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A CONTINUUM THEORY OF HEAT CONDUCTING
FLUID SOLID MIXTURES WITH APPLICATIONS TO
TRANSIENT WAVE PROPAGATION

by

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1. Introduction

Problems involving porous solids permeated by fluids or mixtures of fluids have long been of interest to engineers and chemists. Until recently, analytical solutions have been limited to those problems which can be sufficiently modeled by assuming that the porous matrix is rigid. For a discussion of such problems, see, for example, Scheidegger. [1]*

A more complex and interesting class of problems arises when the deformation of the porous matrix itself, as well as that of the interstitial fluid, must be considered. Important examples of such problems include the flow of a fluid through a highly elastic porous matrix, the propagation of seismic waves through fluid saturated porous strata and the settling of structures situated on fluid saturated porous foundations.

In analyzing the latter problem, known in soil mechanics as the consolidation problem, M. A. Biot [2] made the first serious attempt to derive a linear constitutive theory for a medium consisting of a perfectly elastic porous solid satura-

*Numbers in brackets refer to references at the end of the paper.
ted by a single incompressible fluid. He postulated partial stress tensors in the porous solid and fluid constituents and, using largely intuitive arguments, related the partial stress tensors to the states of deformation in the solid and fluid constituents. By requiring the strain energy in the medium to be independent of path, he showed that the number of constants required to characterize the medium can be reduced to four. In a second, very important paper, Biot [3] addressed himself to the problem of the propagation of infinitesimal waves through a fluid saturated porous elastic medium. He assumed first that the quasi-static stress-strain relations he had derived earlier for the consolidation problem were also applicable to dynamic problems. He derived the equations of motion for the medium by using Lagrange's equations with the infinitesimal displacement vectors in the fluid and solid constituents as generalized coordinates. He was then able to extend the theory to a viscous fluid simply by assuming the presence of Rayleigh dissipation in the medium. Using the equations thus derived, he demonstrated the presence of three characteristic wave velocities and derived attenuation coefficients as a function of frequency.

Following the publication of Biot's second paper, several authors have used his equations to extend his investigation of steady-state wave propagation in a fluid saturated porous medium. Jones [4] has demonstrated the presence of surface
waves in the medium which are similar to the Rayleigh waves in an elastic medium. Deresiewicz [5,6,7,8] has exhaustively investigated the reflection and refraction of steady-state plane waves at an interface, and Kosachevskii [9] has briefly examined the propagation of steady-state plane waves in a layered, fluid saturated porous medium.

In a parallel development, several authors have investigated the constitutive theory of a fluid saturated porous medium from the continuum viewpoint of fluid-solid mixtures. The basis for a continuum theory of mixtures of interdiffusing materials was originally established by Truesdell [10,11], who used the concept of superimposed continua and derived balance and conservation equations for each constituent by introducing supply, or transfer terms for mass, momentum, energy and entropy. Using the work of Truesdell as a foundation, Adkins [12,13,14,15,16,17,18], Green and Adkins [19], and Green and Naghdi [20] have formulated a series of theories of nonlinear and linear diffusion for both fluid and fluid-solid mixtures. As pointed out by Truesdell and Noll [21] in a recent review of the state of the subject, the continuum theories have been marked by uncertainties concerning the proper pattern of the constitutive variables. In particular, if an attempt is made to derive equations analogous to Biot's equations from the continuum theories cited above, paradoxical and inadmissible results are obtained. Truesdell and Noll also note that the resolution of the difficulty almost
certainly requires a proper generalization of the Clausius-Duhem inequality.

In a series of papers on the theory of chemically reacting fluid mixtures, Eringen and Ingram [22,23] have achieved this generalization of the Clausius-Duhem inequality by the inclusion of explicit transfer terms which account for the dependence of the internal energy and entropy of each constituent on independent variables associated with each other constituent. By this means they have obtained a constitutive theory of fluid mixtures in which the inconsistencies of the earlier theories are absent.

In this thesis, the fundamental postulates and laws of continuum mechanics and the Ingram-Eringen generalization of the Clausius-Duhem inequality are used to obtain a rational constitutive theory for a heat-conducting, porous elastic solid permeated by a mixture of \( n \) heat-conducting, compressible viscous fluids. Both the full nonlinear functional forms and the first-order isotropic expansions for the medium are presented. It is demonstrated that, when heat transfer and velocity gradient terms are neglected, equations constituting an extension to \( n \) fluids of the equations of Biot are obtained.

As an example application of the derived theory, a modification of the results of the linearized continuum theory is used to solve boundary-value problems in a porous elastic medium saturated by a single compressible fluid.
The problems solved include the transient response of a fluid-saturated, porous half-space due to normal and lateral excitation at the boundary and the transient response of a fluid-saturated, porous space due to excitation at a point source.

In this section, the field equations and constitutive theory for a heat conducting porous elastic solid permeated by a mixture of \( n \) heat conducting compressible viscous fluids are derived. The order of the development and the notation generally follow that of Eringen and Ingram [22,23]. First, the concept of superimposed continua is introduced, and the associated concepts of constituent velocity and density and the constituent material derivative are defined. The field equations, which embody the fundamental laws and postulates of mechanics and thermodynamics, are then derived for a mixture of \( n \) continua in the manner of Truesdell [10,11]. Fundamental constitutive variables are then postulated, and the restrictions placed upon these variables by objectivity and coordinate invariance requirements and by the Ingram-Eringen generalization of the Clausius-Duhem inequality are used to obtain a rational set of nonlinear constitutive forms for the medium. Finally, linearized constitutive equations are derived for two types of material symmetry.

2.1 Preliminaries

Consider two coordinate systems, a \textit{spatial} and a \underline{material} coordinate system, with position vectors designated by \( \mathbf{x} \) and \( \mathbf{x}^\prime \), respectively. The two coordinate systems need not be identical. They are both assumed to be stationary in the Newtonian sense. The coordinates of a material particle of
a continuum in the material coordinate system are assumed to be invariant with time. For this reason, it is sometimes referred to as the \textit{reference} coordinate system. The spatial coordinate system corresponds to the one familiar from particle mechanics.

Now let it be assumed that one solid and \( n \) fluid constituents are superimposed such that a spatial point \( \mathbf{x} \) may be occupied simultaneously by a material particle of each constituent. Let a material point in the \( \xi \)th constituent be denoted by \( \mathbf{X}(\xi) \) such that at time \( t \), the motion \( \mathbf{X}(\xi) \) of the \( \xi \)th constituent carries \( \mathbf{X}(\xi) \) into \( \mathbf{x} \).

\[
\mathbf{x} = \mathbf{X}(\xi)(\mathbf{X}(\xi), t)
\]  

(1)

The motions \( \mathbf{X}(\xi) \) are assumed to be smoothly invertible functions except possibly at certain singular points, lines and surfaces. Each constituent has an associated \textit{constituent velocity} defined by

\[
\mathbf{V}(\xi) = \frac{\partial \mathbf{X}(\xi)}{\partial t} \bigg|_{\mathbf{X}(\xi)}
\]  

(2)

and an associated \textit{constituent mass density} \( \rho(\xi) \) such that the mass density of the medium is given by

\[
\rho = \sum_{\xi} \rho(\xi)
\]  

(3)

Using equation (2), the material derivative of a tensor function \( \Psi \) following the motion of the \( \xi \)th constituent is defined by

\[
(\xi) \frac{D \Psi}{Dt} = \mathbf{V}(\xi) \bigg|_{\mathbf{X}(\xi)} + v_k(\xi) \Psi, k
\]  

(4)
where a semicolon denotes covariant differentiation.

Two fundamental results from field theory which will be used in deriving the field equations for the medium are stated here without proof. They are derived, for example, in Eringen [24].

A **material volume** \( V(\xi) \) associated with the \( \xi \)th constituent is defined as a volume whose surface has no normal component of flux of the mass of the \( \xi \)th constituent. Now consider a material volume \( V(\xi) \) which is intersected by a **discontinuity surface** \( \sigma \) propagating with velocity \( \bar{v}(d) \).

The first result is that the time derivative of the integral of a tensor function \( \Psi \) over a material volume \( V(\xi) \) can be written as

\[
\frac{d}{dt} \int_{V(\xi)} \Psi \ dV = \int_{V^{*}(\xi)} \left[ \frac{\partial \Psi}{\partial t} + (\nabla \bar{v}(\xi)) ; \hat{\kappa} \right] dV \\
+ \int_{\sigma} \left[ \Psi(\bar{v}(\xi) - \bar{v}(d)) \right] n \hat{\kappa} d\sigma
\]  

(5)

where \( V^{*}(\xi) \) denotes that part of \( V(\xi) \) not containing the neighborhood of the discontinuity surface, \( \hat{n} \) is a unit vector normal to \( \sigma \) in the direction of propagation and heavy brackets denote the jump in value of a quantity across the discontinuity surface in the direction of propagation. The second result is that the generalized Green-Gauss theorem for a vector function \( \bar{A} \) over the surface \( S(\xi) \) of \( V(\xi) \) can be written as
\[ \int_{\mathcal{S}(\xi)} A^R n^R dS = \int_{V(\xi)} A^R \delta^R dV + \int_{\sigma} \left[ A^R \right] n^R d\sigma \]  

The definitions and results of this section will now be used to derive the field equations for the medium.

2.2 The Field Equations

The scope of the theory being developed will be limited to classical mechanical and thermal phenomena. Electromagnetic phenomena, relativistic effects and quantum effects will not be considered. The fundamental laws and postulates of mechanics and thermodynamics for a continuum include the laws of conservation of mass and energy, balance of linear and angular momentum and the Clausius-Duhem inequality. In the following sections, the equations embodying these laws and postulates will be derived for a mixture of \( n \) continua.

2.2.1 Conservation of Mass

In addition to the assumptions stated above, it will be assumed that no chemical reactions or other mass transfer mechanisms, such as dissolution of the solid constituent, occur in the medium. It can therefore be stated that the time derivative of the integral of the mass density of each constituent over a material volume of that constituent must vanish.

\[ \frac{d}{dt} \int_{V(\xi)} \rho(\xi) dV = 0 \]
Using equation (5) and the independence of \( v^*_{\xi} \), then, the local equations of mass conservation can be written for each constituent as

\[
\frac{\partial \rho(\xi)}{\partial t} + \rho(\xi) v^*_\xi (\xi) \cdot \mathbf{n} = 0
\]  
(8)

\[
\left[ \rho(\xi) \left( v^*_\xi (\xi) - v^*_d (\xi) \right) \right] n^\xi = 0
\]  
(9)

2.2.2 Balance of Momentum

The balance of momentum equation is an extension to a continuum of Newton's second law. For the case of a single continuum, it states that the time derivative of the integral over a material volume of the momentum per unit volume must be equal to the total force acting on the material volume. The total force consists of surface forces, or stresses, and volume, or body forces. In the case of \( n \) continua, the equation remains valid if each term is summed over all the constituents. Assuming that there is a partial stress tensor \( t(\xi) \) and an extensive body force \( f(\xi) \) associated with each constituent, the total balance of momentum equation can be written as

\[
\frac{d}{dt} \sum_{\xi} \int V(\xi) \rho(\xi) v^*_\xi (\xi) dV = \sum_{\xi} \int_{\mathcal{S}(\xi)} t^{j\xi} n_j dS
\]

\[
+ \sum_{\xi} \int V(\xi) \rho(\xi) f^*_\xi (\xi) dV
\]

(10)

Applying equations (5) and (6) to equation (10) and using
the conservation of mass equations, the local balance of momentum equations become

\[
\sum_{\xi} \left[ t^{i,j}_{\xi} + \rho(\xi) \left( f^{i}_{\xi} - a^{i}_{\xi} \right) \right] = 0
\]  \hspace{1cm} (11)

\[
\sum_{\xi} \left[ t^{i}_{\xi} - \rho(\xi) v^{i}_{\xi} \left( v^{j}_{\xi} - v^{j}_{d(\xi)} \right) \right] n_{j} = 0
\]  \hspace{1cm} (12)

where the constituent acceleration \( a_{\xi}(\xi) \) is defined by

\[
a_{\xi}(\xi) = \frac{Dv}{Dt_{\xi}}(\xi)
\]  \hspace{1cm} (13)

Equations (11) and (12) are single equations, being summed over all constituents. Employing a very useful technique introduced by Truesdell [10], let momentum transfer vectors \( p_{\xi}(\xi) \) and \( p_{\xi}(\xi) \) be introduced so that momentum equations can be written for each constituent as

\[
t^{i,j}_{\xi} + \rho(\xi) \left( f^{i}_{\xi} - a^{i}_{\xi} \right) = -p^{i}_{\xi}
\]  \hspace{1cm} (14)

\[
\left[ t^{i}_{\xi} - \rho(\xi) v^{i}_{\xi} \left( v^{j}_{\xi} - v^{j}_{d(\xi)} \right) \right] n_{j} = -p^{i}_{\xi}
\]  \hspace{1cm} (15)

where

\[
\sum_{\xi} p^{i}_{\xi}(\xi) = \sum_{\xi} p^{i}_{\xi}(\xi) = 0
\]  \hspace{1cm} (16)

### 2.2.3 Balance of Moment of Momentum

The balance of moment of momentum equation is an exten-
sian to a continuum of the law of conservation of angular momentum in particle mechanics. It has been postulated that additional terms, alien to particle mechanics, may appear when this principle is applied to a continuum. These are the couple stresses, surface couples and body couples. For a discussion of these terms and their possible existence, see Eringen [25]. They will not be considered in this treatment.

In a manner analogous to classical mechanics, then, the integral form of the balance of moment of momentum equation can be obtained by taking the cross product of the balance of momentum equation (10) with a position vector \( \vec{r} \) to equate the rate of change of angular momentum to the total torque.

\[
\frac{d}{dt} \sum \int_{\xi} e_{n_{1}k} r_{j}^{i} \rho (\xi) \mathbf{v}^{\mathbf{h}} (\xi) dV = \sum \int_{\xi} S (\xi) e_{n_{1}k} r_{j}^{i} t^{i} n_{1} dS + \sum \int_{\xi} V (\xi) e_{n_{1}k} r_{j}^{i} \mathbf{f}^{\mathbf{h}} (\xi) dV
\]

(17)

\( e_{n_{1}k} \) is the alternating tensor. [25] Applying equations (5) and (6) to equation (17) and using the conservation of mass and balance of momentum equations, the balance of moment of momentum equation reduces to

\[
\sum_{\xi} e_{n_{1}k} \overrightarrow{t} (\xi) = 0
\]

(18)
Note that equation (18) implies that the sum of the partial stress tensors is a symmetric tensor, while the individual partial stress tensors can have anti-symmetric components. This interesting fact was stressed by Eringen and Ingram [22]. Indeed, it will be seen that the linear constitutive equations for the medium actually contain anti-symmetric terms, terms linear in the relative constituent vorticity tensors.

