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by

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I. INTRODUCTION

The boundary layer concept, first introduced by Prandtl (31), divides the flow past a body into two regions; a very thin layer in the neighborhood of the body (boundary layer) where viscous forces are important, and the remaining "external" flow outside this boundary layer where viscous forces are negligible. The external region is governed by the Euler equations of motion for a non-viscous fluid while the boundary layer is governed by the classical boundary layer equations. The Navier-Stokes equations are the complete equations of motion for a viscous fluid while the boundary layer equations are simplifications of these complete equations based on experimental observations. The boundary layer equations as derived by Prandtl are sometimes called the "first order" equations. Van Dyke (40,41,42) considered the "second order" boundary layer equations, deriving their name from the fact that they include the "second order" effects of transverse curvature, longitudinal curvature, slip, temperature jump, entropy gradient, stagnation enthalpy gradient, and displacement thickness interaction. As pointed out by Van Dyke, all second order effects should be considered concurrently, but this is beyond the scope of this work and the work of all previous authors using numerical or analytical methods of solution.

Although the boundary layer equations are a simplified form of the Navier-Stokes equations, no closed form analytical
solutions exist, even for the simplest flow geometries. Analytical solutions have been obtained for flows with special external pressure distributions, similar flows and etc., by using power series expansions. These methods are cumbersome except for the simplest cases so some approximate techniques were developed such as van Kármán's momentum integral equation (43), which is much less laborious to solve, but also less accurate.

Recently, considerable effort has been put forth to accurately solve the boundary layer equations by numerical methods. Two general methods have been explored, finite difference analogues and numerical integration techniques. Rouleau and Osterle (33) considered the explicit and implicit finite difference analogues to numerically evaluate the solutions for suction flow along and flow in the wake of a flat plate. The implicit difference equations were solved by relaxation since no computer was available but iterative methods were suggested for computer usage. The explicit equations were found to be much simpler to solve, but the severe restrictions of the step sizes imposed by the stability criteria caused some numerical problems. Bodoia and Osterle (4) obtained the numerical solution for development of plane Poiseuille and Couette flows.

Leigh (26) replaced the normal velocity component present in the momentum equation by the integral form of the continuity equation before differencing the partial differential equation. Although the result is an integro-differential equation, the
continuity equation has been removed and only one equation is to be solved. Implicit finite difference analogues were used to approximate the partial derivatives and the "trapezium" rule (14a) was used to approximate the integral. The resulting difference equation was solved iteratively and Richardson's extrapolation procedure was used to reduce the truncation error associated with the finite difference analogue. The extrapolation procedure was only applied in the normal direction but the truncation error is order of \( O(\Delta x, \Delta y^2) \) so the tangential step size must be very small to gain any accuracy.

Baxter and Flügge-Lotz (1,2) numerically solved the Crocco transformed equations for compressible flow. An explicit analogue was used with its inherent small step sizes. The Crocco transformed equations were chosen because the continuity equation is eliminated without the introduction of an integral and the range of the independent variable normal to the body is finite instead of infinite as is the case for the physical plane equations. These equations are valid for any first order boundary layer flow except in regions where the shear is zero. When the shear is zero, the Jacobian of the transformation is zero. These regions include the outer edge of the boundary layer, separation, and "overshoot"\(^1\). The stability criteria derived by Baxter and Flügge-Lotz has been found by this author to be incorrect. This error is due to the technique

\(^1\)Definition of "overshoot": The tangential velocity component within the boundary layer exceeds the exterior velocity.
employed to linearize the nonlinear partial differential equations. Kramer and Lieberstein (25) applied the Mangler transformation (28) to the Crocco equations and solved the resulting equations with an implicit analogue. The additional transformation has the advantage of reducing the partial differential equation to an ordinary differential equation along the stagnation line from which initial profiles can be obtained. The authors presented two example problems but did not prove the validity of the solution by comparison with known solutions.

Wu (46) and Flügge-Lotz and Yu (12) both used explicit analogues to solve the physical plane boundary layer equations. Yu considered flows with pressure gradients, but the step size restrictions were so severe that solutions could not be obtained in some cases. A significant portion of Wu's paper discusses possible difference analogues for the continuity equation. For compressible flow problems, Wu uses the Howarth-Dorodnitsyn transformation which removes the explicit density dependence from the continuity equations. The transformation, as considered by Wu, was not valid for flows with external pressure gradients.

Flügge-Lotz and Blottner (10) considered the compressible boundary layer equations. In addition, they considered one second order effect, displacement thickness interaction. Explicit analogues were only considered to evaluate the effect of initial profiles and were then immediately dropped based on past experience of one of the authors. The physical plane
equations and the Howarth-Dorodnitsyn transformation of these equations were both solved numerically. The ordinary implicit analogue was used to obtain the physical plane solutions and the Crank-Nicolson implicit analogue was used for the transformed plane solutions. No indication was given as to why both methods were not applied to each plane, since they are both applicable in each coordinate system. The flow geometries considered were either of similar type or their initial portion was of similar type. This removed the problems associated with initial profiles since the similar solutions obtained by Low (27) could be used to obtain these profiles at some finite distance from the leading edge. Except for one difference analogue of the continuity equation, the numerical procedures were found to be stable, accurate, and fast.

Recently, Flügge-Lotz and Davis (11) applied the numerical methods used by Flügge-Lotz and Blottner (10) to the boundary layer equations with the second order effect of vorticity interaction\(^2\) included. They obtained the starting profiles using the Blasius Series (3) and used Blottner's numerical method to propagate the solution downstream. Their most significant result is that a pressure gradient due to vorticity interaction cannot be neglected. No new numerical procedure was introduced.

\(^2\)The second order effect of vorticity interaction is the combination of the entropy gradient and stagnation enthalpy gradient effects.
The second class of numerical solutions, integration techniques, are based on a method proposed by Hartree and Womersley (18). The boundary layer equations, being two dimensional and initial value type equations, are generally solved by propagating a known initial solution downstream. This method differs only the derivatives with respect to the tangential (downstream) independent variable which reduces the partial differential equations to a system of ordinary differential equations at each grid column since the upstream values are known. Beginning with the second grid column (the initial profiles are known), these systems of ordinary differential equations are solved consecutively until the entire flow field has been propagated. The solution at each grid column is obtained by numerical integration techniques, the latter varying with authors.

Hartree (17) initially applied this procedure to an equation similar in form to that used by Leigh (26). A computer was not available and the accuracy was limited. Manohar (29) defined a stream function which would satisfy the continuity equation and then defined "similar" type independent variables similar to those used by Meksyn (30a) and Göertler (15). The result is a third order nonlinear partial differential equation which was solved using the Hartree-Womersley procedure. Adam's integration method was used to solve the ordinary differential equations. Smith and Clutter (36,37,38) slightly modified the equation used by Manohar and studied this numerical technique rather thoroughly.
The boundary conditions for this third order partial differential equation are the specification of the stream function at the wall and the first derivative in the normal direction of the stream function at the wall and at the edge of the boundary layer. Since one boundary condition is not located at the wall, interpolation and iteration must be used to obtain a third "initial condition" at the wall such that the solution at the edge of the boundary layer will satisfy the boundary condition at this position. This involved procedure is caused by using initial value numerical methods to solve boundary value equations. When the flow is compressible, an initial enthalpy function must also be determined since the boundary conditions for the second order energy equations are the specification of some enthalpy function at the wall and at the edge of the boundary layer.

After extensive study, a predictor-corrector type method that uses the Falkner multiple integration-extrapolation formulas and the Adams-type multiple integration-interpolation formulas was found by Smith and Clutter to be fairly satisfactory. The wall values were obtained by a three point interpolation procedure which requires that the integration be carried out at least three times at every station. For compressible flows, solutions could only be obtained by the following procedure:

a) Advance the stream function, leaving the enthalpy values constant, until the exterior condition is satisfied with a certain small bound, \( \varepsilon \).

b) Advance the enthalpy profile, leaving the stream function
profile constant, until the exterior enthalpy condition is satisfied within the small bound, \( \varepsilon \).

c) Correct the stream function and enthalpy profiles by repeating a) and b) until the successive change in magnitude of the unknown wall "initial conditions" is less than the small constant \( \varepsilon \).

The method has a distinct disadvantage since the convergence of the numerical solution is dependent upon the ratio of the tangential distance to the tangential step size. Smith and Clutter found, by numerical experiments, that this ratio must be less than twenty-five or the solution would diverge. Therefore, the tangential step size could not be allowed to go to zero which is a requirement of stable, consistent, and convergent finite difference analogues. Although this ratio has the above mentioned upper bound, results are presented which indicate the magnitude of the ratio being about one thousand, Table I of (36). Therefore, it is evident the entire problem was solved several times, each using step sizes such that the final distance varied by some small constant. By adhering to the convergence criteria, the authors claimed to have no difficulty with the convergence of the numerical solution.

The Howarth transformation of the boundary layer equations was used which has the advantage of maintaining an essentially constant boundary layer thickness versus the quadratically increasing thickness with distance common to the physical plane boundary layer equations, thereby, eliminating many
unnecessary calculations or tests for the boundary layer thickness. Also, the reduction of the equations to the similar equations at the stagnation point allows accurate determination of initial profiles.

It is evident that considerable effort has been placed on obtaining accurate solutions to the boundary layer equations numerically. At present, both numerical integration and finite difference methods have been used with some success to solve rather difficult boundary layer problems. The purpose of this research is fourfold:

1) Develop and assess new numerical methods for solving the boundary layer equations.

2) Assess numerical methods that have been proposed by previous authors.

3) Assess the various transformations to determine if any are particularly well-suited for the numerical solution of general boundary layer flows.

4) Determine the best numerical method for solving the boundary layer equations in each of the coordinate systems considered as these results may be applicable to other engineering equations.

Since a general numerical method is desirable, the transient incompressible, steady-state incompressible, and the steady-state compressible, nonsimilar flows must be considered.
II. BOUNDARY LAYER EQUATIONS

The differential equations and the necessary boundary and initial conditions which describe the two-dimensional laminar boundary layer flows considered in this work are presented in this chapter. The incompressible steady-state and transient equations, the steady-state incompressible Göertler transformed equations, the compressible steady-state equations, their Howarth-Dorodnitsyn (HD) transformation, and the Göertler transformation of the HD equations are included. Wall shear formulas are given since accuracy is determined by comparison of this quantity or by comparison of velocity and enthalpy profiles. The boundary conditions necessary to obtain a unique solution of the boundary layer equations are the exterior velocity or pressure, the exterior enthalpy, the tangential and normal wall velocity components, and the wall enthalpy or heat flux. The initial conditions are velocity and enthalpy profiles obtained either by the similar equations or by approximation. The upstream (infinite) flow must be specified for the compressible equations.

All equations are presented in nondimensional form due to the advantages of reducing the number of parameters and of working with quantities of the same order of magnitude. The transient problem assumes fully developed potential flow initially. Partial derivatives are denoted by independent variable subscripts.

A. Incompressible Flow
1. Physical Plane. The flow of an incompressible, viscous fluid is mathematically described by the continuity and Navier-Stokes equations. For flow at large Reynolds numbers, Prandtl (31) reduced the Navier-Stokes equations to the classical two dimensional boundary layer equations

Continuity: \( u_x + v_y = 0 \) \hspace{2cm} 2.1

Momentum: \( u_t + uu_x + vu_y = -\frac{1}{\rho_e} \frac{dp}{dx} + u_{yy} \) \hspace{2cm} 2.2

Euler Equation: \( U_e \frac{dU_e(x)}{dx} = -\frac{1}{\rho_e} \frac{dp_e}{dx} \) \hspace{2cm} 2.3

Wall Shear: \( \tau_w = u_y \bigg|_w \) \hspace{2cm} 2.4

where the subscript "e" refers to the exterior flow and the subscript "w" refers to the wall. Although equations (2.1-2.3) are the transient boundary layer equations, the steady-state equations are obtained by dropping time dependent terms and time dependence of the dependent variables.

The boundary conditions necessary to obtain a unique solution of these boundary layer equations are the tangential and normal velocity components at the wall and either the exterior velocity or pressure distribution. The two wall velocity components are assumed to be

\( u(x,0,t) = 0 \quad v(x,0) = 0 \) \hspace{2cm} 2.5

which corresponds to conditions of no slip and no suction or blowing. The exterior velocity is given by

\( U_e(x) = 1 - ax \quad a = 0,1 \) \hspace{2cm} 2.6
which corresponds to a flat plate in parallel flow at zero incidence if "a" is zero and to either a flat plate in a channel with diverging walls or a negatively inclined plate immersed in an infinite parallel flow if "a" is unity. Throughout this work these two flow geometries will be denoted as the zero and adverse pressure gradient flows, respectively. By using the assumption that the normal direction pressure variation is negligible,

\[ p(x,y) = p_e(x) \]  

the Euler equation is used to relate the pressure gradient term in the momentum equation to the exterior velocity distribution.

The physical plane equations, (2.1 and 2.2), require starting profiles, but these equations are singular at the leading edge; thus starting profiles must be approximated at this point. The approximated initial profiles are given by

\[ u(0,y,t) = 1.0 \quad 0 \leq y \] \[ v(0,y) = 0.0 \]  

If the flow region at and slightly downstream of the leading edge corresponds to a similar type flow geometry, the similar equations can be used to obtain "exact" starting profiles at some finite distance from the leading edge, this distance being within the similar flow geometry region.

2. Göertler Transformation. Göertler (15) developed a transformation of the physical plane boundary layer
equations, (2.1-2.3), which is valid for all two-dimensional flows with an arbitrary exterior pressure or velocity distribution. The new independent variables introduced by Göertler are:

$$\alpha = \int_0^x U_e(x)dx$$  

$$\beta = yU_e(x)[2\int_0^x U_e(x)dx]^\frac{1}{2}.$$  

Introducing these variables into the boundary layer equations, (2.1-2.4), along with a stream function of the form

$$\psi(x,y) = [2\int_0^x U_e(x)dx]^{\frac{1}{2}}h(\alpha,\beta)$$  

we are led to the following nonlinear, third-order, partial differential equation for $h(\alpha,\beta)$

$$h_{\beta\beta\alpha} + h_{\beta\alpha} + \lambda(\alpha)[1-(h_\beta)^2] = 2\alpha(h_\beta h_{\beta\alpha} - h_{\beta\beta} h_\alpha)$$  

where the "principal function", $\lambda(\alpha)$, is defined by

$$\lambda(\alpha) = 2\alpha \frac{d\ln U_e(\alpha)}{d\alpha}.$$  

The wall shear relationship is now given by

$$\tau_w = \frac{U_e^2(\alpha)h_{\beta\beta}}{\sqrt{2\alpha}}$$  

The boundary conditions for (2.11) corresponding to those given by equations (2.5) and (2.6) are

$$\beta = 0 : h = 0 \quad h_\beta = 0$$

$$\beta = \infty : h_\beta = 1.$$
The relations between the dependent and independent variables of the two coordinate systems are

\[ U_e(\alpha) = (1 - 2a\alpha)^{\frac{1}{2}} \]

\[ u(x,y) = U_e(x) h_\beta \]

\[ v(x,y) = U_e(x) (2\alpha)^{-\frac{1}{2}} [h + 2a\alpha \alpha - (1 + \lambda(\alpha)) \beta h_\beta] \quad 2.15 \]

\[ y = (2\alpha)^{\frac{1}{2}} \beta (1 - 2a\alpha)^{-\frac{1}{2}} \]

\[ x = 1 - (1 - 2a\alpha)^{\frac{1}{2}}. \]

These relations can be used to transform the results back to the physical plane. At the leading edge, the normal velocity component is infinite corresponding to the singularity present in the physical plane equations, and the normal distance is zero corresponding to zero boundary layer thickness.

The new coordinates are of similar type in that the nonlinear partial differential equation, (2.11), reduces to a nonlinear ordinary differential equation when "a" or "\(a\)" is zero

\[ h_{\beta \beta} + hh_{\beta \beta} = 0 \quad 2.16 \]

with the boundary conditions given by (2.14). This equation is the similar equation which is one advantage of this system for numerical methods. The similar equation allows one to obtain "exact" starting profiles at the leading edge regardless of the flow geometry. If the initial portion of the flow geometry is similar, this equation can be used to obtain the
starting profiles at the "leading edge" of the nonsimilar region thereby eliminating the calculations for this initial similar portion.

B. Compressible Flow

The flow of a compressible viscous, heat-conducting fluid is mathematically described by the continuity, Navier-Stokes, and energy equations plus a heat conductivity law, a viscosity law, and an equation of state for the fluid. The upstream flow conditions, which are independent of the body shape and the coordinate system, and the stagnation state of the fluid must also be defined.

The following quantities are necessary and sufficient to describe the upstream flows considered in this work:

a. Pr, the Prandtl number describes the thermal properties of the fluid.

b. \( \gamma \), the ratio of the specific heats, describes the type of fluid.

c. \( S^* \) describes the temperature dependence of the fluid's viscosity since Sutherland's viscosity law is used.

d. \( T^* \) describes the thermal state of the fluid.

e. \( M_\infty \) describes the dynamic state of the fluid.

Using isentropic, perfect gas relationships (35), other necessary upstream quantities can be determined from these known quantities such as

\[
u_\infty = M_\infty \left[ 1 + \frac{\gamma-1}{2} M_\infty^2 \right]^{-\frac{1}{2}}\]
Upstream Pressure: \[ p_\infty = \left[ \gamma (1 + \frac{\gamma-1}{2} M_\infty^2) \right]^{\gamma/\gamma-1} \] 2.17

Stagnation Temperature: \[ T_0 = T_\infty (1 + \frac{\gamma-1}{2} M_\infty^2). \]

These relations are written in dimensionless form but they are also valid for dimensional quantities.

The dimensionless stagnation state of the fluid is defined by

\[ p_o = \gamma^{-1} \quad \rho_o = 1 \quad \psi_o = (\gamma-1)^{-1} \quad \mu_o = 1 \] 2.18

where \( \rho_o \) is chosen arbitrarily.

1. Physical Plane. If the Reynolds number is large, the Navier-Stokes equations can be approximated by the classical two-dimensional boundary layer equations and the Euler equation. Assuming the Prandtl number and the specific heats are constant, the complete set of equations describing the compressible boundary layer and exterior flow are:

Continuity: \[ (\rho u)_x + (\rho v)_y = 0 \] 2.19

Momentum:

\[ \rho uu_x + \rho v u_y = -\frac{dp}{dx} + f' u i_y + f u_y y \] 2.20

Energy:

\[ \rho u i_x + \rho u i_y = u \frac{dp}{dx} + f(u_y)^2 \]

\[ + \frac{f}{Pr} i_{yy} + \frac{f'}{Pr} (i_y)^2 \] 2.21

Euler Equation:

\[ \frac{dp_e}{dx} = -\rho_e U_e(x) \frac{dU_e(x)}{dx} \] 2.22

\[ U_e(x) = U_e(x=0) - ax \]
Viscosity Law: \[ f = \mu = \frac{1+S/T_o}{i/i_o + S/T_o} (i/i_o)^{3/2} \] 2.23

\[ f' = \frac{du}{d\bar{i}} = \frac{f}{2i} \frac{(i/i_o + 3S/T_o)}{(i/i_o + S/T_o)} \]

Equation of State: \[ \rho = \rho_e \frac{i}{i_o} = \gamma \rho / (\gamma - 1) i \] 2.24

Wall Shear: \[ \tau_w = fu_y \bigg|_w \] 2.25

Sutherland's viscosity law, (2.23), is used throughout this work, and the constant, S, has been determined experimentally (39).

