CAPITAL BUDGETING MODELS BASED
ON CONSTRAINED PROFIT MAXIMIZING
AND NON-PROFIT MAXIMIZING BEHAVIOR

By

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ABSTRACT

CAPITAL BUDGETING MODELS BASED ON CONSTRAINED PROFIT MAXIMIZING AND NON-PROFIT MAXIMIZING BEHAVIOR

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Although economic literature traditionally assumes maximization of total profits to be the only goal of the firm, recently some alternative entrepreneurial motivations have received much attention.

Nevertheless, in the theory of investment practically all the existing models, positive and normative, are based on profit maximization.

Attempting to fill this gap this dissertation develops a series of capital budgeting models based on a few of the best known of these alternative objectives. Since the purpose was not to construct new goals or to argue for the plausibility of some, the goals and motives of the firm were taken as given. Four different goals were used. Two are constrained profit maximizing, one is revenue maximization, and the fourth is the maximization of the manager's own utility.

The capital budgeting problem, broadly interpreted, requires three interrelated decisions: first, the determination of the total volume of investment the firm will undertake,
second, the optimal financing of the proposed investment outlay and third, the selection of the optimal combination of investment projects. These sub-problems are successively analyzed (respectively in Chapters III, IV.1 and IV.2), and a model to solve each for the four different objectives is constructed in each chapter. Optimality conditions, optimality rules and micro investment equations corresponding to the various goals are derived.

The thesis actually begins in Chapter II with a review of well known capital budgeting models which are similar in that all conclude investment should (will) be carried on by the firm until the marginal rate of return equals the marginal cost of capital. But since all define the cost of capital differently the models in fact lead to quite different investment behavior. It is shown that the differences in opinion about the "correct" definition of the cost of capital are due to different interpretation of the profit maximization hypothesis. This chapter serves a threefold purpose. First, it serves as an introduction by presenting some traditional capital budgeting models against which the models of Chapter III and IV can be compared. Second, it helps to clear a confusing discussion that has been going on for years in financial literature by showing that each model is internally
consistent and that differences in outcome are caused by differences in initial assumptions as to what constitutes the goal of the firm. Third, by showing how even minor variations on the profit maximization theme can cause quite different investment behavior, it demonstrates the need for models integrating more radically different objectives with the theory of investment.

The model used in Chapter III is a modified and generalized version of the so-called "wealth-model". The unrealistic and special behavioral assumption of that model is replaced by the more basic "balance-equality constraint" and the resulting model is made to accept the various non-profit maximizing goals. The problem in Chapter IV.1 is structured so that it can be solved by existing programming methods. Although the mathematical technique is known, the application to the financing problem in this form is quite new. In Chapter IV.2 existing programming methods for determining the optimal project mix under profit maximization are proven to be applicable to most of the other objectives as well.
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Siert Edward de Jong
CHAPTER I
INTRODUCTION

I.1. Background of the Problem

"Classical" capital budgeting literature, in general, analyzes the actual investment behavior (in positive models) and the optimal investment behavior (in normative models) of the firm under the following set of assumptions:¹

a. the only goal of the firm is the maximization of total profits;

b. the market for capital (and usually all other markets too) are perfect;

c. there is no uncertainty, future conditions are perfectly known;

d. the investment projects are independent of each other in that the acceptance of one project does not influence the return on any other project;

e. in the "rate of return-method" intermediate cash flows will be reinvested at a rate of return equal to the return of the project itself;

f. in the "present value-method" intermediate cash flows are reinvested at a rate of return equal to the

¹These assumptions are seldom explicitly listed. Indirect references can be found in the works cited in the next following footnotes.
discount rate employed;
g. the firm sets no exogenous limits on the amounts it wants to borrow, lend or invest.

Under these conditions the optimal investment behavior can be described in two different ways.

Using the "rate of return-method" the firm will accept all investment projects with an internal rate of return higher than the market rate of interest and it will reject those proposals for which the rate of interest exceeds the expected rate of return. The optimal position is reached at the point where the marginal rate of return equals the rate of interest. Here the internal rate of return is defined as the discount rate which equates the algebraic sum of the discounted future yearly net incomes of a capital good with the initial price of the capital good. Let the rate of interest be represented by \( \pi_L \) and the internal rate of return of the marginal project \( j \) by \( \pi_j \). Let \( w_{tj} \) be the net income of project \( j \) in year \( t \) \( (t=1, \ldots, T) \) and let the outlay for \( j \) be represented by \( P_{oj} \). Then the optimal position for the firm, given assumptions a - g, is given by:

\[
\pi_L = \pi_j \quad (1.1)
\]

\[
P_{oj} = \sum_{t=1}^{T} \frac{w_{tj}}{(1+\pi_j)^t} \quad (1.2)
\]
Under the alternative *present value-method* the firm will accept all investment proposals for which the present value—defined as the algebraic sum of discounted future yearly incomes of a capital good discounted at the market rate of interest—exceeds the initial outlay for the capital good and it will reject all proposals for which the opposite is true. The optimal point is reached if, for the marginal project, the present value equals the initial cost. Let $D_{oj}$ represent the present value of marginal project $j$, then the optimal position is given by:

$$P_{oj} = D_{oj} \quad (1.3)$$

$$D_{oj} = \sum_{t=1}^{T} \frac{w_i}{(1+\pi L)^t} \quad (1.4)$$

Under assumptions a - g the "rate of return-method" and the "present value-method" are equivalent as can be seen by comparing equations (1.3) and (1.4) with (1.1) and (1.2). The assumptions can be considered rather unrealistic and recent capital budgeting literature shows many attempts to replace them.

For instance imperfections of the capital market (assumption b) were introduced by a number of writers, i. e., Hirshleifer,¹

Lutz and Lutz. 1

Models in which the return on investment is considered a stochastic variable have been developed to give an optimum solution for the case of risk (assumption c). 2

Interdependency between investment projects (d) is treated by Lorie and Savage, 3 Reiter 4 and Weingartner. 5

Other writers 6 have shown the essential nature of assumptions e and f and have developed conditions under which the rate of return method and the present value method are not equivalent.

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Assumption g—no capital rationing—has also been attacked as being unrealistic. The capital budgeting problem—broadly interpreted—requires three interrelated decisions. First, the total volume of investment a firm will undertake. Second, the selection of the optimal combination of investment projects and third the optimal financing of the proposed investment outlay. In the traditional pure profit maximization model, as explained above, total volume of investment is determined by the interplay of rate of return on investment and the cost of the capital used to finance this investment. The selected project-mix determines the rate of return and the selected financial structure determines the cost of capital. A rational application of the profit maximization principle requires therefore the simultaneous solution of these three problems.

In actual business practice the three problems are separately analyzed and solved, very often by different people or different groups in the organizational structure. First,

---


total volume of investment is determined by top management on the base of rough estimates about probably attainable rates of return and presumably available financing opportunities and often with other—non profit maximizing—motives in mind. After that the problems of the optimal financing and the optimal project-mix are solved by lower echelons in the organization who have in their decisions to accept total investment as given.

The reason for this—from a standpoint of pure profit maximization seemingly irrational—behavior could be a lack of detailed data, sufficient decision time or adequate analytical facilities in the top levels of the firm. Top management wants to keep the decision about total volume of investment to itself but lacks necessary detailed information about specific projects or specific financing opportunities. Therefore a rough investment budget is constructed and later split up between the various departments. In the solution of the problems of the composition of expenditures and the composition of financing the size of the budget is then considered as given.

But not only organizational reasons account for this behavior. Sometimes capital market imperfections or other external constraints limit the amount a firm can invest to a given sum. In government investments it is the political process that accounts for given budgets.
For these reasons recent publications on the selection of the optimal combination of projects have dropped assumption g and have started from a given investment budget.  

So far however all writers in the investment field have upheld the profit maximization assumption (a) which—especially in the long-run investment decision—might be a very unrealistic one indeed. In other fields of economic theory alternative assumptions of entrepreneurial behavior are being advanced and theories built upon them have been developed by many writers, notably Baumol, Williamson, Simon, Marris, Scitovsky, etc. Nevertheless in the important area of capital

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budgeting nearly all models, positive as well as normative, are still based on profit maximization and alternative considerations, e.g. that managers might want to maximize sales or are interested in their own utility or are constrained in the attainment of the profit maximum by the existence of financial constraints, have received no attention.

In an attempt to fill this gap this thesis develops a number of capital budgeting models based upon some alternative hypotheses about other managerial goals that have been advanced in the above literature. The models and their resulting optimality rules are compared with each other and with the profit maximization model under identical circumstances. These comparisons give valuable insight into the question of which investment behavior follows from each goal and why.
I.2. Plan of the Study

In the capital budgeting literature there has been a confusing discussion centered around the interpretation of the concept of the cost of capital for the firm in imperfect capital markets. Different writers have proposed different interpretations and the discussion has not been satisfactorily solved. Chapter II reviews this discussion from a special standpoint. It is shown the differences in outcome exist because different writers start from different assumptions about the goal of the firm. Although these goals can all be considered variations on the profit maximization theme and as such do not give the kind of alternatives to profit maximization as the models developed in this dissertation, the differences are large enough to cause different "cut-off" points and therefore different investment behavior. As soon as one realizes the various writers explicitly or implicitly select different goals for the firm, the confusion ends and many comments and re-comments prove to be not relevant.

Chapter III is devoted to the decision about the total volume of investment. Four alternatives to single profit maximization are successively introduced and integrated in the theory of investment of the firm.

In a complete treatment of the investment problem under non-profit maximization objectives it will also be necessary to
show how firms can solve the project mix and the financing problem in those circumstances. Therefore Chapter IV.1 gives a new solution for the determination of the optimal financial structure of the investment budget while Chapter IV.2 discusses methods to determine the optimum mix of projects. In accordance with the argument above, throughout Chapter IV the total size of the investment budget is considered constant.
CHAPTER II
VARIATIONS ON PROFIT MAXIMIZATION
IN INVESTMENT MODELS: A REVIEW

This chapter gives a review of a number of recent micro-investment models. These are similar since all agree the firm should invest up to the point where the marginal rate of return on investment equals a "cut-off" rate or "cost of capital." However, since each model gives a different definition of this cost of capital, the various models describe (or prescribe in normative approaches) basically different investment behavior.

A confusing discussion has resulted in theoretical as well as "practical" literature about which is the "correct" definition, an issue which has not been satisfactorily settled. Much of this confusion disappears in a close and critical analysis and comparison of the structure of the different models especially with respect to the goals explicitly or implicitly imputed to the decision maker in the investment problem. This will be done in the following paragraphs. To show the argument most clearly, the case of pure stock financing will be carefully analyzed.
II.1. The Dividend-Price Ratio

In a static world with certainty, universal perfect competition and general profit maximization a firm using the rate of return-method or the present value-method would maximize total profits but the long-run maximum would just be equal to a "normal" return on investment. Assuming the market value of the firm's shares to be a function of total profits, market values would also be maximum. The firm would just earn and have to pay its stockholders an effective dividend equal to the market rate of interest.

In the investment criteria of Chapter I the interest rate \( \pi_L \) could be replaced by the dividend-price ratio of its stock or by the earnings-price ratio since all three will be equal in such circumstances.

Let \( w_{ts} \) represent net income from investments (before dividend payments) per share, \( R'_{ts} \) total dividends per share, \( P_{ts} \) share price, \( \alpha \) the dividend-price ratio, \( \beta \) the earnings-price ratio and \( \epsilon \) the retention rate. Then the optimum position will be reached when:

\[
\pi_j = \pi_L = \alpha = \beta
\]  \hspace{1cm} (2.1)

where

\[
\alpha = \frac{R'_{ts}}{P_{ts}} \hspace{1cm} (2.2)
\]

\[
\beta = \frac{w_{ts}}{P_{ts}} \hspace{1cm} (2.3)
\]
Thus \[ \varepsilon = 1 - \frac{\gamma}{\beta} = 0 \tag{2.4} \]

Stockholders' and management's interests would be parallel.

In reality the interest rate, the dividend-price ratio and the earnings-price ratio need not necessarily be equal and the cost of capital to use as cut-off rate in the case of pure equity financing is open to interpretation. Conflict of interest between stockholders and management is then possible.

Some writers have chosen the dividend-price ratio\(^1\) and an optimum investment policy is in their opinion to accept all investment proposals for which the return exceeds \(\alpha\) and to reject all proposals with a return lower than \(\alpha\) so that for the marginal investment \(j\):

\[ \pi_j = \alpha , \text{ when } \tag{2.5} \]
\[ \alpha \text{ is not necessarily equal to } \beta \tag{2.6} \]

Explicit reasoning is usually not given or is based on the analogy with the case of static perfect long run competition. The opinion is consistent with the view that the firm is an entity different from the stockholders. Stockholders are considered by management only as suppliers of financial capital. Dividends are the costs of this capital and the

\(^{1}\text{E.g.: J. B. Williams, The Theory of Investment Value, (Cambridge: Harvard University Press, 1938), pp. 57 a.f.}\)
dividend-price ratio represents the marginal cost. Total profits (after dividends) will be maximized if investment is carried on until the marginal rate of return equals the dividend-price ratio. The view implies an independent management which is actively interested in maximum cash profits (for instance because management's income is related to total profits via a profit sharing agreement) and if the earnings-price ratio exceeds the dividend-price ratio, the policy may result in accepting some investments with a return lower than the current earnings-price ratio. As a result, earnings per share may be diluted; thus the policy is clearly not in the best interests of the stockholders.
II.2. The Earnings-Price Ratio

Many writers have selected the earnings-price ratio as "the" cost of capital in the case of stock financing. ¹ This view can be seen as a logical result of identifying firm and stockholders. If the marginal return on new investment equals the current earnings-price ratio, then the average return on new investment which is at least as large as the marginal return (assuming a downward sloping average return-function) will also be at least as large as the current ratio. But since for new investment equals the average return on investment, the new ratio will be at least as large as the old ratio and this guarantees the interests of the current stockholders. The earnings-price ratio and earnings per share will be maximized by this policy, (earnings before dividend), assuming constant share-prices throughout this process.

¹The view that equity costs are measured by earnings-price ratios can be found by many writers, i.a. Hunt and Williams, Basic Business Finance, (Homewood, Ill.: R.D. Irwin, 1958); M.H. Spencer and L.S. Siegelman, Managerial Economics, (Homewood, Ill.: R.D. Irwin, 1959); See also R.P. Soule, "Trends in the Cost of Capital," Harvard Business Review, (March-April 1953) pp. 33-67.

²For new investment equals:

\[
\frac{\text{total earnings}}{\text{number of shares}} \div \text{price per share}
\]

\[
= \frac{\text{average return on investment times total amount invested}}{\text{total amount invested divided by price per share}} \div \text{price per share}
\]

\[
= \text{average return on investment.}
\]
II.3. The Required Rate of Return

D. Durand, writing in 1952,\(^1\) defends an interpretation of the cost of capital called the "required rate of return," R.R., defined as the return an investment has to make in order to keep stock prices constant. Durand states that the objective in investment should be the maximization of discounted future net income of stocks, called the investment value. Later on, investment value is identified with the market value of stocks and supposedly arrived at by multiplying earnings per share with a capitalization factor C. Durand's main argument is that C is a decreasing function of the amount of financial leverage (ratio of debt to equity in the financial structure) which is a proxy for financial risk. Thus he is able to show that the R.R. is higher than the interest rate on bonds if an investment is financed with bonds only. If the marginal rate of return on investment is equal to the bond interest the average return exceeds the bond interest so that net earnings per share increase. This in itself will increase the market value but the capitalization rate C decreases because of increased leverage and it is very well possible (depending on the exact

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relation between C and leverage) that the market value of stocks actually decreases. The investment should therefore be cut off earlier at a higher marginal return, hence the cut-off rate or R.R. exceeds the interest rate on bonds. Actually this is only a roundabout way of saying that if the objective of investment is to maximize the value of the common stock and if the common stock value is a function of the degree of financial leverage, then the cost of capital determining the optimal amount of investment should also be a function of the degree of leverage.

In order to focus attention on the relation between the definition of cost of capital chosen and the objective of investment implied, this chapter will refrain from discussing the complications introduced by debt financing and uncertainty. In the case of stock financing Durand's capitalization rate C turns out to be the inverse of the average return on new investment, so that the valuation theory is really the definitional relationship established in footnote 2, page 15.
II.4. The Future Earnings-Price Ratio

Professor Solomon\(^1\) proposes a cost of capital for equity financing that is defined as the estimated earnings-price ratio which would exist in the absence of the proposed investment as opposed to the current earnings-price ratio. The argument is that:

"... The relevant fact with regard to any item of business policy is not the difference between net earnings after the policy item is introduced and present earnings. Rather it is the difference between net earnings after the policy item is introduced and what earnings would have been had the policy item not been introduced."\(^2\)

The argument is undoubtedly true but it is not relevant. In the static, \textit{ceteris paribus} analysis with no uncertainty it is difficult to see how the future earnings-price ratio will differ from the present one in the absence of new investments. Thus Solomon's cost of capital is really the earnings-price ratio, whose acceptance will lead to the maximization of earnings per share and average return on investment.

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\(^2\)Ibid., p. 244.
II.5. The Lending Rate and the Borrowing Rate

H. Roberts, in imitation of F. and V. Lutz, makes a distinction between two different concepts of the cost of capital. The first is the "borrowing rate," the rate a particular firm actually has to pay for the funds it acquires. The rate is dependent upon the risk and other special circumstances of the firm, usually varies with the amount acquired, and constitutes the actual cost of capital to the firm. The second rate is the "lending rate." This is an opportunity cost concept, the rate one could earn by investing in the market in a similar firm in similar circumstances. This rate is independent of the acts of the firm under consideration.

According to Roberts the cost of capital to the firm must always be the last one if the value of the stockholders' capital (defined as the net present value of future cash streams) is to be maximized, since the lending rate represents the sacrifice the firm makes by investing in its own plant. All the projects promising a return higher than the lending will be accepted and projects with a return lower than this rate will be rejected since the firm could then better invest its money in the market.

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2. Lutz and Lutz, op. cit., p. 22. See also Hirshleifer, op. cit.
If the availability of funds is not restricted and if the lending rate is above the borrowing rate, this policy will indeed maximize profits and stock value. If the borrowing rate and the lending rate are independent of the amount borrowed by the firm the maximum is even infinite since by borrowing at the (low) borrowing rate and lending at the (high) lending rate profits increase indefinitely. To exhaust all potential profit possibilities it is thus necessary for the firm to invest in its own programs as long as the marginal return exceeds the lending rate and to borrow and lend again outside as long as the return (lending rate) exceeds the borrowing rate. This behavior is not often observed in actuality however; firms are hesitant in investing outside their own business.

