HSU, Feng–Hsiang, 1939–
THE INFLUENCES OF MECHANICAL LOADS ON
THE FORM OF A GROWING ELASTIC BODY.

Rice University, Ph.D., 1966
Engineering, mechanical

University Microfilms, Inc., Ann Arbor, Michigan

RICE UNIVERSITY

THE INFLUENCES OF MECHANICAL LOADS ON THE FORM OF A GROWING ELASTIC BODY

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's signature:

Houston, Texas

August 1965

ACKNOWLEDGMENTS

The author is greatly indebted to Dr. Paul R. Paslay for the suggestion of the subject and his valuable advice. He also wishes to thank Shell Companies Foundation, Inc., for financial support during the summer of 1965.

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NOMENCLATURE

This is not a complete list of notations; those terms not included in this list are defined in the context when they are used.

$\underline{\underline{c}}$	the configuration of the body
$\underline{x}$	the position of a particle in the three dimensional Euclidean space
X	material coordinate of a particle in a body
$M_{\underline{\underline{c}}}$	mass of the body in configuration $\underline{\underline{c}}$
$\partial \underline{\underline{c}}$	the boundary of the configuration $\underline{\underline{c}}$ of the body
g	the strength of mass source in the body
g_n	the strength of mass source in normal growth state
$\bar{f}$	the mass influx vector
ρ	mass density
t	time
d	Euler's rate of deformation tensor
δ_{ij}	Kronecker delta
S_{ij}	stress tensor component
$\dot{S}_{ij}$	the stress rate tensor component
f_i	resultant of body forces in i^{th} direction
F	the deformation gradient tensor, i.e., $\left[\frac{\partial x^i}{\partial X^\alpha}\right]$
R	the rotation tensor
U	the right stretch tensor
I_1	the first stress invariant
J_1	the first stress rate invariant
ω_{ij}	the vorticity tensor

$S_{ij,j}$ covariant spatial derivative of stress tensor component

$\dot{x}^i_{,j}$ covariant spatial derivative of velocity component

$\frac{DS_{ij}}{Dt}$ material time derivative of stress tensor component

I

INTRODUCTION

The first major effort to apply the laws of physics and mechanics directly to the biological problem of growth and form is represented by D'Arcy Thompson's treatise "On Growth and Form" [1]*. Later J. S. Huxley [2] introduced the concepts of growth gradient and differential growth aimed to answer the questions why as well as how a biological body grows to its specific form. However, up to the present the complete answers to these basic questions still have not been found. In this thesis the influences on the configuration of a growing body due to certain classes of mechanical loads are studied only for a special type of material. This research is directed to the problem: if the form to which a body grows under no externally applied loads is known, what will be the form of the body if some mechanical loads are applied to it during growth. Some experimental research has been conducted by investigators interested in the growth of bones under mechanical loads [1, 10, 12]. The results of their investigation confirmed that the form of growing bones did change under the bias of externally applied forces. However, very little data is available for quantitative study.

When mechanical loads, either body forces or surface forces, are applied on a material system, whether it is a growing body or inert body, two types of phenomena can be observed. One is the immediate

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*The numbers in the brackets refer to the numbers of references listed at the end of this thesis.

response of the material system to the applied loads; the other is the slow deformation of the material system associated with the loads. The latter usually cannot be observed if the loads are applied for only a relatively short period of time. For metals this slow deformation is generally known as creep, which is thought to be due to the movements of dislocations and vacancies. In a growing body the slow deviation of the body from its original form and state of growth is a manifestation of the adjustments of the arrangements of newly produced cells, the change of the shape of the cells themselves, as well as the change of the ability of the body to grow in the presence of externally applied loads. The immediate response of a growing body in principle is no different from the immediate response of an inert body because the process of growth is normally relatively slow. Likewise, different growing bodies will have different immediate responses to loads. The material constants and constitutive equations which characterize the immediate responses of a material system will be denoted as "the mechanical properties of the material" in this thesis. The slow deviation of a growing body from its normal state of growth due to mechanical loads can also be related by some phenomenological relations. It is to be expected that different types of materials will have different relations. The phenomenological relations and the necessary constants which characterize the slow response of a growing body to mechanical loads will be called "the growth properties of the material" in later context. It is to be noted that the reason for a growing body changing its form and ability to grow may be biological rather than purely mechanical. Nevertheless, if some definite biological-mechanical relations exist, it is assumed possible to formulate some direct functional relation between growth and mechanical loads. The usual procedure for formulation of such relations is to proceed simultaneously with experimental and theoretical analysis. At the present time reliable systematic data are not available. Therefore, in this thesis a general theoretical study is presented rather than considering a specific material. Some of the pertinent basic concepts in formulating constitutive equations are introduced, and the solutions to a few examples are given for a special class of linear material. The solutions given to the examples should furnish a basis for a critical experimental study to evaluate the postulates of this theoretical work.

Temperature is known to have important influences on the growth of a body [1, 2]; however, its main influence is on the rate of growth of the body. Therefore, in general, a given material growing under different temperature enviroments will be expected to grow to different sizes but of geometrically similar forms. However, in this thesis only isothermal cases are considered.