These equations are also singular at the leading edge thus, starting profiles at this point must be approximated. The starting profiles used in this work are:

\[ u(0,y) = U_e(0) \]

\[ v(0,y) = 0.0 \quad 0 \equiv y \] 2.26

\[ i(0,y) = i_e(0) \]

If the flow is similar at and slightly downstream of the leading edge, the compressible similar equations can be used to obtain "exact" starting profiles at some finite distance downstream of the leading edge, this distance being within the similar region.

2. Howarth-Dorodnitsyn Transformation. For some
numerical calculations, it is advantageous to stretch the coordinate normal to the wall. This is accomplished by defining the new coordinate system (6)

\[ \xi = x \]  
\[ \eta = \int_0^y \rho \, dy \]  

and a new velocity

\[ V = (d\eta/dx)u + \rho v. \]  

This transformation is derived in Appendix A. The steady state HD equations analogous to equations (2.19-2.25) are

Continuity: \[ u_\xi + V_\eta = 0 \]  

Momentum: \[ uu_\xi + Vu_\eta = -(1/\rho)(dp_e/d_\xi) \]

\[ + Fu_\eta + F'\eta i_\eta u_\eta \]  

Energy: \[ ui_\xi + Vi_\eta = (u/\rho)(dp_e/d_\xi) + F(u_\eta)^2 \]

\[ + (F/Pr)i_\eta i_\eta + (F'/Pr)(i_\eta)^2 \]  

Euler Equation: \[-(dp_e/dx) = \rho_e U_e(x)(dU_e(x)/dx) \]  

Viscosity Law: \[ F = \rho \mu = \frac{\rho_e i_e}{i_o} \frac{(1 + S/T_o)}{(S/T_o + i/i_o)} \sqrt{1/i_o} \]  

\[ F' = \frac{F}{2i} \frac{(S/T_o - i/i_o)}{(S/T_o + i/i_o)} \]
Equation of State: $\rho = \rho_e i_e / i = \gamma \rho (\gamma - 1) i$

Wall Shear: $\tau_w = F_w u |_{w}$.

The boundary conditions, (2.25), become

$$\eta = 0 : u = 0, \ v = 0, \ i = i_w (\xi) \ \text{or} \ i_{\eta} |_{w} = 0$$

$$\eta = \omega : U_e (\xi) = U_e (\xi = 0) - a \xi, \ i_e (\xi) = i_o - \frac{1}{2} U_e^2 (\xi), \ a = 0, 1.$$ 

The solutions can be transformed back to the physical plane by the two relations:

$$y = \int_{0}^{\eta} \left( \frac{i}{\rho e i_e} \right) d\eta$$

$$v = \rho^{-1} (v - \eta_x u).$$

This system of equations is also singular at the leading edge so starting profiles must be obtained by procedures similar to those prescribed for the physical plane equations. The second disadvantage of this system is the complexity of the back transformation given by (2.38). The normal coordinate transformation can be performed quite accurately by integral methods, but the normal velocity component transformation is rather involved and of limited accuracy. The term $\eta_x$ must be evaluated by an approximation of the form

$$\eta(x + \Delta x, y) - \eta(x, y) \over \Delta x$$

which will require a trial and error solution to obtain the two values of $\eta$ at the same value of $y$. Also, the error of
this difference analogue is $O(\Delta x)$ which will require small tangential step sizes to ensure accuracy.

The advantages of this transformed system of equations can be accredited to the stretching of the boundary layer, the definition of the normal coordinate, and the density. Since the boundary layer is stretched, the magnitude of derivatives will be reduced so that the finite difference approximations for these derivatives should be more exact. The definition of the normal coordinate reduces the truncation error since

$$
\eta_e \approx \rho_e y_e \quad \rho << 1
$$

Therefore, if the same number of points is to be used in each plane, $\Delta \eta << \Delta y$. The truncation error is governed by the step size so the truncation error of the HD equations would be order of $O(\rho^2)$ the truncation error of the physical plane equations. The continuity equation, (2.30), is not an explicit function of the density which allows one to use finite difference analogues valid for the incompressible continuity equation. The relative position of the density in the momentum and energy equations of the physical and HD planes will reduce the round-off error in the latter plane. The friction force term and the heat conduction term are both effectively divided by the density in the physical plane while they are multiplied by the density in the HD plane. Since the density is normally small, the latter should be more accurate. This will be discussed more fully in Results and Discussion.
3. Görtler-Howarth-Dorodnitsyn Transformation. Using coordinates similar to those defined by Görtler (15),

\[ \alpha = \int_{0}^{x} U_e(x) \, dx \]  

2.41

\[ \beta = \frac{yU_e(x)}{\left[2F_{0}^{x} U_e(x) \, dx\right]^{\frac{1}{2}}} \]  

2.42

along with the stream function

\[ \psi(x, y) = \left[2F_{0}^{x} U_e(x) \, dx\right]^{\frac{1}{2}} h(\alpha, \beta) \]  

2.43

and the "principal function"

\[ \lambda(\alpha) = 2[U_e(x)]^{-2} (dU_e(x)/dx), \]  

2.44

the HD equations (2.30-2.31) are transformed to two nonlinear partial differential equations

**Momentum:**

\[ h_{BB} + \left[(F'/F)i_{B} + h\right]h_{BB} + \lambda(\alpha)[(i/i_e) - (h_{B})^{2}] = 2\alpha[h_{B}h_{B\alpha} - h_{BB}h_{\alpha}] \]  

2.45

**Energy:**

\[ i_{BB} + [(F'/F)i_{B} + Pr(h + 2\alpha h_{\alpha})]i_{B} + PrU_e^2(\alpha)[(h_{BB})^2 - \lambda(\alpha)(i/i_e)h_{B}] = 2\alpha Pr h_{B}i_{B\alpha}. \]  

2.46

The boundary conditions, analogous to those given by (2.37), are

<table>
<thead>
<tr>
<th>( \beta = 0 )</th>
<th>( h = 0, h_{B} = 0, i = i_{w}(\alpha) ) or ( i_{B} = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = \infty )</td>
<td>( h = 1, i = i_{e}(\alpha) ).</td>
</tr>
</tbody>
</table>
in this coordinate system.

The viscosity law (2.34) and the equation of state (2.35) are not altered by the transformation but the exterior velocity and enthalpy are now given by

\[ U_e(\alpha) = [U_e^2(\alpha=0) - 2a\alpha]^{\frac{1}{2}} \]  
2.48

\[ i_e(\alpha) = i_o - \frac{1}{2} U_e^2(\alpha=0) + a\alpha \]  
2.49

and the wall shear is given by

\[ \tau_w = (F_w/2\alpha)^{\frac{1}{2}} U_e^2(\alpha) f_{BB} \big|_w \]  
2.50

When "a" or "\alpha" is zero, (2.45) and (2.46) reduce to the similar equations for compressible boundary layer flow

\[ h_{BB} + [(F'/F)i_B + h]h_{BB} = 0 \]  
2.51

\[ i_{BB} + [(F'/F)i_B + Prh]i_B + PrU_e^2(\alpha)(h_{BB})^2 = 0 \]  
2.52

which can be used to obtain "exact" starting profiles. The boundary conditions associated with (2.51) and (2.52) are given by (2.47).

The back transformation of the results to the HD plane is accomplished by the following relations between the variables of the two coordinate systems:

\[ U_e(x) = U_e(\alpha) \]

\[ u(x,y) = U_e(\alpha) h_B \]

\[ v(x,y) = U_e(\alpha) (F/2\alpha)^{\frac{1}{2}} \{ h + 2\alpha h_B + \lambda \bar{h}_B - \bar{h}_B \} \]
\[ i(x,y) = i(\alpha, \beta) \]

\[ x = \{U_e(\alpha=0) - \left[U_e^2(\alpha=0) - 2\alpha \right]^{\frac{1}{2}}\}^{-1} \]

\[ y = \left[2F \int_{0}^{x} U_e(x) dx\right]^{\frac{1}{2}} \frac{d}{dx} U_e^{-1}(x). \]

The system has the advantages of:

a. being nonsingular at the leading edge, and

b. reducing to the similar equations for similar type flows, thereby eliminating the large number of calculations required by the physical and HD equations to obtain the solution in this region.

These advantages plus other advantages and disadvantages associated with the numerical solution are discussed in Results and Discussion.

4. Exterior Flow. The exterior flow variables at any position \( x^{(1)} \) can be determined if the exterior velocity or pressure distribution along the boundary layer is known. In this work, exterior velocity distributions were known so the exterior variable equations are defined in terms of this distribution. Using isentropic perfect gas theory and equations (2.16) and (2.17) the following are obtained:

\[ p_e(x) = p_\infty \left[1 - (\gamma - 1) \frac{M_e^2}{2} \right] \left[1 - \left(U_e(x)/U_\infty \right)^2 \right] \]

\[ i_e(x) = i_\infty - \frac{1}{2} U_e^2(x) \]

\[ i(x,y) = i(\alpha, \beta) \]

Except for the enthalpy relation, "x" is considered a dummy variable and can be replaced by "s" or "\( \alpha \)".
\[ \rho_e(x) = \left( \frac{i_e(x)}{i_o} \right)^i \]

\[ T_e(x) = T_o \left[ 1 + (\gamma - 1) \frac{M_e^2(x)}{2} \right]^{-1} \]

\[ \mu_e(x) = \frac{1 + S/T_o}{\left( \frac{i_e}{i_o} + S/T_o \right)} \left( \frac{i_e}{i_o} \right)^{3/2} \]

\[ M_e^2(x) = \frac{u_e^2(x)}{(\gamma - 1)i_e(x)} \]

The contribution of the exterior normal velocity component has been neglected in equation (2.55) which is consistent with the boundary layer and Euler equations. If the following conditions are satisfied,

a. adiabatic wall, \( i_y|_w = 0 \)

b. \( Pr = 1 \)

c. \( dp_e/dx = -\rho_e U_e(x)/dx = \rho_e d i_e /dx \)

the adiabatic wall temperature is related to the exterior flow conditions by Busemann's relation (5)

\[ i_{ad}(x) = i_e(x) \left[ 1 + \frac{\gamma - 1}{2} M_e^2(x) \right] \]

If the \( Pr \neq 1 \), the above equation can be approximated by inserting a "recovery factor", \( r \),

\[ i_{ad}(x) = i_e(x) \left[ 1 + \frac{\gamma - 1}{2} r M_e^2(x) \right] \]

Emmons and Brainerd (9) found, by experimentation, that equation (2.61) is a very good approximation if

\[ r = Pr^{1/2} \]
Buseman's relation with Prandtl number equal unity will be used to determine the accuracy of the adiabatic wall condition's finite difference approximation. Then the adiabatic wall condition can be used to check the accuracy of equation (2.61) with the approximation (2.62).
III. FINITE DIFFERENCE ANALOGUES OF THE BOUNDARY LAYER EQUATIONS

The boundary layer equations are parabolic nonlinear partial differential equations. Some numerical solutions are obtained by using iterative methods to solve the nonlinear equations while other solutions are obtained by solving the linearized form of the original equations. For finite difference methods, these linearized equations are obtained by either of two methods, both involving the variational technique. The first method applies the variational technique to the nonlinear partial differential equations and the linear difference equations are obtained from the resulting linear partial differential equation. The second method involves writing nonlinear difference equations which are linearized by the variational technique. In both cases, the variational technique involved replacing the dependent variables $u,v,i$, etc., by $u+\bar{u}$, $v+\bar{v}$, $i+\bar{i}$, etc., where $\bar{u},\bar{v},\bar{i}$, etc., are small quantities of first order. All second and higher order quantities are dropped from the resulting equation leaving a linear equation with zero order quantities as coefficients. The linearization procedure used by Baxter and Flügge-Lotz (1) should not be used since their resulting linear difference equation is unstable contrary to their results.

For finite difference solutions, the flow region is divided into a fine grid or mesh and the solutions are obtained at the grid intersections. The parabolic nature
of the boundary layer equations makes this an initial value problem which allows one to solve the finite difference equations along a downstream column of the grid if the solutions are known along the previous (upstream) column. Therefore, the solutions can be propagated downstream until the entire boundary layer flow field is determined.

There are two general classifications of finite difference analogues, explicit and implicit and several variations within each classification. Explicit analogues are written such that the result at a downstream grid intersection is given in terms of a certain small number of upstream values. Implicit analogues, on the other hand, are written such that the result at a downstream grid intersection is given in terms of a certain small number of its neighboring downstream values and known upstream values. These latter methods require that the results along an entire grid column be determined simultaneously by solving a system of algebraic equations which is most often accomplished by the Crout-Banachiewicz algorithm as given by Richtmyer (32a) and Hildebrand (21a).

The finite difference equation's solution truly approximates the solution of the partial differential equation it represents, if the finite difference equation and its solution are stable, consistent with and convergent to the partial differential equation and its solution. Let \( u(x,y,t) \) denote the exact solution of the partial differential equation and let \( u(k\Delta x, j\Delta y, n\Delta t) \) represent the finite difference equation's
solution. The error, $E$, of the difference approximation is defined as $u(x,y,t) - u(k\Delta x, \ell \Delta y, j\Delta t)$. The difference equation is stable if the following two conditions are satisfied:

a. $E$ remains bounded as $j \to \infty$.

b. $E$ remains bounded as the mesh is refined (i.e., as $\Delta x, \Delta y, \Delta t \to 0$) for a fixed value of $j\Delta t$.

Both (a) and (b) must be satisfied for arbitrary initial data and all proper boundary conditions, i.e., for a properly posed initial value problem. For certain classes of problems, the equations are consistent if the truncation error of the difference equation approaches zero as the step sizes approach zero. Convergence requires the difference equation's solution to approach the exact solution of the partial differential equation as the step sizes approach zero. This work assumes the stability, consistency, and the convergence of the linearized equation also applies to the nonlinear equation.

Stability can be proven by two methods for certain classes of problems. These are the maximum principle (21a) and the von Neumann condition (32b). The first method yields sufficient conditions while the second method yields necessary and sufficient conditions. The sufficient conditions are generally more restrictive to the programmer but for some difference equations the criteria given by the first method are identical to those given by the von Neumann condition, i.e., they are necessary and sufficient. Richtmyer (32c) also presents several criteria derivable from the von Neumann
condition which are only sufficient, but in many cases they are much easier to arrive at.

Consistency is proven in this work by showing that the truncation error approaches zero as the step sizes approach zero. Convergence is then assumed based on Lax's Equivalence Theorem (32d):

"Given a properly posed initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence."

The stability analyses for some of the difference equations are presented in Appendix C. Flügge-Lotz and Blottner (10) considered a very simplified form of the compressible boundary layer equations and proved stability. The stability analyses for the complete system of equations is considered in this work (Appendix C) because it was felt that the simplified form of the equations as considered by Flügge-Lotz and Blottner was inadequate. It was not possible to guarantee stability due to the complicated form of the stability criteria.

Previous authors have considered one or two finite difference analogues and either the incompressible or compressible boundary layer equations in one of the several possible coordinate systems. Since the object of this work is to analyze finite difference analogues of the incompressible and compressible boundary layer equations and determine the best analogue and coordinate system, several possible analogues
and coordinate systems must be considered for both types of flows. Due to the similarity of the incompressible and compressible flow equations, the results of the incompressible equations can be used to eliminate some of the possible analogues for the compressible equations. Some of the coordinate systems could be eliminated by theoretical reasoning while others can be eliminated only by numerical evaluation. This chapter presents a few of the difference analogues that were evaluated numerically. The advantages and disadvantages of these methods plus several that are not included in this chapter are presented in Results and Discussion.

The grid of the flow field is divided such that

\[(j-1)\Delta t = t, \ (i-1)\Delta x = x, \ (k-1)\Delta y = y, \ j,k,\ell = 1,2,...\]

where \(x\) and \(y\) are dummy variables representing the tangential and normal directions, respectively, of any of the coordinate systems. When \(j = \ell = k = 1\), this corresponds to zero time, leading edge of the plate, and the wall, respectively. The equations will not be written in the standard difference form, but instead, they will be written as difference equations of derivatives where the difference analogues of the normal direction derivatives are the standard central difference forms which are given in Appendix B. All of the time derivatives and tangential direction space derivatives are differenced.

There are two types of errors associated with finite difference methods, namely, round-off error and truncation error. Round-off error is due to the finite word length of
the computer and can sometimes be reduced by rearranging
the numerical solution procedure or by redefining the
dependent variables. Truncation error is caused by truncating
the Taylor series expansions used to approximate the partial
differentials and it can be reduced by reducing the step size.
Truncation errors are bounded if the difference equation is
stable, consistent, and convergent. For explicit methods
and the ordinary implicit method, the truncation error of
the finite difference equations are order of \([O(\Delta t), O(\Delta x),
O(\Delta y^2)]\) where the first term is absent for steady-state
methods. The Crank-Nicolson implicit analogue has the
truncation error, order of \([O(\Delta t^2), O(\Delta x^2), O(\Delta y^2)]\), which
is considered to be the advantage of this method. The
Crank-Nicolson analogue is "centered", the ordinary implicit
analogue is "forward", and the explicit analogue is "backward";
this terminology stemming from the point about which the
Taylor series expansion is written.

A. Steady-State Incompressible Flow

1. Physical Plane Analogues. Three explicit and two
implicit difference analogues of the steady-state form of
equation (2.2) are considered numerically. Of these, the
ordinary explicit, ordinary implicit, and Crank-Nicolson
implicit analogues can be expressed jointly by the single
equation

\[
\frac{\Delta x}{u_{t+1,k}} \frac{\Delta x}{u_{t,k}} \frac{\Delta x}{u_{t+1,k}^{yy}} - \frac{\Delta x}{u_{t,k}^{yy}}
\]
\[ u_{\ell,k} = \frac{\Delta x (dp_e/dx)_{\ell+1}}{\rho_{\ell+1,k} u_{\ell,k}} - \frac{\Delta x v_{\ell,k} (1-\varrho) (u_y)}{u_{\ell,k}} \]
\[ + \frac{\Delta x (1-\varrho) (u_{yy})}{u_{\ell,k}}. \]

The constant $\varrho$ can be assigned any value between zero and unity but three values, 0, $\frac{1}{2}$, and 1, which correspond to the ordinary explicit, Crank-Nicolson implicit, and the ordinary implicit analogues, respectively, are to be used in this work. The ordinary implicit (4) and the Crank-Nicolson implicit (10) analogues are stable for all step sizes while the ordinary explicit (46) has two stability criteria
\[ \Delta x \equiv u_{\ell,k} \Delta y^2 / 2 \quad \Delta y \equiv 2 / v_{\ell,k}. \]

The second explicit analogue considered is identical in form to the ordinary explicit analogue, $\varrho = 0$, except the u-component of the velocity is taken implicitly. This "explicit-implicit" analogue of equation (2.2) is
\[ u_{\ell+1,k} + \frac{\Delta x (dp_e/dx)_{\ell+1}}{\rho_{\ell+1,k} u_{\ell+1,k}} - A_{\ell,k} = 0 \]
\[ 3.4 \]
where $A_{\ell,k}$ represents the right side of equation (3.2) with the exception of the pressure gradient term. Multiplying (3.4) by $u_{\ell+1,k}$ results in a quadratic equation the solution of which is
The Fourier-type stability analysis cannot be applied to this difference analogue but by assuming the stability criteria, (3.3), are valid for this equation, one can successfully argue that the system is stable for the exterior velocity distribution given by equation (2.6).

