surface causes an increase. To estimate the magnitude of the thickness ratio and the lift-to-drag ratio which should occur if all of these effects are accounted for, it is observed that leading edge blunting and inclining the upper surface are relatively uncoupled effects. On the other hand, the base drag is decreased considerably when the upper surface is inclined. Therefore, for $M = 10$, $Re = 5 \times 10^6$, and a circular arc leading edge of thickness $\tilde{t}/c = 2 \times 10^{-3}$, we should expect a thickness ratio in the neighborhood of $\tau = 0.1$ and a lift-to-drag ratio in the neighborhood of $E = 6$ for laminar boundary layer. Incidentally, this example supports the conclusion reached in Ref. 1.
REFERENCES


APPENDIX A: PRESSURE COEFFICIENTS

In hypersonic flow, the pressure acting on a surface element which "sees" the flow can be approximated by the modified Newtonian law (Ref. 7)

\[ C_p = \frac{2n\dot{y}^2}{(1 + \dot{y}^2)} \] (A-1)

where

\[ n = \left[ \frac{(1 + \dot{y}_{\text{i}}^2)/2\dot{y}_{\text{i}}^2}{C_{\text{pi}}} \right] \] (A-2)

and where \( \dot{y} \) is the local slope of the wing surface and \( \dot{y}_{\text{i}} \) is the slope at the leading edge. Furthermore, \( C_{\text{pi}} \) is the exact pressure coefficient at the leading edge; hence, if \( \gamma \) denotes the ratio of specific heats and \( M \) the freestream Mach number,

\[ C_{\text{pi}} = 2\dot{y}_{\text{i}}^2 \left\{ \frac{\gamma + 1}{4} + \left[ \left( \frac{\gamma + 1}{4} \right)^2 + \left( \frac{1}{M\dot{y}_{\text{i}}} \right)^2 \right]^{1/2} \right\} \] (A-3)

for a sharp leading edge and

\[ C_{\text{pi}} \approx 2 \] (A-4)

for a blunt leading edge (\( \dot{y}_{\text{i}} = \infty \)).

If a surface is inclined with respect to the undisturbed flow direction in such a way that the flow is shock free (for example, the upper surface of a flat plate at an angle of attack), the pressure coefficient can be approximated by the relation (Ref. 7)

\[ C_p = -\frac{(2/\gamma M^2)}{\left\{ 1 - [1 - (\gamma - 1)M\dot{y}/2]^2\gamma/(\gamma - 1) \right\}} \] (A-5)
providing

\[ \dot{y} \leq \frac{2}{(\gamma - 1)M} \]  \hspace{1cm} (A-6)

In the event this inequality is not satisfied, the flow is no longer attached, and the pressure coefficient is given by

\[ C_p = -\frac{2}{\gamma M^2} \]  \hspace{1cm} (A-7)

which is the relation used to approximate the base pressure.
APPENDIX B: SKIN-FRICTION COEFFICIENTS

According to Refs. 8 and 9, the skin-friction coefficient on a flat plate can be approximated by the relation for incompressible flow providing the fluid properties (density and viscosity) in the boundary layer are evaluated at a reference temperature $T_\ast$, which depends on the free-stream temperature $T$, the wall temperature $T_w$, and ratio of specific heats $\gamma$, and the free-stream Mach number $M$. Hence, for a laminar boundary layer, the skin-friction coefficient (the wall shear stress per unit free-stream dynamic pressure) is given by

$$C_f = N(x/c)^{-1/2} \quad (B-1)$$

In this relation, the constant $N$ is defined by

$$N = 0.664(C/Re)^{1/2}(T/T_\ast)^{(1 - \omega)/2} \quad (B-2)$$

with

$$T_\ast/T = (1/2)(1 + T_w/T) + 0.11(\gamma - 1)rM^2 \quad (B-3)$$

where $Re$ is the free-stream Reynolds number based on the chord $c$, where $C$ and $\omega$ are constants associated with the viscosity-temperature law, and where the constant $r$ is the recovery factor.

For an inclined surface, the situation is different in two respects. First, the velocity and the pressure at the outer edge of the boundary layer are no longer the free-stream velocity and pressure, and second, the reference temperature depends on the temperature $T_e$ at the outer edge of the boundary.
layer, that is,

\[ T_e/\tau = (1/2)(1 + T_w/\tau) + 0.11(\gamma - 1)M^2 + (1/2)(T_e/\tau - 1) \]  \hspace{1cm} (B-4)

However, it can be assumed that the velocity at the outer edge of the boundary layer over most of the wing surface is approximately the free-stream velocity.

Furthermore, if the inequality

\[ (1/2)(T_e/\tau - 1) < (1/2)(1 + T_w/\tau) + 0.11(\gamma - 1)M^2 \]  \hspace{1cm} (B-5)

is satisfied over most of the surface, the reference temperature becomes the same as that for a flat plate (B-3). Hence, the skin-friction coefficient can be approximated by

\[ C_f = N(1 + \gamma M^2 C_{pe}/2)^{1/2} \quad (x/c)^{-1/2} \]  \hspace{1cm} (B-6)

where \( N \) is defined by Eq. (B-2) and where the pressure coefficient \( C_{pe} \) at the outer edge of the boundary layer can be obtained from Appendix A.

