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A GENERAL FORMULATION OF THE BOUNDARY CONDITIONS
ON THE VECTOR POTENTIAL IN THREE-DIMENSIONAL HYDRODYNAMICS

by

George Jiro Hirasaki

A THESIS SUBMITTED IN
PARTIAL FULFILLMENT OF THE
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Thesis Director's Signature:

J D Williams

Houston, Texas
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WITH APPRECIATION

TO MY

PARENTS
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NOMENCLATURE

Roman Letters

\( g \)  Determinant of \( g_{ij} \)

\( H \)  Mean curvature of the surface

\( K \)  Gaussian or total curvature of the surface

\( P \)  Pressure

\( r \)  Distance

\( S_i \)  A surface which forms a part of the boundary of \( V \)

\( S \)  The set of all \( S_i \)

\( t \)  Time

\( V \)  A region of space

Roman Letters (Vectors)

\( \vec{A} \)  Vector potential

\( \vec{A}_s \)  A vector derived from \( \vec{B} \)

\( \vec{A}_T \)  Tangential projection of the vector potential on \( S \)

\( \vec{B} \)  A vector normal to \( S \)

\( \vec{E} \)  A vector

\( \vec{F} \)  A non-conservative external body force

\( \vec{n} \)  The normal vector

\( \vec{V} \)  Velocity

\( \vec{W} \)  Vorticity

\( \vec{X} \)  Coordinate vector

Script letters

\( \sigma \)  Determinant of \( \sigma^{\alpha\beta} \)

\( \sigma^{\alpha\beta} \)  Contravariant surface metric tensor
Roman Letters (Indexed)

A_\text{i}  \quad \text{Covariant representation of } \vec{\mathbf{A}}

A(\text{n})  \quad \text{Normal component of } \vec{\mathbf{A}}

B^k  \quad \text{Contravariant representation of } \vec{\mathbf{B}}

B(\text{i})  \quad \text{Physical component of } \vec{\mathbf{B}}

B^k_{,\alpha}  \quad \text{First covariant derivative of } \vec{\mathbf{B}}

B^k_{,\alpha\beta}  \quad \text{Second covariant derivative of } \vec{\mathbf{B}}

b_{\alpha\beta}  \quad \text{Tensor related to the second fundamental form of the surface}

C_{\alpha\beta}  \quad \text{Tensor related to the third fundamental form of the surface}

D^{ij}  \quad \text{Deformation tensor}

g_{ij}  \quad \text{Metric tensor}

h_i  \quad \sqrt{g_{ii}}

n_j  \quad \text{Covariant representation of } \vec{\mathbf{n}}

R^j_{i\alpha\beta}  \quad \text{Riemann-Christoffel tensor of the surface differentiation of a spatial vector}

T^{ij}  \quad \text{Stress tensor}

t^i  \quad \text{Stress vector}

t^i_{\alpha}  \quad \text{Tensor relating surface coordinates to spatial coordinates}

v^i  \quad \text{Contravariant representation of } \vec{\mathbf{V}}

W_i  \quad \text{Covariant representation of } \vec{\mathbf{W}}
Greek Letters

$\alpha_{ij}$ Transformation matrix from 0123 to 01'2'3'

$\beta_{ij}$ Transformation matrix from 01'2'3' to 01''2''3''

$\gamma_{ij}$ Transformation matrix from 0123 to 01''2''3''

$\gamma$ A Harmonic function

$\varepsilon_{ijk}$ Permutation symbol normalized with $g^{\frac{1}{2}}$

$\varepsilon^{\alpha\beta}$ Permutation symbol normalized with $\alpha^{-\frac{1}{2}}$

$\eta$ Outward normal to the edge of a surface element

$\mu$ Viscosity

$\nu$ Kinematic viscosity

$\rho$ Density (also used as a coordinate)

$\sigma$ Surface tension

$\Phi$ Scalar potential, also a harmonic function

$\Psi$ A scalar function

$\Omega_{ij}$ Vorticity tensor

Special Symbols

$\Delta$ Change across $S$

$\Delta_{c}$ Change across a curve in $S$

$\nabla$ "Del" differential operator

$\nabla_{s}$ Surface differential operator

$[ij,k]$ Christoffel symbol of the first kind

$\{i \ j \ k\}$ Christoffel symbol of the second kind

Indexing

Roman letters take the values 1,2,3

Greek letters take the values 1,2
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CHAPTER I

I. Introduction

The problems of viscous incompressible fluid flow in two dimensions have traditionally been solved through the use of the Lagrange stream function or the Stokes stream function for axisymmetric flows. Yih [1] proves that the velocity in an incompressible flow field in three-dimensions may be expressed in terms of two stream functions. However, he does not explain how these stream functions may be computed. Benton [2] gives an example of a problem in which two stream functions are needed.

Lamb and numerous subsequent workers have discussed the use of the vector potential to calculate the velocity from a given vorticity field [3]. A survey of articles on the vector potential is given in the book by Jacob [4]. As the problem of current interest is the solution of the equations of motion, the vorticity field, as well as the velocity, is to be determined rather than to be specified. Surprisingly enough, the hydrodynamics literature on this important problem is extremely sparse in contrast to that on related problems in electromagnetic field theory. Apparently, only two recent papers refer to the boundary conditions on the vector potential. Timman [5] suggested that the vector potential could be required to vanish on solid boundaries. Moreau [6] pointed out that Timman's conditions were incorrect and that when the velocity is tangential to the boundaries, the vector potential is normal to the boundaries. Aziz
and Hellums [7] used the vector potential in numerical solutions of the three-dimensional equations of motion in transient, laminar natural convection.
CHAPTER II

Mathematical Formulation of the Boundary Conditions

II-1. Introduction

The boundary conditions on the vector potential are formulated for an arbitrary specification of velocity on an arbitrary surface. The boundary conditions on the vorticity are given at an interface between two fluids.

In section 2 we discuss the formulation of the equations of motion in terms of the vorticity and a vector potential. In such problems, the velocity is regarded as specified on the boundaries. Hence, we consider the problem of finding boundary conditions on the vector potential which imply the specified velocity distribution.

In section 3, the hypothesis on the region and the velocity field are stated and certain properties of the vector potential are discussed. In section 4 a vector $\vec{B}$, normal to the boundaries, is introduced. We show that $\vec{B}$ satisfies a certain second order differential equation and that finding $\vec{B}$ is equivalent to finding the desired boundary conditions. In section 5 the procedure for finding the boundary condition on the normal component of the vector potential is discussed. The boundary conditions on the vorticity at a free surface or at an interface between two viscous fluids are presented in section 6. Finally, in section 7 the equations are presented for several important special cases.
II-2. **Hydrodynamic Equations and Problem Formulation**

The hydrodynamic equations in a form convenient for solution by numerical methods are given below

\[
\frac{\partial \vec{W}}{\partial t} + \vec{V} \cdot \vec{\nabla} \vec{W} - \vec{W} \cdot \vec{\nabla} \vec{V} = \frac{\vec{V} \times \vec{F}}{\rho} + \nu \vec{V} \cdot \vec{\nabla}^2 \vec{W} \tag{2.1}
\]

\[\vec{V} \cdot \vec{\nabla}^2 \vec{A} = -\vec{W} \tag{2.2}\]

where \(\vec{A}\) is the vector potential, \(\vec{W}\) is the vorticity, \(\vec{V}\) is the velocity, \(\vec{F}\) is a non-conservative body force, \(\rho\) is the density, and \(\nu\) is the kinematic viscosity. Equation (2.1) is the vorticity transfer equation for an incompressible Newtonian fluid, and (2.2) will be discussed below. The vector potential and vorticity are defined by (2.3) and (2.4) respectively.

\[\vec{V} = \vec{V} \times \vec{A} \tag{2.3}\]

\[\vec{W} = \vec{V} \times \vec{V} \tag{2.4}\]

Since the curl of the gradient of a scalar is identically zero, \(\vec{A}\) is arbitrary to a gradient of a scalar. The continuity equation for an incompressible fluid is identically satisfied by (2.3)

Substituting (2.3) into (2.4) we have:

\[\vec{W} = \vec{V} \times (\vec{V} \times \vec{A})
\equiv \vec{V} (\vec{V} \cdot \vec{A}) - \vec{V}^2 \vec{A} \tag{2.5}\]

Equation (2.5) reduces to (2.2) if \(\vec{A}\) is solenoidal; a solenoidal \(\vec{A}\) is found by the Gauge transformation.