The third explicit analogue considered numerically was the Dufort-Frankel method (8). This analogue, although explicit in form, was initially applied to the heat equation and found to be always stable. The results obtained from the heat equation have indicated two disadvantages, the first being that it requires two grid columns of starting profiles, \((j)\)th and \((j-1)\)th, since this is a tri-level analogue; the second being that the conditions for convergence are very restrictive (7). These same two disadvantages are true for the boundary layer equation's analogues. The first disadvantage is not serious since the second initial profile can be obtained by using any of the other explicit or implicit analogues. The second disadvantage is due to the form of the truncation error of the difference equation which is \(0(\Delta x/\Delta y)^2u_{xx}\). If the convergence criteria given by Richtmyer (32e) and Forsythe and Wasow (13)

\[
\Delta x = 0(\Delta y) \quad \text{or} \quad \lim_{\Delta x \to 0} \left( \frac{\Delta x}{\Delta y} \right) = 0
\]

is true for the system, convergence is guaranteed but if
\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y} = 0 \quad \text{(3.7)}

is true, the solution converges to that of a hyperbolic equation of the form

\[ s^2 u_{xx} + uu_x + vu_y = -\frac{1}{\rho} \frac{dp_e}{dx} + u_{yy} \quad \text{(3.8)} \]

To satisfy (3.6), one must let \( \Delta x/\Delta y^2 \) equal some constant, independent of the step sizes and then allow \( \Delta x \) to approach zero. As \( \Delta x \) approaches zero, \( \Delta y \) will tend toward infinity which is a serious disadvantage. Therefore, convergence of the Dufort-Frankel method must be determined by examining numerical solutions. The Dufort-Frankel explicit difference analogue of equation (2.2) is

\[ u_{j+1,k} + \frac{2\Delta x}{u_{j,k+1} \Delta y^2} = u_{j,k+1} + \frac{2\Delta x}{u_{j,k-1} \Delta y^2} - \frac{v_{j,k} \Delta x}{u_{j,k} \Delta y} + u_{j-1,k} + \frac{2\Delta x}{u_{j,k} \Delta y^2} \]

\[ + \frac{2\Delta x}{u_{j,k} \Delta y^2} - \frac{v_{j,k} \Delta x}{u_{j,k} \Delta y} - \frac{2\Delta x (dp_e/dx)_{j+1}}{\rho_{j+1,k} u_{j,k}} \quad \text{(3.9)} \]

2. Görtler Transformed Plane Analogues. The solution of the boundary layer equations in this transformed plane is obtained by solving the similar equation for the starting profiles which in turn are used to initiate propagation of the nonlinear momentum equation's solution downstream. If the initial flow geometry is similar, the nonsimilar momentum
equation's solution is initiated at the "leading edge" of the nonsimilar flow geometry. Iteration is to be used and the following notation will designate the three iteration levels involved:

i) The unprimed quantities will denote the known values at the \((i)\)th (upstream) grid column.

ii) The single primed quantities will denote the last iteration level along the \((i+1)\)th grid column.

iii) The double primed quantities will denote the next iteration level along the \((i+1)\)th grid column, i.e., the unknown quantities.

Also, since iteration is to be used, the coefficients are to be handled differently than for noniterative problems. The difference equation is to approximate as closely as possible the differential equation it represents as the iteration criteria is satisfied at each grid column. To accomplish this, the coefficients are taken at the downstream level in some cases which is not done for noniterative problems, i.e., the equation is not linearized.

a. Similar Equation. The solution of the similar equation can be obtained by series approximation (3), by numerical integration (27), or by finite difference methods. The latter will be used since this is a boundary value problem and finite differences are boundary value methods. Also, finite differences are to be used for the nonsimilar equation's solution
so the two solutions will be "consistent".

The finite difference analogue of (2.16) is

\[ (h''_{SS})_k + h'(h''_{SS})_k = 0. \]

Initially, some profile must be assumed so several different profiles were used. The final solution was essentially independent of these profiles and a linear profile of the form

\[ h = (k-1)\Delta \delta \quad k = 1, 2, \ldots, K \]

is to be used.

b. Nonsimilar Equation. The ordinary implicit analogue of the nonsimilar momentum equation (2.11) is

\[
(h''_{SS})_{t+1,k} + h'(h''_{SS})_{t+1,k} + \alpha(\alpha)_{t+1,k} \left[ (h'_{SS})_{t+1,k} (h''_{SS})_{t+1,k} \right]
\]

\[ = 2\alpha (h'_{SS})_{t+1,k} \left[ \frac{(h''_{SS})_{t+1,k} - (h''_{SS})_{t,k}}{\Delta \alpha} \right] - (h'_{SS})_{t+1,k} \left[ \frac{h''_{t+1,k} - h''_{t,k}}{\Delta \alpha} \right]
\]

This equation reduces to (3.10) when "a" or "c" is zero so the system is self-starting and (3.10) is not required.

The Crank-Nicolson implicit analogue of (2.11) is

\[
(h''_{SS})_{t+1,k} + h'(h''_{SS})_{t+1,k} - \alpha(\alpha)_{t+1,k} (h'_{SS})_{t+1,k} (h''_{SS})_{t+1,k}
\]

\[ - \frac{2}{\Delta \alpha} \left[ \alpha (h'_{SS})_{t+1,k} + \alpha (h'_{SS})_{t,k} \right] (h''_{SS})_{t+1,k}
\]
\[ + \frac{2}{\Delta \alpha} [\alpha \chi_{l+1}(h_{\beta})_{l+1,k} + \alpha \chi(h_{\beta})_{l,k}]h_{l+1,k} \]

\[ = -[\lambda(\alpha)_{l+1} + \lambda(\alpha)_{l}] + \lambda(\alpha)_{l}(h_{\beta})_{l,k} - (h_{\beta})_{l,k} - h_{l,k}(h_{\beta})_{l,k} \]

\[ - \frac{2}{\Delta \alpha} [\alpha \chi_{l+1}(h_{\beta})_{l+1,k} + \alpha \chi(h_{\beta})_{l,k}]h_{l,k} \]

\[ + \frac{2}{\Delta \alpha} [\alpha \chi(h_{\beta})_{l,k} + \alpha \chi_{l+1}(h_{\beta})_{l+1,k}]h_{l,k}. \]

3.13

This equation does not reduce to (3.10) when either "a" or "\alpha" is zero so (3.10) must be used to obtain the initial profiles.

For the two difference equations, (3.12) and (3.13), the wall shear is given by

\[ \tau_w = \frac{2U_e(\alpha)h_2}{(\Delta \beta)^2(2\alpha)^\beta} \]

3.14

which is the central difference form of (2.13) with the proper boundary conditions (2.14). This relation is also used to calculate the shear of similar flows if "\alpha" is non-zero. When "\alpha" is zero, the shear is infinite corresponding to the singularity of the physical plane equations.

B. Transient Incompressible Flow

The transient problem corresponds to the plate at rest and at time zero, the plate is started impulsively (34). The exterior flow is assumed to be fully developed and independent
of time. Initially, the boundary layer thickness is assumed to be zero and the development of the boundary layer is desired. Two different methods are to be used to calculate the boundary layer development. The first assumes the values at the \((i)\)th level are at steady-state while the flow condition at the \((i+1)\)th level corresponds to potential flow (time zero). The transient finite difference equations are solved successively at the \((i+1)\)th level until steady-state is attained. The procedure is then repeated at the \((i+2)\)th level and etc., until the solution is propagated over the desired flow region. This method is called "pseudo-transient". The "true transient" boundary layer development will be obtained by advancing the entire flow region one time level and then repeating the procedure until steady-state is attained.

The difference analogues of equation (2.2) for the two transient flow methods are nearly identical, thus the difference equations will be presented for the "true transient" and then the modifications necessary to obtain the "pseudo-transient" will be given.

One explicit and two implicit analogues of (2.2) can be represented by the general difference equation

\[
 u_{j+1, k+1}^j + \frac{\Delta t}{\Delta x} u_{j+1, k+1}^j (u_{j+1, k}^j - u_{j, k}^j) + \frac{\Delta t}{\Delta y} (u_{y, j+1, k}^j - u_{y, j, k}^j) = u_{j+1, k}^j - \frac{\Delta t}{\rho} \left( \frac{d \rho}{dx} \right)_{j+1, k} + (1-\vartheta) \Delta t (u_{y, j+1, k}^j - u_{y, j, k}^j) + (1-\vartheta) \Delta t (u_{yy, j+1, k}^j - u_{yy, j, k}^j)
\]

\[
= u_{j+1, k}^j - \frac{\Delta t}{\rho} \left( \frac{d \rho}{dx} \right)_{j+1, k} + (1-\vartheta) \Delta t (u_{y, j+1, k}^j - u_{y, j, k}^j) + (1-\vartheta) \Delta t (u_{yy, j+1, k}^j - u_{yy, j, k}^j)
\]

\[\quad - (1-\vartheta) \frac{\Delta t}{\Delta x} (u_{j+1, k}^j - u_{j, k}^j).\]  
3.15
The constant $\theta$ can be assigned any value between zero and unity but only three values, 0, $\frac{1}{2}$, and 1, will be considered. Choosing $\theta = 0$, (3.15) corresponds to the "central difference" explicit analogue; the name arising from the difference analogue which approximates $u_y$, see Appendix B. The central difference equation has two stability criteria

$$\Delta t \leq \left( \frac{u^j_{2+1,k} + \frac{2}{\Delta y^2}}{\Delta x} \right)^{-1} \quad \Delta y \leq \frac{2}{v^j_{2+1,k}}.$$  

When $\theta = \frac{1}{2}$, (3.15) reduces to the Crank-Nicolson implicit analogue; when $\theta = 1$, the ordinary implicit analogue is obtained, the latter two being stable for all step sizes.

The second explicit analogue, valid for positive normal velocity components only, is identical to that given by (3.15) except the difference analogue used to approximate $u_y$ is given by

$$(u_y^j)_{2+1,k}^j = \frac{u^j_{2+1,k} - u^j_{2+1,k-1}}{\Delta y}.$$  

This "backward difference" equation has been shown to be stable (13) if

$$\Delta t \leq \left( \frac{u^j_{2+1,k} + \frac{v^j_{2+1,k}}{\Delta y} + \frac{2}{\Delta y^2}}{\Delta x} \right)^{-1}.$$  

Comparing (3.18) and (3.16), it is apparent that (3.16) will allow larger time steps but on the other hand, the normal direction step size is limited. Therefore, the magnitude of
the normal velocity component will determine which of the 
explicit equations can be used for each particular problem.

The "pseudo-transient" analogues of (2.2) are obtained 
by replacing all dependent variables of (3.15) evaluated at 
the (i)th level with identical dependent variables which 
are independent of time, i.e., $u^j_{l+1,k}$ becomes $u^\infty_{l+1,k}$, $(u_y)^{j+1}_{l,k}$ becomes $(u_y)^\infty_{l,k}$, and etc., where "\$\infty\$" designates steady-
state. The stability criteria of the "true transient"
difference equations remain valid for the "psuedo-transient"
difference equations.

As the solution approaches steady-state,

$$u^j_{l+1,k} \text{ approaches } u^j_{l+1,k} \quad 3.19$$

and the finite difference equations (3.15) should reduce
to the steady-state finite difference equations (3.2). This 
is true of (3.54) except when $\delta = \frac{1}{2}$ (Crank-Nicolson analogue).
Therefore, the proper Crank-Nicolson implicit difference 
equation is

$$u^j_{l+1,k} + \frac{\Delta t}{2 \Delta x} u^j_{l+1,k} \left[ (u^j_{l+1,k} - u^j_{l,k}) + (u^j_{l+1,k} - u^j_{l,k}) \right]$$

$$+ \frac{\Delta t}{4} (u_y)^j_{l+1,k} \left[ (u_y)^j_{l+1,k} + (u_y)^j_{l+1,k} + (u_y)^j_{l,k} + (u_y)^j_{l,k} \right]$$

$$- \frac{\Delta t}{4} \left[ (u_y)^j_{l+1,k} + (u_y)^j_{l+1,k} + (u_y)^j_{l,k} + (u_y)^j_{l,k} \right]$$

$$= u^j_{l+1,k} - \frac{\Delta t (dp_e/dx)_{l+1}}{\rho_{l+1,k}}. \quad 3.20$$
As steady-state is approached, (3.20) reduces to (3.2) with \( \theta = \frac{1}{2} \).

The physical plane wall shear (2.4) is determined by the difference equation

\[ \tau_w = u_2 (\Delta y)^{-1} \]

with \( u_2 \) representing the tangential velocity component at the first point above the wall.

This completes the finite difference analogues of the steady-state and transient equations of incompressible flow. The difference analogues for the continuity equation which is associated with (2.2) are given in section D of this chapter.

C. Compressible Steady-State Flow

The physical and HD plane momentum and energy equations (2.20, 2.21, 2.31, and 2.32) are identical in form and their method of solution is identical in form. Therefore, only the HD plane difference analogues will be given from which the physical plane analogues can be obtained by direct substitution. The compressible equations are more complex than the incompressible equations and consequently more difference analogues can be used to approximate these equations. Since there are two second order interdependent equations they can, in theory at least, be solved simultaneously or separately. The simultaneous solutions can be obtained by either the ordinary implicit or the Crank-Nicolson implicit
analogues. The separate solutions can also be obtained by either of these analogues or by explicit analogues. When considering separate solutions, the order in which the solutions are obtained may be important. For these equations, the following orders may be available:

a. The tangential and normal velocity components are obtained at the \((i+1)\)th column and these are used to obtain the enthalpy along that column.

b. The enthalpy is obtained at the \((i+1)\)th column which is used to obtain the tangential and normal velocity components.

c. An "alternating type" solution may be possible which uses (a) for one column, (b) for the second, (a) for the third and etc.

d. An "iterative" solution may be used to obtain the "true" solution at the \((i+1)\)th column before advancing to the next column. Any of the above methods, (a-c), could be used to obtain the solution at each column.

In the above, the normal velocity has been considered jointly with the tangential velocity since for the HD plane it is an explicit function of the tangential velocity only. For the physical plane, this is not the case and it should be considered a separate variable.

Difference analogues for the separate solution of the HD equations will not be presented since they are relatively straightforward. The Göertler-HD transformed equations
must be solved separately since the momentum equation (2.45) is third order while the energy equation (2.46) is second order. Also, iteration will be used with this system. The ordinary implicit and Crank-Nicolson implicit analogues will be considered for both the HD and the Göertler-HD equations.

1. Howarth-Dorodnitsyn Transformed Plane Analogues. The energy and momentum equations are solved simultaneously which includes the implicit dependence of the enthalpy on the velocity and vice versa while for separate solutions, this interdependence is explicit.

The ordinary implicit difference analogue of the momentum equation (2.31) and the energy equation (2.32) are

\[
u_{\xi \eta \tau+1, k} = V_{\xi \eta \tau+1, k} (u_{\tau+1, k} - x_{\tau+1, k}) - \Delta x_{\tau+1, k} (u_{\tau+1, k}) + (u_{\tau+1, k}) (i_{\tau+1, k})
\]

\[
u_{\xi \eta \tau+1, k}^2 = \Delta x (dp_e / d\xi)_{\tau+1, k} (\rho_{\tau+1, k})^{-1} + u_{\xi \eta \tau+1, k}^2
\]

and

\[
u_{\xi \eta \tau+1, k} = V_{\xi \eta \tau+1, k} (i_{\tau+1, k} + 2x_{\tau+1, k} (u_{\tau+1, k}) (u_{\tau+1, k})
\]
\[ + \Delta \xi F_{\eta, k}(u_{\eta})^2 - \Delta \xi (F_{\eta, k}/Pr)(i_{\eta})_{t+1, k} \]

\[ - 2\Delta \xi (F'_{\eta, k}/Pr)(i_{\eta})_{t, k} (i_{\eta})_{t+1, k} + \Delta \xi (F'_{\eta, k}/Pr)(i_{\eta})^2_{t, k} \]

\[ = u_{\eta, k}^i_{t, k} + \Delta \xi u_{\eta, k}(dp_e/d\xi)_{t+1, k} (\rho_{\eta, k})^{-1} \]

respectively.

The Crank-Nicolson implicit analogue of (2.31) and (2.32) are

\[ 2u_{\eta, k}^i_{t, k} + \Delta \xi V_{\eta, k}(u_{\eta})_{t+1, k} - \Delta \xi F_{\eta, k}(u_{\eta})_{t+1, k} \]

\[ - F'_{\eta, k}\Delta \xi [(i_{\eta})_{t, k} (u_{\eta})_{t, k} (i_{\eta})_{t+1, k} + (u_{\eta})_{t, k} (i_{\eta})_{t, k} (i_{\eta})_{t+1, k}] \]

\[ = 2u_{\eta, k}^2 - 2\Delta \xi (dp_e/d\xi)(\rho_{\eta, k})^{-1} \]

\[ - \Delta \xi V_{\eta, k}(u_{\eta})_{t, k} + \Delta \xi F_{\eta, k}(u_{\eta})_{t, k} \]

3.23

and

\[ 2u_{\eta, k}^i_{t+1, k} + \Delta \xi V_{\eta, k}(i_{\eta})_{t+1, k} - 2F_{\eta, k}\Delta \xi (u_{\eta})_{t, k} (u_{\eta})_{t+1, k} \]

\[ - \Delta \xi (F'_{\eta, k}/Pr)(i_{\eta})_{t+1, k} - 2\Delta \xi (F'_{\eta, k}/Pr)(i_{\eta})_{t, k} (i_{\eta})_{t+1, k} \]

\[ = 2u_{\eta, k}^i_{t, k} + 2\Delta \xi u_{\eta, k}(dp_e/d\xi)_{t+1, k} (\rho_{\eta, k})^{-1} \]

\[ - \Delta \xi V_{\eta, k}(i_{\eta})_{t, k} + \Delta \xi (F'_{\eta, k}/Pr)(i_{\eta})_{t, k} \]

3.24

3.25
respectively.

The difference analogues for the continuity equation are presented in section D of this chapter and the wall shear is given by

$$\tau_w = F_w u_2 (\Delta \eta)^{-1}$$

which is the forward difference form of (2.36) with the wall boundary condition (2.37).

The above two sets of difference equations (3.22 and 3.23) and (3.24 and 3.25) are solved simultaneously by the Crout-Banachiewicz algorithm (32a) since they are both tridiagonal in form. The matrix to be solved is actually a tridiagonal matrix of two-by-two matrices composed of coefficients of the two difference equations. Two of the coefficients are associated with the momentum difference equation, the other two being associated with the energy difference equation. The solution is more involved and slower than the simple tridiagonal matrix solutions common to the separate equations, but the interdependence of the two dependent variables common to the simultaneous method may justify the additional work.