II

SOME FUNDAMENTAL CONCEPTS

A body is a set of material particles which can be mapped into Euclidean 3-dimensional space by a smooth one to one function. The form or configuration of the body is the position occupied by the set of material particles in the Euclidean 3-space. Let the capital Latin letter X denote the material particle, the small Latin letter $\underset{\sim}{x}$ denote the position in Euclidean 3-space, and $\underline{\underline{c}}$ denote a smooth one-to-one function, then the position occupied by particle X is given by

$$x = \underline{\underline{c}}\,(X) \qquad \text{(II-1)}$$

where the function $\underline{\underline{c}}$ is the configuration of the body. The mass of the body is considered as an absolutely continuous function of volume. At a given instant, the form is $\underline{\underline{c}}$ and the mass of the body in the configuration $\underline{\underline{c}}$ can be expressed as

$$M_{\underline{\underline{c}}} = \int_{\underline{\underline{c}}} \rho \; d(\mathrm{Vol}) \qquad \text{(II-2)}$$

where ρ is a non-negative mass density. The change of the mass of the body can be expressed by the general balance equation over its present configuration as

$$\frac{D}{Dt}\left[\int_{\underline{\underline{c}}} \rho \; d(\mathrm{Vol}) \right] = - \oint_{\partial \underline{\underline{c}}} \overline{f} \cdot \overline{d(\mathrm{area})} + \int_{\underline{\underline{c}}} g \cdot d(\mathrm{Vol}) \qquad \text{(II-3)}$$

where $-\oint_{\underline{\underline{c}}} \overline{f} \cdot \overline{d(\mathrm{area})}$ denotes the mass influx across the boundary $\partial \underline{\underline{c}}$

of the configuration $\underline{\underline{c}}$ of the body, and $\int_{\underline{\underline{c}}} g \cdot d(Vol)$ represents mass sources in the interior of the configuration of the body. If the biological details as to how masses are accumulating in a growing body are not to be studied and only the net effect that mass is increasing is noted, the boundary surface of the body can be considered as impenetrable. The net change of mass of the body can be considered as solely contributed by some fictious mass source distribution in the body. With this simplification the equation of mass balance can be written as

$$\int_{\underline{\underline{c}}} \left\{ \frac{D\rho}{Dt} \cdot d(Vol) + \rho \frac{D}{Dt} \left[d(Vol) \right] \right\} - \int_{\underline{\underline{c}}} g \cdot d(Vol) = 0 \qquad \text{(II-4)}$$

where $\frac{D}{Dt}[d(Vol)] = (\mathrm{div}\ \dot{\underline{x}}) \cdot d(Vol)$

If mass density ρ is continuous and differentiable with respect to time, the configuration of the body is continuously differentiable with respect to spatial coordinates, and the mass sources are continuously distributed within the body, then the integration sign can be dropped and the differential relation

$$\frac{D\rho}{Dt} + \rho\, \mathrm{div}\ \dot{\underline{x}} = g \qquad \text{(II-5)}$$

is obtained. Here the function g stands for the production of mass per unit current volume, and is called the strength of mass source in later context. In an ordinary body of growing material the mass increment is always positive. However, negative growth may also occur in some cases of old age and metamorphosis [1].

In order to facilitate further discussions a few definitions are introduced.

Definition 1. A differential element of material is growing if and only if the mass contained in this element is not constant, i.e., $g \neq 0$.

Definition 2. A growing body has homogeneous growth properties only if the strength of mass sources, density, and the orientation of the newly added masses are the same for every element in the body.

In studying the growth of an element two problems are of prime interest; namely, how fast the element increases its mass, and how the newly added masses are oriented. The first question is solely characterized by the mass source function g . As we can see from Equation (II-5), the function g contains two terms, $\frac{D\rho}{Dt}$ and $\rho \operatorname{div} \dot{x}$; both of them may be influenced by stresses as well as by the age of the material. In general they are assumed to be functions of stresses and time. Of the three terms contained in Equation (II-5), g and $\frac{D\rho}{Dt}$ can be measured by experiments, therefore they could be used as known properties of the material in determining the orientation of newly added masses. From definition 2 a homogeneously growing body will remain in the state of homogeneous growth only if the stresses are uniform throughout the body and the age of every part of the body is the same. If a non-uniform stress field is created in the body, growth will not remain homogeneous although in some cases the strength of mass sources are the same everywhere in the body.

In general the sudden removal of the stress field in a body may not ensure the immediate returning of the state of growth of the body to the state previous to the existence of the stress field. From the point of view of continuum mechanics, the material is considered to have a memory in the sense that the influences of loads remain for a certain period of time after the loads are removed. Therefore a time constant τ_o can be defined as the time period throughout which the influences of the loads remain.

Definition 3. If a material element grows without stresses for a period of time τ_o and if no external loads are applied on the element thereafter, then the element remains free of stresses. A material which satisfies this condition is defined as normal material.

By this definition a body composed of normal material cannot create stresses without external agents, nor can it strengthen the existing residual stress field by itself after the external loads are removed. In the following only normal material will be considered.

Definition 4. The growth state of a growing body is called normal at time t if for all previous times and up to time t the body is stress free. The state of growth at time t is called relaxed if stresses exist in the body previous to time t, but at t the stresses are zero. The state of growth at time t is called loaded if stresses at time t are non-vanishing.

If, for a material, the time constant τ_o is infinite, then the relaxed growth is different from normal growth for all future times. In general τ_o is expected to have a finite value, then the relaxed growth will be identical to normal growth for time greater than $t + \tau_o$. If $\tau_o = 0$, then relaxed growth is the same as normal growth.

The rate of change of volume of a differential element depends only on the term div $\dot{x}$, which in tensor notation can be written as $\sum_i \dot{x}^i_{,i}$. It can be observed that the change of volume is only the trace of the Euler's deformation tensor d^i_j .

<u>Definition 5</u>. An element of material is growing isotropically if and only if its rate of change of strain is the same in three mutually orthogonal directions.

