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THE HEAT TRANSFER RESPONSE
OF A HOT-WIRE ANEMOMETER

by

WILLIAM THOMAS STRICKLAND, JR.

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY IN CHEMICAL ENGINEERING

Thesis Director's signature: 

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# TABLE OF CONTENTS

Title Page 1
Acknowledgements ii
Table of Contents iii
List of Figures iv
List of Tables iv
Table of Nomenclature v
I. Introduction 1
II. Theory 6
III. Calculations 26
IV. Results 34
Bibliography 53
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Flow Pattern</td>
<td>44</td>
</tr>
<tr>
<td>2</td>
<td>Steady-State Stream-Function</td>
<td>45</td>
</tr>
<tr>
<td>3</td>
<td>Steady-State Vorticity</td>
<td>46</td>
</tr>
<tr>
<td>4</td>
<td>Steady-State Temperature</td>
<td>47</td>
</tr>
<tr>
<td>5</td>
<td>Steady-State Nusselt Number vs. Reynolds Number</td>
<td>48</td>
</tr>
<tr>
<td>6</td>
<td>Upstream Heat Transfer</td>
<td>49</td>
</tr>
<tr>
<td>7</td>
<td>Fluctuating Nusselt Number vs. Frequency</td>
<td>50</td>
</tr>
<tr>
<td>8</td>
<td>Amplitude Reduction</td>
<td>51</td>
</tr>
<tr>
<td>9</td>
<td>Phase Lag</td>
<td>52</td>
</tr>
</tbody>
</table>

### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Steady-State Nusselt Number</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>Fluctuating Components of the Nusselt Number</td>
<td>38</td>
</tr>
</tbody>
</table>
# TABLE OF NOMENCLATURE

## CAPITAL LETTERS

<table>
<thead>
<tr>
<th>Letter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Amplitude of fluctuation</td>
</tr>
<tr>
<td>A*</td>
<td>Parameter in Equation 1</td>
</tr>
<tr>
<td>B</td>
<td>Parameter in Equation 1</td>
</tr>
<tr>
<td>C</td>
<td>Speed of sound</td>
</tr>
<tr>
<td>D</td>
<td>Diameter of wire</td>
</tr>
<tr>
<td>F_k</td>
<td>Principal dependent variables of Equations 76-84</td>
</tr>
<tr>
<td>L</td>
<td>Differential operator defined by Equation 27</td>
</tr>
<tr>
<td>M</td>
<td>Differential operator defined by Equation 28</td>
</tr>
<tr>
<td>Nu</td>
<td>Nusselt number</td>
</tr>
<tr>
<td>O</td>
<td>Order-of-magnitude</td>
</tr>
<tr>
<td>Pr</td>
<td>Prandtl number</td>
</tr>
<tr>
<td>R_k</td>
<td>Residue</td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>T</td>
<td>Temperature</td>
</tr>
<tr>
<td>U_∞</td>
<td>Main-stream velocity</td>
</tr>
<tr>
<td>V</td>
<td>Velocity</td>
</tr>
</tbody>
</table>

## LOWER CASE LETTERS

<table>
<thead>
<tr>
<th>Letter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>Acceleration of gravity</td>
</tr>
<tr>
<td>h^2</td>
<td>Jacobian of coordinate transformation</td>
</tr>
<tr>
<td>i</td>
<td>( \sqrt{-1} )</td>
</tr>
<tr>
<td>p</td>
<td>Pressure</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
</tr>
<tr>
<td>u</td>
<td>( x ) (or ( \alpha )) - component of velocity</td>
</tr>
</tbody>
</table>
v \quad y \text{ (or } \beta \text{) } - \text{ component of velocity}

\vec{v} \quad \text{Velocity vector}

x \quad \text{Rectangular coordinate}

y \quad \text{Rectangular coordinate}

**GREEK LETTERS**

\alpha \quad \text{Coordinate}

\beta, \beta^* \quad \text{Coordinate}

\gamma \quad \text{Convergence parameter}

\delta \quad \text{Boundary layer thickness}

\varepsilon \quad \text{Convergence criterion}

\mu \quad \text{Viscosity}

\nu \quad \text{Vorticity}

\rho \quad \text{Density}

\phi \quad \text{Phase lag}

\psi \quad \text{Stream function}

\omega \quad \text{Frequency}

\Delta \quad \text{Difference notation}

\nabla \quad "\text{del}" \text{ operator (See Equation 21)}

\nabla^2 \quad \text{Laplacian operator (See Equation 20)}
I. **Introduction**

In the measurement of velocity profiles and fluctuations in the mean velocity of a flowing fluid use is often made of the hot-wire anemometer. This instrument consists of a small, electrically heated wire which is inserted into the fluid. Since the velocity and physical properties of the fluid determine the rate of heat transfer from the wire, it is possible to relate these factors. The heat dissipated by the wire depends, in turn, on the current flow through and the voltage drop across the wire; and, therefore, by measuring these quantities it should be possible to calculate the velocity of the fluid.

Although the basic principles of the hot-wire anemometer seem simple enough, several complications arise in practice which made the actual calculation of velocity from electrical measurements rather difficult. Some of the practical questions which must be answered are:

A. What is the nature of the heat transfer; is it by free or forced convection, or radiation?

B. What is the equation which relates heat transfer and velocity for the situation in question?

C. What effect does the heat generated by the wire have on the physical properties of the fluid, and how does this influence the heat transfer?