When the wall heat flux is specified as a boundary condition rather than the wall enthalpy, the difference equations must be handled differently at and near the wall. Before discussing this, some general formulas must first be presented. The momentum and energy difference equations (3.22 and 3.23) and (3.24 and 3.25) can be written as
\[-A_{\ell,k}^1 u_{\ell+1,k+1} + B_{\ell,k}^1 u_{\ell+1,k}^+ - C_{\ell,k}^1 u_{\ell+1,k-1}^+ \]

\[-D_{\ell,k}^1 i_{\ell+1,k+1} + E_{\ell,k}^1 i_{\ell+1,k}^+ - F_{\ell,k}^1 i_{\ell+1,k-1}^+ = G_{\ell,k}^1 \]

\[-A_{\ell,k}^2 u_{\ell+1,k+1} + B_{\ell,k}^2 u_{\ell+1,k}^+ - C_{\ell,k}^2 u_{\ell+1,k-1}^+ \]

\[-D_{\ell,k}^2 i_{\ell+1,k+1} + E_{\ell,k}^2 i_{\ell+1,k}^+ - F_{\ell,k}^2 i_{\ell+1,k-1}^+ = G_{\ell,k}^2 \]

where the coefficients are functions of known values and the superscripts represent the particular difference equation, "1" being the momentum equation, "2" being the energy equation. Using the Crout-Banachiewicz algorithm, the solution is of the form

\[u_{\ell+1,k} = K_{\ell+1,k}^1 + K_{\ell+1,k}^2 u_{\ell+1,k+1} + K_{\ell+1,k}^3 i_{\ell+1,k+1}^+ \]

\[i_{\ell+1,k} = L_{\ell+1,k}^1 + L_{\ell+1,k}^2 u_{\ell+1,k+1} + L_{\ell+1,k}^3 i_{\ell+1,k+1}^+ \]

with \(K_{\ell+1,k}^i\) and \(L_{\ell+1,k}^i\) being functions of the known coefficients \(A_{\ell,k}^j\), \(B_{\ell,k}^j\), and etc. To solve this system, the \(L_{\ell+1,1}^i\) and \(K_{\ell+1,1}^i\) must be known and are obtained from the boundary conditions. Therefore, if the wall flux is zero, one must find the \(L_{\ell+1,1}^i\) and \(K_{\ell+1,1}^i\) which satisfy (3.29) and (3.30) as well as the boundary condition, i.e., \(i_w\) must be written in the form of (3.30) with \(L_{\ell+1,1}^1\), \(L_{\ell+1,1}^2\), and \(L_{\ell+1,1}^3\) being known constants.
Consider the boundary condition of zero heat flux at the wall
\[ i_{\eta}|_{w} = 0 \] 3.31

as an example of the two methods to be employed. The first, Method A, is identical to that used by Flügge-Lotz and Blottner (10) while the second, Method B, is derived specifically for this work. Aside from these two methods, one can use the simple difference approximation of (3.31)

\[ \frac{i_{,l+1,2} + i_{,l+1,1}}{\Delta \eta} = 0 \quad \text{and} \quad L_{l+1,1}^3 = 0 \] 3.32

but this has been shown to be unsatisfactory for numerical calculations (10).

Method A. Blottner used Taylor series expansions to obtain the approximation for (3.31)

\[ i_{,l+1,1} = \frac{4}{3} i_{,l+1,2} - \frac{1}{3} i_{,l+1,3} \] 3.33

with an associated truncation error order of \( O(\Delta \eta)^2 \). This expression is solved for \( i_{,l+1,3} \) and substituted into (3.27) and (3.28) with \( k = 2 \). Then, \( u_{,l+1,3} \) is eliminated from the two equations and \( u_{,l+1,1} \) is eliminated by the no slip wall condition. After some rearrangement, the desired equation (3.30) is obtained with

\[ L_{l+1,1}^1 = \Delta^{-1} [ A_{l+1,2}^{1} G_{l+1,2}^{2} - A_{l+1,2}^{2} G_{l+1,1}^{1} ] \]
\[
L_{r+1,1}^2 = \Delta^{-1}[(A_{r+1,2}^2 E_{r+1,2}^1 - A_{r+1,2}^1 E_{r+1,2}^2)
+ 4(A_{r+1,2}^1 D_{r+1,2}^2 - A_{r+1,2}^2 D_{r+1,2}^1)]
\]
\[
L_{r+1,1}^3 = \Delta^{-1}[A_{r+1,2}^2 B_{r+1,2}^1 - A_{r+1,2}^1 B_{r+1,2}^2]
\]
\[
\Delta = [(A_{r+1,2}^2 F_{r+1,2} - A_{r+1,2}^1 F_{r+1,2})
- 3(A_{r+1,2}^2 D_{r+1,2}^1 - A_{r+1,2}^1 D_{r+1,2}^2)] . \quad 3.34
\]

These expressions are not in terms of parameters and variables at the wall, but instead are in terms of parameters and variables at the first point above the wall. Therefore, it appears doubtful that the partial differential equations are satisfied at the wall.

Method B. The partial differential equations can be satisfied at the wall by substituting the wall boundary conditions
\[
u_w = V_w = i_\eta \bigm|_w = 0 \quad 3.35
\]
into the partial differential equations (2.20) and (2.21) or (2.31) and (2.32). The result is two nonlinear ordinary differential equations
\[
u_{\eta\eta} \bigm|_w = (\alpha F)^{-1} \bigm|_w (dp_e/d\xi) \quad 3.36
\]
and
\[
Pr [u_\eta \bigm|_w]^2 + i_{\eta\eta} \bigm|_w = 0 \quad 3.37
\]
The momentum equation (3.36) is actually an ordinary differential equation since $\frac{dp_e}{d\xi}$ is a known constant at any position $\xi$. Using the central difference formulas of Appendix B, these two equations can be differenced and the resulting equations include pseudo-velocities and a pseudo-enthalpy, all located below the wall. The pseudo-enthalpy is removed by the central difference form of the wall flux condition (3.35) which relates this term to the enthalpy at the first point above the wall. The difference equation of (3.36) gives a relationship between the pseudo-velocity and the velocity at the first point above the wall (3.41). This expression can be inserted into the difference equation of (3.37) and an expression similar to (3.30) is obtained with

$$L_{1,1}^1 = -\frac{Pr}{4}(u_{\tau,2} - u_{\tau,0})^2 \left[ \frac{(\Delta \eta)^2}{(\rho_1 F_1)} \left( \frac{dp_e}{d\xi} \right)_{\tau+1} + \frac{1}{2}(u_{\tau,2} - u_{\tau,0}) \right]$$

$$L_{2,1}^1 = \frac{Pr}{4}(u_{\tau,2} - u_{\tau,0})$$

$$L_{3,1}^1 = 1$$

with

$$u_{\tau,2} - u_{\tau,0} = 2u_{\tau,2} - \frac{(\Delta \eta)^2}{(\rho_1 F_1)}_{\tau}$$

where the subscript "0" denotes the pseudo-velocity. These coefficients are functions of the wall density and viscosity and the truncation error is order of $O(\Delta \eta)^2$ as was the
case for Method A.

2. Göertler-HD Transformed Plane Analogue. The Göertler-HD transformed equations are nonsingular at the leading edge, thus, difference equations must be written for the similar equations and for the nonsimilar equations. The similar equations are boundary value type equations while the nonsimilar equations are initial value type equations. Iteration is to be used and the following nomenclature will be employed to denote the various iteration levels:

i) the unprimed quantities will represent the known values along the ($i$)th (upstream) grid column.

ii) the single primed quantities will represent the last iteration level along the ($i+1$)th grid column.

iii) the double primed quantities will denote the next iteration level along the ($i+1$)th grid column, i.e., the unknown values.

iv) the term $F'/F$ represents $(dF/di)/F$ and may be evaluated at either the ($i$)th or ($i+1$)th level.

a. Similar Equations. This system is composed of two nonlinear ordinary differential equations, the third order momentum equation (2.51) and the second order energy equation (2.52). The finite difference equations used to approximate these similar equations are

$$
(h''_{k}) + [(F'/F)_k(i'_{k}) + h'_{k}](h''_{k}) = 0
$$

3.42
\[ (i_{BB}^2) + (F'/F)_k (i_3^1) (i_3^1) + Prh'_k (i_3^1) + \sigma P\mu_{\varepsilon}^2(\alpha)(h'_{BB})^2 = 0 \]

3.43

When the proper difference formulas are substituted into (3.43) and the resulting equation is multiplied by the denominator of the difference analogue of \((i_{BB})\), which is \((\Delta \varepsilon)^2\), it appears that the last term on the left hand side is unbounded since it is multiplied by \((\Delta \varepsilon)^{-2}\). On the other hand, the term may be bounded if it is considered in the form

\[ (\Delta \varepsilon)^2 Pr\mu_{\varepsilon}^2(\alpha)(h'_{BB})^2 = 0(\Delta \varepsilon)^2 \]

3.44

because \((h'_{BB})^2\) is known to be bounded since (3.42) is stable. To assure convergence of the energy equation's solution, "\(\sigma\)" is inserted in (3.43) such that it can be used two different ways, the first being that "\(\sigma\)" is always unity so that (3.44) can be proven numerically. The second assumes "\(\sigma\)" is zero initially and iterations are performed until the solutions of (3.42) and (3.43) reach "steady-state". Then "\(\sigma\)" is incremented by some predetermined small constant and a new "steady-state" is attained. This latter procedure is continued until "\(\sigma\)" becomes unity at which the two solutions should agree if (3.44) is valid.

b. Nonsimilar Equations. The ordinary implicit difference analogue of the momentum equation (2.45) is

\[ (h''_{BB})_{l+1,k} + [(F'/F)_{l+1,k} (i_3^1)_{l+1,k} + h'_{l+1,k}](h''_{BB})_{l+1,k} \]
\[ + \lambda(a) \frac{[i_{t+1,k}(i_{e})_{t+1}]}{(i_{e})_{t+1,k}} - \lambda(a) \frac{(h'_{\beta})_{t+1,k}}{(h''_{\beta})_{t+1,k}} \]

\[ = 2\alpha_{t+1} \left[ \frac{(h'_{\beta})_{t+1,k} - (h''_{\beta})_{t,k}}{\Delta\alpha} \right] \]

\[ - (h''_{\beta})_{t+1,k} \frac{h'_{t+1,k} - h'_{t,k}}{\Delta\alpha} \]

\[ = 3.45 \]

and the equivalent analogue of the energy equation (2.46) is

\[ (i''_{\beta})_{t+1,k} + [(F'/F)_{t+1,k} (i'_{\beta})_{t+1,k} \]

\[ + Pr(h'_{t+1,k} + 2\alpha_{t+1} \frac{h'_{t+1,k} - h'_{t,k}}{\Delta\alpha}) (i''_{\beta})_{t+1,k} \]

\[ + PrU_e^2(a)_{t+1} \left[ \frac{(h'_{\beta})_{t+1,k}^2 - \lambda(a)_{t+1} [i_{t+1,k}(i_{e})_{t+1,k}]}{(h''_{\beta})_{t+1,k}} \right] \]

\[ = 2\alpha_{t+1} Pr(h'_{\beta})_{t+1,k} \frac{(i''_{t+1,k} - i'_{t,k})}{\Delta\alpha} \]

\[ = 3.46 \]

Equations (3.45) and (3.46) reduce to difference equations similar to (3.42) and (3.43) when "a" of equation (2.53) or "c" is zero, the difference being that all terms of the former set are multiplied by \(\Delta\alpha\). This should have no effect on the solution since it only changes the magnitude of specific terms but the relative magnitudes of individual terms to one another remains the same. Therefore, this ordinary implicit analogue has the advantage of being complete since it is self-starting.
The Crank-Nicolson implicit analogue of the momentum equation is

\[
\begin{align*}
(h_{BB}^{\prime})_{l+1,k}^{\prime} & + [(F' / F)_{l+1,k}(i_B^{'})_{l+1,k} + h_{l+1,k}^{'}(h_{BB}^{\prime})_{l+1,k}^{\prime} \\
- \lambda(\alpha)_{l+1,k}(h_{BB}^{\prime})_{l+1,k}^{\prime} + h_{l+1,k}^{'}(h_{BB}^{\prime})_{l+1,k}^{\prime} & + \frac{\alpha_l(h_{BB})_{l,k} + \alpha_{l+1}(h_{BB}^{\prime})_{l+1,k}^{\prime}}{\Delta \alpha} h_{l+1,k}^{'} - \lambda(\alpha)_{l}(h_{BB})_{l,k}^{2} \\
+ \frac{\alpha_l(h_{BB})_{l,k} + \alpha_{l+1}(h_{BB}^{\prime})_{l+1,k}^{\prime}}{\Delta \alpha} h_{l+1,k}^{'} & - \lambda(\alpha)_{l}(h_{BB})_{l,k}^{2} \\
- (h_{BB}^{\prime})_{l,k}^{\prime} & - [(F' / F)_{l,k}(i_B)_{l,k} + h_{l,k}(h_{BB})_{l,k}^{2} \\
+ \frac{\alpha_l(h_{BB})_{l,k} + \alpha_{l+1}(h_{BB}^{\prime})_{l+1,k}^{\prime}}{\Delta \alpha} h_{l,k}^{'} & + \frac{\alpha_l(h_{BB})_{l,k} + \alpha_{l+1}(h_{BB}^{\prime})_{l+1,k}^{\prime}}{\Delta \alpha} h_{l,k}^{'} \\
- \frac{\lambda(\alpha)_{l} i_{l,k}^{\prime} + \lambda(\alpha)_{l+1} i_{l+1,k}^{\prime}}{(i_e)_{l}^{\prime} + \frac{\lambda(\alpha)_{l+1} i_{l+1,k}^{\prime}}{(i_e)_{l+1}^{\prime}}} \\
3.47
\end{align*}
\]

and similarly, the Crank-Nicolson difference analogue of the energy equation is

\[
(i_{BB}^{\prime})_{l+1,k}^{\prime} + [(F' / F)_{l+1,k}(i_B^{'})_{l+1,k} \\
+ \frac{\alpha_{l+1} + \alpha_l}{\Delta \alpha} (h_{l+1,k}^{'} - h_{l,k}^{'})][i_B^{\prime}]_{l+1,k}^{\prime} \\
+ Pr[h_{l+1,k}^{'} + \frac{\alpha_{l+1} + \alpha_l}{\Delta \alpha} (h_{l+1,k}^{'} - h_{l,k}^{'})] (i_B^{\prime})_{l+1,k}^{\prime} 
\]
\[-2Pr[\alpha \iota_l+1(h'_{B \iota_l})_{\iota_l+1,k} + \alpha \iota_l(h_{B \iota_l})_{\iota_l,k}]i'_{\iota_l+1,k}\]

\[= (i_{B \iota_l})_{\iota_l,k} + 2Pr[\alpha \iota_l+1(h'_{B \iota_l})_{\iota_l+1,k} + \alpha \iota_l(h_{B \iota_l})_{\iota_l,k}]i_{\iota_l,k}\]

\[+ \{(F'/F)_{\iota_l,k}(i_{e \iota_l})_{\iota_l,k} + Pr[h_{\iota_l,k} + \frac{\alpha \iota_l+1 + \alpha \iota_l}{\Delta \alpha}(h'_{\iota_l+1,k} - h_{\iota_l,k})]\}(i_{B \iota_l})_{\iota_l,k}\]

\[-Pr[U^2_e(\alpha)_{\iota_l+1}(h'_{B \iota_l})^2_{\iota_l+1,k} + U^2_e(\alpha)_{\iota_l}(h_{B \iota_l})^2_{\iota_l,k}\]

\[\frac{i'_{\iota_l+1,k} \lambda(\alpha)_{\iota_l+1}(h'_{B \iota_l})_{\iota_l+1,k}}{i_{e \iota_l+1}} - \frac{i_{\iota_l,k} \lambda(\alpha)_{\iota_l}(h_{B \iota_l})_{\iota_l,k}}{i_{e \iota_l}}.\] 3.48

These difference equations do not reduce to the similar difference equations when "a" or "\alpha" is zero so (3.42) and (3.43) must be used to start the solution.

For problems with a specified wall heat flux, the difference equations and the differential equations must be satisfied at the wall. A procedure similar to Method B of section C will be used to approximate the difference equations at the wall. Only the energy equation need be considered in this case and its difference equations can be written in the general form

\[-A_{\iota_l+1,k}i_{\iota_l+1,k} + B_{\iota_l+1,k}i_{\iota_l+1,k} - C_{\iota_l+1,k}i_{\iota_l+1,k} - D_{\iota_l+1,k} = 0\] 3.49
where $A_{\tau+1,k}$, $B_{\tau+1,k}$, $C_{\tau+1,k}$, and $D_{\tau+1,k}$ are functions of known dependent and independent variables. The solution is of the form

$$i_{\tau+1,k} = E_{\tau+1,k}i_{\tau+1,k+1} + F_{\tau+1,k} \quad \text{(3.50)}$$

where $E_{\tau+1,k}$ and $F_{\tau+1,k}$ are functions of the coefficients of (3.46 or 3.48). To obtain a solution, $E_{\tau+1,1}$ and $F_{\tau+1,1}$ must be known and are obtained from the boundary conditions. Using the boundary conditions (2.47) with the adiabatic wall condition the Goertler energy equation reduces to

$$i_{BB} + PrU_e^2(\alpha)(h_{BB})^2 = 0. \quad \text{(3.51)}$$

Using the central difference analogues of the derivatives and then applying the boundary conditions one obtains an expression similar to (3.50) with

$$E_{\tau+1,1} = 1 \quad \text{(3.52)}$$

$$F_{\tau+1,2} = \frac{2PrU_e^2(\alpha)h_{\tau+1,2}^2}{(\Delta B)^2}h_{\tau+1,2}^2.$$  

Method A is not to be used with this system.

D. Continuity Equation Analogues.

Due to their general form for compressible and incompressible flows, these analogues are presented separately from the difference analogues for the momentum and energy equations. The general procedure for solving the boundary
layer equations (except for the Görtler transformed equations) advances the tangential velocity component and using these values, the normal velocity component is advanced. For the physical plane compressible flow equations, the normal velocity component is also a function of the enthalpy, thus, the enthalpy may be advanced prior to the advancement of the normal velocity component.

The compressible flow continuity equation's difference analogues will be presented from which the incompressible and HD plane continuity equation's analogues can be obtained by omitting the density. The simplest difference analogue is

$$(2\Delta y/\Delta x)[(\rho u)_{t+1,k}^-(\rho u)_{t,k}^-] + (\rho v)_{t+1,k+1}^-(\rho v)_{t,k-1}^- = 0$$

but this has been successfully proven unsatisfactory by Wu (46). Wu proposed an analogue of the form

$$(\Delta y/2\Delta x)[(\rho u)_{t+1,k}^-(\rho u)_{t,k}^+] + (\rho u)_{t+1,k-1}^-(\rho u)_{t,k-1}^- + (\rho v)_{t+1,k}^- (\rho v)_{t+1,k-1} = 0$$

which has a truncation error order of $O(\Delta x, \Delta y^2)$. Blottner (10) considered this analogue and also a "centered" analogue

$$(\Delta y/\Delta x)[(\rho u)_{t+1,k}^- (\rho u)_{t,k}^+] + (\rho u)_{t+1,k-1}^- (\rho u)_{t,k-1}^- + (\rho v)_{t+1,k}^- (\rho v)_{t,k-1}^- = 0$$

with a truncation error order of $O(\Delta x^2, \Delta y^2)$. This analogue
should be used with Crank-Nicolson analogues of the momentum and energy equations since the truncation error is of the same order. Also, (3.55) should be more accurate than (3.54).

