67-13,052

BEAUBOUEF, Richard Troy, 1936-
FREEZING OF FLUIDS IN FORCED FLOW.

Rice University, Ph.D., 1967
Engineering, mechanical

University Microfilms, Inc., Ann Arbor, Michigan

RICE UNIVERSITY

FREEZING OF FLUIDS IN FORCED FLOW

by

RICHARD TROY BEAUBOUEF

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's signature:

Alan J. Chapman

Houston, Texas

July, 1966

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

The author wishes to use this opportunity to express his appreciation to the members of his advisory committee, Dr. A.J. Chapman, Dr. F.A. Wierum and Dr. J.D. Hellums, for their assistance.

In addition, a debt of gratitude is owed Mrs. Mary Wheeler of the Rice University Research Computations Laboratory for her unflagging enthusiasm and the valuable assistance rendered in obtaining the numerical solutions.

Also, the author wishes to acknowledge the sacrifices patiently borne by his wife, Joan, and son, Richard, Jr., over the duration of the academic program, of which this thesis is the culmination.

Thanks are due the author's mother, Mrs. Eleanor Y. Beaubouef, for her conscientious efforts in typing the preliminary manuscript.

I. INTRODUCTION

The freezing of a liquid probably first received analytical treatment in Stefan's (1)* now-classical work on formation of polar ice. Since, phenomena of freezing and melting of liquids and solids as well as their analogues in the solid and vapor phases have come to be recognized as comprising a class of "Stefan-like" problems.

Typically, Stefan-like problems involve solution of the unsteady diffusion equation in a region partly bounded by an unknown moving surface. It is usually the configuration of this surface, as a function of time, which is of primary concern.

In recent years, this class of problems has attracted attention in diverse quarters. Those works reviewed by the author may be grouped in three broad categories. First, the exact closed-form solutions of Stefan and Neumann (2) and of Rosenthal (3). These solutions are available, of course, only when certain restrictions are imposed on the conditions at the moving boundary. According to Ingersoll, Zobel and Ingersoll (4), for example, the solutions of Stefan and Neumann apply only when the

* Numbers in parentheses refer to references listed in the Bibliography.

freezing liquid is stagnant, and Stefan's solution further requires the liquid to be everywhere at its fusion temperature. The analysis of Rosenthal is specifically concerned with the quasi-steady state in melting of the solid. The second classification includes those contributions to the literature which offer approximate solutions to problems of freezing and melting. The works reviewed which fall into this group deal exclusively with an integral approach, analogous to the von Karman-Pohlhausen integral method in boundary layer theory, with which the name of Goodman (5,6) is most prominently associated. With certain boundary conditions this method yields closed-form solutions, as demonstrated by Goodman in (5). More recently, however, Libby and Chen (7) applied Goodman's method to the problem of the growth of a solid deposited from a gaseous stream, and were forced to a numerical solution of the rather complicated ordinary differential equation which resulted. The third, and largest, group of publications contains finite-difference solutions, among which the works of Douglas (8) and Landau (9) are probably most prominent.

Additionally, the electrical network analogy of Kreith and Romie (10), and the series solution of Evans, Isaacson and MacDonald (11) should be mentioned, although they fall outside the classifications described above. A brief, but comprehensive, survey of analytical and numerical methods

was published by Murray and Landis (12) in 1959, who also introduced two new numerical procedures.

II. ANALYSIS

II. A) Statement of Problem and Basic Equations

The problem to be considered herein is that of freezing of a fluid in steady plane flow over a cold surface. By freezing is meant deposition of a solid phase.

The objective of the present work will be the determination of the thickness of the deposited solid phase, as a function of time and location on the cold surface, during the transient freezing process.

The basic assumptions are:

a) The thickness of the deposited layer is sufficiently small that conduction of heat within the solid phase may be assumed spatially one-dimensional in an orthogonal coordinate system constructed normal and parallel to the cold surface.

b) The convective heat flux, q_c, transferred from the fluid to the solid phase is known as a function of location on the cold surface, and is independent of time.

Additional assumptions are:

c) Physical properties of both fluid and solid phases are constants, but not necessarily the same constants.

d) Temperature of the cold surface, upon which the solid phase is deposited, is assumed uniform and constant.

e) It is assumed that there exists a definite interface between the fluid and solid phases.

Hereinafter, the cold surface upon which the solid phase is deposited will be referred to as the "plate"; the interface between the fluid and solid phases will subsequently be referred to as the "free boundary."

Figure I following shows the physical system and some of the notation employed.

The equations governing the flow of heat in the solid phase and motion of the free boundary are:

The heat diffusion equation:

1) $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}$$

The energy balance at the free boundary:

2) $$q_c + \rho \lambda \frac{ds}{dt} = k \frac{\partial T(s,t)}{\partial y}$$

Boundary and initial conditions are:

3) $$\begin{aligned} y &= 0; & T &= T_P \\ y &= s(t); & T &= T_F \\ t &= 0; & s &= 0 \end{aligned}$$