2.2.4 Conservation of Energy

The conservation of energy equation is an extension to a continuum of the first law of thermodynamics, which introduces the concept of internal energy. For the case of a single continuum, it states that the time derivative of the integral over a material volume of the sum of the kinetic and internal energies per unit volume must be equal to the total rate at which energy is being transmitted to the material volume. The sources of energy include work done by the forces acting on the material volume, heat transmitted through the surface of the material volume and distributed energy from external sources such as radiation. In the case of n continua, the equation remains valid if each term is summed over all the constituents. Note that the rate at which work is done by the forces acting on the material volume can be obtained in a straightforward way by taking the inner product of the force terms appearing in the balance of momentum equation (10) with the constituent velocities. There-
fore, assuming that there is a specific internal energy $e(\xi)$, a heat flux vector $q(\xi)$ directed into the material volume and a distributed external energy source $h(\xi)$ associated with each constituent, the total conservation of energy equation can be written as

$$\frac{d}{dt} \sum_{\xi} \int \left[ \frac{1}{2} \rho(\xi) V(\xi) \dot{V}(\xi) + \rho(\xi) e(\xi) \right] \, dV$$

$$= \sum_{\xi} \int_{S(\xi)} t^j_{\dot{\xi}} \dot{V}(\xi) n_j \, dS + \sum_{\xi} \int \rho(\xi) V(\xi) \dot{V}(\xi) \, dV$$

$$+ \sum_{\xi} \int_{S(\xi)} q_{\dot{\xi}} n_j \, dS + \sum_{\xi} \int h(\xi) \, dV$$

(19)

Applying equations (5) and (6) to equation (19) and using the conservation of mass and balance of momentum equations, the local conservation of energy equations become

$$\sum_{\xi} \left[ \rho(\xi) \frac{D e(\xi)}{D t} - t^j_{\dot{\xi}} V(\xi) n_j; n_j - h(\xi) \right] = 0$$

(20)

$$\sum_{\xi} \left[ \rho(\xi) \left( \frac{1}{2} V(\xi) \dot{V}(\xi) + e(\xi) \right) \left( V^j - V^j d \right) \right] n_j = 0$$

(21)
In a manner similar to that employed for the momentum equations, let energy transfer terms \( e(\xi) \) and \( E(\xi) \) be introduced so that equations (20) and (21) can be written for each constituent as

\[
\begin{align*}
\rho(\xi) \frac{D\mathbf{e}(\xi)}{Dt} - t_j^{\mathbf{h}}(\xi) \mathbf{v}(\xi) \mathbf{h}_j - \mathbf{h}(\xi) - q_j^{\mathbf{h}}; \mathbf{h} = e(\xi) \\
\left[ \rho(\xi) \left( \frac{1}{2} \mathbf{v}(\xi) \mathbf{v}(\xi) \mathbf{h} + e(\xi) \mathbf{v}(\xi) \mathbf{v}(d) \right) - t_j^{\mathbf{h}}(\xi) \mathbf{v}(\xi) \mathbf{h}_j - q_j^{\mathbf{h}} \right] \mathbf{n}_j
\end{align*}
\]

where

\[
\sum_{\xi} \left( e(\xi) + \mathbf{v}(\xi) \mathbf{h} \right) = \sum_{\xi} E(\xi) = 0
\]  

(23)

(24)

2.2.5 The Clausius-Duhem Inequality

The Clausius-Duhem inequality is an extension to a continuum of the second law of thermodynamics, which introduces the concept of entropy and constrains the dissipation of a system to be nonnegative. Postulating the inequality for a single continuum in the form suggested by Coleman and Mizel [26] and summing each term over all constituents, let it be assumed that there is an absolute temperature \( \theta(\xi) \) and a state variable, the specific entropy \( \eta(\xi) \), associated with each constituent such that

\[
\frac{d}{dt} \sum_{\xi} \int_V \rho(\xi) \eta(\xi) dV = \sum_{\xi} \int_S \frac{q_j^{\mathbf{h}}(\xi)}{\theta(\xi)} \mathbf{n}_j \mathbf{d}S
\]
\[ + \sum \int_{\xi} V \left( \frac{\eta(\xi)}{\theta(\xi)} \right) dV \]  

Applying equations (5) and (6) to equation (25), using the mass conservation equations and introducing entropy transfer terms \( \eta(\xi) \) and \( N(\xi) \), the local Clausius-Duhem inequalities can be written for each constituent as

\[ \rho(\xi) \frac{D\eta(\xi)}{Dt} - \left( \frac{q(\xi)}{\theta(\xi)} \right) ; \eta = \frac{\eta(\xi)}{\theta(\xi)} - n(\xi) \geq 0 \]  

\[ \rho(\xi) \eta(\xi) \left[ \left( v(\xi) - v(d) \right) - \frac{q(\xi)}{\theta(\xi)} \right] n_{\eta} = N(\xi) \geq 0 \]  

where

\[ \sum_{\xi} n(\xi) = \sum_{\xi} N(\xi) = 0 \]  

This completes the derivation of the field equations. Note that the derived equations are applicable to any \( n \) superimposed continua; they do not intrinsically restrict or specify the nature of the medium. In addition, an examination of the field equations shows that they do not constitute a determinate system, i.e., there are more unknown quantities than there are equations. In the next section, the constitutive relations which embody the characteristics of the medium and provide the additional equations required to obtain a determinate system are considered.
2.3 The Nonlinear Constitutive Forms

The constitutive equations for a medium cannot be obtained from continuum mechanics. The continuum concept automatically precludes consideration of the molecular, or, in the case of the particular medium under consideration, the pore structure of the solid matrix which would determine the mechanical and thermal behaviour of the medium. However, although the actual equations cannot be obtained, a large amount of information concerning which variables may be present in the constitutive relations and the forms in which those variables must appear can be gotten from continuum theory. In particular, it provides a rational approach to obtaining linear and higher-order constitutive approximations which are sufficient for the theoretical treatment of many important problems.

In the following sections, the fundamental principles of neighborhood, equipresence, objectivity and material invariance and the thermodynamic restrictions arising from the Ingram-Eringen generalization of the Clausius-Duhem inequality are used to determine the nonlinear constitutive forms for a heat-conducting fluid-solid mixture.

2.3.1 Dependent Variables

The equations of conservation of mass, balance of momentum and conservation of energy constitute differential equations for the constituent densities, velocities and tempera-
turers, respectively. An examination of the field equations therefore indicates that constitutive relations are required for the partial stress tensors \( t^{ij\xi} \), the specific internal energies \( \varepsilon(\xi) \), the heat flux vectors \( q(\xi) \), the specific entropies \( \eta(\xi) \) and the transfer terms \( \beta(\xi) \), \( e(\xi) \) and \( n(\xi) \). Variables uniquely associated with the jump conditions will not be considered in this treatment. For convenience, the specific entropies will be replaced in the constitutive relations by the specific free energies

\[
\psi(\xi) = \varepsilon(\xi) - \theta(\xi)\eta(\xi) \tag{29}
\]

In addition, the essential step in obtaining the Ingram-Eringen generalization of the Clausius-Duhem inequality involves replacing the energy and entropy transfer terms by \( \tilde{e}(\xi) \) and \( \tilde{n}(\xi) \), defined by

\[
\tilde{e}(\xi) = e(\xi) + \sum_{\lambda} \sum_{\mu} \frac{\partial \varepsilon(\xi)}{\partial \nu(\mu)} \nu(\lambda) \left( \frac{\partial \eta(\xi)}{\partial \nu(\mu)} \right) \left( \frac{\partial \eta(\xi)}{\partial \nu(\lambda)} \right) \tag{30}
\]

\[
\tilde{n}(\xi) = n(\xi) + \sum_{\lambda} \sum_{\mu} \frac{\partial \eta(\xi)}{\partial \nu(\mu)} \nu(\lambda) \left( \frac{\partial e(\xi)}{\partial \nu(\mu)} \right) \left( \frac{\partial e(\xi)}{\partial \nu(\lambda)} \right) \tag{31}
\]

where \( \nu(\lambda) \): \( \mu = 1, 2, \ldots \) are the independent state variables associated with the \( \lambda \)th constituent upon which \( e(\xi) \) and \( \eta(\xi) \) are functionally dependent and \( \nu(\lambda, \xi) \) denotes the relative constituent velocity.
\[ \nu(\lambda \xi) = \nu(\lambda) - \nu(\xi) \]  

(32)

The expression of the energy and entropy transfer terms in the forms (30) and (31) represents a departure from previous theoretical treatments of fluid-solid mixtures, and was first utilized in deriving a constitutive theory of chemically reacting fluid mixtures by Eringen and Ingram [23]. The second terms in equations (30) and (31) can be interpreted as explicit transfer terms which account for the fact that:

1. \( \varepsilon(\xi) \) and \( \eta(\xi) \) are functions of independent variables associated with each other constituent*;
2. a material particle of the \( \xi \)th constituent traverses gradients in those variables at a rate equal to the relative constituent velocity.

It should be noted that if the energy and entropy transfer terms are not written in the explicit forms (30) and (31), one of the consequences arising from the Clausius-Duhem inequality is that the equilibrium parts of the constituent partial stress tensors must be independent of the state of deformation of the other constituents. This is clearly an inadmissible result for a theory of fluid-solid mixtures.

2.3.2 Independent Variables

The medium under consideration consists of a heat con-

*This is a consequence of the principle of equipresence, to be discussed in section 2.3.3.
ducting elastic solid permeated by a mixture of heat con-
ducting viscous compressible fluids. It will be postulated
that the behavior of the medium at time \( t \) is determined by
the state of the medium at time \( t \). That is, there are no
"hereditary" effects. In addition, it will be assumed that
the principle of neighborhood [25], which stipulates that
the behavior of the medium at a material point is dependent
on the state of the medium only in the neighborhood of the
point, applies. The primary consequence of the principle of
neighborhood is that gradients in the state variables of the
medium of order higher than one need not be included among
the independent variables for the medium.

Therefore, the independent variables to be considered
should include all mechanical and thermodynamical variables
associated with the medium together with their first-order
spatial gradients. They are the deformation gradient in the
porous solid constituent \( \mathbf{X}_s^K \), where \( K \) denotes partial
differentiation with respect to \( \mathbf{X}_s^K \), the fluid densities \( \rho_{(\lambda)} \),
the constituent velocities \( \mathbf{v}_{(\xi)} \), the constituent velocity
gradients \( \mathbf{v}_{(\xi)} \); \( j \), the constituent temperatures \( \theta_{(\xi)} \) and the
constituent temperature gradients \( \theta_{(\xi)} \); \( j \). The spatial coor-
dinates and base vectors have been disregarded because they
are nonobjective, a requirement that will be discussed in
section 2.3.5. By disregarding the material coordinates
and base vectors, the theory has been restricted to a homo-
geneous isotropic medium.
2.3.3 Equipresence

The principle of equipresence as stated by Coleman and Mizel [26] requires that the same independent variables appear in each constitutive relation for a medium unless it is in violation of the restrictions imposed by the field equations, objectivity, invariance requirements or the Clausius-Duhem inequality. Based on the results of sections 2.3.1 and 2.3.2, the tentative constitutive forms for the medium prior to the application of restrictions can be written as

\[
\begin{align*}
t^m_{ij}(\xi) &= t^m_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n) \\
\varepsilon_{ij}(\xi) &= \varepsilon_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n) \\
\sigma_{ij}(\xi) &= \sigma_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n) \\
\psi_{ij}(\xi) &= \psi_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n) \\
\rho_{ij}(\xi) &= \rho_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n) \\
\tilde{\varepsilon}_{ij}(\xi) &= \tilde{\varepsilon}_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n) \\
\tilde{n}_{ij}(\xi) &= \tilde{n}_{ij}(x^m(s), K, \rho(\lambda), v^m(\lambda), n; \theta(\lambda), \theta(\lambda), n)
\end{align*}
\]  

In the following sections, the restrictions on the variables appearing in these functional forms will be considered.
2.3.4 The Thermodynamic Restrictions

In this section, the Clausius-Duhem inequality is used in the manner of Coleman and Mize [26] to determine the thermodynamic restrictions on the functional forms (33) through (39). The procedure consists of substituting the field equations into the Clausius-Duhem inequality* in such a way as to obtain an inequality in which the restrictions placed on the independent variables by the field equations are automatically satisfied. Since the resulting inequality must be satisfied for arbitrary values of any independent variable (subject to restrictions such as $\theta(\xi) > 0$) while all other independent variables are held constant, the coefficients of independent variables which can be isolated so that they appear nowhere else in the inequality must be identically zero. The equations which result from this process provide the sought-for restrictions.

Eliminating the distributed external energy source term $\dot{n}(\xi)$ between the energy equation (22) and the Clausius-Duhem inequality (26) and introducing $\psi(\xi)$ and the explicit transfer terms $\ddot{e}(\xi)$ and $\ddot{n}(\xi)$, the Ingram-Eringen generalization of the Clausius-Duhem inequality can be written as

*Here, of course, the Ingram-Eringen generalization of the Clausius-Duhem inequality is employed.
\[ -\rho(\xi) \frac{D\psi(\xi)}{Dt} - \rho(\xi) \eta(\xi) \frac{D\theta(\xi)}{Dt} + t \eta(\xi) \nu(\xi) \hat{\nu}(\xi) + \frac{q(\xi)}{\theta(\xi)} \hat{t}(\xi) \quad (40) \]

where

\[ s(\xi) = \bar{e}(\xi) - \theta(\xi) \bar{\eta}(\xi) \quad (41) \]

The objective is to write equation (40) in a form in which the independent variables appear explicitly. To this end, let the material derivative of \( \psi(\xi) \) appearing in equation (40) be expanded by the chain rule in terms of the state variables appearing in equation (36).

\[ \frac{D\psi(\xi)}{Dt} = \sum_{\lambda} \sum_{\mu} \frac{\partial \psi(\xi)}{\partial \nu(\lambda)} \frac{D\nu(\lambda)}{Dt} \quad (42) \]

Here \( \nu(\lambda) \) is again the \( \mu \)th variable associated with the \( \lambda \)th constituent. Note that the material derivative of \( \nu(\lambda) \) following the motion of the \( \xi \)th constituent is given by

\[ \frac{D\nu(\lambda)}{Dt} = \frac{\partial \nu(\lambda)}{\partial t} + \nu(\lambda) \hat{\nu}(\xi) \quad (43) \]

The material derivative of \( \nu(\lambda) \) following the motion of the \( \lambda \)th constituent is given by

\[ \frac{D\nu(\lambda)}{Dt} = \frac{\partial \nu(\lambda)}{\partial t} + \nu(\lambda) \hat{\nu}(\lambda) \quad (44) \]
Subtracting equation (44) from equation (43),

\[ \frac{D}{Dt}(\xi) \mathbf{D}_{\nu}(\lambda)(\mu) - \frac{D}{Dt}(\lambda) \mathbf{D}_{\nu}(\lambda)(\mu) = \psi(\xi;\mathbf{\kappa}; \xi\lambda) \]

so that

\[ \frac{D}{Dt}(\xi) \mathbf{D}_{\nu}(\lambda)(\mu) = \frac{D}{Dt}(\lambda) \mathbf{D}_{\nu}(\lambda)(\mu) + \psi(\lambda;\mathbf{\kappa}; \xi\lambda) \]

Substituting equation (46) into equation (42), the material derivative can be expressed as

\[ \frac{D}{Dt}(\xi) \mathbf{D}_{\nu}(\xi) = \sum_{\lambda} \sum_{\mu} \frac{\partial}{\partial \nu(\lambda)} \frac{D}{Dt}(\lambda) \mathbf{D}_{\nu}(\lambda)(\mu) \]

\[ + \sum_{\lambda} \sum_{\mu} \frac{\partial}{\partial \nu(\lambda)} \psi(\lambda;\mathbf{\kappa}; \xi\lambda) \]

Substituting this expression for the material derivative into the inequality (40), it becomes

\[ -\sum_{\lambda} \sum_{\mu} \psi(\xi;\mathbf{\kappa}; \xi\lambda) \frac{D}{Dt}(\lambda) \mathbf{D}_{\nu}(\lambda)(\mu) - \psi(\xi;\mathbf{\kappa}; \xi\lambda) \frac{D}{Dt}(\xi) \]

\[ + j^{\xi\kappa}(\xi) \mathbf{\kappa}; j + q^{\xi\theta}(\xi) \mathbf{\kappa}; \theta(\xi) + s(\xi) \geq 0 \]

Expanding the double summation in expression (48) using the independent variables from equation (36), (48) can be written as
\[ - \frac{\partial \psi (\xi)}{\partial x^m (s), K} \frac{D \psi (s)}{D t} - \sum_\lambda \sigma (\xi) \frac{\partial \psi (\xi)}{\partial \rho (\lambda)} \frac{D \rho (\lambda)}{D t} - \frac{\partial \psi (\xi)}{\partial v^m (\lambda)} \frac{D \psi (\xi)}{D t} \]

\[ - \sum_\lambda \frac{\partial \psi (\xi)}{\partial \theta (\lambda), n} \frac{D \theta (\lambda), n}{D t} - \delta (\xi) \eta (\xi) \frac{D \psi (\xi)}{D t} \]

\[ + \frac{s (\xi)}{\rho (\xi)} \geq 0 \quad (49) \]

where the subscript \( s \) refers to the solid constituent and

\[ \sigma (\xi \lambda) = 1 - \delta (\xi \lambda) \quad (50) \]

where \( \delta (\xi \lambda) \) is defined to be equal to one when the \( \xi \)th constituent is the same constituent as the \( \lambda \)th constituent and is equal to zero otherwise. The symbols \( \delta (\xi \lambda) \) and \( \sigma (\xi \lambda) \) will appear frequently in what follows.