If the borrowing rate exceeds the lending rate, the most obvious occurrence, Roberts' rule fails to maximize either profits or stock value. The policy of accepting all proposals with a return higher than the lending rate will result in accepting some proposals with a return lower than the borrowing rate. If the money is available investing in the firm is still better than not investing or investing outside, but it certainly does not pay to acquire additional money at the (high) borrowing rate and then invest it at returns lower than this rate. Those projects would mean losses and borrowing money for them is not justified. They should be rejected. Only for available retained
earnings the lending rate is still to be considered as the cost of capital and the above policy will minimize losses for this category.

Instead of a general rule for investing and financing, taking the lending rate as the cut-off point below which no funds will be invested but above which every investment will be accepted can be best considered as a criterion for a retention policy pursuing the best interests of the stockholders. It is not applicable to a bond financing for instance and will not always maximize profits or stock values.
II.6. The Required Rate of Profit

In an article written with E. Shapiro\(^1\) and in his book\(^2\) M. J. Gordon proposed as cost of capital the "required rate of profit," \(k\). In his basic model with no debt and no new stock issues the problem is to determine the optimal retention rate \(e^*\); that is, the retention rate which will maximize stock values. Under the assumption that a corporation earns a constant \(\pi_A\) per cent on its book value and retains a constant fraction \(e\) of its earnings it follows from the stock valuation theory accepted that \(k\) equals the dividend-price ratio \(\alpha\) plus a growth factor equal to \(e\pi_A^*\).

The valuation theory proposed is that the value of a share is equal to the discounted present value of the future, steadily increasing, dividend stream, discounted at \(k\). Using the previously employed notation:

\[
\begin{align*}
\omega_0 &= \omega_0 (1 + \pi_A e) \\
\omega_t &= \omega_0 (1 + \frac{\pi_A e t}{t}) \\
\omega_T &= \omega_0 e^{\pi_A t}
\end{align*}
\]

\[(2.7)\]


\[ R_{ts} = \omega_{ts} (1-\epsilon) = \omega_{os} e^{\pi_{A} \cdot e \cdot t} \]  
(2.8)

\[ P_{os} = \int_{0}^{\infty} R_{ts} e^{-kt} dt = \int_{0}^{\infty} \omega_{os} e^{\pi_{A} \cdot e \cdot t} (1-\epsilon) e^{-kt} dt \]

\[ = \omega_{os} (1-\epsilon) \int_{0}^{\infty} e^{-t(k-\pi_{A} \cdot e)} dt \]  
(2.9)

Assuming the price of a share to be finite \( k > \pi_{A} \cdot e \)
and integration gives:

\[ P_{os} = \frac{(1-\epsilon)\omega_{os}}{k-\pi_{A} \cdot e} \]  
(2.10)

from which it follows that the required rate of profit

\[ k = \frac{R_{os}^t}{P_{os}} + \pi_{A} \cdot e = \alpha + \pi_{A} \cdot e \]  
(2.11)

Assuming \( \pi_{A} \), \( k \) and \( \omega_{os} \) to be independent of \( e \) and setting
the derivative of \( P_{os} \) with respect to \( e \) equal to zero results in

\[ \frac{\partial P_{os}}{\partial \epsilon} = \frac{\omega_{os}}{(k-\pi_{A} \cdot e)^2} \cdot (\pi_{A} \cdot k) = 0 \]  
(2.12)

Hence for \( \pi_{A} > k \) price \( P \) increases if \( \epsilon \) increases but for \( \pi_{A} < k \) \( P \) decreases if \( \epsilon \) increases. The second derivative which needs to be negative for a maximum disappears for \( \pi_{A} = k \) so it is not correct to say that \( P \) is maximized for \( \pi_{A} = k \).

\[ \text{1See David Durand, "Growth Stocks and the St. Petersburg Paradox," Journal of Finance (September 1957), pp. 348-363.} \]
Economically the conclusion is that a corporation should retain all of its income or liquidate depending on whether $\pi_A > k$. As Gordon explains this result stems from the assumed independence of $\pi_A$ and $\varepsilon$. \textit{A Priori} more realistic is to write $\pi_A$ as a downward sloping function of $\varepsilon$. But if this is done maximum share prices require $\pi_A > k$. This can be seen by introducing the equation $\pi_A = c_1 - c_2 \cdot \varepsilon$ ($c_1$ and $c_2$ are constants) as a constraint to the maximization problem as done by Lerner and Carleton\textsuperscript{1} in a recent article or simply by differentiating $P$ with respect to $\varepsilon$ from eq. (2.10) and assuming $\pi_A$ and $\varepsilon$ to be related. Following this last way results in

$$\frac{\partial P}{\partial \varepsilon} = \frac{\partial \pi_A}{\partial \varepsilon} \cdot [\pi_A - k + \varepsilon (1 - \varepsilon) \frac{\partial \pi_A}{\partial \varepsilon}] = 0$$

Thus

$$\pi_A = k - \varepsilon (1 - \varepsilon) \frac{\partial \pi_A}{\partial \varepsilon}$$

(2.13)

Since $\frac{\partial \pi_A}{\partial \varepsilon} < 0$, $\pi_A > k$.

At the same time, as shown by D. Vickers\textsuperscript{2} the marginal rate of return $\pi_A$, if defined as $\frac{d(\pi_A \cdot \varepsilon)}{d \varepsilon}$ will be smaller than


k at the point where the stockholders' wealth (stock-price) is maximum in the sense that the present value of the future stream of dividends resulting from the investment of a marginal unit of funds just equals that unit of funds.

The importance of this in the present context is that if income is growing at a constant growth rate as indicated and if the stock prices are a function of dividends in the way assumed above, a firm trying to maximize stock prices should not invest to the point where the average return equals k but should stop short of that investment volume so that in the maximum actually exceeds k.

In the model used above with no debt and no stock issues, no out-of-pocket cost is attached to retained earnings. If the manager is totally indifferent to the interests of the stockholders he can increase total cash profits earned inside the firm by retaining profits and investing these as long as the marginal return exceeds zero. Of course, this is neglecting the possibility of investing outside the firm in the stock market where the firm could make at least the return k. Maximum profits, including outside investments will thus be when marginal return equals k. Under the assumptions of the present section this point of maximum short run profits will not coincide with maximum stock prices.
II.7. Some Rules of Thumb

In actual business practice some rules of thumb—criteria for choice, not based upon an explicit theory and not proven to be successful in reaching the objective, but simple to apply and hence appealing to the businessman—have developed as guides to the determination of the investment budget. Some of these can be considered approximations to optimality rules and are intended to maximize profits, market values or average return on investment.

A rule of this type is to accept all investments expected to bring a return higher than the current average return on book value. The policy will result in increasing average return on book value during the years. It was probably intended to be a rule for maximizing average return on investment but since actual value of investment is open to interpretation book value is used as an approximation. Because of "built-in" conservatism and liberalism in determining book values average return to book value is usually different from average return to investment measured at market prices. Using current average return to book values as cut off point will mostly lead to a different

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investment volume than corresponds to maximum profits—or maximum market price investment.

Other rules of thumb reflect quite different goals. Although the use of a "minimum pay-out period"\(^1\) may be considered a rough approximation to profit maximization,\(^2\) it is more probable that this rule is followed to protect against risks by insisting on a large liquidity position for the firm.\(^3\)

Other examples are the "rules" of never borrowing at all, limiting investment to retained earnings or paying constant dividends. These reflect the wish to avoid interference in company policy and practice from stockholders and outsiders. These objectives will be more fully investigated in the next chapter.

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1 About the use of the "pay back period" see Istvan, op. cit. pp. 87-94.


II.8 A Look Ahead

This chapter has reviewed some recent capital budgeting models and rules proposed in literature and followed in business practice. At the same time some light is thrown on the cost of capital controversy. It has been shown that the different conceptions of the cost of capital reflect different goals in the area of investment. Although additional assumptions made for each model (like the valuation theory accepted) have to be considered also, the following rough conclusions can be drawn.

Using the dividend-price ratio as cut-off point for investment reflects the goal of maximizing total profits after dividend payments. Using the earnings-price ratio reflects maximizing average returns. In the case of one special valuation theory the last ratio is also proposed as the cost of capital consistent with the objective of maximizing market value of common stock.

Applying always the lending rate will only under limited conditions maximize profits for the firm and is mainly meant to be a guide for retention policy. The required rate of profit will under the assumptions of the growth model proposed determine the investment rate that maximizes stock prices but only if the rate of return is independent of the amount invested. If this
last condition is dropped the point of maximum market value will not be reached if the required rate of profit is used as cut-off rate.

The rules of thumb included in the analysis appeared to have the aim of respectively maximizing return on book value, preserving a high liquidity and keeping control over the corporation.

Although most implied goals can be considered variants on the profit maximization objective it is important to note that the difference of opinion exists and that even minor variations in the selection of the objective can lead to quite varied investment behavior.

In the next chapter four other managerial objectives will be discussed and introduced into formal investment models. For each model optimality conditions will be derived and investment equations will be constructed.
CHAPTER III

THE DETERMINATION OF TOTAL INVESTMENT VOLUME
UNDER ASSUMPTIONS OF CONSTRAINED PROFIT MAXIMIZING
AND NON-PROFIT MAXIMIZING BEHAVIOR

The aim of this chapter is to analyze the consequences of introducing various non-profit maximizing objectives in the theory of investment. The question asked is: "What are the determinants of investment when the decision maker's goal is different from or includes more than the maximization of the total volume of profits?" The purpose is not to construct new goals or to argue for the plausibility of some hitherto undiscovered special behavior. The primary interest is the theory of investment, specifically what it appears to be when the decision maker's aim is to maximize profits and what it is if he has other goals.

Most of the goals assumed here have been introduced in general economic theory one way or another and arguments for their plausibility can be found in the literature. It is not claimed here that the behavior actually assumed is the behavior predominantly observed or that the objectives assumed include all goals possible. Obviously, only testing can validate the former and it would be very foolish indeed to claim the latter. The aim here is limited: analyzing the investment decision
following from some a priori plausible assumptions as to what the managers in the area of investment might want.

Most traditional investment models such as the ones discussed in Chapter I and II, start with a given relation, (called the internal rate of return function), between investment expenditures and net return (profit before deduction of capital costs). Then, assuming profit maximization to be the sole objective, it can be shown the optimal amount of investment expenditure lies where the marginal rate of return equals the marginal cost of capital used to finance investment.

Because of this direct relation between investment and net return, variables like product prices, production rates, production costs, etc., do not appear explicitly in the analysis. This makes the traditional approach rather unsuited for introducing other goals than profit maximization, since alternative entrepreneurial objectives, e.g. maximization of total revenue, preference for certain expense classes like management salaries, expense accounts and staff, etc., require the explicit introduction of revenues and costs of all types.

A more fundamental objection can also be made against the "classical" approach. Investment in its proper meaning is a rate, the amount of funds invested per time period. Net return is basically a function of the total amount of capital invested—a stock variable—and not of the yearly additions to it in the
form of investment. Hence the point where marginal return equals the marginal cost of capital determines the optimal stock of capital, not the optimal flow of investment expenditures. Only by adding explicit assumptions about the relation between optimal capital stock and the time flow of additions to actual stock is it possible to derive an investment equation.

Another investment model, not containing the above mentioned flaws, was recently published by F. Hammer. Hammer introduced a so-called "wealth-model"; variables like product prices and product costs are explicitly introduced and directly related to the firm's stock of assets and liabilities. Working with profit maximization as the end towards which the investment behavior is directed, the optimality procedure results in an optimal stock of assets. Desired investment is thus the

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difference between optimal and actual stock of assets (assets conveniently equalized with capital). Using the heuristic that actual investment will be a linear function of desired investment an investment equation can be constructed.

Following is the development of a modified and generalized version of this wealth-model. The "asset-model," as the generalized version will be called, will be used to introduce variations on the profit maximization and non-profit maximization goals in the theory of investment of the firm.
III.1. The Basic Model

III.1.1. The Wealth Model

In the wealth-model the firm is considered having a portfolio of stocks of assets and stocks of liabilities. In Hammer's notation: a is the stock of assets and l the stock of liabilities. The analysis is static and long-run. As a result a liability unit may be regarded as a consol. Thus the prices of a and l, respectively $P_a$ and $P_l$, equal the capitalized value of the income stream generated by the units:

$$
P_l = \frac{r_l}{\rho_1} \quad \text{and} \quad P_a = \frac{r_a}{\rho_a}
$$

where $r_a$ and $r_l$ represent the nominal yield measured in dollars per unit per time and $\rho_a$ and $\rho_l$ stand for the true yield, measured in dollars per dollar per time. $P_a$ and $P_l$ are assumed constant.

Wealth or net worth of the firm is assumed constant. Thus

$$
P_a \cdot a - P_l \cdot l = w \quad \text{is constant} = P_a \cdot a_o - P_l \cdot l_o
$$

Unlimited availability of inputs, constant coefficients and constant returns to scale are assumed. Under these conditions production can be written as:

$$
Q = k_2 \cdot a \quad (k_2 = \text{constant})
$$

\(^1\text{Ibid.}, \text{pp. 27-36 and 43-47.}\)
and production cost as:

\[ C_c = k_1 k_2 a \quad (k_1 = \text{constant}) \]

Output price is \( P_d = P_d(Q) \).

Debt costs are \( C_r = r_1 l \).

Nominal rate of return on assets is profits before debt costs divided by assets, thus:

\[ r_a = \frac{P_d(Q) k_2 a - k_1 k_2 a}{a} = P_d(Q) k_2 - k_1 k_2 \]

Total profits are: \( r_a a - r_1 l \).

Maximum profits under the assumed condition that net worth is constant are derived by using the Lagrangian Multiplier Technique:

\[ \text{Max } L = r_a a - r_1 l - \lambda [P_a (a_0 - a) - P_1 (l_0 - l)] \]

The first order equilibrium conditions are:

\[ \frac{\partial L}{\partial a} = r_a + ar'_a - \lambda P_a = 0 \]

\[ \frac{\partial L}{\partial \lambda} = r_1 + l r'_1 - \lambda P_1 = 0 \]

\[ \frac{\partial L}{\partial \lambda} = P_a (a_0 - a) - P_1 (l_0 - l) = 0 \]

From these conditions the following equilibrium requirement results:

\[ \lambda = \frac{r_a + r'_a a}{P_a} = \frac{r_1 + l r'_1}{P_1} \]
And solving the conditions for the optimal stock of assets \( a^* \) results in
\[
a^* = \alpha_1 \frac{w}{p} + \beta (\rho - \rho_1)
\]
where
\[
\beta_1 = \left( \frac{\rho_1 - \rho}{\rho} \right)^{-1} \quad \alpha_1 = \beta_1 \rho_1
\]

Adding the assumptions that the amount of depreciation in period \( t \) is proportional to the actual stock of capital at the beginning of the period and that in period \( t \) only a fraction of desired investment (= the difference between the optimal stock and the actual stock of assets) is to be completed, an investment equation of the following form results:
\[
I_t = \omega_t + \beta (\rho_K - r)_t - \gamma (1-d)K_t
\]
where
\[
\omega_t \text{ is the real value of net worth}
\]
\[
\rho_K \text{ is the expected long-run return on capital}
\]
\[
r \text{ is the true rate of interest}
\]
\[
K_t \text{ is the actual stock of capital at the start of period } t
\]

This wealth-model has a number of deficiencies which will be analyzed in the next section.
II.1.2 The Wealth Constraint Reconsidered

Net worth of the firm is assumed constant in the "wealth" model. The only reason given is a reference to the static framework of the analysis but it is certainly not clear why a static analysis dictates a constant net worth.

The belief is widely held¹ that firms do want to maximize profits but are constrained in doing so by external or self-imposed constraints; the first are predominantly of a legal, social or institutional nature, and the second mostly moral, habitual or psychological. The character of the constraints depends also on the length of the time period considered. In the short run they are mainly of a technical nature like bottlenecks in production and in the availability of inputs. In the long run the constraints concern more the financial structure of the organization. Schematically speaking there are three categories of financial constraints: those concerning the owners capital, those concerning liabilities, and those concerning both.

In the analysis of investment—typically a long run decision—consideration should be given to the influence of these financial constraints. An investment model which assumes net worth to be

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constant is thus justifiable, but nevertheless special and limited. Constant net worth means that no stock issues are allowed and no earnings may be retained. It seems paradoxical in a model where maximum profits are assumed to rule out any additions to the owners' new worth beforehand.

To be sure, managers sometimes deliberately avoid floating stock and stockholders often protest any plan to retain part or all of the profits. Especially smaller firms—often if they use the incorporated form—do not use stock flotation as a regular source for financing investments. The funds, originally brought in by the owners are in the form of stocks but issuing more stocks is not considered at all. There may be various reasons behind this behavior. First there are institutional regulations: certain minimum amounts are necessary for a stock issue to be accepted by the regulations of the stock market. Also the firm has to have a certain size and the issue has to exceed certain minimum bounds to succeed in the market. Second the constraint may be self-imposed. Floating stock means introducing new owners, if not de facto, then at least de jure. Many times existing owners and management do not want interference from new members in the firm's policy and decisions. Thus the amount of stocks is kept constant. Third it is often believed that the uncertainty inherently connected with investment does not warrant
a permanent and irrevocable measure like a stock issue.

Thus it seems worthwhile to investigate the consequences of this special behavior for the theory of investment but only as one special case. Other financial constraints that limit profit maximizing as a goal as well as non-profit maximizing objectives should also be integrated in the theory of investment and for that purpose the wealth-model, being specially geared to constant net worth and allowing only one source to finance investment, is too limited.

However as demonstrated in the next section this wealth-model can be modified and generalized to accept equity and in principle any other source of financing. This generalized stock model can then be used to introduce other constrained profit maximizing goals as well as non-profit maximizing goals in the formal theory of investment of the firm. It appears then that the wealth-model is itself a special case of this generalized asset-model.
III.2. A More General Asset-Model

This asset-model works with the variables found in the balance sheet of the firm. The left side of a standard balance lists assets: cash and securities, accounts receivable, inventories outside investments, equipment and buildings. The right side of the T-account shows how the assets on the left side are financed: stocks, --common and preferred--, retained profits, bonds, bank credit, accounts payable and other liabilities. Schematically speaking it can be said that the left side total shows the value of the firm's assets (and these are for the majority to be considered capital goods) and the right side lists the sum of the values of the stock of liabilities and the stock of equity capital. These stocks are measured in standard units or real terms. In symbols:

- \( X_A \) is the stock of assets
- \( X_E \) is the stock of equity
- \( X_L \) is the stock of liabilities.

These can be bought and sold in the capital market at prices of \( P_A \), \( P_E \) and \( P_L \) respectively. Since every asset has to be financed one way or another a basic constraint for each firm is that

\[
P_A X_A = P_L X_L + P_E X_E \tag{3.1}
\]

This constraint reflects the accounting convention that the left side total of any balance sheet always equals the right
side total.