D. If the flow is fluctuating, how much does this change the temperature of the wire and what influence does this have on its resistance?
E. How rapidly does the temperature of the wire respond to a change in velocity?

F. What influence do the end supports that hold the wire have on the flow pattern and heat transfer?

G. What is the effect of a non-uniform velocity distribution along the length of the wire?

These and other questions must be answered before meaningful measurements can be made. For a more thorough discussion of these and other points see Hinze (7).

Most of the above questions have been considered and are taken into account in the selection of the wire diameter, length, type of end supports, wire material, and associated electronic measuring devices. There is one question of a fundamental nature, however, that apparently has not been adequately considered. This concerns the influence of fluctuations of the velocity on the nature of the heat transfer relationship.

For forced convection the relationship between the Nusselt number and the Reynolds number is of the form (7)

$$Nu = A^* + B \ Re^{1/2}$$  \hspace{1cm} (1)

where $A^*$ and $B$ are dependent upon the physical properties of the fluid. However, little is known about the relationship when the flow is fluctuating; and, furthermore, experiments designed to measure this effect are particularly difficult to conduct, especially at higher frequencies of
fluctuation. Theoretical approaches are also difficult owing to the complex nature of the differential equations describing the flow and heat transfer. Therefore, it has been the usual practice to replace the Reynolds number in Equation 1 by an average component, $\overline{Re}$, and a fluctuating component, $Re'$. This gives

$$Nu = A^* + B (\overline{Re} + Re')^{\frac{1}{2}}$$  \hspace{1cm} (2)

By assuming $Re' \ll \overline{Re}$, the above equation is usually linearized. Thus,

$$\left(\overline{Re} + Re'\right)^{\frac{1}{2}} = \overline{Re}^{\frac{1}{2}} \left(1 + \frac{Re'}{\overline{Re}}\right)^{\frac{1}{2}}$$

$$= \overline{Re}^{\frac{1}{2}} \left[1 + \frac{1}{2} \frac{Re'}{\overline{Re}} - \frac{1}{8} \left(\frac{Re'}{\overline{Re}}\right)^2 + \frac{1}{16} \left(\frac{Re'}{\overline{Re}}\right)^3 + \ldots \right]$$  \hspace{1cm} (3)

so that

$$Nu \approx A^* + B \overline{Re}^{\frac{1}{2}} \left(1 + \frac{1}{2} \frac{Re'}{\overline{Re}}\right)$$  \hspace{1cm} (4)

An equation of this form is commonly used for the actual calculations, according to Hinze (7).

Suppose that the velocity fluctuates sinusoidally about some mean value. That is,

$$V = V_o + V_i e^{i\omega t}$$  \hspace{1cm} (5)

Then,

$$Re = Re_o + Re_i e^{i\omega t}$$  \hspace{1cm} (6)
and, from Equation 4

$$Nu = A^* + B \frac{Re}{Re_0}^\frac{1}{2} \left( 1 + \frac{1}{2} \frac{Re}{Re_0} e^{i\omega t} \right).$$

(7)

Equation 7 shows that the use of Equation 4 implies that the heat transfer fluctuation is in phase with the velocity fluctuation. That is, the response of the heat transfer should be instantaneous with no amplitude reduction if Equation 4 is correct.

It would be most desirable to know if the heat transfer response is actually instantaneous. Lighthill (9) has analyzed this problem for the case where the boundary layer assumptions are valid and has shown that a finite thermal lag exists above certain frequencies of fluctuation. Schlichting (12) has shown that

$$\delta \sim \frac{1}{\sqrt{Re}} \ll 1$$

(8)

for the boundary layer assumptions to apply. This restricts Lighthill's analysis to Reynolds numbers greater than 100, say. Unfortunately, for Reynolds numbers of this magnitude boundary layer separation occurs on the back of the cylinder and the assumptions are invalid in this region; and, furthermore, hot-wire anemometers are often operated at lower Reynolds numbers.

The problem as described to this point is quite general since the velocity of the fluid approaching the wire may fluctuate in magnitude and/or in direction, the temperature
of the wire surface may remain constant or the heat flux may be constant, and the Reynolds number and Prandtl number may assume a wide range of values. Unfortunately, it is not possible to solve such a general problem mathematically and these variable factors must be precisely specified. In this paper the case will be considered where the wire surface temperature is constant, the velocity of the oncoming fluid is fluctuating in magnitude only, the Reynolds number is ten, and the Prandtl number is 0.7. For this case the basic equations governing the fluid flow and heat transfer in the vicinity of the wire will be attacked numerically.

This direct approach to the problem is taken because it involves the least number of approximations. Nevertheless, it is not without its own difficulties. The accuracy of the numerical solution is limited by the computer memory size and the computing time for the solution of partial differential equations is quite large with present computers. These problems as well as the results of the calculations will be discussed in detail in the following sections.
II. Theory

It is the purpose of this section to develop the equations needed to simulate the behavior of a hot wire under conditions listed in the introduction. The differential equations which describe the fluid mechanics and heat transfer in the vicinity of the wire will be used to develop a set of finite difference equations amenable to solution with the aid of a digital computer. In so doing it will be necessary to take into account such practicalities as the size of the computer memory and the time required to complete the calculations. Every effort will be made to justify the approximations and assumptions but, because of theoretical difficulties, it will not always be possible. In such cases the only possible justification for the approximations must come from a comparison of the calculations with experimental values.