Note from Eringen [25] that the material derivative of the solid constituent deformation gradient appearing in (49) can be expressed in terms of the velocity gradient and the deformation gradient as

\[ (s) \frac{D x^m (s)}{D t}, K = x^j (s), K^m (s); j \quad (51) \]

In addition, by using the conservation of mass equation (8), the partial time derivative of the fluid densities \( \rho (\lambda) \) can
be expressed in terms of the velocity gradients as

$$\frac{\partial \rho(\lambda)}{\partial t} = -\left(\rho(\lambda) \dot{\mathbf{v}}(\lambda)\right); \mathbf{v}$$

(52)

The material derivative of the fluid densities is

$$\frac{(\lambda)D \rho(\lambda)}{Dt} = \frac{\partial \rho(\lambda)}{\partial t} + \dot{v}(\lambda) \rho(\lambda); \mathbf{v}$$

(53)

Eliminating $\frac{\partial \rho(\lambda)}{\partial t}$ between (52) and (53), the material derivative of the fluid densities in terms of the velocity gradients and fluid densities is given by

$$\frac{(\lambda)D \rho(\lambda)}{Dt} = -\rho(\lambda) \dot{\mathbf{v}}(\lambda); \mathbf{v}$$

(54)

Substituting relations (51) and (54) into the inequality (49), it becomes

$$- \frac{\partial \psi(\xi)}{\partial \mathbf{v}^m(s), \mathbf{K}} \times j(s), K^m(s); j + \sum_{\lambda} \frac{\partial \psi(\xi)}{\partial \mathbf{v}^m(\lambda)} \rho(\lambda) \dot{\mathbf{v}}(\lambda); \mathbf{v}$$

$$- \sum_{\lambda} \frac{\partial \psi(\xi)}{\partial \mathbf{v}^m(\lambda)} \mathbf{a}(\lambda) - \sum_{\lambda} \frac{\partial \psi(\xi)}{\partial \mathbf{v}^m(\lambda)} \frac{(\lambda)D \mathbf{v}^m(\lambda); n}{Dt}$$

$$- \sum_{\lambda} \left( \frac{\partial \psi(\xi)}{\partial \theta(\lambda)} + \delta(\lambda, \xi) \eta(\xi) \right) \frac{(\lambda)D \theta(\lambda)}{Dt}$$

$$- \sum_{\lambda} \frac{\partial \psi(\xi)}{\partial \theta(\lambda)} \frac{(\lambda)D \theta(\lambda), n}{Dt} + \frac{1}{\rho(\xi)} t^j \mathbf{v}(\xi) \mathbf{v}(\xi); j$$

$$+ \frac{q(\xi) \theta(\xi), \mathbf{v}}{\rho(\xi) \theta(\xi)} + \frac{s(\xi)}{\rho(\xi)} \approx 0$$

(55)
Note that the material derivative of the constituent temperatures appears only once in the inequality (55). Since the inequality must be satisfied for arbitrary values of these independent variables while the values of all the other independent variables are held fixed, it can be inferred that the coefficients of these variables must be identically zero. Therefore, the following equation is obtained.

\[
\frac{\partial \psi(\xi)}{\partial \theta(\lambda)} + \delta(\lambda \xi) \eta(\xi) = 0
\]  

(56)

For \( \xi \neq \lambda \), equation (56) gives

\[
\frac{\partial \psi(\xi)}{\partial \theta(\lambda)} = 0
\]  

(57)

so that \( \psi(\xi) \) is not a function of the temperatures of the other constituents. For \( \xi = \lambda \), equation (56) gives

\[
\frac{\partial \psi(\xi)}{\partial \theta(\xi)} = -\eta(\xi)
\]  

(58)

so that the entropy of each constituent is given by the negative partial derivative of the constituent free energy with respect to the constituent temperature.

Note that the material derivative of the constituent temperature gradients appears in only one place in the inequality (55), from which it can be inferred that

\[
\frac{\partial \psi(\xi)}{\partial \theta(\lambda)} = 0
\]  

(59)
so that \( \psi(\xi) \) is not a function of the constituent temperature gradients.

The material derivative of the velocity gradients appears only once in the inequality (55), resulting in the following equation.

\[
\frac{\partial \psi(\xi)}{\partial \nu^m(\lambda);n} = 0 \tag{60}
\]

\( \psi(\xi) \) is therefore not a function of the constituent velocity gradients.

The material derivative of the constituent accelerations appears only once in the inequality (55), resulting in the following equation.

\[
\frac{\partial \psi(\xi)}{\partial \nu^m(\lambda)} = 0 \tag{61}
\]

The constituent free energy \( \psi(\xi) \) is therefore not a function of the constituent velocities.

In summary, the Ingram-Eringen generalization of the Clausius-Duhem inequality has been written in a form in which the postulated independent variables of the medium and their gradients and derivatives appear explicitly. The resulting inequality has then been expressed so that certain of the independent variables appear only once. Since the independent variables can take on arbitrary positive or negative values while the remaining variables are held constant, the coefficients of those independent variables which appeared
in only one place were assumed to be identically zero in order that the inequality be satisfied. The equations thus obtained are summarized below.

\[ \frac{\partial \psi(\xi)}{\partial \theta(\lambda)} + \delta(\lambda \xi) \nabla(\xi) = 0 \] \hspace{1cm} (62)

\[ \frac{\partial \psi(\xi)}{\partial \theta(\lambda), n} = 0 \] \hspace{1cm} (63)

\[ \frac{\partial \psi(\xi)}{\partial v_m(\lambda); n} = 0 \] \hspace{1cm} (64)

\[ \frac{\partial \psi(\xi)}{\partial v_m(\lambda)} = 0 \] \hspace{1cm} (65)

As a result of equations (62), (63), (64) and (65), the inequality (55) reduces to

\[ t^k(\xi) j \nu(\xi); n - \rho(\xi) \frac{\partial \psi(\xi)}{\partial x_j(\xi), K} x^k(\xi), K \nu_j(\xi); n \]

\[ + \sum_{\lambda} \sigma(\lambda s) \rho(\xi) \rho(\lambda) \frac{\partial \psi(\xi)}{\partial \rho(\lambda)} \nabla(\lambda); n + \frac{q(\xi) \theta(\xi)}{\theta(\xi)} \]

\[ + s(\xi) = 0 \] \hspace{1cm} (66)

Now let the constituent partial stress tensors be written as the sum of equilibrium and dissipational parts

\[ t^k(\xi) j = E^k(\xi) j + D^k(\xi) j \] \hspace{1cm} (67)

where it is assumed that \( D^k(\xi) j \) vanishes when the constituent
velocity gradient \( v^j_j \) vanishes, i.e. \( D^t(\xi) j^j \); \( \approx \) is a second-order term in the dissipational variables.

In addition, let \( s(\xi) \) be expanded in a Maclaurin series in these variables.

\[
s(\xi) = s_0(\xi) \left| _\approx + \sum \frac{\partial s(\xi)}{\partial v(\lambda)} \right| _\approx v^j_j(\lambda) \; \approx + \sum \frac{\partial s(\xi)}{\partial v(\lambda)} \right| _\approx v^j_j(\lambda) \\
+ \sum \frac{\partial s(\xi)}{\partial \theta(\lambda)} \right| _\approx \theta(\lambda) \; \approx + \cdots \tag{68}
\]

The notation \( \right| _\approx \) is to emphasize that the coefficients are evaluated for zero values of the dissipational variables.

Substituting equations (67) and (68) into the inequality (66), the result can be written as

\[
\left( B^t(\xi) j - \delta(\xi s)^p j(s) \right) \left( \frac{\partial \psi(s)}{\partial x} \right) \left( s, K \right) \left( \sigma(\xi s) \right) \left( s, K + \frac{\partial s(\xi)}{\partial v(s)} \right) v^j_j(\xi) \; \approx + \right| _\approx v(\xi) \; \approx \\
+ \left( -\sigma(\xi s) \rho(\xi) \right) \left( x(s), K + \sigma(\xi s) \right) v^j_j(\xi) \; \approx \\
+ \sum \left( \sigma(\xi s) \sigma(\xi s) \right) \left( s, \right) \left( s, K + \frac{\partial s(\xi)}{\partial v(s)} \right) v^j_j(\xi) \; \approx \\
+ \left( -\sigma(\xi s) \sigma(\xi s) \right) \left( s, \right) \left( s, K + \sigma(\xi s) \right) v^j_j(\xi) \; \approx \\
+ \left\{ \frac{q(\xi) \theta(\xi)}{\theta(\xi)} + \sum \frac{\partial s(\xi)}{\partial \theta(\lambda)} \right| _\approx \theta(\lambda) \; \approx + D^t(\xi) j^j \; \approx + \sum \frac{\partial s(\xi)}{\partial v(\lambda)} \right| _\approx v(\lambda) \\
+ s(\xi) \right| _\approx + \cdots \cdots \geq 0 \tag{69}
\]
where \( f \) refers to an arbitrary fluid constituent. Note that when the medium is at equilibrium, i.e. the dissipational variables equal zero, the inequality (69) must reduce to an equality. Therefore it can immediately be seen that

\[
S^s_\sigma(\xi) \bigg|_0 = 0
\]  

(70)

Now consider the inequality as the medium approaches equilibrium. In order for the inequality to hold for sufficiently small values of the dissipational variables, the coefficients of the dominant first order terms must vanish, resulting in the following equations.

\[
E^t(\xi)_j = \delta(\xi_\sigma) \rho(s) \frac{\partial \psi(s)}{\partial x(s), K} x(s), K
\]

\[+ \delta(\xi f) \rho^s(f) \frac{\partial \psi(f)}{\partial p(f)} \frac{\partial \psi(s)}{\partial v(s), \xi} \bigg|_0\]

\[\sigma(\xi_\sigma) \frac{\partial x(s)}{\partial v(s), \xi} \bigg|_0 = \sigma(\xi_\sigma) \rho(\xi) \frac{\partial \psi(s)}{\partial x(s), K} x(s), K\]

\[\sigma(\xi_\lambda) \rho^s(\lambda) \frac{\partial x_\lambda(s)}{\partial v(\lambda), \xi} \bigg|_0 = -\sigma(\xi_\lambda) \rho(\xi) \rho(\lambda) \frac{\partial \psi(s)}{\partial x(s), K} x(s), K\]

\[\frac{\partial s(\xi)}{\partial v(\lambda)} \bigg|_0 = \frac{\partial s(\xi)}{\partial \theta(\lambda), \xi} \bigg|_0 = 0\]

(74)

Since it has been shown that \( \psi(\xi) \) is not a function of the dissipative variables, equation (71) provides the important
result that the equilibrium part of the partial stress tensors is also not a function of the dissipative variables. Using equations (71), (72), (73) and (74), the final form of the inequality becomes

\[
\frac{q \theta}{\theta} + \lambda \frac{\theta}{\theta} + D \lambda j^j \theta \kappa + \cdots \geq 0
\]  

(75)

where \( \cdots \) denotes terms of order two and higher in the dissipative variables.

Note from equations (62) through (65) that the functional form (36) for \( \psi(x) \) has been reduced to

\[
\psi(x) = \psi(x) \left( \kappa^m(s), K, \rho(\lambda), \theta(x) \right)
\]  

(76)

From equation (62),

\[
\lambda(x) = - \frac{\partial \psi(x)}{\partial \theta(x)}
\]  

(77)

This implies that

\[
\lambda(x) = \lambda(x) \left( \kappa^m(s), K, \rho(\lambda), \theta(x) \right)
\]  

(78)

From equation (29),

\[
e(x) = \psi(x) + \theta(x) \lambda(x)
\]  

(79)

This provides the important result that the functional form (34) for \( e(x) \) reduces to

\[
e(x) = e(x) \left( \kappa^m(s), K, \rho(\lambda), \theta(x) \right)
\]  

(80)

This completes the determination of the thermodynamic
restrictions on the variables appearing in the constitutive functional forms for the medium. The results obtained allow a theorem on the necessary and sufficient conditions for the existence of thermal equilibrium between the constituents of the medium to be proved. This theorem was proved for a fluid mixture by Eringen and Ingram [23].

From equations (78) and (80), the differentials of \( \eta(\xi) \) and \( \varepsilon(\xi) \) can be written as

\[
\begin{align*}
\frac{d\eta(\xi)}{d\nu} &= \left. \frac{\partial \eta(\xi)}{\partial \theta(\xi)} \right|_\nu \frac{d\theta(\xi)}{d\nu} + \left. \frac{\partial \eta(\xi)}{\partial \nu} \right|_\theta \frac{d\theta(\xi)}{d\nu} \\
\frac{d\varepsilon(\xi)}{d\nu} &= \left. \frac{\partial \varepsilon(\xi)}{\partial \theta(\xi)} \right|_\nu \frac{d\theta(\xi)}{d\nu} + \left. \frac{\partial \varepsilon(\xi)}{\partial \nu} \right|_\theta \frac{d\theta(\xi)}{d\nu} 
\end{align*}
\] (81) (82)

where \( \nu \) represents the state vector of variables \( x^m_{(s)}, K \) and \( \rho(\lambda) \). From thermodynamics,

\[
\left. \frac{\partial \eta(\xi)}{\partial \theta(\xi)} \right|_\nu = \frac{1}{\theta(\xi)} \left. \frac{\partial \varepsilon(\xi)}{\partial \theta(\xi)} \right|_\nu 
\] (83)

Equations (81), (82) and (83) can be combined to obtain

\[
\frac{d\eta(\xi)}{d\nu} - \left. \frac{\partial \eta(\xi)}{\partial \nu} \right|_\theta \frac{d\theta(\xi)}{d\nu} - \frac{1}{\theta(\xi)} \left. \frac{d\varepsilon(\xi)}{d\nu} \right|_\theta - \left. \frac{\partial \varepsilon(\xi)}{\partial \nu} \right|_\theta \frac{d\theta(\xi)}{d\nu} = 0 
\] (84)

At constant \( \nu \) this gives

\[
\left. \frac{\partial \varepsilon(\xi)}{\partial \eta(\xi)} \right|_\nu = \theta(\xi) 
\] (85)
Now let the average internal energy and average entropy be defined as

\[ \bar{\varepsilon} = \sum_{\xi} c(\xi) \varepsilon(\xi) \]  
\[ \eta = \sum_{\xi} c(\xi) \eta(\xi) \]  

(86)  
(87)

where

\[ c(\xi) = \frac{\rho(\xi)}{\rho} \]  

(88)

Then if \( \theta(\xi) = \theta(\lambda) = \theta \), (85), (86) and (87) give

\[ \frac{1}{c(\xi)} \frac{\partial \bar{\varepsilon}}{\partial \eta(\xi)} \bigg|_\lambda = \theta \]  

(89)

The solution of this set of partial differential equations is

\[ \bar{\varepsilon} = \bar{\varepsilon}(\eta, \lambda) \]  

(90)

Further, given equation (90), differentiating gives

\[ \frac{\partial \bar{\varepsilon}}{\partial \eta(\xi)} \bigg|_\lambda = c(\xi) \frac{\partial \bar{\varepsilon}}{\partial \eta(\xi)} \bigg|_\lambda \]  

(91)

and since

\[ \frac{\partial \bar{\varepsilon}}{\partial \eta(\xi)} \bigg|_\lambda = c(\xi) \frac{\partial \varepsilon(\xi)}{\partial \eta(\xi)} \bigg|_\lambda = c(\xi) \theta(\xi) \]  

(92)

it is evident that

\[ \frac{\partial \bar{\varepsilon}}{\partial \eta} \bigg|_\lambda = \theta(\xi) \]  

(93)
so that all the constituent temperatures are equal. These results can be stated in the following

**Theorem:** The necessary and sufficient condition for thermal equilibrium to exist between the constituents of a mixture is that there exist an equation of state of the form

\[ \bar{e} = \bar{e}(\eta, \gamma) \]  

(94)

2.3.5 Objectivity

The requirement of objectivity specifies that the constitutive functional forms (33) through (39) must be independent of the observer. Mathematically, this requirement can be satisfied by requiring that the independent variables appearing in them satisfy their respective tensor transformation laws when the spatial reference frame is subjected to a rigid motion

\[ x'_{\alpha} = Q_{\alpha j}(t)x_j + b_\alpha(t) \]  

(95)

where equation (95) is written in Cartesian tensors, \( Q_{\alpha j} \) is a time varying orthogonal transformation and \( b_\alpha \) is a time varying vector.

The tensor transformation law for the spatial coordinates is given by

\[ x'_{\alpha} = Q_{\alpha j}x_j \]  

(96)

Comparing this with equation (95), it is seen that, as mentioned previously, the spatial coordinates are not objective quantities. As a second example, the tensor transformation
law for the constituent velocity is given by

\[ v'(\xi) = Q_{ij} v(\xi) j \]  

(97)

Taking the time derivative of equation (95), the constituent velocity transformation under a rigid motion of the spatial frame is

\[ v'(\xi) = \frac{dQ_{ij}}{dt} x(\xi) j + Q_{ij} v(\xi) j + \frac{db_{i}}{dt} \]  

(98)

Comparing equation (98) with equation (97), it is seen that the constituent velocity is also not an objective quantity. However, using equation (98), let the transformation under a rigid motion of the relative velocity between two constituents be written as

\[ v'(\xi) - v'(\lambda) = \frac{dQ_{ij}}{dt} (x(\xi) j - x(\lambda) j) \]

\[ + Q_{ij} (v(\xi) j - v(\lambda) j) + \frac{db_{i}}{dt} - \frac{db_{i}}{dt} \]  

(99)

If the relative velocity at a spatial point is considered so that \( x(\xi) j = x(\lambda) j \), (99) reduces to

\[ v'(\xi\lambda) = Q_{ij} v(\xi\lambda) j \]  

(100)

Since this is the proper transformation law, it is seen that the relative constituent velocity is an objective quantity, and therefore it is this variable rather than the constituent velocity which must appear in the constitutive functional forms.