The effective yield (cost rate) \( \eta_i \) is related to the prices \( P_i \) by the nominal yield (cost rate) \( p_i \), in the following way:

\[
\eta_i = \frac{p_i}{P_i} \quad \text{for} \quad i = A, L, E
\]  \( (3.2) \)

For example in the case of stocks, \( P_E \) is the market price, \( p_E \) is the dividend per share and \( \pi_E \) is the return as a percentage. If in the analysis \( P_E \) is taken to be constant this means that stocks can be floated at constant value.

Since the analysis is micro and long run it is acceptable to assume unlimited availability of inputs especially labor and constant coefficients. Those conditions make that the rate of production \( Q \) can be written as a function of the stock of capital only:

\[
Q = Q(X_A)
\]  \( (3.3) \)

If in addition constant returns to scale are assumed the production function is linear:

\[
Q = a.X_A
\]  \( (3.4) \)

where \( a \) is a constant.

Long run unit production cost will then be constant too. Hence total production costs \( K_Q \) are proportional to output and to the stock of capital:

\[
K_Q = b.Q = b.a.X_A
\]  \( (3.5) \)
where $b$ is a constant.

The average revenue of output is assumed to be a function of the rate of production:

$$ P = P(Q) = P(aX_A) \quad (3.6) $$

Total financial costs are the sum of debt cost and equity costs:

$$ K_F = P_E X_E + P_L X_L \quad (3.7) $$

where the latter can be considered either as an "opportunity cost" or as a "normal" return to capital in the form of dividends.

Profits $\omega$ are:

$$ P(Q) a X_A - abX_A - P_E X_E - P_L X_L \quad (3.8) $$

or since $p_A$ equals per definition profits before financing costs divided by assets or

$$ p_A = \frac{P(Q) a X_A - abX_A}{X_A} = P(Q) a - ab, \quad (3.9) $$

profits can also be written as:

$$ \omega = p_A X_A - P_E X_E - P_L X_L \quad (3.10) $$
III.3. Profit Maximizing and the Asset-Model

III.3.1. Unconstrained Profit Maximizing

It will be instructive and will facilitate later comparisons if the traditional profit maximizing postulate is applied to the generalized model developed above. The traditional profit model does not recognize any behavioral constraints. Of course, the basic financing identity (3.1) must be included to assure that any asset acquired is also financed. Thus an increase in assets must be accompanied by a corresponding increase in liabilities or equity capital. As long as (3.1) is satisfied the firm can freely manipulate $X_A$, $X_L$ and $X_E$ in any direction and independent of each other. Hence maximizing profits implies maximizing equation (3.8) or (3.10) subject to equation (3.1). This can be done using the Lagrangian multiplier technique:

$$\max L = P(Q) a X_A - ab X_A - p L X_L - P X_E - \lambda [P X_A - P L X_L - P X_E]$$

Assuming the second order conditions for a maximum to be satisfied, the optimum is determined by setting the partial derivatives of $L$ with respect to the decision variables $X_A$, $X_E$ and $X_L$ and with respect to $\lambda$ equal to zero. This produces the following system of four equations in four unknowns:

$$\frac{\partial L}{\partial X_A} = P(Q) a + a X_A \cdot \frac{d(P(Q))}{dQ} a - ab - \lambda P_A = 0$$
\[
\frac{\partial L}{\partial X_L} = p_L + X_L p'_L - \lambda p_L = 0
\]

\[
\frac{\partial L}{\partial X_E} = p_E + X_E p'_E - \lambda p_E = 0
\]

\[
\frac{\partial L}{\partial \lambda} = p_A X_A - p_L X_L - p_E X_E = 0,
\]

When \( p_L \) and \( p_E \) are assumed to be functions of \( X_L \) and \( X_E \) respectively, such that \( p'_L \) and \( p'_E \) are \( \geq 0 \).

From equation (3.11) it follows that the optimal solution is characterized by the situation that:

\[
\lambda = \frac{a \cdot P(Q)-ab+2 X_A}{p_A} \frac{d(P(Q))}{dQ} = \frac{p_L + p'_L X_L}{p_L} = \frac{p_E + p'_E X_E}{p_E}
\]

(3.12)

Since \( p_A = a \cdot P(Q) - ab \) (3.9) and thus \( \frac{dp_A}{dx_A} = p'_A = a^2 \cdot \frac{d(P(Q))}{dQ} \), expression (3.12) can also be written as:

\[
\frac{P_A + p'_A X_A}{P_A} = \frac{p_L + p'_L X_L}{p_L} = \frac{p_E + p'_E X_E}{p_E}
\]

(3.13)

Thus the profit maximizer will shift his portfolio of assets, liabilities and equity in such a way that the net marginal return on assets divided by the price of assets equals the marginal cost of liability capital over the price of liabilities equals the marginal return to equity over the price of equity.

This is analogous to the "marginal return on capital equals the
marginal cost of capital" criterion of the conventional investment model\textsuperscript{1} in flow terms.

Solving equation (3.10) for $X_A$ will give the optimal stock of assets (= capital) $X^*_A$. This proceeds as follows. Solve the second equation of (3.10) for $X_L$ and the third for $X_E$.

Substitute the resulting expressions for $X_L$ and $X_E$ which both contain $\lambda$ in the last equation. Solve this one for $\lambda$ and put the resulting expression for $\lambda$ in the first equation, then solve for $X_A$. This gives (see Appendix A.1):

$$X^*_A = \frac{P_A P_E P_L P^*_E P_E + P_A P^*_L P^*_E P_L - P^*_E P^*_A P_L}{P^*_E P^*_L P^*_A + P^*_L P^*_E P^*_A - P^*_A P^*_E P^*_L}$$ \hspace{1cm} (3.14)

In the optimization procedure $X_A$, $X_L$ and $X_E$ are the decision variables; they can be varied at will by the manager of the firm. The nominal rates $p_A$, $p_L$ and $p_E$ are related to the effective yield $\pi_A$, $\pi_L$ and $\pi_E$ by the unit prices $P_A$, $P_L$ and $P_E$. One of the three variables $p_i$, $\pi_i$ or $\pi_i$ can be defined as constant to the firm. Changes in the $X_i$ will then be reflected in the other two. In the following the $P_i$ are taken as constant.

Moreover the units of measurement are defined so that $P_A = P_L = P_E = \Pi$. By substituting this in (3.14) and writing $\pi_i P_i$ for $p_i$ for $i = A, L, E$ in (3.2) the following result is gotten.

\textsuperscript{1}See Chapter I.
\[ x_A^* = \frac{\pi_L^t \pi_L^t \pi_L^t + \pi_L^t \pi_L^t \pi_L^t}{\pi_L^t \pi_L^t \pi_L^t + \pi_L^t \pi_L^t \pi_L^t} = \pi_A \pi_L^t \pi_L^t - \pi_A \pi_L^t \pi_L^t \]

Then, we can simplify (3.15) as:

\[ X_A^* = \frac{\pi_L^t \pi_E^t + \pi_L^t \pi_L^t - \pi_A (\pi_L^t + \pi_E^t)}{\pi_A (\pi_L^t + \pi_E^t) - \pi_E^t \pi_L^t} \quad (3.15) \]

Let \( \frac{-\pi_L^t}{\pi_A (\pi_L^t + \pi_E^t) - \pi_E^t \pi_L^t} \) be defined as \( c_1 \), and

\( \frac{-\pi_E^t}{\pi_A (\pi_L^t + \pi_E^t) - \pi_E^t \pi_L^t} \) be defined as \( c_2 \).

Then \( c_1 \) and \( c_2 \) \( \geq 0 \) because \( \pi_L^t, \pi_E^t \geq 0 \) and \( \pi_A^t < 0 \).

Equation (3.15) can then be simplified to

\[ X_A^* = c_1^*(\pi_A - \pi_E^t) + c_2^*(\pi_A - \pi_L^t), \quad (3.16) \]

where the \( c \)'s are assumed to be constant.

Thus it can be concluded that the optimal amount of assets (capital) is positively related to the difference between the rate of return on capital and the cost of financing in case the interest rate on liabilities and the "normal" return to equity.

With the help of two additional assumptions a micro
investment behavior equation can be derived. The first assumption is that net investment in a certain period ($B_t$) is proportional to the difference between the optimal stock of capital at the beginning of the period ($X_{At}^*$, given by eq. (3.16)) and the actual stock of capital at the same moment ($\bar{X}_{At}$). Thus it is assumed that

$$B_t = c_3^i (X_{At}^* - \bar{X}_{At}) \tag{3.17}$$

In (3.17) $c_3^i$ is a constant for a given firm, representing the speed with which the firm reacts to a profitable investment opportunity, $0 \leq c_3^i \leq 1$. If $c_3^i = 1$ the reaction is immediate and actual investment equals desired investment (the difference between the optimal and the actual stock of capital). There is no lag in that case. On the other hand if $c_3^i = 0$ there is no investment at all.

The second assumption is that replacement investment contemplated during the period is proportional to the actual stock of capital at the beginning of the period. Let $F_t$ be replacement investment in period $t$, then it is assumed

$$F_t = c_4^i \cdot \bar{X}_{At} \tag{3.18}$$

In (3.18) $c_4^i$ is a constant, in size inversely related to the

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1See Jorgenson, op. cit., pp. 47-53 for empirical and theoretical justification of these assumptions.
economic life of the actual capital stock, \( 0 \leq c_4^2 \leq 1 \). If 
\( c_4^2 = 0 \) no capital is replaced; if \( c_4^2 = 1 \) all capital is 
replaced during the period.

Gross investment during a period \( (B_c^2) \) the sum of net 
investment and replacement investment is then equal to (3.17) 
plus (3.18) or,

\[
B_c^2 = B_c + F_c = c_2^2(c_1^2(\tau_A - \pi_E) + c_2^2(\tau_A - \pi_L)) - [c_2^2 - c_4^2]X_A t^3
\]
or by redefining the constants in the following way:

\[
\begin{align*}
c_1 &= c_1^2.c_3^2 \geq 0 \\
c_2 &= c_2^2.c_3^2 \geq 0 \\
c_3 &= c_3^2 - c_4^2 \geq 0 \text{ if the reaction speed is} \\
& \text{sufficiently large and the} \\
& \text{economic life of the capital stock} \\
& \text{is sufficiently long,}^1
\end{align*}
\]

\[
B_c^2 = c_1(\tau_A - \pi_E) + c_2(\tau_A - \pi_L) - c_3 X_A t^3 \tag{3.19}
\]

Using (3.7) and (3.2) an alternative form for (3.19) is

\[
B_c^2 = c_1\left[\frac{a.P(O)-ab}{n} - \pi_E\right] + c_2\left[\frac{a.P(O)-ab}{n} - \pi_L\right] - c_3 X_A t^3 \tag{3.19'}
\]

^1Of course only statistical testing can give the actual sign of 
\( c_3 \) for a macro-investment equation of this type.
For the firm whose only goal is the maximization of total profits gross investment during a period is positively related to the rate of return on capital, negatively to the cost of financing capital (interest rate and dividend rate) and negatively or positively to the actual stock of capital at the beginning of the period, depending on whether the "net investment effect" (negative) or the "replacement investment effect" is strongest. That the "net investment effect" is negative may seem hard to believe at a first glance. But it becomes reasonable if one realizes that investment is a rate dependent on the actual stock. The larger the actual stock in relation to the optimal stock, the less addition to actual stock in the form of investment is needed. ¹

Comparing eq. (3.19) with Hammer's model it can be seen that the latter is actually a special case of the general asset-model, arrived at by eliminating equity financing \( X_E = 0 \) from eq. (3.1). In the next sections other alternatives will be successively substituted in the asset-model and equilibrium conditions and investment equations will be derived for each case.

III.3.2 A Constraint on Liabilities

It is not uncommon for some firms to exclude in their investment plans all external financing such as bonds, mortgages, bank credit, etc. The motive may be a general aversion against indebtedness or it may be that the outlook for the next investment planning period is such that the increase in risk caused by increasing liabilities is felt to be unjustified. The aversion against debt is apparent in statements that "the corporation's policy is to finance all expansions internally" and similar expressions.

By using debt in the financial structure a firm can profit from the phenomenon of "trading on the equity," or equivalently "leverage" or "gearing." This is the effect that if the return on capital exceeds the interest rate the firm has to pay on its liabilities, the net return to the owners equity capital will be increased proportional to the debt/equity ratio with the difference. If the net return to the owners equity capital (net profit as percentage of new worth) is \( \omega_E \), then

\[
\omega_E = \omega_A + (\omega_A - \omega_L) \cdot \frac{X_L \cdot P_L}{X_E \cdot P_E}.
\]

(3.20)

This is a double edged sword however; if the return on capital

---

1 See e.g. Heller, op. cit., p. 98. Also Spencer and Siegelman, op. cit., p. 401 and bibliography listed.
falls below the interest rate on liabilities, net return to equity will be proportionally lower. Thus leverage may increase profits; it also increases the possibility of losses. In addition debt brings with it the burdens of fixed yearly interest charges, fixed repayments at fixed dates plus priority in case of bankruptcy or liquidation. Hence only firms with a rather constant and large enough income to be able to pay the fixed charges even in unfavorable years, can consider taking on a large amount of debt. Financing with debt, especially long term debt, therefore involves an increase in financial risk.

Another argument against going into debt apart from the extra risk involved is the resistance many firms show against "outsiders" in the firm. In most cases no large amount of funds can be raised without giving detailed information, sufficient collateral and often a voice in company policy to the prospective lenders. Firms resent this and their fears for interference also tend to cause an aversion to indebtedness.

Of course, instead of a self-imposed behavioral constraint, the absolute bounds on borrowing may result from capital market imperfections.

This aversion may show in the wish not to use any debt at all, in the wish to use only a fixed maximum amount or in the policy to keep a certain maximum ratio between debt and equity.
capital in the financial structure. The consequences of these financial constraints for investment behavior will be analyzed below, the first two in this and the last in the next sub-

chapter.¹

A Maximum Bound on Liabilities

This case is very similar to the previous one in its development. The objective is to maximize total profits: $p_A X_A - p_L X_L - p_E X_E$ subject to the constraint: $p_L X_L \leq \bar{U}$ where $\bar{U}$ is the given amount of liabilities, not to be exceeded. If the constraint is not binding, the case degenerates into unconstrained profit maximization treated before. However for the most of the time the return on capital $p_A$ exceeds the interest cost of liabilities $p_L$. Thus it will be profitable for the firm to use the principle of trading on the equity to the maximum extent admitted by the constraint. Thus the constraint can usually be written in equality form:

$$P_L X_L = \bar{U}. \quad (3.21)$$

The interest cost of using liabilities $p_L X_L$ will be constant, hence irrelevant for the maximization procedure.

Combining eq. (3.1) and (3.21) the constraint can be written:

$$P_A X_A - P_E X_E = \bar{U} \quad (3.22)$$

¹Werkema, op. cit., p. 122.
Setting the problem up as a Lagrangian

$$\text{max } L = p_A X_A - p_E X_E - \lambda [P_A X_A - P_E X_E - \bar{U}]$$  \hspace{1cm} (3.23)

Again assuming the second order conditions for a maximum to be satisfied, the optimum is determined by setting the first partial derivatives of $L$ with respect to $X_A$, $X_E$ and $\lambda$ equal to zero. (Since $P_L X_L$ is constant $X_L$ ceases to be a decision variable). Hence

$$\frac{\partial L}{\partial X_A} = \frac{\partial L}{\partial X_E} = \frac{\partial L}{\partial \lambda} = 0.$$  

In matrix form:

$$\begin{bmatrix} p_A' & 0 & p_A \\ 0 & p_E' & -p_E \\ -p_A' & -p_E' & 0 \end{bmatrix} \begin{bmatrix} X_A \\ X_E \end{bmatrix} + \begin{bmatrix} \lambda \end{bmatrix} = \begin{bmatrix} -p_A' \lambda - p_E \bar{U} \end{bmatrix}$$  \hspace{1cm} (3.24)

The equilibrium condition for the firm in this case can be derived from (3.24) and expressed as:

$$\lambda = \frac{p_A' p_A X_A}{P_A} = \frac{p_E' p_E X_E}{P_E}$$  \hspace{1cm} (3.25)

Thus the equilibrium position of the firm that maximizes profits while keeping the amount of debt in the financial structure constant will be reached when the ratio of marginal return to capital over the price of capital equals the ratio of marginal return to equity over the price of equity. Proportionality of
rate of return to capital with the interest cost on liabilities
(as in the unconstrained case eq. (3.13)) will no longer exist
if the constraint is satisfied as an equality, as in (3.21).

The Lagrangian multiplier $\lambda$ can be given an economic
interpretation.

$$\lambda = \frac{\lambda P_A dX_A - \lambda P_E dX_E}{P_A dX_A - P_E dX_E}$$

$$= \frac{(P_A + p'X_A) dX_A - (P_E + p'X_E) dX_E}{P_A dX_A - P_E dX_E}$$

$$= \frac{d\omega}{dU}, \quad (3.26)$$

where $d\omega$ is the total differential of profits (eq. (3.8))
and $dU$ is the total differential of eq. (3.22). If $\lambda = 0$,
the constraint is not binding. This could be the case for
instance if credit is so expensive in relation to other sources
of finance that the absolute profit maximum would require the
application of less credit than the maximum amount the firm is
willing to accept. Thus the constraint is redundant and the
case degenerates into the previous unconstrained one. On the
other hand if $\lambda > 0$ the constraint is binding and the firm is
prevented from reaching the profit maximum. In this way $\lambda$,
representing the marginal profit of a prospective increase in
liabilities, functions as the "shadow price" of debt in the
financial structure and as such gives important information about the price the firm has to pay in profits foregone for its aversion of indebtedness.

Solving eq. (3.24) for $X_A$ (in the same way as done for eq. (3.10)) results in the following equation for the optimal stock of capital $X^*_A$:

$$X^*_A = g_1^2 (n_A - n_E) + g_2^2 \frac{\bar{u}}{n},$$

(3.27)

where $g_1^2$ and $g_2^2$ are positive constants.

Using as before assumptions (3.17) and (3.18) the equation for gross investment becomes:

$$B^*_t = B_t + F_t = c_3^2 [ g_1^2 (n_A - n_E) + g_2^2 \frac{\bar{u}}{n}] - [c_3^2 - c_4^2] \bar{X}_A t$$

or by redefining

$$g_1 = g_1^2 c_3^2 \geq 0$$

$$g_2 = g_2^2 c_3^2 \geq 0,$$

$$B^*_t = g_1^2 (n_A - n_E) + g_2^2 \frac{\bar{u}}{n} - c_3^2 \bar{X}_A t$$

(3.28)

or

$$B^*_t = g_1 [ \frac{(a.F(Q) - ab)}{n} - n_E ] + g_2 \frac{\bar{u}}{n} - c_3^2 \bar{X}_A t$$

(3.28')

Thus gross investment during any period is a positive function of the difference between the rate of return on capital and the "normal" rate required by the stock market on equity capital.
and a negative or positive function of the actual stock of capital at the beginning of the period, depending on whether the "net" effect or the "replacement" effect is strongest. So far the conclusion is parallel to the one derived for the unconstrained profit maximizing case. In contrast with the latter gross investment in the former case is also a positive function of the real value of the given stock of liabilities while the interest on liabilities is no longer a determining factor.