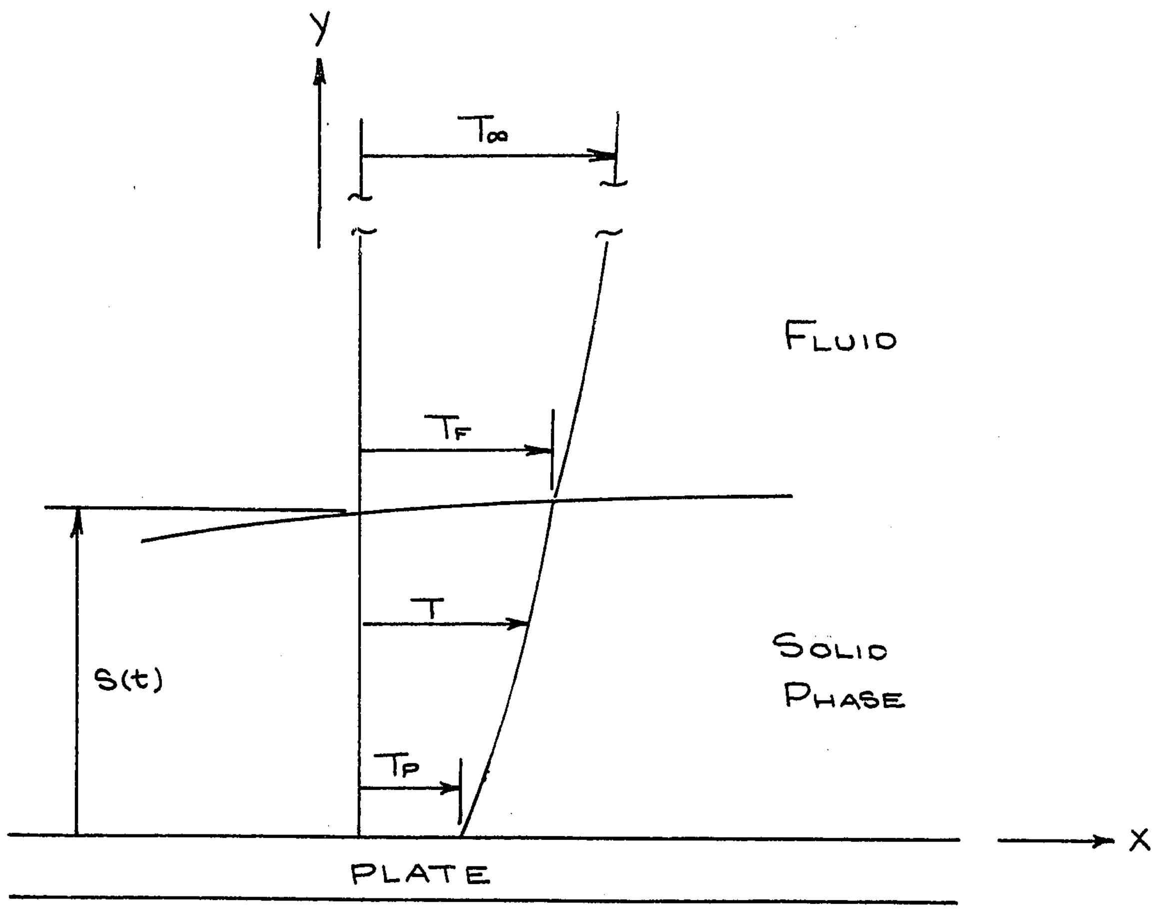

FIGURE I

Equation 1) needs no explanation. Equation 2), as noted, expresses a balance between the flux of heat from the fluid to the solid phase and the flux of heat conducted away from the free boundary in the solid. The convective heat flux, q_c, will be present in all flows in which the fluid is not at its fusion temperature.

In the example calculations presented, freezing of water in two laminar flows will be considered, these flows being:

a) Plane stagnation flow

b) Flow over a flat plate at zero incidence.

The variation of q_c with streamwise distance, x, in these two flows is well-known (see Schlichting (18), for example).

II. B) Transformation of Basic Equations

The essential difficulty in solving the system of equations 1), 2) and 3) evolves from the fact that the region in which the solutions must be found is both of unknown configuration and time-variant. The necessity of satisfying the energy balance, 2), and the boundary condition, 3), at the moving free boundary, $y = s(t)$, complicates an otherwise simple problem because:

a) It is precisely the position of the moving free boundary which must be determined as a function of time, as a part of the problem; and

b) The fact that $s(t)$ depends upon the temperature gradient at the free boundary, as seen from 3), renders the problem non-linear, although the heat diffusion equation, 1), is linear.

It would be advantageous if by suitable transformation the moving boundary could be eliminated; that is, fixed in space for all time. This is accomplished by introduction of the dimensionless space variable

$$4) \qquad \eta = \frac{y}{s(t)} \quad .$$

As y ranges from zero to $s(t)$, η varies from zero to unity; thus, the boundaries of the region are fixed at $\eta = 0$ and $\eta = 1$ for all time.

Complete non-dimensionalization of the system 1), 2) and 3) will be effected by introduction of the dimensionless temperature, θ , defined by

$$5) \quad \theta = \frac{T - T_P}{\Delta T} \quad , \quad \Delta T = T_F - T_P \quad ,$$

and a dimensionless time variable, τ , defined by

$$6) \quad \tau = \frac{\alpha}{S^2} t \quad .$$

S is the limiting, or steady-state, value of $s(t)$. It is determined by the requirements that, at steady-state,

$$\frac{ds}{dt} = 0 \qquad \text{and} \qquad \frac{\partial T}{\partial y} = \frac{\Delta T}{S} \quad ,$$

throughout the region $0 \leq y \leq S$. Thus, from equation 2), at steady-state,

$$7) \quad q_c = \frac{k \Delta T}{S} \quad ,$$

or

$$8) \quad S = \frac{k \Delta T}{q_c} \quad .$$

Transformations similar to equations 4), 5) and 6) have been applied to equation 1), and to equation 2) both with and without a term corresponding to the convective heat flux q_c , by Landau (9), Horvay (13) and Hamill and Bankoff (14). The development of the transformed equations is presented here in brief detail for completeness of presentation and because it was not provided by the authors

just named.

The following "transforming operators" are applied to $T(y,t)$:

9) $$\frac{\partial}{\partial t} = -\frac{\eta}{s}\frac{ds}{dt}\left(\frac{\partial}{\partial \eta}\right) + \frac{\alpha}{s^2}\left(\frac{\partial}{\partial \tau}\right)$$

10) $$\frac{\partial^2}{\partial y^2} = \frac{1}{s^2}\left(\frac{\partial^2}{\partial \eta^2}\right)$$

When results of the indicated operations on

5) $$T(y,t) = T_P + (\Delta T) \cdot \theta(y,t)$$

are substituted into equation 1), the dimensionless heat diffusion equation

11) $$H(\tau)\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\eta}{2}\frac{dH}{d\tau}\frac{\partial \theta}{\partial \eta}$$

is obtained, after simplification. The variable $H(\tau)$ is defined by

12) $$H(\tau) = \left(\frac{s}{S}\right)^2 \quad .$$

Similar operations on equation 2) yield its dimensionless form:

13) $$\frac{dH}{d\tau} = \frac{2k\Delta T}{\rho\lambda\alpha}\left[\frac{\partial \theta(1,\tau)}{\partial \eta} - (H)^{\frac{1}{2}}\right]$$

The boundary and initial conditions, equations 3), may be seen to become

14) $\theta(0,\tau) = 0; \quad \theta(1,\tau) = 1; \quad H(0) = 0 .$

Transformation of equations 1), 2) and 3) to the desired dimensionless forms is thus completed.

II. C) Limiting Solution for Time t Near Zero

The objective in this section is to obtain a closed-form solution valid near time t equal zero. This "limiting solution" will be used, subsequently, to provide starting values of θ and H at some $\tau = \tau^{*}>0$ for numerical integration of the governing equations. The numerical integration procedure requires a "starting" θ -profile, which occasions the need for a starting solution.

Two observations are pertinent at the outset:

a) Near the beginning of the freezing process, when the solid phase is still very thin, it may be reasoned that the growth rate of the solid will be maximum. This is most readily deduced from equation 13), given that $H(\tau)$ starts from zero. This suggests that the term involving $H(\tau)$ may be neglected for the purpose of obtaining the limiting solution .

b) The dimensionless temperature, $\theta(\eta, \tau)$, assumes constant values along the boundaries $\eta = 0$ and $\eta = 1$ of the solid phase in the transformed $(\theta - \eta - \tau)$ space. This suggests the possibility that θ is a function only of η , at least near time equal zero.

In accordance with a), equation 13) simplifies to

$$15) \quad \frac{dH}{d\tau} = \frac{2k\Delta T}{\rho \lambda \alpha} \frac{\partial \theta(1,\tau)}{\partial \eta}$$

Following b), let

16) $\theta\ (\eta, \tau) = F\ (\eta)$.

Since equation 16) requires that

17) $\frac{\partial \theta}{\partial \tau} = 0$,

equation 11) becomes simply

18) $\frac{d^2F}{d\eta^2} + \frac{\eta}{2} \frac{dH}{d\tau} \frac{dF}{d\eta} = 0.$

If equation 16) is to be valid, then the coefficient of $\frac{dF}{d\eta}$ in equation 18) must be either a function only of η , or a constant. However, $\frac{dH}{d\tau}$ cannot be a function of η alone because $H = H(\tau)$. Therefore, let

19) $\frac{dH}{d\tau}$ = constant = C_1 .

