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A MATHEMATICAL MODEL FOR RELATING EEG

TO CERTAIN STIMULUS FIELDS

by

Ashley James Welch

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INTRODUCTION

The discovery of electrical activity in the brain by Hans Berger in 1928 and the confirmation of his work by Adrian and Matthews in 1935 opened new fields of research for the scientific community. One field, Electroencephalography, is a study of electrical potentials, other than neuron spikes, which are associated with the brain's action. These potentials are assumed to be some epiphenomenon of the brain's processing of information, physiological state, and neurological state. Research in this field is generally related to the problem of determining practical means of utilizing the EEG to answer specific clinical and experimental questions. For this purpose several models have been developed which attempt either to relate the measured electrical activity of the brain to some type of elementary signal source or to use the model as an aid for interpreting the EEG.

This research is concerned with first proposing a mathematical model which relates the brain's response to certain stimulus fields and then testing the usefulness of the model with experimental data. The proposed model, which is based upon certain fundamental concepts of statistical theory of communication and control system theory, describes the brain's processing of information in terms of operations on the input signal (stimulus). The nature of these operations is assumed to be implicit in the output signals (EEG). The electrical activity of the brain is represented by this model as a lumped-parameter system whose transmission operators are members of a multi-parameter random process. This formulation enables us to account for the effects of
the physio-neurological state and the inherent randomness of evoked responses in the multiple parameter transmission operators. The consideration of these operators as members of a random process leads to several interesting input-output relationships in terms of "mathematical expectation" and "time averages". These relations are applied to measurements of the EEG from the visual cortex of an unrestrained cat during photic stimulation tests.
1-1. **Characteristics of the EEG.**

It is not the purpose of this research to present a complete exposition of the nature and origin of electrical activity in the brain. Nevertheless, a brief outline of the signals generated by individual neurons and neurons "en masse" is described in this section in order to provide the reader with an introduction to the literature on neuron theory and its relation to the EEG. The following material was obtained from Ackerman [I-1-1], Whitteridge [I-1-2], Wyke [I-1-3] and Purpura [I-1-4].

The authors named above introduce their discussions of the nature of the EEG, with a review of the fundamental properties of individual neurons. Special emphasis is placed upon the neuron since it is the basic unit for transmitting information at a rapid rate in the central nervous system. Bundles of sensory neurons carry information into the central nervous system and another set called motor neurons carry "command" instructions to the rest of the body. Neurons which are completely within the central nervous system are designated interneurons. Stimulation of a neuron produces a depolarization spike potential which travels from the point of stimulation to the ends of the neuron. Between the junction of two neurons we find the synapse which limits the transmission of information to one direction. The conduction of a spike potential from one neuron to another requires a chemical transmission in the synapse which introduces a time delay at each junction. It is known that in some synapses the chemical transmission produces an inhibition rather than excitation of adjoining neurons.
The known and unknown nature of the synapses and the parameters which control the excitation threshold of neurons has made it impossible to describe analytically a physical or subjective process of the body in terms of the behavior of a population of neurons. Wyke [I-1-5] states "When large numbers of neurons are found in combination as a functional unit, they are also found to possess special properties which cannot as yet be explained by simple extrapolation from the theory of the individual neuron model". From the conclusions of Wyke and Ackerman concerning the origin of the EEG, it appears that the EEG is more than a sum of individual neuron spike potentials.

Purpura, in a lengthy and well-documented paper, presents a detailed summary of the past and current theories of "cortical electrogenesis". His work postulates sources for the spontaneous and evoked potentials of the brain. He notes that attempts to define the nature of the EEG have generally been based upon interpreting the net activity of the cortex in terms of the properties and behavior of the neuron. Two major sources of the neuron which contribute to the brain's potential field are "all-or-none" and grade responses. Associated with each all-or-none response is a small, slow after potential which follows the spike response. The graded responses are classified as (1) electrically excitable graded responses or (2) electrically inexcitable graded responses. The first is a depolarization which precedes the development of the all-or-none spike. The second type of graded response may be a signal generated in the post synaptic membrane, which indicates junctional transmissions. These responses are called "electrically inexcitable" since electrical stimulation of the post-
synaptic site does not produce the graded response. These postsynaptic potentials may be due to either electrical or synaptic stimulation of dendrites.

Purpura concludes that the electrical response of excitable cells may be accounted for by all-or-none and synaptic potentials; "the latter arising in membrane which is chemically, but not electrically, excitable. The degree to which conductile (all-or-none) and synaptic responses are detectable in the cortical surface recordings is dependent upon the source and size of an afferent volley as well as the manner in which afferent excitation of cortical neurons is effected. Because of temporal dispersion and algebraic summation, conductile activity is not reflected in the spontaneous cortical rhythms." However, "brain waves are adequately accounted for as summations of post synaptic potentials..." [I-1-6]. He states further that, "Evoked cortical potentials are compounded of depolarizing and hyperpolarizing p.s.p's (post synaptic potentials) of elements synaptically interconnected by different conductile pathways." [I-1-7].
1-2. **Review of Previous Models Used in Connection with the EEG.**

Several models have been proposed in order to relate the form of the electric field of the brain to a set of fundamental signal sources. Walter [I-2-1] postulates that the cortex may be considered as a uniformly conducting sphere with an internal generator. He suggests that the generator may be represented as an alternating dipole. This concept was extended by Bishop to fit experimental data more closely. In a review of Bishop's work, Wyke [I-2-2] states "he (Bishop) envisages three geometrical situations (geometry of sources)...which correspond with three types of cellular arrangement in the central nervous system." Computations and experimental results for the dipole model have been compared by Geisler [I-2-3]. He represented the source of EEG as an equivalent dipole located within a sphere (brain) of uniform conductivity. The sphere was surrounded by a shell (skull) of variable conductivity. Geisler concluded in his thesis that the potential field of an animal does not really have a dipole-like distribution.

Wiener [I-2-4] developed a theory of nonlinear coupled oscillators to account for the predominant \( \alpha \) rhythm of the brain. The system of oscillators are reported to produce a spectrum consisting of a narrow line with single lobes on each side of the line. Although this spectrum is assumed to be similar to the one produced by the spontaneous EEG, it does not seem to fit many of the published "normal" EEG spectra. The concept of an oscillator or pacemaker has also been expressed by Aird and Garoutte [I-2-5]. They postulated the existence of "the cerebral pacemaker" which controls the synchronization of the two
hemispheres at all times.

Models and data representation techniques which have utilized "systems analysis" techniques have been described by Freeman [I-2-6], Adey and Walter [I-2-7], and Sato et al [I-2-8]. Freeman has analyzed average responses due to single shocks in terms of a linear second-order differential equation. Reconstruction of typical average evoked potentials on an analog computer required differentiation of the impulse response of the second-order system and mixing this response with additional signals. These signals consisted of "frequencies having one to three discrete means and a normal distribution about each mean". Adey has used cross-spectral methods to relate the relative amplitude and phase patterns between EEG records to performance of conditioned cats. These patterns are designated by Adey as "transfer functions", although it is not apparent that his methods yield a transfer function in the sense of control system theory.

The work of Sato et al, recognizes that the basic randomness found in evoked responses should be associated with a linear system. They formulated an input-output relation based upon this assumption; however, their mathematical development is untenable and produces results which could not be interpreted.
REFERENCES


II.

Mathematical Background

The most striking feature of biopotentials recorded from gross electrodes in the brain is the randomness of the analog record. Even when every effort is made to maintain the experimental parameters at set values for each presentation of the stimulus field, there is a degree of uncertainty associated with the characteristics of the EEG. Goldstein [II-0-1] notes that "Evoked responses are random in that repeated presentations of a stimulus do not produce identical responses ...". Because of this uncertainty, it is reasonable to consider the various EEG as members of a random process.

The randomness of the EEG and its apparent adherence to the basic principle of "cause and effect" have led to the postulation of a mathematical model for the brain's electrical response to stimulus fields. This model is based upon certain fundamental concepts of probability theory and control theory.

Before describing the model, several mathematical definitions are presented which are necessary for the development of the model. Much of the notation and formal definitions of basic concepts used in this thesis follows the recent text by P. E. Pfeiffer [II-0-2].
2-1. Random Process

The concept of a random process is stated by Bendat [II-1-1] as a "...set of records, which can be described by certain statistical properties...". Bendat proceeds to explain these "statistical properties" with definitions and intuitive logic. If the reader is primarily interested in the application of statistical theory, Bendat's paper provides sufficient background for an understanding of the postulated model and the experimental results of this research. However, for those who wish a more complete development of the model it is essential to define a random process precisely. In general a random process $X(\cdot,\cdot)$ may be considered to be a set of time functions which are basic elements of a probability space. For each selected time $t \in T$, a random variable $X(\cdot,t)$ is defined on the probability space. This may be stated in a formal definition as:

**Definition 2-1:** Given a probability space $[S, \mathcal{E}, P(\cdot)]$, a random process $X(\cdot,\cdot)$ is a family of random variables $\{X(\cdot,t) : t \in T\}$, where $T$ is called the index set or parameter set for the process.

The probability space consists of three items: (1) a basic space $S$ of elements $\xi$, (2) a completely additive class $\mathcal{E}$ of events, and (3) a probability measure $P(\cdot)$ defined for each event in the class $\mathcal{E}$. For example, consider that a stimulus field can produce a number of possible responses. The entire set of possible responses is included in the basic space containing all possible EEG. Each element (response or sequence of responses) of the set is designated by a
choice variable \( \xi \). Thus the selection of a \( \xi \) from the subset of all possible evoked responses, represents the selection of one of the possible evoked responses.

The concept of the probability space also carried with it the idea of a completely additive class \( \mathcal{E} \) of events (subsets). This suitable additive class is assumed to include as members all of the significant subsets associated with the basic space. In the example of a basic space of all the possible responses to some stimulus, an event is characterized by the subset of responses having the specified property of the event. We may be interested in the event \( A \) that the amplitude of the evoked response is greater than 50 \( \mu \) volts. The subset of responses with amplitudes greater than 50 \( \mu \) volts defines the event \( A \). The selection of a \( \xi \) which belongs to the set \( A \) corresponds to the occurrence of the event \( A \).

Other propositions about the evoked response lead to other events and their associated sets of \( \xi \). For this thesis all possible subsets of the basic space make up the class of events \( \mathcal{E} \).

The last part of our probability space is the probability measure \( P(\cdot) \) which is associated with events. Thus probability is a function of sets; that is, it assigns a number \( P(A) \) to each set \( A \) in the class of events. The probability \( P(S) \) of the event whose set is the entire basic space is unity, whereas the probability \( P(A) \) of any event \( A \) is greater than or equal to zero.

Now that we have established the abstract nature of the random process in terms of a probability space, there remains the problem of using these concepts to make useful calculations and decisions in our "real world". Laning and Battin [II-1-2] suggest that a practical set
of statistical parameters should meet three requirements: (1) "... they must be calculatable, either from a theoretical model of our problem or directly from the analysis of empirical data...", (2) "...a knowledge of their values for the inputs to a control system enables us to compute their values for the output as well...", and (3) "...they afford the opportunity for making sufficient statements concerning the quality of performance of the system studied." These are almost exactly the requirements of the statistical parameters needed for this research, except the second requirement needs to be generalized to include the ability to determine their value for any unknown quantity in the input-output relations of a control system type model.

The basic parameter which has been used classically to describe a random process is the distribution function. With the aid of distribution functions, several meaningful statistical parameters in the form of expectations (probability set averages) of the random process may be determined. Once again the reader is referred to the literature in order to supplement the brief definitions which follow.
2-2. Distribution Functions

If we consider a random process $X(\cdot, \cdot)$ with probability space $[S, \mathcal{E}, P(\cdot)]$, the fundamental definitions for the probability distribution functions for the random variable $X(\cdot)$ are easily extended to the random process.

**Definition 2-2a:** The first order distribution functions

$F_X(\cdot, \cdot)$ for a random process $X(\cdot, \cdot)$ is the set $\{F_X(\cdot, t): t \in T\}$

where $F_X(x, t) = P[X(\cdot, t) \leq x]$ for every real $x \in V$ and $t \in T$.

For this thesis, the random processes are functions of time; thus the index set $T$ refers to time and the index set $V$ denotes possible values of the sample functions.

**Definition 2-2b:** The second order distribution functions

$F_X(\cdot, ; ; s)$ for the random process $X(\cdot, \cdot)$ is the set

$\{F_X(\cdot, t; s) : t \in T, s \in T\}$, where $F_X(x, t; y, s) = P[X(\cdot, t) \leq x ; X(\cdot, s) \leq y]$ for every real $x, y \in V$ and $t, s \in T$.

For the special (but important) case in which the first order distribution function is the same for each choice of $t \in T$ and the second order distribution function obeys the relation

$F_X(\cdot, t; s) = F_X(\cdot, t + h; s + h)$

for all $t, t + h, s + h \in T$, the random process is said to be (at least) second order stationary or stationary in the wide sense. This is a property of sets of functions and does not have meaning in terms of a single record.
2-3. **Mathematical Expectations**

Whenever it is necessary to characterize a random process with a statistical parameter, it is natural to consider some type of probability weighted average. These set averages generally take the form of mathematical expectations as illustrated by Bendat [II-3-1], Lee [II-3-2], Laning and Battin [II-3-3] and Pfeiffer [II-3-4]. Pfeiffer's exposition shows with the aid of abstract integration that the expectations may be expressed directly in terms of the probability measure \( P(\cdot) \) or more conveniently in terms of the distribution function for the random process. This is best seen by examining the definition of the mathematical expectation for a suitable function \( g(\cdot) \) of the random variable \( X(\cdot) \), and then extending the definitions to the random process.

**Definition 2-3a:** If \( X(\cdot) \) is a random variable and \( g(\cdot) \) is a function of \( X(\cdot) \), the mathematical expectation \( E[g(X)] = E[Z] \) of the random variable \( Z(\cdot) = g[X(\cdot)] \) is given by

\[
E[g(X)] = \int g(X) \, dP = \int g(x)dF_X(x) = \int z \, dF_Z(z) = \int z \, dP = E[Z]
\]

for the general case.