On a surface which is planar, the pressure coefficient \( C_{pe} \) is constant so that the average skin-friction coefficient for this surface becomes

\[ C_{fa} = (1/c) \int_0^c C_f \, dx = (1 + \gamma M^2 C_{pe}/2)^{1/2} \tilde{C}_{fa} \]  \hspace{1cm} (B-7)

where

\[ \tilde{C}_{fa} = 2N \]  \hspace{1cm} (B-8)

is the average skin-friction coefficient of a flat plate. Rather than calculating the quantity \( N \), the value of \( \tilde{C}_{fa} \) can be taken from the graphs presented in Ref. 10.
APPENDIX C: DATA AND FORMULAS FOR EXAMPLES

To determine the actual geometry and lift-to-drag ratio of an optimum wing, it is necessary to specify the ratio of specific heats, the Mach number, and the Reynolds number. In this connection, the following values have been chosen:

\[ \gamma = 1.4 \quad , \quad M = 10 \quad , \quad Re = 5 \times 10^6 \]  \hspace{1cm} (C-1)

In Sections 6 and 9, the optimum shape is a wedge with a sharp leading edge whose upper surface is parallel to the undisturbed flow direction. Therefore, since the slope of the lower surface equals the thickness ratio \( \tau \), the pressure modifying factor \( n \) and the average skin-friction coefficient for the entire wing are as follows:

\[ n = 0.6 + (0.36 + 0.01/\tau^2)^{1/2} \]  \hspace{1cm} (C-2)

\[ C_{fa} = (1/2)[1 + (1 + 140n\tau^2)^{1/2}] \tilde{C}_{fa} \]

where \( \tilde{C}_{fa} \) is the average skin-friction coefficient on a flat plate for the conditions (C-1). For a laminar boundary layer, Ref. 10 shows that, for wall temperatures between the free-stream temperature and the recovery temperature, \( \tilde{C}_{fa} \) takes the values

\[ 0.486 \times 10^{-3} \geq \tilde{C}_{fa} \geq 0.430 \times 10^{-3} \]  \hspace{1cm} (C-3)

and varies only 6% with respect to the mean value

\[ \tilde{C}_{fa} = 0.458 \times 10^{-3} \]  \hspace{1cm} (C-4)
Since $C_{fa}$ is directly proportional to $\tilde{C}_{fa}$, the skin-friction drag varies 6% with respect to the value obtained using Eq. (C-4). However, for an optimum wing, the skin-friction drag is less than or equal to one-third of the total drag. Therefore, a 6% change in $C_{fa}$ causes less than a 2% change in the drag, or in other words, less than a 2% change in the lift-to-drag ratio. This means that we can neglect the effect of wall temperature and use the mean $\tilde{C}_{fa}$ as given by Eq. (C-4).

In Section 10, the optimum shape is a wedge with a slightly blunt leading edge whose upper surface is parallel to the undisturbed flow direction. However, since the drag and the lift of that part of the airfoil excluding the leading edge have been calculated as if the leading edge were sharp, the pressure modifying factor $n$ and the average skin-friction coefficient are given by Eqs. (C-2).

In Section 11, the optimum shape is a wedge with a sharp leading edge at an angle of attack. Hence, the function $P(\dot{z})$, the pressure modifying factor, and the average skin-friction coefficient are given by

$$P(\dot{z}) = (1/140n)[1 - (1 - 2\dot{z})^{7/2}]$$

$$n = 0.6 + (0.36 + 0.01\dot{y}^{2})^{1/2}$$

$$C_{fa} = (1/2) \left\{ \left[ 1 - 140nP(\dot{z}) \right]^{1/2} + \left[ 1 + 140n\dot{y}^{2} \right]^{1/2} \right\} \tilde{C}_{fa}$$

where $\tilde{C}_{fa}$ is given by Eq. (C-4).
LIST OF CAPTIONS

Fig. 1  Coordinate system.

Fig. 2  Maximum lift-to-drag ratio for given thickness.

Fig. 3  Optimum shape for given profile area.

Fig. 4  Optimum thickness ratio for given profile area.

Fig. 5  Maximum lift-to-drag ratio for given profile area.

Fig. 6  Optimum thickness ratio with base drag included.

Fig. 7  Maximum lift-to-drag ratio with base drag included.

Fig. 8  Coordinate system for blunt leading edge problem.

Fig. 9  Optimum thickness ratio with leading edge blunting.

Fig. 10  Maximum lift-to-drag ratio with leading edge blunting.

Fig. 11  Coordinate system for inclined upper surface problem.

Fig. 12  Optimum slope of lower surface.

Fig. 13  Optimum slope of upper surface.

Fig. 14  Optimum thickness ratio.

Fig. 15  Maximum lift-to-drag ratio.