It is easy to show that every other variable appearing
in the constitutive forms (33) through (39) satisfies the criterion of objectivity except the constituent velocity gradients \( \dot{\mathbf{v}}(\lambda); j \). When dealing with a single constituent, it is the requirement of dealing with objective linear combinations of the velocity gradients that gives rise to the familiar deformation rate and spin tensors. In multi-component systems, Adkins [16] has shown that the linearly independent objective combinations of the constituent velocity gradients can be reduced to

\[
\dot{\mathbf{v}}(\xi)j = \frac{1}{2} \left( \dot{\mathbf{v}}(\xi); j + \mathbf{v}(\xi)j; \right) \tag{101}
\]

\[
\dot{\mathbf{v}}(\xi\lambda)j = \dot{\mathbf{v}}(\xi)j - \dot{\mathbf{v}}(\lambda)j \tag{102}
\]

\[
\dot{\mathbf{v}}(\xi\lambda)j = \frac{1}{2} \left( \dot{\mathbf{v}}(\xi); j - \mathbf{v}(\xi)j; \right) - \frac{1}{2} \left( \dot{\mathbf{v}}(\lambda); j - \mathbf{v}(\lambda)j; \right) \tag{103}
\]

The constituent velocity gradients must therefore be replaced in the constitutive functional forms by relations (101), (102) and (103).

2.3.6 Material Invariance

It is apparent that the constitutive functions (33) through (39) must transform as scalar-valued functions in the material reference frame. These functions have only one argument which is not a scalar in the material frame. This is the solid constituent deformation gradient \( x_{(s),K} \), which transforms as a contravariant vector in the material.
frame. Therefore, according to the Cauchy representation theorem [21], the constitutive functions must be expressible as functions of the material scalar and triple products of $x(s), K$. Only the scalar product

$$c_j = G^{KL} x(s), K^j(s), L$$

(104)

is independent. [27] The solid constituent deformation gradient must therefore be replaced in the constitutive functional forms by $c_j$.

Based on the results of sections 2.3.4, 2.3.5 and 2.3.6, the final nonlinear constitutive forms for the medium can be written as

$$E^t(\xi) = E^t(\xi) \left( c_{n}, \rho(\lambda), \theta(\lambda) \right)$$

(105)

$$D^t(\xi) = D^t(\xi) \left( c_{n}, \rho(\lambda), \nu(\xi), d_n, d_m, (\xi), (\xi)_n, (\xi)_m, \theta(\lambda), \theta(\lambda), \theta(\lambda), \theta(\lambda) \right)$$

(106)

$$\varepsilon(\xi) = \varepsilon(\xi) \left( c_{n}, \rho(\lambda), \theta(\xi) \right)$$

(107)

$$q(\xi) = q(\xi) \left( c_{n}, \rho(\lambda), \nu(\xi), d_n, d_m, (\xi), (\xi)_n, (\xi)_m, \theta(\lambda), \theta(\lambda), \theta(\lambda), \theta(\lambda) \right)$$

(108)

$$\psi(\xi) = \psi(\xi) \left( c_{n}, \rho(\lambda), \theta(\xi) \right)$$

(109)

$$p(\xi) = p(\xi) \left( c_{n}, \rho(\lambda), \nu(\xi), d_n, d_m, (\xi), (\xi)_n, (\xi)_m, \theta(\lambda), \theta(\lambda), \theta(\lambda), \theta(\lambda) \right)$$

(110)
\[ \tilde{\varepsilon}(\xi) = \tilde{\varepsilon}(\xi) \left( c_m n, \rho(\lambda), \nu(\xi \lambda), d_m n, \theta(\lambda), \theta(\lambda), n \right) \]

(111)

\[ \tilde{\eta}(\xi) = \tilde{\eta}(\xi) \left( c_m n, \rho(\lambda) \nu(\xi \lambda), d_m n, \theta(\lambda), \theta(\lambda), n \right) \]

(112)

As mentioned previously, the determination of the functions represented by equations (105) through (112) is outside the scope of continuum mechanics. For large variations of their arguments, functions of the type (105) through (112) cannot usually be determined for real materials by any means. Therefore, for the purpose of obtaining theoretical results, approximations must be resorted to.

The most common technique for approximating relations (105) through (112) consists of expanding the relations in Taylor or Maclaurin series in their arguments and retaining only a finite number of terms. In the next section, this technique will be demonstrated by obtaining the full first-order constitutive equations for the medium.

Another approximation commonly applied to the constitutive forms consists of neglecting certain independent variables altogether. This technique is familiar, for example, from the theory of fluids where real fluids are approximated by various "ideal" fluids by assuming that velocity gradients, temperature gradients, etc. can be neglected in the constitutive relation for the stress tensor.

In the medium under consideration here, intuitive argu-
ments can be given for neglecting certain of the independent variables under certain circumstances. Since the porous solid matrix is assumed to be elastic and not viscoelastic, for example, there is some motivation for neglecting the deformation rate tensor in the solid constituent. In addition, if the porosity of the solid matrix is small, that is, the volume of fluid per unit volume of the medium is small, there is some justification for neglecting the deformation rate tensors of the fluid constituents also. However, there is no such apparent justification for neglecting the relative deformation rate and spin tensors. This point will be touched upon again in a later section.

An important point is that, although terms such as the deformation rate tensor in the solid constituent may be neglected as being small, this is not the same as assuming that these terms are absent altogether. If this is done prior to obtaining the constitutive theory, it is found that the thermodynamic restrictions again give paradoxical and unacceptable results.

In part 3 of the thesis, both of these techniques of approximation are applied to arrive at equations similar to the well-known Biot equations, which are then used to solve several illustrative boundary-value problems.

2.4 The Linearized Constitutive Equations

Due to the relative complexity of the medium, only the
first-order expansions of the constitutive relations will be presented. Second and higher order theories based on relations (105) through (112) can be obtained in a straightforward, though tedious, manner.

The first-order equations are obtained by taking Maclaurin expansions of the constitutive relations in terms of their arguments. The resulting equations must transform under rotations of the spatial coordinate system in a manner consistent with the symmetry properties of the medium. Although many types of material symmetry have been identified [21], only holohedral materials, which are isotropic under all orthogonal transformations including reflections, and hemihedral materials, for which reflections are not allowed, will be considered here. It is easy to show that the requirement of isotropy is satisfied if the coefficients of the constitutive expansions are isotropic tensors of the proper type. The general isotropic tensors through fourth order for the two types of material symmetry under consideration are shown below, where \( \alpha, \beta, \gamma, \mu \) and \( \nu \) are arbitrary scalars. [28]
In order to obtain an approximate theory based on small principal extensions of the solid constituent and small fluid dilatations, Finger's deformation tensor \( c^m_n \) will be replaced in the constitutive functional forms by its first order term

\[
\tilde{\varepsilon}^m_n = \frac{1}{2} \tilde{\varepsilon}^m_n \cdot \tilde{\varepsilon}^m_n = \frac{1}{2} \left( \tilde{u}^m_n + \tilde{u}^m_n \right) \tag{113}
\]

where \( \tilde{u}^m_n \) is the infinitesimal displacement vector in the solid, and the fluid densities will be replaced by their first order terms \( \tilde{\rho}^m_n \). [25] In addition, in order to obtain completely linear constitutive equations, the constituent temperatures \( \theta^m_n \) will be replaced by a first-order constituent temperature variation \( \tilde{\theta}^m_n \) about a reference equilibrium temperature. With these modifications, the constitutive functional forms for the dependent variables appearing in the field equations can be written as

\[
\bar{E}^i = \tilde{E}^i \left( \tilde{\varepsilon}^m_n, \tilde{\varepsilon}^m_n, \tilde{\rho}^m_n, \tilde{\theta}^m_n \right) \tag{114}
\]
\[ D^{t_j} = D^{t_j}(\xi) = e^{(s)mn, \sigma(\lambda s) e^{(\lambda)m, v(\xi \lambda)n}}, \]
\[ \mathcal{d}(\xi \lambda mn, w(\xi \lambda mn, \tilde{\theta}(\lambda), \bar{\theta}(\lambda), n) \]
\[ \varepsilon(\xi) = e^{(s)mn, \sigma(\lambda s) \tilde{e}^{(\lambda)m, \bar{\theta}(\xi)}} \]
\[ q^{\tilde{\Phi}} = q^{\tilde{\Phi}}(\xi) = e^{(s)mn, \sigma(\lambda s) \tilde{e}^{(\lambda)m, v(\xi \lambda)n}}, \]
\[ \mathcal{d}(\xi \lambda mn, w(\xi \lambda mn, \tilde{\theta}(\lambda), \bar{\theta}(\lambda), n) \]
\[ \rho^{\tilde{\Phi}} = \rho^{\tilde{\Phi}}(\xi) = e^{(s)mn, \sigma(\lambda s) \tilde{e}^{(\lambda)m, v(\xi \lambda)n}}, \]
\[ \mathcal{d}(\xi \lambda mn, w(\xi \lambda mn, \tilde{\theta}(\lambda), \bar{\theta}(\lambda), n) \]
\[ \bar{\varepsilon}(\xi) = \varepsilon^{(s)mn, \sigma(\lambda s) \tilde{e}^{(\lambda)m, v(\xi \lambda)n}}, \]
\[ \mathcal{d}(\xi \lambda mn, w(\xi \lambda mn, \tilde{\theta}(\lambda), \bar{\theta}(\lambda), n) \]

where the intuitive arguments of section 2.3.6 have been invoked to disregard the individual deformation rate tensors while retaining the relative deformation rate and vorticity tensors.

Expanding relations (114) through (119) in Maclaurin series in terms of their arguments and retaining terms through first order only, they become
\[ E_t^{j \xi} = E_0^{j \xi} + E_{11}^{j \xi \mu n} \varepsilon^{mn} + \sum_{\lambda} \varepsilon^{j \xi}(\xi) (\lambda) (\xi) \varepsilon^{(\lambda) m} + \sum_{\lambda} \varepsilon^{j \xi}(\xi) (\lambda) \varepsilon^{(\lambda) m} \] 

\[ D_t^{j \xi} = \sum_{\lambda} D^{j \xi}(\xi) (\lambda) v^{(\lambda) n} + \sum_{\lambda} D_{4}^{j \xi}(\xi) (\lambda) \tilde{v}^{(\lambda) m} + \sum_{\lambda} D_{5}^{j \xi}(\xi) (\lambda) w^{(\lambda) m} + \sum_{\lambda} D_{7}^{j \xi}(\xi) (\lambda) \tilde{\theta}^{(\lambda) n} \] 

\[ \varepsilon^{(\xi)} = \varepsilon_{0}^{(\xi)} + \varepsilon_{1}^{(\xi)} (\xi) \varepsilon^{(\lambda) m} + \sum_{\lambda} \varepsilon_{2}^{(\xi)} (\xi) (\lambda) \varepsilon^{(\lambda) m} + \varepsilon_{3}^{(\xi)} (\xi) \tilde{\theta}^{(\lambda)} \] 

\[ q^{k \xi} = q_{0}^{k \xi} (\xi) + q_{1}^{k \xi} (\xi) \varepsilon^{(\lambda) m} + \sum_{\lambda} q_{2}^{k \xi} (\xi) (\lambda) \varepsilon^{(\lambda) m} + \sum_{\lambda} q_{3}^{k \xi} (\xi) (\lambda) \varepsilon^{(\lambda) m} + \sum_{\lambda} q_{4}^{k \xi} (\xi) (\lambda) \tilde{v}^{(\lambda) m} + \sum_{\lambda} q_{5}^{k \xi} (\xi) (\lambda) \tilde{w}^{(\lambda) m} + \sum_{\lambda} q_{6}^{k \xi} (\xi) (\lambda) \tilde{\theta}^{(\lambda) m} + \sum_{\lambda} q_{7}^{k \xi} (\xi) (\lambda) \tilde{\theta}^{(\lambda) m} \]
\[ p_0 (\varepsilon) = \sigma_0 (\varepsilon) + \frac{\sigma_1 (\varepsilon)}{1} \tilde{e} (s) mn + \sum_{\lambda} \frac{\sigma_2 (\varepsilon_\lambda) \sigma (\lambda s)}{2} \tilde{e}^m (\lambda) \]

\[+ \sum_{\lambda} \frac{\sigma_3 (\varepsilon_\lambda) v (\varepsilon_\lambda)}{3} n + \sum_{\lambda} \frac{\sigma_4 (\varepsilon_\lambda) d (\varepsilon_\lambda)}{4} mn \]

\[+ \sum_{\lambda} \frac{\sigma_5 (\varepsilon_\lambda) w (\varepsilon_\lambda)}{5} mn + \sum_{\lambda} \frac{\sigma_6 (\varepsilon_\lambda) \tilde{\theta} (\lambda)}{6} \]

\[+ \sum_{\lambda} \frac{\sigma_7 (\varepsilon_\lambda) \tilde{\theta} (\lambda)}{7} , n \]

(124)

\[ \tilde{e} (\varepsilon) = \tau_0 (\varepsilon) + \tau_1 (\varepsilon) \tilde{e} (s) mn + \sum_{\lambda} \frac{\tau_2 (\varepsilon_\lambda) \sigma (\lambda s)}{2} \tilde{e}^m (\lambda) \]

\[+ \sum_{\lambda} \frac{\tau_3 (\varepsilon_\lambda) v (\varepsilon_\lambda)}{3} n + \sum_{\lambda} \frac{\tau_4 (\varepsilon_\lambda) d (\varepsilon_\lambda)}{4} mn \]

\[+ \sum_{\lambda} \frac{\tau_5 (\varepsilon_\lambda) w (\varepsilon_\lambda)}{5} mn + \sum_{\lambda} \frac{\tau_6 (\varepsilon_\lambda) \tilde{\theta} (\lambda)}{6} \]

\[+ \sum_{\lambda} \frac{\tau_7 (\varepsilon_\lambda) \tilde{\theta} (\lambda)}{7} , n \]

(125)

where \( \alpha, \beta, \gamma, \zeta \) and \( \tau \) are tensor constants and the fact that the dissipative stress tensors are assumed to vanish when the dissipative variables vanish has been used in equation (121). Requiring the tensor coefficients in equations (120) through (125) to be isotropic tensors, the linear constitutive equations for a heat conducting, elastic
porous solid permeated by a mixture of n heat conducting, viscous compressible fluids can be written as

\[ E^t(j) = E^0_0(\xi) e^{j\theta} + E^1_1(\xi) g^{j \theta} e^m(s) m \]

\[ + E^1_2(\xi) \delta(\xi s) e^{j\theta} e^m(s) + \sum_{\lambda} E^\lambda_2(\xi \lambda) g^{j \theta} \sigma(\lambda s) e^m(\lambda) m \]

\[ + \sum_{\lambda} E^\lambda_3(\xi \lambda) g^{j \theta} \tilde{e}(\lambda) \]

(126)

\[ D^t(j) = \sum_{\lambda} D^\lambda_3(\xi \lambda) e^{j \theta n}(\xi \lambda)n + \sum_{\lambda} D^\lambda_4(\xi \lambda) g^{j \theta d} m(\xi \lambda) m \]

\[ + \sum_{\lambda} D^\lambda_2(\xi \lambda) d^{j \theta} e(\xi \lambda) + \sum_{\lambda} D^\lambda_3(\xi \lambda) w^{j \theta} e(\xi \lambda) \]

\[ + \sum_{\lambda} D^\lambda_7(\xi \lambda) e^{j \theta n \tilde{e}}(\lambda),n \]

(127)

\[ \xi(\xi) = \xi_0(\xi) + \xi_1(\xi) \tilde{e}^m(s) m + \sum_{\lambda} \xi_2(\xi \lambda) \sigma(\lambda s) e^m(\lambda) m \]

\[ + \xi_3(\xi) \tilde{\xi}(\xi) \]

(128)

\[ q^\theta(j) = \sum_{\lambda} \gamma_3(\xi \lambda) v(\xi \lambda) + \sum_{\lambda} \gamma_5(\xi \lambda) e^{j \theta m n}(\xi \lambda) m n + \sum_{\lambda} \gamma_7(\xi \lambda) \tilde{\xi}(\xi \lambda) \]

(129)
\[ p(\xi) = \sum_{\lambda} \zeta_{3}(\xi_{\lambda}) v(\xi_{\lambda}) + \sum_{\lambda} \zeta'_{5}(\xi_{\lambda}) v(\xi_{\lambda})^{m} w(\xi_{\lambda})^{m} \]
\[ + \sum_{\lambda} \zeta_{7}(\xi_{\lambda}) \theta(\lambda), \] (130)

\[ \bar{e}(\xi) = \tau_{0}(\xi) + \tau_{1}(\xi) \bar{e}^{m} s^{m} + \sum_{\lambda} \tau_{2}(\xi_{\lambda}) \sigma(\lambda) \bar{e}^{m}(\lambda)^{m} \]
\[ + \sum_{\lambda} \tau_{4}(\xi_{\lambda}) d^{m}(\xi_{\lambda})^{m} + \sum_{\lambda} \tau_{6}(\xi_{\lambda}) \bar{\theta}(\lambda) \] (131)

In arriving at equations (126) through (131) the same symbol originally used as a tensor constant has occasionally been used for the corresponding scalar constant. This should cause no confusion. In addition, extensive use has been made of the relations

\[ g^{jk} \hat{\xi}[jk] = 0 \] (132)

\[ e^{ijk} \hat{\phi}(jk) = 0 \] (133)

\[ (g^{ik} g^{jl} + g^{il} g^{jk}) \hat{\xi}[jk] = 0 \] (134)

\[ (g^{ik} g^{jl} - g^{il} g^{jk}) \hat{\phi}(jk) = 0 \] (135)

where \( \hat{\phi}(jk) \) and \( \hat{\xi}[jk] \) denote arbitrary symmetric and antisymmetric tensors, respectively. Equations (126) through (131) are written for a hemihedral material. The equations for a holohedral material can be obtained by equating primed coefficients to zero.
3. One-Dimensional Transient Wave Propagation in a Fluid Saturated Porous Medium

In this section, the equations derived in part 2 of the thesis are simplified by neglecting heat transfer and velocity gradient terms and by considering a porous elastic solid saturated by a single viscous compressible fluid. The resulting simplified equations are then used to examine three one-dimensional transient wave propagation problems in such a medium.