**No Liabilities in the Financial Structure**

A special case of the previous one is a maximum bound on liabilities equal to zero. No liabilities are allowed at all in the financing of investment. Because the maximum allowed is none at all, one would expect the real value of debt, a determinant of investment behavior in the preceding case to disappear from the investment equation. That this is true can be seen from the development of the basic model under the condition that:

\[ X_L P_L = 0 \]

Substituting this in (3.1) results in:

\[ X_A P_A = X_E P_E \]  \hspace{1cm} (3.29)

Thus all capital has to be financed inside, with the owners' funds. The only way of financing is with stocks and retained
earnings. (These two can, in the simple model used so far, with no discriminatory tax effects, no irrational stockholders behavior and no underwriting and flotation expenses, considered to be identical, having the same cost and benefit for the firm.)

Setting up the Lagrangian and deriving the maximum:

Max \( L = X_A \cdot P_A - X_E \cdot P_E - \lambda [X_A \cdot P_A - X_E \cdot P_E] \) \hspace{1cm} (3.30)

\[
\begin{align*}
\frac{\partial L}{\partial X_A} &= P_A + P_A^l \cdot X_A - \lambda P_A = 0 \\
\frac{\partial L}{\partial X_E} &= P_E + P_E^l \cdot X_E - \lambda P_E = 0 \\
\frac{\partial L}{\partial \lambda} &= X_A \cdot P_E - X_E \cdot P_E = 0
\end{align*}
\] \hspace{1cm} (3.31)

The equilibrium condition can again be expressed as:

\[
\lambda = \frac{P_A + P_A^l \cdot X_A}{P_A} = \frac{P_E + P_E^l \cdot X_E}{P_E}
\]

and is thus a general condition for profit maximization with a constraint on borrowing.

Solving (3.31) for \( X_A \) gives (see Appendix A.2):

\[
X_A^* = h_1^2(\pi_A - \pi_E) \hspace{1cm} (3.32)
\]

Applying eq. (3.17) and (3.18) as before and redefining \( h_1 = h_1^2 \cdot c_3 \) results in the following equation for gross investment:
\[ B'_c = B_c + F_c = c_3 [h'_1(\pi_A - \pi_E)] - [c_3^2 - c_4^2] \bar{X}_A t \]

\[ = h'_1(\pi_A - \pi_E) - c_3 \bar{X}_A t \]  \hspace{1cm} (3.33)

or,

\[ B'_c = h'_1 \left( \frac{aP(Q) - ab}{\pi} \right) - \pi_E - c_3 \bar{X}_A t \]  \hspace{1cm} (3.33')

Comparison with (3.28) and (3.28') shows the determinant \( \frac{U}{\pi} \) of (3.28) does not appear in (3.33) as was to be expected. Hence investment is only determined by the return on capital, the market return to equity and the initial stock of capital.
III.3.3. A Constraint on Leverage

A different type of financial constraint is the fixed debt-equity ratio. Instead of an absolute limit on the size of one source of financing, now a ratio between two sources is established, which cannot be exceeded. The arguments behind the ratio are usually the same as before. First to limit the extra financial risk associated with debt-financing. Second to prevent the interference from outsiders. Equity capital, representing ownership as well as risk-taking participating functions as a buffer against the dangers of debt. If more equity is introduced, the amount of debt a firm can carry will also increase. Hence as long as the debt-equity ratio stays below certain limits, debt can be used in the financial structure. The actual size of the maximum ratio cannot be objectively established. It depends on the particular risks of the industry the firm is in, or the state of the business cycle and other variables influencing the variability of the firm's net income. And of course it depends on the willingness of the management to assume risk in return to the promise of profit. The debt-equity ratio accepted functions as a measure of the firm's subjective 'trade-off' between risk and return. The risk preferring-type management will accept a higher leverage ratio than the risk-avoiding type because high leverage gives the
probability of large profits and alternatively the opportunity of large losses.

The argument then is that firms maximize profits subject to the constraint that the ratio of actual debt over actual equity is smaller than a given constant $\gamma$:

$$\frac{P_{LX}}{P_{EX}} \leq \gamma,$$  \hspace{1cm} (3.34)

where $\gamma$ is based on the estimated uncertainty of the firm's net income and the firm's aversion to risk. As to the approximate size of $\gamma$, since both $P_{LX}$ and $P_{EX} > 0$, $\gamma > 0$. Also because of fear of insolvency not many firms would accept more debt than they have equity capital, thus nearly always:

$P_{EX} < P_{LX}$. Therefore it can be said that

$$0 \leq \gamma \leq 1$$ \hspace{1cm} (3.35)

If the constraint (3.34) is not binding, the problem becomes one of unconstrained profit maximization as treated in Chapter III.3.1.

However over the range of values of $\gamma$ as established in (3.35), the price of liability capital $p_L$ is usually smaller than the price of equity $p_E$. Therefore it pays the firm to use debt to the fullest amount as permitted by (3.34) and no slack will develop in (3.34). The constraint becomes:

$$\frac{P_{LX}}{P_{EX}} = \gamma$$ \hspace{1cm} (3.36)
Combining this result with the basic equality (3.1) we now have two equations which can be used to express the variables $X_L$ and $X_E$ in terms of the third variable $X_A$ and the given constraints $\bar{v}$ and $P_i$ $(i=A,E,L)$. The resulting expressions can be substituted in the profit equation and the equilibrium conditions are derived in the usual way. Solving the equilibrium conditions for $X^*_A$, (see Appendix A.3) results in:

$$X^*_A = l^*_1 \left( \frac{v}{\bar{v}+1} \cdot \eta_A - \frac{1}{\bar{v}+1} \cdot \eta_L - \frac{1}{\bar{v}+1} \cdot \eta_E \right).$$  \hspace{1cm} (3.37)

Applying eq. (3.17) and (3.18) and redefining $l_1 = c_3'1'$ gives the following investment equation:

$$B_t^* = B_t + F_t = c_3' \left[ l^*_1 \left( \frac{v}{\bar{v}+1} \cdot \eta_A - \frac{1}{\bar{v}+1} \cdot \eta_L - \frac{1}{\bar{v}+1} \cdot \eta_E \right) \right]$$

$$- [c_3'-c_4']X_{At}, \text{ or,}$$

$$B_t^* = l^*_1 \left( \frac{v}{\bar{v}+1} \cdot \eta_A - \frac{1}{\bar{v}+1} \cdot \eta_L - \frac{1}{\bar{v}+1} \cdot \eta_E \right) - c_3'X_{At}, \hspace{1cm} (3.38)$$

and in its alternative form:

$$B_t^* = l^*_1 \left[ \frac{(a_p-u)-ab}{\bar{v}} \cdot \eta_A - \frac{1}{\bar{v}+1} \cdot \eta_L - \frac{1}{\bar{v}+1} \cdot \eta_E \right] - c_3'X_{At}. \hspace{1cm} (3.38')$$

As before in the unconstrained profit maximization case the optimum stock of capital and the rate of investment are positively related to the rate of return on capital $\eta_A$ and
negatively to the cost of equity \( \pi_E \) and the cost of using debt \( \pi_L \). But as a result of the leverage constraint a fixed proportional relationship exist between the coefficients of the cost variables of equity and debt capital in the investment equation. This proportional relationship is:

\[
\frac{\bar{V}}{\bar{V}+1} / \frac{1}{\bar{V}+1} = \bar{V}
\]  

(3.39)

In the next section the attention will be directed to a radically different objective, n.l. the maximization of total sales revenue, and the consequences of accepting this objective for the theory of investment.
III.4. Maximization of Total Revenue

Many writers have suggested that large firms in oligopolistic market situations are as much interested in sales as they are in profits. An eloquent case for sales maximization as the goal of the firm has been made by Baumol.\(^1,2\)

Although the consequences of adopting sales maximization as the objective of business behavior have been analyzed for the theory of production and taxation, the implications for investment behavior have not been pursued at all.

In the revenue maximization hypothesis the inclusion of a profit constraint is essential. In the theory of production for instance maximum sales without a constraint on profits or costs requires marginal revenue to be zero.

If the market situation is one of perfect competition, marginal revenue will not become zero because the price elasticity of demand is per definition infinite large and thus the optimal rate of production will be infinitely large too.\(^3\) But even if the market form is one of imperfect competition, the existence of sales promotion policies (e.g. advertising)

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1W. J. Baumol, op. cit., esp. Ch. 6.
2"Revenue" and "sales" are used interchangeably to denote the dollar value of gross income of the firm.
3Although sales maximization and perfect competition must be considered inconsistent, see infra.
with always positive marginal revenue will make the rate of production and the size of the firm infinite, unless the firm is otherwise constrained. A minimum profit constraint will under most circumstances do just this, but because of the positive marginal revenue of sales promotion the profit constraint will always be binding.¹

¹See Baumol, op. cit., p. 71 for formal proof.
III.4.1. The Revenue Maximization Model

The previously employed general model contains three variables. Applying this model unchanged to the sales maximizing hypothesis is not immediately possible. Revenue, the maximand, is a function of one of the variables only: geometrically speaking \( R = R(X_A) \) can be represented by a curve in a two dimensional space. Profit, the constraining relation, is a function of all three variables. Geometrically \( \Pi = \Pi(X_A, X_E, X_L) \) is a configuration in a multi-dimensional space. The optimum must be on the line as well as on the surface of the configuration. But in this case there is no a priori guarantee that both will cross. Even if they do, \( X_E \) and \( X_L \) will have to be zero.

To overcome these existence problems without having to specify special cases, the general model is simplified. Thus the Lagrangian technique can still be used and the existence of an optimum is guaranteed. It is believed that not much generality, if any, is lost this way.

It is assumed the firm in question wants to maximize sales subject to a minimum profit constraint and the market form leaves the firm discretion to do so. It is also assumed only one source of funds to finance investment is available. Say this is equity as in Ch. III.3.2. because the firm has an aversion
against going into debt. (But also any other source could be
selected for instance only borrowing because equity is held
constant.) Thus eq. (3.29) applies here. The value of sales,
\[ R = P(Q).Q = P(Q).a.X_A \]  \hspace{1cm} (3.40)
The profit constraint can be written as \( \pi = \pi \) or according to
the argument that the constraint is always satisfied as an
equality:
\[ \pi = P(Q).a.X_A - abX_A - p_EX_E = \bar{\pi} \]  \hspace{1cm} (3.41)
in which \( \bar{\pi} \) is the minimum amount of profit required.

As discussed before (See III.3) in this model it is possible
to define one of the three related variables \( p_i, \pi_1 \) and \( p_i, \)
(\( i=A, E \)) as constant for the firm and let variations in \( X_i \)
be reflected in the remaining two. So \( p_i \) will be kept constant
which means the firm can buy assets at a fixed price and can
float stocks at constant par value. Moreover the units of
measurement can be arbitrarily defined and will be taken so as
to have \( P_A = P_E \) (=1). With this last result substituted in
eq. (3.29) and the outcome of the substitution in (3.41), the
profit constraint becomes:
\[ \pi = P(Q).a.X_A - abX_A - p_EX_E = \bar{\pi} \]

Here it should be apparent that price \( P(Q) \) is a function
of Q, hence of $X_A$ by the relation $Q = aX_A$. (Only in the market form of perfect competition is $P$ independent of $X_A$, but in such a market a sales maximizer could in the long run not exist since maximum profit is a necessary condition of survival in perfect competition.) If we assume for simplicity a linear, "normally" sloped demand curve written as $P = f_1 - f_2 Q$ with $f_1$ and $f_2$ positive constants, the profit constraint becomes:

$$\omega = f_1 aX_A - f_2 a^2 x_A^2 - abX_A - pE_A x_A = \bar{w} \quad (3.42)$$

This quadratic equation for profit has two real roots but only one is consistent with the objective assumed. This is shown in Appendix A.4.

Thus the optimal stock of assets (capital) for the sales maximizing firm is represented by:

$$X_A^* = 1/(2f_2) \left[ f_1 a - ab - p_E + \left( (f_1 a - ab - p_E)^2 - 4f_2 a^2 \bar{w} \right)^{1/2} \right] \quad (3.43)$$

If $f_2 a^2 > 0$ is redefined as $m_1$, and $f_1 a - ab$ is redefined $m_2$, eq. (3.43) can be simplified to

$$X_A^* = \frac{1}{2m_1} (m_2 - p_E) + \frac{1}{2m_1} \left( (m_2 - p_E)^2 - 4m_1^2 \bar{w} \right)^{1/2} \quad (3.44)$$

The conclusion is that for the sales maximizing firm faced with a linear downward sloping demand curve the optimal stock of capital is a declining function of both the cost of capital.
and the amount of profit required. The first relation is of the same direction as it would be for a profit maximizing firm. The second is adverse to profit maximizing behavior since the normal reaction for a profit maximizer is to increase the stock of capital parallel with profits. This reaction of the sales maximizer is caused of course by the desire to maintain a given amount of profits. With a downward sloping profit function (downward because of declining demand function) a higher amount of profits required brings the sales maximizer closer to the profit maximizing point and requires lesser production and a smaller volume of capital.

As can be seen from (3.44), the optimal stock of capital increases with increasing $m_2$ and decreasing $m_1$, hence with an increase in the rate of return on capital $P_A$. (Since $P_A = P(Q)\cdot a - ab$

$$= (f_1 - f_2 a X_A) a - ab$$
$$= f_1 a - f_2 a^2 X_A - ab$$
$$= m_2 - m_1 X_A$$

As a result an autonomous increase (upward shift) in the rate of return function (increase in $m_2$) or a smaller absolute slope of this function ($m_1$ decreases) will result in more investment, a "normal" reaction.

An investment equation is derived from (3.44) in the usual
way with (3.17) and (3.18) and with redefining the constant terms as follows:

\[
\frac{c_3}{2m_1^3} = m_1 \\
m_2^2 = m_2 \\
m_3^2 = m_3.
\]

This results in:

\[
B_t^2 = B_t + F_t = m_1(m_2 - \pi_E^n) + m_1[(m_2 - \pi_E^n)^2 - 4m_3 \bar{w}]^{1/2} - c_3 \bar{x}_{At} \tag{3.45}
\]

Contrasting this equation with the one for the profit maximizing firm it appears that in both cases the investment is positively influenced by an increase in the rate of return to capital and a decrease in the cost of financing. Unlike the profit maximizing firm, the sales maximizer's investment is negatively influenced by a profit constraint.\(^1\)

\[^1\text{Assuming that after aggregation the corresponding macro equation has the same form, this relation could explain the negative correlation between profitability and retention rate and between profitability and leverage, found by Myron Gordon and which he tentatively attributed to the: "...Influence of other objectives subordinating the maximization of stock value". (op. cit., p. 233)\]
III.4.2 Introducing Sales Promotion

Increasing production and sales by increasing the productive capacity is not the only way to increase sales revenue. It is not even a certain way because greater production can under imperfect competition only be sold by cutting prices and the accumulated revenue loss from lower unit prices may outweigh the gain from selling more units, as it will if the product demand curve is price inelastic in the relevant region.

Another way to increase sales revenue is by sales promotion techniques, e.g. advertising, etc. These can be thought of as shifting the demand curve for the product upwards, hence every quantity can be sold at a higher unit price than before. The marginal revenue from sales promotion will nearly always be positive as a result of which total revenue will monotonically increase with increased sales promotion expenses (although of course the increase in sales revenue may not be large enough to cover the increased sales costs).

If a firm has two or more instruments available which serve the same purpose he has to find a balance between them. Thus a choice has to be made between increasing productive capacity by investing in buildings and other equipment and increasing advertising expenditures.¹

¹It is customary to classify expenses for "durable" goods (those with an economic life longer than one production process
Introducing advertising $S$ as a variable changes the revenue and the profit equations into:

\[ R = Q \cdot P(Q, S) = P(Q, S) \cdot a \cdot X_A \quad (3.46) \]

\[ \omega = P(Q, S) \cdot a \cdot X_A - abX_A - p_E X_A \quad (3.47) \]

The optimum is found by setting up the Lagrangian and finding the values for which the partials with respect to $X_A$, $S$ and $\lambda$ are equal to zero. This results in

---

or because of accounting convenience lasting longer than one year) as investment and the rest as current expenses, although there are exceptions for small equipment as tools, etc., which are usually seen as current costs but last longer than one year. Advertising expenses are normally booked as cost the same year they are made. If their effect lasts longer than one year—which may very well be the case, e.g. with a large advertising campaign—they ought to be put on the balance sheet as an asset and gradually be depreciated. In this sense advertising is an investment, different in degree but not in principle from other investments. The problem of striking a balance between expenses for sales promotion and for capital goods can then be looked upon as the problem of selecting between two different investment projects.

Since the problem under consideration is basically an inequality constrained maximization problem ($X_A, S \geq 0, \omega_{\lambda S}$) applying programming methods seems to be obvious. (Note that both $R$ and $\omega_{\lambda S}$ are concave quadratic.) But in the light of the available information that really $\omega = \omega_{\lambda S}$, using the conventional Lagrange multiplier technique is much more convenient.
Max \( L = P(Q,S) . a . X_A + \lambda [P(Q,S) . a . X_A - abX_A - p_E X_A - S - \bar{w}] \)

\[
\begin{align*}
\frac{\partial L}{\partial X_A} &= \frac{\partial P}{\partial Q} . a^2 X_A + p . a + \lambda \left[ \frac{\partial P}{\partial Q} . a^2 X_A + p . a - ab - p_E \right] = 0 \quad (3.48) \\
\frac{\partial L}{\partial S} &= \frac{\partial P}{\partial S} . a . X_A + \lambda \left[ \frac{\partial P}{\partial S} . a . X_A - 1 \right] = 0 \\
\frac{\partial L}{\partial \bar{w}} &= P . a . X_A - abX_A - p_E X_A - S - \bar{w} = 0
\end{align*}
\]

From (3.48) it follows that:

\[
\frac{\partial P}{\partial Q} . a^2 X_A + p . a = \frac{1}{\lambda + 1} \left[ ab + p_E \right] \quad (3.49)
\]

In equation (3.49) the expression on the left side of the equality sign represents \( \frac{\partial R}{\partial X_A} \), the first derivative of total revenue with respect to assets, while the expression in brackets on the right side is the sum of marginal production cost and the marginal cost of capital, both with respect to assets.