These two difference analogues can be used to obtain the general solution for v at any point (x,y). Using the incompressible forms of (3.54) and (3.55), it can be shown that the normal velocity obtained by (3.54) is

\[ v_{x,k} = f\left[ \sum_{i=1}^{k} a_i (u_{x,i} - u_{x,i-1}) \right] \]

where \( a_i \) is a constant multiplier for each \( i \) and "f" means "a function of". Similarly, the normal velocity obtained by (3.55) is

\[ v_{x,k} = f\left[ \sum_{j=1}^{2} \sum_{i=1}^{k} a_{ij} (u_{x,i} - u_{x,i-1}) \right] \]

where \( a_{ij} \) is a constant multiplier for a given \( i,j \). These two equations show that the error of the normal velocity given by (3.54) is only affected by the errors of the tangential velocities along the column in question and the adjacent upstream column. On the other hand, the error of the normal velocity given by (3.55) is a function of the error of all the tangential velocities in the region \( 0 \leq x \leq (l-1) \Delta x \), \( 0 \leq y \leq (k-1) \Delta y \). Therefore, if initial profiles are assumed, the results obtained using (3.55) may be less accurate than those using (3.54) due to this error growth. The author attempted to prove boundedness of (3.56) and (3.57). A rather loose bound could be obtained for (3.56) but no bound could
be obtained for (3.57).

For compressible flow in the physical plane, Wu (46) also proposed the difference analogue

\[(\rho v)_{t+1,k} - (\rho v)_{t+1,k-1} + \frac{\Delta v}{4\Delta x} (\rho_{t,k} + \rho_{t,k-1})(u_{t+1,k} - u_{t,k})
+ u_{t+1,k-1} - u_{t,k-1}\] \[+ \frac{\Delta v}{4\Delta x} (u_{t,k} + u_{t,k-1})(\rho_{t+1,k} - \rho_{t,k})
+ \rho_{t+1,k-1} - \rho_{t,k-1} = 0 \] 3.53

which is obtained by expanding \((\rho u)_x\) and using averages for the coefficients. The truncation error is the same as that of (3.54) but the extra calculations may increase the rounding error significantly.

E. Starting Profiles.

The boundary layer equations, being initial value type, require a known solution initially, i.e., initial profiles. Most of the numerical solutions present in the literature are for similar type flow geometries or at least geometries that are of similar type initially. This has been done because "exact" starting profiles can be obtained from the similar equations, thereby, essentially eliminating any initial error. The Görtler transformed equations reduce to the similar equations initially, regardless of the flow geometry, so the following does not apply to these equations.

There are many flow geometries whose initial sections are not of the similar type so one must assume initial profiles. Considerable effort has been given to the determination
of initial profiles which will yield accurate results after
the solution has been propagated downstream some finite
distance.

Leigh (26) claimed the finite difference analogues
removed irregularities of the initial profiles and claimed
six place accuracy at downstream stations when starting with
five place initial profiles of unknown accuracy. Smith and
Clutter (36) found that small errors at the upstream station
could be greatly magnified as the solution was propagated
downstream. They used an iterative integration technique
to solve the boundary layer equations and the amplification
of errors is indicative of instability of the numerical
procedure since, in general, the numerical solution should
converge to the analytical solution of the partial differential
equation as the step size is reduced to zero if the
numerical method is stable and consistent. Of course, the
analytical and numerical solutions are both dependent on
boundary and initial data.

Consider the simple flat plate flow geometry which has
a similar solution. This solution is the exact mathematical
solution of the boundary layer equations for all positive
tangential distance, x. At the leading edge, the similar
solution, if transformed back to the physical plane, results
in an infinite normal velocity component which is physically
impossible. However, one realizes that the fluid along the
line y = 0, -∞ ≤ x ≤ 0, is initially at the inviscid fluid
velocity and gradually approaches zero as x → 0. Therefore,
the flow in the vicinity of this stagnation line looks like a boundary layer type flow and at the leading edge of the plate, the boundary layer has some finite thickness. Based on considerations of this type several different approximate initial profiles have been proposed. Wu (46) used profiles of the form

$$u(x = 0, y > 0) = U_e(0)$$

$$u(x = 0, y = 0) = 0$$

$$i(x = 0, y > 0) = i_e(0)$$

$$i(x = 0, y = 0) = i_w(0)$$

$$v(x = 0, y) = 0$$

and claimed the solution at a downstream station was in good agreement with the similar solution. Flügge-Lotz and Blottner (10) used an explicit difference analogue, similar to that used by Wu, with the initial profiles, (3.59), to propagate the flat plate solution downstream a predetermined distance, $x_f$. These results were not in agreement with the similar solutions obtained by Low (27) and reduction of the step size did not improve the accuracy, which is contrary to Wu's results. Realizing that the numerical solution was that of the boundary layer equations and did not necessarily represent the true physical phenomena, these authors proposed the following profiles based on the mathematically correct similar solution:
\[ u(x = 0, y > 0) = U_e(0) \]
\[ u(x = 0, y = 0) = 0 \]
\[ i(x = 0, y > 0) = i_e(0) \quad 3.60 \]
\[ i(x = 0, y = 0) = i_w(0) \]
\[ v(x = 0, y = 0) = v_e(\Delta x). \]

The value of \( v_e(\Delta x) \) was obtained by using the predetermined step size in the back transformation of the similar solution. Using these initial profiles, agreement with Low's similar solution was obtained at \( x_f \).

If one considers the finite difference analogues for the incompressible momentum equation (3.2) at the first downstream station \( (t = 2) \) and the first point above the wall \( (k = 2) \), they are of the form:

Explicit:
\[ u_{2,2} = 1 - \left[ \frac{\Delta x}{(\Delta y)^2} + \frac{v_{1,2} \Delta x}{2 \Delta y} \right] + A(x) \]

Ordinary Implicit:
\[ u_{2,2} = 1 + A(x) + f_1(u_{2,1}) \quad 3.61 \]

Crank-Nicolson Implicit:
\[ u_{2,2} = 1 - \left[ \frac{\Delta x}{(\Delta y)^2} + \frac{v_{1,2} \Delta x}{2 \Delta y} \right] + A(x) + f_2(u_{2,1}) \]

where \( U_e(0) = 1 \) has been substituted into the equations and the starting profiles given by (3.59) or (3.60) have
been used.\footnote{1} Also, $f(u_{2,i})$ represents implicit dependent variable terms and $A(x)$ represents the pressure gradient term. From (3.61) it is obvious that the solution at this point will vary significantly depending on the finite difference analogue used. Using this as a basis, the initial profiles given by (2.8) and (2.26), i.e.,

\[
\begin{align*}
  u(x = 0, y) &= U_e(0) \\
  i(x = 0, y) &= i_e(0) \\
  v(x = 0, y) &= 0
\end{align*}
\]

are proposed by this author. All of these initial profiles were considered numerically and will be discussed in the next chapter.

\footnote{1}{For the starting profile (3.59), $v_{1,2} = 0$.}
IV. RESULTS AND DISCUSSION

The main purpose of this investigation is to determine the best "system" (form of the boundary layer equations and a finite difference analogue of that form) for numerically solving each of the following general boundary layer flow distributions:

i) incompressible transient flows,

ii) incompressible steady-state flows and,

iii) compressible steady-state flows.

For each of the three general flow distributions, the possible forms of the boundary layer equations were analyzed to determine which are applicable to general flow distributions and then, the finite difference analogues of each of these general forms were solved numerically. The accuracy of, the generality of, and the computational time required by each system was compared to determine the best system for solving that particular flow distribution. This chapter describes the results of this analysis and compares the best systems determined in this work with systems proposed by previous authors.

A. General Boundary Layer Equations.

The physical plane equations are of course applicable to any boundary layer geometry. Their solutions possess singularities at the leading edge and at separation (if present in the flow region). They have no other direct disadvantages. The von Mises transformation for incompressible
flow (44) reduces the physical plane momentum equation to an equation similar to the well-known heat equation. The latter equation has been studied extensively both numerically and theoretically which is a definite advantage. The boundary layer form of this equation unfortunately has a singularity at the wall as well as at the point of separation and at the leading edge. The transformation is not valid for compressible flow, so the system cannot be considered general. Crocco introduced a transformation which combines the momentum and the continuity equation into a single nonlinear partial differential equation. The transformation becomes invalid at any point in the flow region where the shear is zero or infinite. Therefore, the outer edge of the boundary layer, the separation point, and "overshoot" are all positions at which the transformation is invalid. The singularity at the leading edge is still present. It is possible, however, to apply the Mangler transformation (28) to the Crocco equations resulting in equations which are not singular at the leading edge. The equations reduce to nonlinear ordinary differential equations which can be solved for the initial velocity profiles. None of the other singularities are removed by this second transformation.

Howarth and Göertler both proposed transformed equations that are applicable to general boundary layer flow geometries. The Howarth transformation only defines a new normal direction variable while the Göertler transformation defines new normal and tangential direction variables. If the exterior
velocity is constant, the tangential distance in the Göertler transformed plane is identical to that of the Howarth transformed plane and the physical plane. Howarth's and Göertler's normal direction variables are not identical but can be related to each other quite simply. Both of these transformations reduce to the similar equations at the leading edge so "exact" initial profiles can be obtained. Their solutions only singularity exists at the separation point.

The Howarth-Dorodnitsyn transformation is used for compressible flow problems in which it is desirable to stretch the normal distance to the wall. The transformation is applicable to general flow geometries and its advantage lies in its removal of the implicit dependence of the density from the equations, especially the continuity equation. The solution is singular at the leading edge and at separation.

Therefore, the physical plane equations and the Howarth and Göertler transformed equations are considered applicable to general, incompressible or compressible, transient or steady-state, boundary layer flows. The Howarth-Dorodnitsyn transformed equations are considered applicable to general, compressible, transient or steady-state boundary layer flows. Only the physical plane, transient, incompressible boundary layer equations were considered in this work but the principal results indicate that the transient form of the Göertler transformed equations should have been considered. All other forms of the boundary layer equations are not considered general due to their additional limitations. The Howarth
transformed equations were not considered numerically since they are nearly identical in form to the Göertler equations.

B. Stability of the Difference Equations.

A finite difference equation should be shown to be convergent before it is used numerically. The explicit difference equations possess stability criteria which place limitations on the propagating direction step size and in some cases on the normal direction step size. Two exceptions to this are the Dufort-Frankel analogue, which has no stability criteria but does possess a very restrictive convergence criteria (Chapter III), and the Göertler transformed equations which are always unstable (Appendix C).

The implicit difference analogues of the boundary layer equations all have stability criteria some of which are of little consequence. For example, the implicit forms of the incompressible physical plane equations have the criterion that the tangential velocity component must remain positive. This means that the difference equations are not valid past separation but it is known that the boundary layer equations are not valid at and beyond separation (16).

The implicit analogues of the Göertler transformed equations have the stability criterion (Appendix C) that the flow cannot have a region of "overshoot". This criterion is the same for the compressible and incompressible equations. Also, for flows with adverse pressure gradients, the momentum equation has an inherent growth term aside from the known growth term—the principal function. This inherent growth term is
bounded (Appendix C). The third conclusion from the stability analysis is that the Crank-Nicolson implicit momentum equation is "less stable" than the ordinary implicit equation. This is the same conclusion one obtains for the heat equation. The HD transformed equations were not proven stable under all conditions due to the complicated equations involved in the stability analysis (Appendix C). The author was able to obtain the criterion that the tangential velocity component must remain positive but no other criteria could be obtained. From the analysis, it appears that additional criteria may be necessary.

Based on the stability criteria, it is obvious that the Göertler transformed equations are not general since flows with "overshoot" cannot be considered. To the author's knowledge, no other method has been used to calculate nonsimilar flows with "overshoot". The flows with "overshoot" that have been reported (1, 38) were similar type flows for which the Göertler transformed equations (2.45 and 2.46) reduce to ordinary differential equations with a constant "principal function" (30b). Therefore, the stability analysis of Appendix C does not apply.

The finite difference analogues of the HD equations will definitely be inadequate for totally nonsimilar flows with "overshoot" since they will be shown to be inadequate for totally nonsimilar flows. If the flow is similar initially, thereby enabling one to obtain "exact" starting profiles from the similar equations, it appears to the author
that they still may be inadequate for the following reasons. The continuity equation is implicit within the Göertler transformed equations while it is explicit with respect to the HD transformed momentum and energy equations. Therefore, the stability analysis, Appendix C, for the Göertler equations includes the continuity equation while the stability analysis, Appendix C, for the HD equations does not. In fact, from the derivation of the Göertler transformed momentum equation, the term which is responsible for the "nonovershoot" criterion is directly related to the continuity equation, i.e., the definition of V (30a). Although the stability analysis for the HD equations neglects the continuity equation, numerical calculations by this author have shown that positive $u_\xi$ or negative V will cause instability. Either of these latter conditions might occur with "overshoot".

Secondly, as mentioned previously, the stability analysis indicates additional criteria which, in affect, may invalidate flows with "overshoot".

No analyses were performed to determine if the numerical integration technique of Smith and Clutter would be valid for flows with "overshoot" and no numerical examples were considered which included "overshoot".

C. General

Before discussing the results of each of the systems, the problems considered as well as some general results applicable to all or many of the systems will be presented. The difference equations, except for the nonsimilar Göertler
equations, were checked by considering the simple flat plate for which the "exact" solution is easily obtained. Also, the results of one analogue were always checked against all other results. The nonsimilar problem considered corresponded to a flat plate with an adverse pressure gradient, the exterior velocity distribution given by equations (2.6) and (2.22). This flow does possess the separation phenomenon; it is clearly nonsimilar for all positive values of the tangential distance; and it has been studied quite extensively by previous authors. To this author's knowledge, the transient flow solution corresponding to this geometry has not been obtained previously.

Except for the transient equations, the propagating derivative has a leading coefficient which in certain regions of the flow is very small. Tests were conducted using implicit analogues to determine the effect of leaving this coefficient in its proper position or dividing through by this coefficient before the difference analogues were written. If the coefficient is divided through, the pressure gradient term in difference form is \( \Delta x (dp_e/dx)/(\rho u) \). At the wall the tangential velocity component is zero so this term, a growth term, is unbounded. Since the pressure gradient is a known value at every point throughout the flow region, this term can easily be shown to be order of \( O(\Delta x) \) if the tangential velocity component is absent. At a given distance down the plate, the variation of the wall shear with tangential step size is shown in Figure 1 for the difference equation with and without division by the leading coefficient. For the
Crank-Nicolson implicit analogues and the ordinary implicit analogues, the variation of the wall shear is much less for the difference equations without division by the leading coefficient. The effect of dividing through by this leading coefficient is greater for the compressible flow equations than for the incompressible equations. The leading coefficient, \( \mu \), of the physical plane compressible equations, (2.20 and 2.21), was divided through and the solution differed from the solution obtained without division by as much as a factor of nine. On the other hand, the solutions of the HD transformed plane compressible equations, (2.31 and 2.32), were much less sensitive to division by the leading coefficients. From this analysis, three important conclusions are demonstrated:

i) Growth terms must be proven bounded before one can be assured of obtaining accurate solutions (32c).

ii) The relative position of the density term in the HD equations is advantageous and division by a small quantity will significantly affect the round-off error.

iii) The results shown in Figure 1 converge to one another as the step size approaches zero since the finite difference analogues with and without division are convergent.

The continuity equation was studied quite thoroughly due to the difficulty it caused the numerical calculations. Two difference analogues, equations (3.54) and (3.55), were
Figure 1. Wall shear variation with tangential step size—physical plane, incompressible, steady-state equations.
used to describe this equation numerically. The first, equation (3.54), can be thought of as a forward difference analogue and it is similar to that used by Wu (46). The second, equation (3.55), is centered and similar to that used by Blottner (10). Both were used with "exact" starting profiles obtained from the similar solution and with the approximate starting profiles given by equations (3.59), (3.60), and (3.61). The "central difference" analogue was satisfactory if and only if the following two conditions were met:

a. very accurate initial profiles obtained from the similar solution were used and

b. the step sizes were sufficiently small.

If condition (a) was not fulfilled, the normal velocity component would oscillate and then diverge while if (a) was fulfilled but (b) was not, the normal velocity component would oscillate slightly and then become "stable". Both the "centered" and the "forward" analogues were used with "exact" starting profiles and the solution downstream varied by less than one percent. Therefore, since nonsimilar flows whose initial profiles are not known are to be calculated only the "forward difference" analogue was used. The "centered" analogue was always included to determine if cases existed which would not require the fulfillment of conditions (a) and (b) but none were found.

The implicit difference analogues of the HD transformed
plane equations\textsuperscript{2} were unstable if iteration was attempted. The procedure used was to "advance" the tangential velocity component using the old values of the normal velocity component and the enthalpy. Using these "new" tangential components, the normal velocity component would be "advanced" using the continuity equation. Then the enthalpy would be advanced using these "new" velocity components. The first values from the wall would be tested against their previous value and if they did not agree within $10^{-4}$, the procedure would be repeated. The values at the previous upstream level were considered "exact" and were not changed during the iteration. Although unsuccessful for these equations, iteration was used successfully to obtain "exact" solutions of the Göertler transformed equations.

There are three major differences in the HD plane equations and the Göertler equations:

a. the continuity equation is not a separate equation in the Göertler system.

b. "exact" initial profiles are always known for the Göertler equations.

c. the difference analogues for the square of normal direction derivatives are different for the two systems.

To explain (c) more fully, one can compare the ordinary implicit difference analogues (3.22 and 3.23) of the HD

\textsuperscript{2}The physical plane equations were also unstable when iterative methods were used.
equations with the ordinary implicit difference analogues (3.45 and 3.46) of the Göertler equations from which it is obvious that the former represent \((u_y')^2\) by the relation
\[
(u_y')^2 \approx -(u_y')_{i,k}^2 + 2(u_y')_{i,k} (u_y')_{i+1,k}
\]
while the latter represents this term by the relation
\[
(u_y')^2 \approx (u_y')_{i+1,k} (u_y'')_{i+1,k}
\]
where \(u\) and \(y\) are dummy variables and the primes denote iteration levels. Both relations were used with the Göertler incompressible equation and the solutions did not vary significantly. When applied to the Göertler compressible equations, the first difference analogue resulted in incorrect solutions. Therefore, the second difference analogue should be used with iterative methods, but this does not fully explain the reason for the divergence. The reason being that the Göertler incompressible solutions, using both difference analogues, agreed fairly well while the Göertler compressible equations did not. Also, the incompressible physical plane equations diverged and they do not include a term of the form \(u_y^2\).

The effect of "exact" starting profiles was determined by starting the solution in the HD plane downstream at some finite distance where exact starting profiles could be determined. The solution would not converge so apparently the continuity equation is also partially at fault. Although
no numerical proof can be given, it appears an iterative solution of the HD equations might be possible if only applied to the momentum and energy equations and once these reached a constant value, the continuity equation would be used to advance the normal velocity component after which the procedure would be advanced downstream one level, i.e., no iteration is performed with the continuity equation.

The Dufort-Frankel explicit analogue of the physical plane, incompressible, steady-state boundary layer equations was solved numerically. The solutions were highly oscillatory and did not correspond to the solutions obtained by other numerical methods. Therefore, the author concludes that the Dufort-Frankel analogue should not be used to calculate boundary layer flow distributions. This conclusion is in agreement with that obtained by most authors (7) when the Dufort-Frankel analogue was applied to other engineering equations.

D. Transient Flow--Physical Plane.

The four difference analogues, given by (3.15) and (3.17), were used to solve the pseudo-transient problem with the adverse pressure gradient. They all converged to the same values and the stability criterion for the allowable time step was found to be necessary for the two explicit analogues. The central difference analogue, equation (3.15), also has the additional stability criterion governing the normal step size which may be due to "low order terms" (32f) and therefore
could be neglected. This criterion was violated by about twenty percent and no instability was apparent. This was also the case for the steady-state explicit methods. Although the methods all converged to the same values, the Crank-Nicolson analogue required the least number of "iterations" and the least amount of time to reach the steady-state value; therefore, it is considered the best analogue for this boundary layer flow problem. It required from thirty to sixty percent fewer iterations than the ordinary implicit analogue while the time per "iteration" was only about fifteen percent more.