3.1 The Biot Equations

Consider a holohedral medium consisting of a porous elastic medium saturated by a viscous compressible fluid. Dropping terms involving heat transfer and velocity gradients in equations (126) and (130), the partial stress tensors and momentum transfer vectors in the solid and fluid constituents are given by

\[ \tau_{j}^{s}(s) = \varepsilon_{11}(s) \varepsilon_{j}^{s} - \varepsilon_{12}(s) \varepsilon_{j}^{f} + \varepsilon_{2}(s) \varepsilon_{j}^{f} \varepsilon_{m}^{f} \varepsilon_{m}^{f} \]  \hspace{1cm} (136)

\[ \tau_{j}^{f}(f) = \varepsilon_{11}(f) \varepsilon_{j}^{s} - \varepsilon_{12}(f) \varepsilon_{j}^{s} + \varepsilon_{2}(f) \varepsilon_{j}^{s} \varepsilon_{m}^{s} \varepsilon_{m}^{s} \]  \hspace{1cm} (137)

\[ P^{s}(s) = \frac{c}{3(s f)} v^{s}(s f) \]  \hspace{1cm} (138)

\[ P^{f}(f) = \frac{c}{3(f s)} v^{f}(f s) \]  \hspace{1cm} (139)

It should be noted that it is inconsistent to drop the
relative deformation rate and spin tensors while retaining terms such as the relative velocity. Only experimental results could provide justification for doing so.

Equations (136) through (139) are formally identical to the constitutive equations of Biot [3] with one exception. Biot's method of derivation predicted the presence of an additional momentum transfer term proportional to the relative constituent accelerations. The derivation of this paper can neither verify nor deny the presence of such a term. This question would have to be examined by including the constituent accelerations among the independent variables for the constitutive relations. When this was attempted in this study, it was found that the thermodynamic restrictions could not be determined. In order to facilitate comparison of the problem solutions in this paper with solutions for steady-state wave propagation which have been obtained based on Biot's equations, this additional term will be included in what follows. Therefore let equations (138) and (139) be written as

\[ p_s = \frac{C}{3}(sf) \nabla (sf) + \frac{C}{8}(sf) \nabla (sf) \] (140)

\[ p_f = \frac{C}{3}(fs) \nabla (fs) + \frac{C}{8}(fs) \nabla (fs) \] (141)

In order to conform to the notation of Biot, let the following definitions be made.
\[ A = E^\alpha_{11}(s) \quad \quad 2N = E^\alpha_{12}(s) \]
\[ Q = E^\alpha_{2}(sf) \quad \quad Q' = E^\alpha_{11}(f) \]
\[ R = E^\alpha_{2}(ff) \quad \quad b = -\frac{c}{3}(sf) \]
\[ \rho(a) = -\frac{c}{8}(sf) \quad \quad b' = -\frac{c}{3}(fs) \]  
\[ \rho'(a) = -\frac{c}{8}(fs) \quad \quad (142) \]

Biot \[3\] proved that
\[ Q' = Q \quad \quad (143) \]

and from equation (16) it is clear that
\[ b' = b \quad \quad (144) \]
\[ \rho'(a) = \rho(a) \quad \quad (145) \]

Substituting equations (136), (137), (140) and (142) into the momentum balance equations (18), making the change of notation (142) and using (143), (144) and (145), the Biot displacement equations for the medium in vector form can be written as

\[ (A+N)\nabla(\nabla \cdot \underline{u}(s)) + NV^2 \underline{u}(s) + Q\nabla(\nabla \cdot \underline{u}(f)) \]
\[ = \rho(s) \frac{\partial^2 \underline{u}(s)}{\partial t^2} + b \frac{\partial}{\partial t}(\underline{u}(s) \cdot \underline{u}(f)) + \rho(a) \frac{\partial^2 (\underline{u}(s) \cdot \underline{u}(f))}{\partial t^2} \quad (146) \]

\[ Q\nabla(\nabla \cdot \underline{u}(s)) + R\nabla(\nabla \cdot \underline{u}(f)) \]
\[ = \rho(f) \frac{\partial^2 \underline{u}(f)}{\partial t^2} - b \frac{\partial}{\partial t}(\underline{u}(s) \cdot \underline{u}(f)) - \rho(a) \frac{\partial^2 (\underline{u}(s) \cdot \underline{u}(f))}{\partial t^2} \quad (147) \]
Biot showed that $A$, $N$, $Q$ and $R$ must be positive constants. In order to include velocity gradient phenomena in his theory, he used a macroscopic model to derive the term $b$ as a function of the frequency of motion of the medium. This fact renders transient wave propagation problems intractable when they are attacked using Biot's equations. However, when equations (146) and (147) are obtained as the linear equations arising from the continuum theory, the term $b$ is clearly a constant and transient problems are tractable. The term $b$ is obviously positive, as is $\rho(a)$.

3.2 The Boundary Conditions at a Fluid-Saturated Porous Solid-Fluid Interface

The jump conditions derived as an integral part of the field equations in section 2.2 provide an elegant way of obtaining the mechanical boundary conditions at an interface between a fluid saturated porous solid and a fluid.

From equations (9) and (12), the mechanical jump conditions across a discontinuity surface when terms of order higher than one are neglected are given by

\[
\left[ \rho(\xi) \left( v^h(\xi) - v^d(\xi) \right) \right] n^h = 0 \tag{148}
\]

\[
\sum_{\xi} \left[ t_{jk}^\xi \right] n_j = 0 \tag{149}
\]

Now assume that there is only one solid and one fluid constituent. Equation (148) for the fluid constituent becomes
\[
\left[ \rho(f) \left( \vec{v}(f) - \vec{v}(d) \right) \right]_{n_{\mathcal{A}}} = 0 \quad (150)
\]

Now let the discontinuity surface be identified with the surface of the porous solid matrix at a fluid saturated porous solid-fluid interface so that

\[
\vec{v}(d) = \vec{v}(s) \quad \text{surface} \quad (151)
\]

Equation (150) thus becomes

\[
\rho(f) \left[ \begin{array}{c|c}
\text{free} & \vec{v}(f) - \vec{v}(s) \\
\hline
\text{fluid} & \text{surface}
\end{array} \right]_{n_{\mathcal{A}}} = 0 \quad (152)
\]

\[
\quad - \rho(f) \left[ \begin{array}{c|c}
\text{porous} & \vec{v}(f) - \vec{v}(s) \\
\hline
\text{solid} & \text{surface}
\end{array} \right]_{n_{\mathcal{A}}} = 0
\]

where the subscript "free fluid" denotes the condition on the fluid side of the interface, and the subscript "porous solid" denotes the condition on the porous solid side of the interface. If the fluid density is continuous across the interface, it can be stated that

\[
\rho(f) \left|_{\text{porous}} \right. = \bar{\rho} \rho(f) \left|_{\text{free}} \right.
\]

\[
\quad \bar{\rho} = \left( \frac{\rho_{\text{porous}}}{\rho_{\text{fluid}}} \right)
\]

where \( \bar{\rho} \) is the porosity of the porous solid. Substituting this into equation (152), the velocity boundary condition across a fluid saturated porous solid-fluid interface is given by
\[
\begin{pmatrix}
  v(f) \\
  \text{free fluid}
\end{pmatrix}
= \begin{pmatrix}
  v(s) \\
  \text{surface}
\end{pmatrix}
\begin{pmatrix}
  n_f \\
  \text{solid}
\end{pmatrix}
\]  

(154)

If the interface is impermeable, so that

\[
\begin{pmatrix}
  v(f) \\
  \text{free fluid}
\end{pmatrix}
= \begin{pmatrix}
  v(s) \\
  \text{surface}
\end{pmatrix}
\]  

(155)

equation (154) gives as the velocity boundary condition at an impermeable surface

\[
\begin{pmatrix}
  v(f) \\
  \text{porous}
\end{pmatrix}
= \begin{pmatrix}
  v(s) \\
  \text{solid}
\end{pmatrix}
\begin{pmatrix}
  n_f \\
  \text{surface}
\end{pmatrix}
\]  

(156)

Writing equation (149) across a fluid saturated porous solid-fluid interface, it becomes

\[
\begin{pmatrix}
  t_{ij}(f) \\
  \text{free fluid}
\end{pmatrix}
= \begin{pmatrix}
  t_{ij}(s) \\
  \text{porous solid}
\end{pmatrix}
\begin{pmatrix}
  n_j \\
  \text{porous solid}
\end{pmatrix}
+ \begin{pmatrix}
  t_{ij}(f) \\
  \text{free fluid}
\end{pmatrix}
\begin{pmatrix}
  n_j \\
  \text{porous solid}
\end{pmatrix}
\]  

(157)

If the stress tensor in the fluid on the fluid side of the interface is assumed to be continuous tangential to the interface, equation (157) provides the following two stress boundary conditions at a fluid saturated porous solid-fluid interface.
\[
(1 - \delta) t^j_{(f)} \begin{bmatrix} n_j \\ \text{free fluid} \end{bmatrix} \quad = \quad t^j_{(s)} \begin{bmatrix} n_j \\ \text{porous solid} \end{bmatrix} \quad (158)
\]

\[
\delta t^j_{(f)} \begin{bmatrix} n_j \\ \text{free fluid} \end{bmatrix} \quad = \quad t^j_{(f)} \begin{bmatrix} n_j \\ \text{porous solid} \end{bmatrix} \quad (159)
\]

3.3 The Potential Equations

Let the displacement vectors in the solid and fluid constituents be written as sums of the gradient of a scalar potential and the curl of a vector potential in a manner similar to that used in solving problems in elastic wave propagation. [29]

\[
\mathbf{u}_s = \nabla \phi_s + \nabla \times \mathbf{\psi}_s \quad (160)
\]

\[
\mathbf{u}_f = \nabla \phi_f + \nabla \times \mathbf{\psi}_f \quad (161)
\]

Substituting these expressions into equations (146) and (147) and taking the divergence and the curl of the resulting equations, it is easy to show that equations (146) and (147) are satisfied if the following equations hold.

\[
P \nabla^2 \phi_s + Q \nabla^2 \phi_f = (\rho_s + \rho_a) \frac{\partial^2 \phi_s}{\partial t^2} - \rho_a \frac{\partial^2 \phi_f}{\partial t^2} + b \frac{\partial}{\partial t}(\phi_s - \phi_f) \quad (162)
\]

\[
Q \nabla^2 \phi_s + R \nabla^2 \phi_f = -\rho_a \frac{\partial^2 \phi_s}{\partial t^2} + (\rho_f + \rho_a) \frac{\partial^2 \phi_f}{\partial t^2} - b \frac{\partial}{\partial t}(\phi_s - \phi_f) \quad (163)
\]
\[ Nv^2 \psi_s = \left( \rho_s + \rho_a \right) \frac{\partial^2 \psi_s}{\partial t^2} - \rho_a \frac{\partial^2 \psi_f}{\partial t^2} + b \frac{\partial}{\partial t} (\psi_s - \psi_f) \quad (164) \]

\[ 0 = -\rho_a \frac{\partial^2 \psi_s}{\partial t^2} + (\rho_f + \rho_a) \frac{\partial^2 \psi_f}{\partial t^2} - b \frac{\partial}{\partial t} (\psi_s - \psi_f) \quad (165) \]

where

\[ P = A + 2N \quad (166) \]

and parentheses on subscripts have been dropped. Equations (162) through (165) were obtained in a different form by Biot [3]. In the next section, these potential equations will be used to investigate one-dimensional transient wave propagation in a fluid saturated porous medium.

3.4 Solutions

Three problems in one-dimensional transient wave propagation in a fluid saturated porous medium will be presented. 1) Lateral excitation at the boundary of a fluid saturated porous half-space; 2) normal excitation at the boundary of a half-space; and 3) spherical waves propagating from a point source in a fluid saturated porous space.

3.4.1 Lateral Excitation of a Half-Space

Consider a fluid saturated porous half-space with a Cartesian coordinate system situated so that the surface of the half-space coincides with the x-y plane and the z-axis extends into the half-space. Let the boundary of the half-
space be subjected to a time-varying lateral displacement so that
\[ u(s) \bigg|_{z=0} \right. = \tilde{H}(t)f(t) \quad (167) \]

where \( u(s) \) is the x-component of \( \mathbf{u}_{(s)} \), \( f(t) \) is some function of time and \( \tilde{H}(t) \) is Heaviside's step function. (Figure 1)

Note at this point that Biot's equations are somewhat paradoxical. They are ostensibly for a viscous fluid. For such a fluid, the "sticking" boundary condition
\[ u(f) \bigg|_{z=0} = \tilde{H}(t)f(t) \quad (168) \]

would have to be satisfied for a lateral excitation problem as well as condition (167). However, in solving the problem, it will be found that only condition (167) can be satisfied. Biot's equations are therefore limited to application to fluids of small viscosity so that the "boundary layer" at the interface will be of negligible thickness compared with distances of importance in the problem under consideration.