The flow of a fluid around the outside of a circular cylinder, as shown in Figure 1, may be described by the dimensionless Navier-Stokes equation (2)

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \left( \frac{\mu}{\rho \omega} \right) \nabla^2 \mathbf{u} + \left( \frac{\rho g}{\omega^2} \right) \mathbf{g}
\]  

(9)

and the equation of continuity (2)

\[
\nabla \cdot \mathbf{u} = 0
\]

(10)

if it is assumed that the fluid is (1) Newtonian, (2) laminar, (3) incompressible, and (4) the viscosity is constant. Since
the fluid to be considered is air, the first assumption is valid. In order for the flow to be laminar the Reynolds number, \( \frac{DU_\infty \rho}{\mu} \), must be less than about \( 5 \times 10^5 \) (12). The values of the Reynolds number to be considered in this paper will be on the order of ten. Thus, the laminar flow assumption is also valid.

The assumption that the flow is incompressible is not so obvious because air (indeed, all gases) is usually thought of as being quite compressible. However, if flow conditions are such that \( \frac{\Delta \rho}{\rho} \) is small, one may consider the flow to be incompressible. For \( \frac{\Delta \rho}{\rho} \) to be small Schlichting (12) has shown that

\[
\frac{1}{2} \left( \frac{U_\infty}{C} \right)^2 \ll 1, \tag{11}
\]

or more specifically

\[
\frac{1}{2} \left( \frac{U_\infty}{C} \right)^2 \leq 0.05. \tag{12}
\]

The velocity, \( U_\infty \), can be estimated from the Reynolds number. Let the Reynolds number be ten, say. Hinze (7) states that \( \frac{\mu}{\rho} \) for air is about \( 0.2 \) cm\(^2\)/sec., and that an average wire diameter is \( 5 \times 10^{-4} \) cm. Thus, \( U_\infty \) is about 4000 cm/sec. Since the speed of sound, \( C \), is about 33,500 cm/sec

\[
\frac{1}{2} \left( \frac{U_\infty}{C} \right)^2 = 0.0072 < 0.05. \tag{13}
\]

Therefore, the incompressibility assumption is also valid.
The viscosity may be considered to be a function of the density and the temperature, in general. The above argument has shown that the density may be considered constant. Whether or not the temperature dependence must be considered is determined by the magnitude of the temperature difference between the surface of the cylinder and the oncoming fluid. As will be seen later, it will never be necessary to specify the magnitude of this temperature difference and so it will place no restrictions on the solution to let it be small enough to justify the constant viscosity assumption.

Equations (9,10) adequately describe the fluid flow, but it is more convenient to use a change of variables which will reduce the number of equations. Following Allen and Southwell (1), the vorticity

$$\mathbf{j}^* = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$$

(14)

and the stream-function

$$\frac{\partial \psi^*}{\partial y} = u , \quad - \frac{\partial \psi^*}{\partial x} = v$$

(15)

are introduced into the Navier-Stokes and the continuity equations. The continuity equation is identically satisfied by the stream-function and the Navier-Stokes equation becomes

$$\frac{\partial \mathbf{j}^*}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{j}^* = \frac{1}{Re} \nabla^2 \mathbf{j}^*$$

(16)
Since a rectangular finite-difference grid will not mesh with a cylindrical boundary it is convenient to make the following conformal transformation of the coordinate system:

\[
\alpha = x \left\{ 1 + \frac{1}{4(x^2 + y^2)} \right\} \\
\beta^* = y \left\{ 1 - \frac{1}{4(x^2 + y^2)} \right\}
\]

This transformation was, apparently, first suggested by Thom (13). It has also been employed by Allen and Southwell (1). It is clear that

\[
\frac{\partial \alpha}{\partial x} = \frac{\partial \beta^*}{\partial y}, \quad -\frac{\partial \alpha}{\partial y} = \frac{\partial \beta^*}{\partial x}
\]

and by defining

\[
h^2 = \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 = \left( \frac{\partial \beta^*}{\partial x} \right)^2 + \left( \frac{\partial \beta^*}{\partial y} \right)^2
\]

\[
h^2 = 1 + \frac{1 - 8(x^2 - y^2)}{16(x^2 + y^2)^2}
\]

the following relationships hold:

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = h^2 \left[ \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \beta^*}{\partial x^2} \right] = h^2 \nabla^2_{\alpha \beta^*},
\]

\[
\vec{U} \cdot \nabla = U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} = h^2 \left[ \frac{\partial \psi^*}{\partial \beta^*} \frac{\partial}{\partial \alpha} - \frac{\partial \psi^*}{\partial \alpha} \frac{\partial}{\partial \beta^*} \right]
\]
Thus, Equation 16 becomes
\[
\frac{1}{\hbar^2} \frac{\partial \xi^*}{\partial t} + \frac{\partial \psi^*}{\partial \beta^*} \frac{\partial \xi^*}{\partial \alpha} - \frac{\partial \psi^*}{\partial \alpha} \frac{\partial \xi^*}{\partial \beta} = \frac{1}{Re} \nabla_{\alpha \beta}^2 \xi^*,
\]
(22)

and Equation 14 becomes
\[
- \xi^* = \hbar^2 \nabla_{\alpha \beta}^2 \psi^*.
\]
(23)