Upon integration,

20) $H(\tau) = C_1\ \tau$; $H(0) = 0$.

Equation 18) possesses a simple solution if

21) $C_1 = 4\,b$,

and that solution is

$$22)\quad F(\eta) = \frac{\int_0^{\eta} e^{-bz^2} dz}{\int_0^{1} e^{-bz^2} dz}$$

The constant, b, will be determined such that equations 19) and 22) satisfy equation 15).

After the indicated substitutions, equation 15) is

$$23)\quad 4b = \frac{2k\Delta T}{\rho \lambda \alpha} \frac{e^{-b}}{\int_0^{1} e^{-bz^2} dz} .$$

Equation 23) may now be solved for the constant, b. This equation is transcendental and must be solved by trial-and-error for a given set of physical constants.

Solutions of the governing equations 11) and 15) have been found which satisfy the boundary and initial conditions, equations 14). These solutions are valid sufficiently near $\tau = 0$, and therefore constitute a "limiting solution" for the temperature field in the solid phase and the thickness of the solid. These solutions are seen to be:

$$24) \quad \theta(\eta) = \frac{\int_0^{\eta} e^{-bz^2} dz}{\int_0^{1} e^{-bz^2} dz} \quad ,$$

and

$$25) \quad H(\tau) = 4\, b\, \tau \quad .$$

It will be of importance later to know a reasonable value of τ, say τ^*, for which equations 24) and 25) are valid and at which these equations may be applied to provide starting solutions for numerical integration of equations 11) and 13).

In order that equation 15) be valid at $\tau = \tau^*$, it is necessary that

$$26) \quad \left(H(\tau^*) \right)^{\frac{1}{2}} << \frac{\partial \theta(1)}{\partial \eta}$$

This may be insured by requiring that

$$27) \quad (H(\tau^*))^{\frac{1}{2}} = \epsilon \frac{\partial \theta(1)}{\partial \eta} \quad ,$$

where ϵ is an arbitrarily small positive number. Substituting into equation 27) from equations 24) and 25), and simplifying,

28) $$\tau^* = \left[\frac{\varepsilon}{2\sqrt{b}\; e^{b} \int_0^1 e^{-bz^2} dz} \right]^2 .$$

Equation 28) will be used to determine the value of dimensionless time, $\tau = \tau^*$, at which the numerical integration procedure will be started.

II. D) Limiting Solution for Large Time t

As will be discussed in a subsequent section, the steady-state distribution of $\theta(\eta, \tau)$ in the range $0 \leq \eta \leq 1$ is achieved much more rapidly than is the steady-state thickness of the solid phase, denoted here by the condition $H(\infty) = 1$. Actually, certain physical systems will be characterized by longer θ-transients than others; this characteristic behavior is dependent upon physical properties of the solid phase and upon the "plate" temperature, and is discussed in Section IV. However, it suffices here to recognize that, for τ greater than some τ^{**}, the dimensionless temperature gradient at $\eta = 1$ may be assumed to have reached its steady-state value of unity and to maintain that value as $H(\tau) \to 1$, $\tau \to \infty$.

Therefore, let it be assumed for purposes of this "asymptotic" analysis, that

29) $$\frac{\partial \theta(1, \tau)}{\partial \eta} = 1 , \qquad \tau \geq \tau^{**} .$$

The value of $\tau = \tau^{**}$ must be determined from results of the numerical integration in the range $\tau^{*} \leq \tau \leq \infty$, and it will, of course, be different for every value of the parameter $\frac{2k\Delta T}{\rho \lambda \alpha}$.

Combining equations 13) and 29) yields

30) $$\frac{dH}{d\tau} = \frac{2k\Delta T}{\rho\lambda\alpha}\left[1 - (H)^{\frac{1}{2}}\right], \quad \tau \geq \tau^{**} \quad .$$

Further, let $H(\tau)$ be represented by

31) $$\tilde{H}(\tau) = H(\infty) - H(\tau) = 1 - H(\tau), \quad \tau \geq \tau^{**} \quad ,$$

where

32) $$\tilde{H}(\tau) << H(\infty) = 1 \quad .$$

With the proper substitutions, equation 30) becomes

33) $$-\frac{d\tilde{H}}{d\tau} = \frac{2k\Delta T}{\rho\lambda\alpha}\left[1 - (1 - \tilde{H})^{\frac{1}{2}}\right] \quad .$$

Boundary conditions on $\tilde{H}(\tau)$ are:

34) $$\begin{aligned} \tilde{H}(\tau^{**}) &= 1 - H(\tau^{**}) = 1 - H^{**} \\ \tilde{H}(\infty) &= 0 \quad . \end{aligned}$$

It may be shown that equations 33) and 34) are satisfied by

35) $$\tau = \tau^{**} + \frac{\rho\lambda\alpha}{k\Delta T}\left[(H^{**})^{\frac{1}{2}} - (1 - \tilde{H})^{\frac{1}{2}} - \mathrm{Ln}\,\frac{1 - (1 - \tilde{H})^{\frac{1}{2}}}{1 - (H^{**})^{\frac{1}{2}}}\right]$$

Equation 35), with H^{**} evaluated from the numerical solution should predict the asymptotic behavior of $H(\tau)$ for $\tau \to \infty$.

II. E) The Numerical Solution

Exact closed-form solutions of the system of equations 11), 13) and 14), or their dimensional antecedents, have been obtained in the case of freezing of fluids only with the aid of restrictive assumptions concerning the convective heat flux to the solid phase (these are the previously mentioned solutions of Stefan and Neumann). No such solution has been reported for cases in which the convective heat flux is non-zero and independent of time. When it is considered that the term involving $(H(\tau))^{\frac{1}{2}}$ in equation 13) is not present when the convective heat flux is zero, the importance of that parameter as a major obstacle to solution is apparent. Of course, the dimensionless variables $H(\tau)$ and τ are invalid for q_c identically zero, but this is of no consequence. As demonstrated, an analytical solution is then available which need not involve those variables in dimensionless form.

This section will describe a finite-difference procedure devised for solution of the system of equations 11), 13) and 14) . These equations are repeated here for convenience.

11) $$H \frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\eta}{2} \frac{dH}{d\tau} \frac{\partial \theta}{\partial \eta}$$

13) $$\frac{dH}{d\tau} = \frac{2k\Delta T}{\rho\lambda\alpha}\left[\frac{\partial\theta(1,\tau)}{\partial\eta} - (H)^{\frac{1}{2}}\right]$$

14) $$\begin{aligned}\theta(0,\tau) &= 0\\ \theta(1,\tau) &= 1\\ H(0) &= 0\end{aligned}$$

Equations 11) and 13) are obviously coupled. Their simultaneous solution will be achieved in the following manner.

First, the $\eta - \tau$ plane in the interval $\eta = 0,1$ is divided into a number of "spatial" strips, each of width $\Delta\eta$, the borders of these strips being lines of constant η.