In tensor notation <u>definition 5</u> can be written as $d^i_j = C\delta^i_j$ where δ^i_j is the Kronecker delta, C is a function of time. Isotropy of growth is a phenomena instead of a property because in many cases materials grow more rapidly in one direction than in the other directions. However, a state of isotropic growth could conceivably be achieved for a material which originally grows anisotropically by the application of proper external loads. Likewise an originally isotropically growing body may not remain so if some arbitrary loads are applied to it. It is this geometry of growth that determines the form of a growing element.

In order to define the influences of mechanical loads on the ability of the growing body to produce new mass, a growth stimulation factor P is defined as

$$P = \frac{\text{Strength of mass source in the present state of growth}}{\text{Strength of mass source in normal growth}} \quad \text{(II-6)}$$

The strength of mass source in normal growth may usually be determined experimentally. Therefore, if the strength of mass source in the present state is measured, P can be determined. In general, according to previous restrictions, P will be a function of stresses and time. P is a function which depends on the specific material considered. The specific form of the function for a specific material can be determined by a series of experiments. Details in this respect will be discussed in later context.

The process of growth under consideration is a relatively slow process, therefore with steady or slow varying stress fields the inertia effect will be neglected. For such cases, at any instant, the forces acting on a differential element will have to satisfy the equation of equilibrium

$$S_{ij,j} + \rho f_i = 0 \quad \text{(II-7)}$$

where S_{ij} denotes the components of the stress tensor, and f_i denotes the resultant of the body forces. The determination of stresses will require the knowledge of the mechanical properties, growth properties, load conditions, and the constitutive equation.

A constitutive equation is a relation which defines the mechanical behavior of a specific material. By mechanical behavior we mean the behavior of stresses, motion, mass, and time. The class of equations which may be used as constitutive equations must satisfy the two requirements: (1) Principle of material objectivity, and (2) Principle of determinism for stress [3, 4, 5]. Therefore the equations must be invariant with respect to orthogonal transformations of spatial frames, and only involve the local motions and the history of local motions.

If the motion is described by the deformation gradient tensor F, $F = \left[\frac{\partial x^i}{\partial X^\alpha}\right]$, then the most general form of a constitutive formulation can be written as

$$S(t) = R(t)\,\phi(U^t)\,R(t)^T$$

where $\phi(U^t)$ is an arbitrary function of U^t, $R(t)$ is the rotation tensor, U^t denotes the history of the strength tensor U up to time t with respect to an arbitrary reference configuration. The deformation tensor is related to the rotation tensor and the stretch tensor by the relation $F = RU$. The specific form of constitutive equations depends on special constitutive assumptions employed for the material.

III

CONSTITUTIVE EQUATIONS FOR A SPECIAL CLASS OF GROWING MATERIALS

In this section the general constitutive formulation for the class of materials satisfying the following requirements will be discussed.

i) The material grows isotropically in normal growth.

ii) The material is normal.

iii) The state of growth of the material is completely determined by the current mechanical conditions. In other words the material has time constant $\tau_0 = 0$, or the current material behavior does not depend on the history of the motion.

iv) The growth is relatively slow and the stress field in the body is quasi-static.

From (i), (ii), (iii), an element of this class of material in normal growth will change its dimension but induce no stresses. Therefore it is not the absolute deformation of the element with respect to some reference configuration that is related directly to stresses. Instead, the influences of stresses tend to deviate the growth from its normal courses by changing the rate of change of dimensions in different directions. A constitutive equation of the rate type [4, 5] will be the most suitable type to serve the purpose.

Let lower case Latin letters above a quantity denote the order of time derivative of the quantity, then a constitutive equation of the rate type for the class of materials under consideration can be written as

$$\phi\left[t , \overset{(k)}{\rho} , \overset{(m)}{S} , \overset{(n)}{U}\right] = 0 \qquad \text{(III-1)}$$

where

$$k = 0 , 1 , 2 ,\ldots.$$

$$m = 0 , 1 , 2 ,\ldots.$$

$$n = 1 , 2 , 3 ,\ldots.$$

By assumption (ii), the values of all quantities in Equation (III-1) are the present values with respect to the current configuration and do not involve the history. Here the natural time t is included in Equation (III-1) because growth properties of growing bodies depend on the age of the material. Density is an independent property because its value cannot be derived from the motion by constancy of mass. Equation (III-1), however, does not completely define the behavior unless a set of initial values of stresses for some initial time are given. The choice of an initial time is completely arbitrary. By requirement (iv) all orders of accelerations could be deleted. The general formulation can be simplified as

$$\phi\left[t , \rho , \dot{\rho} , S , \dot{S} , \dot{U}\right] = 0 \qquad \text{(III-2)}$$

It will be shown that the material derivative of the stretch tensor, $\dot{U}$, is the same as the well known Euler's deformation rate tensor. By the theory of polar decomposition of linear transformation, the deformation gradient tensor $F = \left[\frac{\partial x^i}{\partial X^\alpha}\right]$ can be written as the product of a rotation tensor R and a positive definite symmetric stretch tensor U as $F = RU$. The material derivative of the relation $F = RU$ is $\dot{F} = \dot{R}U + R\dot{U}$. The transpose of the relation $F = RU$ is $F^T = U^T R^T$, and because R is orthogonal, $R^T = R^{-1}$, and U is symmetric, $U^T = U$. Therefore $F^T = U R^{-1}$.