This definition is the basis for expectations of a random process \( X(\cdot, \cdot) \), since the random process is a random variable \( X(\cdot,t) \) for every \( t \in T \). The mean value \( \mu_X(t) \) of the random process \( X(\cdot, \cdot) \) is the average value of the set of sample functions at time \( t \),

\[
\mu_X(t) \triangleq \bar{X}(t) = E[X(t)] = \int X(\xi,t) \, dP(\xi).
\]  \hspace{1cm} (2-3-1)

If the probability density function \( \frac{dF_X(x)}{dx} = f_X(x) \) exists, then

\[
\mu_X(t) = \int x f_X(x,t) \, dx.
\]  \hspace{1cm} (2-3-ii)
Whenever measurements are made on a physical system, it may be necessary to consider the resolution associated with the measurement. That is, when a value \( X(\cdot, t) \) of the system is within the interval \((x_{j-1} < x \leq x_j)\) it is measured as \( x_j \). In general, the resolution is small enough that computations based upon the measured values of the process are good approximations of the actual statistics of the process. However, the resolution or quantitizing associated with some digital representations of the data may make it necessary to consider computations based on measured data as statistics of the functions \( Q(\cdot, t) \), which are step approximations of the sample functions \( X(\cdot, t) \), that is, \( Q(\cdot, t) = g(X) \). The mean value of \( Q(\cdot, t) \) is

\[
\mu_Q(t) = \sum_{j=1}^{\infty} x_j \ P(x_{j-1} < X(t) \leq x_j).
\]

Other common expectations associated with random processes are the mean-squared value, variance, and standard deviation. Each of these statistical parameters is a measure of how the values of the random variables are dispersed or distributed at time \( t \in T \).

Mean Squared Value of \( X(\cdot, \cdot) \)

\[
X^2(t) = \mathbb{E}[X^2(t)] = \int X^2(\xi, t) dP(\xi).
\]  \hspace{1cm} (2-3-iii)

If the density function \( f_X(\cdot, \cdot) \) exists, then

\[
X^2(t) = \int x^2 f_X(x, t) \ dx.
\]  \hspace{1cm} (2-3-iv)

For Quantitized Data,

\[
Q^2(t) = \sum_{j=1}^{\infty} x_j^2 \ P(x_{j-1} < X(t) \leq x_j).
\]

One distribution of the data about the mean value \( \mu_X(t) \) is defined as the mean square value of \( X(\cdot, \cdot) \) about the mean, that is
\[ E[(X(t) - \mu_X(t))^2] = \sigma_X^2(t). \quad (2-3-v) \]

The square root of the variance is called the standard deviation \( \sigma(\cdot) \) of the random process \( X(\cdot, \cdot) \).

Further properties of the process may be determined by the ensemble autocorrelation function \( \varphi_{XX}(\cdot, \cdot) \) which is defined as

\[ \varphi_{XX}(s, t) = E[X(s) X(t)] = \int X(\xi, S) X(\xi, t) dP(\xi). \quad (2-3-vi) \]

Whenever the second-order density function \( f_X(\cdot, \cdot; \cdot, \cdot) \) exists

\[ \varphi_{XX}(s, t) = \int yX f_X(x, s; y, t) dy dx. \quad (2-3-vii) \]

If the process \( X(\cdot, \cdot) \) is stationary, then

\[ \varphi_{XX}(s, t) = \varphi_{XX}(0, t-s) = \varphi_{XX}(s-t, 0) = \varphi_{XX}(\tau) \quad (2-3-viii) \]

where \( \tau \) is the time differential \( s-t \).

In the application of these techniques to an experimental situation, we often find that only a few measurements of the process are possible. That is, rather than having the entire set of responses available for our use, there may be only a part of a single record with which to characterize the random process. If there is only one sample function available, we generally assume that the time averages such as the average value and time autocorrelation function \( R(\cdot, \cdot) \) of the record are related to the mathematical expectations of the random process. Under certain conditions, each of the possible functions of the process may be regarded as typical. That is, "intuition and experience indicate the possibility that \( \varphi_{XX}(\tau) \) and \( R(\xi, \tau) \) should be the same... at least for typical functions from the process." Pfeiffer [II-3-5].

Processes of this type are called ergodic processes.
**Definition 2-3b:** The process \( X(\cdot, \cdot) \) is ergodic in a wide sense if and only if \( X(\cdot, \cdot) \) is stationary in a wide sense and \[
M[X(\xi, t)] \overset{D}{=} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(\xi, t) dt = E[X(t)]
\]
and \( R(\xi, t) \overset{D}{=} M[X(\xi, t) X(\xi, t + \tau)] \)

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(\xi, t) X(\xi, t+\tau) dt = \varphi_{XX}(\tau).
\]

The convergence of the time averages and expectations in terms of certain probability laws is described by Pfeiffer [II-3-6]. For a treatment of the intuitive logic associated with the ergodic hypothesis, see Lee [II-3-7] and Bendat [II-3-8].

In many real situations, it is **not reasonable** to consider **all** of the measurable sample functions of a process to be representative of the ensemble averages. Those samples whose time averages yield the same statistics as the underlying ensemble expectations are designated as **typical** functions.

**Definition 2-3c:** A sample function of a stationary random process is **typical** in a wide sense if

(a) \( M_T [X(\xi, t)] = E[X(t)] \)

and

(b) \( M_T [X(\xi, t) X(\xi, t+\tau)] = E[X(t) X(t+\tau)] \).

It is possible that a finite interval of a sample function is sufficient to characterize the process. In this case a sample function is said to be **locally typical** in a wide sense if its time averages over the interval \((t_1, t_1 + P)\) are equal to certain set averages. These are
\[ M_{p}\{X(\xi, t)\} = E[X(t)] \quad \text{for} \quad t_i \leq t_i + p \]

and

\[ M_{p}\{X(\xi, t) X(\xi, t + \tau)\} = E[X(t) X(t + \tau)] . \]

Once again the reader must remember that we are attempting to characterize a set of sample functions which are defined over the interval \((-\infty, \infty)\) with data from a finite interval. Therefore, it becomes extremely important to select functions which experience indicates to be typical and to reject those functions which are non-typical.
REFERENCES


3-1 Bendat, Julius S., Principles and Applications of Random Noise Theory, John Wiley and Sons, Inc., New York, 1958, p. 84.


3-3 Laning and Battin, Op. Cit.

3-4 Pfeiffer, Op. Cit.

3-5 Pfeiffer, Ibid.

3-6 Pfeiffer, Ibid.


III

Description of the Mathematical Model

The measurement of the electrical activity of the brain yields at best a very small sampling of information from what is known to be a complex continuous volume. Even with this restriction, a great amount of research in neurophysiology is concerned with relating fragmentary data to two basic questions: how does the brain process information? or what does the measured EEG tell us about the psycho- or neurophysiological state of the subject?

The desire to understand the significance of the EEG has created a need for mathematical representations of certain neurophysiological processes. In particular, models are needed which do not require massive amounts of data and computing time in order to characterize the "state" or "mode of processing" of the brain. These models are not required to be complete mathematical representations of how the brain really works. In fact the model does not even need to be remotely related to the actual situation. It is important that the computations based upon the model are useful in predicting and classifying experimental results.
3-1. **Mathematical Model**

The particular model developed for this study assumes that the brain may be considered as a lumped-parameter system. That is, the electrical activity measured at a recording electrode is representative of the brain's state within some neighborhood of the electrode. The complete model consists of an input, processing section, and output. Electrode placements below the skull define discrete nodes within the processing section, whereas electrodes on the skull (or scalp) are designated as output nodes of the system. The input is connected through the processing nodes by a set of unidirectional transmission operators. The input to the system may be a single stimulus or a stimulus field consisting of a set of individual sources of stimulation. A graphical representation of the nodes is shown in Figure 3-1-1. The symbol at each node (for example $x_j$) represents the location as well as the signal at the node. The transmission operator symbols (for example $T_{x_i y_k}$) represent some operation of the system on the signal transmitted from one node ($x_i$) to another node ($y_k$). The addition of a superscript prime to the node symbol indicates a component of the measurable signal at the node. The signal received at node $y_k$ due only to $s_1$ is

$$y'_k = \sum_{i=1}^{n} T_{x_i y_k} x_i(s_1).$$

which may be reduced to a single effective transmission between $y_k$ and $s_1$ as

$$y'_k = T_{s_1 y_k s_1}. \quad (3-1-i)$$
DISCRETE PARAMETER MODEL OF THE BRAIN

Figure 3-1-1.
In addition to the effect of the input \( y^I_k \), the actual signal measured at the output \( y_k \) contains a component which is assumed to be independent of the input stimulus. This component is due to the brain's normal ongoing activity determined by the psychophysiological state and it is symbolized by \( z \). Thus the total signal measured at node \( y_k \) is \( y_k = y^I_k + z_k \). (3-1-ii)

A review of EEG literature reveals that most contemporary researchers assume the electrical activity of the brain may be described by members of some random process, and the major problem of analysis is extracting the signal (hopefully related to the desired information) from a large noise background. However, in this study the emphasis is placed upon a representation of the processes in terms of the transmission operators.

The uncertainty of the evoked responses with each presentation of the stimulus field is assumed for this thesis to be associated with the transmission operators. Thus the transmission operators are members of a random process \( T_{x_i, y_j} (\cdot, \cdot) \) with a choice variable \( \xi \). With each presentation of the stimulus pattern a \( \xi \) is chosen; this means that a particular operator is selected from the basic space of all possible transmission operators for the process under consideration.

An additional parameter which affects the transmission operators and "ongoing activity" is the psycho-neurophysiological state of the subject. The effect of a "state" variable is indicated by a multiparameter set notation for the random process. The random process \( X(\xi, t, a) \) has parameters \( t \in T, a \in R_a \), where \( T \) and \( R_a \) are the
parameter sets. If the parameter set \( R_a \) has discrete values (that is a set of definite known states), then the set may be written as

\[ R_a \in \{ a_i : i \in J \} \] where \( J \) is a finite index set, and

\[ X_i(\xi, t) = X(\xi, t, a_i). \]

It is possible to represent the process as a function \( X(\cdot, \cdot, \cdot) \) on a product space \( S \times T \times R_a \), which may be illustrated by a cartesian axis in three dimensional space (see Figure 3-1-2). For a fixed \( \xi \), we may consider \( X(\xi, \cdot, \cdot) \) as a one parameter family of time functions, where \( a \) is the parameter. Then if we pick an \( a \), \( X(\xi, a, \cdot) \) is a function of \( t \), and for any \( t \), a single value \( X(\xi, a, t) \) is defined on the real line \( x = X(\xi, a, t) \). The block diagram in Figure 3-1-3 illustrates the notation involved when these principles are applied to equations (3-1-I) and (3-1-II).

In this figure:

(i) \( t \) designates time.

(ii) \( \xi \) is the choice variable.

(iii) "\( a \)" indicates the parameter value associated with the subject's psychophysiological state.

(iv) \( s(t) \) is a controlled or known input to the system under study.

(v) \( T_{sy_k}^{(\cdot, \cdot, \cdot)} \) is a member of a random process which is the systems operation on the input signal.

(vi) \( z_k^{(\cdot, \cdot, \cdot)}, y_k^{(\cdot, \cdot, \cdot)}, \) and \( y_k^{(\cdot, \cdot, \cdot)} \) are random processes which describe various components of the measurable signal at \( y_k \).
PRODUCT SPACE OF CHOICE VARIABLE, TIME AND FUNCTION PARAMETER.

Figure 3-1-2.
BASIC TWO PORT MODEL OF THE BRAIN

Figure 3-1-3.
Experience with any physical device illustrates the principle that we live in a nonlinear world. The brain is no exception, and even when we deal with the EEG recorded from gross electrodes there is an apparent nonlinear relation between the EEG and the stimulus field. Nevertheless, the application of linear time-invariant theory to real systems has been very useful in characterizing these nonlinear devices under certain restrictions. Within the framework of these restrictions, it is usually possible to compute results which may not be possible or practical for a more exacting nonlinear system.

Therefore, the initial assumption of this thesis is that the transmission operators are linear within certain limits. That is, for a restricted set of inputs the superposition principle is valid. The assumption that the system is time-invariant states that

\[ P[s_1(t + \tau)] = y_1(t + \tau) \]

for every choice of \( \tau \).
3-2. System Response

For a linear, time-invariant system, the transmission operator \( T_{s_i} y_j \) is designated in the time domain by the impulse response or weighting function \( g_{s_i} y_j \) and the operation which relates the input signal and impulse response to the output is convolution. In order to avoid difficult integrals which occur when evaluating the response of the system in the time domain, it is usually more convenient to work in the complex frequency domain, \( s = \sigma + j \omega \). The transmission operator \( T_{s_i} y_j \) in the complex frequency domain is the transfer function \( G_{s_i} y_j \) of the system. For example, consider the two-port system of Figure 3-2-1.

The output of the system may be expressed in the a) Time Domain

\[
y(t) = \int_{0}^{\infty} x(t - \tau) g(\tau) \, d\tau \tag{3-2-i}
\]

or in the b) Complex frequency domain as

\[
Y(s) = X(s) G(s) \tag{3-2-ii}
\]

These domains are related by means of the

a) Direct Laplace Transform

\[
G(s) = \int_{0}^{\infty} g(t) \, e^{-st} \, dt
\]

\[
\L[g(t)] \tag{3-2-iii}
\]

and

b) Inverse Laplace Transform

\[
g(t) = \frac{1}{2\pi} \int_{c-j\omega}^{c+j\omega} G(s) \, e^{st} \, ds.
\]

\[
\L^{-1}[G(s)] \tag{3-2-iv}
\]

The more familiar Fourier transforms may be obtained from these equations by setting \( s = j\omega \) and changing the limits of integration for the
I
Time Domain

x(t) \rightarrow g(t) \rightarrow y(t)

II
Complex Frequency Domain

X(s) \rightarrow G(s) \rightarrow Y(s)

BLOCK DIAGRAM OF TWO PORT SYSTEM
Figure 3-2-1.
direct transform; then the

(a) Direct Fourier transform is

\[ G(jw) = \int_{-\infty}^{\infty} g(t) e^{-jwt} \, dt \]

and the

(b) Inverse Fourier transform is

\[ g(t) = \frac{1}{2\pi j} \int G(jw) e^{jwt} \, dw \]

For a detailed review of the properties, theorems, and application of Laplace transforms to control system theory, the reader is referred to Kuo [III-2-1], Pfeiffer [II-2-2], and Truxal [III-2-3].

So far, the uncertainty associated with the response of the brain to certain stimuli has been related to transmission operators (impulse response or transfer function) of a lumped multi-parameter system.

These operators are considered to be members of a random process, where \( g(\xi, \alpha, t) \) is specified by the selection of a choice variable \( \xi \) from the basic space, state parameter \( \alpha \), and time \( t \).