At $\tau = \tau^*$, the time at which the numerical procedure is to be started, values of $\theta(i\Delta\eta, \tau^*)$, $(i = 1,2,.., \frac{1}{\Delta\eta})$ and of $H(\tau^*)$ are calculated from equations 24) and 25).

Next, equation 11) is written in finite-difference form. The time-derivative is replaced by a forward-difference quotient evaluated at mesh points (i,j) and $(i,j+1)$; the second order spatial derivative is replaced by a linear combination of central second-difference quotients evaluated at the j'th and $(j+1)$'th time levels. The first order spatial derivative of θ is similarly replaced

by a linear combination of central differences quotients evaluated at the j'^{th} and $(j+1)'^{th}$ time levels. See Figure II following for description of the $\eta - \tau$ grid space.

Equation 11) is thus replaced by

$$36)\quad \overline{H}(\tau)\,\frac{\theta(i,j+1)-\theta(i,j)}{\Delta\tau}$$

$$= \sigma\,\frac{\theta(i+1,j+1)-2\theta(i,j+1)+\theta(i-1,j+1)}{(\Delta\eta)^2}$$

$$+ (1-\sigma)\,\frac{\theta(i+1,j)-2\theta(i,j)+\theta(i-1,j)}{(\Delta\eta)^2}$$

$$+ \sigma\,\frac{\eta}{2}\,\overline{\frac{dH}{d\tau}}\,\frac{\theta(i+1,j+1)-\theta(i-1,j+1)}{\Delta\eta}$$

$$+ (1-\sigma)\,\frac{\eta}{2}\,\overline{\frac{dH}{d\tau}}\,\frac{\theta(i+1,j)-\theta(i-1,j)}{\Delta\eta}$$

In this scheme, $H(\tau)$ and $\frac{dH}{d\tau}$ are regarded as constants during integration of equation 11) between τ and $\tau + \Delta\tau$; therefore, in equation 36), $H(\tau)$ is replaced by $\overline{H}(\tau)$,

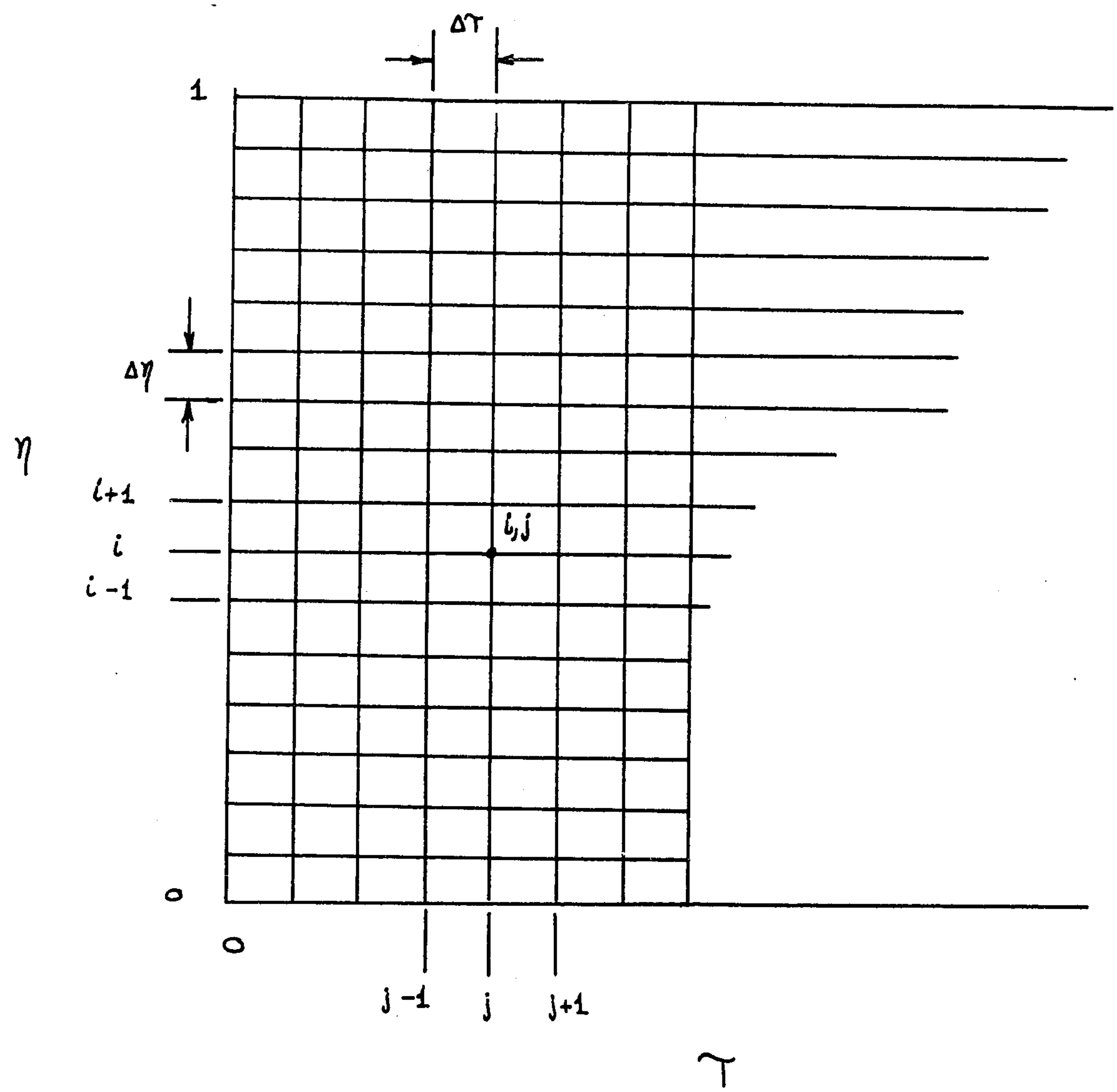

FIGURE II

its mean value over the interval $\Delta\tau$, and $\frac{dH}{d\tau}$ by $\overline{\frac{dH}{d\tau}}$

With $\sigma = \frac{1}{2}$, equation 36) assumes the well-known Crank-Nicholson form (see for example Forsythe and Wasow (15)). The Crank-Nicholson finite-difference representation of the unsteady diffusion equation is shown in (15) to be both stable with respect to error propagation and convergent as the grid spacings $\Delta\eta$ and $\Delta\tau$ tend to zero.

Equation 13) is integrated over the interval $\Delta\tau$ from the j'^{th} to the $(j+1)'^{th}$ time level by a modified fourth-order Runge-Kutta numerical integration procedure. The modification of the standard Runge-Kutta procedure is occasioned by the mutual coupling existing between equations 11) and 13). This modification, although essential for accuracy, requires only minor changes to the standard procedure. Since the Runge-Kutta method requires evaluation of the derivative of $H(\tau)$ at both end-points and at the mid-point of the interval, $\Delta\tau$, it was necessary to utilize an iterative procedure wherein intermediate values of the dimensionless temperature gradient were determined from equation 36) for use in equation 37), following.