For the central difference explicit analogue, which was found to be much better than the backward difference analogue, the number of iterations required was in some cases ten times that of the Crank-Nicolson analogue. The Crank-Nicolson analogue required about twice the time per iteration.

The "true" transient problem was then considered but the backward difference analogue was not used based on the pseudo-transient results. The transient problem assumed the potential flow was fully developed and the boundary layer began to form at zero time. Each of the finite difference analogues was used to calculate the flow development to a given point in time for various time steps and the wall shear was evaluated at that point. The variation of wall shear with the time step is shown in Figure 2. From this figure, it is obvious that the ordinary implicit analogue is the best method since the accuracy is nearly independent of the step size for the range given. After the work was completed,
FIGURE 2. WALL SHEAR VARIATION WITH TIME STEP--PHYSICAL PLANE, INCOMPRESSIBLE, TRANSIENT EQUATIONS
it was realized that the Crank-Nicolson analogue, equation (3.15), did not reduce to the steady-state analogue as time approached infinity. Therefore, the difference analogue given by equation (3.20) should be used. It is felt by the author that this would prove to be the best analogue of the three. The Göertler transformed equations were not considered for time dependent problems, although they are applicable.

E. Steady-State Incompressible Flow.

1. Physical Plane. The explicit and explicit-implicit analogues were first applied to the flat plate to determine the applicability of the latter difference analogue. The solutions agreed very well so the two were applied to the adverse pressure gradient flow but neither are applicable since the stability criteria, determined valid for both systems, requires an infinite number of steps to reach separation. Also, near the leading edge, the solutions using the explicit-implicit method were more accurate than those obtained by the explicit method. On the other hand, the square-root computation requires additional time per step, but the better accuracy appeared to more than compensate for this time since larger step sizes could be used for a given accuracy.

The ordinary implicit analogue's constant exterior velocity solution was checked against Howarth's similar solution and then applied to the adverse pressure gradient flow. This method was found to be very stable until the vicinity
of the separation point was reached. The wall shear would be decreasing and then suddenly increase. After a short distance, it would again decrease and eventually become negative corresponding to separation. If this latter region were ignored, the extrapolated separation point would be relatively close to that predicted by other authors, \( x_s \approx 0.125 \). The method's accuracy was very sensitive to the tangential step size as shown in Figure 1.

The Crank-Nicolson analogue, equation (3.2), proved to be a good method for calculating incompressible flows. The accuracy was not largely dependent on the step size as shown in Figure 1. It requires about fifteen percent more work than the ordinary implicit analogue but a much larger step size can be used. The separation point predicted by this analogue was \( x_s = 0.1245 \) (Figure 4).

2. Görtler Transformed Plane. The finite difference analogue of the similar equation was used to obtain the initial profiles in both cases. Iteration techniques were used and the accuracy of the iteration was assumed satisfied when the absolute value for the difference of two successive values of the wall shear were within \( 10^{-6} \). The value of \( \frac{\partial^2 h}{\partial \alpha^2} \) evaluated at the wall was compared with that given by Smith and Clutter (36) and the two agreed to five decimal places, i.e.,

\[
\text{Smith and Clutter} \quad f_{BB} \bigg|_w = 0.332057
\]

\[
\text{This work} \quad h_{BB} \bigg|_w = 0.332053
\]
The Göertler transformed equations were then solved using the ordinary implicit and the Crank-Nicolson implicit analogues, equations (3.12) and (3.13) respectively. If iteration was used the exact solution was assumed when the absolute value of the difference of two successive values of the wall shear was within $10^{-6}$. Figure 3 shows the accuracy of the two difference analogues versus step size both with and without iteration. This is for the adverse pressure gradient flow and at a distance downstream quite close to the separation point. The variation is much less at about one-fourth the distance to the separation point. For example the step size was varied by a factor of forty and the wall shear near the leading edge varied in the fourth decimal place which corresponds to an error of about one part in ten thousand. The Crank-Nicolson difference equation requires nearly twice as much time per iteration but the number of iterations required was apparently reduced by about one-half since the two methods required essentially the same amount of time per step. From Figure 3, it is obvious that the Crank-Nicolson analogue is superior to the ordinary implicit analogue. Also, it was evident from the numerical results that many problems could be solved fairly accurately without iteration if small step sizes are used. For flows with an adverse pressure gradient, two important results of the stability analysis (Appendix C), i.e.,

i) the difference equations have an inherent growth term excluding the known growth terms such as the
Figure 3. Wall shear variation with tangential step size—Görtler incompressible, steady-state equations.
"principal function" and,

ii) the difference equations become more stable as
$\delta$ approaches unity,

are verified by the noniterative curves of Figure 3. The noniterative Crank-Nicolson wall shear variation with tangential step size shows that the accuracy is highly dependent upon the step size which verifies (i). Comparing the noniterative ordinary implicit analogue's wall shear variation curve with the similar curve for the Crank-Nicolson analogue one notes that the accuracy of the ordinary implicit analogue is much less dependent upon the step size than the Crank-Nicolson analogue which verifies (ii). The noniterative solution for a flow with a favorable pressure gradient was obtained and this further validated (i) since in this case the Crank-Nicolson implicit analogue's accuracy was less sensitive to step size variation than the accuracy of the ordinary implicit analogue.

3. General Results. Figure 4 shows the wall shear variation along the plate as calculated by the Crank-Nicolson and ordinary implicit analogues of the physical and Göertler transformed plane equations. The solutions of the two planes do not agree and this disagreement is apparently due to the starting profiles. The Göertler transformed plane has no discontinuity at the leading edge and this solution is correct. The physical plane equations have the discontinuity at the leading edge so the profiles given by equations (3.59), (3.60),
FIGURE 4. WALL SHEAR VARIATION WITH TANGENTIAL DISTANCE—INCOMPRESSIBLE, STEADY-STATE EQUATIONS

FINITE DIFFERENCE FORMS:

- Physical Plane
- Ordinary Implicit
- Crank-Nicolson Implicit

(a) Gönter Transformed Plane
- Ordinary Implicit
- Crank-Nicolson Implicit

Δx = 3.1250 x 10^{-4}, Δy = 0.1
Δt = 3.1250 x 10^{-4}, Δβ = 0.20, β_e = 10.0
Δx = 3.1250 x 10^{-3}, Δβ = 0.20, β_e = 10.0

X, Tangential Distance
T, Wall Shear Stress
and (3.62) were used as approximations. It has been found that the magnitude of the normal velocity component is not nearly as important as the effect of the discontinuity caused by assuming the wall value of both velocity components zero and all the other values greater than zero. The perfectly flat profile, equation (3.62), proved to give the more accurate results and the normal velocity component was assumed zero for all values of \( y \). From this result, one can affirm the fact that a stable finite difference analogue will keep the errors bounded and will converge but the converged solution will be dependent on the initial data. Therefore, approximate initial profiles are not adequate for accurate calculation of boundary layer flows.

Based on the above analysis, the Crank-Nicolson implicit analogue, equation (3.13), of the Göertler (or Howarth) transformed equations is the best finite difference analogue for the solution of incompressible, steady-state, boundary layer flows. The separation point, obtained by extrapolation was determined to be \( x_s = 0.121 \) which is in good agreement with Smith and Clutter's value \( x_s = 0.120 \).

F. Compressible Steady-State Flow.

The ordinary implicit and Crank-Nicolson implicit analogues were used to obtain the numerical solution for the physical plane, Howarth-Dorodnitsyn transformed plane, and the Göertler transformed plane flow geometries. Both the zero and adverse pressure gradient flows were considered. The constant wall enthalpy and the adiabatic wall were
considered for the zero pressure gradient flow. Adverse pressure gradient flows were calculated using the adiabatic wall condition and the wall temperature variation specified by a constant ratio to the adiabatic wall temperature obtained by equation (2.61). The parameters (Chapter II, section B) chosen were

\[ M = 9.6 \]
\[ T^* = 82.34^\circ R \]
\[ S^* = 198.6^\circ R \]
\[ Pr = 0.72 \text{ or } 1.0 \]
\[ \gamma = 1.40 \]
\[ \frac{T_w}{T_{ad}} = 1.0 \]
\[ r = Pr^{\frac{1}{2}} \]

which correspond to those chosen by Blottner (10).

1. Howarth-Dorodnitsyn Transformed Plane. These equations have been shown to be adequate for flows whose initial geometry is similar (10). For nonsimilar geometries, the system was found to be totally inadequate. This fact can best be shown by considering flows with the adiabatic wall condition. For these flow geometries, two difference analogues, methods A and B, were presented in Chapter III which were to be used to approximate this adiabatic wall condition. Figure 5 shows the variation of the wall enthalpy with distance along an inclined plate (adverse pressure gradient
flow) using both method A and method B in conjunction with the ordinary implicit difference equations (3.22 and 3.23) and the Crank-Nicolson implicit difference equations (3.24 and 3.25). Since the Prandtl number is unity, the "exact" wall enthalpy variation with distance for this adverse pressure gradient flow should correspond to the curve of Figure 5 representing the variation of the wall enthalpy with distance for the zero pressure gradient flow when method B is used with "exact" starting profiles but the curves do not correspond. The latter curve is the exact solution. From this figure two important results are evident. The first being that the HD equations are inadequate for totally nonsimilar flow geometries and the second being that method B does satisfy the momentum and energy equations at the wall while method A does not. The third important result obtained for this system is not apparent in the figure. The numerical results confirmed that the Crank-Nicolson implicit difference equation was unstable with respect to the approximate starting profiles for "large" tangential step sizes. These "unstable" step sizes were used to propagate "exact" starting profiles downstream and the results remained stable. Although the instability did exist, the Crank-Nicolson difference equations were found to be better than the ordinary implicit difference equations for solving the compressible boundary layer equations. The latter's accuracy was found to be very sensitive to step size while the former was rather insensitive until near the "unstable point". The time required per step
FIGURE 5. ADIABATIC WALL ENTHALPY VARIATION WITH DISTANCE--HOWARTH-DORODNITSYN STEADY-STATE EQUATIONS
was essentially identical for the two difference methods. Therefore, the Crank-Nicolson analogue is considered to be the better by this author.

The physical plane solution was calculated using the Crank-Nicolson analogue. The continuity equation proposed by Wu (46), equation (3.54) as well as the two given by equations (3.55) and (3.58) were used. Again, the "forward" difference analogue given by equation (3.54) was found to be the most accurate and stable. No results are given for these equations since they are identical in form to the HD plane equations. Also, they will not enter into the final conclusions. Separate solutions were obtained for the physical plane equations but these were unsatisfactory due to the inability to iterate as mentioned previously. Accurate solutions could be obtained only by using extremely small step sizes which requires a large amount of computer time and due to the large number of calculations, round-off error becomes prominent.

2. Göertler-HD Transformed Plane. The compressible Göertler transformed plane equations, (2.45) and (2.46) were solved numerically using an ordinary implicit analogue, (3.45 and 3.46), and a Crank-Nicolson implicit analogue, (3.47 and 3.48). The wall shear variation with step size is shown in Figure 6 for both analogues with iteration. Results are included for both the adiabatic wall and the fixed wall temperature-adiabatic wall temperature ratio.
FINITE DIFFERENCE FORMS:

- Crank-Nicolson Implicit
- Ordinary Implicit

$\Delta \beta = 0.20$, $\beta_e = 10.0$, $\alpha = 0.09375$

Pr = 1.0, $i_w = f(\alpha)$
Pr = 1.0, $i_\rho|_w = 0$
Pr = 0.72, $i_\rho|_w = 0$
No Iteration

$f(\alpha)$ given by Eqns. (2.60-2.62)

$\tau_w$, Wall Shear Stress

FIGURE 6. WALL SHEAR VARIATION WITH TANGENTIAL STEP SIZE--
GÖERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS
The Crank-Nicolson analogue is much less sensitive to step size than the ordinary implicit analogue. Also, the number of iterations per step is about one-half the number required of the ordinary implicit analogue. This was not determined precisely since no counter was used to determine the number of iterations required. For each iteration, the Crank-Nicolson analogue would require about twice as many arithmetic operations as the ordinary implicit analogue but the computational time per step was essentially identical for both analogues. Two iteration procedures were used. The first advanced the momentum and energy equations simultaneously until two successive values of the stream function and of the enthalpy agreed within $10^{-6}$. The second would iterate the momentum equation until "steady-state" was obtained and then, using these values, the energy equation was iterated to a "steady-state". The new energy values were then used to "correct" the momentum equation. If the stream function agreed to within $10^{-6}$ of that calculated before the last energy equation calculation, the whole procedure would be advanced to the next station, otherwise the procedure would be repeated for the present station. This second procedure is similar to that used by Smith and Clutter. For the calculations performed, the second procedure appeared to require about twice the computational time of the first procedure while the accuracy of the two procedures was essentially the same.

The adiabatic wall condition, equation (3.52), was used
and the enthalpy variation with distance is shown in Figure 7. In this case, the proper values are obtained starting at the leading edge which is contrary to that obtained for the HD plane equations. From Figure 7, it appears that the adiabatic wall enthalpy values for flows with \( Pr = 0.72 \) are in error but this is due to the inaccuracy of the relation concerning the "recovery factor", \( r \). This has been found by other authors, for example Smith and Clutter (38), and the reasoning can be validated by the good agreement one obtains when the Prandtl number is unity, Figure 7.

Based on the results of Figures 6 and 7, it is evident that the accuracy of the Crank-Nicolson implicit analogue for adiabatic and non-adiabatic flows is essentially independent of the tangential step size for the range considered. Also, the computational time per step required by the Crank-Nicolson implicit analogue and the ordinary implicit analogue is essentially equal. On this basis, the Crank-Nicolson implicit analogue of the Göertler transformed compressible equations should be used to calculate compressible flow distributions.

The variation of wall shear with distance for the known wall enthalpy and adiabatic wall is shown in Figure 8. Comparing the separation points predicted

\[
x_S \approx 0.0655 \quad Pr = 0.72
\]

\[
x_S \approx 0.060 \quad Pr = 1.0
\]

it is evident that the Prandtl number has an effect on the
FINITE DIFFERENCE FORMS:

- Crank–Nicolson Implicit, $\Delta \alpha = 2.50 \times 10^{-3}$
- Ordinary Implicit, $\Delta \alpha = 5.0 \times 10^{-4}$

$\Delta \beta = 0.20$
$\Delta \beta_e = 10.0$

$\text{Pr} = 0.72$, $i_{\beta |w} = 0$

$f(\alpha)$ given by Eqns. (2.60–2.62)

FIGURE 7. ADIABATIC WALL ENTHALPY VARIATION WITH TANGENTIAL DISTANCE--GÖERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS
FIGURE 8. WALL SHEAR VARIATION WITH TANGENTIAL DISTANCE--
GÖERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS

CURVES VALID FOR:
- Crank-Nicolson Implicit
  \( \Delta \alpha = 2.50 \times 10^{-3} \)
- Ordinary Implicit
  \( \Delta \alpha = 5.0 \times 10^{-4} \)
  \( i_{plw} = 0 \)
  \( i_{w} = f(\alpha) \), with \( f(\alpha) \) given by
  Eqns. (2.60-2.62)

\( \Delta \beta = 0.2 \)
\( \beta_e = 10.0 \)
separation point. The Mach number also has an effect (23) but that is not demonstrated specifically in this work. The separation point for the compressible flow has moved upstream from the incompressible flow separation point which is in the proper direction.

The results indicate that the Crank-Nicolson implicit analogue should always be used to calculate the solution of boundary layer problems by finite difference methods. There are two types of equations which could be used, the physical or HD plane equations and the Göertler or Howarth transformed equations. The physical plane solution is very accurate if proper initial values are given and it is faster than the Göertler plane iterative technique. But it has the disadvantage of not being suitable for nonsimilar flow geometries for which "exact" starting profiles cannot be obtained. Therefore, the only method which is applicable for general boundary layer flow problems is the Crank-Nicolson finite difference analogue of the Göertler or Howarth transformed equations.

G. Typical Profiles.

The profiles at specified distances down the plate which were obtained for the adverse pressure gradient incompressible flow are shown in Figure 9. The initial profile is also shown to exemplify the nearly similar type solution. The profiles obtained for the physical plane incompressible flow are shown in Figure 10 to further exemplify the similarity
of the Göertler solutions.

The profile variation with time for the incompressible transient flow with an adverse pressure gradient is shown in Figures 11-14. Each figure represents a different distance along the plate and by comparison, it is evident that the time required for the flow to reach steady-state is proportional to the downstream distance. The solutions given in Figure 14 are not at steady-state since this distance corresponds to a distance past the separation point for the particular step sizes chosen.

The solutions for the Göertler-HD transformed equations are presented in Figures 15-18. These are for adverse pressure gradient flows with the wall temperature-adiabatic wall temperature ratio fixed and with the adiabatic wall condition. The different profiles correspond to different distances along the plate and again the "similarity" of the Göertler equations is evident.

The profiles given in these figures are to be considered as indications of the type of solutions obtained. They were obtained by direct display on an oscilloscope. Table 1 contains the auxiliary data for Figures 9-18. The value of the independent variable (x, \( \alpha \), or \( t \)) for each curve of each figure is given in Table 1 while only the curves corresponding to the first and last independent variable of Table 1 are identified in the figures. The values of the independent variable, from left to right in Table 1, correspond to the curves, from left to right, of the figure. For compressible
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FIGURE 9. STREAM FUNCTION PROFILES--GÖERTLER
INCOMPRESSIBLE, STEADY-STATE EQUATIONS
FIGURE 10. TANGENTIAL AND NORMAL VELOCITY COMPONENT PROFILES--
PHYSICAL PLANE INCOMPRESSIBLE, STEADY-STATE EQUATIONS
FIGURE 11. TANGENTIAL AND NORMAL VELOCITY COMPONENT PROFILES--
PHYSICAL PLANE INCOMPRESSIBLE, TRANSIENT EQUATIONS--
\( x = 0.03125 \)
FIGURE 12. TANGENTIAL AND NORMAL VELOCITY COMPONENT PROFILES--
PHYSICAL PLANE INCOMPRESSIBLE, TRANSIENT EQUATIONS--
\[ x = 0.06250 \]
FIGURE 13. TANGENTIAL AND NORMAL VELOCITY COMPONENT PROFILES--
PHYSICAL PLANE INCOMPRESSIBLE, TRANSIENT EQUATIONS--
\( x = 0.09375 \)
FIGURE 14. TANGENTIAL AND NORMAL VELOCITY COMPONENT PROFILES--
PHYSICAL PLANE INCOMPRESSIBLE TRANSIENT EQUATIONS--
\[ x = 0.12500 \]
FIGURE 15. STREAM FUNCTION AND ENTHALPY PROFILES--
GÖERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS--
$Pr = 0.72, \frac{t_{B}}{w} = 0.0$
FIGURE 16. STREAM FUNCTION AND ENTHALPY PROFILES---
GÖERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS---
Pr = 0.72, \( i_w = f(\alpha) \)
FIGURE 17. STREAM FUNCTION AND ENTHALPY PROFILES--
GOERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS--
Pr = 1.0, $i_{\beta} |_{w} = 0.0$
FIGURE 18. STREAM FUNCTION AND ENTHALPY PROFILES--
GÖERTLER COMPRESSIBLE, STEADY-STATE EQUATIONS--
Pr = 1.0, i_w = f(α)
flows, the specified wall enthalpy variation, $f(\alpha)$, is given by equation (2.60) if the Prandtl number is unity but for the Prandtl number 0.72, $f(\alpha)$ is given by equation (2.61) with equation (2.62). All curves of Figures 9-18 are for the adverse pressure gradient flow.