Equations (2.1), (2.2), and (2.3), together with the specification of \(\vec{V}\) on the boundaries, constitute a complete problem which presumably determines the velocity distribution. For purposes of solving the
equations, it is necessary to express the boundary conditions in terms of the vector potential and vorticity. In numerical work an iterative procedure is used in which the two equations are treated within a given iteration as if uncoupled with the coupling introduced between subsequent iterations. Hence, the conditions on \( \vec{A} \) which imply the specified conditions on \( \vec{V} \) are desired. Finding these conditions for arbitrary \( \vec{V} \) is the subject of this paper.

II-3. Hypothesis and Preliminary Results

Let the region of space for which we wish to determine the flow field be denoted by \( V \), and let it be a regular or an infinite regular region of space as defined by Kellogg [8]. Let \( S \) be the surface(s) that is(are) the boundary(ies) of \( V \). To consider the possibility of multiple disconnected surfaces, denote \( S \) by \( S = \bigcup_{i=1}^{N} S_i \) where each \( S_i \) is regular, closed, and simply connected. Also, denote the surface integral over \( S \) to mean \( \oint_S = \sum_{i=1}^{N} \oint_{S_i} \). The net flux across each of the surfaces \( S_i \) must be equal to zero, i.e.,

\[
\oint_{S_i} \vec{n} \cdot \vec{V} \, da = 0, \quad i = 1, \ldots, N.
\]

For the purpose of this paper, we will have to require more smoothness on \( S \) than that required by Kellogg. Let \( y^3 = f(y^1, y^2) \) be a standard representation for a regular surface element of \( S \). Define a "face" to be a region of \( S \) such that the third derivative of \( f(y^1, y^2) \) satisfies a uniform Hölder condition [8]. \( S \) must consist of a finite number of faces as defined here, and each face must be bounded by a finite number of regular arcs.
Let the velocity $\vec{V}$ have continuous derivatives of the second order in the interior of $V$, continuous one-sided derivatives normal to $S$ on $S$, and continuous first derivatives with respect to the surface coordinates on $S$. The velocity field is solenoidal, $\nabla \cdot \vec{V} = 0$, since the flow is incompressible. If the region is infinite, $\vec{V}$ must be regular at infinity [9].

**Theorem 3.1:** Given a vector $\vec{V}$ with the above conditions in the region $V$, there exists a vector potential $\vec{A}$ such that

\[ \vec{V} = \nabla \times \vec{A} \text{ and } \nabla \cdot \vec{A} = 0. \]

**Proof:** Helmholtz's Theorem [9] states that if $\vec{V}$ is a vector field that is piece-wise differentiable everywhere and regular at infinity, it may be represented as the sum of an irrotational field and a solenoidal field at points where $\vec{V}$ is continuous:

\[ \vec{V} = -\nabla \phi + \nabla \times \vec{A} \quad (3.1) \]

In the present instance $\vec{V}$ may be regarded as the velocity vector, $\phi$ as the scalar potential, and $\vec{A}$ as the vector potential. When $\vec{V}$ is continuously differentiable in the interior of $V$ and in the complement of $V$ but possibly discontinuous across $S$, and is regular at infinity, $\phi$ and $\vec{A}$ may be expressed as:

\[ \phi = \frac{1}{4\pi} \int \frac{\nabla \cdot \vec{V}}{r} \, dv - \frac{1}{4\pi} \int_S \frac{\vec{n} \cdot \Delta \vec{V}}{r} \, da \quad (3.2) \]

\[ \vec{A} = \frac{1}{4\pi} \int \frac{\nabla \times \vec{V}}{r} \, dv - \frac{1}{4\pi} \int_S \frac{\vec{n} \times \Delta \vec{V}}{r} \, da \quad (3.3) \]

The volume integral is taken over all space, $r$ is the distance from the point at which $\phi$ or $\vec{A}$ is evaluated to the point at which
the integrand is evaluated, \( \vec{n} \) is the outward unit normal to \( S \), and \( \Delta \vec{V} \) is the increment in \( \vec{V} \) across \( S \). Notice that for \( \vec{V} \) in a finite region, \( V \), the potentials \( \phi \) and \( \vec{A} \) are not unique. They are functions of \( \vec{V} \) in the complement of \( V \).

Let the extension of \( \vec{V} \) in the complement of \( V \) be the gradient of a harmonic function whose normal derivative on \( S \) is equal to \( \vec{n} \cdot \Delta \vec{V} \). Hence, \( \vec{n} \cdot \Delta \vec{V} = 0 \) on \( S \). Since \( \vec{V} \) is solenoidal in \( V \) by hypothesis and the gradient of a harmonic function is solenoidal, both integrals in (3.2) vanish and \( \phi = 0 \) everywhere. The curl of the gradient of a harmonic function is zero so the volume integral in (3.3) may be evaluated only over \( V \). Equations (3.2) and (3.3) now become:

\[
\phi = 0 \quad (3.2')
\]

\[
\vec{A} = \frac{1}{4\pi} \int_V \frac{\nabla \times \vec{V}}{r} \, dv - \frac{1}{4\pi} \int_S \frac{\vec{n} \times \Delta \vec{V}}{r} \, da \quad (3.3')
\]

Substituting (3.2') and (3.3') into (3.1) proves the first part of the theorem, that \( \vec{V} = \nabla \times \vec{A} \).

Phillips* [9] proves that the conditions sufficient for \( \nabla \cdot \vec{A} = 0 \) are:

\[
\nabla \cdot (\nabla \times \vec{V}) = 0 \quad (3.4)
\]

\[
\vec{n} \cdot \Delta(\nabla \times \vec{V}) + \nabla_s \cdot (\vec{n} \times \Delta \vec{V}) = 0 \quad (3.5)
\]

\[
\Delta_c [\vec{n} \times (\vec{n} \times \Delta \vec{V})] \cdot d\vec{x} = 0 \quad (3.6)
\]

where \( \Delta_c \) denotes an increment across a curve.

Equation (3.4) is identically satisfied since \( \vec{V} \) has continuous second order derivatives. Equation (3.5) is evaluated on \( S \) and \( \nabla_s \)

*Equation (3.6) was incorrectly stated by Phillips [9] if \( S \) has discontinuities in \( n \).
is the surface differentiation operator. We may use the identities

\[ \vec{n} \cdot \Delta (\nabla \times \vec{V}) = \Delta (\vec{n} \cdot \nabla \times \vec{V}) \]
\[ = \Delta (\vec{n} \cdot \nabla_s \times \vec{V}) \]
\[ \nabla_s \cdot (\vec{n} \times \Delta \vec{V}) = -\vec{n} \cdot \nabla_s \times \Delta \vec{V} \]
\[ = -\Delta (\vec{n} \cdot \nabla_s \times \vec{V}) \]

Thus, (3.5) is satisfied. To complete the proof it remains to show that (3.6) is satisfied. First we give Lemma 3.1 which will be useful later.

Lemma 3.1: Let \( \vec{V} \) be continuous in the interior of \( V \) and the complement of \( V \) but have a discontinuity across \( S \) and let \( \vec{n} \cdot \Delta \vec{V} = 0 \). Then \( \Delta \vec{V} \) is tangential to and continuous across an edge.

Proof: \( \Delta \vec{V} \) is by hypothesis tangent to \( S \) on each face. Assume that \( \Delta \vec{V} \) has a non-zero component normal to an edge when evaluating \( \Delta \vec{V} \) on a face adjacent to the edge. Then because of the continuity of \( \vec{V}, \vec{n} \cdot \Delta \vec{V} \neq 0 \) on the adjacent face at the edge. This is a contradiction so \( \Delta \vec{V} \) does not have a non-zero component normal to an edge. Therefore, \( \Delta \vec{V} \) is tangential to an edge. By continuity of \( \vec{V}, \Delta \vec{V} \) is continuous at an edge.