The consideration of the operators as members of a random process requires that a new set of input-output relations be developed in order to characterize the response of the system. The following development will be for a simple two-port system which does not depend upon the state parameter \( \alpha \), see Figure 3-2-2. It is assumed that the input \( x(t) \) is known and an additive noise \( z(\xi, t) \) combines with the output \( y'(\xi, t) \) to produce a measurable signal \( y(\xi, t) \). In order to simplify the notation for the two-port system, the direct output of the system
System with additive noise

Figure 3-2-2.
y'(\xi, t) will be designated as \( w(\xi, t) \) for the remainder of this thesis.

Before proceeding with the input-output relations, let us consider the nature of the impulse response and the additive noise. As the name implies, the impulse response of a linear, time-invariant system is the actual transient response of the system to a unit impulse \( \delta(t) \).

The output of the system in the time domain is

\[
w(t) = \int_0^\infty \delta(t-\tau) \ g(\tau) \ d\tau.
\]

By the sampling theorem [III-2-4] which states that the integral of a unit impulse which occurs at time \( t \) and a function \( g(\tau) \) with respect to \( \tau \) is equal to the value of the function at time \( t \).

That is,

\[
\int_0^\infty \delta(t-\tau) \ g(\tau) \ d\tau = g(t).
\]

In the frequency domain, where the Laplace transform of a unit impulse is unity, the output is

\[
W(s) = X(s)G(s) = G(s).
\]
3-3. Nature of the Impulse Response

For this research, the impulse responses encountered are assumed to possess the following properties:

(g-1) A new \( g(\xi, \cdot) \) is selected with each presentation of the stimulus field. That is, if a single impulse (or other input form) is applied, this is considered as one trial corresponding to the choice of a single \( \xi \). However, in the case in which a train of impulses or a periodic signal is applied, this is also considered one (composite) trial, corresponding to the choice of a single \( \xi \). [See sections III-6-7 for a development of this case.]

(g-2) The impulse response is equal to zero prior to the application of an input to the system at time \( t \),

\[
g(\xi, t) = 0 \quad \text{for} \quad t \leq 0.
\]

(g-3) The impulse response is a member of a non-stationary random process, since

\[
E[g(t_1)] \neq E[g(t_2)].
\]

(g-4) The impulse response is assumed to be absolutely integrable, i.e.,

\[
\int_{-\infty}^{\infty} |g(\xi, t)| dt < \infty,
\]

so that the Fourier transform does exist.

(g-5) A fixed interval \( T \) is assumed to exist such that

\[
g(\xi, t) \overset{\circ}{=} g_T(\xi, t),
\]

where

\[
g_T(\xi, t) = g(\xi, t) \quad \text{for} \quad 0 < t \leq T
\]

\[
= 0 \quad \text{for} \quad t > T.
\]
(g-6) The **expected impulse response**
\[ E[g(t)] = \mu_g(t) \]
is the probability weighted set average of all possible impulse responses from the basic space.

(g-7) The variance function provides a measure of how the values are
\[ E[(g(t) - \mu_g(t))^2] = \sigma_g^2(t) \]
distributed about the mean value \(\mu_g(t)\).

Similar statements could be made about the transfer function \(G(\cdot, \cdot)\).

For example,

(G-8) The expected transfer function is
\[ E[G(s)] = \mu_G(s) \]

and

(G-9) The variance of the transfer function is
\[ E[(G(s) - \mu_G(s))^2] = \sigma_G^2(s). \]

In fact, under very general conditions, the Laplace transform of the expected impulse response is equal to the expected transfer function; i.e.,
\[ \mathcal{L}[\mu_g(t)] = E[G(s)] = \mu_G(s). \]

This may be proved by examining the mathematical expectation of \(G(\cdot, \cdot)\) in terms of the abstract integral on the basic space;
\[ E[G(s)] = E[\mathcal{L}[g(\xi, t)]] = \int [\int g(\xi, t) e^{-st} dt] dP(\xi). \]
By interchanging the order of integration,

$$E[G(s)] = \int_0^\infty \left[ \int g(\xi, t) \, dP(\xi) \right] e^{-st} \, dt,$$

where the symbol \((=)\) indicates a formal operation which is used here without proof \{III-3-1\}. By the basic definition of the mean value (see equation 2-3-1)

$$E[G(s)] = \int_0^\infty \mu_g(t) \, e^{-st} \, dt.$$

By the definition of the Laplace transform Equation 3-2-iii,

$$E[G(s)] = \mathcal{L}[\mu_g(t)].$$
3-4. Nature of the Measured Activity $y(\cdot, \cdot)$

In the absence of a stimulus field, the measured signal is the "ongoing" activity of the brain. This activity is assumed to be represented by a typical sample function $z(\xi, \cdot)$ of the random process $z(\cdot, \cdot)$; thus the time averages of a selected function may be related to the ensemble averages of the process. Also the function $z(\cdot, \cdot)$ is assumed to be uncorrelated in the sense of time averages with the input signal $x(\cdot)$ and the mean value of $z(\cdot, t)$ is

$$E[z(t)] = M_T[z(\xi, t)] = 0$$  \hspace{1cm} (3-4-i)

In a real situation, where measurements are made from gross electrode sites, the capacitive coupling of the EEG preamplifiers insures that the analog record will have a mean value of zero.

The measurable signal $y(\xi, t)$ is determined by the "ongoing" activity $z(\xi, t)$ and the stimulus input $x(t)$ in the time domain as,

$$y(\xi, t) = \int_{-\infty}^{\infty} x(t-u) \ g(\xi, u) \ du + z(\xi, t).$$

The mathematical expectation of $y(\xi, t)$ is

$$E[y(t)] = E[\int_{-\infty}^{\infty} x(t-u) \ g(\xi, u) \ du]$$

since the expectation of $z(\xi, t)$ is assumed to be zero (see equation 3-3-i). By changing the order of integration (remember the mathematical expectation of a function of $\xi$ is an integration) we have,

$$(y-1) \ E[y(t)] = \int_{-\infty}^{\infty} x(t-u) \ E[g(u)] \ du_1$$

$$= \int_{-\infty}^{\infty} x(t-u) \ \mu_g(u) \ du$$

$$= \mu_y(t)$$
Thus the mathematical expectation of the measurable signal is reduced in the time domain to the convolution of the input with the mean impulse response. The Laplace transform of the above equation yields
\[ \mathcal{L}[\mu_y(t)] = \mathcal{L}[\int_{-\infty}^{\infty} x(t-u) \mu_g(u) \, du], \]
which may be written as
\[ \mathcal{L}[\mu_y(t)] = X(s) \mu_g(s). \]
The function above is the expectation,
\[ (y-2) \quad E[X(s)] = X(s) E[G(s)] \overset{\mu}{=} \mu_Y(s), \]
(see equation G-7). A slight rearrangement shows the expected transfer function is the ratio of the expected measurable output to the input of the system;
\[ E[G(s)] = \frac{E[Y(s)]}{X(s)} \overset{\mu}{=} \mu_G(s) \quad (3-3-ii) \]
Obviously, the last equation could have been obtained by the direct application of the mathematical expectation to the following input-output relation for the system in the complex frequency domain;
\[ Y(\xi, s) = X(s) G(\xi, s) + Z(\xi, s). \]
Taking the expectation we retain only
\[ E[Y(s)] = X(s) E[G(s)] \]
since the
\[ E[Z(s)] = E[\mathcal{L}[z(t)]] \]
\[ (\ast) \quad \mathcal{L}[E[z(t)]] = 0 \]
by the assumption that the mean value of the "ongoing" activity is zero.
Although the mean or expected values of the measurable output are independent of the "ongoing" activity, the actual dispersion of \( y(\xi,t) \) about the mean is indicated by the variance,

\[
\sigma_y^2(t) = E[y(t)^2] - E[y(t)]^2 ,
\]

which may be expanded as the sum of the system response and the "ongoing" activity,

\[
\sigma_y^2(t) = E[\{w(\xi,t) + z(\xi,t)\}^2] - E[w(\xi,t) + z(\xi,t)]^2 .
\]

Squaring, we have

\[
\sigma_y^2(t) = E[w(\xi,t)^2 + 2w(\xi,t)z(\xi,t) + z(\xi,t)^2]
- E[w(\xi,t) + z(\xi,t)]^2 .
\]

We suppose that the system response \( w(\xi,t) \) and the "ongoing" activity \( z(\xi,t) \) are uncorrelated; thus the variance of the measurable output is

\[
(y-3) \quad \sigma_y^2(t) = \sigma_w^2(t) + \sigma_z^2 .
\]

The above relations are for ensemble average and care must be exercised in relating these values to measurable EEG, because of the non-stationary nature of the impulse response. As long as the stimulus field is absent, it is possible to pick typical EEG and then equate the time averages to probability set averages of the random process. When a stimulus is presented, the EEG is no longer typical (although it still contains a component \( z(\xi,t) \) which may be typical), but the component associated with \( g(\xi,t) \) represents a segment of the sample function related to the selection of each \( \xi \).

Thus a large number of repetitions of the stimulus are needed in order to provide a "good" estimate of the actual ensemble averages.
3-5. Input-Output Relations

In the analysis of control systems, several relations have been developed which characterize a system in terms of its input and output signals. Laning and Battin [III-5-1] and others have shown that correlation functions and spectral relations are very useful if these signals are members of a random process. However, when the input is non-random and the impulse response is a member of a non-stationary random process, these functions are not as meaningful, except for special inputs such as an impulse. Nevertheless the following relations for the system of Figure 3-2-2 will be stated here in a generalized form. Remember that the mean value of \( z(\xi, t) \) is assumed to be equal to zero so that \( z(\xi, t) \) is uncorrelated with deterministic input functions.

**Correlation Functions.** The complete derivation of the correlation functions of the mathematical model are presented in Appendix A.

(c-1) The autocorrelation function of the measurable output is

\[
\varphi_{yy}(t, t + \tau) = E[y(t) y(t + \tau)]
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_{gg}(t-u, t+\tau-v) \varphi_{xx}(u) \varphi_{xx}(v) \, du \, dv
\]

\[+ \varphi_{zz}(\tau).\]

Since the "ongoing" activity is assumed to be typical, the ensemble autocorrelation may be replaced by the time average \( R_z(\xi, \tau) \). The effect of the additive signal \( z(\cdot, \cdot) \) may be eliminated (see Appendix A) by using the cross-correlation function.
The ensemble cross-correlation function of the measurable output and input signal is

\[ \phi_{xy}(t, t + \tau) = E[x(t) y(t + \tau)] \]

\[ = x(t) \int_{-\infty}^{\infty} x(t + \tau - u) \mu_g(u) \, du . \]

**Spectral Densities.** Perhaps the most common way of describing a signal is in terms of its behavior in the frequency domain. If the signal is periodic, it is possible to relate the terms of its Fourier series representation to measurements of the signal's harmonic components. Many times the "power" associated with each frequency is used to describe the signal. In the case of a random signal which is non-periodic and has a Fourier transform, it may be meaningful to describe the signal by its power or energy spectrum [III-5-2]. Basically these spectra are associated with a particular sample function of the random process \( y(\xi, \cdot) \). If the sample function is typical, then its spectrum is congruent with the spectrum for the random process. In this research, the measurable output \( y(\xi, \cdot) \) is typical only if the stimulus field is not present. During stimulation \( y(\xi, \cdot) \) is non-stationary and spectral measurements are difficult to interpret. For example, the energy spectrum due to a signal over a finite interval is

\[ (E-1) \quad E(\xi, \omega, Y) = E(\xi, \omega, W) + E(\xi, \omega, \overline{W}) \]

\[ + \ E(\xi, \omega, \overline{W} Z) + E(\xi, \omega, Z) \]

That is, the total energy consists of components due to the direct output of the system, the "ongoing" activity, and the cross spectral energies of these two parameters. The value of the cross terms is determined by the time cross correlation equation; if
\[ R_{WZ}(\xi, \tau) = \frac{1}{2T} \int_{-T}^{T} w(\xi, t) z(\xi, t+\tau) \, dt = 0 \]

then

\[ \mathcal{E}(\xi, \omega, \overline{WZ}) = \mathcal{E}(\xi, \omega, \overline{W} \overline{Z}) = 0. \]

By introducing the convolution integral, this expression may be related to the input of the system

\[ R_{XZ}(\xi, \tau) = \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{\infty} x(t-u) g(\xi, u) \, du \, z(\xi, t+\tau) \, dt, \]

which by a change in the order of integration is

\[ R_{XZ}(\xi, \tau) = \frac{1}{2T} \int_{-\infty}^{\infty} g(\xi, u) \left[ \int_{-T}^{T} x(t-u) z(\xi, t+\tau) \, dt \right] \, d\tau. \]

If the input signal and the "ongoing" activity are uncorrelated in the sense of time averages, then \( R_{XZ}(\xi, \tau) = 0. \) It would be unreasonable to assume that every stimulus is uncorrelated with \( z(\xi, t) \) over a restricted interval. However, a time correlation does not exist if: (1) the input is an impulse or series of impulses, or (2) the input is deterministic and the corresponding sample of "ongoing" activity is locally typical.