Allen and Southwell (1) have suggested still another change of variables which should reduce the magnitude of the functional dependence of the vorticity and stream-function on the Reynolds number. The transformation is
\[
\beta^* = Re^{-\frac{1}{2}} \beta, \quad \psi^* = Re^{-\frac{1}{2}} \psi, \quad \xi^* = Re^{\frac{1}{2}} \xi.
\]
(24)

Thus, Equation 22 transforms to
\[
\frac{1}{\hbar^2} \frac{\partial \xi}{\partial t} + \frac{\partial \psi}{\partial \beta} \frac{\partial \xi}{\partial \alpha} - \frac{\partial \psi}{\partial \alpha} \frac{\partial \xi}{\partial \beta} = \frac{1}{Re} \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \xi}{\partial \beta^2},
\]
(25)

and Equation 23 to
\[
- \frac{\xi}{\hbar^2} = \frac{1}{Re} \frac{\partial^2 \psi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \beta^2}.
\]
(26)

Now, let
\[
L \equiv \frac{1}{Re} \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2},
\]
(27)

and
\[
M \equiv \frac{\partial \psi}{\partial \beta} \frac{\partial}{\partial \alpha} - \frac{\partial \psi}{\partial \alpha} \frac{\partial}{\partial \beta}.
\]
(28)
Then,
\[
\frac{1}{h^2} \frac{\partial \Psi}{\partial z} + M \Psi = \Lambda \Psi,
\] 
(29)

and
\[
-\frac{\Psi}{h^2} = \Lambda \psi.
\] 
(30)

For steady-state flow, the equations become
\[
\bar{M} \bar{\Psi} = \Lambda \bar{\Psi},
\] 
(31)

and
\[
-\frac{\bar{\Psi}}{h^2} = \Lambda \bar{\psi}.
\] 
(32)

In the \( \alpha - \beta \) plane the cylinder corresponds to that portion of the \( \alpha \)-axis from \( \alpha = -1 \) to \( \alpha = +1 \), and on the surface of the cylinder \( u = v = 0 \). Thus
\[
\frac{\partial \psi}{\partial \alpha} = \frac{\partial \psi}{\partial \beta} = 0 \implies \psi = \text{arbitrary constant}.
\]

For convenience, let this arbitrary constant be zero. On the rest of the \( \alpha \)-axis
\[
\nu = \frac{\partial u}{\partial \beta} = 0,
\] 
(33)

by symmetry. It follows that
\[
\psi = \bar{\psi} = 0.
\] 
(34)

Furthermore, at a large distance from the cylinder
\[
\frac{\partial \psi}{\partial \alpha} \to 0, \quad \frac{\partial \psi}{\partial \beta} \to 1, \quad \bar{\psi} \to 0
\]
or
\[ \psi = \beta, \quad \xi = 0. \]  
(35)

To obtain the unsteady-state solution Equations (29, 30) may be solved directly, but such a solution is difficult to obtain numerically. Therefore, they will be linearized. Let the velocity an infinite distance from the cylinder fluctuate sinusoidally. Thus,
\[ U_\infty = 1 + A_0 e^{i\omega t} \]  
(36)
and
\[ R_e = R_e (1 + A_0 e^{i\omega t}). \]  
(37)

Likewise, the stream-function may be expressed as
\[ \psi_\infty = \beta (1 + A_0 e^{i\omega t}), \]  
(38)
and the vorticity at infinity will remain zero. Near the cylinder the stream-function and vorticity will be assumed to be of the form
\[ \psi = \overline{\psi} (\alpha, \beta) + \psi'(\alpha, \beta) e^{i\omega t}, \]  
\[ \xi = \overline{\xi} (\alpha, \beta) + \xi'(\alpha, \beta) e^{i\omega t}. \]  
(39)

Substituting these into Equations 29, 30 gives
\[ \frac{i\omega \xi e^{i\omega t}}{h^2} + (\overline{R} + Me^{i\omega t})(\overline{\xi} + \xi' e^{i\omega t}) = L (\overline{\xi} + \xi' e^{i\omega t}), \]  
(40)
\[ -\left( \frac{\overline{\psi} e^{i\omega t}}{h^2} \right) = L (\overline{\psi} + \psi' e^{i\omega t}). \]  
(41)
where
\[
\tilde{M} = \frac{\partial \varphi}{\partial \beta} \frac{\partial}{\partial \alpha} - \frac{\partial \varphi}{\partial \alpha} \frac{\partial}{\partial \beta},
\]
(42)

and
\[
\tilde{M}' = \frac{\partial \psi'}{\partial \beta} \frac{\partial}{\partial \alpha} - \frac{\partial \psi'}{\partial \alpha} \frac{\partial}{\partial \beta}.
\]
(43)

In order to linearize Equation 40 it is necessary to assume that \( e^{i \omega t} \tilde{M}' \tilde{\varphi} \) is negligible. This assumption requires that the amplitude of the fluctuations be much less than unity. Therefore, the following analysis will be valid only when
\[ A_0 \ll 1 \]