Equation (3.6) is evaluated only on curves where there are discontinuities in \( \vec{n} \times \Delta \vec{V}. \Delta \vec{V} \) is continuous on \( S \) so that the only discontinuities can occur at the edges where \( \vec{n} \) is discontinuous. Using an identity we have:
\[ \vec{n} \times (\vec{n} \times \Delta \vec{V}) = \vec{n}(\vec{n} \cdot \Delta \vec{V}) - \Delta \vec{V}(\vec{n} \cdot \vec{n}) \]
\[ = -\Delta \vec{V} \]

Thus,
\[ \Delta_c [\vec{n} \times (\vec{n} \times \Delta \vec{V})] = -\Delta_c (\Delta \vec{V}) \]
\[ = 0 \]

since \( \Delta \vec{V} \) is continuous across an edge. Therefore, (3.6) is satisfied and \( \nabla \cdot A = 0 \).

The existence of the vector potential and proofs similar to the one given here have been known for a long time. The proof given here is of interest not only because it is part of a complete development, but also because it shows that the vector potential as defined by (3.3') is already solenoidal under the hypothesis of this work without recourse to the Gauge transformation.

**Lemma 3.2:** Let \( \vec{E} \) be continuously differentiable on each face of \( S_1 \), have a continuous tangential component across an edge, and satisfy the equation
\[ \vec{n} \cdot \nabla \times \vec{E} = 0 \]
on each face. Then the tangential projection of \( \vec{E} \) on each face of \( S_1 \) is:
\[ \vec{E}_T = \nabla_s \Psi \]

where
\[ \Psi(P) = \int \limits_{P_0}^P \vec{E} \cdot d\vec{x} \]
The line integral is taken on \( S_1 \) where \( P_0 \) is an arbitrary point on \( S_1 \).
Proof: If \( \vec{E} \) is defined only on \( S_i \), define an extension, \( \vec{E}' \), of \( \vec{E} \) that is continuously differentiable in a region containing \( S_i \). The lemma is proven for a face of \( S_i \) by a theorem by Brand [10]. It remains to prove that \( \Psi \) is single valued on all of the faces.

If the component of \( \vec{E} \) tangential to an edge is continuous across the edge, the value of \( \Psi \) is continuous across the edge. Stokes' theorem will apply to any region of \( S_i \) by evaluating the line integrals on both sides of each edge in opposite directions as a part of the circuit. Thus, by Stokes' theorem \( \Psi \) is single valued on \( S_i \).

Theorem 3.2: Let \( \vec{A}_s \) be a vector tangent to \( S \) with continuous tangential values across an edge, continuously differentiable on each face of \( S \) and satisfying

\[
\vec{n} \cdot \nabla_s \times \vec{A}_s = \vec{n} \cdot \vec{V} \tag{3.7}
\]

Then there exists a vector potential, \( \vec{A}' \), having the same divergence and curl as \( \vec{A} \) and having \( \vec{A}_s \) as its tangential projection on \( S \).

Proof: The original vector potential, \( \vec{A} \), also satisfies (3.7) on \( S \) since:

\[
\vec{n} \cdot \nabla_s \times \vec{A} = \vec{n} \cdot \nabla \times \vec{A} = \vec{n} \cdot \vec{V}
\]

Thus:

\[
\vec{n} \cdot \nabla_s \times (\vec{A} - \vec{A}_s) = 0
\]
We may now express the tangential projection of \( \vec{A} - \vec{A}_s \) as:

\[
(\vec{A} - \vec{A}_s)_T = A_T - A_s
\]

\[
= \nabla_s \psi_i
\]

where

\[
\psi_i(p) = \int_{\vec{P}}^{\vec{P}_o} (\vec{A} - \vec{A}_s) \cdot d\vec{x}
\]

\( A_T \) is the tangential projection of \( \vec{A} \) on \( S \) and the line integral is evaluated on \( S_i \) with \( P_0 \) as an arbitrary point on \( S_i \). \( \nabla_s \psi_i \)

is unique but \( \psi_i \) is arbitrary by a constant which depends on the location of \( P_0 \).

Let \( \hat{\phi} \) be a function harmonic in \( V \) and having its value equal to \( \psi_i \) on \( S_i \). The existence of \( \hat{\phi} \) is established by the Dirichlet problem. If \( S \) is a single connected surface then \( \hat{\phi} \) is arbitrary by a constant but \( \nabla \hat{\phi} \) is unique. However, if \( S \) is a sum of multiple disconnected surfaces then neither \( \hat{\phi} \) nor \( \nabla \hat{\phi} \) are unique.

Let

\[
\vec{A}' = \vec{A} - \nabla \hat{\phi}
\]

Then on \( S \)

\[
\vec{A}'_T = \vec{A}_T - \nabla_s \hat{\phi}
\]

\[
= \vec{A}_T - \nabla \psi_i
\]

\[
= \vec{A}_s
\]

The divergence and curl of \( \vec{A}' \) are equal to that of \( \vec{A} \) since

\[
\nabla \cdot \nabla \hat{\phi} \equiv \nabla^2 \hat{\phi} = 0
\]

\[
\nabla \times \nabla \hat{\phi} \equiv 0
\]
We have proven that $A'$ has the same divergence and curl as $\vec{A}$ so it may replace $\vec{A}$ in the differential equation (2.2). $A'$ is a vector potential with $\vec{A}_s$ as its tangential projection on $S$.

II-4. Formulation as a Differential Equation in a Normal Vector $\vec{B}$

Lemma 4.1: Let $\vec{B}$ be a vector normal to $S$ and satisfying the differential equation

$$\vec{n} \cdot \nabla_s \times (\nabla_s \times \vec{B}) = \vec{n} \cdot \vec{V} \quad (4.1)$$

on each face. Let

$$\vec{A}_s = \nabla_s \times \vec{B} \quad (4.2)$$

Then $\vec{A}_s$ satisfies (3.7) on each face and is tangential to each face.

The surface curl of a spatial vector may be expressed in the tensor notation such as that used by Aris [11].

$$\left(\nabla_s \times \vec{B}\right)_i = \varepsilon_{ijk} \partial^\alpha \beta \ t^i_{\rho} \ t^j_{\beta} \ B^k_{s,\alpha} \quad (4.3)$$

Equation (4.1) may be written in terms of the components of $\vec{B}$ by two applications of the surface curl operator and the scalar product with the normal vector. After some simplification the results are:

$$\vec{n} \cdot \nabla_s \times (\nabla_s \times \vec{B}) = n_j \partial_{\rho}^\alpha \partial_{\beta}^\gamma \ t^m_{\rho} \ t^j_{\beta} \ v^k_{l,\rho} \ v^k_{l,\gamma} \ B^k_{s,\alpha} - n_k \partial_{\rho}^{\gamma \alpha} B^k_{s,\alpha} \quad (4.4)$$

Since $B^k$ is normal to $S$, let

$$B^k = (B^r n_r) n^k$$

$$B^k_{s,\alpha} = \frac{\partial (B^r n_r)}{\partial x^\alpha} n^k + (B^r n_r) n^k$$
\[ g_{km} t^m n^k = 0 \]
\[ g_{km} t^m n^k, \alpha = -b \alpha \rho \]
\[ n_j t^j \beta, \gamma = b \beta \gamma \]
\[ n_j \partial^\gamma \partial^\alpha \partial^\beta \partial^\rho \partial^m t^j \beta, \gamma \ g_{km} B^k, \alpha, \beta, \gamma = -\partial^\gamma \partial^\alpha \partial^\beta b \alpha \rho \beta \gamma (B^r n_r) \]
\[ = (2K - 4H^2)(B^r n_r) \]

where \( K \) is the Gaussian curvature and \( H \) is the mean curvature of the surface. Therefore:

\[ (2K - 4H^2)(B^r n_r) - n_k \partial^\alpha B^k_r \partial^\alpha = n_j \partial^j \]  \( (4.5) \)

If one of the coordinate surfaces is made to coincide with the face of \( S \) and the other two coordinates are orthogonal with the first, \( \vec{n} \) will have only one non-zero component on that face. Equation \( (4.5) \) will then result in a partial differential equation of the second order for the normal component of \( \vec{B} \).

The partial differential equation is valid only on the faces of \( S \) and boundary conditions must be specified for the normal component of \( \vec{B} \) on the edges or, if there are no edges, on any closed curve on \( S_1 \). The boundary conditions must satisfy two conditions:

(a) the component of \( \nabla_s \times \vec{B} \) tangential to the edge must be continuous across the edge, and

(b) on each face \( \nabla_s \times \vec{B} \) must satisfy

\[ \int_{S'} \vec{n} \cdot \partial \ d\alpha = \int_{S'} \vec{n} \cdot \nabla_s \times (\nabla_s \times \vec{B}) \ d\alpha \]
\[ = \vec{b} \cdot (\nabla_s \times \vec{B}) \cdot d\vec{x} \]  \( (4.6) \)

where \( S' \) is a face of \( S \) and \( C' \) is the boundary of \( S' \).
The conditions (a) and (b) are boundary conditions on the derivative of \( \vec{B} \) normal to an edge or a linear combination of \( \vec{B} \) and the derivative of \( \vec{B} \) normal to an edge, i.e., boundary condition of the second or third kind. Condition (b) is a necessary condition for the existence of a solution to (4.5).