The cross spectral energy for a sample function of the model is

\[ (E-2) \quad \mathcal{E}(\xi, \omega, X, Y) = \frac{1}{2\pi} \overline{X(j\omega)} \left| Y(\xi, j\omega) \right|^2. \]

Substituting for \( Y(\xi, j\omega), \) we have

\[ \mathcal{E}(\xi, \omega, X, Y) = \frac{1}{2\pi} \left| X(j\omega) \right|^2 G(\xi, j\omega) + \overline{X(j\omega)} Z(\xi, j\omega). \]

The last term is zero whenever the input signal and the "ongoing" activity are uncorrelated in the sense of time averages (see E-1). If

\[ \mathcal{E}(\xi, \omega, X, Y) = \frac{1}{2\pi} \left| X(j\omega) \right|^2 G(\xi, j\omega), \]
then the spectral energy for the process is

\[ E[\mathcal{E}(\omega, X, Y)] = \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \frac{|X(j\omega)|^2 \mu_G(j\omega)}{X(j\omega)} \]

Thus the expected transfer function is

\[ \mu_G(j\omega) = \frac{2\pi \mathcal{E}_{XY}(\omega)}{|X(j\omega)|^2} \]

It is also possible to compute the cross spectral energy in terms of the time series correlation function for sample functions. The cross energy spectrum \( \mathcal{E}_{XY}(\omega) \) for the process is the expectation of the Fourier transform of a time correlation function of \( x(t) \) and \( y(\xi, t) \); that is

\[ \mathcal{E}_{XY}(\omega) = E\left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T[x(t) y(\xi, t+\tau)] e^{-j\omega \tau} d\tau \right], \]

where \( I_T \) denotes the time integral

\[ \int_0^T \] \]

By changing the order of integration, we obtain

\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T \{E[x(t) y(\xi, t+\tau)]\} e^{-j\omega \tau} d\tau. \]

Since \( x(t) \) is deterministic, its expectation is simply \( x(t) \);

\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T \{x(t) E[y(\xi, t+\tau)]\} e^{-j\omega \tau} d\tau. \]

The measurable output \( y(\xi, t+\tau) \) may be stated in terms of the other system parameters as

\[ y(\xi, t+\tau) = \int_{-\infty}^{\infty} x(t+\tau - u) g(\xi, u) \, du + z(\xi, t+\tau). \]

Now the energy spectral density may be written

\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T \{x(t) E[\int_{-\infty}^{\infty} x(t+\tau - u) g(\xi, u) \, du + z(\xi, t+\tau)]\} e^{-j\omega \tau} d\tau. \]
Once again, the order of integration is changed, so that
\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T(x(t)) \left[ \int_{-\infty}^{\infty} x(t + \tau - u) \, E[g(\xi, u)] \, du + E[z(\xi, u)] \right] e^{-j\omega \tau} \, d\tau. \]

The expectation of \( g(\xi, u) \) is symbolized as
\[ E[g(\xi, u)] \equiv \mu_g(u) \]
and the expectation of the "ongoing" activity \( z(\xi, t) \) is assumed to be zero. Thus, the cross energy equation reduces to
\[ (\mathcal{E}-3) \quad \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T(x(t)) \int_{-\infty}^{\infty} x(t + \tau - u) \, \mu_g(u) \, du \, e^{-j\omega \tau} \, d\tau. \]

If the input \( x(t) \) of the system is an impulse, the above equation reduces to a very interesting and useful form for a non-stationary random process,
\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T(\delta(t)) \int_{-\infty}^{\infty} \delta(t + \tau - u) \, \mu_g(u) \, du \, e^{-j\omega \tau} \, d\tau. \]

This may be evaluated using the sampling property of impulses as follows
\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T(\delta(t) \mu_g(t+\tau)) e^{-j\omega \tau} \, d\tau. \]

Performing the time integration \( I_T \), we have
\[ \mathcal{E}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mu_g(\tau) \, e^{-j\omega \tau} \, d\tau. \]

Thus the cross-spectral energy is the Fourier transform of the expected impulse response;
\[ \mathcal{E}_{XY}(\omega) = F[\mu_g(t)] = \mu_g(j\omega) \]
which is the expected transfer function in the frequency domain.

The foregoing development of the relation between the input signal and measurable output signal provides the theoretical basis for interpreting the EEG in terms of the mathematical expectation of the impulse response or transfer function of a lumped parameter model.
of the brain. Although this research is directed toward a model for the brain's response to a stimulus field, these equations are suitable for any linear, time-invariant system which has a degree of uncertainty associated with the response of the system to a deterministic input signal.

Thus far we have shown that the expected impulse response or expected transfer function may be defined in terms of expectations of the measurable output, cross-correlation function, and spectral energies. However, some of these solutions would be difficult to evaluate unless the input signal were impulsive. These equations are summarized in Table 3-5-1 for the case of a generalized input \( x(t) \) and the special case of an impulse input \( \delta(t) \).
TABLE 3-5-1

Summary of Input-Output Relations

Expected Measurable Response (Time Domain)

(y-1) $E[y(t)] = \int_{-\infty}^{\infty} x(t-u) E[g(u)] \, du$

$= E[g(u)] \frac{n}{n} \mu_g(u) \text{ for } x(t) = \delta(t)$

Expected Measurable Response (Complex Frequency Domain)

(y-2) $E[Y(s)] = X(s) E[G(s)]$

$= E[G(s)] \frac{n}{n} \mu_G(s) \text{ for } x(t) = \delta(t)$

Ensemble Autocorrelation Function

(c-1) $\phi_{yy}(t, t + \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{gg}(t-u, t+\tau-v) x(u) x(v) \, du \, dv$

$= \phi_{zz}(\tau) = \phi_{gg}(t, t+\tau). + \phi_{zz}(\tau) \text{ for } x(t) = \delta(t)$

Ensemble Cross-correlation Function

(c-2) $\phi_{xy}(t, t + \tau) = x(t) \int_{-\infty}^{\infty} x(t+\tau-u) \mu_g(u) \, du$

$I_T \phi_{xy}(t, t+\tau) = \mu_g(t) \text{ for } x(t) = \delta(t)$

Energy Spectral Density of the Function $y(\xi, \cdot)$

(\xi-1)* $E(\xi, \omega Y) = E(\xi, \omega \bar{W}) + E(\xi, \omega, Z)$

Cross-Energy Spectral Density for the Functions $y(\xi, \cdot)$ and $x(t)$

(E-2)* $E(\xi, \omega, XY) = \frac{1}{2\pi} |X(j\omega)|^2 G(\xi, j\omega)$

$= \frac{1}{2\pi} G(\xi, j\omega) \text{ for } x(t) = \delta(x)$

Cross-Energy of the Random Process

(E-3) $E_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} I_T[x(t)] \int_{-\infty}^{\infty} x(t+\tau-u) \mu_g(u) \, du \, e^{-j\omega \tau} \, d\tau$

$= F[\mu_g(t)] \frac{n}{n} \mu_G(j\omega) \text{ for } x(t) = \delta(t)$

* The input signal $x(t)$ and the ongoing activity are assumed to be uncorrelated in the sense of time averages.
3-6. \textit{Response to a Train of Impulses}

Let us now consider the situation in which a train of impulses is applied to the two-port system illustrated in Figure 3-2-2. The response of this system was shown in section 3-2 to be

\[ y(\xi, t) = z(\xi, t) + \int_{-\infty}^{\infty} x(t - u) g(\xi, u) \, du. \]

For an input signal consisting of a train of impulses such as

\[ x(t) = \sum_{i=0}^{\infty} \delta(t - T_i), \]

the measurable output of the system is

\[ y(\xi, t) = z(\xi, t) + \int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \delta(t + T_i - u) g(\xi, u) \, du. \]

By interchanging the order of summation and integration we have

\[ y(\xi, t) = z(\xi, t) + \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} \delta(t + T_i - u) g(\xi, u) \, du \]

\[ = z(\xi, t) + \sum_{i=0}^{\infty} g(\xi, t - T_i). \]

This expression assumes that the system has one particular impulse response during the application of the stimulus. That is, each impulse in the train produces the same impulse response. However, this does not agree with experimental observations. Therefore, the following approach has been taken. When a train of impulses is applied to the system, the sample function associated with the choice of a particular elementary event \( \xi \) is the total system response. We suppose that if the impulses are not too closely spaced, the sample function has components consisting of a sequence of impulse responses. We thus set up a model analogous to the one that is used to represent the selection of a random statistical sample of size \( n \) from a given population. An
observation of a random sample of size \( n \) is represented as the single observation of \( n \) independent random variables \( X_1(\cdot), X_2(\cdot), \ldots, X_n(\cdot) \). Each random variable has the same distribution; that is, the distribution of \( X_1(\cdot) \) is the same as that of the given population. In the case of the impulse response, we assume the observation of a sequence of impulse responses is congruent to an observation of the independent class of random processes \([g_i(\cdot, \cdot) : 0 \leq i < \infty]\), where each process has the same distribution (as long as the animal is in a given state). In order to account for the time of application of the \( i \)th impulse \( \delta(t - T_i) \), we suppose the response to be \( g_i(\xi, t - T_i) \). We assume further that once the impulse response is sampled, it continues in its normal manner (usually decaying exponentially). Thus, we assume that applying a given train of impulses and observing the system response corresponds to one sampling of the basic space. The measurable response \( y(\xi, t) \) in terms of the individual random processes \( g_i(\cdot, \cdot) \) is taken to be

\[
y(\xi, t) = z(\xi, t) + \sum_{i=0}^{\infty} g_i(\xi, t - T_i).
\]

A particular averaging process, applicable to experimental results, is represented by the function

\[
A_n(\xi, t) = \frac{1}{n} \sum_{j=0}^{n-1} [z(\xi, t + T_j) + \sum_{i=0}^{\infty} g_i(\xi, t - T_i + T_j)].
\]

If we restrict our interest to the interval \( 0 \leq t \leq T_{\text{min}} \), where \( T_{\text{min}} \) is the minimum elapsed time between impulses, the above equation is analogous to the following three operations; (1) the record is cut at time \( t = T_j \) for each \( 0 \leq j \leq n - 1 \); (2) then each piece is shifted \( T_j \) units to the left along the time axis; and (3) the
aligned sections are averaged at each $t$ in the interval $0 < t < T_{min}$.

The function $A_n(\cdot, \cdot)$ is a new random process derived from the response process. We may speak of $A_n(\cdot, \cdot)$ as the \textit{average process}. The mathematical expectation of this average process, after interchanging the order of integration and summation, is

$$E[A_n(\xi, t)] = \frac{1}{\eta} \sum_{j=0}^{n-1} \{E[z(\xi, t + T_j) + \sum_{i=0}^{\infty} g_t(\xi, t - T_i + T_j)]\}.$$

The expectation of the "ongoing" activity $z(\xi, t + T_j)$ is zero and since $[g_t(\xi, t - T_j): 0 \leq i < \infty]$ is an independent class of random variables with the same distribution, we may write

$$E[A_n(t)] = \frac{1}{\eta} \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} \mu_g(t - T_i + T_j).$$

Within the interval $0 \leq t \leq T_{min}$ the predominant term of $E[A_n(t)]$ is $\mu_g(t)$, which is obtained by setting $i = j$. In addition to the expected impulse response, the "tails" of the decaying responses which were shifted to the left of $t = 0$ contribute to the value of $E[A_n(t)]$ in the interval $[0, T_{min}]$. That is

$$E[A_n(t)]_{T_{min}} = \frac{1}{\eta} \sum_{j=1}^{n-1} \sum_{i=0}^{n-1} \mu_g(t - T_i + T_j) + \mu_g(t)$$

$$i > j$$

Since the average is over a finite number of terms $n$, there are only a finite number of non-zero terms shifted to the left of $t = 0$. If the value of $g_t(\xi, t - T_i)$ for $t > T_{i+1}$ is small for all $i < n-1$, then the expectation of $A_n(\cdot, t)$ within the interval of restriction is approximately

$$E[A_n(t)]_{T_{min}} \approx \mu_g(t).$$
The strong law of large numbers insures that for large \( n \) the average response \( A_n(\xi, t) \) approximates closely the expected impulse response \( \mu_g(t) \) over the interval \([0, T_{\text{min}}]\).

The variation of \( A_n(\xi, t) \) about its mean value \( \mu_g(t) \) is

\[
\sigma_A^2(t) = E[A(t)^2] - E^2[A(t)].
\]

Substituting for \( A_n(\xi, t) \) we have

\[
\sigma_A^2(t) = E[\frac{1}{n} \sum_{j=0}^{n-1} \left( z(\xi, t + T_j) + \sum_{i=0}^{\infty} g_i(\xi, t - T_i + T_j) \right)^2] - \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} \mu_g(t - T_i + T_j)^2
\]

Expanding this equation, we find

\[
\sigma_A^2(t) = \frac{1}{n^2} E[\sum_{j=0}^{n-1} z(\xi, t + T_j)]^2
\]

\[
+ 2 \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} z(\xi, t + T_j) g_i(\xi, t - T_i + T_j)
\]

\[
+ \left[ \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} g_i(\xi, t - T_i + T_j)^2 \right] - \frac{1}{n^2} \left[ \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} \mu_g(t - T_i + T_j)^2 \right]
\]

Before interchanging the order of summation and integration, we recall:

(A-1) \( z(\xi, t) \) and \( g_i(\xi, t) \) are independent.

(A-2) \( g_i(\xi, t - T_i) \) and \( g_k(\xi, t - T_k) \) are independent for \( i \neq k \).

(A-3) The mean value of \( z(\xi, t) \) is zero.
Further, let us assume that the intervals are long enough that

\[(A-4) \ z(\xi, t + T_i) \text{ and } z(\xi, t + T_k) \text{ are uncorrelated random variables for each } t \text{ and each } i \neq k.\]

Making the interchange and writing the squared terms as a double summation, we find

\[
\sigma_A^2 = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} E[z(t + T_j) \cdot z(t + T_k)] \\
+ 2 \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} E[z(t + T_j) \cdot g_i(t - T_i)] \\
+ \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} E[g_h(t - T_h + T_j) \cdot g_i(t - T_i + T_k)] \\
- \frac{1}{n^2} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \mu_g(t - T_h + T_j) \mu_g(t - T_i + T_k)
\]

Before expanding this expression, let us examine some of the terms.

First according to \((A-4)\)

\[E[z(t + T_j) \cdot z(t + T_k)] = 0 \quad \text{for } j \neq k.\]

By applying first \((A-1)\) and then \((A-3)\) to the cross correlation terms involving \(z(\xi, t)\) and \(g_i(\xi, t)\), we have

\[E[z(t + T_j) \cdot g_i(t - T_i + T_j)] = E[z(t + T_j)]E[g_i(t - T_i + T_j)] = 0\]

for all \(i, j\). Also using \((A-2)\) we see that the expectation of the cross terms of the impulse response are

\[E[g_h(t - T_h + T_j) \cdot g_i(t - T_i + T_k)] = E[g_h(t - T_h + T_j)]E[g_i(t - T_i + T_k)] \quad \text{for } h \neq i.\]

Since the individual impulse responses are assumed to have the same distribution, the above equation reduces to

\[E[g_h(t - T_h + T_j)]E[g_i(t - T_i + T_k)] = \mu_g(t - T_h + T_j)\mu_g(t - T_i + T_k)\]

for \(h \neq i.\)
Terms of this form for the cross correlation of the individual impulse responses will cancel with corresponding terms in $E^2[A_n(t)]$. For these conditions we may write the variance as

$$\sigma^2_A(t) = \frac{1}{n} \sum_{j=0}^{n-1} \left[ E[z^2(t + T_j)] \right]$$

$$+ \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} E[g_i(t - T_i + T_j)g_i(t - T_i + T_k)]$$

$$- \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \mu_g(t - T_i + T_j) \mu_g(t - T_i + T_k)$$

Once again let us restrict our interest to the interval $0 \leq t \leq T_{\text{min}}$, and write the variance as the sum of terms for which $i = j = k$ and a summation for $j, k > i$. We have

$$\sigma^2_A(t)|_{T_{\text{min}}} = \frac{1}{n} \left[ E[z^2(t)] \right]$$

$$+ n E[g^2(t)] + \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \sum_{i=0}^{n-1} E[g_i(t-T_i + T_j)g_i(t-T_i + T_k)]$$

$$- n \mu_g^2(t) - \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \sum_{i=0}^{n-1} \mu_g(t-T_i + T_j) \mu_g(t-T_i + T_k)$$

where the triple summations contain a finite number of non-zero terms.