The material derivative of F and F^T are given as

$$\dot{F} = \dot{R} U + R \dot{U}$$

$$\dot{F}^T = \dot{U} R^{-1} - U R^{-1} \dot{R} R^{-1}$$

If a quantity Q is defined by the relations

$$\dot{F} = Q F$$

and

$$\dot{F}^T = F^T Q^T$$

then

$$Q = (\dot{R} U + R \dot{U}) F^{-1}$$

$$Q^T = (F^T)^{-1} (\dot{U} R^{-1} - U R^{-1} \dot{R} R^{-1})$$

If the current configuration is chosen as the reference configuration so that

$$F = F^T = F^{-1} = U = R = R^{-1} = I$$

then the expressions for Q and Q^T can be simplified as

$$Q = \dot{U} + \dot{R}$$

$$Q^T = \dot{U} - \dot{R}$$

Comparing Q and Q^T with $\dot{F}$ and $\dot{F}^T$ it is clear that

$$Q = \dot{F} \quad , \quad Q^T = \dot{F}^T$$

From the expressions for Q and Q^T, $\dot{U}$ and $\dot{R}$ can be solved as

$$\dot{U} = \tfrac{1}{2} (Q^T + Q) = \tfrac{1}{2} (\dot{F}^T + \dot{F})$$

$$\dot{R} = \tfrac{1}{2} (Q^T - Q) = \tfrac{1}{2} (\dot{F}^T - \dot{F})$$

It is observed that $\dot{U}$ is just the Euler's deformation rate tensor d whereas $\dot{R}$ is the vorticity tensor ω.

Since it is the deformation rate tensor that is directly related to the form of a body, the constitutive equation, for sake of convenience, can be expressed as

$$d = f (t , \rho , \dot{\rho} , S , \dot{S}) \qquad \text{(III-3)}$$

and a set of initial conditions of stresses $[S]_{t=0}$.

Equation (III-3) is in a symbolic notation. Some general properties that the functional f should have will be discussed in the fofowing.

1) The constitutive equations are continuous relations, therefore as the stresses and the stress rate tend to zero the equation will denote the state of normal growth.

2) The influences of the age of the material on the constitutive equation can be considered in two different ways. First, consider that certain parameters which characterize the properties of growth are functions of time. In other words define some parameters to account for the microstructural change of the material with respect to time. Another way to view this phenomenon is by considering that the influences of stresses are changing with respect to time. Thus if some "effective stresses" can be defined as a function of true stresses and time, then the constitutive equation formulated with the "effective stresses" and motion is explicitly independent of natural time. From either consideration the influence of age is represented by

terms like $A_1 S$, $A_2 S^2$, etc., where A_1 , A_2 , etc., are functions of time only. But if the normal growth is also a function of time, terms like t , t^2 , etc., may also enter the equation explicitly. By the same consideration the stimulation factor P should also contain some time elements in the same category.

3) If the density of the material has a constant value over a notable period of time, then the terms ρ and $\mathring{\rho}$ can be deleted from the equation, i.e.,

$$d = f\ (\ t\ ,\ S\ ,\ \dot{S}) \qquad \text{(III-4)}$$

The validity of such an equation is within the regime that density is essentially constant.

4) If the external surface loads on the element are kept constant, and the strength of mass source is positive, then the stress field never increases its strength. If the element is constrained to a constant configuration, then the stress field never decreases its strength.

The general constitutive equation can be written as the sum of a series of functions as

$$d = f = f_0 + f_1 + f_2 + \ldots..$$

where f_i denotes the homogeneous function of i^{th} order of variables ρ, $\dot{\rho}$, S , $\dot{S}$.

The first few f_i's are listed in the following:

f_0 = function of time only

$$f_1 = A_1 \rho + A_2 \dot{\rho} + A_3 S + A_4 \dot{S} + A_5 I_1 + A_6 J_1$$

$$f_2 = B_1 \rho^2 + B_2 \rho\dot{\rho} + B_3 \rho S + B_4 \rho\dot{S} + B_5 \rho I_1 + B_6 \rho J_1 + B_7 \dot{\rho}^2$$
$$+ B_8 \dot{\rho} S + B_9 \dot{\rho} \dot{S} + B_{10} \dot{\rho} I_1 + B_{11} \dot{\rho} J_1 + B_{12} S^2 + B_{13} S \dot{S}$$
$$+ B_{14} S I_1 + B_{15} S J_1 + B_{16} \dot{S}^2 + B_{17} \dot{S} I_1 + B_{18} \dot{S} J_1 + B_{19} I_1^2 + B_{20} I_1 J_1$$
$$+ B_{21} J_1^2 + B_{22} I_2 + B_{23} J_2$$

where I_1, I_2, I_3 denote the stress invariants,

J_1, J_2, J_3 denote the invariants of the stress rate,

the coefficients A_i's and B_i's are functions of time only.

A linear equation is given by

$$d = f_0 + f_1 \tag{III-5}$$

A quasi-linear constitutive equation can be written as

$$d = G_0 + G_1 \rho + G_3 \dot{\rho} + G_4 S + G_5 \dot{S} \tag{III-6}$$

where G_i's are scalar functions of the stress invariants, stress rate invariants, and time.

In the same manner, the stimulation factor P can also be written as

$$P = \overline{P}_0 + \overline{P}_1 + \overline{P}_2 + \ldots\ldots$$

with P_i denoting the i^{th} homogeneous function.

The relation between the stimulation factor and the strength of mass source is

$$P = \frac{g}{g_n} = \frac{\rho(\sum_i d^i{}_i) + \dot{\rho}}{g_n}$$

Therefore P is a function of time, density, and the invariants of stress and stress rate.