In a plane surface with straight edges, condition (a) requires that the derivative of \( \vec{B} \), normal to the edge, be continuous across the edge and condition (b) requires that

\[
\int_{S_i} \vec{n} \cdot \vec{V} \, d\Gamma = - \oint_{C_i} \frac{\partial (\vec{n} \cdot \vec{B})}{\partial \eta} \, ds
\]

where \( \frac{\partial}{\partial \eta} \) is the derivative in the direction of the outward normal to the edge. This second condition is a necessary condition for the existence of the Neumann problem for \( \vec{n} \cdot \vec{B} \).

**Lemma 4.2:** Boundary conditions (a) and (b) can both be satisfied on all of \( S \) if and only if \( \vec{V} \) satisfies

\[
\oint_{S_i} \vec{n} \cdot \vec{V} \, d\Gamma = 0 \quad i = 1, \ldots, N
\]  

(4.7)

**Proof:** Stokes' theorem may be applied to \( \vec{A}_s \) over all of \( S \) by taking the circuit integrals along both sides of the edges. Because of condition (a) the net sum of the circuit integrals is equal to zero. Since \( \vec{n} \cdot \nabla_s \times \vec{A}_s = \vec{n} \cdot \vec{V} \), the net sum of the surface integrals is the left side of (4.7). Thus, conditions (a) and (b) can be satisfied on all of \( S \) only if (4.7) is satisfied.
If (4.7) is satisfied and the boundary conditions (a) and (b) are evaluated in a sequence of faces such that the faces for which the boundary conditions have not yet been specified are connected then conditions (a) and (b) can be satisfied on all of the faces. Equation (4.7) has been adopted as a hypothesis in this paper.

**Lemma 4.3:** Equation (4.5) with only one non-zero component of \( \bar{B} \) is an elliptic partial differential equation.

**Proof:** The second order terms of the equation are

\[
n_k \frac{\partial^{\alpha\gamma}}{\partial u^\alpha} \frac{\partial^2 \bar{B}^k}{\partial u^\alpha \partial u^\gamma}
\]

where \( \bar{B}^k \) is non-zero for only one \( k \). A necessary and sufficient condition for the equation to be elliptic is that the matrix \( (\partial^{\alpha\gamma}) \) be positive definite [12].

A regular surface element has a standard representation

\[ y^3 = f(y^1, y^2) \]

where in this case let \( u^1 = y^1 \) and \( u^2 = y^2 \). Then

\[
\det(\partial^{\alpha\gamma}) = 1 + \left( \frac{\partial f}{\partial y^1} \right)^2 + \left( \frac{\partial f}{\partial y^2} \right)^2 > 0
\]

**Lemma 4.4:** The second order derivatives of every solution to (4.5) satisfy a Hölder condition.

**Proof:** Equation (4.5) is elliptic by Lemma 4.3. Let

\[ y^3 = f(y^1, y^2) \]

be the standard representation of a face of \( S \) as before. Derivatives of the third order of \( f \) satisfy a Hölder condition by hypothesis and are the highest order that appear in the coefficient in the left side of (4.5). The right
side, \( \vec{n} \cdot \vec{V} \), is continuously differentiable by hypothesis. If the coefficients and the right side of the elliptic equation (4.5) satisfy Hölder conditions, then the second order derivative of every solution also satisfies a Hölder condition [12].

**Theorem 4.1:** Let \( \vec{B} \) be a solution of (4.5) with boundary conditions (a) and (b) and let \( \vec{A}_s \) be calculated from \( \vec{B} \) by (4.2), then \( \vec{A}_s \) satisfies the hypothesis of Theorem 3.2.

**Proof:** That \( \vec{A}_s \) is tangent to \( S \) and satisfies (3.1) on each face has been proven in Lemma 4.1. Condition (a) of the boundary conditions requires that the component of \( \vec{A}_s \) tangential to an edge must be continuous across the edge.

By Lemma 4.4, \( \vec{A}_s \) is continuous differentiable on each face.

**Theorem 4.2:** If \( S \) is a single connected surface the vector potential \( \vec{A}' \) with the tangential projection \( \vec{A}_s \) is unique in \( V \).

**Proof:** Let \( \vec{A}'' \) be another vector potential satisfying

\[ \nabla \times \vec{A}'' = \vec{V}, \quad \nabla \cdot \vec{A}'' = 0, \text{ in } V, \text{ and } \vec{A}''_T = \vec{A}_s \text{ on } S. \]

Then

\[ \vec{A}' - \vec{A}'' = \nabla \gamma \]

where \( \gamma \) is a harmonic function. On \( S \)

\[ (\vec{A}' - \vec{A}'')_T = \vec{A}_s - \vec{A}_s \]

\[ = \vec{0} \]

\[ = \nabla_s \gamma \]
Since $\nabla_\gamma = 0$ on $S$, $\gamma$ is a constant on $S$. By the maximum principle a harmonic function that is constant on the boundary of a region is constant in the region. Thus, $\nabla_\gamma = 0$ and $\vec{A}'' = \vec{A}'$.

**Theorem 4.3:** If $S$ is a sum of multiple disconnected surfaces, and $\vec{A}'$ is a vector potential such that the net flux of $\vec{A}'$ on $S_i$ is zero, i.e.

$$\oint_{S_i} \vec{A}' \cdot \vec{n} \, d\Omega = 0, \quad i = 1, \ldots, N$$

and has the tangential projection $\vec{A}'_s$ on $S$, then $\vec{A}'$ is unique in $V$.

**Proof:** The proof is the same as for Theorem 4.2 to the point where it states that $\gamma$ is constant on $S$. Here we have that $\gamma$ is a constant, $c_i$ on $S_i, \ i = 1, \ldots, N$ but it may be a different constant on each $S_i$. The condition that the net flux is zero implies

$$\oint_{S_i} (\vec{A}' - \vec{A}'') \cdot \vec{n} \, d\Omega = \oint_{S_i} \nabla_\gamma \cdot \vec{n} \, d\Omega = \oint_{S_i} \frac{\partial \gamma}{\partial n} \, d\Omega = 0, \quad i = 1, \ldots, N.$$

The Green's identity for $\gamma$ gives

$$\int_V (\nabla_\gamma)^2 \, dv = \int_S \gamma \frac{\partial \gamma}{\partial n} \, d\Omega = \sum_{i=1}^{N} \oint_{S_i} c_i \frac{\partial \gamma}{\partial n} \, d\Omega.$$
\[
= \sum_{i=1}^{N} C_i \oint_{s_i} \frac{\partial \psi}{\partial n} d\sigma
\]

= 0

Thus, \( \nabla \psi = 0 \) and \( \vec{A}'' = \vec{A}' \).

II-5. **Boundary Condition on the Normal Component**

The boundary condition on the normal component of the vector potential may be expressed as a boundary condition of the third kind by using tensor notation. This boundary condition is found by applying the condition \( \nabla \cdot \vec{A}' = 0 \) on \( S \).