Thus \( \frac{\partial R}{\partial X_A} = \frac{1}{\lambda + 1} \left[ \frac{\partial K_d}{\partial X_A} + \frac{\partial K_F}{\partial X_A} \right] \quad (3.50) \)

A similar observation can be made for sales promotion:

\[
\frac{\partial R}{\partial S} = \left[ \frac{\partial P}{\partial S} . a . X_A \right] = \frac{1}{\lambda + 1} \cdot \frac{\partial S}{\partial S} \quad (3.51)
\]

The derivation is on the assumption that \( p_E \) is independent of \( X_A \).
Since $\lambda > 0 \rightarrow \frac{\lambda}{\lambda + 1} < 1$.

Thus from (3.50) it follows that:

$$\frac{\partial R}{\partial X_A} < \frac{\partial P}{\partial X_A} + \frac{\partial K}{\partial X_A} \quad \text{or} \quad \frac{\partial R}{\partial X_A} - \frac{\partial K}{\partial X_A} < \frac{\partial K}{\partial X_A}$$

But $\frac{\partial R}{\partial X_A} - \frac{\partial P}{\partial X_A} = \frac{\partial m}{\partial X_A}$ is the marginal rate of return on capital. Thus it is proven that in the optimal position investment is carried on past the point where the marginal return on capital equals the marginal cost of capital into the region where the marginal cost of capital exceeds the marginal return to capital.

Eq. (3.51) states in the same way that the firm in its optimal position uses advertising in the region where the marginal revenue of advertising is less than its marginal cost, a well-known conclusion.\(^1\)

It is instructive to compare the sales-maximizers behavior with that of the profit-maximizer. As is well-known production theory states that the profit-maximizer will produce up to the point where the marginal revenue of his product equals the marginal cost of his product. Similarly he will invest to the point where marginal return to capital just equals the marginal cost to capital.

\(^1\)Baumol, op. cit., pp. 59-61.
As was shown above (Ch. III.3) these last results still apply for the profit maximizer whose behavior is constrained by financial limitations.

Baumol first proved that the sales maximizer's optimal position with respect to the rate of production shows a marginal revenue of his product lower than the marginal cost of the product. The argument above indicates that a similar conclusion follows for the sales maximizer's investment behavior: he will invest past the point where the marginal return to capital is equal to the marginal cost of capital so that the optimal investment position shows a marginal return to capital lower than the marginal cost of capital.

As to the interpretation of \( \lambda \) in economic terms it follows from eq. (3.50) and eq. (3.51) that:

\[
\lambda = \frac{\partial R}{\partial X_A} = \frac{\partial S}{\partial S} = \frac{\partial R}{\partial S} \tag{3.52}
\]

The sales maximizing firm bound by a profit constraint will according to (3.52) balance its expenditures between advertising and capacity-expansion in such a way that at the margin the decline in profits caused by increasing the stock of

\(^{1}\text{Ibid.},\ p.\ 55.\)
capital with one unit will be accompanied by the same revenue
gain as will the loss caused by an increase in advertising
expenditures by one unit.

\( \lambda \) itself is the ratio of the total differential of revenue
and the total differential of profit as can be shown as follows:

\[
(3.46) \Rightarrow dR = \left( \frac{3P}{3Q} a^2 X_A + P a \right) dx_A + \left( \frac{3P}{S} a X_A \right) dS
\]

\[
(3.47) \Rightarrow dw = \left[ \frac{3P}{3Q} a^2 X_A + P a - ab - p_E \right] dx_A + \left[ \frac{3P}{S} a X_A - 1 \right] dS
\]

Substituting from (3.52) in \( dR \) gives:

\[
dR = \lambda \left[ \frac{3P}{3Q} a^2 X_A + P a - ab - p_E \right] dx_A + \lambda \left[ \frac{3P}{S} a X_A - 1 \right] dS
\]

Thus

\[
\frac{dR}{dw} = \frac{\lambda \left( \frac{3P}{3Q} a^2 X_A + P a - ab - p_E \right) dx_A + \lambda \left( \frac{3P}{S} a X_A - 1 \right) dS}{\left( \frac{3P}{3Q} a^2 X_A + P a - ab - p_E \right) dx_A + \lambda \left( \frac{3P}{S} a X_A - 1 \right) dS} = \lambda
\]

Thus the interpretation of \( \lambda \) is that \( \lambda \) represents the
"shadow price" associated with the profit constraint, \( \lambda \)
measures per unit profit the extra revenue attained by relaxing
the profit constraint.
III.5  The Utility Maximization Model

In his thesis\(^1\) O. E. Williamson introduced managerial objectives such as salary, security, power, status, prestige and professional excellence into the formal theory of the firm by using the notion of "expense preference": managers are not indifferent towards costs of all types but have distinctive preference for certain classes of expenses. In this way the essentially non-pecuniary goals listed above are built into models of the firm by way of the means by which these objectives can be realized, such as salary and other emoluments, personal staff and discretionary profits.

Williamson's most complete model is the s.c. "Staff and Emoluments Model" in which the manager maximizes his own utility (in contrast to that of the owners of the firm, i.e. the stockholders) which is a function of staff expenditures, emoluments and discretionary profits. In the following section these three managerial preferences will be substituted in the general investment model developed before to analyze the investment behavior following from these imputed goals.

\(^{1}\) O. E. Williamson, op. cit., Ch. 3.
III.5.1. The Staff and Emoluments Model

The decision maker tries to maximize his own utility

\[ U = U^D(M,S) \]

where

\[ ^D = R - K_F - K_F - M - S - \bar{m}, \]

discretionary profit

\[ M = \text{emoluments, the discretionary portion of management} \]

salaries and perquisites. They have the character

of economic rent, hence zero productivities.

\[ S = \text{staff expenses or approximately general administrative} \]

and selling expense; \( P = P(Q,S) \)

\[ \bar{m} = \text{minimum profit demanded by stockholders.} \]

Since \( R - K_F - K_F \) is also defined as

\[ ^D = p_A X_A - p_L X_L - p_E X_E - p_{E E} - p_{L L}, \]

\[ \bar{m} \text{ can be written as: } p_A X_A - p_L X_L - p_E X_E - M - S \bar{m} \]  \hspace{1cm} (3.53)

or as \[ p_A X_A - p_L X_L - p_E X_E - M - S \bar{m} \]  \hspace{1cm} (3.54)

Combining (3.53) with the basic constraint (3.1), the goal of

the firm in this model can be written

\[ \text{Max } L = U^D + \lambda [p_A X_A - p_L X_L - p_E X_E] \]  \hspace{1cm} (3.55)

The optimality conditions are:

\[ \frac{\partial L}{\partial X_A} = \frac{\partial U}{\partial D} \cdot (p_A + p_A X_A) + \lambda P_A = 0 \]  \hspace{1cm} (3.56)
From the first three it follows that:

\[
\frac{p_A + p_A' X_A}{P_A} = \frac{p_L + p_L' X_L}{P_L} = \frac{p_E + p_E' X_E}{P_E}
\]  

(3.62)

Thus the equilibrium position is determined by the condition that the marginal net revenues, the marginal debt costs and the marginal equity costs are proportional to the price of assets, liabilities and equity capital respectively. Thus the optimal stock of assets for this case is determined in the same way as it is for the single profit maximizer under the same circumstances. (Compare Ch. III.3.1.)

This conclusion is analogous to Williamson's conclusion for production theory that the utility-maximizing manager will
determine the optimal rate of production in the same way as the profit-maximizing manager, n.l., production will be continued until the marginal revenue of selling equals the marginal cost of producing.\(^1\)

W! second conclusion, n.l. that the utility maximizer will use staff to a larger extend than the profit maximizer,\(^2\) can be found back in eq. (3.59) which can be written:

\[
\frac{\partial R}{\partial S} = 1 - \frac{\partial U/\partial W}{\partial S/\partial W_D} < 1
\]

According to eq. (3.60) the marginal utility of emoluments equals the marginal utility of discretionary profits in equilibrium. However the introduction of a profit tax would have discriminatory effects. This can be seen by adding the tax rate \(t\) to eq. (3.53).

\[(3.53') \Rightarrow_D = (1-t)(p_A X_A - p_L X_L - p_E X_E - M - S) - \bar{w}
\]

Substituting (3.53') in (3.55) will change equilibrium condition (3.60) into:

eq. (3.60') \[
\frac{\partial U}{\partial M} = (1-t)\frac{\partial U}{\partial W_D}.
\]

\(^1\)Ibid., p. 53.

\(^2\)Ibid.
No longer is the marginal utility of emoluments equal to the marginal utility of profits. Instead, as Williamson concluded, the firm will absorb some part of actual profit as emoluments, the amount being dependent on the tax rate.

The Investment Equation

As shown in (3.62), the investment behavior of the utility maximizing manager (with profit one of the determinants of utility) of the above model does not differ from the investment behavior of the single profit maximizer as described in Ch. III. Equations (3.56), (3.58) and (3.61) completely determine $X^*_A$, the optimal stock of capital. The investment equation for this case is similar to the one of the unconstrained profit maximizer as developed in Ch. III.

Additional Remarks

In the model used here the manager expresses his desires for power, prestige, etc. in excess current expenditures for staff and emoluments. Investment, being closely related to production, is just as production not directly effected, witness the fact that the optimal position requires the equality between marginal rate of return and the marginal cost of capital. It

\[ \text{Ibid.} \]
is true that total investment will in all likelihood be larger than it would for the profit maximizer because the excess staff expenditures shift the marginal revenue curve to the right. Thus production, productive capacity hence investment will be larger than they otherwise would be but apart from this investment policy is not different from the profit maximizer's.

This is no longer true if the managerial motives assumed in this section are expressed in investment itself instead of in current expenditures. Examples may be an unneeded plush office building, executive limousines, a company yacht, etc. Often these investments are done to satisfy the manager's own tastes instead of contributing to the firm's overall profit. In these cases investment will be pushed beyond the point where marginal return equals marginal cost into the region where the marginal cost exceeds the marginal return. Not every executive limousine and not every plush office building is the result of a manager's whim and is submarginal from a pure profit maximizing standpoint. As long as the potential customers of a company, for reasons of ignorance, irrationality or inability to judge a product on its merits, associate the firm's expensive office surroundings with good quality products, dependable service, etc. and as a result are willing to pay a higher price, it may well pay the firm to invest in these
luxury items.

Another example where the rate of return is pushed below the cost of capital is when the firm is extremely interested in growth, so that potential profit is sacrificed for attaining a larger growth rate of sales. This can be considered the dynamic equivalent of the sales maximizer and the conclusions will be similar.¹

III.5.2. Aversion Against Leverage

In a recent article J. L. Bouma and H. Willems applied Williamson's approach to an investment model. Their main conclusion is that in the utility maximizing firm the marginal rate of return on capital exceeds in equilibrium the marginal cost of financing. Since this conclusion is at variance with the results obtained in the preceding section a close look at their model seems appropriate.

In the article two types of firms are distinguished. First the so-called "Stockholders Controlled Firm," where the manager/decision maker's only interest is the maximization of the stockholders' welfare, in casu the maximization of share prices. Share prices are assumed to be a function of profits, dividends and the debt-equity ratio. Second, the "Management Controlled Firm" in which the manager takes the necessary decisions so as to maximize his own utility which is this time assumed to be a function of share prices as well as "preferred expenses" (a lumping together of staff and emoluments) and financial leverage directly. The model used is a two period one with a choice of retaining earnings, paying dividends and

---

using debt financing. The results of the two cases are compared with each other and with the "Traditional Firm" in which both the manager's and the stockholders' only interest is the maximization of profits.

Using Bouma and Willems' own notation, if the subscript refers to the period then:

- $w_1$ is the actual profit
- $s_1$ is preferred expenses (a lumping together of staff and emoluments)
- $w_1 - s_1$ is published profit
- $u_1$ is total dividend paid
- $v_1$ is total amount borrowed from banks or bondholders
- $i$ is interest rate on $v_1$
- $I$ is total amount invested
- $Y(I)$ is net income from investment before interest payments
- $w_2$ is actual profit after repayment of loan and interest
- $s_2$ is preferred expenses, subtracted from $w_2$ by management
- $w_2 - s_2$ is published profits after 2 periods
- $M$ is the manager's utility and $B$ is share price

The "Manager Controlled Firm" model is the following:
Max \( M = M(B, S_1, S_2, F_1) \)

\[
B = B(w_1 - S_1, U_1, w_2 - S_2, F_1)
\]

\[
F_1 = V_1/(w_1 - S_1 - U_1)
\]

\[
w_2 = Y(I) - (1+i)V_1
\]

\[
I = w_1 - S_1 - U_1 + V_1
\]

The maximum position is determined by setting the partials from \( M \) with respect to \( S_1, S_2, U_1 \) and \( V_1 \) equal to zero and substituting from the equations for \( B, F_1, w_2 \) and \( I \). It then follows that:

\[
\frac{\partial M}{\partial F_1} \cdot \frac{1}{w_1 - S_1 - U_1} = \frac{\partial M}{\partial S_2}
\]

Marginal rate of return = \( \frac{\partial V}{\partial I} \cdot -1 \) = \( i - \frac{\partial M}{\partial S_2} \) \( (3.64) \)

In the "Stockholder Controlled Firm" model, in which per definition \( M = B \) and \( S_1 = S_2 = 0 \) the corresponding equilibrium situation appears to be

\[
M,R,R = \frac{\partial V}{\partial I} \cdot -1 \) = \( i - \frac{\partial B}{\partial F_1} \cdot \frac{1}{w_1 - U_1} \)
\]

\( (3.65) \)

In the "Traditional Firm Model" the analogous expression becomes

\[\text{Ibid., p. 187.}\]
M.R.R. = \( \frac{d^Y}{dI} - 1 \) = i \hspace{1cm} (3.66)

It is assumed that \( \frac{\partial M}{\partial F_1} \) and \( \frac{\partial B}{\partial F_1} < 0 \) because managers and stockholders dislike a high leverage ratio, apart from the interest and repayment obligation.

Comparing (3.64) and (3.65) and (3.66) it can be seen that in the "Management Controlled" and in the "Stockholder Controlled" firm the M.R.R. exceeds i. A close look learns that this difference however is not so much caused by the existence of the so-called preferred expenses (\( \equiv 0 \) in (3.65)) but by the debt aversion as expressed in \( \frac{\partial M}{\partial F_1} \) and \( \frac{\partial B}{\partial F_1} < 0 \). If both management and stockholders would feel indifferent between debt and non-debt financing (apart from the interest and principal repayment condition) in both models M.R.R. would equal i, as we found before (Ch. III.5.1).

That managers and stockholders dislike a high leverage per se is a phenomenon often observed and a few times integrated in a (profit-maximizing) investment model. The result of this behavior is (as above) that a smaller amount is being invested than otherwise would be the case so that the marginal rate of return on capital exceeds the interest rate.

It is not however a result of managerial objectives like status, power, prestige, etc. as these are expressed in a preference for staff and emoluments but mostly a defense against unfavorable outcomes of the basically uncertain processes of investing and financing, and sometimes it represents a dislike of interference from "outsiders" like banks, mortgage and bondholders.

The same results as obtained by the Bouma and Willems article will be gotten if the aversion against leverage is added to the model of the previous section as one of the determinants of managerial utility.

This gives:

\[
(3.55) \quad \text{Max } L = U(D, M, S, F) + \lambda [P_A X_A - P_L X_L - P_E X_E]
\]

where \( F = \frac{P_L X_L}{P_E X_E} \)

\[
\frac{\partial U}{\partial D}, \frac{\partial U}{\partial M}, \frac{\partial U}{\partial S} > 0
\]

\[
\frac{\partial U}{\partial F} < 0
\]

\[
(3.56) \quad \frac{\partial L}{\partial X_A} = \frac{\partial U}{\partial D} \cdot (P_A + p_A X_A) + \lambda P_A = 0
\]

\[
(3.57) \quad \frac{\partial L}{\partial X_L} = \frac{\partial U}{\partial D} \cdot (-P_L - p_L X_L) + \frac{\partial U}{\partial F} \cdot \frac{P_L}{P_E X_E} - \lambda P_L = 0
\]
(3.58") \[ \frac{\Delta L}{\Delta X_E} = \frac{\Delta U}{\Delta w} \cdot (-p_E - p_E^X_E) + \frac{\Delta U}{\Delta F} \cdot \frac{P_L X_L}{P E X_E} - \lambda P_E = 0 \]

And

(3.62") \[ \frac{p_A + p_A^X A}{P_A} = \frac{p_L + p_L^X L}{P_L} - \frac{\Delta U}{\Delta F} \cdot \frac{1}{P E X_E} \]

\[ = \frac{P_E + P_E^X E}{P E} + \frac{\Delta U}{\Delta F} \cdot \frac{P_L X_L}{(P E X_E)^2} \]

Since \( \frac{\Delta U}{\Delta F} < 0 \), the first part of (3.62") states that:

\[ \frac{p_A + p_A^X A}{P_A} > \frac{p_L + p_L^X L}{P_L} \]

Thus the marginal rate of return on assets exceeds the marginal interest rate on debt capital. At the same time the second part of (3.62") states that

\[ \frac{p_A + p_A^X A}{P_A} < \frac{p_E + p_E^X E}{P_E} \]

Hence the marginal cost of equity capital exceeds the rate of return on physical capital.

It appears therefore that the result of the manager's dislike for an increase in the debt/equity ratio is that more equity capital and less debt is attracted than would be the case if the manager felt indifferent between the two types of
financing investment. There is a trade-off between profit and leverage: some potential profit is sacrificed in order to arrive at a lower debt/equity ratio.

In the next Chapter the attention is shifted from the decision over the total volume of investment to the related decision over the optimal financing of the proposed investment budget.
CHAPTER IV

OPTIMAL FINANCIAL STRUCTURE AND OPTIMAL SELECTION OF PROJECTS

IV.1. The Optimal Financing of the Proposed Investment Budget

As is well known in a world of perfect capital markets there is no optimal capital structure for the firm.¹ The investment decision can be solved completely independent of any financial considerations. Under imperfect capital markets this is no longer true and a rational application of the profit maximization principle then requires the simultaneous solution of three interdependent problems; the determination of the total amount to be invested, the optimal combination of investment projects and the optimal financing of the investment budget. Variables on which this solution must be based are the size and the variability of the return on the various investment projects and the time structure, the cost and the risk associated with the financial instruments used. Models in which the optimal financial structure, though not detailed, is determined together with the total size of the budget and/or together with the optimal project mix are

available in the literature.¹

In reality firms first determine the total amount they want to invest without using any information about specific projects or specific financing opportunities, hence independent of the optimal project mix and the optimal financing problem. The latter are then separately analyzed and solved, often by different people or different departments in the organizational structure. In the solution of these last two problems the size of the budget is then a given constant.²

This Chapter will give a pragmatic operational solution to the problem of the optimal financing of the investment budget in which, in accordance with the actual business practice and its assumed rationalization mentioned above the size of the budget, the demand for funds, is considered as given. In addition, some remarks are made about the solution of the problem of the optimal project mix for the various other objectives considered under the same assumptions.