H. Comparison with a Previous Method and Summary.

There has only been one other method presented in the literature which is comparable to the one proposed in this work. This is the method used by Smith and Clutter (36-38) which also uses the Göertler and Howarth transformed equations.\(^3\) These authors use a very complicated interpolation and integration technique to obtain their solution. Since integration is used, they must use extrapolation to obtain $(\partial^2 h/\partial \alpha^2)_w$ for the incompressible flows and, in addition, $(\partial i/\partial \alpha)_w$, for compressible flows.\(^4\) These values must be continually guessed until the exterior conditions are satisfied. A procedure similar to this was attempted in this work to obtain the solution of the compressible similar equations. The two nonlinear ordinary differential equations were written as five simultaneous nonlinear first order differential equations which were to be solved simultaneously by the Runge-Kutta numerical integration technique (20b). For this case, these same two initial values must be

\(^3\)Smith and Clutter used only the Howarth equations, but their method is valid for the Göertler equations.

\(^4\)If the flow is adiabatic, then $i_w$ must be obtained.
assumed and the exterior values were to be satisfied. One of the initial values, \((\partial^2 h/\partial \phi^2)_W\) did not appear too sensitive but the enthalpy function, \((\partial i/\partial \phi)_W\), was so sensitive that a variation of one in the sixth decimal place caused the numerical solution to diverge. Some very complicated numerical techniques based on linear programming were used to try to obtain convergence but no method would work. Smith and Clutter (38) also found sensitivity of this initial value but through extrapolation procedures and a different numerical integration procedure, they were able to get their stepwise solutions to converge. No indication was given as to how their initial solution was obtained which corresponds to that attempted in this work. In this work these ordinary differential equations were solved by using finite difference analogues. As pointed out by Fox (146), boundary value problems are not integral problems and should be solved by boundary value methods, i.e., finite difference methods. By using the finite difference technique, the guessing of initial values is eliminated which is a big advantage.

Smith and Clutter found that they must use the more complicated technique of advancing the momentum equation based on the old enthalpy values and then the energy equation was advanced. This new enthalpy profile was used to correct the momentum equations, etc. In this work, it was determined that the difference methods are not as sensitive and the two equations can be simultaneously brought to their "exact" solutions.

Smith and Clutter claim their method is similar to an
ordinary implicit method but this work has shown that ordinary implicit analogues are much less accurate than Crank-Nicolson implicit analogues. The incompressible flow separation point predicted by Smith and Clutter was \( x_s = 0.120 \) which is essentially equal to that obtained in this work, \( x_s = 0.121 \). There are two differences (neglecting the different numerical procedures) in the procedure used by this author and the procedure used by Smith to obtain this value. The parameters of the problem as defined by Smith and Clutter forced them to consider a flow region eight times the length used by this author. Secondly, according to the values given in their report and based on their stability criteria, they were forced to traverse the entire flow field more than once while the results of this work show that the flow field is traversed only once. To further explain this latter difference, their method has a stability criterion involving the ratio of the tangential distance to the tangential step size, i.e., \( x/\Delta x \). By experimentation, they found that this ratio must be less than twenty-five for convergence but according to Table I of (36), they used values of this ratio as high as one thousand. Therefore, they evidently traversed the flow field several times, choosing step sizes which would enable them to end their calculations at minutely different downstream stations. This stability criterion also affects the convergence and consistency of this "ordinary implicit" analogue since the tangential step size cannot go to zero without divergence of
the solution. With respect to finite difference analogues, this is sufficient to consider the method unstable. Smith and Clutter claim little sensitivity of the solution with step size which is not in accordance with Figure 4. The flow geometry they based their results on may not have been highly nonsimilar as is the case for the results of Figure 4. Based on Figure 4, it is evident their method may be inadequate for highly nonsimilar problems due to their method's restrictive stability criterion.

To solve the equations, Smith and Clutter found it necessary to transform the Howarth equations in an attempt to reduce round-off error. This may be due to the numerical methods employed and/or to the short word-length of the computer employed. In this work, the initial profiles, accurate to $10^{-6}$, were propagated downstream along a flat plate sixty steps using the nonsimilar difference equations. The solution at this distance agreed within $10^{-5}$ of the initial solution. Therefore, no additional transformations were required. It should be noted that the additional transformation used by Smith and Clutter slightly increases the amount of computation per equation.

Smith and Clutter claimed the calculation time per step for incompressible flows was one-eighth of a minute using the IBM 7090 computer. In this work, the maximum time per step was one-twentieth of a minute. Said authors

---

5 For the flat plate, the solution should remain constant with distance.
did not calculate a problem in compressible flow similar to that considered in this work but they claim the time per step was about one-quarter of a minute. The present author found the maximum time per step to be less than one-quarter of a minute. Considering that the IBM 7090 is at least five times faster than the Rice Computer, which is a conservative estimate, the author feels that the present method is superior to that proposed by Smith and Clutter. The method is believed to be much more flexible, it is easier to use, faster, at least as accurate, probably more accurate. The method proposed has no stability criteria\(^6\) while that of Smith and Clutter has a very restrictive stability criteria. It has the advantage that the stability analysis is complete which is also apparently true for Smith and Clutter's method\(^7\). No other general finite difference method has this advantage. As previously mentioned, it is not known whether the method of Smith and Clutter is applicable to totally non-similar flows with "overshoot" but it is known (Appendix C) that the present method is not. The normal direction step size used by Smith and Clutter was \(\Delta \beta = 0.05\). In this work, this step size was \(\Delta \beta = 0.20\). When this was reduced by one-half, the change in the wall shear near separation varied by less than two percent. Therefore, this method may have the possible additional advantage of requiring fewer

\(^6\)For flows without "overshoot".

\(^7\)Smith and Clutter state the criteria, but do not include the analysis.
normal direction grid points. For example, the incompressible wall shears at the leading edge obtained by the two methods were compared, see section E2, and although the two agree, the normal step size used in this work was four times that used by Smith and Clutter.

The advantages of this method over other numerical methods are essentially the same that were found by Smith and Clutter for their method. The boundary layer thickness of the transformed plane remains nearly constant while the physical plane thickness increases quadratically. This eliminates unnecessary calculations or unnecessary testing for the edge of the boundary layer, depending on the method employed. Accurate initial profiles are known, and although the flow geometry is nonsimilar, the solutions do not vary significantly with distance, i.e., the solution is nearly similar. Due to the similar-type transformation, the tangential step sizes can be relatively large. Also, it should be mentioned that the time required per step did not increase directly with an increase of the step size. Therefore, one should be able to find an optimum "step size-time per step" ratio for each flow or perhaps for each class of flows. The parameter for this optimum would be accuracy. If the initial region of the flow geometry is similar, the calculations can be started at the beginning of the non-similar region thereby eliminating the calculations within this region. The same would apply to flows with sections of similar geometries within them. These could be by-passed if their presence is known.
In summary, the equations which could possibly be used to calculate general boundary layer flows have been investigated. The finite difference methods that have been employed by previous authors as well as some derived specifically for this work were analyzed theoretically and in some cases numerically. The result is a very accurate, simple, and efficient method for calculating general boundary layer flows without "overshoot". Also, a new difference approximation is used to satisfy the heat flux wall conditions. This approximation has the advantages of satisfying the differential and difference equations and simultaneously having the same order of truncation error as the difference equation.
**TABLE OF NOMENCLATURE**

<table>
<thead>
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<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>A,B,C,D,E,F,G</td>
<td>Coefficients used in the Crout-Banachiewicz algorithm and defined by equations (3.27) and (3.28).</td>
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<tr>
<td>A,B,C,D,E,F,H,K,L,M,R</td>
<td>Variable coefficients associated with the stability analyses. These coefficients are defined in Appendix C.</td>
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<tr>
<td>a</td>
<td>Constant associated with the exterior velocity distribution; its value prescribes the slope of the plate with respect to the mainstream velocity or the slope of the channel walls with respect to the plate.</td>
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<tr>
<td>a*</td>
<td>Dimensional pressure gradient for the &quot;Howarth&quot; exterior velocity distribution.</td>
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<tr>
<td>a,b,c,d,e</td>
<td>Variable coefficients associated with the stability analyses. These are defined in Appendix C.</td>
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<td>c</td>
<td>Velocity of sound.</td>
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<td>$c_p$</td>
<td>Heat capacity of the fluid, $c_p = c_p^<em>/c_p^</em> = 1$.</td>
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<td>F</td>
<td>Function representing the viscosity-enthalpy relation in the Howarth-Dorodnitsyn and Görtler planes, $F(i) = \rho \mu$; (equation 2.34).</td>
</tr>
<tr>
<td>f</td>
<td>Function representing the viscosity-enthalpy relation in the physical plane, $f(i) = \mu$; (equation 2.23).</td>
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f Represents "function of" in equations (3.56) and (3.57).

f(α) Represents equations (2.60-2.62) when these equations specify the wall enthalpy.

h Stream function of the Görtler plane for incompressible flows—\( \psi(x, y) = (2\alpha) \frac{k}{h(\alpha, \beta)} \).

h Stream function of the Görtler plane for compressible flows—\( \psi(\xi, \eta) = (2Fa) \frac{k}{h(\alpha, \beta)} \).

i Enthalpy of the perfect fluid—\( i = i^*/c_o^2 \).

k Thermal conductivity of the fluid—\( k = k^*/(c_o^* \mu_o^*) \).

L* Dimensional length in the compressible plane—\( L^* = c_o^*/a^* \).

M Mach number.

p Static pressure of the flow at any position along the plate.

Pr Prandtl number of the fluid—\( Pr = c_p \mu/k = c_p^* \mu^*/k^* = Pr^* \).

r Recovery factor used in approximation of Busemann's relation and defined as \( r = \sqrt{Pr} \).

\( \Re_o^* \) Reynolds number at the stagnation state of the fluid—\( \Re_o^* = c_o^* L^* \rho_o^*/\mu_o^* \).

S Constant specifying the fluid's viscosity-temperature relation using Sutherland's viscosity law—\( S = S^* c_o^*/c_o^2 \).

T Temperature of the fluid—\( T = T^* c_p^*/c_o^2 \).
**t**  
Time variable, incompressible flow—
\[ t = t^* \]

**u**  
Tangential velocity component within the boundary layer—
incompressible flow,
\[ u = u^* / U_e^* (x^* = 0), \text{ compressible flow, } u = \frac{u^*}{c^*}. \]

**U_e**  
Exterior velocity component, incompressible and compressible flow.

**V**  
Normal velocity component within the boundary layer of the Howarth-Dorodnitsyn plane;
\[ V = \eta_x u + \rho v. \]

**v**  
Normal velocity component within the boundary layer—
incompressible flow,
\[ v = v^* \sqrt{v^* a^*} \text{; compressible flow, } v = \frac{v^*}{\sqrt{Re^*}} c^*. \]

**x**  
Tangential distance along the plate in the physical plane—
incompressible flow,
\[ x = \frac{x^* a^*}{U_e^* (x^* = 0)} \text{; compressible flow, } x = \frac{x^*}{L^*}. \]

**y**  
Normal distance to the plate—
incompressible flow,
\[ y = y^* \sqrt{y^* a^*}, \text{ compressible flow, } y = \frac{y^*}{\sqrt{Re^*}} L^*. \]

**α**  
Tangential distance along the plate in the Görtler incompressible and compressible planes—
\[ \alpha = \frac{x}{U_e}\int_0^x dx. \]

**α_i, α_ij**  
Constant multipliers in equations (3.56) and (3.57).

**s**  
Normal distance to the plate in the Görtler
plane--incompressible flow, $\beta = yU_e(x)/(2\alpha)^{1/2}$; compressible flow, $\beta = yU_e(x)/(2F\alpha)^{1/2}$.

Ratio of the specific heats of the perfect fluid.

Frequency function for the Fourier series common to the stability analyses, Appendix C.

Denotes the denominator function in the difference representation of the differential equations at the wall.

Distance between mesh points in the $t$-direction.

Distance between mesh points in the $x$-, $\alpha$-, and $\xi$- directions, respectively.

Distance between mesh points in the $y$-, $\beta$-, and $\eta$- directions, respectively.

Eigenvalue of the amplification matrix used in the stability analyses.

Small quantity used to specify accuracy of iteration methods.

Normal direction to the plate in the Howarth-Dorodnitsyn plane, $\eta = \int_0^y \rho dy$.

Parameter for generalizing difference equations, $0 \leq \varrho \leq 1$ is the range of $\varrho$.

"Principal function" of the Görtler transformation, $\lambda(\alpha) = 2\alpha d\ln U_e(\alpha)/d\alpha$.

Viscosity of the fluid--$\mu = \mu^* / \mu^*_0$.

Largest eigenvalue of the product of the
conjugate transpose amplification matrix with the amplification matrix.

\( \nu^* \)
Dimensional kinematic viscosity of the fluid, \( \nu^* = \mu^*/c^* \).

\( \xi \)
Tangential distance along the plate in the Howarth-Dorodnitsyn plane -- \( \xi = x \).

\( \rho \)
Density of the fluid -- \( \rho^* = \rho_0^* \).

\( \sigma \)
Restraining parameter used in solving the compressible similar equations, \( 0 \leq \sigma \leq 1 \).

\( \tau \)
Shear stress -- incompressible flow, \( \tau = \tau^* U_e^* (x=0) \sqrt{a_0^* c_0^* \mu^*} \); compressible flow, \( \tau = \tau^* \sqrt{Re_0^*} / (\rho_0^* c_0^* 2) \).

\( \psi(x,y), \psi(\xi,\eta) \)
Stream function of the physical plane or Howarth-Dorodnitsyn equations; \( \psi_x = -v \), \( \psi_y = u \); \( \psi_\xi = -V \), \( \psi_\eta = u \).

Subscripts:

\( \text{ad} \)
Adiabatic wall condition.

\( e \)
Local flow outside the boundary layer, exterior flow.

\( f \)
Fixed distance downstream at which solutions obtained by various methods are compared for accuracy.

\( j \)
Designation of mesh points in time, \( t = (j-1) \Delta t \).

\( k \)
Designation of mesh points in \( y^- \), \( \xi^- \), or
\( \eta \) - directions; \( y = (k-1)\Delta y, \ \bar{\xi} = (k-1)\Delta \bar{\xi}, \)
or \( \eta = (k-1)\Delta \eta. \)

Designation of mesh points in \( x-, \alpha-, \) or \( \xi- \) directions; \( x = (i-1)\Delta x, \ \alpha = (i-1)\Delta \alpha, \)
or \( \bar{\xi} = (i-1)\Delta \bar{\xi}. \)

\( o \)
Stagnation state of the fluid.

\( w \)
Wall or surface value.

\( \infty \)
Infinite flow value or free stream value.

Superscripts

\( \ast \)
Dimensional quantities with consistent units.

\( \infty \)
Denotes steady-state flow for the pseudo-transient flow problem.

Other notations

\( |_{y} \)
Means a function evaluated at constant \( y. \)

\( |_{w} \)
Means a function evaluated at the surface or the wall.

HD
Abbreviation for Howarth-Dorodnitsyn.
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APPENDIX A

Derivation of the Howarth-Dorodnitsyn Equations

The HD transformation is generally assumed straightforward in the literature but an attempt to validate it proved that it was not. The momentum and energy equations are straightforward so only the momentum equation will be transformed in this appendix. The continuity equation caused difficulties, so it will be considered first. The continuity equation for compressible flow in the physical plane is

\[(\rho u)_x + (\rho v)_y = 0.\]  \hspace{1cm} A.1

The new space variables

\[\xi = x\]  \hspace{1cm} A.2

\[\eta = \int_0^y \rho(x,y)dy\]  \hspace{1cm} A.3

and the new normal velocity component

\[V = (\eta_x)_y u + \rho v\]  \hspace{1cm} A.4

were defined by Howarth and Dorodnitsyn. Using (A.2) and (A.3), one obtains the relations between the differential operators of the two coordinate systems, i.e.,

\[\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + (\eta_x)_y \frac{\partial}{\partial \eta}\]  \hspace{1cm} A.5

and

\[\frac{\partial}{\partial y} = \rho \frac{\partial}{\partial \eta}\]  or  \[\rho = \frac{\partial \eta}{\partial y}.\]  \hspace{1cm} A.6
Using these relationships and defining

\[ a = \frac{\eta_x}{y} \]  

the continuity equation is now written as

\[ (\rho u)_{\xi} + a(\rho u)_{\eta} + \rho(\rho v)_{\eta} = 0. \]  

Taking the derivative with respect to \( \eta \) of \( V \) and substituting for \( \rho(\rho v)_{\eta} \) in (A.8), one obtains

\[ (\rho u)_{\xi} + a(\rho u)_{\eta} + \rho V_{\eta} - \rho(au)_{\eta} = 0. \]  

Expanding these derivatives and substituting (A.6) for \( \rho \) except in the terms \( \rho u_{\xi} \) and \( \rho V_{\eta} \), one obtains

\[ \rho u_{\xi} + \rho V_{\eta} + u_{\xi} y_{\xi} + a_{\eta} y_{\eta} + a_{\eta} y_{\eta} - a_{\eta} y_{\eta} - u_{\eta} y_{\eta} = 0. \]  

Noting that

\[ \eta y_{\xi} = \eta_{\xi} y = 0 \]  

\[ \eta y_{\eta} = \eta_{\eta} y = 0 \]  

\[ a_{y} = (\frac{\eta_x}{y} y) = 0 \]  

and that

\[ \eta y a_{\eta} = a_{y} \]  

it becomes obvious that (A.10) reduces to the proper form of the continuity equation for the HD plane, i.e.,

\[ u_{\xi} + V_{\eta} = 0. \]
The physical plane momentum equation is

$$\rho u_u + \rho v_{u_y} = -\rho e' + f_{u_{yy}} + f'u_{y_{yy}}$$  \hspace{1cm} A.14

with

$$f = \left(\frac{1+S/T_0}{i/i_o + S/T_0}\right)^{3/2}$$  \hspace{1cm} A.15

$$f' = f\left[\frac{i/i_o + 3S/T_0}{2i(i/i_o + S/T_0)}\right].$$

Using the differential operator relationships, (A.5 and A.6), and the definition of the velocity function, (A.4), one can show that (A.14) becomes

$$u_u + V_{u_{\eta}} = -\rho e'/\rho + f_{\rho u_{\eta}} + f_{\rho u_{\eta}} + f'u_{\eta}i_{\eta}.$$  \hspace{1cm} A.16

Then, by defining

$$F = \rho f$$  \hspace{1cm} A.17

and

$$F' = \rho e_i e d/di(\mu/i)$$  \hspace{1cm} A.18

and using the fact that

$$\rho = \rho e i_e / i$$

it is easy to show that

$$f_{\rho u_{\eta}} + f'_{\rho u_{\eta}}i_{\eta} = F'_{u_{\eta}}i_{\eta}$$

thereby reducing (A.16) to the momentum equation for the HD plane, i.e.,

$$u_u + V_{u_{\eta}} = -\rho e'/\rho + F_{\eta}u_{\eta} + F'_{u_{\eta}}i_{\eta}.$$  \hspace{1cm} A.19
with

\[ F = \rho f = \frac{\rho e i e_f}{i} = \frac{\rho e i e}{i_0} \frac{1+S/T_0}{1/i_0 + S/T_0} \sqrt{i/i_0} \]  \hspace{1cm} A.20

and

\[ F' = \rho e i e \frac{d}{di}(f/i) = \frac{F}{2i} \frac{(S/T_0 - i/i_0)}{(S/T_0 + i/i_0)}. \]  \hspace{1cm} A.21

The only assumption involved in this derivation is the relation between the density and the enthalpy since \( f \) represents the viscosity. Having obtained the momentum equation along with the new definition of the viscosity function (A.20) and its derivative (A.21), the energy equation's transformation is straightforward.
APPENDIX B

Finite Difference Analogues

The finite difference analogues of derivatives with respect to tangential distance and time were included in the difference equations of Chapter III. The normal direction derivatives were written as derivatives evaluated at a particular grid point of the flow field and the proper difference analogues for these derivatives are

\[
(H_y)_{i,k} = \frac{H_{i,k+1} - H_{i,k-1}}{2\Delta y} \tag{B.1}
\]

\[
(H_{yy})_{i,k} = \frac{H_{i,k+1} - 2H_{i,k} + H_{i,k-1}}{(\Delta y)^2} \tag{B.2}
\]

\[
(H_{yyy})_{i,k} = \frac{H_{i,k+2} - 2H_{i,k+1} + 2H_{i,k-1} - H_{i,k-2}}{2(\Delta y)^2} \tag{B.3}
\]

where \(H\) and \(y\) are dummy variables. These difference formulas are valid for explicit and implicit analogues but the truncation error of the explicit and ordinary implicit analogues is order of \(O(\Delta t, \Delta x, \Delta y^2)\) while that of the Crank-Nicolson implicit analogue is order of \(O(\Delta t^2, \Delta x^2, \Delta y^2)\). The subscript \(i+1\) can be substituted for \(i\) when indicated by the difference equation of Chapter III.
APPENDIX C

Stability Analyses

Based on the assumptions given in Chapter III, the finite difference equation's solution converges in the limit to the exact solution of the partial differential equation it represents if the difference equation is consistent and stable. Except for the Dufort-Frankel difference analogue, which was discussed in Chapter III, the truncation error of the difference equations is order of $O(\Delta a', \Delta b^m, \Delta c^n)$ where $\Delta a$, $\Delta b$, and $\Delta c$ are dummy step increments and $\lambda$, $m$, and $n$ are positive integers. $\Delta a$ represents the step increment corresponding to the propagation (time or distance) direction of the partial differential equation and for steady-state problems, $\Delta c$ is absent. Based on the definition given in Chapter III, the equations are consistent.