From equations (160), (161) and (167), it is clear that only the y-components of \( \psi_s \) and \( \psi_f \) are required. Letting the y-components of these vectors be denoted by \( \psi_s \) and \( \psi_f \), respectively, equations (160) and (161) reduce to
\[ u(s) = -\frac{\partial \psi_s}{\partial z} \quad (169) \]
\[ u(f) = -\frac{\partial \psi_f}{\partial z} \quad (170) \]
and the potential equations (164) and (165) become

\[
\frac{\partial^2 \psi_s}{\partial z^2} = (\rho_s + \rho_a) \frac{\partial^2 \psi_s}{\partial t^2} - \rho_a \frac{\partial^2 \psi_f}{\partial t^2} + b \frac{\partial}{\partial t} (\psi_s - \psi_f)
\]  
(171)

\[
0 = -\rho_a \frac{\partial^2 \psi_s}{\partial t^2} + (\rho_f + \rho_a) \frac{\partial^2 \psi_f}{\partial t^2} - b \frac{\partial}{\partial t} (\psi_s - \psi_f)
\]  
(172)

Defining the Laplace transform in time by

\[
\psi^* = \int_0^\infty \psi e^{-st} dt
\]  
(173)

the transformed equations (171) and (172) become

\[
N \frac{d^2 \psi_s^*}{dz^2} = \left[ (\rho_s + \rho_a) s^2 + bs \right] \psi_s^* - (\rho_a s^2 + bs) \psi_f^*
\]  
(174)

\[
0 = -(\rho_a s^2 + bs) \psi_s^* + \left[ (\rho_f + \rho_a) s^2 + bs \right] \psi_f^*
\]  
(175)

Equation (175) can immediately be solved for \( \psi_f^* \) in terms of \( \psi_s^* \).

\[
\psi_f^* = \left[ \frac{\rho_a + \frac{b}{s}}{(\rho_f + \rho_a) + \frac{b}{s}} \right] \psi_s^*
\]  
(176)

Substituting this into equation (174), it becomes

\[
\frac{d^2 \psi_s^*}{dz^2} - \frac{s^2}{c_s^2} \psi_s^* = 0
\]  
(177)

where

\[
c_s = \sqrt{\frac{N \left[ (\rho_f + \rho_a) + \frac{b}{s} \right]}{\rho_o^2 + (\rho_s + \rho_f)b}}
\]  
(178)
and

\[ \rho_o = \left[ \left( \rho_s + \rho_a \right) \left( \rho_f + \rho_a \right) - \rho_a^2 \right]^{\frac{1}{2}} \quad (179) \]

Note from equation (178) that for s real and positive, \( c_s \) is real and positive. Therefore, solving the ordinary differential equation (177) and retaining only that part of the solution representing an outward propagating wave, the solution for \( \psi_s \) is

\[ \psi_s = \text{const.} \, e^{-\frac{s}{c_s} z} \quad (180) \]

Laplace transforming the boundary condition (167) and using equation (169), the boundary condition for \( \psi_s \) becomes

\[ \frac{d\psi_s}{dz} \bigg|_{z=0} = -f^*(s) \quad (181) \]

Using this relation to evaluate the constant in equation (180), the solution for the solid constituent displacement potential in Laplace transform space is found to be

\[ \psi_s = f^* \frac{c_s}{s} e^{-\frac{s}{c_s} z} \quad (182) \]

From equation (176), the corresponding solution for the fluid constituent displacement potential is

\[ \psi_f = \left[ \frac{\rho_a + \frac{b}{s}}{(\rho_f + \rho_a) + \frac{b}{s}} \right]^f \psi_s \frac{c_s}{s} e^{-\frac{s}{c_s} z} \quad (183) \]
In order to obtain the solutions, the Laplace transforms (182) and (183) must be inverted. The classical method of inverting the Laplace transform can be written formally as

$$\psi = L^{-1}(\psi^*) = \frac{1}{2\pi i} \oint_{Br} \psi^* e^{st} ds$$  \hspace{1cm} (184)$$

where $Br$ denotes integration over the Bromwich contour in the s-plane. [30] Due to the relatively complicated nature of $c_s$, the inversion of the transforms (182) and (183) by the relation (184) would in general be quite difficult. In this case, a perturbation technique can be utilized which greatly simplifies the inversion procedure.

Note that equations (182) and (183) are of the form

$$\psi^* = F(s,b) e^{G(s,b)}$$  \hspace{1cm} (185)$$

where the functions corresponding to $F$ and $G$ are analytic in $b$ for all positive real values of $b$ and $s$. These functions can therefore be expressed as Maclaurin series in terms of $b$.

$$F(s,b) = F(s,0) + \frac{\partial F}{\partial b}(s,0)b + \ldots$$  \hspace{1cm} (186)$$

$$G(s,b) = G(s,0) + \frac{\partial G}{\partial b}(s,0)b + \ldots$$  \hspace{1cm} (187)$$

Let it now be assumed that the dissipation parameter $b$ is small so that terms of order higher than one in equations (186) and (187) can be neglected. Substituting equations
(186) and (187) into equation (185) and dropping higher order terms in \( b \), an approximate solution for \( \psi \) based on a first-order perturbation in the dissipation parameter can be obtained. Formally,

\[
\psi = L^{-1} \left[ G(s,0) + \frac{\partial G(s,0)}{\partial b}(s,0)b \right] e^{\left( F(s,0) + \frac{\partial F(s,0)}{\partial b}(s,0)b \right)}
\]  

(188)

In applying this procedure, care must be exercised to insure that the neglected higher-order terms in the exponential function are not such that Jordan's lemma [30] cannot be applied to the closure of the Bromwich contour when performing the integration (184). In all the problems treated here, it has been found that the higher-order exponential terms are proportional to \( s^{-N} \), where \( N \) is a positive integer, so that no problem arises. However, this point should be noted carefully in using this procedure for other problems.

Applying the procedure, note from equations (178), (182) and (183) that the function \( G(s,b) \) for both the fluid and solid constituents is given by

\[
G(s,b) = -sz\left[ \rho_o^2 + \frac{b}{s}(\rho_s + \rho_f) \right]^{\frac{1}{2}} \left\{ \sqrt{N(\rho_f + \rho_s)} + \frac{b}{s} \right\}^{-\frac{1}{2}}
\]  

(189)

Therefore

\[
G(s,0) = -s\frac{z}{c_s}
\]  

(190)

where
\[ c_s = \sqrt{\frac{N(\rho_f + \rho_a)}{\rho_o^2}} \]  

(191)

Taking the derivative of equation (189) with respect to \( b \),

\[
\frac{\partial G}{\partial b} = -sz\frac{2}{\rho_o} - \frac{b}{s}(\rho_s + \rho_f)\left( \sqrt{\frac{1}{s} (\rho_s + \rho_f) N^2 (\rho_f + \rho_a) + \frac{b}{s}} \right)^{-\frac{1}{2}}
\]

\[ + sz\rho_o^2 + \frac{b}{s} (\rho_s + \rho_f) \left( \frac{1}{2} N^2 (\rho_f + \rho_a) + \frac{b}{s} \right)^{-\frac{3}{2}} N \]  

(192)

Therefore,

\[
\frac{\partial G}{\partial b}(s, o) = -sz\frac{2}{\rho_o} (\rho_s + \rho_f) \frac{z}{\sqrt{N(\rho_f + \rho_a)}} + \rho_o^2 \frac{Nz}{[N(\rho_f + \rho_a)]^2} \]  

(193)

With some manipulation, this can be reduced to

\[
\frac{\partial G}{\partial b}(s, o) = -sz\frac{2}{c_s} \frac{\rho_f^2}{\rho_o^2 (\rho_f + \rho_a)} z \]  

(194)

For the solid constituent, \( F(s, b) \) is given by

\[
F(s, b) \bigg|_{\text{solid}} = \frac{f^*(s)}{s} \left\{ \frac{N^2 (\rho_f + \rho_a) + \frac{b}{s}}{2} \left( \frac{1}{2} \rho_o^2 + \frac{b}{s} (\rho_s + \rho_f) \right) \right\}^{-\frac{1}{2}} \]  

(195)

Therefore

\[
F(s, o) \bigg|_{\text{solid}} = c_s \frac{f^*(s)}{s} \]  

(196)

Taking the derivative of equation (195) with respect to \( b \),
\[ \frac{\partial F}{\partial b} \bigg|_{\text{solid}} = \frac{f^*}{s} \left\{ \frac{1}{2} \frac{N}{\sqrt{N(\rho_f + \rho_a)}} \right\} \frac{1}{\rho_0} \frac{1}{2} \left( \rho_0 + \frac{b}{s}(\rho_s + \rho_f) \right)^{-\frac{3}{2}} \frac{1}{s} \left( \rho_s + \rho_f \right) \]  

Therefore

\[ \frac{\partial F}{\partial b}(s, o) \bigg|_{\text{solid}} = \frac{f^*}{s} \frac{1}{2} \frac{1}{\sqrt{N(\rho_f + \rho_a)}} \frac{N}{\rho_0} \frac{1}{\rho_0} \frac{1}{2} \left( \rho_0 + \frac{b}{s}(\rho_s + \rho_f) \right)^{-\frac{3}{2}} \frac{1}{s} \left( \rho_s + \rho_f \right) \]  

With some manipulation this becomes

\[ \frac{\partial F}{\partial b}(s, o) \bigg|_{\text{solid}} = -\frac{f^*}{s} \frac{N\rho_f^2}{2c_s \rho_0} \frac{1}{4} \]  

Substituting equations (190), (194), (196) and (199) into equation (188), the first-order perturbation solution for the solid constituent displacement potential can be written as

\[ \psi_s = e^{-bD_s \frac{z}{c_s}} \left[ c_s L^{-1} \left( \frac{f^*}{s} e^{\frac{z}{c_s}} \right) - bD_s c_s 2L^{-1} \left( \frac{f^*}{s} e^{\frac{z}{c_s}} \right) \right] \]  

where

\[ c_s = \left[ \frac{N(\rho_f + \rho_a)}{\rho_0^2} \right]^{\frac{1}{3}} \]
\[
D_s = \frac{N \rho_f^2}{2 c_s \rho_o^3 \rho_o^4}
\]

(202)

Applying an identical procedure to equation (183), the first-order perturbation solution for the fluid constituent displacement potential can be obtained in a straightforward manner as

\[
\psi_f = e^{-b D_s z} \left[ \frac{\rho_a}{\rho_f + \rho_a} \right] c_s L^{-1} \left( \frac{f^*}{s} e^{-s c_s} \right)
+ b \frac{c_s}{(\rho_f + \rho_a)^2} \left( \frac{\rho_f}{2} - \frac{\rho_f^2}{2 \rho_o} \right) L^{-1} \left( \frac{f^*}{s^2} e^{-s c_s} \right)
\]

(203)

From equations (169), (170), (200) and (203), the perturbation solutions for the displacement histories in the solid and fluid constituents are given by

\[
u_s = e^{-b D_s z} L^{-1} \left( \frac{f^*}{s} e^{-s c_s} \right)
\]

(204)

\[
u_f = e^{-b D_s z} \left[ \frac{\rho_a}{\rho_f + \rho_a} \right] c_s L^{-1} \left( \frac{f^*}{s} e^{-s c_s} \right)
+ b \frac{\rho_f}{(\rho_f + \rho_a)^2} L^{-1} \left( \frac{f^*}{s} e^{-s c_s} \right)
\]

(205)

For the case of an impulsive step displacement at the boundary, \( f(t) = 1 \), the Laplace transform inversions appearing in equations (204) and (205) can be readily evaluated.
either by relation (184) or by recourse to a table of transform inversions. The resulting displacement histories are given by

\[ u(s) = e^{-bd_s z} H(t - \frac{z}{c_s}) \]  

(206)

\[ u(f) = e^{-bd_s z} \left[ \frac{\rho_a}{\rho_f + \rho_a} + b \frac{\rho_f}{(\rho_f + \rho_a)^2} \right] \frac{1}{(c_s)^2} H(t - \frac{z}{c_s}) \]  

(207)

where \( H(\cdot) \) is again Heaviside's step function.

The response of the solid constituent of a fluid saturated porous medium to an impulsive lateral step displacement at the boundary of a half-space is therefore seen to consist of a step displacement propagating in the z-direction with velocity \( c_s \) and with amplitude attenuation exponentially proportional to the dissipation parameter \( b \) and to the wave travel distance \( z \). The fluid constituent response consists of a similar step displacement propagating at the same velocity. However, in the fluid component, there is a residual component of velocity in the x-direction following the passage of the wave. The magnitude of this velocity has the same attenuation coefficient as the wave amplitude and is proportional to the dissipation parameter. It is clear from equation (207) that the approximate solutions (206) and (207) are valid for small values of time following the passage of the wave.
3.4.2 Normal Excitation of a Half-Space and Spherical
Propagation from a Point Source

Consider a fluid saturated porous half-space with a Cartesian coordinate system oriented as in section 3.4.1. Let the boundary of the half-space be subjected to a time varying normal displacement so that

\[ w(s) \bigg|_{z=0} = w(f) \bigg|_{z=0} = \tilde{H}(t)f(t) \]  \hspace{1cm} (208)

where \( w(s) \) is the z-component of \( u(s) \), \( w(f) \) is the z-component of \( u(f) \) and \( f(t) \) is some function of time. (Figure 2)

From equations (160), (161) and (208) it is clearer that only \( \phi_s \) and \( \phi_f \) are required so that equations (160) and (161) reduce to

\[ w(s) = \frac{\partial \phi_s}{\partial z} \]  \hspace{1cm} (209)

\[ w(f) = \frac{\partial \phi_f}{\partial z} \]  \hspace{1cm} (210)

and the potential equations (162) and (163) become

\[ p \frac{\partial^2 \phi_s}{\partial z^2} + q \frac{\partial^2 \phi_f}{\partial z^2} = (\rho_s + \rho_a) \frac{\partial^2 \phi_s}{\partial t^2} - \rho_a \frac{\partial^2 \phi_f}{\partial t^2} + b \frac{\partial}{\partial t}(\phi_s - \phi_f) \]  \hspace{1cm} (211)

\[ q \frac{\partial^2 \phi_s}{\partial z^2} + r \frac{\partial^2 \phi_f}{\partial z^2} = -\rho_a \frac{\partial^2 \phi_s}{\partial t^2} + (\rho_f + \rho_a) \frac{\partial^2 \phi_f}{\partial t^2} - b \frac{\partial}{\partial t}(\phi_s - \phi_f) \]  \hspace{1cm} (212)
From equations (208), (209) and (210), the potential boundary conditions are given by

\[
\frac{\partial \phi_s}{\partial z} \bigg|_{z=0} = \frac{\partial \phi_f}{\partial z} \bigg|_{z=0} = \bar{H}(t)f(t)
\]  

(213)

Now consider instead the problem of a spherical cavity in a fluid saturated porous space which is subjected to a time varying radially symmetric normal displacement at the surface of the cavity. Let \( r \) be the radial coordinate and let \( u_r^r(s) \) and \( u_r^r(f) \) be the radial components of the constituent displacement vectors. (Figure 3) Let the displacement boundary condition be given by

\[
\left. u_r^r(s) \right|_{r=r_0} = \left. u_r^r(f) \right|_{r=r_0} = \frac{\bar{H}(t)f(t)}{\mu}
\]

(214)

where \( r_0 \) is the cavity radius and \( \mu \) is a parameter. Writing equations (162) and (163) in radially symmetric spherical coordinates and making the changes of variable

\[
\phi_s = \frac{1}{r} \Omega_s
\]

(215)

\[
\phi_f = \frac{1}{r} \Omega_f
\]

(216)

equations (162) and (163) become

\[
P \frac{\partial^2 \Omega_s}{\partial r^2} + Q \frac{\partial^2 \Omega_f}{\partial r^2} = (\rho_s + \rho_a) \frac{\partial^2 \Omega_s}{\partial t^2} - \rho_a \frac{\partial^2 \Omega_f}{\partial t^2} + b \frac{\partial}{\partial t} (\Omega_s - \Omega_f)
\]

(217)
\[ Q \frac{\partial^2 \Omega_s}{\partial r^2} + R \frac{\partial^2 \Omega_f}{\partial r^2} = -\rho_a \frac{\partial^2 \Omega_s}{\partial t^2} + (\rho_f + \rho_a) \frac{\partial^2 \Omega_f}{\partial t^2} - b \frac{\partial}{\partial t}(\Omega_s - \Omega_f) \quad (218) \]

Note that equations (217) and (218) are formally identical to equations (211) and (212). Writing equations (160) and (161) in radially symmetric spherical coordinates and using equations (214), (215) and (216), the potential boundary conditions are

\[ \left. \frac{1}{r_o} \frac{\partial \Omega_s}{\partial r} \right|_{r=r_o} - \left. \frac{1}{r_o^2} \Omega_s \right|_{r=r_o} = \frac{\tilde{H}(t)f(t)}{\mu} \quad (219) \]

\[ \left. \frac{1}{r_o} \frac{\partial \Omega_f}{\partial r} \right|_{r=r_o} - \left. \frac{1}{r_o^2} \Omega_f \right|_{r=r_o} = \frac{\tilde{H}(t)f(t)}{\mu} \quad (220) \]

In order to reduce the finite cavity to a point source, let equations (219) and (220) be multiplied by \( r_o^2 \) and let \( r_o \) and the parameter \( \mu \) each approach zero in such a way that \( r_o^2/\mu = 1 \) uniformly. The potential boundary conditions then reduce to

\[ \left. \Omega_s \right|_{r=0} = -\tilde{H}(t)f(t) \quad (221) \]

\[ \left. \Omega_f \right|_{r=0} = -\tilde{H}(t)f(t) \quad (222) \]