If the overlap of $g_i(\xi, t - T_i)$ into the next interval is small, then the terms in the triple summations are negligible and

$$\sigma^2_A(t)|_{T_{\text{min}}} \approx \frac{1}{n} \sigma^2_z(t) + \frac{1}{n} \sigma^2_g(t).$$

Since the "ongoing" activity $z(\xi, t)$ is assumed to be stationary its variance does not depend on $t$; thus

$$\sigma^2_A(t)|_{T_{\text{min}}} \approx \frac{1}{n} \sigma^2_z + \frac{1}{n} \sigma^2_g(t).$$
As the number of samples \( n \) increase, the spread of the values about the mean decreases since \( \sigma_A \) varies as \( \frac{1}{\sqrt{n}} \).

In order to aid the reader to visualize the significance of this section, a graphical representation of the time shifting and averaging process is shown in Figure 3-6-1.
\[ y(t, e) = a(t, e) + \sum_{i=0}^{\infty} b_i(t, e - T_1) \]

\[ y_{f}(t, e) = a(t, e + T_2) + \sum_{i=0}^{\infty} b_i(t, e + T_1 + T_2) \]

\[ y_{f}(t, e) = a(t, e + T_2) + \sum_{i=0}^{\infty} b_i(t, e - T_1 + T_2) \]

\[ y_{f}(t, e) = a(t, e) + \sum_{i=0}^{\infty} b_i(t, e - T_1 + T_2) \]

\[ y_{f}(t, e) = a(t, e + T_2) + \sum_{i=0}^{\infty} b_i(t, e - T_1 + T_2) \]

\[ y_{f}(t, e) = a(t, e) + \sum_{i=0}^{\infty} b_i(t, e - T_1 + T_2) \]

Summation of the above functions for each \( t \)

\[ A_y(t, e) = \frac{1}{n+1} \sum_{j=0}^{n} y_{f}(t, e) \]

\[ + \frac{1}{n+1} \sum_{j=0}^{n} [a(t, e + T_2) + \sum_{i=0}^{\infty} b_i(t, e - T_1 + T_2)] \]

\[ A_y(t, e) = \frac{1}{n+1} \sum_{j=0}^{n} y_{f}(t, e) \]

Example of averaging process \( A_y(t, e) \)

Figure 3-6-1
3-7. **Response of the System to a Periodic Input Signal**

Let us now consider the case where the input signal to the system shown in Figure 3-2-2 is periodic. The impulse response of the system is still assumed to be an observation of the independent class of random processes \( \{g_i(t, u) : -\infty < i < \infty \} \), as indicated in section 3-6. Since each \( g_i(\xi, t - iT) \) may be different, the output of the system does not reach a steady state condition. For any input \( x(t) \) the measurable signal may be written as

\[
y(\xi, t) = z(\xi, t) + \int_{-\infty}^{\infty} x(t - u) g(\xi, u) \, du.
\]

If \( x(t) \) is periodic with period \( T \), let us define \( x_T(t) \) as a restriction of \( x(t) \) within an interval \( T \); that is,

\[
x_T(t) = \begin{cases} 
  x(t) & 0 \leq t < T \\
  0 & \text{elsewhere.}
\end{cases}
\]

Writing \( x(t) \) as a series involving the summation of \( x_T(t) \), we have

\[
x(t) = \sum_{i=-\infty}^{\infty} x_T(t - iT).
\]

Rewriting the measurable output in terms of the periodic input, we obtain

\[
y(\xi, t) = z(\xi, t) + \sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} x_T(t - iT - u) g(\xi, u - iT) \, du.
\]

The averaging process \( A_n(\xi, t) \) for a periodic input signal may be written as
\[ A_n(\xi, t) = \frac{1}{n} \sum_{j=0}^{n-1} [z(\xi, t + jT) + \sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} x_T(t - iT + jT - u) g_i(\xi, u - iT + jT) \, du]. \]

The expectation of \( A_n(\xi, t) \) is
\[ E[A_n(t)] = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=-\infty}^{\infty} x_T(t - iT + jT - u) E[g_i(u - iT + jT)] \, du. \]

Since the \( g_i(\xi, t) \) have the same distribution, the expectation of \( A_n(\cdot, t) \) reduces to
\[ E[A_n(t)] = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=-\infty}^{\infty} x_T(t - iT + jT - u) \mu_g(u - iT + jT) \, du. \]

Expanding the above equation into a series we have
\[ E[A_n(t)] = \ldots \int_{-\infty}^{\infty} x_T(t + T - u) \mu_g(u + T) \, du + \int_{-\infty}^{\infty} x_T(t - u) \mu_g(u) \, du + \ldots \]

Recombining terms yields
\[ E[A_n(t)] = \int_{-\infty}^{\infty} x(t - u) \mu_g(u) \, du, \]
which is a periodic function determined by the convolution of the periodic input and the expected impulse response of the system.

Although the Fourier transform does not exist for a periodic function, it is possible to characterize the entire function from the value of the function in any period \( T \). Therefore, we may take the Fourier transform of \( E[A_n(t)] \) within the interval \( 0 \leq t < T \) and completely define the function in the frequency domain, that is,
\[ F\{E[A_n(t)]\}_T = X(j\omega) \mu_G(j\omega) = \mu_{\mathcal{F}}(j\omega). \]
By the strong law of large numbers there exists an $n_0$ such that with probability one

$$|A_n(\xi,t) - \int_{-\infty}^{\infty} x(t-u) \mu_g(u) \, du| < \varepsilon \quad \text{for all} \quad n > n_0.$$  

Thus for some $n > n_0$

$$F[A_n(\xi,t)]_{\mu_g} \equiv X(j\omega) \mu_g(j\omega).$$

This expression indicates that the expected transfer function $\mu_g(j\omega)$ of the process may be estimated at the fundamental and harmonic frequencies of the input signal by taking the Fourier transform of the summation function $A_n(\xi,t)$. For intuitive reasons, this summation function is designated as the expected periodic response whenever the input signal to the system is periodic.
3-8. REFERENCES


IV. EXPERIMENTAL DESIGN TO TEST THE MODEL.

In the previous sections, a mathematical model was developed from an intuitive notion that the EEG is in some way related to a random process. This relation was hypothesized to be through a lumped parameter system which was a random process. From this assertion several mathematical formuli were presented in Chapter III. However, none of this work has meaning for the EEG until the model is applied to a neurophysiological problem. Even then, if the analysis dictated by the model does not relate significantly to the problem, this model must be discarded as a technique for interpretation of the EEG.

This research does not attempt to make a complete defense or show all of the restrictions associated with the model; however, it is the purpose of this research to examine a particular problem and point the way for future programs.
4-1. **Relationship of the Mathematical Model to a Particular Neurophysiological Problem.**

An experimental study of the visual cortex of an unrestrained alert cat to photic stimulation was chosen because:

1. The standardized structure of the cat's brain increases the probability that chronic electrodes placed with the aid of a stereotaxic device will rest in a specified response center.
2. A photic stimulus closely approximates an impulse; thus many of the equations of Chapter III are reduced to more convenient forms.
3. It is possible to check the reasonableness of the experimental results with the vast amount of previous experimental work associated with the response of a cat to photic stimulation.

In addition to testing the model, the experimental phase of the research was directed toward a better understanding of the electrical activity of the visual cortex. The study was designed to permit an examination of the relative coupling of the EEG between the primary receptor and associative areas of the cortex with the cat at different states of alertness.

The model of a system of unidirectional transmission operators shown in Figure 4-1-1 describes the passage of the photic stimulus from the eye to the skull. The block N.L. represents a nonlinear element which is inserted between the input and linear processing system. For a high intensity photic stimulus, this element is assumed to be a limiting device. That is, for any strength input pulse, the output
MODEL FOR PHOTIC STIMULATION
Figure 4-1-1.
of the limiter is a fixed height and a short duration. When the photic stimulus is not present, the measured potentials represent the cat's "ongoing" activity.
4-2. Review of Literature Related to the Evoked Responses of a Cat to Photic Stimulation.

Various aspects of cortical reactions to photic and electrical stimulation have been investigated since the 1930's. When researchers noticed that the spontaneous rhythms of the cortex could be modified by photic stimulation, a systematic effort was started to determine the extent of the area giving these responses. The early research of Gerard, Marshall and Saul [IV-2-1] in 1936 employed concentric needle electrodes and a loud speaker as an indicating device to explore the cortex. From this research they ascribed large evoked responses at certain Horsely-Clark coordinates. In 1941 O'Leary [IV-2-2] defined the Area Striata (visual cortex) as a large posterior section of the cortex. Marshall, Talbot, and Ades [IV-2-3] published in 1943 a continuation of their observations of responses in the visual cortex of anesthetized cats. In addition to mapping the response of a large section of the cortex to photic stimulus they concluded:

(1) Single or multiple cortical responses are evoked by a photic stimulus, whereas "a single electrical stimulus applied to the optic nerve does not evoke a multiple response of the type which follows photic stimulus."

(2) The initial response is positive which may be followed by a large negative component in light anesthesia.
(3) "With picrotoxin it can be demonstrated that the negative wave is associated with activation of association areas and tectal regions. Applications of nembutal to the same pial surfaces abolishes the negative wave and the projected reaction."

The characteristics and a mapping of the cortical response evoked by diffuse photic stimulation in the cat under pentobarbital was presented by Doty [IV-2-4] in 1958. Doty detailed the effect of the length of time the light is left on and the effect of the position of the light with respect to the retina of the eye. He believed that secondary responses, occurring 60-100 msec. after the flash of light, were not related to concurrent optic nerve activity. The evoked responses were considered to be the result of the interaction between the response produced when the light was turned on and a separate response which occurred when the light was turned off.

A study of the evoked responses in man to repetitive photic stimulation and the possible use of this technique in clinical situations has been published by Walter and Walter [IV-2-5]. They described the records obtained during photic stimulation in terms of components such as:

(1) "A series of discrete elementary evoked responses".

(From the curves shown in the article these responses seemed to be associated with a 3.5 flash/sec. stimulus which was the minimum repetition rate used in their study.)
(2) "Fusion of evoked responses giving an accidental appearance of rhythmicit."

(3) A summation of evoked responses and spontaneous activity.

(4) A "driving" of local rhythms at the frequency of the stimulus.

They noted that these responses were dependent upon the subject's "somatic, mental, and emotional state." Also they found it was possible to alter the subject's state under certain stimulus conditions.

Kimura [IV-2-6] reported in 1961 typical wave forms of multiple responses recorded from the visual cortex of anesthetized (nembutal) and unanesthetized rats. He indicated that responses recorded from the rat were similar to those recorded from the cat and rabbit. The "late" response in the unanesthetized rat was characterized by "a series of high amplitude negative waves with sharp initially positive activity interposed between the waves" which was found to be sensitive to the state of alertness of the animal.

Further evidence of the effect of the state of alertness upon photically evoked potentials was offered by Mundy-Castle [IV-2-7] in 1953. His work substantiated many of the results of Walter and Walter and significantly related the harmonic content of evoked responses due to a periodic photic stimulus to certain clinical situations. The further use of harmonic content of evoked responses in man as a clinical diagnostic aid for detecting anxiety-proneness was recorded by Ulett, Gleser, Winokur and Lawler [IV-2-8].
Much of the significant research concerning the characteristics of evoked responses from the visual cortex of the cat has been reported by Brazier [IV-2-9,10,11]. In these references she has used the average response technique to determine: (1) latency times, (2) the effect of anesthesia upon evoked responses, and (3) the effect of photic stimulus upon the electrical activity of the reticular formation of the brain. She has shown that the response of the visual cortex of an unanesthetized cat has an initial positive response 12 msec. after the flash of light. This response is followed 11 msec. later by a negative wave. These results were substantiated by additional data which showed an averaged response with an initial positive wave followed by a sharp negative response that had a peak at 26 msec. Next there was a positive recovery and another negative peak at 40 msec. After these sharp transients, a long-lasting surface-positive wave occurred followed by a surface-negative wave. These articles also illustrate the effect of anesthetic agents upon the averaged responses and the histograms of the evoked potential's amplitude distribution. Another result which is important to this research showed that the averaged responses to slow rhythmic (0.7 flashes/sec.) and averaged response to random light flashes were essentially the same.

The linear assumption necessary for the model proposed in this research implies that the superposition principle is valid. This proposition is demonstrated to a certain degree by the experimental results of Geisler [IV-2-12] for paired stimuli. His data indicates that the superposition of two average evoked responses separated by
\[ \tau \] seconds, results in a response similar to the average evoked response for paired stimuli with a separation of \[ \tau \] seconds.

Although the present research did not consider the effect of light intensity upon the evoked responses, the importance of this aspect was recognized. The work of Danilov [IV-2-13] presents a detailed account of the effect of light intensity upon the evoked responses in man for flicker frequencies in the \( \alpha \)-rhythm range. He stated that "Increase of the intensity of the light flicker stimulus with the subject in the waking state was usually accompanied by parallel intensification of all the harmonics recorded." Other results in his paper show that the harmonic composition of the evoked potentials are an indication of the subject's state of alertness. The presence of high harmonics were correlated with an excited state of the central nervous system.

The problem of variation in intensity and frequency has been considered by Il'Yanok [IV-2-14]. A Gray Walter harmonic analyzer, modified to measure frequencies up to 480 cps. was used by Il'Yanok to measure the frequency responses for stimulus rates up to 120 flashes per second. He found that an increase in the intensity brought about a rise in the amplitudes of the high frequencies.

While one group of researchers has examined the EEG, another distinct group has investigated the structure of the brain at the level of the neuron. Hubel [IV-2-15] traced the interconnected set of nerve cells from the retina to the visual cortex. The so-called "visual pathway" extends from the retina, along the optic nerve,
to the lateral geniculate bodies in each hemisphere. From here new fibers transmit visual messages to the visual area of the cerebral cortex.
4-3. Preparation of the Cat and Data Gathering

In order to study the relative coupling both horizontally and laterally in the visual cortex, an array of 12 monopolar electrodes was arranged to provide four localities with cortical, cortical surface, and cranial surface electrodes. The position of the array in terms of Horsely-Clark coordinates was based upon the results of Gerard, Marshall and Saul [IV-3-1]. They observed that large evoked potentials were recorded at position OA. 8L. For cortical recordings, they noted strong responses at a depth of 3mm. This area corresponds to O'Leary's [IV-3-2] broad definition that the visual cortex of the cat is "The dorsomedial portion of the posterior one-third of the hemisphere, and extends from the base of the Splenial Sulcus to the middle of the Lateral Gyrus anteriorly and to the base of the Post-lateral Sulcus posteriorly."