Then, Equations 40, 41 become
\[
\frac{i \omega \tilde{\varphi} e^{i \omega t}}{h^2} + \tilde{M} \tilde{\varphi} + e^{i \omega t} (\tilde{M}' \tilde{\varphi} + \tilde{M}' \tilde{\varphi}) = \Lambda (\tilde{\varphi} + \tilde{\varphi}' e^{i \omega t})
\]
(44)

and
\[
-\frac{(\tilde{\varphi} + \tilde{\varphi}' e^{i \omega t})}{h^2} = \Lambda (\tilde{\varphi} + \tilde{\varphi}' e^{i \omega t}).
\]
(45)

Subtracting Equations 31, 32 from Equations 44, 45, respectively, gives
\[
\frac{i \omega \tilde{\varphi}'}{h^2} + \tilde{M} \tilde{\varphi}' + \tilde{M}' \tilde{\varphi} = \Lambda \tilde{\varphi}'
\]
(46)

and
\[
\frac{-\tilde{\varphi}'}{h^2} = \Lambda \tilde{\varphi}'.
\]
(47)
Since $\xi'$ and $\psi'$ are complex, they may be written as
\[ \xi' = \xi^R + i \xi^I \]
\[ \psi' = \psi^R + i \psi^I \] (48)

Then, Equations 46, 47 become
\[ -\frac{\omega \xi^I}{\hbar^2} + \overline{M} \xi^R + M^R \xi = \Xi \xi^R \] (49)
\[ \frac{\omega \xi^R}{\hbar^2} + \overline{M} \xi^I + M^I \xi = \Xi \xi^I \] (50)
\[ -\frac{\xi^R}{\hbar^2} = \Xi \psi^R \] (51)
\[ -\frac{\xi^I}{\hbar^2} = \Xi \psi^I \] (52)

where
\[ M^R = \frac{\partial \psi^R}{\partial \beta} \frac{\partial}{\partial \alpha} - \frac{\partial \psi^R}{\partial \alpha} \frac{\partial}{\partial \beta} \] (53)
\[ M^I = \frac{\partial \psi^I}{\partial \beta} \frac{\partial}{\partial \alpha} - \frac{\partial \psi^I}{\partial \alpha} \frac{\partial}{\partial \beta} \] (54)

These are the equations, together with Equations 31, 32, which are to be replaced by finite-differences and solved. First, however, the equations describing the temperature distribution in the fluid around the cylinder will be developed.

The temperature distribution in the fluid flowing around the cylinder may be described by the dimensionless
energy equation (2)

\[ \frac{\partial T}{\partial t} + \vec{U} \cdot \nabla T = \frac{1}{Re Pr} \nabla^2 T \]  \hspace{1cm} (55)

if it is assumed that the fluid is (1) Newtonian, (2) laminar, (3) incompressible, (4) the thermal conductivity is constant, and (5) viscous heating is negligible. The first three of these assumptions have been justified in connection with the Navier-Stokes equation. The thermal conductivity is dependent upon the temperature so that if the maximum temperature difference is small the thermal conductivity may be considered constant. As was argued in the case of viscosity, such an assumption is valid.

Viscous heating depends upon the magnitude of the velocity gradients in the system. Bird, Stewart, and Lightfoot (2) state that viscous dissipation is not important except in problems such as rocket re-entry or lubrication between fast-moving parts where the velocity gradients are extremely large. Since such large gradients are not likely in this problem the viscous heating will be neglected.

By replacing the velocity in Equation 55 by the stream-function and introducing the change of variables used on the Navier-Stokes equation, the energy equation becomes

\[ \frac{1}{h^2} \frac{\partial T}{\partial t} + \nabla T = \frac{1}{Pr} \nabla T \]  \hspace{1cm} (56)

and for the steady-state

\[ \bar{\nabla} T = \frac{1}{Pr} \nabla T \]  \hspace{1cm} (57)
On that portion of the $\alpha$-axis corresponding to the cylinder, from $\alpha = -1$ to $\alpha = +1$,
$$T = 1.$$ 
On the rest of the $\alpha$-axis
$$\frac{\partial T}{\partial \beta} = 0$$
by symmetry, and at a large distance from the cylinder
$$T \to 0.$$

The unsteady-state energy equation is linearized by introducing
$$T = T(\alpha, \beta) + T'(\alpha, \beta) e^{i\omega t}$$
(58)

into Equation 56. This gives
$$\frac{i \omega e^{i\omega t}}{h^2} T' + (\bar{M} + M' e^{i\omega t})(\bar{T} + T' e^{i\omega t}) = \frac{1}{Pr} L(T + T' e^{i\omega t})$$
(59)

Neglecting $e^{i\omega t}$ and subtracting Equation 57 from Equation 59 gives
$$\frac{i \omega T'}{h^2} + \bar{M} T' + M' \bar{T} = \frac{1}{Pr} L T'.$$
(60)
Since $T'$ is complex it may be written as
$$T' = T^R + i T^I$$
(61)