To complete the convergence proof, the difference equations must be shown to be stable. The von Neumann method (32b) will be used which involves obtaining an amplification matrix for the difference equation(s) and showing that the spectral radius or the largest eigenvalue of the amplification matrix remains bounded by the relation

$$\delta \not= 1 + O(\Delta a). \quad C.1$$

The condition (C.1) is necessary and sufficient for two level difference equations but it is only necessary for higher level difference equations. Several sufficient conditions are available (32c) but one of importance in this
work is

\[ |\mu_i| \leq 1 + O(\Delta a) \]  \hspace{1cm} \text{(C.2)}

where \( \mu_i \) is the largest eigenvalue of \( G^*(\Delta a, m, n)G(\Delta a, m, n) \) with \( G^*(\Delta a, m, n) \) being the conjugate transpose of \( G(\Delta a, m, n) \).

The difference equations contain growth terms such as the pressure gradient term common to the physical plane equations and the "principal function" common to the Göertler transformed equations. If these terms are bounded, they will not affect the stability since they can easily be shown to be order of \( O(\Delta a) \). The bounds of these terms are discussed in Chapter IV.

The two level difference equation's amplification matrix is a one by one matrix so for these cases only the eigenvalue relation will be presented. For the HD plane equations, the difference equations could be considered two level but due to the simultaneous solution, they should be considered a tri-level system. The amplification matrix and the eigenvalue relation are generally complicated to work with mathematically so calculus is not always used to determine the maximum of the eigenvalues.

A. Incompressible Göertler Equation.

For stability analyses, the ordinary implicit (3.12) and the Crank-Nicolson implicit (3.13) difference equations can be written as

\[ \tilde{\varepsilon}(h_{zzz})_{l+1,k} + (1-\varepsilon)(h_{zzz})_{l,k} + \varepsilon h(h_{zzz})_{l+1,k} + (1-\varepsilon)h(h_{zzz})_{l,k} \]
\[
- \delta \lambda h_\delta (h_\delta)_{\ell+1,k} - (1-\delta) \lambda h_\delta (h_\delta)_{\ell,k}
\]

\[
= \frac{2\alpha h_\delta}{\Delta \delta} [(h_\delta)_{\ell+1,k} - (h_\delta)_{\ell,k}] - \frac{2\alpha h_\delta}{\Delta \delta} [h_{\gamma+1,k} - h_{\gamma,k}]
\]

where the coefficients are assumed constant and are evaluated at the proper point within \( \gamma, k \leq \delta \leq \gamma + 1, k \) depending on the value of \( \delta \). The relation (C.3) reduces to the ordinary implicit difference equation when \( \delta = 1 \) and to the Crank-Nicolson implicit difference equation when \( \delta = \frac{1}{2} \). Theoretically the range of \( \delta \) is zero to unity but the above mentioned values are the two of most interest.

Using the von Neumann method, the magnitude of the eigenvalue can be expressed as

\[
|\delta|^2 = \frac{(A+(1-\delta)B(1-\cos \gamma))^2 + \sin^2 \gamma \left((1-\delta)[C(1-\cos \gamma) + D] - E\right)^2}{(A-\delta B(1-\cos \gamma))^2 + \sin^2 \gamma \left[\delta [C(1-\cos \gamma) + D] + E\right]^2}
\]

with

\[
\gamma = k \Delta \delta
\]

\[
A = 2\alpha h_{\delta \delta}
\]

\[
B = 2\Delta \alpha / (\Delta \delta)^2
\]

\[
C = 2\Delta \alpha / (\Delta \delta)^3
\]

\[
D = \Delta \alpha h_{\delta} / \Delta \delta
\]

\[
E = 2\alpha h_{\delta} / \Delta \delta
\]

If one assumes the difference equations (C.3) have no
"growth" terms, the relation (C.1) is modified to

\[ 2AB(1 - \cos \gamma) + (1 - 2\varepsilon)B^2(1 - \cos \gamma)^2 \]

\[ \equiv \sin^2 \gamma [2E[C(1 - \cos \gamma) + D] - (1 - 2\varepsilon)[C(1 - \cos \gamma) + D]^2]. \]

When \( \cos \gamma \) equals unity, this inequality becomes an identity but when \( \cos \gamma \) equals minus unity, (C.6) reduces to

\[ A \equiv B(2\varepsilon - 1) \]

which is the first stability criterion. This expression states that the difference equation (C.3) is unstable when \( \varepsilon \) is zero since \( A \) must be allowed to approach zero at the edge of the boundary layer. Since \( \Delta \alpha \) must be allowed to approach zero, (C.7) remains valid for positive \( \Delta \alpha \) and \( \Delta \beta \) if

\[ h_0 \beta \equiv 0 \text{ and } \frac{1}{2} \equiv \theta \equiv 1. \]

If (C.8) remains satisfied and \( D \) remains positive, which corresponds to a positive "principal function", it becomes obvious that the system is always stable, consistent, and therefore convergent.

Assume (C.8) is satisfied and \( D \) is nonpositive. Rearranging (C.6) to the form

\[ (1 - 2\varepsilon)[B^2(1 - \cos \gamma)^2 + \sin^2 \gamma[C(1 - \cos \gamma) + D]^2 \]

\[ \equiv 2|A|B(1 - \cos \gamma) + 2E \sin^2 \gamma[C(1 - \cos \gamma) + D] \]

which can be expressed in the form

\[ \equiv \kappa_t + \varepsilon. \]
For the range of $\gamma$ between one-half and unity, it is obvious that
\[
\max_{\gamma} \varepsilon = 0 \quad \text{C.11}
\]
and
\[
\min_{\gamma} \tau = 0 \quad \text{C.12}
\]
so it is necessary and sufficient that
\[
\min \varepsilon \equiv 0 \quad \text{C.13}
\]
if (C.5) is to be satisfied. Since $E$ is always positive, define $\chi$ to be
\[
\chi = (2E)^{-1} \varepsilon = (C - |D| - C \cos \gamma) - \cos^2 \gamma (C - |D| - C \cos \gamma). \quad \text{C.14}
\]
Using calculus, the maxima of $\chi$ occur at
\[
\cos \gamma = +1 \quad \text{C.15}
\]
\[
\cos \gamma = (3C)^{-1} (C - |D| - [3C^2 + (C - |D|)^2]^{1/2}) \quad \text{C.16}
\]
and the minima occur at
\[
\cos \gamma = -1 \quad \text{C.17}
\]
\[
\cos \gamma = (3C)^{-1} (C - |D| + [3C^2 + (C - |D|)^2]^{1/2}). \quad \text{C.18}
\]
Assuming $D$ to be very small, one finds the following approximate positions for minima and maxima in the range $0 \leq \gamma \leq \pi$:
\[
\begin{align*}
\gamma = \pi & \quad \text{minima} \\
\gamma \approx (5/9)\pi & \quad \text{maxima} \quad \text{C.19} \\
\gamma \approx 0 & \quad \text{minima} \\
\gamma = 0 & \quad \text{maxima}.
\end{align*}
\]
Knowing these positions and knowing that $\chi$ is zero when $\gamma = 0$ it becomes obvious that $\chi$ and therefore $\varepsilon$ must be negative when $\gamma$ is given by (C.18). Thus, (C.13) is false. If one defines

$$X = \cos \gamma$$

and uses the fact that $\partial \chi / \partial \gamma$ is zero when $\gamma$ is given by (C.18), the relation for $\chi$ can be expressed in the form

$$\chi = (1 - X)C[2X^2 + X + 1] - |D|[1 + X^2] \quad \text{C.20}$$

from which it is obvious that the minimum value of $\chi$ must be greater than $-2|D|$. Thus,

$$\varepsilon \equiv -4E|D| = -4\lambda|\Delta \alpha(\Delta \beta^B)^2 \leq 0(\Delta \alpha) \quad \text{C.21}$$

so (C.4) can be expressed in the form

$$|\delta| \equiv \frac{1}{1 - 0(\Delta \alpha)} \leq 1 + 0(\Delta \alpha) \quad \text{C.22}$$

which proves the system to be stable, consistent, and converger for positive or negative $D$ if the stability criterion (C.8) is not violated.

**B. Compressible Göertler Equations**

These equations, momentum (2.45) and energy (2.46), were solved separately, thus, stability criteria must be given for both systems. Growth terms will be ignored but it should be mentioned that the energy equation's growth term involves $h_{BB}$. If the momentum difference equation's solution is convergent, then $h_{BB}$ is bounded and the growth term is order of $O(\Delta \alpha)$. 

1. Momentum Equation. For the stability analysis, the ordinary implicit equation (3.45) and the Crank-Nicolson implicit difference equation (3.46) can be expressed jointly in the form

\[
(1-\xi)(h_{\xi\xi \xi})_{\xi,j,k} + \xi(h_{\xi\xi \xi})_{\xi+1,j,k} + (1-\xi)[(F'/F)i_{\xi}+h](h_{\xi\xi})_{\xi,j,k} \\
+ \xi[(F'/F)i_{\xi}+h](h_{\xi\xi})_{\xi+1,j,k} - (1-\xi)\lambda h_{\xi}(h_{\xi})_{\xi,j,k} - \xi\lambda h_{\xi}(h_{\xi})_{\xi+1,j,k} \\
= \frac{2\Delta h_{\xi}}{\Delta \xi} [(h_{\xi})_{\xi+1,j,k} - (h_{\xi})_{\xi,j,k}] - \frac{2\Delta h_{\xi\xi}}{\Delta \xi} (h_{\xi+1,j,k} - h_{\xi,j,k}) \quad C.23
\]

where the coefficients are assumed constant and are evaluated at the proper point within \( \xi,j,k \) depending on the value of \( \xi \). The relation (C.23) reduces to the ordinary implicit difference equation when \( \xi \) equal unity and to the Crank-Nicolson implicit difference equation when \( \xi \) equal one-half. The range of \( \xi \) is zero to unity but the values one-half and unity are of primary interest.

Using the von Neumann method, the magnitude of the eigenvalue is given by (C.4) with

\[
\gamma = k\Delta \xi \\
A = 2\alpha h_{\xi\xi} \\
B = 2\Delta \alpha [(F'/F)i_{\xi}+h]/(\Delta \xi)^2 \\
C = 2\Delta \alpha/(\Delta \xi)^3 \\
D = \Delta \alpha \lambda h_{\xi}/\Delta \xi
\]
\[ E = \frac{2\alpha h}{\Delta \dot{\sigma}} \]

Since the eigenvalue relation for this system is identical in form to (C.4), the stability analysis will yield the same result, i.e.,

\[ A \equiv B(2\dot{\sigma} - 1) \]  

C.24

from which one obtains the criteria

\[ \alpha h \dot{\sigma} \equiv \left[ \frac{\Delta \alpha}{(\Delta \dot{\sigma})^2} \right] \left[ (F'/F)i + h \right] (2\dot{\sigma} - 1). \]  

C.25

Since \( \Delta \alpha \) must be allowed to approach zero, this criteria reduces to

\[ h \dot{\sigma} \equiv 0 \quad \frac{1}{2} \equiv \theta \equiv 1. \]  

C.26

The other conclusions of the preceding section also apply.

2. Energy Equation. For this stability analysis, the known growth terms will be neglected and the ordinary implicit difference equation (3.46) and the Crank-Nicolson implicit difference equation (3.48) can be expressed jointly in the form

\[ (1-\theta)(i_{\dot{\sigma}\theta})_{\dot{\sigma},k} + \theta(i_{\dot{\sigma}\theta})_{\dot{\sigma}+1,k} + (1-\theta)[(F'/F)i_{\dot{\sigma}}_{\dot{\sigma},k} + \Pr(h+2\alpha h)](i_{\dot{\sigma}} + \theta [(F'/F)i_{\dot{\sigma}} + \Pr(h+2\alpha h)](i_{\dot{\sigma}}_{\dot{\sigma},k} + 1,k) = \frac{2\alpha \Pr h}{\Delta \alpha} \frac{\dot{\sigma}}{\dot{\sigma}}(i_{\dot{\sigma},k} - i_{\dot{\sigma},k}). \]  

C.27

where the coefficients are assumed constant and are evaluated
at the proper point within \( \xi, k \leq \xi \leq r+1, k \) depending on the value of \( \theta \). The relation (C.2) reduces to the ordinary implicit difference equation when \( \theta \) is unity and to the Crank-Nicolson implicit difference equation when \( \theta \) equal one-half.

Using the von Neumann method, the magnitude of the eigenvalue can be expressed as

\[
|\delta|^2 = \frac{[A-(1-\theta)B(1-\cos \gamma)]^2 + C^2(1-\theta)^2 \sin^2 \gamma}{[A+\theta B(1-\cos \gamma)]^2 + C^2 \theta^2 \sin^2 \gamma}
\]

with

\[
\gamma = k\Delta y
\]

\[
A = 2\alpha Pr h_y
\]

\[
B = 2\Delta x / (\Delta y)^2
\]

\[
C = (\Delta x / \Delta y)[(F'/F) i_y + Pr(h+2\alpha h_y)].
\]

Assuming the finite difference equation (C.27) has no growth terms, the relation (C.5) is used to express (C.28) as

\[
-2AB(1-\cos \gamma) + (1-2\theta)[B^2(1-\cos \gamma)^2 + C^2 \sin^2 \gamma] \leq 0.
\]

(C.29)

The second term on the LHS of (C.29) is always negative for \( \frac{1}{2} \leq \theta \leq 1 \) and its maximum value is zero. Therefore the necessary and sufficient condition for stability is

\[
-2AB(1-\cos \gamma) \leq 0
\]

(C.30)

which requires that

\[
A \geq 0 \quad \text{or} \quad h_y \geq 0.
\]

(C.31)

This is guaranteed by considering flow with positive tangential
velocity components \((u)\).

C. Compressible Howarth-Dorodnitsyn Equations.

Using a simplified form of the momentum equation, Blottner (10) used the von Neumann method to show that these equations are always stable if the tangential velocity component remains positive. The energy and momentum equations were solved simultaneously so one should determine the conditions under which the simultaneous system is stable. Since the equations, matrices, and eigenvalue relations are rather complicated, only the Crank-Nicolson difference equations will be considered. For this system, one obtains

\[
G^*G = \begin{bmatrix}
(A^2 + F^2 + E^2) & (H + Ki) \\
(H - Ki) & (A^2 + B^2 + D^2)
\end{bmatrix} \frac{L^2 + M^2}{R^*R} \tag{C.32}
\]

with

\[
A = 1 + 2c(1 - \cos \gamma) \\
B = 2(a - b)\sin \gamma \\
D = 2d \sin \gamma \\
E = 2e \sin \gamma \\
F = 2(a - 2b) \sin \gamma \\
H = DF + EB \\
K = A(D - E) \\
L = [1 - 2c(1 - \cos \gamma)]
\]
\[ M = 2a \sin \gamma \]
\[ R^R = (A^2 + B^2) (A^2 + F^2) + D E [D E + 2A^2 - 2BF] \]

and

\[ a = V_{\Delta \xi} / 4u_{\Delta \eta} \]
\[ b = F^{'} i_{\eta} \Delta \xi / 4u_{\Delta \eta} \]
\[ c = F_{\Delta \xi} / 2u_{\Delta \eta} \]
\[ d = F^{'} u_{\eta} \Delta \xi / 4u_{\Delta \eta} \]
\[ e = F u_{\eta} \Delta \xi / 2u_{\Delta \eta}. \]

The assumptions necessary to obtain (C.23) are:

a) the Prandtl number is unity, and

b) the Fourier series used in the von Neumann method to represent the velocity and enthalpy are in phase with each other.

Although the above matrix is difficult to work with, one can show that the system is always stable when \( \cos \gamma \) equal unity and, when \( \cos \gamma \) is minus unity, one obtains the criterion Blottner obtained, i.e., the tangential velocity must remain positive. For other values of \( \cos \gamma \), no simplified criteria could be obtained.

The simplified equation Blottner considered for the stability analysis assumed the following:

a) \( F' \) is zero,

b) the difference equations are solved separately, and

c) the energy equation is stable if the momentum equation is stable.
These assumptions do not consider the interdependence of the two equations and the author feels that the simultaneous solution has additional criteria. This can only be shown by determining the conditions under which the eigenvalue of (C.32) is less than unity. To do so, one must find the conditions under which (21b)

$$\left( L^2 + M^2 \right)^2 \neq R^* R $$  \hspace{1cm} C.33

and

$$ \frac{L^2 + M^2}{R^* R} \left[ 2A^2 + B^2 + D^2 + E^2 \right] \neq 1 + \left( \frac{L^2 + M^2}{R^* R} \right)^2 $$  \hspace{1cm} C.34

are always true. Therefore, one cannot guarantee the system to be always stable but instead one must assume stability since checking the expressions (C.33) and (C.34) at each step would consume a prohibitive amount of computer time.