Using the coordinate OA. 8L. as a reference, electrodes were placed at the reference and 3mm anterior, medial and lateral to this coordinate. In this system of coordinates, the midsagittal ridge is the lateral origin, and the interaural line is the anterior-posterior origin. A plan and section view of the electrode locations are shown in Figure 4-3-1. With this configuration it was hoped to have electrodes in the primary and associative receptor areas. Grossly, from the coordinate graph the anterior electrode is in the medial portion of the Suprasylvian Gyrus and the medial electrode is in the lateral portion of the Lateral Gyrus.

Electrodes. The electrodes were made from #14/1000" silver wire in a special form for each level. A point electrode was constructed
ELECTRODE LOCATION

Figure 4-3-1
for the locations within the cortex by coating the tip of the wire with epoxy. When the epoxy had hardened, a grinder was used to expose a sharp 0.5mm tip. The cortical surface electrode, a 1.0 mm diameter ball, was made by melting one end of the silver wire until the desired diameter globule was formed. The cranial surface electrode, a 2mm length of wire, was placed around the lip of the hole drilled in the skull for inserting the cortical electrodes. All electrodes were insulated by slipping a teflon sleeve stripped from #26 teflon coated wire over the exposed wire. The wires were then soldered to a 14 pin sub-minature Winchester # SMRE 14 SG connector. In order to maintain a 3 mm vertical displacement between the cortical surface and cortical electrodes, these two electrodes were attached to each other. A 4mm diameter flat electrode was used as a ground and placed anterior to the array. The general details of the electrodes are presented in Figure 4-3-2.

Preparation of the Cat. The technique recommended by Horovitz and Chow [IV-3-3] was used as a guide for implanting chronic electrodes in the cats used in this research. The anesthetized animal was placed in a stereotaxic instrument and midline incision of about two inches was made in the scalp. After the temporal muscles were reflected to give the desired skull exposure, a dental burr drill was used to rough the cranial surface. Next the electrode entry locations were marked and drilled with a #1 inverted cone #37 drill. Sterile water was used as a drilling fluid to prevent excessive heating and possible damage to the cortex. The cortical electrodes were then inserted and held in place with dental cement packed into each hole. The surface electrodes
ELECTRODE CONFIGURATION

Figure 4-3-2
were bent to fit the lip of each hole and were also cemented into place. Next the ground plate was attached and the entire array was covered with a thick coat of cement. Two machine screws were placed in the skull to support the preparation and provide an additional surface for the cement to adhere. After the initial layers of cement had set, the connector was placed on the midline and bridged to the skull with the cement.

During this phase of the research, three cats were prepared with chronic electrodes generally following this procedure. Two of the implants were very successful; however, one cat which had only one support screw, quickly pulled out his array since a bond was not achieved between the cement and the cranial surface.
4-4. **Recording of Data**

Magnetic tape recordings and strip chart records of the animal's EEG were made using the data recording system illustrated in Figure 4-4-1. The use of untrained animals in the study made it necessary to confine the cat's movements to a relatively small area. A special clear plastic bubble proved to be an excellent cage since it allowed an unrestricted observation of the animal. The cage was placed in one of the screened subject rooms at the Houston State Psychiatric Institute and a 14 conductor cable was placed between the room's "signal conjunction box" and the connector on the cat's head. The cable was made from a very light and flexible Alpha Corp. 1110 stereo wire which did not appear to impair the cat's activities or produce movement noise. In fact the animal did not make any effort to extract the cable after it had been put in place a few minutes.

Four channels of low level EEG (25 to 75 µV) were connected from the subject room to a model Grass 1307 Electroencephalograph located in the next room. The Grass amplified the EEG and provided a visual record for the study. Single ended high level signals were obtained from the second stage of the Grass power amplifiers which were capacity coupled through a bank of isolation-filter amplifiers to the f.m. inputs of the Precision Instruments model PS-207A magnetic tape recorder. In order to reduce the high frequency noise associated with the tape recorder's playback system, the isolation amplifiers frequency response was set to enhance the EEG signal between 30 cps and 150 cps. On playback, a reproduce filter with 6db/octave attenuation at 30 cps restored the signals frequency response to a 0.5 cps to 150 cps bandwidth.
Fig. 4-4-1.
The enhancing of the high frequencies of the EEG did not appear to over-
drive the record system, since above 30 or 40 cycles the EEG amplitude
is generally inversely proportional to frequency [IV-4-1].

A Buffington Electronics photic stimulator was employed to provide
single light flashes or flashes at repetition rates from 0.5 flashes
per second to 50 flashes per second. An external electrical output of
the stimulator provided a 2μ second pulse synchronized with the flashing
of the light. This pulse was widened to a few milliseconds by a one shot
and the output of the one shot was connected to one of the f.m. channels.
Throughout the study, four f.m. channels were used for EEG. One f.m.
channel was used for synchronizing pulses, and a direct channel re-
corded voice instructions.

Magnetic tape recordings were obtained from 6 electrodes; which
were the cranial surface, cortical surface, and within the cortex at
locations OA. 5L. and OA. 11L. Simultaneous recordings were made
from the three electrodes at one location and the subcortical electrode
of the other location. The limitation of four recording channels made
it necessary to repeat the entire sequence of tests in order to achieve
a uniform set of data. Two sets of data of the cat's EEG response to
photic stimulation were obtained. For one set of data the cat was nom-
inally alert, for the other set of data the cat was given 25 mg i.m.
of Thorazine twenty minutes prior to the recording of data. This
tranquilizer reduced the alertness of the cat to such an extent that
the states of alertness for the two sets of data were grossly different.
That is, the cat's reflexes were much slower under the influence of the
tranquilizer. These two conditions shall be designated by the terms
control and tranquil. The data in each set recorded when the tests were conducted the first time are designated as Run 1. The data obtained for a different electrode configuration by repeating the test is labeled Run 2.

The amount of time devoted to each test was not the same; however, the following sequence was adhered to:

(1) Control (2 to 5 minutes)

(2) Approximately one flash/second photic stimulation
   (2 to 3 minutes)

(3) Control (3 minutes)

(4) Approximately 8 flashes/second photic stimulation
   (2 to 3 minutes)

(5) Control (3 minutes)

(6) Approximately 32 flashes/second photic stimulation
   (2 to 3 minutes)

(7) Control (2 to 5 minutes)

(8) Random light flashes (5 minutes)
4-5. **Analysis of Data**

In Chapter III, we developed a rational scheme for representing the cat's response to photic stimulation. In order to test the usefulness of this scheme, data processing procedures were selected which performed the specified operations of the recorded EEG. Also, several techniques were employed as independent measures of: (1) the degree of coupling between electrodes and (2) the animal's state of alertness. The results of the independent measures provided a basis for evaluating the usefulness of the concepts described in this research.

**Averaging of Evoked Potentials.** The evoked responses caused by the photic stimulation were considered to represent the animal's impulse response. Whenever the repetition rate of the stimulus is equal to or less than one flash per second, we assume that the evoked response due to one flash of light reaches a small value before the next flash occurs. That is, the scheme for estimating \( \mu_g(t) \) by the summation \( A_n(\xi, t) \) described in Section III-6 is valid if \( n > n_0 \). This summation process illustrated in Figure 3-6-1 may be realized by the "Computer of Average Transients" [IV-5-1] or similar devices which have the capability of "averaging" by adding successive intervals of data [IV-5-2, 3, 4]. In the compute mode, the CAT computer samples and stores 400 equally spaced values of an analog signal immediately after it receives an external trigger pulse. When the next trigger pulse is received the computer samples another 400 values and adds them sequentially to the previously stored data. The sampling rate of the computer is determined by the "analysis time" selection knob which has settings from 0.0625 seconds to 32 seconds.
In this research the EEG and the pulses which were synchronized with the occurrence of the photic stimulus were respectively the "analog input" and the trigger pulse for the CAT computer. A schematic diagram of the equipment used in this phase of the research is shown in Figure 4-5-1. After the summation of a desired number of evoked potentials, the average response was recorded on an x-y plotter. In order to avoid summing together evoked responses which might be from different "states", averages of 10 and 30 responses were performed at an analysis time of 0.5 seconds on all of the EEG which were recorded during the random and the one-flash-per-second photic stimulation. Additional analysis time and the number of responses to be averaged were used in connection with the analysis of several sections of the data.

At stimulation frequencies greater than one flash per second, the overlap of the individual impulse responses becomes significant. However, the average response computed by the "Cat" approaches the expected periodic response described in section III-7. The analysis sweep for the 8 and 32 flashes per second data was 0.0625 seconds; the maximum sampling rate of the computer. According to section III-7 the expected periodic response is meaningful, in terms of its Fourier series representation, which yields values of the expected transfer function at the fundamental and harmonic frequencies.

**Fourier Transform of Averaged Responses.** Several of the estimates of the expected impulse responses and expected periodic responses were digitalized, and a standard sine-cosine transform [IV-5-5] was performed on this data with a digital computer. The
sine-cosine transform of the expected impulse response provides values of amplitude and phase of the Fourier transform at any desired frequency. However, if the time function is periodic, the sine-cosine transforms should be evaluated only at the fundamental and harmonic frequencies.

The particular program used on the Rice Computer contained a provision for shifting the time origin of the data vector. Thus the phase computations could be corrected for the phase produced by the transport lag (latency) between the light flash and the response of the brain. The details of this problem are discussed in Chapter V.

Measure of the Distribution of the Evoked Response. Although the summation process of the "Computer of Average Transients" provides an easy method of determining the expected impulse response, it does not indicate the variation of the individual response about the mean response. According to section III-6, the variance of the summation process $A_n(\cdot, \cdot)$ is the sum of the variance of the impulse response $g(\cdot, \cdot)$ and variance of the "ongoing" activity $z(\cdot, \cdot)$. However, the slight variation in the time delay between the photic stimulus and each evoked response impairs the meaning of measurements at a specified time $t$. Whenever the system response is much larger than the background activity, measurements of time intervals and peak to peak values are assumed to be estimates of the distribution of the impulse response.

These measurements were performed by hand on pictures of the individual responses taken with a 16mm movie camera. The system
for obtaining the pictures is shown in Figure 4-5-2. The filtered EEG is displayed both on the monitoring scope and the scope fitted with a 16mm camera. A pulse synchronized with the occurrence of a light flash triggered both scopes and the first 0.2 of a second of the response was displayed. At the end of the 0.2 of a second, an "end of sweep" pulse set a flip-flop which in turn controlled the film advance system of the camera. The film advanced one frame at which time the flip-flop was reset and the shutter of the camera opened ready to record the next evoked response. Since the film advance cycle was about 0.4 of a second, pictures were taken only of the response to one flash per second stimulation.
SYSTEM FOR OBTAINING PICTURES OF INDIVIDUAL RESPONSES.

Figure 4-5-2.
4-6. **Independent Measures.**

In the interpretation of the EEG records, gross changes in the state of alertness are usually indicated by corresponding changes in the dominant frequency of the EEG [IV-6-1]. Since the dominant frequency is a traditional measure of alertness, independent measures were selected which would be able to describe this activity in the EEG. The three methods used for this part of the analysis each considered a different aspect of the signal: that is, (1) **visual inspection** considered the entire wave form in a somewhat qualitative manner; (2) **low frequency resonant filters** provided a quantitative measure of the energy associated with a particular frequency and time interval; (3) **period analysis** detected certain critical points, such as, the zero crossings of the signal.

The visual interpretation of the data consisted of scanning the EEG record of a test and then classifying sections of the data according to slow, medium, or fast activity. Wherever a section of the data appeared to be nontypical because of artifact, limiting, or baseline drift, the section was omitted from further consideration. The classical definition presented in Hill and Parr [IV-6-2] for the EEG frequency bands was generally used to interpret the dominant frequency. That is, slow activity included the delta and theta bands, medium activity corresponded to alpha rhythm, and fast activity included the beta and gamma frequencies. Relatively high amplitude, disorganized waves which appeared to have a fundamental frequency between 2.0 cps and 7.0 cps were considered to be slow activity. The alpha or medium activity was set as the 8.0 cps to 13.0 cps
band and the fast activity consisted of low amplitude signals above 14.0 cps.

Two low-frequency resonant filters provided a semi-automatic indication of the presence of any slow activity. The analog system shown in Figure 4-6-1 provided a method of determining the energy in the EEG for successive epochs at any desired frequency. A resonant filter with a $Q > 10$ was simulated with two operational amplifiers as shown in this figure. Although the gain of the simulated filter required adjustment each time the system was used, the simple scheme had the advantage that a low loop gain could be used for any resonant frequency. This may be seen from the transfer function for the simulated filter which is

$$\frac{C}{R} = \frac{-K}{1 - \frac{AS}{T_1 S + 1}} \frac{-K(T_1 S^2 + 2T_1 S + 1)}{T_1 S^2 + T_1 S(2-A) + 1}$$

for $R_0 = R_{FB}$

$$C_0 = C_{FB}$$

$$R_0 C_0 = T_1$$

From this equation we can see that if all the components associated with the time constant $T_1$ are perfectly matched, the gain $A$ may be set for a desired $Q$ and not changed when the components determining $T_1$ are changed. In this research, center frequencies of 2.65 cps and 6.55 cps were employed to ascertain the magnitude of the slow activity. A plot of the band pass characteristics of the simulated filter and the cascade Krohn-Hite filter is shown in Figure 4-6-2.

The period analysis system [IV-6-3] at the Houston State Psychiatric Institute provided not only a measure of the predominant frequency but it also examined the organization, and structure of
SIMULATED RESONANT FILTER SYSTEM

Figure 4-6-1
BAND PASS OF SIMULATED FILTERS

Figure 4-6-2.
the EEG. This system located certain critical points of the EEG such as the function's crossings of an arbitrary base line, points of zero slope, and points of zero acceleration.