²See Chapter I.
IV.1.1. The Problem

The need for an optimal capital structure of the investment budget arises because there are various ways to finance investment expenditures. The various forms available differ in time, structure, cost, obligation of periodic and final repayment, voting rights, etc. Since, especially in a world of uncertainty the objectives of the firm with respect to the capital structure can be many, an optimum has to be found between the multiple instruments used to satisfy the often conflicting needs.

Early financial literature, close to business practice, usually considered the demand for funds as given, but was limited to predominantly qualitative discussions about the relative merits of the various forms of financing.

A notable exception can be found in early Continental literature in the theory of the so-called "critical period analysis,"¹ which is one of the first quantitative solutions to the capital optimization problem. Usually applied to the total capital structure, its consequences for the financing of the additional need for funds caused by investment will be analyzed here, after which a more general solution of the optimal financing problem will be developed.

¹See e.g. J. L. Meij, Leerboek der Bedrijfseconomie II, (Den Haag, Delwel) pp. 99-106.
IV. 1.2. The Critical Period

In this theory three forms of financing are assumed. First there is permanent capital, equity. Second there is short term capital, i.e., bank credit. Third, the, from the standpoint of the borrowing firm negative, source of investing temporary idle cash into a time or savings account. Short term capital is mostly more expensive for the firm than permanent capital but it is more flexible; it can be repaid as soon as it is no longer needed. Permanent capital is mostly never repaid. Hence it can lay idle and to prevent this there is always the possibility to lend it out at the, mostly very low, credit interest earned on time and savings accounts.

If the demand for funds $B_t$, caused by the net proposed investments in the future planning period fluctuates per period $t$, ($t = 1, ..., n$) but the fluctuations are known and if the prices (interest) of permanent, short, and credit capital are respectively $p_p$, $p_H$, and $p_c$, the question is asked for which period, the critical period $T^*$, it is indifferent from a cost standpoint if the demand for funds is satisfied by permanent or by short term funds. All money needed for a time period longer than $T^*$ is then cheapest provided for in permanent and all funds needed shorter than $T^*$ are best supplied in short term capital. To determine $T^*$ it is necessary to realize that from the definition of $T^*$ it follows that:
\[
\bar{B}_t \cdot T^* \cdot p_H = \bar{B}_t \cdot T^* \cdot p_S + (n-T^*) (p_S - p_c)
\]

hence \( T^* p_H - T^* p_c = n \cdot p_S - n \cdot p_c \)

or \( T^* = n \frac{p_S - p_c}{p_H - p_c} \) \hspace{1cm} (4.1)

An optimum will exist \( T^* \leq n \), which requires that:

\[
\frac{p_S - p_c}{p_H - p_c} \leq 1
\] \hspace{1cm} (4.2)

which requires in turn that:

either \( p_H - p_c > 0 \) and also \( p_s \leq p_h \) \hspace{1cm} (4.3)

or \( p_H - p_c < 0 \) and also \( p_s \geq p_h \) \hspace{1cm} (4.4)

the first case (4.3) is usually satisfied.

An example heroically simplified may illustrate the point.

Assume: the number of time periods of the firm's investment planning horizon \( n = 3 \)

: the investment budget for the net three periods \( \bar{B}_t = 50, 70, 60 \)

: the following sources of capital and their associated prices are available:

Stocks \( X_s \) with price \( p_s = 4\% \) per period

Bank credit \( X_h \) with interest \( p_h = 8\% \) per period

Savings accounts \( X_c \) with interest revenue \( p_c = 2\% \) per period.
Hence an amount of 50 is needed for 3 periods
an amount of 10 is needed for 2 periods
an amount of 10 is needed for 1 period.

\[ T^* = n \cdot \frac{P_s - P_c}{P_H - P_c} = 3 \cdot \frac{.04 - .02}{.08 - .02} = 1 \text{ period} \]

Therefore all money longer needed than 1 period (50 + 10 = 60) is financed by stocks, and funds needed shorter or equal to 1 period (10) are financed by bank credit. In the first period 10 more is raised than is needed, which 10 will then be put in a savings account or will otherwise be invested outside at the credit interest \( p_c \).

The analysis gives an optimum decision rule for choosing between permanent and nonpermanent funds. It also explains the often observed phenomenon that because of its low price more permanent capital is acquired than is actually needed on the base of the long term needs of funds earned by permanent or long term investments plus the permanent nucleus in the sequence of short term capital investments. Permanent funds are according to this theory even used to finance temporary peaks in the demand for funds and are then in the time they are not needed invested outside the firm at the credit interest \( p_c \).

The explanatory power of the theory is limited however and the many objections which can be brought against it make
the theory unattractive for normative use.

The first objection is that the rule signals an optimum only under the implicit assumption that permanent capital, if used at all, must be used in the same amount all during the planning period. This can be seen if we compare the above solution which has a total cost of \((3 \times 60 \times .04 + 10 \times .08 - 10 \times .02 = 7.8)\), with a solution in which in the first period 50 is raised by a stock issue and an additional 20 is raised by stocks the second period which will be kept throughout the third period so that an amount of 10 will have to be invested outside. This solution costs \(.04(50 + 70 + 70) - .02 \times 10 = 7.4\). Although in reality there are limits as to the minimum amount that can be floated and the minimum time period that is required between stock issues, an increase in equity during the planning period is in principle entirely feasible, hence the above solution is unrealistically limited.

A second problem lies in the assumed knowledge of \(\bar{B}_c\). In the real world \(\bar{B}_c\) can at best be an estimate, based on anticipated costs and returns. These anticipations might prove to be wrong and after some time management might want to change \(\bar{B}_c\) for the remainder of the planning period. The financial structure is then fixed and can seldom and only at high cost be changed. The critical period analyses gives no clues as to the consequences of differences between outcomes and estimates.
and allows no way to incorporate in the solution buffers to mitigate these consequences.

A third objection is the assumed constancy of the unit costs \( (p_j) \). If the structure of the interest rates changes, the critical period changes and the actual financial structure is no longer optimal. Apart from some vague generalizations over the direction of the change in \( T^* \) the analysis cannot give much insight into the outcome for the cost minimum of changes in the structure of relative financial prices. This results from the indirect character of the solution; although the purpose is to derive a financial structure which fits the fluctuating needs for funds with minimum costs, neither the structure nor the cost minimum is determined directly.

The third objection is that only two different sources of funds are allowed and that the choice is consequently limited to a combination of short term and permanent funds only.

Reality shows more forms and since more or less from each form can be accepted, in fact an extremely large number of combinations is feasible.

A final objection is that only two criteria are considered; the time structure of needs and supplies and the cost. The analysis allows no additional criteria and leaves no room to build in alternatives.

Hereafter a less limited model will be developed which
permits one to find the optimum combination of financial instru-
ments which satisfies the fluctuating capital demand with
minimum cost and which simultaneously leaves options to extend
the analysis to include additional considerations and alterna-
tive motivations.
IV. 1.3. A Programming Solution

The economic problem under consideration can be structured in such a form that it may be solved with relatively simple programming methods. The following symbols are employed:

- $E_t$ is the net investment budget adopted for period $t$, $t = 1, \ldots, n$.
- $X_{st}$ is the volume of stocks\(^1\) outstanding during period $t$.
- $X_{gt}$ is the volume of retained earnings\(^2\) during period $t$.
- $X_{qt}$ is the volume of bonds outstanding during period $t$.
- $X_{ht}$ is the volume of bank credit during period $t$.
- $X_{jt}$ stands for any other financial instruments used during period $t$ (e.g. preferred stock, convertible bonds, etc.)

\(^1\)In Chapter III, equity capital $X_E$ represents both stocks $X_s$ and retained earnings $X_r$. Liabilities is used in III for both bank credit $X_H$ and bonds $X_Q$.

\(^2\)Usually retained earnings are imputed an opportunity cost of capital, to realize that investing them means foregoing the revenue they could have earned outside the firm. However, this possibility is explicitly introduced in the model as $X_{gt}$ with opportunity revenue $p_c$. Thus retained earnings in the model above are considered as available funds that can be invested at no additional cost, hence $p_g = 0$. Another solution could be to subtract $X_{gt}$ from the corresponding $E_t$ and to finance the adjusted $E_t$ with the available sources exclusive of $X_{gt}$. 
$X_{ot}$ is the amount of money invested outside the firm (e.g., time and savings account, securities, etc.)

$p_S$ is the unit cost of equity funds per period

$p_Q$ is the unit cost of bonds

$p_H$ is the bank interest rate on loans per period

$p_G$ is the cost of retained earnings per period

$p_J$ is the cost of $J$ per period

$p_c$ is the interest revenue on a unit $X_{ot}$ per period

The problem is then to determine the $X_{jt}$ and $X_{ot}$ where $J = S, G, Q, L, J$ in such a way that in each period $t$ the need for funds $E_t$ is satisfied at minimum overall total cost $K$, or:

Min. $K = \sum \sum X_{jt} \cdot p_J - \sum X_{ot} \cdot p_c \quad (4.5)$

subject to $\sum X_{jt} - X_{ot} \geq E_t$ for $t=1, \ldots, n \quad (4.6)$

$X_{jt}, X_{ot} \geq 0$ for $J=S, G, Q, H \quad (4.7)$

and as before $p_G = 0$.

So far it is assumed possible to borrow or acquire each type of fund at any amount at the beginning of any period and pay back at the end of each period. Obviously the optimal solution is then to fill the demand of each period exactly with the cheapest type. If internally generated funds are not available, the cheapest source is usually stocks and the optimal solution of the problem (4.5), (4.6) and (4.7) above will be:
If internally generated funds are available, they will always be the cheapest source and the optimal solution, although more complicated, can be seen immediately. The amount of internal funds $X_{gt}$ invested in any period cannot exceed the available amount, say $X_{gt}$, which is the sum of accumulated earnings and accumulated depreciation. Let $F_i$ represent depreciation, $W_i$ profits and $\epsilon$ the retention rate, the following $n$ constraints should be added:

$$X_{gt} = X_{gt} \leq X_{gt} \text{ for } t=1, \ldots, n \quad (4.8a)$$

where

$$X_{gt} = \sum_{i=1}^{t-1} F_i + \sum_{i=1}^{t-1} \epsilon W_i.$$

But since $p_0 = 0$ and there always exist the possibility to invest funds outside at the positive revenue rate of $p_c$, every optimal solution must make maximum use of $X_{gt}$, hence eq. $(4.8a)$ simplifies to

$$X_{gt} = \bar{X}_{gt} \quad (4.8b)$$

for $t=1, \ldots, n$.

Without further constraints the optimal solution involving internally generated funds will be as follows. If in any period $X_{gt}$ happens to be just equal to $\bar{X}_t$, no other type funds are needed, hence for that period:
\[ X_{Gt} = \bar{X}_{Gt} = \bar{B}_t \]  
(4.8c)

while \[ X_{St}, X_{Qt}, X_{ct} \text{ and } X_{Ht} \text{ are 0}. \]

If in any period \( X_{Gt} \) exceeds the demand for funds \( \bar{B}_t \), the surplus will be invested outside so that for that period:

\[ X_{Gt} = \bar{X}_{Gt} \]
\[ X_{St} = \bar{X}_{Gt} - \bar{B}_t \]  
(4.8d)

while \[ X_{Jt} = X_{St} = X_{Qt} = X_{Ht} = 0. \]

If the cheapest source of funds \( \bar{X}_{Gt} \) cannot satisfy \( \bar{B}_t \), the rest is financed with the next cheapest source \( X_{St} \), hence for that period for which \( \bar{X}_{Gt} < \bar{B}_t \),

\[ X_{Gt} = \bar{X}_{Gt} \]
\[ X_{St} = \bar{B}_t - \bar{X}_{Gt} \]  
(4.8e)

and \[ X_{Jt} = X_{Qt} = X_{Ht} = X_{ct} = 0. \]

In eq. (4.8) and (4.8e) the assumption is that stocks can be issued and stock funds be paid back to the stockholders at the beginning and the end of any period. This is not realistic. Resources acquired by issuing stocks are usually not repaid unless the firm liquidates and then there are no resources left anyway. A first possible constraint could be the requirement that stocks if used at all must be used in the same amount throughout the planning period, a constraint which is implicit in the solution of the critical period analysis above. This adds the following constraint to the model:
\[ x_{s1} = x_{s2} = \ldots = x_{sn} = x_s \]  \hspace{1cm} (4.9)

More realistic would be to assume that equity capital can go up but not down, hence

\[ x_{s1} \leq x_{s2} \leq \ldots \leq x_{sn} \]  \hspace{1cm} (4.10)

Now the optimal solution can no longer be seen immediately and no longer is \( X_{jt} = X_{qt} = X_{lt} = X_{ct} = 0 \). But if we assume enough competition in each capital market to let the unit interest costs be independent of the amounts acquired by the firm, eq. (4.5) will be a linear polynomial. Because eqs. (4.6), (4.7), (4.8), (4.9), and (4.10) are also linear, the model can be solved by standard linear programming techniques.

For illustration the previous 3-period example will be put in the above form and solved by the simplex method.\(^1\)

**Case 1.** If there is no dependency between the \( X_{st} \) of the various time periods the model is:

\[
\text{Min: } K = 0.04 \, x_{s1} + 0.04 \, x_{s2} + 0.04 \, x_{s3} + 0.08 \, x_{l1} + 0.08 \, x_{l2} \\
+ 0.08 \, x_{l3} - 0.02 \, x_{c1} - 0.02 \, x_{c2} - 0.02 \, x_{c3}
\]

Subject to: \[ X_{S1} + X_{H1} - X_{c1} \geq 50 \]
\[ X_{S2} + X_{H2} - X_{c2} \geq 70 \] (4.11)
\[ X_{S3} + X_{H3} - X_{c3} \geq 60 \]
all \( X_{jt} \geq 0 \)

Because of the special form of the model as evident in the pattern of coefficients, a basic feasible solution which is also optimum is immediately available in \( X_{S1}, X_{S2}, \) and \( X_{S3} \) at values of respectively 50, 70 and 60, with a minimum total cost of 7.2. This is the solution of eq. (4.8) above.

Case 2. If we add the weak constraint (4.9) to the problem:

\[(4.6) + (4.5) + (4.7),\] this general model simplifies to:

Min:
\[ K = n \cdot p_s \cdot X_s + \sum_j \sum_t X_{jt} \cdot p_j - \sum_t X_{ct} \cdot p_c \] (4.12)

Subject to:
\[ X_s + \sum_j X_{jt} - X_{ct} \geq B_t \text{ for } t = 1, \ldots, n \] (4.13)

\[ X_s, X_{ct}, X_{jt} \geq 0 \] (4.14)

and \( j = Q, H, G, J \)

\( (p_G = 0). \)

Applied to the numerical example under consideration, (4.11) simplifies into:

Min:
\[ K = .12 X_s + .08 X_{H1} + .08 X_{H2} + .08 X_{H3} - .02 X_{c1} \]
\[ - .02 X_{c2} - .02 X_{c3} \]
Subject to: \[ \begin{align*}
X_s + X_{H1} - X_{c1} &\geq 50 \\
X_s + X_{H2} - X_{c2} &\geq 70 \\
X_s + X_{H3} - X_{c3} &\geq 60 \\
X_s, X_{Ht}, X_{ct} &\leq 0
\end{align*} \] (4.15)

Via the simplex method the following optimal solution emerges:

\[ X_s = X_{s1} = X_{s2} = X_{s3} = 60 \]
\[ X_{H2} = 10 \]
\[ X_{c1} = 10 \] (4.16)
\[ K = 7.8 \]

This is of course the same solution as derived above when eq. (4.1) was applied to the example. This confirms the argument that the critical period analyses implicitly assumes the constancy of \( X_{st} \) over time, an unnecessarily limiting condition.

Case 3. A more realistic assumption for stocks is eq. (4.10); the amount of stocks outstanding can be raised but not lowered over time. The basic model then consists of eqs. (4.5), (4.6), (4.7), plus (4.10). Applied to the example we have:

\[ \text{Min. } K = 0.04 X_{s1} + 0.04 X_{s2} + 0.04 X_{s3} + 0.08 X_{H1} + 0.08 X_{H2} + 0.08 X_{H3} - 0.02 X_{c1} - 0.02 X_{c2} - 0.02 X_{c3} \]
Subject to:  
\[ \begin{align*} 
X_{s1} + X_{h1} - X_{c1} & \geq 50 \\
X_{s2} + X_{h2} - X_{c2} & \geq 70 \\
X_{s3} + X_{h3} - X_{c3} & \geq 60 \\
- X_{s1} + X_{s2} & \geq 0 \\
- X_{s2} + X_{s3} & \geq 0 \\
X_{ct}, X_{jt} & \geq 0 \\
\end{align*} \]

(4.17)

Again applying the simplex method, will give the optimal solution of:

\[ \begin{align*} 
X_{s1} &= 50 \\
X_{s2} = X_{s3} &= 70 \\
X_{c3} &= 10 \\
K &= 7.4 \\
\end{align*} \]

Hence the cheapest solution which will still satisfy all constraints is to float stocks at the beginning of the first period at an amount of 50, float an additional 20 at the beginning of the second period and invest the 10 which are not needed during the third period outside the firm.

The advantages of structuring the problem this way are numerous. Although for purposes of demonstration only five variables were included in the model and only three in the numerical example, more may be added. There is in principle no limit to the number of variables or constraints that can be included.
As explained before, \( \bar{B}_t \) is the result of a management decision, based on incomplete information about uncertain future events. Management after some time might want to change its decision on \( B_t \) if more or different information becomes available. As long as the future is uncertain this can never be avoided.

But existing programming techniques in case the methods of "parametric programming"\(^1\) will permit to derive in advance the consequences of different values for \( B_t \) (the "parametric right side") and to select a compromise based on the (naturally subjective) probability distribution for \( B_t \).

The programming solution also tells that if all \( p \)'s change in the same direction and in the same proportion, the value of the optimum (total cost) might change but not the optimum position itself, (the financial structure). If prices do not change in the same proportion, "parametric cost row" programming tells if the optimum will change and by how much. Since interest rates usually move up and down together the changes will probably not be large anyway and it might very well happen that the optimal solution is virtually unaffected for the range of interest conditions which are likely to prevail.

---

\(^1\)Charnes and Cooper, op. cit., Ch. IX.
IV. 1.4. **Financial Flexibility and Endurance In the Capital Structure**

The uncertainty in income and cost influences the optimal capital structure via $B_t$ and $p_j$. It is often not possible to construct subjective probability distributions to use in decisions about $B_t$ and the optimal financial structure. In those cases firms are not able to prevent the results of uncertainty but they still want to mitigate the unfavorable consequences. In financial literature this is described as building into the capital structure enough "flexibility" and "resistance" or "endurance". If the actual development in costs and prices shows a different pattern from what was anticipated, flexibility in the structure will leave open the possibility to adapt the structure to the new circumstances. Financial endurance can be defined as the ability to receive financial blows without danger for the liquidity, solvability or control within the corporation. These considerations will have to be built into the model.