Because of the non-linearity of equation 13), an iterative procedure was included in the integration which used

successive determinations of the θ -profile at the $(j+1)'^{th}$ time level to refine the calculated value of H at that time level. The iteration was continued, at each time level, until the dimensionless temperature gradient evaluated at $\eta = 1$ changed by less than 10^{-4} from one iteration to the next. H(j+1) was then calculated using this "latest" estimate of the temperature gradient.

The finite-difference form of equation 13) is:

37) $$H(j+1) = H(j) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = \Delta\tau \cdot f(\tau, H(\tau))$$

$$K_2 = \Delta\tau \cdot f(\tau + \frac{\Delta\tau}{2}, H(\tau) + \frac{K_1}{2})$$

$$K_3 = \Delta\tau \cdot f(\tau + \frac{\Delta\tau}{2}, H(\tau) + \frac{K_2}{2})$$

$$K_4 = \Delta\tau \cdot f(\tau + \Delta\tau, H(\tau) + K_3)$$

$$f(\tau, H(\tau)) = \frac{dH}{d\tau}$$ from equation 13).

With an assumed $\Delta\tau$, $H(j+1)$ is found from equation 37) and $\overline{H}(\tau)$ and $\overline{\frac{dH}{d\tau}}$ are determined for that interval, $\Delta\tau$, as

$$\overline{H}(\tau) = (H(j) + H(j+1)) \cdot \frac{1}{2}$$

$$\overline{\frac{dH}{d\tau}} = (K_1 + 2K_2 + 2K_3 + K_4)\frac{1}{6\Delta\tau}$$

Then, the system of linear algebraic equations resulting from application of equation 36) to mesh points at the $(j+1)$'th time level is solved using the Thomas algorithm, a systematized Gaussian elimination procedure, as described by Young (16).

This procedure yields the θ-profile at the $(j+1)$'th time level, and the value of $H(\tau)$ at that time level, given the θ-profile and $H(\tau)$ at the j'th time level.

It may be noted that the finite-difference scheme employed here avoids the awkward situation, encountered by Ehrlich (17), of estimating the position of the moving boundary when it lies between the end-points of a spatial interval, $\Delta\eta$. That is because the free boundary is not moving in the $(\theta - \eta - \tau)$ space, but is fixed at $\eta = 1$, and its position need not be estimated.

III. AN APPROXIMATE SOLUTION FOR POSITION OF THE FREE BOUNDARY

Results of the numerical procedure have shown that the duration of the θ -transient is negligible, or nearly so, in comparison with the time required for $H(\tau)$ to reach its steady-state value of unity. In addition, the time, τ^*, at which the numerical procedure is started is essentially zero in relation to the duration of the transient in $H(\tau)$.

This circumstance suggests that $H(\tau)$ may be calculated approximately by neglecting the transient in θ, beginning at $\tau = 0$.

Recalling equation 13),

$$13)\quad \frac{dH}{d\tau} = \frac{2K\Delta T}{\rho\lambda\alpha}\left[\frac{\partial\theta(1,\tau)}{\partial\eta} - (H)^{\frac{1}{2}}\right]$$

and giving the temperature gradient its steady-state value,

$$38)\qquad \frac{\partial\theta}{\partial\eta}(1,\infty) = 1,$$

there results

$$39)\quad \frac{dH}{d\tau} = \frac{2K\Delta T}{\rho\lambda\alpha}\left[1 - (H)^{\frac{1}{2}}\right]$$

which possesses a closed-form solution. The solution of equation 39) which satisfies the conditions

$$H(0) = 0 \quad ; \; H(\infty) = 1$$

is

40) $$\tau = -\frac{\rho\lambda\alpha}{k\Delta T}\left[\mathrm{Ln}\ (1 - (H)^{\frac{1}{2}}) + (H)^{\frac{1}{2}} \right] .$$

As discussed previously, the duration of the θ-transient is strongly dependent upon the degree to which the "starting" θ-profile at $\tau = \tau^{*}$ deviates from the linear steady-state θ-profile. The more nearly linear the starting profile, the shorter the θ-transient. The dependence of the duration of this transient on physical properties is contained in the dimensionless parameter $\frac{2k\Delta T}{\rho\lambda\alpha}$. The smaller this parameter, the shorter the θ-transient.

IV. RESULTS

The non-dimensionalized governing equations, 11) and 13), permit numerical evaluation of $H(\tau)$ with only the physical parameter $\frac{2k\Delta T}{\rho\lambda\alpha}$ needing specification; that is, the solution $H(\tau)$ is dependent upon a single physical parameter.

Certain generalizations concerning the behavior of $H(\tau)$ may be made conveniently in terms of this physical parameter. First, however, note that

41) $$\gamma = \frac{2k\Delta T}{\rho\lambda\alpha} = \frac{2c\Delta T}{\lambda},$$

where the γ-notation has been adopted for convenience. This dimensionless grouping may be interpreted as the ratio of the heat required to change the temperature of unit mass of the solid phase by an amount ΔT to the latent heat of fusion of unit mass. It may be anticipated, therefore, that when this ratio is small, say less than unity, the solid phase will exhibit a short θ-transient; for larger values of the parameter, long thermal transients may be expected. This is complemented by the fact that, when γ is unity or less, the "starting" θ-profile departs very little from the linear steady-state profile. The following table was taken from results of the numerical integration, with γ = 0.10, 1.0 and 10.0.

Duration of the θ-transient was approximated by the time required for the θ-profile to reach a maximum deviation of 1.0% from the linear steady-state profile. It may be seen that duration of the θ-transient is essentially zero for $\gamma = 0.10$, 1.0, becoming slightly longer as that parameter is increased to 10.0.

γ	τ AT END OF TRANSIENT
0.10	0 (initially less than 1.0% deviation)
1.0	0 (initially less than 1.0% deviation)
10.0	0.3

Next, it may be observed that materials with small latent heat of fusion will form "ice" more rapidly, other factors being equal. With λ very small, γ will be large, and equation 13) shows that $\frac{dH}{d\tau}$ will be proportionately large. The following table, showing times required for $H(\tau)$ to become 99% of its steady-state value,

demonstrates that the steady-state thickness of the solid phase is achieved more rapidly the larger the parameter $\frac{2k\Delta T}{\rho \lambda \alpha}$.

γ	τ WHEN $H = 0.99$
0.10	86
1.0	10
10.0	2

In fact, the figures tabulated above show that γ and the time required for $H(\tau)$ to reach 0.99 vary approximately in an inverse ratio; specifically, it may be seen that as γ is increased by two orders of magnitude, duration of the H-transient is diminished by two orders of magnitude.

Comparisons of the numerical solutions, limiting solutions and approximate solutions for $H(\tau)$ are shown in Figures III, IV and V for $\gamma = 0.10$, 1.0 and 10.0, respectively.