**Definition:** The Riemann-Christoffel tensor of the surface differentiation of a spatial vector is \( R^j_{i\alpha\beta} \) where

\[
B_{i,\alpha\beta} - B_{i,\beta\alpha} = R^j_{i\alpha\beta} B_j
\]

By performing the covariant differentiating and simplifying \( R^j_{i\alpha\beta} \) may be expressed as

\[
R^j_{i\alpha\beta} = \frac{\partial}{\partial u^{i}} \{ t^j_{\alpha} \} t^k_{\beta} - \frac{\partial}{\partial u^{j}} \{ t^j_{\alpha} \} t^k_{\beta} + \{ i m \} \{ j l \} ( t^m_{\beta} c^{k} - t^m_{\alpha} c^{k} )
\]

**Definition:** \( R_{ri\alpha\beta} = g_{rj} R^j_{i\alpha\beta} \)

**Lemma 5.1:** \( R_{ri\alpha\beta} = -R_{ir\alpha\beta} \)

**Proof:** By using the equations

\[
g_{rj,\alpha} = 0
\]

\[
g_{rj} \{ i k \} = [ i k, r ]
\]

\[
= \frac{\partial g_{ir}}{\partial x^k} + \frac{\partial g_{rk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^r}
\]
and after some manipulation, we obtain the formula

$$R_{\rho \sigma \beta \gamma} = \left( t^k_{\beta} \frac{\partial}{\partial u^\alpha} - t^k_{\alpha} \frac{\partial}{\partial u^\beta} \right) \left( \frac{\partial g_{\rho k}}{\partial x^1} - \frac{\partial g_{\beta k}}{\partial x^r} \right)$$

$$- \{ p \}_{i k} \{ p \}_{r n} \left( t^n_{\alpha} t_k^{\beta} - t^n_{\beta} t_k^{\alpha} \right)$$

from which the antisymmetry is obvious.

**Lemma 5.2**: Let $\vec{B}$ be a vector normal to $S$ and have continuous derivatives of the second order and define an orthogonal coordinate system such that $x^1 = u^1$, $x^2 = u^2$, and $S$ is the coordinate surface $x^3 = 0$. Then $B_{3, \alpha \beta} = B_{3, \beta \alpha}$.

**Proof:**

$$B_{i, \alpha \beta} - B_{i, \beta \alpha} = R_{i \alpha \beta}^{\rho} B_j$$

$$= g^{\rho j} R_{i \alpha \beta} B_j$$

Since the coordinates are orthogonal

$$g^{ij} = 0, \ i \neq j$$

The only non-zero component of $\vec{B}$ is $B_3$.

$$B_{3, \alpha \beta} - B_{3, \beta \alpha} = g^{33} R_{33 \alpha \beta} B_3$$

$$= 0$$

since $R_{33 \alpha \beta} = 0$ because of antisymmetry.

**Lemma 5.3**: Let $\vec{B}$ be a vector normal to $S$. Then $\nabla_S \times \vec{B}$ is tangential to $S$ or is zero.

**Proof**: Since $\vec{B}$ is normal to $S$
\[ \varepsilon_{ijk} n^i B^k = 0 \]

\[ (\varepsilon_{ijk} n^i B^k)_{,\alpha} = \varepsilon_{ijk} (n^i B^k + n^i B^k)_{,\alpha} = 0 \] (5.2)

\[ \nabla_S \times \vec{B} \text{ is tangential to } S \text{ or zero if and only if} \]

\[ \vec{n} \cdot \nabla_S \times \vec{B} = 0. \]

Writing \( \vec{n} \cdot \nabla_S \times \vec{B} \) in tensor notation and substituting in (5.2) we have after some manipulation:

\[ \vec{n} \cdot \nabla_S \times \vec{B} = \varepsilon^{\rho\alpha} b_{\alpha \rho} n^k B^k \]

\( b_{\alpha \rho} \) is a symmetric tensor associated with the second fundamental form of the surface so \( \varepsilon^{\rho\alpha} b_{\alpha \rho} = 0. \)

Therefore, \( \vec{n} \cdot \nabla_S \times \vec{B} = 0. \)

**Theorem 5.1:** Let \( \vec{B} \) be a vector normal to \( S \) with continuous second order derivatives. Then \( \nabla_S \cdot \nabla_S \times \vec{B} = 0. \)

**Proof:** By Lemma 5.3, \( \nabla_S \times \vec{B} \) is tangential to \( S \).

\[ (\nabla_S \times \vec{B})_{,\gamma} = t_{i}^{\gamma} (\nabla_S \times \vec{B})_{,i} = \]

\[ = n_k \varepsilon_{\gamma \beta} \partial^{\alpha \beta} g^{mk} B_{m,\alpha} \]

\[ \nabla_S \cdot \nabla_S \times \vec{B} = \partial^{\rho} (\nabla_S \times \vec{B})_{,\rho} = \]

\[ = \varepsilon_{\gamma \beta} \partial^{\gamma \rho} \partial^{\alpha \beta} g^{mk} (n_k,\rho B_{m,\alpha} + n_k B_{m,\alpha \rho}) \]

Let \( B_m = (B_r n^r)_m \)

\[ B_{m,\alpha} = \frac{\partial (B_r n^r)}{\partial u^\alpha} n_m + (B_r n^r) n_{m,\alpha} \]

\[ g^{mk} n_{k,\rho} n_m = 0 \]
\[ g^{mk} n_{k, \rho} n_{m, \alpha} = c_{\rho \alpha} \]

\[ \varepsilon_{\gamma \beta} A^\gamma \rho \alpha \beta c_{\rho \alpha} = \varepsilon_{\rho \alpha} c_{\rho \alpha} \]

\[ = 0 \]

since \( c_{\rho \alpha} \) is symmetric. Thus,

\[ \nabla_s \cdot \nabla_s \times \vec{B} = \varepsilon_{\gamma \beta} A^\gamma \rho \alpha \beta g^{mk} n_k B_{m, \alpha \rho} \]

\[ = \varepsilon_{\rho \alpha} n^m B_{m, \alpha \rho} \]

Let the coordinate system be such as that in Lemma 5.2. Then

\[ n^m B_{m, \alpha \rho} = n^3 B_3, \alpha \rho \]

By Lemma 5.2 the above expression is symmetric in \( \alpha \) and \( \rho \).

Thus

\[ \nabla_s \cdot \nabla_s \times \vec{B} = 0. \]

**Lemma 5.4:** Let \( \vec{A} \) be a spatial vector defined on \( S \) and let

the coordinate system be the one defined in Lemma 5.2. Then on \( S \)

\[ \nabla \cdot \vec{A} = g^{33} A_{3, 3} + \nabla_s \cdot \vec{A} \quad (5.3) \]

**Proof:**

\[ t^i_\gamma = \delta^i_\gamma, \quad g_{\alpha \beta} = \delta_{\alpha \beta} \]

\[ \nabla \cdot \vec{A} = g^{11} A_{1, 1} + g^{22} A_{2, 2} + g^{33} A_{3, 3} \]

\[ = \alpha_{\alpha \beta} t^i_\alpha A_{i, \beta} + g^{33} A_{3, 3} \]

\[ = \nabla_s \cdot \vec{A} + g^{33} A_{3, 3} \]
**Theorem 5.2:** Let the coordinate system be the one defined in Lemma 5.2 and let the tangential projection of the vector potential be given by (4.2). Then the boundary condition for the normal component of the vector potential may be expressed as:

\[
g^{33} \frac{\partial_{3} h_{\alpha} A(3)}{\partial x^{3}} - (g^{33} \{_{3}^{3}\} h_{3} + 2\mathbf{H}) A(3) = g^{33} \{_{3}^{\alpha}\} h_{\alpha} A(\alpha) \tag{5.4}
\]

**Proof:**

\[
\nabla \cdot \mathbf{A} = \nabla_{s} \cdot \mathbf{A} + g^{33} A_{3,3}
\]

\[
\nabla_{s} \cdot \mathbf{A} = -2\mathbf{H} A_{(n)} + \nabla_{s} \cdot \vec{A}_{s} \tag{11}
\]

\[
\vec{A}_{s} = \nabla_{s} \times \vec{B}
\]

Thus,

\[
\nabla_{s} \cdot \mathbf{A}_{s} = \nabla_{s} \cdot \nabla_{s} \times \vec{B} = 0
\]

\[
A_{3,3} = \frac{\partial A_{3}}{\partial x^{3}} - \{_{3}^{3}\} A_{3} - \{_{3}^{\alpha}\} A_{\alpha}
\]

\[
A_{1} = h_{1} A(1)
\]

\[
A_{(n)} = A(3)
\]

\[
\nabla \cdot \mathbf{A} = -2\mathbf{H} A(3) + g^{33} \frac{\partial h_{3} A(3)}{\partial x^{3}} - \{_{3}^{3}\} h_{3} A(3)
\]

\[- \{_{3}^{\alpha}\} h_{\alpha} A(\alpha) \]

\[= 0
\]

**II-6. Boundary Conditions at a Fluid Interface**

A stationary interface between two viscous fluids or between a viscous fluid and an inviscid fluid will be considered. The
boundary condition on the vector potential may be found as previously discussed but the vorticity boundary condition cannot be calculated from (2.4) since the tangential components of the velocity are not specified on the interface.