Then, Equation 60 becomes
$$-\frac{\omega T^I}{h^2} + \bar{M} T^R + M^R \bar{T} = \frac{1}{Pr} L T^R$$
(62)
$$\frac{\omega T^R}{h^2} + \bar{M} T^I + M^I \bar{T} = \frac{1}{Pr} L T^I$$
(63)
These equations, together with Equation 57, must now be converted to their finite-difference forms. To do this a rectangular grid will be superimposed on a section of the positive $\beta$ half-plane. The values of the functions will then be calculated at the grid points and outside of the grid the flow will be assumed inviscid. The size of this grid had to be determined empirically so that the best representation could be obtained from a given number of grid points. It was decided that

$$0 \leq \beta \leq 7.5,$$
$$-2 \leq \alpha \leq 3.5.$$  

By defining
$$\beta = j \Delta \beta,$$
$$\alpha = i \Delta \alpha - 2$$
and since $\Delta \alpha = \Delta \beta = \frac{1}{4},$
$$\beta = j/4,$$
$$\alpha = i/4 - 2.$$

Therefore,
$$0 \leq j \leq 30,$$
$$0 \leq i \leq 22.$$

This requires 609 internal grid points—about the practical limit for the computing facilities available.

It is now necessary to replace the differential operators, $L$ and $M,$ with finite-difference operators. To do so the following Taylor series expansions are needed.
\[
T(\alpha + \Delta \alpha, \beta) = T(\alpha, \beta) + \Delta \alpha \frac{\partial^2 T(\alpha, \beta)}{\partial \alpha^2} + \frac{(\Delta \alpha)^2}{2} \frac{\partial^2 T(\alpha, \beta)}{\partial \alpha^2} \\
+ \frac{(\Delta \alpha)^3}{6} \frac{\partial^3 T(\alpha, \beta)}{\partial \alpha^3} + \frac{(\Delta \alpha)^4}{24} \frac{\partial^4 T(\alpha, \beta)}{\partial \alpha^4}, \quad \alpha \leq \alpha' \leq \alpha + \Delta \alpha
\]

\[
T(\alpha - \Delta \alpha, \beta) = T(\alpha, \beta) - \Delta \alpha \frac{\partial^2 T(\alpha, \beta)}{\partial \alpha^2} + \frac{(\Delta \alpha)^2}{2} \frac{\partial^2 T(\alpha, \beta)}{\partial \alpha^2} \\
- \frac{(\Delta \alpha)^3}{6} \frac{\partial^3 T(\alpha, \beta)}{\partial \alpha^3} + \frac{(\Delta \alpha)^4}{24} \frac{\partial^4 T(\alpha', \beta)}{\partial \alpha^4}, \quad \alpha \geq \alpha'' \geq \alpha - \Delta \alpha
\]

Subtraction of (65) from (64) gives

\[
\frac{\partial T(\alpha, \beta)}{\partial \alpha} = \frac{T(\alpha + \Delta \alpha, \beta) - T(\alpha - \Delta \alpha, \beta)}{2 \Delta \alpha}
\]

\[
- \frac{(\Delta \alpha)^2}{12} \left[ \frac{\partial^3 T(\alpha, \beta)}{\partial \alpha^3} + \frac{\partial^3 T(\alpha', \beta)}{\partial \alpha^3} \right]
\]

Addition of (64) and (65) gives

\[
\frac{\partial^2 T(\alpha, \beta)}{\partial \alpha^2} = \frac{T(\alpha + \Delta \alpha, \beta) + T(\alpha - \Delta \alpha, \beta) - 2 T(\alpha, \beta)}{(\Delta \alpha)^2}
\]

\[
- \frac{(\Delta \alpha)^2}{24} \left[ \frac{\partial^4 T(\alpha, \beta)}{\partial \alpha^4} + \frac{\partial^4 T(\alpha', \beta)}{\partial \alpha^4} \right]
\]

and similarly for the \( \beta \) -derivatives. Using the order-of-magnitude notation these derivatives may be written as

\[
\frac{\partial T}{\partial \alpha} = \frac{T(\alpha + \Delta \alpha, \beta) - T(\alpha - \Delta \alpha, \beta)}{2 \Delta \alpha} + O(\Delta \alpha^2)
\]

\[
\frac{\partial^2 T}{\partial \alpha^2} = \frac{T(\alpha + \Delta \alpha, \beta) + T(\alpha - \Delta \alpha, \beta) - 2 T(\alpha, \beta)}{(\Delta \alpha)^2} + O(\Delta \alpha^2)
\]

By replacing the functional notation with subscript notation, these become
\[
\frac{\partial T}{\partial \alpha} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta \alpha} + O(\Delta \alpha^2) \quad (70)
\]

\[
\frac{\partial^2 T}{\partial \alpha^2} = \frac{T_{i+1,j} + T_{i-1,j} - 2T_{i,j}}{(\Delta \alpha)^2} + O(\Delta \alpha^2) \quad (71)
\]

Thus,

\[
LT = 16 \left[ \frac{T_{i+1,j} + T_{i-1,j} - 2T_{i,j}}{Re} + T_{i,j+1} + T_{i,j-1} - 2T_{i,j} \right] + O(\Delta \alpha^2 + \Delta \delta^2) \quad (72)
\]

and

\[
MT = 4 \left[ (\psi_{i+1,j} - \psi_{i-1,j})(T_{i,j+1} - T_{i,j-1}) \right. \\
\left. - (\psi_{i+1,j} - \psi_{i-1,j})(T_{i,j+1} - T_{i,j-1}) \right] + O(\Delta \alpha^2 + \Delta \delta^2) \quad (73)
\]