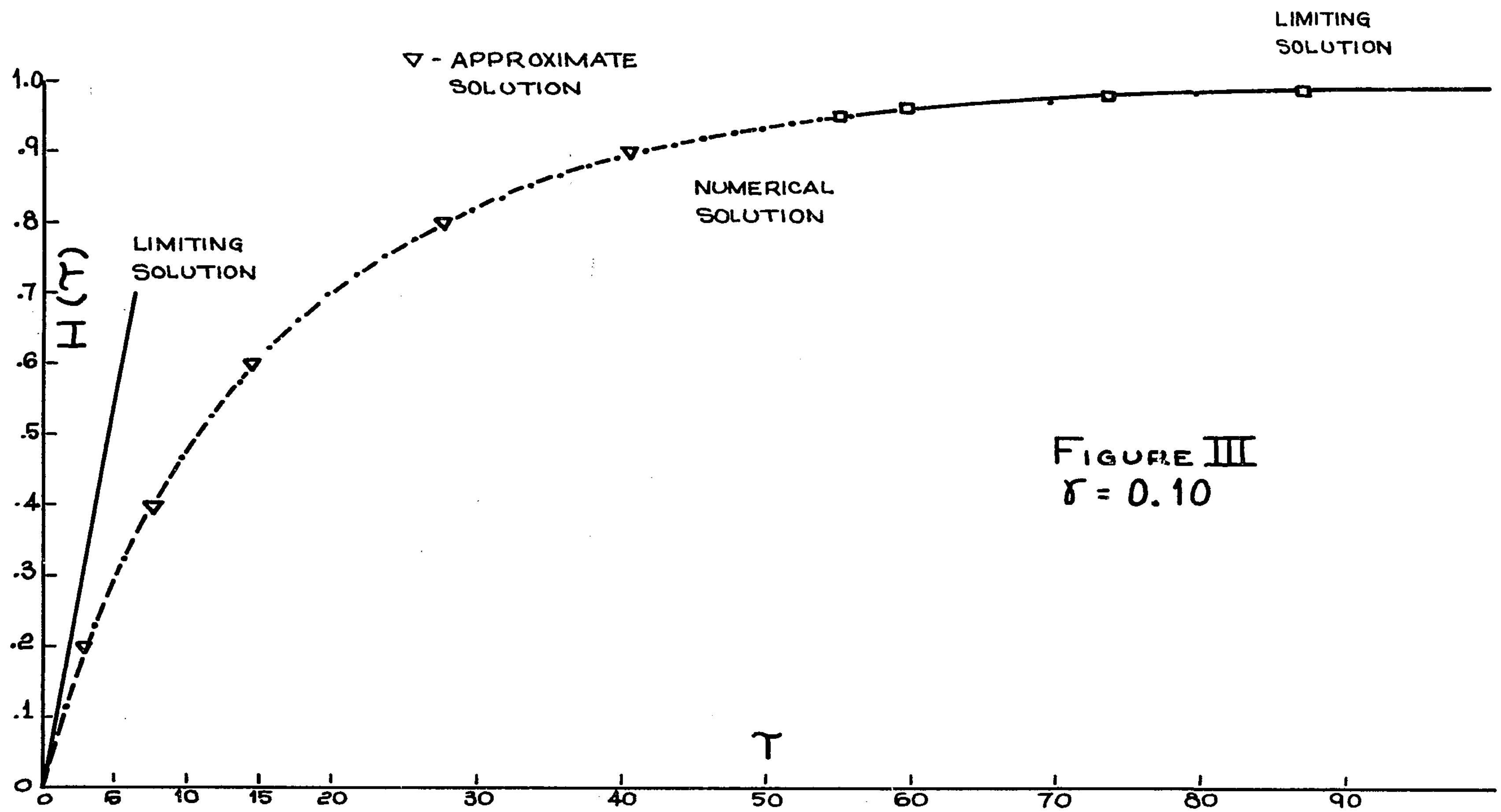

FIGURE III
$\gamma = 0.10$

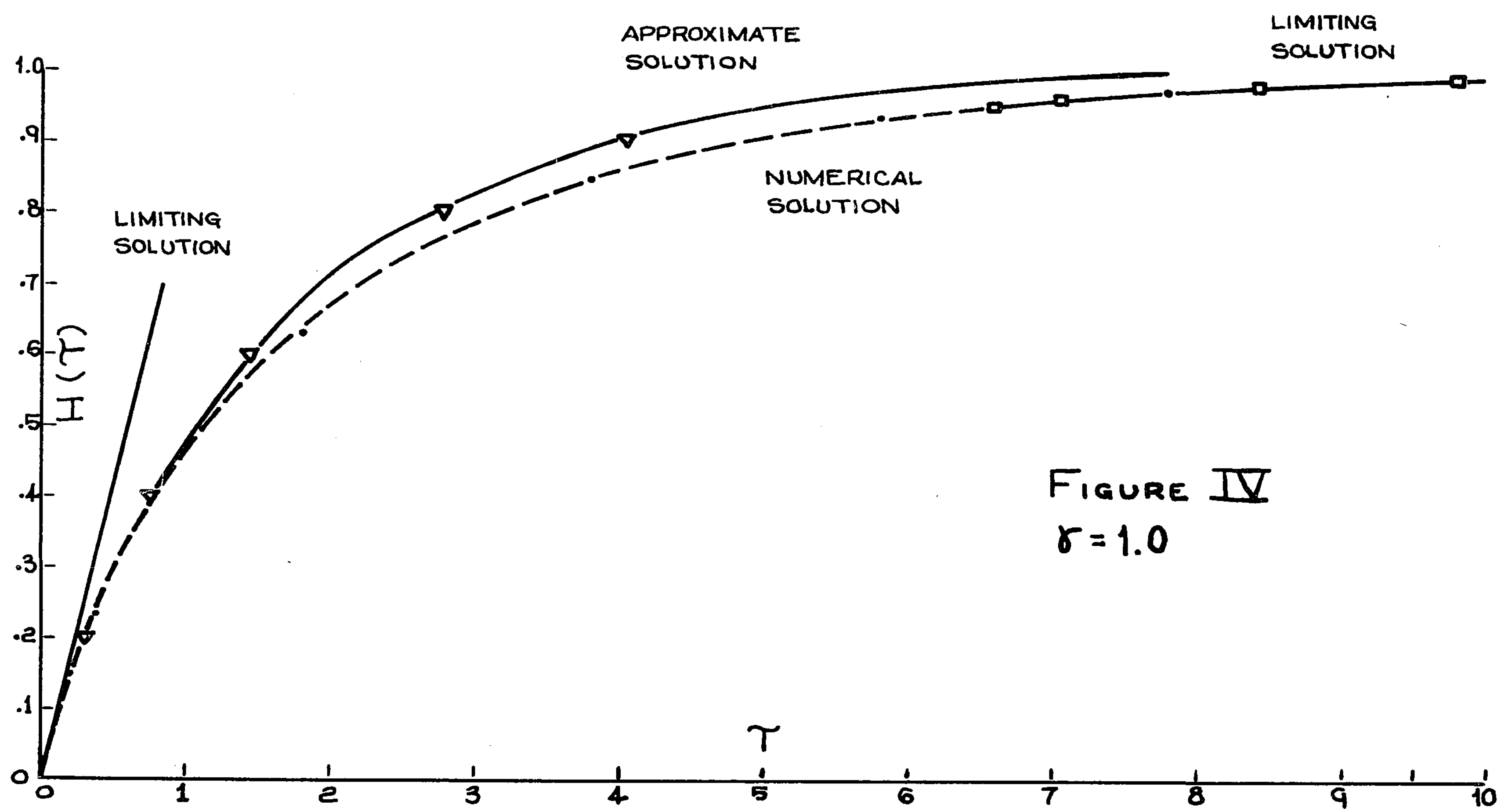

FIGURE IV

$\gamma = 1.0$

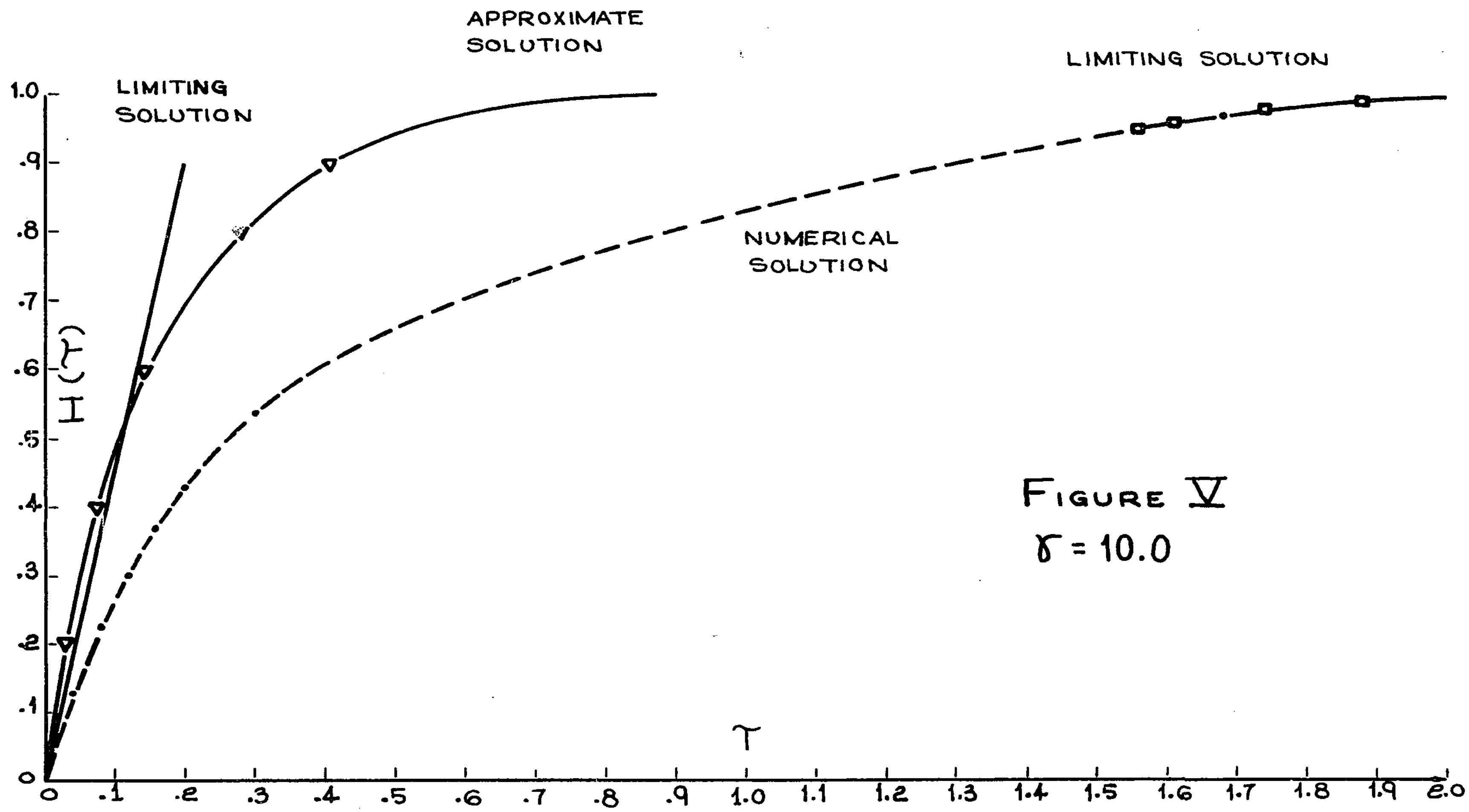

FIGURE V

$\gamma = 10.0$

These curves demonstrate the agreement between limiting and numerical solutions, and the degree to which the approximate solution is valid. It may be observed in the Figures that the validity of the approximate solution suffers as γ is increased. Specifically, the approximate solution should be regarded as inapplicable when $\gamma > 1$, as Figure V demonstrates. It appears that the assumption of negligible θ-transient, upon which the approximate solution is founded, is not realized when $\gamma > 1$. As shown just previously, with $\gamma = 10.0$, the θ-profile had essentially achieved steady-state at $\tau = 0.3$, but freezing was completed at $\tau = 2.0$, approximately; thus, in this case, the θ-transient occured during 15% of the duration of freezing, and was not negligible.

Figure III, for $\gamma = 0.10$, shows excellent agreement, the approximate and numerical solutions being indistinguishible, as plotted. In Figure IV, for $\gamma = 1.0$, the deviation of the approximate from the numerical solution is on the order of 5% or less.

V. NUMERICAL EXAMPLES

Results of the numerical integration procedure for the value $\gamma = 0.10$ will be employed in calculating the thickness, $s(t)$, of the deposited solid phase in the two cases of: a) plane stagnation flow; and b) flow over a flat plate at zero incidence. In these examples, physical properties of water-ice will be used, as found in Ingersoll, Zobel and Ingersoll (4). The necessary physical properties of the deposited solid phase are, therefore:

$k = 1.28$ BTU/(hr-ft-°F)

$c = 0.49$ BTU/(lb-°F)

$\lambda = 144$ BTU/lb

$\alpha = 0.046$ $(ft)^2$/hr

Case a): Plane Stagnation Flow

The convective heat flux, q_c, in steady laminar plane stagnation flow is independent of the streamwise coordinate, x. Therefore, the thickness of the deposited solid phase will be uniform over the "plate" at every instant. For purpose of these calculations, it will be assumed that:

$$q_c = 1000 \ \text{BTU/(hr-ft}^2) .$$

With $\gamma = 0.10$, and c and λ as listed above, the temperature difference across the solid phase is found to be:

$$\Delta T = (0.10)\ (144/0.49)\ ^\circ F = 29.6^\circ F \quad .$$

The steady-state thickness of the solid may now be calculated, and it is found to be:

$$S = \frac{k\ \Delta T}{q_c} = (1.28)\ (29.6)/1000\ \text{ft} = 0.0379\ \text{ft.}$$

The dimensional thickness of the solid phase, $s(t)$, is thus given by:

$$s(t) = S\ (H)^{\frac{1}{2}} = 3.79 \times 10^{-2}\ (H^{\frac{1}{2}})\ \text{ft.}$$

Similarly, t is related to τ by

$$t = \frac{S^2 \tau}{\alpha} = \frac{14.39 \times 10^{-4}}{4.6 \times 10^{-2}}\ (\tau)\ \text{hr} = 3.13 \times 10^{-2}\ (\tau)\ \text{hr.}$$

Using corresponding values of $H(\tau)$ and τ as given by the numerical integration for $\gamma = 0.10$, a table of corresponding values of $s(t)$ and t may be constructed, and is shown following:

t , hr	s(t) , ft
0.0	0.0
0.04989	0.0130
0.1745	0.0216
0.2990	0.0259
0.5500	0.0308
0.8600	0.0339
1.1700	0.0355
1.4900	0.0365
2.5800	0.0378

From this table, it may be seen that the example system would reach steady-state in just over 2.58 hours.