A momentum balance on the interface will provide a relation to specify the boundary conditions on the tangential components of the vorticity. Assume that the surface tension is constant and that it is possible to neglect the surface density and the coefficients of dilational and shear surface viscosity. The momentum equation with these assumptions is:

\[ t^i - \hat{t}^i = -(2\pi\sigma) n^i \quad [11] \]
\[ t^i = T^{ij} n_j \]

where \( \sigma \) is the surface tension, \( t^i \) is the stress vector, and \( T^{ij} \) is the stress tensor. If \( f \) is a function defined on both sides of the interface, denote \( f \) as the value on one side of the interface and \( \hat{f} \) as the value on the other side. Denote the jump in \( f \) as:

\[ f - \hat{f} = [f] \]

If the bulk fluid is Newtonian and incompressible, the stress tensor may be expressed as:

\[ T^{ij} = -P g^{ij} + 2\mu D^{ij} \]

where \( D^{ij} \) is the deformation tensor

\[ D^{ij} = \frac{1}{2} (g^{iq} v^i_{,q} + g^{iq} v^j_{,q}) \]

The vorticity tensor \( \Omega^{ij} \) is defined as:

\[ \Omega^{ij} = \frac{1}{2} (g^{jq} v^i_{,q} - g^{iq} v^j_{,q}) \]
The deformation tensor may be expressed in terms of the vorticity tensor as:

\[ D_{ij} = \Omega_{ij} + \epsilon_{ij}^q v^q. \]

The scalar product of the vorticity tensor with the normal vector gives:

\[ \Omega_{ij} n_j = \frac{1}{2} (\vec{W} \times \vec{n})^i. \]

Thus, the stress vector may be expressed as:

\[ t^i = -P n^i + \mu (\vec{W} \times \vec{n})^i + 2\mu \epsilon_{ij}^q v^j n_j, \]

since the equation for the stress vector is linear its difference across the interface may be expressed as the sum of the difference of its individual parts.

\[ [t^i] = -[P] n^i + ([\mu \vec{W}] \times \vec{n})^i + 2\epsilon_{ij}^q n_j [\mu v^j], \quad (6.2) \]

The jump in the stress vector may be eliminated by substituting (6.1) into (6.2) and the equation may be rearranged to give

\[ ([\mu \vec{W}] \times \vec{n})^i = ([P] - 2\mu \sigma) n^i - 2\epsilon_{ij}^q n_j [\mu v^j]. \]

The cross product of this equation with the normal vector results in

\[ [\mu \vec{W}]_m - n_m (\vec{n} \cdot [\mu \vec{W}]) = -2 \epsilon_{mnk} n_j g_{ij} [\mu v^j]. \]

Define an orthogonal coordinate system such that the coordinate lines of \( x^1 \) and \( x^2 \) lie in the interface and the coordinate line of \( x^3 \) is normal to the interface. Then

\[ g_{ij} = 0, \quad i \neq j \]

\[ n_1 = (0, 0, e_{33}). \]
On the interface \( n \cdot \vec{V} = 0 \), so
\[
\frac{\partial v^3}{\partial x^\alpha} = 0, \quad \alpha = 1, 2
\]
\[
\varepsilon_{mnj} n^n n_j g^{iq} [\mu v^j, q] = \varepsilon_{m31} (g^{33} \delta^i_3 [\mu \frac{\partial v^3}{\partial x^3}] + g^{iq} \delta^i_q [\mu v^q])
\]
(6.3)

Written in terms of its components (6.3) is
\[
\begin{pmatrix}
[\mu W_1] \\
[\mu W_2] \\
0
\end{pmatrix} = \begin{pmatrix}
2g^{\frac{1}{2}} g^{22} \{^3_2 \} [\mu v^2] \\
-2g^{\frac{1}{2}} g^{11} \{^3_1 \} [\mu v^1] \\
0
\end{pmatrix}
\]

If the interface is between two viscous fluids the normal component of the vorticity is continuous across the interface [13]. If the interface is between a viscous fluid and an inviscid fluid, the boundary condition on the normal component of the vorticity may be found from the solenoidal property of the vorticity.

\[
\frac{\partial w^3}{\partial x^3} + \{^i_3 \} w^3 = -\frac{\partial w^\alpha}{\partial x^\alpha} - \{^i_1 \} w^\alpha
\]

In the special case of a plane interface
\[
W_1 = W_2 = \frac{\partial w_3}{\partial x^3} = 0
\]

II-7. Discussion of Applications

In the preceding sections it has been shown that in general a second order partial differential equation must be solved to establish the boundary conditions on the vector potential of three-dimensional
hydrodynamics. The most general form of this partial differential equation is given by (4.5). Equation (4.2) specifies the tangential components of the vector potential and the boundary condition on the normal component is given by (5.4).

Equations (4.5) and (5.4) reduce to a relatively simple form for the important cases of surfaces which coincide with a coordinate surface. Three of these are given below:

(a) The surface is a plane with \( x^3 = \text{constant} \) in Cartesian coordinates.

\[
\frac{\partial^2 B(3)}{\partial x^1^2} + \frac{\partial^2 B(3)}{(\partial x^2)^2} = -V(3)
\]

\[
\frac{\partial A(3)}{\partial x^3} = 0
\]

(b) The surface is spherical in spherical coordinates

\[
\frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial B(\rho)}{\partial \theta} \right) + \frac{1}{(\rho \sin \theta)^2} \frac{\partial^2 B(\rho)}{\partial \theta^2} = -V(\rho)
\]

\[
\frac{\partial A(\rho)}{\partial \rho} + \frac{2}{\rho} A(\rho) = 0
\]

(c) The surface is cylindrical in cylindrical coordinates.

\[
\frac{1}{\rho} \frac{\partial^2 B(\rho)}{\partial \theta^2} + \frac{\partial^2 B(\rho)}{\partial z^2} = -V(\rho)
\]

\[
\frac{\partial A(\rho)}{\partial \rho} + \frac{1}{\rho} A(\rho) = 0
\]

If the normal component of velocity is everywhere zero on \( S \) such as on a solid or free surface, a solution to (4.5) is \( \vec{B} = 0 \).
The vector potential will then be normal to the surface. An example of the former case was computed by Aziz and Hellums [7] and the latter case was studied by Moreau [6]. However, if the normal component velocity is not zero everywhere on $S$, $\mathbf{\nabla} \cdot \mathbf{V} = 0$ may not be zero on the parts of $S$ where $\mathbf{n} \cdot \mathbf{V} = 0$.

An example of a case where the velocity is either zero or tangential to the boundaries of the region is a cube with one face sliding in a direction parallel to one of the axis. Since $\mathbf{n} \cdot \mathbf{V} = 0$ everywhere the vector potential is normal to the surface.

A simple example in which the normal component of velocity is not zero everywhere on the boundaries of the region is a unit cube with solid walls everywhere except on the face $x_3 = 1$ and on this face $V_2 = \sin (2\pi x_1)$. Since the net flux on the face $x_2 = 1$ is zero, we may specify $\partial B/\partial \eta = 0$ on the edges. On that face $B_2$ must satisfy the equation

$$\frac{\partial^2 B_2}{(\partial x_1)^2} + \frac{\partial^2 B_2}{(\partial x_3)^2} = -V_2 = -\sin (2\pi x_1).$$

On all other faces $\partial B(n)/\partial \eta = 0$ and $\nabla_s^2 B(n) = 0$ so a solution is $B(n) = 0$ on these faces. Thus, the vector potential has tangential components only on the face $x_2 = 1$. In this case the normal derivative of the normal component of the vector potential is everywhere zero.
If a portion of the boundary is a fluid-fluid interface that is stationary with respect to position in space, then \( \vec{n} \cdot \vec{V} = 0 \) since the interface is a material surface. If all of the surfaces bounding the region is either a fluid-fluid interface or a solid surface then the vector potential is normal to the boundaries.

It is interesting to note that only the normal component of velocity is used to determine the boundary conditions on the vector potential. All components of velocity are used through (2.4) to calculate the boundary values of the vorticity.
CHAPTER III

A Numerical Example of the Application of the

Vector Potential

III-1. Introduction

The physical problem and the coordinate transformations, as well as a cubical region of the flow field are defined in section 2. In section 3 the procedure for computing the boundary conditions for the vector potential is discussed. The boundary conditions on $B(n)$ at the edges are specified for a given flux on the face, and a necessary and sufficient condition for the existence of a numerical solution is presented. The procedure for the numerical solution of the equations in the interior of the cube is discussed in section 4. In section 5 the results are presented as a comparison between the computed solution and the exact solution and the value of the divergence of the vector potential.