IV

LINEAR MATERIALS WITH CONSTANT DENSITY

In this section, materials with linear constitutive equations and constant density will be studied in detail. The basic constitutive relation is

$$d^i_{\ j} = \delta^i_{\ j}(A + BI_1 + CJ_1) + GS^i_{\ j} + K\dot{S}^i_{\ j} \qquad \text{(IV-1)}$$

where A, B, C, G, K are functions of time only. The state of normal growth for this class of material is characterized by the relation

$$d^i_{\ j} = \delta^i_{\ j}\ A$$

which means a body in normal growth merely enlarges itself with the same geometric proportion, while the rate of such changes are given by the time function A, which is one of the properties of the material. The stimulation factor is

$$P = \frac{3(A + BI_1 + CJ_1) + GI_1 + KJ_1}{3A} = \bar{P}_o + \bar{P}_1 I_1 + \bar{P}_2 J_1 \qquad \text{(IV-2)}$$

hence $\bar{P}_o = 1\ ,\ \bar{P}_1 = \frac{B}{A} + \frac{G}{3A}\ ,\ \bar{P}_2 = \frac{C}{A} + \frac{K}{3A}$.

Once the arbitrary function A, B, C, G, K, and $\bar{P}_1$, $\bar{P}_2$ are determined the deviation of the growth pattern from the state of normal growth can be determined.

The values of $\bar{P}_1$ and $\bar{P}_2$ can be measured directly from experiments. For example, if in an experiment J_1 is kept zero, then the net increase of mass measured is P, the value of $\bar{P}_1$ is given by

$$\bar{P}_1 = \frac{P - 1}{I_1}$$

On the other hand, if I_1 is kept constant, then the value of $\bar{P}_2$ is given by

$$\bar{P}_2 = \frac{P - 1 - \bar{P}_1 I_1}{J_1}$$

where $\bar{P}_1$ is known. However, attention has to be paid to the influence of the age of the material. The measured P may be a function of time. Thus the determination of $\bar{P}_1$ and $\bar{P}_2$ are not as straightforward as it may seem.

The basic relations at our disposal are

i) Equation of mass balance

$$g = \rho \dot{x}^i{}_{,j} \tag{IV-3}$$

ii) Equations of equilibrium

$$S_{ij,j} + \rho f_i = 0 \tag{IV-4}$$

iii) Constitutive equations of the material

$$d^i{}_j = \delta^i{}_j \, (A + BI_1 + BJ_1) + GS^i{}_j + K\dot{S}^i{}_j \tag{IV-5}$$

where the stress rate, by Jaumann's definition [8, 9], is

$$\dot{S}^i{}_j = \frac{DS^i{}_j}{Dt} + {}_m \, (S^i{}_m \omega^m{}_j + \omega^i{}_m S^m{}_j)$$

The known quantities are density, boundary conditions, the initial conditions, and the state of normal growth.

To illustrate the type of considerations these constitutive equations lead to, consider a differential element of material. At time $t = 0$ the stress is given by

$$[S]_{t=0} = \begin{bmatrix} S_{10} & 0 & 0 \\ 0 & S_{20} & 0 \\ 0 & 0 & S_{30} \end{bmatrix}$$

The boundary condition is such that a uniformly distributed force is acting normal to the surface which is perpendicular to the x^1-axis. The easier way to determine the form of the element under the described boundary and initial conditions is by the inversed method. By assuming a kinematically admissible velocity field, the corresponding stress field which satisfies the described conditions is determined.

Let the velocity components be denoted by

$$\dot{x}^1 = \alpha x^1$$

$$\dot{x}^2 = \beta x^2$$

$$\dot{x}^3 = \gamma x^3$$

then the constitutive equation consists of the following six differential equations

$$\alpha = A + BI_1 + CJ_1 + GS_{11} + K \frac{\partial S_{11}}{\partial t}$$

$$\beta = A + BI_1 + CJ_1 + GS_{22} + K \frac{\partial S_{22}}{\partial t}$$

$$\gamma = A + BI_1 + CJ_1 + GS_{33} + K \frac{\partial S_{33}}{\partial t}$$

$$0 = GS_{12} + K \frac{\partial S_{12}}{\partial t}$$

$$0 = GS_{13} + K \frac{\partial S_{13}}{\partial t}$$

$$0 = GS_{23} + K \frac{\partial S_{23}}{\partial t}$$

From the given initial and boundary conditions the solution for the stress can be expressed as

$$[S] = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix}$$

Then, if $S_{22} = S_{33} = S_{20} = S_{30} = 0$, we have

$$\beta = \gamma = A + BS_{11} + C \frac{\partial S_{11}}{\partial t}$$

and

$$\alpha = (A + BS_{11} + \frac{C\partial S_{11}}{\partial t}) + GS_{11} + K \frac{\partial S_{11}}{\partial t}$$

If, furthermore, $S_{11} = S_{10}$ = constant, then

$$\alpha = A + (B + G)S_{10}$$

$$\beta = \gamma = A + BS_{10}$$

where α and β could be measured. From the mass balance equation

$$g = (\alpha + \beta + \gamma)\rho$$

thus

$$P = \frac{\alpha + 2\beta}{3A} = 1 + \bar{P}_1 S_{10}$$

Therefore

$$\bar{P}_1 = (\frac{\alpha + 2\beta}{3A} - 1) \frac{1}{S_{10}}$$

$$B = \frac{\beta - A}{S_{10}}$$

$$G = \frac{\alpha - \beta}{S_{10}}$$

The axial force required to render $S_{11} = S_{10}$ is

$$F = F_o e^{2\beta t}$$

If, on the other hand, the axial force is kept constant, then

$$S_{11} = \frac{F_o}{x^2 x^3} = S_{10} e^{-2\beta' t}$$

and

$$\beta' = \gamma' = A + BS_{10} e^{-2\beta' t} - 2\beta' K e^{-2\beta' t}$$

$$\alpha' = A + (B + G) S_{10} e^{-2\beta' t} - 2\beta'(K + C) e^{-2\beta' t}$$

where α' and β' can be measured. If B and G are already determined, the values of C and K can be obtained as