It is therefore apparent that equations (211) and (212) for wave propagation in a fluid saturated porous half-space due
to normal excitation at the boundary are also applicable to
the propagation of spherically symmetric waves due to normal
excitation at a point source if the boundary conditions (213)
are replaced by

$$\phi_s \bigg|_{z=0} = \phi_f \bigg|_{z=0} = -\bar{H}(t)f(t)$$  \hfill (223)

and displacements are determined from the relations

$$u_r^s(s) = \frac{1}{z} \frac{\partial \phi_s}{\partial z} - \frac{1}{z^2} \phi_s$$ \hfill (224)

$$u_r^f(f) = \frac{1}{z} \frac{\partial \phi_f}{\partial z} - \frac{1}{z^2} \phi_f$$ \hfill (225)

Until the boundary conditions are applied, therefore,
the problems of normal excitation of a half-space and spheri-
cal propagation from a point source can be solved concurrent-
ly. This result is familiar from the similar procedure used
to solve problems based on the classical wave equation. [31]

Proceeding to solve equations (211) and (212), let
the Laplace transform in time again be defined as in equa-
tion (173). The transformed equations (211) and (212) are
given by

$$P \frac{d^2 \phi_s^*}{dz^2} + Q \frac{d^2 \phi_f^*}{dz^2} = \left[ (\rho_s + \rho_a)s^2 + bs \right] \phi_s^* - (\rho_a s^2 + bs) \phi_f^*$$ \hfill (226)

$$Q \frac{d^2 \phi_s^*}{dz^2} + R \frac{d^2 \phi_f^*}{dz^2} = -(\rho_a s^2 + bs) \phi_s^* + \left[ (\rho_f + \rho_a)s^2 + bs \right] \phi_f^*$$ \hfill (227)
Now let a second Laplace transform in the spatial coordinate $z$ be defined by

$$
\tilde{\phi} = \int_0^\infty \phi e^{-\lambda z} \, dz
$$
(228)

The doubly transformed equations (226) and (227) become

$$
\begin{align*}
&\left[ P^2 - (\rho_s + \rho_a) s^2 - bs \right] \tilde{\phi}_s + (Q^2 + \rho_a s^2 + bs) \tilde{\phi}_f \\
&= P \left( \frac{d\phi_s^*}{dz} \right|_{z=0} + \frac{d\phi_f^*}{dz} \right|_{z=0} + Q \left( \frac{d\phi_f^*}{dz} \right|_{z=0} + \frac{d\phi_f^*}{dz} \right|_{z=0}
\end{align*}
$$
(229)

$$
\begin{align*}
&\left[ Q^2 + \rho_a s^2 + bs \right] \tilde{\phi}_s + \left[ R^2 - (\rho_f + \rho_a) s^2 - bs \right] \tilde{\phi}_f \\
&= Q \left( \frac{d\phi_s^*}{dz} \right|_{z=0} + \frac{d\phi_f^*}{dz} \right|_{z=0} + R \left( \frac{d\phi_f^*}{dz} \right|_{z=0} + \frac{d\phi_f^*}{dz} \right|_{z=0}
\end{align*}
$$
(230)

Equations (229) and (230) are two linear algebraic equations in the doubly transformed variables $\tilde{\phi}_s$ and $\tilde{\phi}_f$. Solving these equations for $\tilde{\phi}_s$ and $\tilde{\phi}_f$, the results can be written as

$$
\tilde{\phi}_s = \frac{1}{\Sigma(\lambda, s)} \left\{ - \left[ K^2 - P \rho_f s^2 - (P+Q)(\rho_a s^2 + bs) \right] \lambda \phi_s^* \right|_{z=0}
$$

$$
+ \left[ K^2 - P \rho_f s^2 - (P+Q)(\rho_a s^2 + bs) \right] \frac{d\phi_s^*}{dz} \right|_{z=0}
$$

$$
+ \left[ -Q \rho_f s^2 - (Q+R)(\rho_a s^2 + bs) \right] \lambda \phi_f^* \right|_{z=0}
$$

$$
+ \left[ -Q \rho_f s^2 - (Q+R)(\rho_a s^2 + bs) \right] \frac{d\phi_f^*}{dz} \right|_{z=0}
\right\}
$$
(231)
\[ \ddot{\phi}_f = \frac{1}{\tilde{\Sigma}(\lambda, s)} \left\{ \left[ -Q \rho_f s^2 - (Q+P)(\rho_a s^2 + bs) \right] \phi_s^{*} \right\}_{z=0} \\
+ \left[ -Q \rho_s s^2 - (Q+P)(\rho_a s^2 + bs) \right] \frac{d\phi_s^{*}}{dz} \bigg|_{z=0} \\
+ \left[ K \lambda^2 - R \rho_s s^2 - (R+Q)(\rho_a s^2 + bs) \right] \left[ \phi_f^{*} \right] \bigg|_{z=0} \\
+ \left[ K \lambda^2 - R \rho_0 s^2 - (R+Q)(\rho_a s^2 + bs) \right] \frac{d\phi_f^{*}}{dz} \bigg|_{z=0} \right\} \] (232)

where

\[ K = PR - Q^2 \] (233)

and \( \tilde{\Sigma}(\lambda, s) \) is defined by

\[ \tilde{\Sigma}(\lambda, s) = \left[ P \lambda^2 - (\rho_s + \rho_a) s^2 - bs \right] \left[ R \lambda^2 - (\rho_f + \rho_a) s^2 - bs \right] \\
- (Q \lambda^2 + \rho_a s^2 + bs)^2 \] (234)

In order to obtain the solutions, the doubly transformed equations (231) and (232) must be subjected to two inverse transformations. The transformation in \( z \) will be inverted first. This can be accomplished in a relatively straightforward way by using the inversion formula (184). In this case, the transformations can be written as

\[ \phi_s^{*} = \frac{1}{2\pi i} \int_{B} \tilde{\phi}_s e^{\lambda z} d\lambda \] (235)
\[ \phi_f^* = \frac{1}{2\pi i} \oint_{\mathcal{C}} \Phi_f e^{\lambda z} d\lambda \]  

Biot [3] showed that \( K \) must be positive. An examination of equations (231) and (232) therefore shows that the denominators of \( \Phi_s \) and \( \Phi_f \) are of order one higher in \( \lambda \) than the numerators. Since only positive real values of \( z \) are being considered, Jordan's lemma applies to the closure of the Bromwich contour when the closure is in the left-hand side of the \( \lambda \)-plane. Note also that the functions \( \Phi_s e^{\lambda z} \) and \( \Phi_f e^{\lambda z} \) have no branch-points in the \( \lambda \)-plane. Cauchy's theorem therefore allows equations (235) and (236) to be written as

\[ \Phi_s^* = \sum \text{Residues } (\Phi_s e^{\lambda z}) \]  

\[ \Phi_f^* = \sum \text{Residues } (\Phi_f e^{\lambda z}) \]  

The poles of the functions \( \Phi_s e^{\lambda z} \) and \( \Phi_f e^{\lambda z} \) occur at the roots of

\[ \sum (\lambda) = 0 \]  

From equation (234), these roots are as follows.

\[ \lambda_1 = -s \left( \frac{\sqrt{(J+H_s b_s)^2} - 4K[\rho_o^2 + (\rho_s + \rho_f) b_s]}{2K} \right) \]
\[
\lambda_2 = -s\left\{ \frac{(J+Hb_s)}{2K} - \sqrt{\left(\frac{(J+Hb_s)}{2K}\right)^2 - 4K\left[\rho_o^2 + (\rho_s + \rho_f)\frac{b}{s}\right]} \right\} \frac{1}{2} \tag{241}
\]

\[
\lambda_3 = -\lambda_1 \tag{242}
\]

\[
\lambda_4 = -\lambda_2 \tag{243}
\]

where

\[
H = P + R + 2Q \tag{244}
\]

\[
J = P\rho_f + R\rho_s + H\rho_a \tag{245}
\]

and \(\rho_o\) is defined as in equation (179).

It can be shown that, for \(s\) real and positive, \(\lambda_1\) and \(\lambda_2\) are real and negative, \(\lambda_3\) and \(\lambda_4\) are real and positive, and there can be no repeated roots. Therefore, the functions \(\tilde{\varphi}_s e^{\lambda z}\) and \(\tilde{\varphi}_f e^{\lambda z}\) are of the form

\[
\tilde{\varphi}_s e^{\lambda z} = \frac{\Gamma_s(\lambda)}{\Sigma(\lambda)} \tag{246}
\]

\[
\tilde{\varphi}_f e^{\lambda z} = \frac{\Gamma_f(\lambda)}{\Sigma(\lambda)} \tag{247}
\]

where the functions \(\Gamma_s(\lambda), \Gamma_f(\lambda)\) and \(\Sigma(\lambda)\) are everywhere analytic in \(\lambda\) and \(\Sigma\) has simple zero's at \(\lambda = \lambda_1, \lambda_2, \lambda_3\) and \(\lambda_4\). The residues of the functions \(\tilde{\varphi}_s e^{\lambda z}\) and \(\tilde{\varphi}_f e^{\lambda z}\) can therefore be obtained from the relations \([32]\)
\[
(248) \quad \text{RESIDUE}(\bar{\phi}_s e^{\lambda z}) \bigg|_{\lambda = \lambda_k} = \frac{\Gamma_s}{\frac{d\Sigma}{d\lambda}} \bigg|_{\lambda = \lambda_k}
\]

\[
(249) \quad \text{RESIDUE}(\bar{\phi}_f e^{\lambda z}) \bigg|_{\lambda = \lambda_k} = \frac{\Gamma_f}{\frac{d\Sigma}{d\lambda}} \bigg|_{\lambda = \lambda_k}
\]

and equations (237) and (238) become

\[
(250) \quad \phi^*_s = \sum_{k=1}^{4} \frac{\Gamma_s}{\frac{d\Sigma}{d\lambda}} \bigg|_{\lambda = \lambda_k}
\]

\[
(251) \quad \phi^*_f = \sum_{k=1}^{4} \frac{\Gamma_f}{\frac{d\Sigma}{d\lambda}} \bigg|_{\lambda = \lambda_k}
\]

Performing the indicated differentiation on equation (234) and using equations (231), (232), (250) and (251), the solutions for \( \phi^*_s \) and \( \phi^*_f \) are given by

\[
\phi^*_s = \sum_{k=1}^{4} \left[ \left( K \frac{\lambda_k^2}{s^2} - P\rho_f - (P+Q)(\rho_a + \frac{b}{s}) \right) \lambda_k \phi^*_s \bigg|_{z=0} \right. \\
+ \left. \frac{\lambda_k^2}{s^2} - P\rho_f - (P+Q)(\rho_a + \frac{b}{s}) \right] \frac{d\phi^*_s}{dz} \bigg|_{z=0}
\]

\[
\phi^*_f = \sum_{k=1}^{4} \left[ \left( K \frac{\lambda_k^2}{s^2} - P\rho_f - (P+Q)(\rho_a + \frac{b}{s}) \right) \lambda_k \phi^*_f \bigg|_{z=0} \right.
\\
+ \left. \frac{\lambda_k^2}{s^2} - P\rho_f - (P+Q)(\rho_a + \frac{b}{s}) \right] \frac{d\phi^*_f}{dz} \bigg|_{z=0}
\]
\[
\phi^*_f = \sum_{k=1}^{4} \left\{ \left[ -Q \rho_s - (Q+P)(\rho_a_s + \frac{b}{s}) \right] \lambda_k \phi^*_s \right|_{z=0}^{\frac{\lambda_k z}{M_k}} \right\} + \left\{ \left[ -Q \rho_f - (Q+R)(\rho_a_s + \frac{b}{s}) \right] \frac{d\phi^*_f}{dz} \right|_{z=0}^{\frac{\lambda_k z}{M_k}} \right\} \\
\] (252)

\[
\phi^*_s = \sum_{k=1}^{4} \left\{ \left[ -Q \rho_s - (Q+P)(\rho_a_s + \frac{b}{s}) \right] \lambda_k \phi^*_s \right|_{z=0}^{\frac{\lambda_k z}{M_k}} \right\} + \left\{ \left[ -Q \rho_s - (Q+P)(\rho_a_s + \frac{b}{s}) \right] \frac{d\phi^*_s}{dz} \right|_{z=0}^{\frac{\lambda_k z}{M_k}} \right\} \\
\] (253)

where

\[
M_k = 2\lambda_k \left( 2K \frac{\lambda_k}{s^2} - J - H \frac{b}{s} \right) \] (254)

Since there are two real positive and two real negative \(\lambda_k\)'s for \(s\) real and positive, it is clear that the four terms obtained by the summation over \(k\) in equations (252) and (253) represent two outward propagating and two inward propagating disturbances. Note that equations (252) and (253) contain four boundary conditions, \(\phi^*_s \left|_{z=0}^{\frac{\lambda_k z}{M_k}} \right.\) and \(\frac{d\phi^*_s}{dz} \left|_{z=0}^{\frac{\lambda_k z}{M_k}} \right.\)
\[ \phi_f^* \bigg|_{z=0} \quad \text{and} \quad \frac{d\phi_f^*}{dz} \bigg|_{z=0} \]  

while from equations (213) and (223) it is seen that the problem specification supplies only two of these. The two remaining unknown boundary conditions appearing in equations (252) and (253) will be determined by requiring terms representing inward propagating disturbances to vanish. Although it may at first appear that this procedure provides four equations for determining only two unknown boundary conditions, since there are two inward propagating disturbances for each constituent, it is clear from equations (231) and (232) that for \( \lambda = \lambda_k \), only two of the four equations can be linearly independent.

Consider first the problem of normal excitation of a half-space. From equations (213), the transformed potential boundary conditions are

\[ \frac{d\phi_s^*}{dz} \bigg|_{z=0} = \frac{d\phi_f^*}{dz} \bigg|_{z=0} = f^*(s) \quad (255) \]

Substituting equations (255) into equations (252) and (253) and solving for the boundary conditions \( \phi_s^* \bigg|_{z=0} \) and \( \phi_f^* \bigg|_{z=0} \) by requiring terms containing the roots \( \lambda_3 \) and \( \lambda_4 \) to vanish, the results can be written as

\[ \phi_s^* \bigg|_{\text{normal excitation}} = \sum_{k=1}^{2} \left( K \frac{\lambda_k^2}{s^2} - (P+Q) \rho_f - H(\rho_a^b) \right) \frac{2f^* \cdot \lambda_k z}{M_k} \quad (256) \]
\[ \phi_f^* = \sum_{k=1}^{2} \left[ K \frac{\lambda_k^2}{s^2} - (Q+R) \rho_s - H(\rho_a + b_s) \right] \frac{2f_e^* \lambda_k^z}{M_k} \] (257)

This completes the inversion of the spatial transform for the problem of normal excitation of a half-space.

For the problem of spherical propagation from a point source, the transformed displacement boundary conditions are obtained from equations (223) as

\[ \phi_s^* \bigg|_{z=0} = \phi_f^* \bigg|_{z=0} = -f^*(s) \] (258)

Substituting these equations into equations (252) and (253) and solving for the boundary conditions \( \frac{d\phi_s^*}{dz} \bigg|_{z=0} \) and \( \frac{d\phi_f^*}{dz} \bigg|_{z=0} \) by requiring terms containing the roots \( \lambda_3 \) and \( \lambda_4 \) to vanish, the results can be written as

\[ \phi_s^* \bigg|_{spherical} = \sum_{k=1}^{2} \left[ K \frac{\lambda_k^2}{s^2} - (P+Q) \rho_f - H(\rho_a + b_s) \right] \frac{2\lambda_k f_e^* \lambda_k^z}{M_k} \] (259)

\[ \phi_f^* \bigg|_{spherical} = \sum_{k=1}^{2} \left[ K \frac{\lambda_k^2}{s^2} - (R+Q) \rho_s - H(\rho_a + b_s) \right] \frac{2\lambda_k f_e^* \lambda_k^z}{M_k} \] (260)

This completes the inversion of the spatial transform for the problem of spherical propagation from a point source.