For example, let the primary signal of Figure 4-6-3 represent an analog function to be analyzed. The points where the function \( f(t) \) intersects a preselected base line (normally, its mean value) are designated as zero crossings of the primary wave. In a similar manner the points at which \( \frac{df(t)}{dt} \) and \( \frac{d^2f(t)}{dt^2} \) are equal to zero, are called respectively zero crossings of the first derivative and the zero crossings of the second derivative. The information contained in an epoch (that is, a particular time interval) is reported as being related to the time between consecutive zero crossings of the signal and its derivatives. In particular, period analysis determines for the primary and secondary derivative:

(1) The total number of positive-going zero crossings per epoch.

(2) A period spectrum of the number of occurrences that the time interval \( T_i \) between positive going zero crossings is \( T_n \leq T_i < T_{n+1} \) where the interval \( (T_n, T_{n+1}) \) represents a preselected band in the period spectrum [IV-6-4].

The period spectrum is usually divided into ten intervals which are designated in terms of equivalent frequency from zero to one hundred cycles per second.
PERIOD ANALYSIS CODING
Figure 4-6-3
REFERENCES


6-2 Walter, Op. Cit., p. 83


PRESENTATION AND DISCUSSION OF EXPERIMENTAL RESULTS

In the previous chapters a model for relating the EEG to certain stimulus fields is developed and an experimental situation for testing the reasonableness of the model is detailed. We are now in a position to examine the usefulness of the model for providing a meaningful interpretation of the evoked responses of a cat to photic stimulation. The following results were obtained from EEG recorded from six electrodes. These electrodes were located in the primary receptor area and the associative area of the cat's cortex. In each area chronic implants of a cranial, a cortical surface, and a cortical electrode were made.

Evoked responses recorded from each electrode provided data for evaluating the overall transmission operator (expected impulse response or expected transfer function) from the stimulus to the electrode in question. From plots of the expected impulse response and expected transfer function we may ascertain certain characteristics of the linear, time-invariant system postulated in the model.

The usefulness of the model is examined in terms of the following two experimental questions: (1) does the model provide a means for classifying psycho-neurophysical states or detecting changes in these states? (2) may the model be used to designate the degree of coupling between EEG recorded from different locations? In order to answer the first experimental question, the evoked responses are examined for changes during control conditions and examined for differences between the control and drugged responses. The effect of the drug upon the evoked response may be compared to published data in order to affirm our
confidence that the limited number of evoked responses analyzed in this research are "typical". Since we do not have documented results of the effect of changes in alertness upon evoked responses over short intervals of time, three independent measures were employed to classify each ten-second epoch of the control data according to the dominant frequency. These results may then be compared with corresponding averaged responses in order to determine if there is a correlation between changes in the dominant frequency and changes in the evoked responses.

The second experimental question concerning the degree of coupling between EEG is investigated by comparing evoked responses of the primary receptor area with those of the associative area during control and drug conditions.
5-1. **General Description of the Evoked Response**

During the control portion of the tests, evoked responses recorded from the primary receptor area were generally distinguishable above the ongoing activity (see Figure 5-1-1). These responses were similar to the classical "multiple responses" described by Marshal, Talbot and Ades [V-5-1] which have an initial "surface positive" wave followed by a secondary response. In this research, the complete evoked response was considered in its entirety and not as the sum of elementary wave forms. Therefore some of the nomenclature normally used in conjunction with evoked responses has been set off by quotation marks.

Evoked responses from the far lateral cortical electrode B-3 (see Figure 5-1-1) were not as well defined and the first major "wave form" was "surface negative" suggesting that these signals were not from the visual cortex [V-5-2] but responses from an area associated with the visual cortex. The reader should note that the classic system of displaying "positive going" biological potentials as a downward deflection has been used throughout this chapter.
Occurrence of Light Flashes

EEG RECORDED DURING PHOTIC STIMULATION

Figure 5-1-1
5-2. Expected Impulse Response and Expected Transfer Function

In sections III-6 and III-7 the summation $A_n(\cdot, \cdot)$ is described which under certain conditions is approximately equal to the expected impulse response (see section III-6) or the expected periodic response (see section III-7). Let us examine some of these conditions in relation to the evoked responses measured in the cortex of the primary receptor area D-3. If it is reasonable to conclude that the summations and the expectations are approximately equal, we may determine some of the properties of the system from the expected impulse response and expected transfer function.

It was shown in section III-6 that the value of each previous evoked response must be small before the next response occurs if the average response to a train or periodic flashes of light will approximate the expected impulse response $\mu_0(t)$. From Figure 5-2-1a we see that the average of the first 30 responses to a one flash per second stimulus reaches a small value 0.5 seconds after the stimulus has occurred. In addition to decay in the transient response, the effect of the summation process on the ongoing activity $z(\cdot, \cdot)$ is illustrated in the figure. Notice that the mean value of $z(\xi, t)$ is small compared to the average response.

By expanding the time scale, the details of the average response are more apparent as shown in Figure 5-2-1b. A visual comparison of the average of the first 10 responses to one flash per second stimulation and the average of the first 30 responses shown in Figure 5-2-1 indicates that the 20 additional responses do not greatly change the average response. Therefore, the average of 10 or more evoked responses
Primary Receptor Area of Cortex, D-3

\[ \text{Light} \]

\[ 50 \mu V \]

\[ +0 \quad 0.5 \quad 1.0 \]

Time in Seconds

A
Average of First 30

\[ +0 \quad 125 \quad 250 \]

Time in Milliseconds

B
Average of First 30

\[ +0 \quad 125 \quad 250 \]

Time in Milliseconds

C
Average of First 10

AVERAGED EVOKED RESPONSES OF UNRESTRAINED CAT
TO ONE FLASH PER SECOND STIMULATION
Figure 5-2-1
measured at D-3 approximates the expected impulse response of the system between the eye and D-3 whenever the animal is in the same state of alertness. In general, an average of 10 samples of the measured signal appears to be sufficient for $A_n(\cdot,\cdot)$ to approximate $W(\cdot,\cdot)$, if the individual evoked responses $w(\cdot,\cdot)$ are distinguishable from the ongoing activity $z(\cdot,\cdot)$. When $w(\xi,t)$ is not visible in the EEG, a larger number of samples are needed.

If the repetition rate of the stimulus is greater than two flashes per second, the summation process $A_n(\cdot,\cdot)$ must be considered as an average periodic response (see section III-7). Examples of the average periodic response for photic stimulation at 7.33 flashes per second are shown in Figure 5-2-2. By setting the interval over which the average is determined, to a value greater than twice the period of the stimulus, more than one period of the expected periodic response may be obtained. For example, four cycles of the averaged response to 7.33 flashes per second are shown in Figure 5-2-2. Notice that there is a good resemblance between each period of the averaged data, whereas a randomness is visible in the measured EEG presented in the same figure. This feature may be also seen in the EEG recorded during 30.2 flashes per second stimulation and in the associated average periodic response (see Figure 5-2-3.)

By examining the initial and final values of the expected impulse response, we may ascribe to the associated system certain properties. Although we shall focus our attention on the average response at D-3 shown in Figure 5-2-1b, similar results may be obtained using the responses measured at the other electrodes. The general properties
Primary Receptor Area of Cortex, D-3

Light

25 μV

Average of First 240 Responses

0 62.5
Time in Milliseconds

25 μV

Average of 146 sequences of 4 Responses Each

0 250 500
Time in Milliseconds

50 μV

Sample of EEG Averaged

0.1 Second

RESPONSE OF UNRESTRAINED CAT TO 7.33 FLASH PER SECOND STIMULATION

Figure 5-2-2
Primary Receptor Area of Cortex, D-3

Average of 480 Sequences of 2 Responses Each

Time in Milliseconds

Response of unrestrained cat to 30.2 flashes per second stimulation

Figure 5-2-3
of the linear system connecting the eye and the cortex of the primary receptor area D-3 are described below.

First let us notice that there is a finite time $\tau$ between the occurrence of the stimulus and the response of the system. The effect of the transport lag may be incorporated in the proposed model by separating the system into a time delay section in series with a linear system without transport lag. The overall expected transfer function $\mu_g(s)$ may be expressed as the product of functions in $s$ for the transport lag $\tau$ and the linear system. The transfer function for the linear system may be expressed as the ratio of two polynomials in $s$, which do not contain common factors, and the time delay or transport lag may be indicated in the complex frequency domain by the factor $e^{-\tau s}$; that is

$$(p-1) \quad \mu_g(s) = e^{-\tau s} \frac{q(s)}{p(s)}.$$ 

From the final value theorem

$$\lim_{t \to \infty} \mu_g(t) = \lim_{s \to 0} s \mu_g(s),$$

we see that for the expected impulse response under consideration

$$\lim_{t \to \infty} \mu_g(t) = 0 = \lim_{s \to 0} s \frac{q(s)}{p(s)},$$

implies the denominator $p(s)$ does not contain any factorable $s$ terms. Therefore, if we write $\mu_G(s)$ in the form

$$\mu_G(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0},$$

the final value theorem implies

$$(p-2) \quad |b_n| > 0.$$
If we apply the initial value theorem

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

to the expected impulse $\mu_g(t)$ at time $\tau$, we see that

$$\lim_{t \to \tau} \mu_g(t) = 0 = \lim_{s \to \infty} s \mu_G(s).$$

As $s$ approaches infinity, the expected transfer function reduces to the form

$$\lim_{s \to \infty} \mu_G(s) = \frac{s^{m+1}}{s^n} = 0$$

which implies

(p-3) $n - m > 2$ for $\mu_g(\tau) = 0$.

That is, the order of the denominator is at least two greater than the numerator. Since the impulse response appears to have zero slope at $t = \tau$, the initial value theorem may be used again to show

(p-4) $n - m > 3$ for $\frac{d\mu_g(t)}{dt} \bigg|_{t = \tau} = 0$.

Further properties of the system may be obtained from the transfer function in the frequency domain by computing the Fourier transform of $\mu_g(t)$. A plot of the absolute value of amplitude and phase versus frequency for the transfer function relating the response at D-3 to the photic stimulus, is shown in Figure 5-2-4. Although the resonant peaks and wide bandwidth seen in the Figure are associated only with sharp evoked responses (for example the impulse response of Figure 5-2-1b), the excessive phase shift (2260 degree between 2.0 cps and 160.0 cps for this example) is common to all of the transfer functions relating the photic stimulus to the six recording locations. The appearance of
AMPLITUDE AND PHASE OF EXPECTED TRANSFER FUNCTION

Figure 5-2-4
more phase shift than might be expected by an inspection of the amplitude plot suggest the phase shift is due to either transport lag or a non-minimum phase system or a combination of these two. First we may use the general property $p-1$ to assert that the transport lag produces a phase shift of $-\omega \tau$ radians at each frequency. From a listing of the values of the data vector $\mu_g(t)$ upon which the Fourier transform was computed, it is possible to estimate the transport lag for the signal measured at D-3 as 0.0075 seconds. A plot of the phase of the transfer function compensated for the transport lag is presented in Figure 5-2-5. We may now associate the amplitude and the compensated phase plots with the rational portion of the transfer function, that is $\frac{q(j\omega)}{p(j\omega)}$. Since this function is analytic except at its critical points, the imaginary and real parts are not independent. Further, the logarithm of the rational part of transfer function is analytic [V-2-1]. If none of the poles and zeros of the rational transfer function are in the right hand plane (i.e., $\text{Re } s \geq 0$) the log amplitude and the phase for $s = j\omega$ are uniquely related. A system of this type is said to be minimum-phase. In particular, the difference in phase angle $\Delta \theta$ between the phase at $s = 0$ and $s = \infty$ is related to log amplitude as

$$\Delta \theta = -(n_0 + n_\infty) \frac{\pi}{2},$$

where $n_0$ is the order of $\frac{q(s)}{p(s)}$ at $s = 0$

and $n_\infty$ is the order of $\frac{q(s)}{p(s)}$ at $s = \infty$. 
Test Parameters
Location D-3
Control
First 30 responses
One flash per second

Phase in
Degrees

(n + 1) 360
n = 0
n = 1
n = 2
n = 3
n = 4
n = 5

(n + 1) 180

0 20 40 60 80 100 120 140 160

Frequency in cps

PHASE OF EXPECTED TRANSFER FUNCTIONS
COMPENSATED FOR TRANSMISSION LAG
OF 0.0075 SECONDS

Figure 5-2-5
Let us now estimate $n_o$ and $n_\infty$ from the log amplitude plot of the transfer function between the eye and D-3 which is shown in Figure 5-2-6. By noting that each unit slope (6 db/octave) requires a corresponding critical point, and approximating $n_o$ and $n_\infty$ by the slopes of the log amplitude plot at 2.0 cps and 160.0 cps, we may indicate that $n_o \leq 2$ and $n_\infty \leq 4$. For a minimum-phase system the change in phase should be equal to or less than 540°; therefore, since the system has 1800 degrees change in phase, the system is said to be non-minimum-phase. The response is bounded; thus the non-minimum phase character is due only to zeros in the right hand plane (i.e., roots of $q(s)$ with positive real parts). Using Bode's procedure of phase reduction, the rational non-minimum-phase transfer function may be written as the product of a minimum-phase transfer function and a stable all-pass function. The all-pass function does not contribute to the amplitude of the system, but it does account for the additional phase shift introduced by the zeros in the right hand plane [V-2-2].

For each zero in the right hand plane, 180 degrees of phase lag is added to the system. The system under consideration may be expected to have at least 10 zeros in the right hand plane.

It is reasonable to suppose that rather than one transmission path from the stimulus to the brain, there are multiple paths and the evoked response at some location is a summation of the responses from each path. If we stipulate that the transfer function of each path is minimum phase, then the overall transfer function is stable. However, large differences in the phase of the individual responses may cause an actual substraction of signals and thus produce the
TEST PARAMETERS

Location D-3
Control

First 30 Responses
One Flash per Second

Amplitude in
DB

12 DB/Oct (Max)
(2 Unit Slope)

24 DB/Oct (Max)
(4 Unit Slope)

Frequency in cps

20 \log_{10} \text{AMPLITUDE OF EXPECTED TRANSFER FUNCTION}

Figure 5-2-6
associated non-minimum-phase condition for the overall system.

Let us now turn from the general nature of the transmission systems to some of the specific properties of the impulse response measured at D-3. From Figure 5-2-4 we see predominant complex poles at 2, 10, and 44 cps. Except for the peaking which occurs at the poles, there is less than 6 db attenuation between 1.0 and 100.0 cps. Beyond 100.0 cps there is a rapid (4 unit slope) attenuation, which suggests that the bandwidth of the data processing system should not be less than 100.0 cps.