By neglecting the terms of second order it may be said that

\[
LT \approx L_4 T = 16 \left[ \frac{T_{i+1,j} + T_{i-1,j} - 2T_{i,j}}{Re} + T_{i,j+1} + T_{i,j-1} - 2T_{i,j} \right] \quad (74)
\]

and

\[
MT \approx M_4 T = 4 \left[ (\psi_{i+1,j} - \psi_{i-1,j})(T_{i,j+1} - T_{i,j-1}) - (\psi_{i+1,j} - \psi_{i-1,j})(T_{i,j+1} - T_{i,j-1}) \right] \quad (75)
\]

The difference equations to be solved are

\[
R_1 \equiv \frac{1}{16} \left( L_4 \overline{\psi} + \frac{\overline{\delta}}{\kappa^2} \right) = 0 \quad (76)
\]

\[
R_2 \equiv \frac{1}{16} \left( L_4 \overline{\delta} - M_4 \overline{\delta} \right) = 0 \quad (77)
\]
\[ R_3 = \frac{1}{16} \left( \frac{1}{\hat{p}_R} L_d \bar{T} - \bar{M}_d \bar{T} \right) = 0 \quad (78) \]

for the steady-state, and

\[ R_4 = \frac{1}{16} \left( L_d \psi^R + \frac{5}{h^2} \right) = 0 \quad (79) \]
\[ R_5 = \frac{1}{16} \left( L_d \psi^I + \frac{5}{h^2} \right) = 0 \quad (80) \]
\[ R_6 = \frac{1}{16} \left( L_d \bar{R}^R + \frac{\omega \bar{R}^I}{h^2} - \bar{M}_d \bar{R}^R - \bar{M}_d \bar{R}^I \right) = 0 \quad (81) \]
\[ R_7 = \frac{1}{16} \left( L_d \bar{T}^I - \frac{\omega \bar{T}^R}{h^2} - \bar{M}_d \bar{T}^R - \bar{M}_d \bar{T}^I \right) = 0 \quad (82) \]
\[ R_8 = \frac{1}{16} \left( \frac{1}{\hat{p}_R} L_d T^R + \frac{\omega T^I}{h^2} - \bar{M}_d T^R - \bar{M}_d T^I \right) = 0 \quad (83) \]
\[ R_9 = \frac{1}{16} \left( \frac{1}{\hat{p}_R} L_d T^I - \frac{\omega T^R}{h^2} - \bar{M}_d T^I - \bar{M}_d T^R \right) = 0 \quad (84) \]

for the unsteady-state, where \( R_k \) is termed the residue.

For convenience let \( F_k \) represent the principal dependent variable of the \( k \) th residue; i.e., \( F_1 = \bar{\psi} \), \( F_2 = \bar{R} \), \( F_3 = \bar{T} \), etc. Then, an initial approximation, \( F_k^{(0)} \), to \( F_k \) will be assumed and a better approximation may be calculated by

\[ F_k^{(n+1)} = F_k^{(n)} + \gamma R_k^{(p)} \quad (85) \]

where \( \gamma \) is, at present, an arbitrary constant. If \( p = n \), then the equation is explicit. Although this is the simplest
form of the equation, experience showed that it converged much too slowly to be of practical value. If \( p = n + 1 \), the equation is implicit and it is necessary to invert a 609 x 609 matrix—a job much too difficult for the computing facilities available. To overcome these difficulties the alternating-direction method, developed by Douglas (5) and Peaceman and Rachford (11), will be employed. This technique divides the iteration procedure into two parts—one where the derivatives (actually, their finite-difference approximations) in the \( \alpha \)-direction are implicit and the other where the \( \beta \)-derivatives are implicit. Thus,

\[
F_k^{(n+\frac{1}{2})} = F_k^{(n)} + \gamma R_k \left( \frac{\partial F_k^{(n+\frac{1}{2})}}{\partial \alpha}, \frac{\partial^2 F_k^{(n+\frac{1}{2})}}{\partial \alpha^2}, \frac{\partial F_k^{(n)}}{\partial \beta}, \frac{\partial^2 F_k^{(n)}}{\partial \beta^2} \right) \tag{86}
\]

\[
F_k^{(n+\frac{1}{2})} = F_k^{(n+\frac{1}{2})} + \gamma R_k \left( \frac{\partial F_k^{(n+\frac{1}{2})}}{\partial \alpha}, \frac{\partial^2 F_k^{(n+\frac{1}{2})}}{\partial \alpha^2}, \frac{\partial F_k^{(n+\frac{1}{2})}}{\partial \beta}, \frac{\partial^2 F_k^{(n+\frac{1}{2})}}{\partial \beta^2} \right) \tag{87}
\]

This method has the advantage of being faster converging than the explicit method and requires only that a tri-diagonal 29 x 29 matrix be inverted.