In order to reduce the steps in completing the solution, equations (256), (257), (259) and (260) can be written as the
single equation

\[ \phi^* = \sum_{k=1}^{2} \left[ K \frac{\lambda_k}{s^2} - \tilde{\sigma} - H\left(\rho_a + \frac{b}{s}\right) \right] \frac{(-1)^{\xi+1} \lambda_k^* e^{\lambda_k z}}{M_k^{\xi-1}} \]  

(261)

where

\[ \xi = \begin{cases} 1, & \text{normal excitation of a half-space} \\ 0, & \text{spherical propagation from a point source} \end{cases} 

(262)

\[ (P+Q)\rho_f, \text{ solid constituent} \]

(264)

\[ (Q+R)\rho_s, \text{ fluid constituent} \]

(265)

In order to complete the solution, the Laplace transform in time (261) must be inverted. Note that equation (261) is of the form of equation (185), where the functions corresponding to F and G are again analytic in b for all positive real values of b and s. The inversion of (261) can therefore be accomplished by the same perturbation technique used in section 2.4.1. After lengthy but straightforward algebra, the first order perturbation solution for \( \phi \) can be shown to reduce to

\[ \phi = \sum_{k=1}^{2} \frac{c_k}{d_k} e^{-b c d_k d k} \left[ E_k L^{-1}\left( \frac{f_*}{s^\xi} e^{-\frac{z}{c} d k} s \right) \right] \]

\[ + b F_k L^{-1}\left( \frac{f_*}{s^\xi+1} e^{-\frac{z}{c} d k} s \right) \]

(266)

where
\[ c_{d1} = \left( \frac{2K}{J + \sqrt{J^2 - 4K\rho_o^2}} \right)^{\frac{1}{2}} \]  
(267)

\[ c_{d2} = \left( \frac{2K}{J - \sqrt{J^2 - 4K\rho_o^2}} \right)^{\frac{1}{2}} \]  
(268)

\[ D_{d1} = \frac{H + (J^2 - 4K\rho_o^2)^{\frac{1}{2}} [JH - 2K(\rho_s + \rho_f)]}{4K} \]  
(269)

\[ D_{d2} = \frac{H - (J^2 - 4K\rho_o^2)^{\frac{1}{2}} [JH - 2K(\rho_s + \rho_f)]}{4K} \]  
(270)

\[ E_k = \frac{K}{2} - \bar{\sigma} - H\rho_a \]  
(271)

\[ F_k = \frac{2KD_{dk} - H}{2K - J} - \frac{(4KD_{dk} - H)(\frac{K}{c_{dk}^2} - \bar{\sigma} - H\rho_a)}{(\frac{2K}{c_{dk}^2} - J)^2} \]  
\[ \left(\xi_{dk}c_{dk}^2\right) \left(\frac{K}{c_{dk}^2} - \bar{\sigma} - H\rho_a\right) \]  
\[ - \frac{\bar{\sigma}}{2K - J} \]  
(272)

Applying equations (209) and (210) to equation (266) with \( \xi = 1 \), the first-order perturbation solution for the
displacement history in a fluid-saturated porous half-space due to normal excitation at the boundary is determined as

\[
w = \sum_{k=1}^{2} e^{-bc_{dk}D_{dk}z} \left[ E_k L^{-1}\left( f^* e^{z/c_{dk}} \right) \right. \\
+ \left. b(c_{dk}^2D_{dk}E_kF_k) L^{-1}\left( f^* e^{z/c_{dk}} \right) \right] \tag{273}
\]

Note that this result is valid for either the solid or the fluid constituent, depending on whether \( \sigma \) is evaluated by equation (264) or by equation (265).

For the case of an impulsive step displacement, \( f(t)=1 \), the inverse transforms in equation (273) can be readily evaluated to give

\[
w = \sum_{k=1}^{2} e^{-bc_{dk}D_{dk}z} \left[ E_k H(t - \frac{z}{c_{dk}}) \right. \\
+ \left. b(c_{dk}^2D_{dk}E_k+F_k)(t - \frac{z}{c_{dk}}) H(t - \frac{z}{c_{dk}}) \right] \tag{274}
\]

The response of a fluid-saturated porous half-space to a normal impulsive step displacement at the boundary is therefore seen to consist of two propagating step displacements, one with velocity \( c_{d1} \) and the other with velocity \( c_{d2} \), the magnitude of each wave attenuating exponentially with the exponent proportional to the dissipation parameter and to...
and \( c_{d1} D_{d1} \) and \( c_{d2} D_{d2} \), respectively. In addition, a residual velocity term proportional to the dissipation parameter is present following the passage of each wave.

Applying equation (224) and (225) to equation (226) with \( \kappa = 0 \), the first-order perturbation solution for the displacement history in a fluid-saturated porous space due to normal excitation at a point source is determined as

\[
u^{r} = \sum_{k=1}^{2} e^{-b_{c} \frac{D_{d} D_{d}}{c_{d} k}} \left[ \frac{1}{z} \frac{E_{k}}{c_{d}} \right] L^{-1} \left( s \ast \frac{z}{c_{d} k} \right)
+ \frac{b}{z} \left( c_{d} D_{d} D_{d} E_{k} + \frac{F_{k}}{c_{d} k} + \frac{1}{b z} E_{k} \right) L^{-1} \left( s \ast \frac{z}{c_{d} k} \right)
+ \frac{b}{z} \frac{F_{k}}{L^{-1} \left( s \ast \frac{z}{c_{d} k} \right)}
\]

(275)

For an impulsive displacement, equation (275) becomes

\[
u^{r} = \sum_{k=1}^{2} e^{-b_{c} \frac{D_{d} D_{d}}{c_{d} k}} \left[ \frac{b}{z} \left( c_{d} D_{d} D_{d} E_{k} + \frac{F_{k}}{c_{d} k} \right) + \frac{E_{k}}{z^2} \right] \tilde{H}(t - \frac{z}{c_{d} k})
+ \frac{b}{z} \frac{F_{k}}{L^{-1} \left( s \ast \frac{z}{c_{d} k} \right)} \tilde{H}(t - \frac{z}{c_{d} k})
\]

(276)

Note again that this solution applies to both constituents depending on the value given to \( \bar{\sigma} \). The response of a fluid-saturated porous space to excitation at a point source is seen to be qualitatively similar to the normal half-space.
solution except that the wave and residual velocity magnitudes attenuate with inverse powers of the wave travel distance. The wave magnitudes have two terms, one attenuating as $z^{-1}$ and the other attenuating as $z^{-2}$, while the residual velocity magnitude attenuates as $z^{-2}$.

This completes the problem solutions. Note that the solutions consisting of propagating step displacements obtained above are completely a consequence of the perturbation approximation employed. It is clear from the fact that the exact transform solutions contain only branch points and no poles that the exact inverted solutions would not have stable step displacements, but would exhibit scattering. In order to obtain solutions presenting a reasonable approximation to an actual case from the perturbation solutions determined above, the perturbation solutions would have to be convolved with a high-frequency, short duration excitation pulse with no d.c. component.

3.4.3 Numerical Examples

In this section, example waveforms based on the solutions presented in sections 3.4.1 and 3.4.2 are given using experimental data for a particular fluid-saturated porous medium.

The data used is that for kerosene-saturated sandstone presented by Fatt [33] and quoted by Deresiewicz [7].
\[ R = 0.0473 \times 10^6 \text{ lb/in}^2 \]
\[ Q = 0.1078 \times 10^6 \text{ lb/in}^2 \]
\[ A = 0.6453 \times 10^6 \text{ lb/in}^2 \]
\[ N = 0.3996 \times 10^6 \text{ lb/in}^2 \]
\[ \rho_s = 1.924 \text{ gm/cm}^3 \]
\[ \rho_f = 0.2132 \text{ gm/cm}^3 \]
\[ \beta = 0.26 \]

The data of Fatt was limited to static measurements. The values of \( \rho_a \) and \( b \) are therefore unspecified. In the manner of Deresiewicz, it will be assumed that \( \rho_a = 0.001 \rho \). In addition, the results will be presented in terms of the variables

\[ z' = bz \quad (278) \]
\[ t' = bt \quad (279) \]

so that the value of the dissipation parameter \( b \) can remain unspecified.

Using (277) through (279), equations (206) and (207) have been used to compute the solid and fluid constituent displacement histories due to an impulsive lateral displacement at the boundary of a fluid-saturated porous elastic half-space. The solid displacement as a function of the displacement parameter \( z' \) is plotted for various values of the time parameter \( t' \) in figure 4. The corresponding fluid displacement history is plotted in figure 5.
In figures 6 and 7 the fluid and solid displacement histories due to impulsive normal excitation at the boundary of a fluid-saturated porous half-space are plotted. The existence of two propagating waves is apparent.

It should again be emphasized that the solutions represented by figures 4 through 7 should be convolved with a high frequency, short duration pulse to obtain physically realizable results.
REFERENCES


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<td>$\mathbf{e}(\xi)$</td>
<td>Constituent acceleration.</td>
<td>(11)</td>
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<td>$A$</td>
<td>Constitutive constant.</td>
<td>(142)</td>
</tr>
<tr>
<td>$\mathbf{A}$</td>
<td>Arbitrary vector function.</td>
<td>(6)</td>
</tr>
<tr>
<td>$b, b'$</td>
<td>Constitutive constants.</td>
<td>(142)</td>
</tr>
<tr>
<td>$b_k$</td>
<td>Cartesian time-varying vector.</td>
<td>(95)</td>
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<tr>
<td>$c_s$</td>
<td>Solenoidal propagation velocity.</td>
<td>(191)</td>
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<td>$c(\xi)$</td>
<td>Constituent density fraction.</td>
<td>(88)</td>
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<td>$c_{d1}, c_{d2}$</td>
<td>Irrotational propagation velocities.</td>
<td>(267), (268)</td>
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<td>$\mathbf{d}^{kj}$</td>
<td>Finger's deformation tensor.</td>
<td>(104)</td>
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<tr>
<td>$\mathbf{d}^k(\xi)$</td>
<td>Constituent deformation rate tensor.</td>
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<td>$\mathbf{d}^k(\xi,\lambda)$</td>
<td>Constituent relative deformation rate tensor.</td>
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<td>$D_s$</td>
<td></td>
<td>(202)</td>
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<td>$D_{d1}, D_{d2}$</td>
<td>(269), (270)</td>
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<tr>
<td>$\mathbf{e}(\xi)$</td>
<td>Constituent energy transfer term.</td>
<td>(22)</td>
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<td>$\mathbf{e}(\xi)$</td>
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<td>$\mathbf{e}^k(\xi)$</td>
<td>Constituent infinitesimal strain tensor.</td>
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<td>The alternating tensor.</td>
<td>(17)</td>
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<td>$E_k$</td>
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<tr>
<td>$E(\xi)$</td>
<td>Constituent energy transfer across $\sigma$.</td>
<td>(23)</td>
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*The numbers in parentheses refer to the first equation in which the particular symbol appears. Where a symbol has been defined for algebraic convenience and has no intrinsic physical significance, only the equation number is given.
f(t) Function of time. (167)
\( \dot{f}(\xi) \) Constituent extensive body force. (10)
F Analytic function of b. (185)
\( F_k \) (272)
\( g_{kj}^{kl} \) Spatial fundamental metric tensor. (126)
G Analytic function of b. (185)
\( G_{KL} \) Material fundamental metric tensor. (104)
h(\xi) Constituent external energy source. (19)
H (244)
\( \tilde{H}(t) \) Heaviside's step function, = 1 for \( t \geq 0 \),
= 0 for \( t < 0 \). (167)
i Square root of -1. (184)
J (245)
K (233)
L^{-1} Inverse Laplace transform. (184)
\( M_k \) (254)
n(\xi) Constituent entropy transfer term. (26)
\( \bar{n}(\xi) \) Generalized entropy transfer term. (31)
\( \bar{n} \) Unit normal vector. (5)
N Constitutive constant. (142)
\( N(\xi) \) Constituent entropy transfer across \( \sigma \). (27)
P(\xi) Constituent momentum transfer vector. (14)
P (166)
\( P(\xi) \) Constituent momentum transfer vector across \( \sigma \). (15)
\( Q(\xi) \) Constituent heat flux vector. (19)
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<td>Radial coordinate. (214)</td>
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<td>Constituent partial stress tensor. (10)</td>
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<td>Equilibrium part of the constituent partial stress tensor. (67)</td>
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<td>$u(\xi)$</td>
<td>x-component of $\mathbf{u}(\xi)$. (167)</td>
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<td>Radial component of the constituent displacement vector. (214)</td>
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<td>$\mathbf{v}(\xi)$</td>
<td>Constituent velocity. (2)</td>
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<td>$\mathbf{v}(\xi, \lambda)$</td>
<td>Constituent relative velocity. (32)</td>
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<td>$V(\xi)$</td>
<td>Constituent material volume. (5)</td>
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<td>$V^*(\xi)$</td>
<td>Constituent material volume not containing the neighborhood of $\sigma$. (5)</td>
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<tr>
<td>Symbol</td>
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<td>----------</td>
<td>-----------------------------------------------------------------------------</td>
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<tr>
<td>$w(\xi)$</td>
<td>Z-component of $u(\xi)$.</td>
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<tr>
<td>$w^k(\xi\lambda)j$</td>
<td>Constituent relative spin tensor.</td>
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<td>$x$</td>
<td>Cartesian coordinate.</td>
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<tr>
<td>$\xi$</td>
<td>Spatial coordinate vector.</td>
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<td>$X(\xi)$</td>
<td>Constituent material coordinate vector.</td>
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<tr>
<td>$z$</td>
<td>Cartesian coordinate.</td>
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<td>$z'$</td>
<td></td>
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<tr>
<td>$\alpha$</td>
<td>Scalar or tensor constant.</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Scalar or tensor constant.</td>
</tr>
<tr>
<td>$\bar{\beta}$</td>
<td>Porosity of the solid constituent, void volume per unit volume.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Scalar or tensor constant.</td>
</tr>
<tr>
<td>$\delta(\xi\lambda)$</td>
<td>Defined to be equal to one when the $\xi$th constituent and the $\lambda$th constituent are the same constituent and equal to zero otherwise.</td>
</tr>
<tr>
<td>$\epsilon(\xi)$</td>
<td>Constituent internal energy.</td>
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<tr>
<td>$\bar{\epsilon}(\xi)$</td>
<td>Average internal energy.</td>
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<td>$\zeta$</td>
<td>Scalar or tensor constant.</td>
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<td>$\eta$</td>
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<td>Laplace transform variable in $z$.</td>
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<td>Roots of $\Sigma(\lambda) = 0$.</td>
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μ
Scalar or tensor constant.

ν
Scalar or tensor constant.

χ
State vector of the variables \( x_{(s),k}^m \), \( \rho(\xi) \). (81)

ρ
Medium mass density. (3)

\( \rho_o \)
(179)

\( \rho(\xi) \)
Constituent mass density. (3)

\( \rho(a), \rho'_a, \rho_a \)
Constitutive constants. (142)

σ
A propagating discontinuity surface. (5)

\( \bar{\sigma} \)
(264), (265)

\( \sigma(\xi,\lambda) \)
Defined as \( 1 - \delta(\xi,\lambda) \).

\( \bar{\Sigma} \)
(234)

τ
Scalar or tensor constant. (125)

\( \phi_{\xi} \)
Irrotational constituent displacement potential.

(160)

\( \phi \)
Laplace transform in \( z \) of \( \phi^* \). (228)

\( \phi^* \)
Laplace transform in time of \( \phi \). (226)

\( \phi(jk) \)
Arbitrary symmetric tensor. (133)

\( \phi[jk] \)
Arbitrary antisymmetric tensor. (132)

\( \chi(\xi) \)
Motion function of the \( \xi \)th constituent. (1)

\( \psi \)
Arbitrary tensor function. (4)

\( \psi(\xi) \)
Constituent free energy. (29)

\( \psi^*(\xi) \)
Laplace transform in time of \( \psi \). (173)

\( \mathcal{L}(\xi) \)
Solenoidal constituent displacement potential.

(160)

\( \psi_{\xi} \)
y-component of \( \mathcal{L}_{\xi} \). (169)

\( \Omega_{\xi} \)
(215)
\( (\xi)_D \frac{D}{Dt} \) Constituent material derivative. (4)

\( \nabla \) Del operator. (146)

\( \nabla \times \) Curl operator. (160)

\( \nabla^2 \) Laplacian operator. (162)

\[ ] \] The jump in value of a quantity across \( \sigma \) in the direction of propagation. (5)

\( ; \) Covariant differentiation. (4)

\( , \) Partial differentiation. (33)
Figure 1. Lateral Excitation of a Fluid Saturated Porous Half-Space

Figure 2. Normal Excitation of a Fluid Saturated Porous Half-Space
Figure 3. Normal Excitation of a Spherical Cavity in a Fluid Saturated Porous Space
Figure 4. Solid Displacement History due to Impulsive Lateral Displacement of a Half-Space
Figure 5. Fluid Displacement History due to Impulsive Lateral Displacement of a Half-Space
Figure 6. Solid Displacement History due to Impulsive Normal Displacement of a Half-Space
Figure 7. Fluid Displacement History due to Impulsive Normal Displacement of a Half-Space