$$K = -\left[(\beta' - A) e^{2\beta' t} - BS_{10}\right] / 2\beta'$$

$$C = -\left[(\alpha' - \beta') e^{2\beta' t} - GS_{10}\right] / 2\beta'$$

The same type of measurements can be made for the cases where the body is growing under hydrostatic pressure. If the hydrostatic pressure is kept constant,

$$\alpha'' = A + (3B + G) S_o = \beta'' = \gamma''$$

If the hydrostatic pressure is regulated to vary linearly with respect to time, i.e., $S = \epsilon t + S_o$, then

$$\alpha''' = A + (3B + G)(\epsilon t + S_o) + (3C + K)\epsilon$$

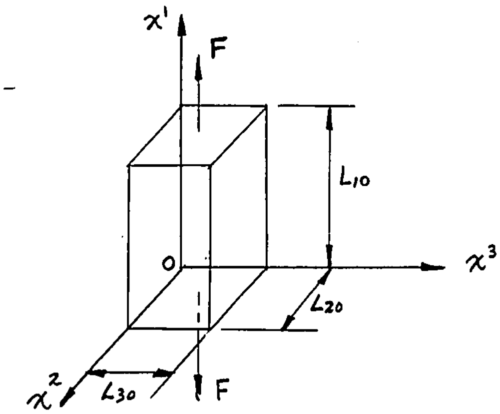
the value of α'' and α''' can be determined from measurements. Therefore by a series of systematic experiments the coefficients in the constitutive equation as well as in the stimulation factor can be determined.

In the following a few examples are treated assuming the constitutive equation and the stimulation factor are known.

(a) Simple tension acting on a rectangular bar.

Consider a rectangular column at time t = 0 which has the dimension $L_{10}L_{20}L_{30}$. The stress field at time t = 0 is

$$[S]_{t=0} = \begin{bmatrix} S_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



The boundary condition is specified by a uniformly distributed force F acting normal to the surface perpendicular to the x^1-axis. If the force is given by $F = F_o e^{2(A + BS_o)t}$, then the stress field and the motion which satisfy the constitutive equation, the boundary condition, the initial condition, and the equilibrium equations can be written as

$$[S] = \begin{bmatrix} S_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$x^1 = x_o^1 \, e^{(A + (B + G)S_o)t}$$

$$x^2 = x_o^2 \, e^{(A + BS_o)t}$$

$$x^3 = x_o^3 \, e^{(A + BS_o)t}$$

The above shows that the application of force in the x^1-direction will not only influence the growth rate in the x^1-direction but also in the other two directions. This is similar to the phenomena represented by Poisson's ratio in the linear elasticity. In general it is anticipated that a tension force will increase the growth rate in the direction of force. The tension will not necessarily reduce the growth rate in the other two directions because the stimulation factor P may be a function of stresses too. It is possible that the growth rates in the other two directions are also increased. If the stimulation factor $P \equiv 1$, then the relation $3B + G = 0$ must hold, and the motion is given by

$$\dot{x}^1 = (A + \frac{2}{3} G) \; x^1$$

$$\dot{x}^2 = (A - \frac{1}{3} G) \; x^2$$

$$\dot{x}^3 = (A - \frac{1}{3} G) \; x^3$$

which shows that the increase of growth in x^1-direction will cause the reduction of growth rate in the other two directions.

Instead of force the motion on part of the boundary can be specified in a manner that is kinematically and biologically admissible. By biologically admissible we mean that the kind of motion being specified will not result in the destruction of the growth ability. Consider the case that the motion in the x^1-direction is completely stopped, i.e., $\dot{x}^1 = 0$. The initial condition is

$$[S]_{t=0} = \begin{bmatrix} S_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Then the solution which satisfies the conditions is

$$[S] = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$S_{11} = (S_o + \frac{A}{B + G})\, e^{\frac{-(B + G)}{K + C} t} - \frac{A}{B + G} \qquad \text{(IV-6)}$$

and the motion is given by

$$\dot{x}^1 = 0$$

$$\dot{x}^2 = \left[\frac{GA}{B + G} + (S_o + \frac{A}{B + G}) \frac{(BK - CG)}{C + K}\, e^{\frac{-(B + G)}{K + C} t} \right] x^2$$

$$\dot{x}^3 = \left[\frac{GA}{B + G} + (S_o + \frac{A}{B + G}) \frac{(BK - CG)}{C + K}\, e^{\frac{-(G + G)}{K + C} t} \right] x^3$$

From the expression for stresses, Equation (IV-6), if we let $S_{11} = 0$, then there exists a time τ

$$\tau = \frac{K + C}{B + G} \ln \left[1 + \frac{B + G}{A} S_o \right]$$

for

$$1 + \frac{B + G}{A} S_o > 0 \quad .$$

The value of τ determines the time at which the stresses are zero. If $1 + \frac{B + G}{A} S_o < 0$ then the value of τ does not exist, which means that under no circumstances would the stresses in the body vanish. Only the positive values of τ are physically meaningful because time can only proceed in one direction. In order to see clearly what τ means, consider the following cases:

(A) If $S_o + \frac{A}{G + B} = 0$, then $S_{11} \equiv S_o = \frac{-A}{B + G}$, and $\tau = -\infty$. This implies that if the present constraint conditions are kept constant there does not exist an instant, in the region $t > 0$, at which the stresses in the body are zero. As a matter of fact this is the case of constant stresses state.

(B) If $\ln \left[1 + \frac{B + G}{A} S_o \right] > 0$, and $\frac{G + B}{K + C} < 0$, then τ has a negative value. Therefore for all time $t > 0$ the stresses in the body are non-vanishing.