An adequate financial resistance requires in the first place the formation of liquid reserves, cash or near cash. From the standpoint of the investment decision cash is an investment project, in this case with an internal rate of return equal to zero but with also zero risk (assuming no price inflation or deflation).

As such the creation of liquid reserves belongs to the problem of the selection of an optimal project mix and for the decision over an optimal financial structure these reserves
are to be considered as part of the demand for funds, hence as given.

In the second place flexibility and endurance are preserved by maintaining certain ratio properties between financial categories and between the asset and financial structure.\(^1\) The optimal size of these ratios are subjectively set, depending on the riskiness of the particular business of the firm and the decision maker's preference between risk and return. Larger ratios usually mean higher safety but lower return and the firm sets the minimum required ratio so as to strike a balance between safety and monetary return required.

Two well known and widely used ratios are the current ratio and the acid test on quick ratio, the former usually but not always being defined as the ratio of current assets to current liabilities and the latter as the ratio of current assets minus inventories (hence cash, marketable securities, and accounts receivable) to current liabilities. Values often accepted for the minimum constraints on these ratios are respectively 2 and 1, although it follows from the above argument that this is a purely subjective matter and actual values can be at most crude rules of thumb. Both ratios cannot be included here since they are not limited to the financial structure alone but relate certain financial categories with the asset structure. Also, because of their current character they belong to the management of working capital and not to the capi-

tal budgeting problem. However, if the problem is to determine the total financial structure instead of the structure of the investment budget, current liabilities can be considered a source of funds (e.g. trade credit) and when the asset structure is known constraints for current liabilities may be constructed.

Thirdly, maintaining sufficient flexibility requires that stocks do not constitute too large a proportion of the total supply of funds. The total amount of capital raised by issuing stocks is rather constant for any firm; the amount is difficult to increase because the conditions for a successful stock issue are seldom met; in normal operations the amount of stock capital can rarely ever be decreased. Stocks can therefore only infrequently be used and will almost never be considered as a constantly available source of funds. Stock issues bring also with them a problem of internal control as explained in Ch. III.

Apart from its influence on the decision over the total amount of investment to be undertaken (see Ch. III.3.1.), a constraint on equity will also dictate a specific structure of the optimal financial budget. This will be considered in Ch. IV.1.5.

Also, for reasons explained in Ch. III.3.2. firms put constraints on the amount of debt they want to carry. The integration of these constraints in the financing of the investment budget is discussed in Ch. IV.1.6.

Finally, as explained in Ch. III.3.3. the ratio between
debt and equity, called financial leverage or gearing, relating two basically different financial categories, represents the decision maker's preference between risk and return. This constraint will be included in the model in Ch. IV.1.7.
Ch. IV. 1.5. A Constraint on Net Worth

If the manager, for reasons touched upon before, decides to refrain from using stocks in the financing of the investment budget the following constraint should be added:

\[ X_{st} = 0 \quad (4.18) \]

If no retained earnings are available, then

\[ X_{Q_t} = 0 \quad (4.19) \]

but this last constraint is redundant because of (4.8b).

If it is required that stocks form a certain fraction, \( \bar{s} \), or alternatively do not exceed a certain fraction of the total resources used to finance the investments budget, the constraint becomes:

\[ X_{st} = \bar{s} B_t \text{ or resp. } X_{st} \leq \bar{s} B_t \quad (4.20) \]
Ch. IV. 1.6. **Constraints on Liabilities**

Absolute or relative size constraints on liabilities can likewise be introduced. If no bank credit is to be used,

\[ X_{Ht} = 0 \]  \hspace{1cm} (4.21)

If no bonds are considered,

\[ X_{Qt} = 0 \]  \hspace{1cm} (4.22)

If bank credit or bonds should not exceed a certain portion, say respectively \( \bar{h} \) and \( \bar{q} \) of the total investment budget,

\[ X_{Ht} \leq \bar{h} \bar{E}_t \text{ and } X_{Qt} \leq \bar{q} \bar{E}_t \]  \hspace{1cm} (4.23)
Ch. IV. 1.7. **Constraints on Leverage**

The two arguments for the relevance of a leverage constraint were explained above (Ch. III.3.3.). Except for the initial financing of new firms the financing of the investment budget means an addition to the financial structure in existence. If the current debt, equity ratio constraint shows "slack", the existing structure has to be explicitly introduced. Suppose this is the case.

If $X_{SO}$ represents the total amount of stock capital outstanding before the current financing is contemplated, $X_{Ho}$ *mutatis mutandis* the amount of bank credit, $X_{Go}$ invested retained earnings and $X_{Qo}$ the amount of bonds outstanding, then if bank credit is included in the debt-equity ratio (a matter of dispute), the leverage requirement gives rise to the following (n) constraints:

\[
\frac{X_{Ho} + X_{Qo} + \sum_{i=1}^{t} X_{Hi} + \sum_{i=1}^{t} X_{Q1}}{\sum_{i=1}^{t} X_{G0} + \sum_{i=1}^{t} X_{Q0}} \leq \bar{\gamma} \quad (4.24)
\]

for $t = 1, \ldots, n$

where $\bar{\gamma}$ is the maximum debt-equity ratio allowed.

If bank credit is not included, the constraint becomes:

\[
\frac{X_{Qo} + \sum_{i=1}^{t} X_{Q1}}{\sum_{i=1}^{t} X_{G0} + \sum_{i=1}^{t} X_{Q0} + \sum_{i=1}^{t} X_{S1} + \sum_{i=1}^{t} X_{G1}} \leq \bar{\gamma} \quad (4.25)
\]

for $t = 1, \ldots, n$
If originally, before the additional financing was considered, there was no slack in the leverage constraint; eq. (4.24) and (4.25) reduce to resp:

\[ \gamma_t = \frac{t}{\sum_{i=1}^{t} X_{HI} + \sum_{i=1}^{t} X_{G1}} \leq \bar{\gamma} \]  
(4.26)

\[ \gamma_t = \frac{\sum_{i=1}^{t} X_{G1}}{\sum_{i=1}^{t} X_{S1} + \sum_{i=1}^{t} X_{G1}} \leq \bar{\gamma} \]  
(4.27)

for \( t = 1, \ldots, n \).

Of course with constraints on the amounts of equity and debt (Ch. IV, 1.5 and IV, 1.6) as well as on the debt-equity ratio (Ch. IV, 1.7) care must be taken to avoid inconsistencies and redundancies. For example, suppose that before the new financing is considered the leverage constraint is satisfied as an equality \( \gamma_o = \bar{\gamma} \). Suppose further that \( t = 1 \) and that the only sources to finance investment are \( X_H, X_Q \) and \( X_S \).

Let also \( h = \frac{X_H}{B_t} \leq \bar{h} \) and \( q = \frac{X_Q}{B_t} \leq \bar{q} \), where \( \bar{h} \) and \( \bar{q} \) are given. From eq. (4.6) it follows that \( X_H + X_Q + X_S \geq B_t \), hence \( s \geq 1 - (h + q) \).

As a result:

Setting \( s \) in eq. (4.20) \( < 1 - (\bar{h} + \bar{q}) \) means inconsistency and setting \( s \geq 1 - (\bar{h} + \bar{q}) \) means redundancy.

At the same time it follows from eq. (4.26) that
\[ \gamma = \frac{X_H + X_0}{X_S} = \frac{X_H}{B_t} + \frac{X_0}{B_t} = \frac{h + q}{s}. \]

But given (4.28) that \( s > 1 - (h + q) \),

\[ \gamma \leq \frac{h+q}{1-(h+q)} \leq \frac{\bar{h} + \bar{q}}{1-(\bar{h} + \bar{q})} \]  

(4.29)

Hence setting \( \bar{\gamma} \) in \( \gamma > \frac{\bar{h} + \bar{q}}{1-(\bar{h} + \bar{q})} \) gives inconsistency and setting

\[ \gamma \leq \frac{\bar{h} + \bar{q}}{1-(\bar{h} + \bar{q})} \]  

gives redundancy .

Thus the combination of the assumptions of this case \( (\gamma_0 = \bar{\gamma}, t = 1, x_1 = 0 \text{ for } i = c, g, j) \) and eqs. (4.6), (4.26), (4.20) and (4.23) results in 2 degrees of freedom for \( \bar{s}, \bar{h}, \bar{q} \) and \( \bar{\gamma} \). Selecting any two of these determines the remaining two.
IV.1.8. Other Constraints

Many restrictions externally imposed on the firms by fund suppliers, law or other regulations concerning the amounts, cost and timing of the various financial instruments used to finance investment can easily be put in the form of a linear inequality and introduced into the model above.

As an example consider the practical impossibility of floating very small amounts of stocks in the open market. Assume that either because of the regulations of the S.E.C., the N.Y.S.E., and A.S.E. or the W.C.S.E., or because of the attitudes and customs of the underwriters and the market a minimum amount of $X_S$ for a stock issue is required. This gives rise to the following $(n)$ constraints:

$$X_{st} - X_{st-1} \geq X_S \quad (4.30)$$

for $t=1, \ldots, n$.

Of course also maximum bounds due to the attitude of underwriters and markets may exist for stocks and also for bonds. For instance if the maximum amount of bonds that can be floated at any time is, say, $X_Q$, the corresponding constraints become:

$$X_{Qt} - X_{Qt-1} \leq X_Q \quad (4.31)$$

for $t=1, \ldots, n$.

Other external limitations on the use of funds by fund sup-
pliers can similarly be transformed in linear inequalities and added to the model\textsuperscript{1}.

\textsuperscript{1}Beranek \textit{op. cit.}, p. 14.
Ch. IV.1.9. The Revenue Maximizer

As shown in Ch. III.4, a firm that wants to maximize total revenue will usually invest more than the firm which maximizes total profits. Thus $B_t$ will be larger for the former. Since economies of scale and limiting constraints are nearly always present, also the optimal financial structure of the investment budget will be different from that of the profit maximizer under otherwise similar circumstances.

Apart from this scale effect no basic differences are to be expected between the financing methods of the sales maximizer and the profit maximizer. The reason for this is the existence of the profit constraint. If no profit constraint exists, the financial structure will be indeterminate since the sales maximum is independent of financial costs and can be reached at an infinite number of possible financial combinations. The profit constraint forces the firm to economize on its resources. The argument is much the same as Baumol's when he states that a sales maximizer, bound by the profit constraint, will use the same input combinations for the same output as a profit maximizer would.\(^1\) The difference between maximum attainable profit and the amount of profit the rates maximizer is satisfied with can be considered a fund of resources which is spent by the firm in its effort to increase rates.

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\(^1\) Baumol op. cit., p. 56.
In order to get maximum benefit (maximum sales) from this fund it should be spent so as to equalize the marginal revenue yield of a dollar of profit sacrificed in all directions. Investment will be carried on past the point of maximum profits and the marginal profit yield will be negative. The cost of financing investment will not be defrayed at the margin by the revenue of the investment projects. But the sales maximizer will still have to economize on his resources in the sense that the marginal revenue yield of a dollar sacrificed by financing investment with, say, stocks should be equal to the marginal revenue yield of a dollar spent by financing investment with bonds. In other words, the sales maximizer bound by the profit constraint will finance each given investment so that total financial costs, given the constraints, are minimum. The model developed in Ch. IV.1.3 is still applicable in this case.
IV.1.10. The Staff and Emoluments Model

In Ch. III.5.1. the conclusion was drawn that the manager with distinct preferences for certain classes of current expenditures, such as staff and emoluments, which enable him to satisfy personal motives like power, prestige, etc., will spend more on staff than the profit maximizer will. He also will absorb some profits as emoluments so that both his reported and his unreported profits are likely to be lower than the profit-maximizer's in the same circumstances. But his interest in profits, one of the determinants of his utility function, makes that he uses the same financing method as the profit maximizer and the optimal size of the investment budget will be reached when the marginal net rate of return (taking into account staff expenses) equals the marginal cost of capital. The total volume of investment will be larger than it would be for his profit-maximizing counterpart. This is because the larger staff expenditures (general, administrative and selling staff) will shift the demand curve to the right, increase marginal revenue, production and hence productive capacity and total assets. So the marginal rate of return curve will be above the one for the profit maximizer, however, the average rate of return curve will be lower and so will be total profits.

As in the previous section $B_t$ will exceed the $B_t$ of the profit maximizer and because of economies of scale a different financial structure will result.
Another difference with the pure profit maximization case, where the manager is only interested in a minimum cost financial structure, is that an utility maximizing manager will select a financial structure that will give him the greatest opportunity for discretion. This may result in a preference for bonds in place of stocks to avoid stockholders' interferences, issuance of preferred stock with limited voting rights, etc.

Still another difference follows from his emoluments preference. Very often managers share in the firm's profits, mostly net profits after subtraction of dividends, etc. Changing the financial structure often changes the de facto profit distribution system and the utility maximizing manager may be expected to use this to his advantage.

The management will be able to integrate these preferences into the optimal financing model of Ch. IV.1.3. with the kind of constraints on net worth, bonds, leverage, etc., introduced and discussed in Ch. IV.1.5. and following. Noticeable effects will be that the financial sources favored will be attracted to a larger extent than would be the case if these preferences were absent and that the marginal cost of those favored sources will be higher than the marginal return on investment as a whole. The opposite will be true for the financial sources avoided.
IV.2. Some Remarks About the Selection of the Optimal Combination of Investment Projects

IV.2.1. Proposed Solutions

The problem of selecting the optimal project mix when the objective is to maximize profits (discounted to take into account the time factor) and when the total amount to be invested is given (capital rationing) is not a new one.

Various solutions have been proposed but not all are optimal.

The first, in analogy of the case of no rationing\(^1\), is to rank all investment projects in order of decreasing rate of return and all financial capital available in order of increasing unit costs and to cut off at the point where the two cross. The method is not optimal since there is no guarantee the total outlay for the accepted projects will fall within the budget.

A second method is to rank projects in a similar way and to accept projects from the top of the list until the accumulated outlay equals the budget. As can be shown\(^2\), this method is not optimal either.

The method advocated most is to rank projects on the basis of their present worth per dollar of outlay required.

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\(^1\) See Ch. I.

\(^2\) Lorge and Savage *op. cit.*., p. 23. See also Hirshleifer, *op. cit.*., p. 17.
and to cut off when the budget is just exhausted. The method is optimal, that is selects the set of projects with maximum total present worth given the budget ceiling only if the budget is for one period.¹ If the budget is set up for more periods than it may happen that each period requires a different ranking because the projects' outlays change from period to period. Then the method breaks down.

¹Lorie and Savage op. cit., p. 231.
IV.2.2. A Programming Solution For Maximum Present Value

In his thesis, H. M. Weingartner\(^1\) first showed how the powerful techniques of linear and integer programming can be applied to the problem under consideration.

Let the budget ceiling for year \(t\) \((t = 1, \ldots , T)\) be denoted by \(B_t\) and let \(b_j\) represent the present value of project \(j\) \((j = 1, \ldots , n)\). Furthermore let \(c_{tj}\) be the cost of project \(j\) in period \(t\). Then the model for selecting among independent projects the set whose present value is maximum but whose accumulated outlay in each period stays within the budget constraint is:

\[
\text{Max } \sum_{j=1}^{n} b_j x_j
\]

Subject to \(\sum_{j=1}^{n} c_{tj} x_j \leq B_t\) for \(t = 1, \ldots , T\)

\[0 \leq x_j \leq 1\]

where \(x_j\) is the fraction of project \(j\) accepted.

The advantages of structuring the problem this way are the following:\(^2\)

In the first place the solution can be derived with relative simple computational methods (e.g. adjacent extreme

\(^1\) Weingartner, Mathematical Programming, op. cit., p. 17

\(^2\) Ibid, p. 24-27.
points methods). The upperbound on $X_j$ assures that the total budget is not spent on multiples of the best projects. It may lead to the acceptance in the optimal solution of fractional projects but because the number of fractional projects will not exceed the (limited) number of time periods, this is not too much of a problem. (Fractional projects will not appear if the problem is set up as an integer programming problem by requiring the $X_j$ to be integer.)

In the second place proper values given to the dual variables will allow to make a sharp distinction between integrally accepted projects, partially accepted projects, integrally rejected projects and marginally rejected projects.

In the third place questions of dependent projects, manpower bottlenecks, etc., can conveniently be put in linear constraint form and added to the problem.

In this way the programming solution proves to be very general. The same approach can also be used to introduce other objectives into project selection as demonstrated in the next section.
IV.2.3. The Influence of Alternative Objectives

As shown in Ch. IV.1. the financial constraints introduced in Ch. III have a direct influence on the financing of the investment budget. They also have a direct influence on the size of the budget itself as demonstrated in Ch. III. For if the capital market is imperfect the various financial instruments are not perfectly substitutable for each other and an upperbound on one type of capital may limit the size of the total budget below what it otherwise would be. Since projects are usually not perfectly divisible the optimal set of projects when financial constraints are introduced may differ from the optimal set in the unconstrained case. Apart from this size-effect the model of IV.2.2. can still be used to determine the optimal project mix in these constrained profit maximizing cases.

At a first glance the sales maximization hypothesis seems to lead to a different conclusion. Surely it is true that some projects, although not more profitable than others, are better revenue producers and *vice versa*. Therefore it seems reasonable to expect a different project mix if the objective is maximizing revenue rather than maximizing profits.

Due to the budget ceilings this is not true. We can see this easily if the budget is for one period only. For maximum discounted profits the projects should then be ranked on the basis of the ratio of present value to outlay required and the optimal set will be the top part of the list (highest ratio's) with an accumulated outlay that will just ex-
haust the budget. For maximum sales revenue the projects should be ranked on the basis of the ratios of revenue to total outlay and the optimal set will be the group with the highest sales - outlay ratios that will just exhaust the budget. It appears that in both cases the ranking is the same which can be demonstrated as follows:

Let \( i \) and \( j \) be two arbitrarily selected projects from the first (profit to outlay) ranking and let it be assumed that \( i \) has a higher ranking than \( j \). Thus

\[
\frac{b_i}{c_i} > \frac{b_j}{c_j}
\] (4.33)

where \( b \) is the profits and \( c \) is the cost associated with each project. Let total revenue of \( i \) and \( j \) be represented respectively by \( a_i \) and \( a_j \), then

\[
b_i = a_i - c_i, \text{ and}
\]

\[
b_j = a_j - c_j.
\] (4.34)

Then if \( \frac{b_i}{c_i} > \frac{b_j}{c_j} \), it is also true that

\[
\frac{a_i}{c_i} = \frac{b_i + c_i}{c_i} > \frac{b_j + c_j}{c_j} = \frac{a_j}{c_j}
\] (4.35)

Thus in the second ranking (sales to revenue) project \( i \) will also rank higher than project \( j \). Since \( i \) and \( j \) were arbitrarily selected, the conclusion must be that both cri-
teria rank the projects in the same way. Since the cut off point (accumulated outlay equals budget) is in both cases also the same, it is clear that the optimal set of projects for the sales maximizer will be the same set of projects that is optimal for the profit maximizer if both have the same one period budget to start with.