III-2. Definition of the Physical System

The application of the vector potential may best be seen by a numerical example. An example for which the exact solution of the flow field is known is provided so that the computed solution may be compared with the exact solution.

The flow field that was studied in the example is the parallel flow between two parallel plates separated by a distance of two units of length. The velocity profile in this system is parabolic, and the maximum velocity was specified to have a magnitude of $\sqrt{2}$ (Fig. 1).
A unit cube of this flow field was studied for this example. Orient a rectangular cartesian coordinate system so that the plane $x^3 = 0$ coincides with one plate and the direction of positive $x^3$ is into the flow field.

Rotate the coordinate system about the 03 axis so that

$$V_1 = V_2 = 1 - (1 - x_3)^2 \quad \text{(Figure 2)}.$$ 

Rotate the coordinate system 45° about the 01 axis to define a new coordinate system 01'2'3' (Figure 3). The matrix of the direction cosines for the transformation is

$$\alpha_{ij} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} \\
0 & -\frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2}
\end{pmatrix}$$

and

$$x'_i = \alpha_{ij} x_j .$$

Perform another rotation of 45° about the 03' axis (Figure 4). The matrix of the direction cosines of this transformation is

$$\beta_{ij} = \begin{pmatrix}
\frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} & 0 \\
-\frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

and

$$x''_i = \beta_{ij} x'_j .$$

The rotation from 0123 to 01'2'3' to 01''2''3'' may be thought of as a composite transformation from 0123 to 01''2''3'' (Figure 5). The matrix of the directional cosines which is given by
\[ \gamma_{ij} = \beta_{ik} \alpha_{kj} \]

is

\[
(\gamma_{ij}) = \begin{pmatrix}
\frac{1}{2} \sqrt{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} \sqrt{2} & \frac{1}{2} & \frac{1}{2} \\
0 & -\frac{3}{2} \sqrt{2} & \frac{3}{2} \sqrt{2}
\end{pmatrix}
\]

and

\[ x''_i = \gamma_{ij} x'_j. \]

The velocity in terms of the transformed coordinates 01''2''3'' with the primes dropped is given by

\[ V_1 = \frac{1}{2} \left( 1 + \sqrt{2} \right) (x_1 + x_2 + \sqrt{2} x_3) - \frac{1}{8} \left( 1 + \sqrt{2} \right) (x_1^2 + x_2^2 + 2 x_3^2 + 2 x_1 x_2 + 2 \sqrt{2} x_1 x_3 + 2 \sqrt{2} x_2 x_3) \]  

(2.1a)

\[ V_2 = \frac{1 - \sqrt{2}}{1 + \sqrt{2}} V_1 \]  

(2.1b)

\[ V_3 = \frac{-\sqrt{2}}{(1 + \sqrt{2})} V_1 \]  

(2.1c)

Henceforth we will speak only in terms of the transformed 01''2''3'' coordinate system. Let the unit cube to be studied be the region such that

\[ 0 \leq x_i \leq 1, \quad i = 1, 2, 3. \]

In the transformed coordinate system no component of the velocity or vorticity is zero and there is dependence on all coordinates so the system may now be considered as a three-dimensional system. The exact value of velocity will be specified on the surface of the cube as boundary conditions for computing the flow in the interior of the cube. Since the exact solution satisfies the equations of motion, the
computed solution should be the same as the exact solution except for error due to the approximate method of computation.

III-3. Computation of the Boundary Conditions

In discussing the boundary conditions on the surface of the cube it is necessary to label the faces. They will be labeled as follows:

<table>
<thead>
<tr>
<th>Number</th>
<th>Coordinate Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_1 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( x_1 = 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( x_2 = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>( x_2 = 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( x_3 = 0 )</td>
</tr>
<tr>
<td>6</td>
<td>( x_3 = 1 )</td>
</tr>
</tbody>
</table>

The vector \( \vec{b} \), normal to the surface on each face, is to satisfy the differential equation

\[
\nabla_s^2 \vec{b}(n) = -V(n)
\]  
(3.1)

The value of \( V(n) \) is given by the exact value of velocity on the faces of the cube. The derivative of \( b(n) \) normal to the edges must satisfy the equation

\[
\oint_{S'} \frac{\partial b(n)}{\partial n} \, ds = - \int_{S'} V(n) \, d\alpha
\]  
(3.2)

where \( S' \) is a face, \( C' \) is the edges bounding the face \( \frac{\partial}{\partial n} \) is the derivative in the direction tangential to \( S' \) and outward normal to \( C' \). The right side of the equation is the flux across that face and may be computed from the exact value of the velocity which is
given on the face. The net fluxes on each face, which is the integral of the normal component of the velocity on that face, were calculated to be as follows:

<table>
<thead>
<tr>
<th>Face</th>
<th>Flux</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-(\frac{1}{2} - 5/16\sqrt{2})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{1}{2} + 7/16\sqrt{2})</td>
</tr>
<tr>
<td>3</td>
<td>1/4 - 1/16\sqrt{2}</td>
</tr>
<tr>
<td>4</td>
<td>1/4 - 5/16\sqrt{2}</td>
</tr>
<tr>
<td>5</td>
<td>17/48\sqrt{2}</td>
</tr>
<tr>
<td>6</td>
<td>-(\frac{1}{2} - 5/48\sqrt{2})</td>
</tr>
</tbody>
</table>

The derivative of \(B(n)\) normal to the edge must also satisfy the condition that it be continuous across the edge. If the normal is considered to change sign as it crosses the edge then the normal derivative on one side must equal the negative of the normal derivative on the other side of the edge. This condition on \(B(n)\) assures that the component of the vector potential tangential to the edge will be continuous across the edge.

The boundary condition on \(B(n)\) will be specified as follows:

1. On faces #1, #2, and #3 let \(\frac{\partial B(n)}{\partial \eta} = 0\) on edges \(x_1 = 0,1\) and \(x_2 = 0,1\). The line integral of \(\frac{\partial B(n)}{\partial \eta}\) along one edge, if \(\frac{\partial B(n)}{\partial \eta}\) is constant, is just equal to the value of \(\frac{\partial B(n)}{\partial \eta}\) since the length of the edge is unity. Let \(\frac{\partial B(n)}{\partial \eta}\) be constant and equal on the edges \(x_3 = 0,1\) and be equal to one-half of the surface integral of \(V(n)\) on that face.
\[ \frac{\partial B(n)}{\partial \eta} = 0 \ , \ x_1, x_2 = 0,1 \]

\[ \frac{\partial B(n)}{\partial \eta} = -\frac{1}{2} \int_{S_i} V(n) \ d\sigma \ , \ x_3 = 0,1 \]

(2) On face #5 at the edges adjacent to faces #1, #2, and #3 let

\[ \left| \frac{\partial B(n)}{\partial \eta} \right|_{#5:#i} = - \left| \frac{\partial B(n)}{\partial \eta} \right|_{#i:#5}, \ i = 1, 2, 3 \]

where \( \left| \frac{\partial B(n)}{\partial \eta} \right|_{#i:#j} \) denotes the derivative on face #i adjacent to face #j. The derivative on the edge adjacent to face #4 must be such that (3.2) is satisfied.

\[ \left| \frac{\partial B(n)}{\partial \eta} \right|_{#5:#4} = - \int_{#5} V(n) \ d\sigma - \sum_{i=1,2,3} \left| \frac{\partial B(n)}{\partial \eta} \right|_{#5:#i} \]

(3) Face #4

\[ \left| \frac{\partial B(n)}{\partial \eta} \right|_{#4:#5} = - \left| \frac{\partial B(n)}{\partial \eta} \right|_{#5:#4} \]

\[ \left| \frac{\partial B(n)}{\partial \eta} \right|_{#4:#i} = 0 \ , \ i = 1,2 \]

\[ \left| \frac{\partial B(n)}{\partial \eta} \right|_{#4:#6} = - \int_{#4} V(n) \ d\sigma - \left| \frac{\partial B(n)}{\partial \eta} \right|_{#4:#5} \]