(C) If $\ln \left[1 + \frac{B + G}{A} S_o \right] < 0$, and $\frac{G + B}{K + C} < 0$, then the value of τ is positive. This means that after the constraints are imposed on the body at $t = 0$ there is a time $t = \tau > 0$ at which the stresses are vanishing.

The physical picture is as follows: the material is under positive growth and is under tension at time $t = 0$. Due to the restriction on growth in the x^1-direction, the stresses gradually change from tension to compression. The time $t = \tau$ is the time when tension is reduced to zero but compression has not yet been developed. If the bar is initially under compression, then for all time $t > 0$ it is under compression with increasing intensity.

Another case which can be considered for a bar is that motions are specified on part of the boundary, and forces are specified on other parts of the boundary. For example, let us consider a cylindrical bar of circular cross-section subjected to the boundary conditions:

i) the radius of the bar is kept constant with no torsional rotation,

ii) an axial force of constant magnitude is acting in the longitudinal direction.

In cylindrical coordinates condition (i) can be written as

$$\dot{r} = 0$$
$$\dot{\theta} = 0$$

Condition (ii) is $S_{33} \equiv S_o$; the remainder of the stress components are to be determined. The solution to the constitutive equation is

$$[S] = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & S_o \end{bmatrix}$$

where

$$S_{11} = S_{22}/r^2 = \frac{(A + BS_o)}{2B + G} \left[e^{\frac{-(2B + G)}{2C + K} t} - 1 \right]$$

The motion is

$$\dot{r} = 0$$
$$\dot{\theta} = 0$$
$$\dot{z} = \alpha z$$

where

$$\alpha = G \left[S_o + \frac{A + BS_o}{2B + G} \right] + \frac{2(A + BS_o)(BK - CG)}{(2B + G)\ (2C + K)} \quad e^{\frac{-(2B + G)}{2C + K} t}$$

It can be observed that all boundary conditions, initial conditions, and equilibrium conditions are satisfied.

If no axial force is applied, i.e., $S_{33} \equiv 0$, then the solution can be expressed as

$$S_{11} = S_{22}/r^2 = \frac{A}{2B + G}\left[e^{\frac{-(2B + G)}{2C + K} t} - 1 \right]$$

and

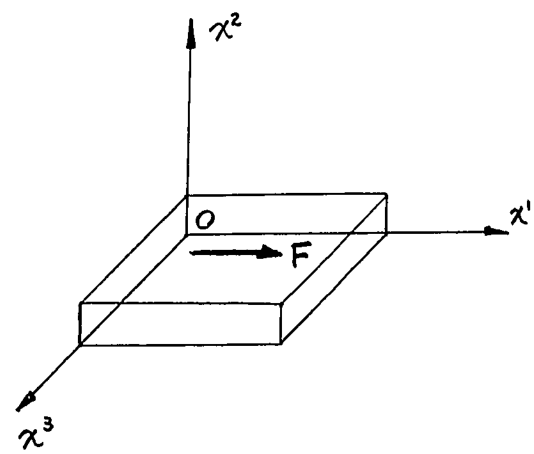
$$\alpha = \frac{AG}{2B + G} + \frac{2A\ (BK - CG)}{(2B + G)(2C + K)}\ e^{\frac{-(2B + G)}{2C + K} t}$$

which implies that the rate of change of length is changed and a plane hydrostatic stress is created in the body.

(b) Simple shear

Consider a flat body as shown in the figure subjected to the initial condition of pure shear stress, i.e.,

$$[S]_{t=0} = \begin{bmatrix} 0 & S_o & 0 \\ S_o & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



The boundary condition is specified by a shear force applied to the surface perpendicular to the x^2-axis, and the magnitude of the shear force is a time function such that the shear stress is kept constant. Then the solution can be written as

$$[S] = \begin{bmatrix} 0 & S_o & 0 \\ S_o & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the motion is

$$\dot{x}^1 = \left[A - 2KG(S_o)^2\right] x^1 + GS_o x^2$$
$$\dot{x}^2 = \left[A + 2KG(S_o)^2\right] x^2$$
$$\dot{x}^3 = A\ x^3$$

The required force is

$$F = F_o\ e^{2At}\left\{\left[1 - \frac{x_o^2/x_o^1}{4KG(S_o)^2}\right] e^{-2KG(S_o)^2 t} + \frac{x_o^2/x_o^1}{4KG(S_o)^2}\ e^{+2KG(S_o)^2 t}\right\}$$

where F_o is the initial force on the surface.

If the shear motion of the body is specified as

$$\dot{x}^1 = \alpha x^1 + \beta x^2$$
$$\dot{x}^2 = \eta x^2$$
$$\dot{x}^3 = \xi\ x^3$$

where α, η, ξ are undetermined functions but β is prescribed. The solution for the constitutive equation satisfying the initial condition

$$\left[S\right]_{t=0} = \begin{bmatrix} 0 & S_o & 0 \\ S_o & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{can be written as} \quad \left[S\right] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$S_{11} = -S_{22} = \frac{GS_o - \beta}{2K\beta}\left[e^{-\frac{G}{K}t}\cos\ (2\beta t) - 1\right]$$

$$S_{12} = \frac{GS_o - \beta}{2K\beta}\left[e^{-\frac{G}{K}t}\sin\ (2\beta t)\right] + S_o$$

The motion for this case is

$$\dot{x}^1 = \left(A - \frac{G(GS_o - \beta)}{2K\beta} - 2K\beta S_o\right) x^1 + x^2$$

$$\dot{x}^2 = \left(A + \frac{G(GS_o - \beta)}{2K\beta} + 2K\beta S_o\right) x^2$$

$$\dot{x}^3 = A x^3$$

From the expression for stresses it is clear that in order to keep the body under a prescribed shear motion some normal forces in the direction of the x^1-axis and x^2-axis must be supplied in addition to the shear force acting on the surface perpendicular to the x^2-axis. This type of effect is also observed in non-linear elastic materials.