If the projects have costs and returns spread out over many periods and varying from period to period while each period has a different budget ceiling, the ranking procedure no longer applies. But as long as the budgets are completely exhausted (and with the possibility of fractional projects this will always be the case since it will pay to let no slack develop in the budget constraint) the conclusion that for a given budget the sales maximizer and the profit maximizer will select the same set of projects still stands. This can be shown by using the L.P. formulation.

Let $c_j$ be defined as $\sum_{t=1}^{T} c_{tj}$. Thus $c_j$ represents total discounted cost of project $j$ ($j = 1, \ldots, n$). In the same way let $a_j$ represent total discounted revenue of project $j$, defined as $\sum_{t=1}^{T} a_{tj}$. Hence,

$$b_j = a_j - c_j.$$ 

The model for selecting the set of projects whose total discounted profits are maximum but whose accumulated outlay in each period stays within the budget constraint is given in (4.32) as
Max \[ \sum_{j=1}^{n} b_j X_j \]

Subject to \[ \sum_{j=1}^{n} a_{tj} X_j \leq \bar{B}_t \] for \( t = 1, \ldots, T \)
\[ 0 \leq X_j \leq 1 \] .

The model for selecting the set of projects whose total present value of sales is maximum and whose total outlay in each period stays within the budget ceilings is:

Max \[ \sum_{j=1}^{n} a_j X_j \]

Subject to \[ \sum_{j=1}^{n} c_{tj} X_j \leq \bar{B}_j \] for \( t = 1, \ldots, T \) \( (4.36) \)
\[ 0 \leq X_j \leq 1 \] .

The objective function of \((4.32)\) differs from the one in \((4.36)\) but the set of constraints is the same. But if all budget ceilings are exhausted:

\[ \sum_{j=1}^{n} c_{tj} = \bar{B}_t \] for \( t = 1, \ldots, T \)

and \[ \sum_{j=1}^{T} c_j = \sum_{t=1}^{T} \sum_{j=1}^{n} c_{tj} = \sum_{t=1}^{T} \bar{B}_t \] .

Hence, \[ \sum_{j=1}^{n} b_j X_j = \sum_{j=1}^{n} X_j (a_j \otimes c_j) = \sum_{j=1}^{n} (a_j X_j - c_j X_j) = \]
\[ \sum_{j=1}^{T} a_j X_j - \sum_{j=1}^{n} c_j X_j = \sum_{j=1}^{T} a_j X_j - \sum_{j=1}^{T} (X_j \sum_{t=1}^{T} c_{tj}) = \]
\[ \sum_{j=1}^{T} a_j X_j - \sum_{t=1}^{T} \bar{B}_t \] .
where $\sum_{t=1}^{T} E_t$ is constant.

Hence the objective function of (4.32) differs from the one of (4.36) only by a constant. As a result the value of the optimum of (4.32) will be lower than the value of the optimum of (4.36), the difference being $\sum_{t=1}^{T} E_t$, but the program, the set of projects selected, will be the same.

Basically what is proven here is the statement that given total cost, that group of projects that maximizes revenue will maximize profits also for the simple reason that profits is the difference between revenue and costs.\(^1\)

Of course, if indivisibilities are present which prevent the budgets from being exhausted, or if not all costs associated with a project are caught in the investment budget (as will nearly always happen in practice where capital budgets carry only limits for long-term expenses and current budgets regulate short-term expenses) differences will develop.

In the case of the utility maximizing manager the conclusion is not much different.

The manager has a preference for certain types of costs that enable him to satisfy personal motives like power, prestige, etc. But in the model adopted above expense preference is for the class of current expenses like staff and emoluments. Thus the selecting of the optimal set of investment projects is not directly influenced. As mentioned before the size of the investment budget will be affected and if

\(^1\)See Baumol op. cit., p. 58 for a similar case.
indivisibilities are present, project selection is affected too. But for the same size budget the same optimal set of projects will be selected by following the same set of rules.

Of course, much depends on the utility function in question. The one accepted here has staff, emoluments and discrete profits as determinants and it is the last one that links the model with the more conventional constructions. A utility function can no doubt be constructed that will result in quite a different selection of projects for instance one in which the manager has an aversion against profits. But this seems a priori too special a case to warrant attention.

Another complication is introduced if the motives attributed to the manager can be expressed by certain long term investment projects; luxury office buildings, executive limousines and company yachts being cases in question (see Ch. III.5.).

The model of Ch. IV.2.2. can still be used by leaving these projects outside the model and deduct their cost from the budget. Another method would be to deliberately overestimate the present value of these projects until they are forced into the optimal solution.
CHAPTER V
CONCLUDING REMARKS

V.5.1. Summary

Capital budgeting literature so far has focused its attention exclusively on models which assume the decision maker's only goal is to maximize profits. This dissertation is an attempt to introduce alternative objectives into micro-investment theory.

Chapter II reviewed some well-known capital budgeting models which seem to be similar in that all conclude investment should (will) be carried on by the firm until the marginal rate of return equals the marginal cost of capital. But since all define the cost of capital differently the models in fact lead to quite different investment behavior. It was shown the differences in opinion about the "correct" definition of the cost of capital are due to different interpretation of the profit maximization hypothesis. In this way some light was shown on a confusing discussion that has been going on for years. At the same time the importance of introducing alternative goals into the theory of investment became evident. If the small variations on the general profit maximization theme analyzed in Chapter II can account for significant differences in investment behavior, then the construction and analysis of non-profit maximization investment models becomes the more worthwhile.

In addition, some rules of thumb often used by business as guidelines in investment decisions were analyzed in the
same framework. It was shown these can be considered either as approximations to optimality rules following from some variation on the profit maximization rule or as representing financial constraints.

In Chs. III and IV various capital budgeting models based on constrained profit maximization and non-profit maximization objectives were introduced. Chapter III discussed the decision over the total size of investment. Chapter IV.1. was devoted to the treatment of the problem of the optimal financing of the proposed investment budget while Ch. IV.2. analyzed the consequences of the different objectives for the selection of the optimal combination of investment projects. Since it was not the intention to develop new theories about managerial motives but only to analyze the influence on investment behavior of some existing ones, the motives as such were taken as given.

Four different models were developed, two based on constrained profit maximization, one on revenue maximization and one on the maximization of the manager's own utility.

The basic model type used in Ch. III is a modified version of the so called "Wealth Model". The special and rather unreal behavioral constraint of that model was replaced by the balance equality constraint and the resulting generalized model was made to accept non-profit maximization objectives. For each of the objectives accepted the maximization conditions were developed and analyzed. Comparisons were made between the results in each case and the
traditional profit maximization case. In addition, a micro-investment equation was constructed for each objective.

In Ch. IV.1, a connection was sought with the "critical period" solution to the problem of the optimal financial structure. However, apart from being very special, the critical period solution proved to be non-optimal. Therefore a new solution was developed using programming techniques. This new method was first applied to the pure profit maximization model but it proved to be general enough to accept the other alternatives under consideration as well. In Ch. IV.2, programming methods were used again to solve the problem of the determination of the optimal project mix under the non-profit maximization assumptions.
V.2. Directions for Further Research

The objectives integrated in investment theory in this dissertation were selected because they recently have received much attention and because convincing arguments have been made for their relevance. The conclusions reached in Chapter III and IV are therefore interesting in their own right. But the applicability of the models of Chapters III and IV is not limited to these four cases. Other, less well known objectives could be substituted into the models. For example in the theory of oligopolistic competition maintaining market shares is sometimes thought to be an important consideration. "Translating" this in a minimum sales constraint and substituting into the models will make possible the derivation of the corresponding investment behavior.

Apart from some incidental remarks nothing could be said about the empirical relevance of the investment behavior discussed. In order to do this the various models should be tested. The investment equations derived are all micro, thus an aggregation problem will result. But since all but one of the investment equations are linear, there is hope the aggregation problem can be solved without too much trouble using existing aggregation techniques. Another problem associated with statistical testing is the acquisition of relevant data. Finally some allowance has to be made for uncertainty. The aggregation procedure may introduce a stochastic variable, convenient for testing. Also Ch. IV.2
discusses the incidence of financial risk to a certain extent. But it seems advisable to introduce uncertainty directly into the models of Ch. III and IV, for instance along the lines discussed in Ch. I.
APPENDIX A

1. **Derivation of the Optimal Stock of Assets for the Unconstrained Profit Maximizing Case**

The equilibrium conditions were: (Eq. (3.11))

\[ P(Q) \cdot a + a \cdot X_A \cdot \frac{d(P(Q))}{dQ} \cdot a - ab - \lambda P_A = 0 \quad \text{I} \]

\[ P_L + P'_L X_L - \lambda P_L = 0 \quad \text{II} \]

\[ P_E + P'_E X_E - \lambda P_E = 0 \quad \text{III} \]

\[ P_A X_A - P_L X_L - P_E X_E = 0 \quad \text{IV} \]

From II it follows: \( X_L = \frac{\lambda P_L - P_L}{p'_L} \quad \text{II'} \)

From III it follows: \( X_E = \frac{\lambda P_E - P_E}{p'_E} \quad \text{III'} \)

Substitution of II' and III' in IV gives:

\[ P_A X_A = P_E \cdot \left[ \frac{\lambda P_L - P_L}{p'_E} \right] + P_L \cdot \left[ \frac{\lambda P_L - P_L}{p'_L} \right] \]

\[ P_A X_A + \frac{P_E P_E}{p'_E} + \frac{P_L P_L}{p'_L} = \lambda \left[ \frac{P_E^2}{p'_E} + \frac{P_L^2}{p'_L} \right] \]
\[
\lambda = \frac{P_A X_A + P_E \cdot \frac{P_E}{p'_E} + P_L \cdot \frac{P_L}{p'_L}}{P_E^2/p'_E + P_L^2/p'_L}
\]

This result in Eq. (I) —

\[
p'_A \cdot X_A = P_A \cdot \frac{P_A X_A + P_E \cdot \frac{P_E}{p'_E} + P_L \cdot \frac{P_L}{p'_L}}{P_E^2/p'_E + P_L^2/p'_L} - p_A
\]

\[
p'_A \cdot X_A - P_A^2 \cdot X_A \cdot \left(\frac{P_E^2}{p'_E} + \frac{P_L^2}{p'_L}\right) = P_A \cdot \frac{P_E \cdot \frac{P_E}{p'_E} + P_L \cdot \frac{P_L}{p'_L}}{P_E^2/p'_E + P_L^2/p'_L} - p_A
\]

\[
X_A \left[ p'_A - P_A^2 \cdot \frac{p'_E \cdot p'_L}{p'_L \cdot P_E^2 + p'_E \cdot P_L^2} \right] =
\]

\[
= P_A \cdot \frac{p'_L \cdot P_E^2 + P_L \cdot P_E \cdot P_L}{p'_L \cdot P_E^2 + p'_E \cdot P_L^2} - p_A
\]

\[
X_A \left[ \frac{-p'_A \cdot p'_L \cdot P_E^2 + p'_A \cdot P_E \cdot P_L^2}{p'_L \cdot P_E^2 + p'_E \cdot P_L^2} \right] =
\]

\[
= \frac{P_A \cdot p'_L \cdot P_E + p'_E \cdot p'_L \cdot P_L}{p'_L \cdot P_E^2 + p'_E \cdot P_L^2}
\]

From this (3.14) is derived by dividing the right side of the above expression by the quantity inside the square brackets on the left.
2. Derivation of the Optimal Stock of Assets When No Liabilities Are Accepted in the Financial Structure

The equilibrium conditions were: (Eq. (3.31))

\[ P_A + P_A' X_A - \lambda P_A = 0 \]  \hspace{1cm} I

\[ P_E + P_E' X_E - \lambda P_E = 0 \]  \hspace{1cm} II

\[ X_A P_A - X_E P_E = 0 \]  \hspace{1cm} III

From III it follows that

\[ X_E = \frac{X_A \cdot P_A}{P_E} \] \hspace{1cm} III'

III' in II:

\[ \lambda = \frac{P_E + P_E' \left( \frac{X_A \cdot P_A}{P_E} \right)}{P_E} = \frac{P_E}{P_E} + \frac{P_E'}{P_E^2} \left( X_A \cdot P_A \right) \]

The last result substituted in I gives:

\[ P_A' \cdot X_A = P_A \left[ \frac{P_E}{P_E} + \frac{P_E'}{P_E^2} \left( X_A \cdot P_A \right) \right] - P_A \]

\[ = \frac{P_A \cdot P_E}{P_E} + \frac{P_E'}{P_E^2} \left( X_A \cdot P_A \right) \]

\[ X_A \left[ P_A' - P_E \frac{P_A^2}{P_E^2} \right] = \frac{P_A \cdot P_E}{P_E} - P_A \]

\[ X_A' = \frac{P_A \cdot P_E}{P_E} - P_A \]

\[ X_A^* = \frac{P_A \cdot P_E}{P_E} - P_A \]
Again substitute (3.2) and take the units of measurements such that \( P_A = P_E = \Pi \rightarrow \)

\[
X_A^* = \frac{\pi_E \cdot \Pi - \pi_A \cdot \Pi}{\pi_A \cdot \Pi - \pi_E \cdot \Pi} = \frac{\pi_E - \pi_A}{\pi_A - \pi_E}
\]

Setting \( \frac{-1}{\pi_A - \pi_E} = h_2 \geq 0 \),

and substituting in the previous expression for \( X_A^* \) will give Eq. (3.32).

3. Derivation of \( X_A^* \) when Financial Leverage Is Constant

From Eq. (3.1) it follows that \( P_A X_A - P_L X_L = P_E X_E \) \( \text{I} \)

From Eq. (3.36) it follows that \( P_L X_L = \gamma \cdot P_E X_E \) \( \text{II} \)

\( \text{I} + \text{II} \rightarrow \)

\[
\frac{P_A X_A}{P_E X_E} = \frac{P_A X_A}{P_A X_A - P_L X_L} = \frac{\gamma + 1}{\gamma}
\]

\[
P_E X_E = \frac{P_A X_A}{\gamma + 1} \quad \text{or} \quad X_E = \frac{P_A X_A}{P_E} \cdot \frac{1}{\gamma + 1} \quad \text{III}
\]

and
\[ P_L X_L = P_A X_A \cdot \frac{\gamma}{\gamma + 1} \quad \text{or} \quad X_L = \frac{P_A X_A}{P_E} \cdot \frac{\gamma}{\gamma + 1} \]  

Maximizing total profits subject to Eq. (3.1) and (3.36) is equivalent to:

\[ \max W = P_A X_A - P_L \left( \frac{\gamma}{\gamma + 1} \cdot \frac{P_A X_A}{P_L} \right) - P_E \left( \frac{P_A X_A}{P_E \cdot (\gamma + 1)} \right), \]

where III and IV were used to eliminate \( X_E \) and \( X_L \) as explicit variables from the profit function.

Maximum profits subject to the constraint on the debt - equity ratio will occur when:

\[ \frac{dW}{dX_A} = 0, \quad \text{or} \]

\[ P_A + P_A' X_A - \frac{P_L \cdot P_A}{P_L} \cdot \frac{\gamma}{\gamma + 1} - \frac{P_E \cdot P_A}{P_E (\gamma + 1)} = 0 \]

Solving for the optimal stock of capital \( X_A^* \)

\[ X_A^* = \frac{1}{P_A'} \left[ \frac{P_L \cdot P_A \cdot \gamma}{P_L (\gamma + 1)} + \frac{P_E \cdot P_A}{P_E (\gamma + 1)} - P_A \right] \]

Substitute (3.2) and take the units of measurement so as to equate \( P_A = P_E = P_L = \Pi \).
Derivation of $x^*_A$ for the Sales Maximizing Firm

The profit equation (with linear demand curve) is quadratic, namely

$$w = f_1 \cdot a \cdot x_A - f_2 \cdot a^2 \cdot x_A^2 - ab \cdot x_A - p_E \cdot x_A = \bar{w}$$

Solving this quadratic in the standard way gives two real roots, namely

$$x^*_A, x_A, = \frac{1}{2f_2^2} \left[ \left( f_1 \cdot a - ab - p_E \right) + \left( f_1 \cdot a - ab - p_E \right)^2 - 4 \cdot f_2 \cdot a^2 \cdot \bar{w} \right]^{1/2}$$
\[ X_{A,2}^* = \frac{1}{2 \pi^2} \left[ \frac{1}{(f_1 \cdot a - ab - \mu)} - \frac{1}{(f_1 \cdot a - ab - \mu)^2 - 4 f_2 \cdot a^2 \sigma^2} \right]^{1/2} \]

But only \( X_{A,1} \) is consistent with the objective of maximizing sales.

This can be shown as follows.

Assume there exist two points \( X_{A,1} \) and \( X_{A,2} \) such that \( X_{A,1} > X_{A,2} \). The corresponding values for total sales \( R_1 \) and \( R_2 \) will have the relation \( R_1 > R_2 \) if and only if:

\[ f_1 \cdot a X_{A,1} - f_2 \cdot a^2 X_{A,1}^2 > f_1 \cdot a X_{A,2} - f_2 \cdot a^2 X_{A,2}^2 \]

\[ f_1 \cdot a X_{A,1} - f_1 \cdot a X_{A,2} > f_2 \cdot a^2 X_{A,1}^2 - f_2 \cdot a^2 X_{A,2}^2 \]

\[ f_1 \cdot a (X_{A,1} - X_{A,2}) > f_2 \cdot a^2 (X_{A,1}^2 - X_{A,2}^2) \]

\[ f_1 \cdot a (X_{A,1} - X_{A,2}) > f_2 \cdot a^2 (X_{A,1} - X_{A,2}) (X_{A,1} + X_{A,2}) \]

\[ \frac{f_1 \cdot a}{f_2 \cdot a^2} > X_{A,1} + X_{A,2} \]

Comparing I and II it can be seen that

\[ X_{A,1}^* > X_{A,2}^* \]

Adding \( X_{A,1}^* \) and \( X_{A,2}^* \) gives:
From III it follows that therefore \( R_L^* > R_2^* \) if \( ab + p_E > 0 \).

Since \( ab + p_E \) represents the marginal cost of production plus the marginal cost of capital, which both are positive, the amount of assets \( X_{A,1}^* \) produces a higher total revenue than \( X_{A,2}^* \), and is this the only point consistent with the assumed objective. Thus the optimal stock of assets (capital) for the sales maximizer in this model is given by (3.43).
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