(4) Face #6

\[ \left| \frac{\partial B(n)}{\partial \eta} \right|_{#6:#i} = - \left| \frac{\partial B(n)}{\partial \eta} \right|_{#i:#6}, \ i = 1,2,3,4 \]

If the net fluxes across all of the faces is zero and the derivative of \( B(n) \) across the edges are continuous then face #6 should satisfy (3.2).
\[ \sum_{i=1,2,3,4} \frac{\partial B(n)}{\partial \eta} \bigg|_{#6:#i} = - \int_{#6} V(n) \, d\alpha \]

To illustrate the procedure in computing \( \bar{B} \), the procedure will be discussed in detail on face \( #4 \). This is the face \( x_2 = 1 \) so \( B(n) = B_2 \). On edges adjacent to faces \( #1 \) and \( #2 \) we have

\[ \frac{\partial B_2}{\partial x_1} \bigg|_{x_1=0} = 0, \quad \frac{\partial B_2}{\partial x_1} \bigg|_{x_1=1} = 0, \quad \text{respectively.} \]

On the edge adjacent to face \( #5 \):

\[ \frac{-\partial B_2}{\partial x_3} = -\frac{\partial B(n)}{\partial \eta} \bigg|_{#5:#4} = 1/8 + \frac{37}{96} \sqrt{2} \]

On the edge adjacent to face \( #6 \):

\[ \frac{\partial B_2}{\partial x_3} = - \int_{#4} V(n) \, d\alpha - \frac{\partial B(n)}{\partial \eta} \bigg|_{#4:#5} = -3/8 - \frac{7}{96} \sqrt{2} \]

The differential equation on this face is

\[ \frac{\partial^2 B_2}{(\partial x_1)^2} + \frac{\partial^2 B_2}{(\partial x_3)^2} = - V_2(x_2=1) \]

Now that the differential equation and the boundary conditions have been established, \( B(n) \) may be computed on each face. Equation (4.5) is an elliptic partial differential equation so a successive over-relaxation (SOR) or an alternating direction implicit (ADI) method may be used to obtain an approximate finite difference solution. The boundary conditions are of the Neumann type so (4.6) is a necessary condition for the existence of a solution to (4.5).
The existence of a solution to the algebraic finite difference equation also must be established. First, a lemma and a theorem will be stated.

**Lemma:** An $n \times n$ matrix whose row sums are all equal to zero is singular.

**Theorem:** A solution to $Ax = f$ exists if and only if all solutions $z$ to $A^Tz = 0$ are orthogonal to $f$ [14].

In the theorem $A$ is an $n \times n$ matrix, $A^T$ is the transpose of $A$, and $x$, $z$, and $f$ are vectors.

Because of the Neumann boundary conditions the coefficient matrix of the finite difference equation for (3.1) has all of its row sums equal to zero and by the lemma it is singular. Let the matrix $A$ in the theorem represent the coefficient matrix. Since $A$ is singular, $A^T$ is also singular and a nontrivial solution $z$ exists such that $A^Tz = 0$. If a finite difference lattice is defined with equal spacing and the boundary points are included in the set of points satisfying the finite difference equations, then the vector $z$ is given by $z_i = 1/4$ for each $i$ that corresponds to a corner point, $z_i = 1/2$ for each $i$ that corresponds to a boundary point but not a corner, and $z_i = 1$ for all interior points. Let $f$ be the right side of the finite difference equations. In general $z$ is not orthogonal to $f$, i.e. $z^Tf \neq 0$, because of the discretization the algebraic analog of (3.2) is not satisfied. $f$ may be adjusted by making small changes in the flux or boundary conditions until $z^Tf$ is reduced to the order of the round-off error of the computer.
A simple but effective way to make the adjustment in the boundary condition is to multiply $\frac{\partial B(n)}{\partial \eta}$ by the factor

$$1 + \frac{z_T f}{\int_C \frac{\partial B(n)}{\partial \eta} \, ds}$$

The boundary conditions were adjusted until $z_T f$ was less than $10^{-6}$. It is true that adjusting the boundary conditions destroys the continuity of $\frac{\partial B(n)}{\partial \eta}$ but the change is only of the order $10^{-3}$ and it is much simpler to change the boundary conditions than it is to change the flux on the face.

Once the $f$ was adjusted, the finite difference equations were solved by using an ADI method [15]. The convergence was accelerated by solving the equation

$$\nabla^2_B(n) - \sigma B(n) = -v(n)$$

and taking the limit as $\sigma \to 0$ as the iteration proceeded.

The tangential projection of the vector potential on the faces of the cube, $\vec{A}_s$, is calculated by taking the curl of $\vec{B}$. The boundary condition on the normal component of the vector potential is calculated from (II-4.2). For this problem the normal derivative of the normal component is zero.

The vorticity boundary conditions are computed by taking the curl of the velocity on the faces. Since the curl involves a normal derivative of velocity which is not known a priori, the vorticity boundary condition must be calculated at each time step.
III-4. Computation of the Vector Potential and Vorticity in the Interior of the Cube

The vorticity and vector potential are computed in the interior of the cube with the finite difference analogs of (II-2.1) and (II-2.2) respectively. The vorticity equation is parabolic so it may be solved by using an alternating direction explicit (ADE) [16] or an alternating direction implicit (ADI) [15] procedure. The vector potential equation is elliptic so unless very fine grid spacing is used the SOR procedure is the best method.

In this present study only the steady state solution was of interest so not much care was taken to insure an accurate transient solution. The computational procedure used was as follows:

(1) Specify initial conditions on the vorticity and velocity.
(2) Advance the vorticity one step in time.
(3) Compute the vector potential.
(4) From the vector potential compute the velocity
(5) Compute the vorticity boundary conditions.
(6) Return to step (2).

When there was no significant change in the vorticity the solution was accepted as the steady state solution.

III-5. Results

The computed velocity was compared with the exact velocity given by (2.1a), (2.1b), and (2.1c). The average values of the magnitude of the velocity and the magnitude of the difference between the computed
and exact velocities were 1.24 and $2.33 \times 10^{-3}$ respectively, which is a relative error of $1.88 \times 10^{-3}$. This value of the error may be less than that which would be expected in more complex problems because in this problem the velocity profile is quadratic and thus results in a smaller discretization error than would be obtained in general.

The vector potential is solenoidal in principle if an exact solution were obtained. Hence, the divergence of the vector potential may be of value in indicating the discretization error. The average magnitude of the computed vector potential is $4.05 \times 10^{-1}$, and the average magnitude of divergence of the vector potential is $1.60 \times 10^{-3}$. More meaningful may be a comparison of the average magnitude of $\nabla (\nabla \cdot A)$ with the average magnitude of the vorticity. The values are $1.33 \times 10^{-2}$ and $6.76 \times 10^{-1}$ respectively.

The example was solved using a $41 \times 41$ grid on the faces for the computation of the vector potential boundary conditions and a $11 \times 11 \times 11$ grid for the computation in the interior of the cube.
CHAPTER IV

IV. Conclusions

The theory presented here gives a general method for determining the boundary conditions on the vector potential required to imply a specified velocity distribution on the boundary. The method makes possible the formulation of three-dimensional problems in a way well suited for solution using digital computers.

The boundary conditions take a relatively simple form for several special cases, especially those involving solid surfaces. In the more general case of non-zero normal component of velocity on the boundary, a second order elliptic differential equation must be solved in two dimensions on the boundary, before one can attack the three-dimensional problem of primary interest. In the example given here, a numerical procedure was used to find the solution to the elliptic problem on the boundary. As is well known, methods of solving two-dimensional elliptic problems are highly developed.

It should be emphasized again that this approach using the vector potential seems to be the only highly promising approach to three-dimensional problems of the general type considered here. There is, of course, considerable research effort on new methods for this class of problems.
REFERENCES


\[ v_{MAX} = \sqrt{2} \]

**Figure 1**

Velocity Profile Between Parallel Plates
FIGURE 2

PLANE $X_3 = 0$ COINCIDES WITH PLATE

$V_1 = V_2 = 1 - (1 - X_3)^2$
FIGURE 3
TRANSFORMATION FROM O123 TO O1'2'3'
FIGURE 4

TRANSFORMATION FROM $01'2'3'$ TO $01''2''3''$
FIGURE 5
TRANSFORMATION FROM 0123 TO 01''2''3''