Case b): <u>Flow Over A Flat Plate At Zero Incidence</u>

If it is assumed that the presence of the solid phase deposited on the plate induces no change in the fluid flow, which is consistent with the assumption of a thin layer of deposited solid, the variation in convective heat flux, $q_c(x)$, with the streamwise coordinate, x, may be taken to be that for steady laminar flow over a flat plate with no streamwise pressure gradient. As is well known (see Schlichting (18), for example),

$$q_c \sim 1/\sqrt{x}$$

in such a flow. For the purpose of this example, it will be assumed that

$$q_c = 1000/\sqrt{x} \ \text{BTU/(hr-ft}^2) \ .$$

Calculations will be performed for $0 \leq x(\text{ft}) \leq 1$. Thus, q_c will range from ∞ to 1000 BTU/(hr-ft^2) in this example.

Now,

$$S = \frac{k\Delta T}{q_c(x)} = (1.28)(29.6)\sqrt{x}/1000 \text{ ft} = (3.79 \times 10^{-2})\sqrt{x} \text{ ft.}$$

Therefore, the thickness of the deposited solid phase is given by:

$$s(x,t) = S(H)^{\frac{1}{2}} = (3.79 \times 10^{-2})\sqrt{x}\sqrt{H} \text{ ft.}$$

It is interesting to note here that while $s(x,t)$ is a function of the streamwise coordinate, x, $H(\tau)$ is independent of x. As may be seen, the x-dependence of $s(x,t)$ is contained in the function $S(x)$. Therefore, the one solution, $H(\tau)$, is valid for all x.

The configuration of the solid phase, $s(x,t)$, will be calculated at two times; $t = 0.299$ hours, and 1.49 hours, during the transient, and at steady-state, in the range $0 \leq x(ft) \leq 1$. The results are shown in the following table:

x , ft	t , hr	τ	H(τ)	s(t) , ft
0.0	0.299	9.56	0.470	0.0
0.25	"	"	"	0.130
0.50	"	"	"	0.0184
0.75	"	"	"	0.0226
1.0	"	"	"	0.0260

0.0	1.49	47.56	0.928	0.0
0.25	"	"	"	0.0182
0.50	"	"	"	0.0258
0.75	"	"	"	0.0315
1.0	"	"	"	0.0364

0.0	Steady-State	1.0	0.0
0.25	"	"	0.0189
0.50	"	"	0.0268
0.75	"	"	0.0328
1.0	"	"	0.0379

BIBLIOGRAPHY

1) Stefan, J., Ann. Phys. and Chem. 42, (1801), p. 269

2) Carslaw, H.S. and J.C. Jaeger, CONDUCTION OF HEAT IN SOLIDS, Oxford, second edition

3) Rosenthal, D., THE THEORY OF MOVING SOURCES OF HEAT AND ITS APPLICATION TO METAL TREATMENTS, Trans. ASME 68, (1946), p. 849

4) Ingersoll, L. , O. Zobel and A. Ingersoll, HEAT CONDUCTION, WITH ENGINEERING, GEOLOGICAL AND OTHER APPLICATIONS, Univ. of Wisconsin Press, (1954)

5) Goodman, T., THE HEAT BALANCE INTEGRAL AND ITS APPLICATION TO PROBLEMS INVOLVING CHANGE OF PHASE, Trans. ASME 80, (1958), p. 335

6) Goodman, T. and J. Shea, THE MELTING OF FINITE SLABS, Trans. ASME, (1960), p. 16

7) Libby, P. and S. Chen, THE GROWTH OF A DEPOSITED LAYER ON A COLD SURFACE, Int. J. Heat Mass Trans. 8, (1965), p. 395

8) Douglas, J. and T. Gallie, ON THE NUMERICAL INTEGRATION OF A PARABOLIC DIFFERENTIAL EQUATION SUBJECT TO A MOVING BOUNDARY CONDITION, Duke Math. J. 22, (1955), p. 557

9) Landau, H., HEAT CONDUCTION IN A MELTING SOLID, Quart. Appl. Math. 8, (1950), p. 81

10) Kreith, F. and F. Romie, A STUDY OF THE THERMAL DIFFUSION EQUATION WITH BOUNDARY CONDITIONS CORRESPONDING TO SOLIDIFICATION OR MELTING OF MATERIALS INITIALLY AT THE FUSION TEMPERATURE, Proc. Phys. Soc. 68, (1955), p. 277

11) Evans, G. , E. Issacson and J. MacDonald, STEFAN-LIKE PROBLEMS, Quart. Appl. Math. 8, (1950), p. 312

12) Murray, W. and F. Landis, NUMERICAL AND MACHINE SOLUTIONS OF TRANSIENT HEAT CONDUCTION INVOLVING MELTING OR FREEZING, Trans. ASME, (1959), p. 106

13) Horvay, G. , FREEZING OF A GROWING LIQUID COLUMN, Trans. ASME 82c, (1960), p. 37

14) Hamill, T. and S. Bankoff, MAXIMUM AND MINIMUM BOUNDS ON FREEZING-MELTING RATES WITH TIME-DEPENDENT BOUNDARY CONDITIONS, AIChE J. 9, (1963), p. 74

15) Forsythe, G. and W. Wasow, FINITE-DIFFERENCE METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS, John Wiley and Sons, (1965)

16) Young, D. , SURVEY OF NUMERICAL ANALYSIS, (J. Todd, Editor), McGraw-Hill, (1962)

17) Ehrlich, L.W. , NUMERICAL METHOD OF SOLVING A HEAT FLOW PROBLEM WITH MOVING BOUNDARY, J. Assoc. Comput. Mach. 5, (1958), p. 161

18) Schlichting, H. , BOUNDARY LAYER THEORY, McGraw-Hill, fourth edition

NOMENCLATURE

x: Dimensional coordinate tangent to the cold surface

y: Dimensional coordinate normal to the cold surface

T: Temperature

T_F: Fusion temperature of the solid phase

T_P: Temperature of the cold surface (plate)

t: Time

$s(t)$: Local thickness of the deposited solid phase

S: Local steady-state thickness of the deposited solid phase

ρ: Mass density of the solid phase

k: Thermal conductivity of the solid phase

c: Specific heat of the solid phase

λ: Heat of fusion

α: Thermal diffusivity of the solid phase

q_c: Convective heat flux from fluid to solid phase

η: Dimensionless space variable (Eqn. 4)

θ: Dimensionless temperature (Eqn. 5)

H: Dimensionless thickness of the deposited solid (Eqn. 12)

τ: Dimensionless time (Eqn. 6)

τ^*: Dimensionless time at which numerical integration is started

τ^{**}: Dimensionless time at which limiting solution for large time is started

γ: Dimensionless physical parameter (Eqn. 41)