From section III-7 we note that the expected transfer function \( \mu_G(j\omega) \) may be computed from our knowledge of the input signal and average periodic response \( A_n(\xi,t) \). That is

\[
\mu_G(j\omega) = \frac{F[A_n(\xi,y)_T]}{F[x(t)_T]} = \frac{A_n(\xi,j\omega)}{X(j\omega)}.
\]

The Fourier transform over one period of a series of rectangular pulses of height \( E_m \) and duration \( b \) is

\[
X(j\omega) = \frac{E_m b}{2\pi} \sin \frac{\omega b}{2}.
\]

In this research the "on" time of the photic stimulus was about 0.5 msec.; thus the amplitude spectrum of the input spectrum is essentially constant over the frequency range of interest [0,160 cps]. Since the amplitude of signals at the output of the model's nonlinear element have not been specified, we shall indicate that the amplitude of the spectrum \( \frac{E_m b}{2\pi} \) at the input to the linear system is equal to some constant \( K \). Now we may write the expected transfer function as
\[ \mu_G(j\omega) = \frac{\hat{A}_n(\xi, j\omega)}{K} \]

which may be evaluated at the fundamental and harmonic frequencies of the input signal. If the system is linear and time-invariant, values of the expected transfer function determined from a series of periodic impulses should be equal to corresponding values of the frequency response obtained from the expected impulse response \( \mu_g(t) \). We may test the linear, time-invariant assumption to a limited degree with frequency response data by superimposing values of the frequency spectrum of the expected periodic response upon a plot of the continuous frequency response of the system. A curve of this type is shown in Figure 5-2-7 of the system measured at D-3. We can see that the responses to 7.33 and 30.1 flashes of light per second, produced spectra similar to the basic amplitude plot of the Fourier transform of the expected impulse response \( \mu_g(t) \).

A measure of the distribution of the evoked response recorded at D-3 was computed from the first 30 evoked potentials to one flash per second. Distributions of the time intervals and peak to peak values are shown in Figure 5-2-8.
AMPLITUDE PLOT FOR FOURIER TRANSFORM OF AVERAGED PERIODIC RESPONSES TO PHOTIC STIMULATION OF 1.0, 7.33, AND 30.1 FLASHES PER SECOND.

Figure 5-2-7.
DISTRIBUTION OF TIME INTERVALS AND PEAK TO PEAK VALUES

Figure 3-2-8
5-3. **Effect of Changes in State Upon the Expected Impulse Response.**

Associated with any indicator of the state of alertness, there is a degree of uncertainty. First we may not know the resolving power of the indicator. That is, how great a change in alertness is required before the indicator signals a change in state? Secondly, we may not be certain of the "false alarm" nature of the indicator. Therefore, the measurements of the dominant frequency in this research are not regarded as an absolute measure of state. However, the dominant frequency is construed to be a reasonable indicator of gross changes in the animal's state of alertness.

An inspection of the EEG records revealed that the animal did not remain in one state throughout a sequence of tests; for example, analysis of the first set of controlled data using the three independent measures revealed several major changes of state. The original strip chart records of the EEG recorded at locations B-3, D-1, D-2, and D-3 were coded at ten second intervals according to slow, medium and fast rhythms. Prior to the one flash per second stimulation, a single flash of light produced an abrupt change in the nature of the EEG. Previously, slow ongoing activity increased in frequency and decreased in amplitude. This high frequency rhythm was sustained for the first 40 seconds of the one flash per second stimulus. This apparent increase in alertness was also signaled in the other two independent measures by a decrease in the output power of the low frequency filters and by an increase in the total major period count over 10 seconds (see Figure 5-3-1). A visual comparison of the results
ANALYSIS OF CONTROL RUN 1

Classification per Epoch

Sequence of Tests

Control Precontrol 1 flash 10 flash Post Pre- 20 flash Post Dark Light sec sec Control Control Control sec Control

Relative Energy per Epoch

Low Frequency Resonant Filters

Time in 10 Second Epochs

MEASURE OF DOMINANT FREQUENCY

Figure 5-3-1
of the three independent measures indicated that they were generally in agreement.

These measures pointed to another change of state which occurred during the one-flash-per-second stimulation. In Figure 5-3-2, we can see that this change of state produced a dramatic change in the evoked response. The latency increased, the sharp transient disappeared, and a large, slow, negative, secondary wave developed as the dominant frequency of the EEG decreased. These changes in the expected impulse response are reflected in the frequency domain as a decrease in the high frequency content of the corresponding expected transfer functions $\mu_G(j\omega)$ (see Figure 5-3-3). Under drug and control conditions, sharp evoked responses were associated with fast EEG activity; and as the EEG dominant frequency decreased, the sharpness of the evoked potentials decreased.

The above mentioned figures help illustrate the importance of the state parameter $a$ in the model. Thus it may be possible to use the expected impulse response as a relative gauge for detecting various levels of alertness. As a further example of the importance of the state parameter, let us examine the effect of the tranquilizer upon the evoked response. A comparison of the response of the animal to photic stimulation under drug and control conditions is presented in Figure 5-3-4. Even though response transients are sharp, the tranquilizer increased the time delay and decreased the rate of decay of the evoked response. These results are similar to those of other researchers noted in section IV-2.
EFFECT OF CHANGES IN STATE UPON EVOKED RESPONSE

Figure 5-3-2
EFFECT OF TRANQUILIZER UPON EVOKED RESPONSE

Figure 5-3-4
5-4. Relative Coupling of the EEG.

In studies involving normal human subjects, measurements of the brain's electrical activity are restricted to scalp potentials. These signals are usually interpreted in terms of the overall activity of the cortex. However, we must determine or assume: (1) the relation between scalp and depth EEG and (2) the number of electrodes needed to define or note the occurrence of a psycho- or neurophysiological event.

DeLucchi, et al, [V-4-1] concluded that the scalp acts as an averager of the signals at the surface of the cortex. The cortical potential immediately below the scalp electrode was found to account for approximately 60 percent of surface signal. Examination of the strip chart records and evoked responses obtained in this research did not confirm their work. Although simultaneous recordings from the cranial surface, cortical surface and cortex of the primary receptor area were very similar, corresponding measurements from the associative area were dissimilar. Signals recorded at the cranial surface and cortical surface of the associative area appeared to be strongly coupled to the activity of the primary receptor area rather than the activity of their immediate cortex. This coupling is quite apparent in the control evoked responses shown in Figure 5-4-1. From this figure, we may note several interesting points. First, the close agreement of the potentials measured at the three levels of location D suggest that activity at D-3 and in the neighborhood of D-3 are synchronized. This close agreement is also found in the frequency plot of the expected transfer functions. Thus the attenuation of the passive skull does not alter the frequency response of the EEG below it. Second, comparison of the
average responses from the associative area indicates that although the signal at the cranial surface is predominantly due to the activity of the primary receptor area, it contains certain low frequency response features of the signal generated at B-3. This is readily confirmed by a visual comparison of the frequency response plots shown in Figure 5-4-2.

In Figure 5-4-3 we can see that the degree of coupling between the primary receptor area and B-1 (cranial surface of associative area) is decreased by the tranquilizer. The surface electrode reflects the activity of the cortex of the associative area to a large degree. However, the cortical surface electrode still resembles the response from the primary receptor area. From this apparent paradox we might suggest that: (1) skull or scalp electrodes have a stronger coupling to cortical activity than cortical surface potentials, and (2) the drug effect of the tranquilizer causes a decoupling between the primary and associative areas.
Test Parameters:

Content:
One Flash per second
First 10 responses

These expected transfer functions have been computed from the averaged responses $B_1$, $B_2$, and $B_3$ of Run 2 shown in Figure 3-4-1.

Amplitude Ratio

Comparison of expected transfer functions for the primary receptive area and associative area: Figure 3-4-2
REFERENCES


1-2 Marshall, Talbot, and Ades, Ibid.


2-2 Pfeiffer, Ibid, pp. 366-370

CONCLUSIONS AND RECOMMENDATIONS

The core of this research has been the mathematical formulation of a model which describes the brain's response to certain stimulus fields. Experimental data of the evoked responses from the cat's visual cortex produced by photic stimulation have been used to test the reasonableness and usefulness of the model. The results of the limited test described in Chapter V, and a review of the literature, imply that the system between the input signal and each electrode consists of the following three factors: (1) a nonlinear element, (2) a transport lag, (3) a linear, time-invariant, nonminimum phase system. These results also demonstrate the importance of the state parameter \( a \), not only for denoting drug effects but also for signifying changes in state of alertness during control or drug test.

Because of the randomness of the evoked responses, enough terms must be averaged for the average response \( A_n(\xi, t) \) to approximate the expected impulse response or expected periodic response. On the other hand, changes in the animal's state of alertness impose an upper bound on the number of terms which should be averaged together. If \( n_o \) is the number of terms required to yield some desired approximation of the expected response and if \( n_a \) is assumed to be the number of responses which occur while the subject remains in one state, the averaging process \( A_n(\xi, t) \) will not produce the desired approximation for \( n_a < n_o \). Since the purpose of this research has been to propose a model and perform only limited tests, absolute values of \( n_o \) and \( n_a \) should not be inferred from the experimental data presented in this thesis. Likewise, measurements of the degree of coupling between electrodes should be
construed as an interesting example of the usefulness of the model and not viewed as reporting a hypothesis for the cat's processing of visual information.

The mathematical model presented in these pages represents in some respects an unsophisticated approach to the problem of describing the brain's response to stimulus fields. However, it is felt that the concepts and the mathematical foundation developed in this thesis provide an intuitively pleasing translation of the physical problem into the abstract world of mathematical models. This work should be viewed as a "sign post" for related research in physical systems which appear to be random processes. Further investigation of the basic model (primarily in relation to the brain) in the following areas is needed before a final evaluation of the concept can be made.

**Nonlinear series element.** The incorporation of a series nonlinear element with a linear factor is a simple way of handling nonlinear systems. However, there is a limit to the practicality of this method. Therefore, if the model is to be used with input signals other than high intensity flashes of light, appropriate nonlinear elements are needed for each class of input signals. For example, a model of the brain's response to large variations in the intensity of a flashing light should not use a simple limiter as a nonlinear element. From our knowledge of previous research, an element consisting of a combination of threshold, linear gain, and saturation would seem to be more suitable.

**Simulation.** The ability to simulate the average evoked response with an electrical network or analog computer provides a visual proof that the essential nature of the system is known. That is, knowledge
of major poles and zeros of the expected transfer function $\mu_G(s)$ provides sufficient information for simulating the linear system. The impulse response of the simulation is the expected impulse response $\mu_g(t)$. With the aid of the simulation it may be possible to correlate changes in the simulation coefficients to changes in the state of alertness. A simulation which includes intermediate nodes may allow the researcher to study possible feedback loops which are not apparent when simulating a single overall transmission factor.

**Independence assumptions.** In the computation of the summation process $A_n(\xi,t)$, we assumed that successive evoked responses were independent. This may not be a valid assumption for all input stimuli. In situations where this is not true, we must determine what type of dependence exists and the effect it has upon the model.

**Transport lag.** Slight differences in the delay time of each evoked response introduces an error in the computation of the expected impulse response. Further research is needed to determine the extent of this error and the relation of the delay time to the state parameter $a$.

For data sampling rates of 1000 points per second, a 1 msec. variation in delay time is not expected to produce a noticeable error.

**Branch transmission factors.** The flow graph representation of the brain's processing of information may be viewed as the flow of signals from one node to another through branch transmission factors. Although the results of this research considered only overall transmission factors between the input and each electrode, it is possible to use these factors to determine the nature of each individual branch factor. For example, comparison of expected impulse responses at each electrode in section V-4 indicates that the responses in the primary receptor area
are identical. We postulate that the signal measured at D-3 is representative of the potential field generated in this area of the cortex. Since the cortical response D-2 is smaller, we would further postulate that this signal could be represented as an attenuation of the signal at D-3 by a simple constant $k$. Also, since there does not appear to be any transmission of signal from the associative area to D-2, we may assume that there are no significant transmission branches between the associative area and the surface of the cortex in the primary receptor area. Observations of this type are needed in carefully controlled and designed experiments in order to provide reliable data for evaluating the branch transmission factors. A knowledge of these factors may enable us to describe the brain's processing of information in terms of feedback through the transmission factors.
APPENDIX A

Correlation Functions for the Model

The autocorrelation of the measurable output is

$$\varphi_{yy}(t, t + \tau) = E[y(t) y(t + \tau)].$$

Replacing $y(t)$ by $w(t) + z(t)$, we have

$$\varphi_{yy}(t, t + \tau) = E[(w(t) + z(t))(w(t + \tau) + z(t + \tau))]$$

$$= E[w(t)w(t + \tau)] + E[w(t)z(t + \tau)]$$

$$+ E[z(t) w(t + \tau)] + E[z(t) z(t + \tau)].$$

Since the system response $w(t)$ and the ongoing activity $z(t)$ are assumed to be independent, and the mean value of $z(t)$ is zero, the above equation reduces to

$$\varphi_{yy}(t, t + \tau) = \varphi_{ww}(t, t + \tau) + \varphi_{zz}(t, t + \tau).$$

The convolution integral may be used to express $\varphi_{ww}(t, t + \tau)$ in terms of the input signal and the system impulse response. Thus the output autocorrelation function is

$$\varphi_{yy}(t, t + \tau) = E[ \int_{-\infty}^{\infty} g(\xi, t-u) x(u) \, du \int_{-\infty}^{\infty} g(\xi, t + \tau - v) x(v) \, dv]$$

$$+ \varphi_{zz}(\tau).$$

Interchanging the order of integration we find

$$\varphi_{yy}(t, t + \tau) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} E[g(t-u) g(t + \tau - v)] x(u) \, du \right] x(v) \, dv$$

$$+ \varphi_{zz}(\tau)$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \varphi_{gg}(t-u, t + \tau - v) x(u) \, du \right] x(v) \, dv.$$
Replacing $y(t)$ with $w(t) + z(t)$ we have,

$$
\phi_{xy}(t, t + \tau) = E[x(t) (w(t + \tau) + z(t + \tau))]
= E[x(t) w(t + \tau)] + E[x(t) z(t + \tau)].
$$

Since $x(t)$ and $z(t)$ are independent, we may write

$$
\phi_{xy}(t, t + \tau) = E[x(t) w(t + \tau)].
$$

Expanding in terms of the input signal and impulse response we obtain

$$
\phi_{xy}(t, t + \tau) = E[x(t) \int_{-\infty}^{\infty} x(t + \tau - u) g(u) \, du]
$$

Interchanging the order of integration we have

$$
\phi_{xy}(t, t + \tau) = x(t) \int_{-\infty}^{\infty} x(t + \tau - u) E[g(u)] \, du.
$$
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