(c) Torsion of a circular cylinder

Consider a circular cylinder at time t = 0 subjected to the stresses

$$[S]_{t=0} = \begin{bmatrix} f & 0 & 0 \\ 0 & 0 & \frac{M_o}{I_o} r_o^2 \\ 0 & \frac{M_o}{I_o} r_o^2 & 0 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & 0 & S_o \\ 0 & S_o & 0 \end{bmatrix}$$

where r_o denotes the point at a distance r_o from the center of the cylinder at time t = 0 , and f is an arbitrary function of time. By the inverse method the motion can be described in cylindrical coordinates as

$$\dot{r} = \alpha r$$

$$\dot{\theta} = \beta z$$

$$\dot{z} = \eta z$$

where α, β, η are to be determined from boundary and initial conditions.

For the given motion the constitutive equation yields the following six differential equations

$$\alpha = (A + BI_1 + CJ_1) + GS_{11} + K \frac{\partial S_{11}}{\partial t}$$

$$\alpha r^2 = r^2(A + BI_1 + CJ_1) + GS_{22} + K \left(\frac{\partial S_{22}}{\partial t} - 2\alpha S_{22} - r^2\beta S_{23}\right)$$

$$\eta = (A + BI_1 + CJ_1) + GS_{33} + K \left(\frac{\partial S_{33}}{\partial t} + \beta S_{23}\right)$$

$$0 = GS_{12} + K \left(\frac{\partial S_{12}}{\partial t} - \alpha S_{12} - \frac{r^2\beta}{2} S_{13}\right)$$

$$0 = GS_{13} + K \left(\frac{\partial S_{13}}{\partial t} + \frac{\beta}{2} S_{12}\right)$$

$$\frac{r^2\beta}{2} = GS_{23} + K \left(\frac{\partial S_{23}}{\partial t} - \alpha S_{23} + \frac{\beta}{2} S_{22} - \frac{r^2\beta}{2} S_{33}\right)$$

where $I_1 = S_{11} + \frac{S_{22}}{r^2} + S_{33}$

$$J_1 = \frac{\partial S_{11}}{\partial t} + \frac{1}{r^2}\left(\frac{\partial S_{22}}{\partial t} - 2\alpha S_{22}\right) + \frac{\partial S_{33}}{\partial t} = \frac{\partial}{\partial t}\left(S_{11} + \frac{S_{22}}{r^2} + S_{33}\right)$$

If the boundary condition is specified such that the cylinder is subjected to an end moment only, then the six equations can be reduced to four equations by assuming the stress tensor has the form

$$\begin{bmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{23} & S_{33} \end{bmatrix}$$

The equations are

$$\alpha = (A + BI_1 + CJ_1) + GS_{11} + K \frac{\partial S_{11}}{\partial t}$$

$$\alpha r^2 = r^2(A + BI_1 + CJ_1) + GS_{22} + K \left(\frac{\partial S_{22}}{\partial t} - 2\alpha S_{22} - r^2\beta S_{23}\right)$$

$$\eta = (A + BI_1 + CJ_1) + GS_{33} + K \left(\frac{\partial S_{33}}{\partial t} + \beta S_{23}\right)$$

$$\frac{r^2\beta}{2} = GS_{23} + K \left(\frac{\partial S_{23}}{\partial t} - \alpha S_{23} + \frac{\beta}{2} S_{22} - \frac{r^2\beta}{2} S_{33}\right)$$

From the differential equations, the only kind of motion which will give the stresses satisfying the boundary and initial conditions is

$$\dot{r} = \alpha r = A\, r$$

$$\dot{\theta} = 0$$

$$\dot{z} = \eta z = A z$$

The stresses for this case are

$$[S] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & S_{23} & 0 \end{bmatrix} , \qquad f = 0$$

where $$S_{23} = S_o\, e^{\left(\frac{-G}{K} + A\right) t}$$

The end moment is given by

$$M = \int_0^r S_{23} \cdot 2\pi r\, d r = \frac{\pi S_o r_o^2}{\left(\frac{-G}{K} + 3A\right)} \, e^{\left(\frac{-G}{K} + 3A\right)t}$$

The solution obtained corresponds to the case where the cylinder is given an angle of twist θ_o at time t = 0, and then the angle is kept constant for all time $t > 0$. It is to be noted, however, that for non-homogeneous stress fields determination of appropriate motions which give a stress field satisfying prescribed boundary conditions is in general a difficult task.

(d) Bending of a rectangular beam

Consider a beam of rectangular cross-section subjected to pure bending stress at time $t = 0$ and the boundary condition is such that the bending moment is constant. In general with this kind of boundary condition, an exact solution is difficult to find. However, if the mechanical properties are all known, an approximate solution which is valid only for small time increments can always be obtained. In treating this problem of pure bending, assume the material behaves like a linear elastic material for the immediate response to loads. Consequently, the initial stress is given by

$$[S]_{t=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{M_o x^1}{I_o} \end{bmatrix}$$

where M_o is the moment, I_o is the area moment of inertia of the beam cross-section with respect to its neutral axis at time $t = 0$, x^1 is the distance of a particle from the neutral axis. Since the beam is under pure bending, the cross-section at the middle of the span may be taken as a line of symmetry. Let the coordinates be so chosen that the plane $x^3 = 0$ coincides with the axis of symmetry as shown in the figure.