For \( k = 1 \) and 2, the two equations must be solved simultaneously. Solving for \( \overline{\psi} \) by rows gives

\[
\frac{-\gamma}{Re} \overline{\psi}_{i,j}^{(n+\frac{1}{2})} + \left( 1 + \frac{2\gamma}{Re} \right) \overline{\psi}_{i,j}^{(n+\frac{1}{2})} - \frac{\gamma}{Re} \overline{\psi}_{i+1,j}^{(n+\frac{1}{2})} = \overline{\psi}_{i,j}^{(n)} + \frac{\delta^{(n)}}{16 h^2} \tag{88}
\]

\[
\overline{\psi}_{i,j}^{(n)} + \gamma \left[ \overline{\psi}_{i+1,j}^{(n)}, + \overline{\psi}_{i-1,j}^{(n)}, - 2 \overline{\psi}_{i,j}^{(n)} \right]
\]

\[
\overline{\psi}_{i,j}^{(n)} + \gamma \left[ \overline{\psi}_{i+1,j}^{(n)}, + \overline{\psi}_{i-1,j}^{(n)}, - 2 \overline{\psi}_{i,j}^{(n)} \right]
\]
This is a tri-diagonal $i \times 1$ (21 x 21) matrix which must be inverted $j(29)$ times. Then, to solve for $\overline{\Psi}$ by rows

$$-\gamma \overline{\Psi}_{i,j-1}^{(n+1)} + (1 + 2\gamma) \overline{\Psi}_{i,j}^{(n+1)} - \gamma \overline{\Psi}_{i,j+1}^{(n+1)} =$$

$$\overline{\Psi}_{i,j}^{(n+1/2)} + \gamma \left[ \frac{\overline{\Psi}_{i-1,j}^{(n+1/2)} + \overline{\Psi}_{i+1,j}^{(n+1/2)} - 2 \overline{\Psi}_{i,j}^{(n+1/2)}}{2\epsilon} + \frac{\overline{\Psi}_{i,j}^{(n)}}{2\epsilon} \right]$$

must be solved. This is a tri-diagonal $j \times j$ (29 x 29) matrix which must be solved $i(21)$ times. Now it is necessary to solve for $\overline{\Psi}$ first by rows and then by columns. Then the whole process is repeated until

$$\sum \left| R_{kj} \right| \leq \epsilon$$

where $\epsilon$ is the convergence criterion.

Once the solution to $k = 1$ and 2 is obtained, the equations for $k = 3$ or $k = 4, 5, 6, 7$ may be solved in the same manner. Finally, these can then be used to solve $k = 8, 9$. For the sake of brevity these equations will not be presented as they are the same in form as Equations 88, 89.

On the cylinder, $4 \leq i \leq 12$, the boundary conditions are:

$$\overline{T} = 1, \quad \overline{\Psi} = \psi^R = \psi^I = T^R = T^I = 0$$

$$\overline{\Psi} = -h^2 \left[ \frac{\partial^2 \overline{\Psi}}{\partial \beta^2} \right] = -\frac{h^2}{(\Delta \beta)^2} \left[ \overline{\Psi}_{i+1} + \overline{\Psi}_{i-1} - 2 \overline{\Psi}_{i} \right]$$

but, since

$$u = \frac{\partial \overline{\Psi}}{\partial \beta} = 0$$

$$\overline{\Psi}_{i,1} = \overline{\Psi}_{i,-1} \quad \text{and} \quad \overline{\Psi}_{i,0} = 0.$$
Therefore,
\[
\vec{S}_i = -2 \ h^2 \ \vec{\psi}_{i,1}, \\
\vec{S}_R = -2 \ h^2 \ \psi_{i,1}^R, \\
\vec{S}_I = -2 \ h^2 \ \psi_{i,1}^I,
\]
On the rest of the \( \alpha \)-axis, \( i \leq 3 \) or \( i \geq 13 \),
\[
\vec{T}_{i,1} = \vec{T}_{i-1}, \quad \vec{T}_{i}^R = \vec{T}_{i-1}^R, \quad \vec{T}_{i}^I = \vec{T}_{i-1}^I,
\]
\[
\vec{\psi}^I = \vec{\psi}^R = \vec{\psi}_i = \vec{S}^I = \vec{S}^R = \vec{S}^I = 0
\]
For \( i = 0 \) and for \( j = 30 \)
\[
\vec{\psi} = \beta, \quad \vec{\psi}^R = A_0 \beta,
\]
\[
\vec{\psi}^I = \vec{T} = \vec{T}^R = \vec{T}^I = \vec{S} = \vec{S}^R = \vec{S}^I = 0
\]
For \( i = 22 \)
\[
\vec{\psi} = \beta, \quad \vec{\psi}^R = A_0 \beta,
\]
\[
\vec{\psi}^I = \vec{S} = \vec{S}^R = \vec{S}^I = 0.
\]
The boundary condition on temperature is not apparent. In order to obtain a condition it was assumed that the conduction term \( \frac{1}{\text{Re}} \ \frac{\partial^2 \overline{T}}{\partial \alpha^2} \) in Equation 56 could be neglected relative to the convection in the inviscid flow region downstream from the cylinder. Then,
\[
\overline{T}_{i+1,j} + \overline{T}_{i-1,j} - 2 \overline{T}_{i,j} = 0
\]
or
\[
\overline{T}_{i+1,j} = 2 \overline{T}_{i,j} - \overline{T}_{i-1,j}
\]
Applying this at \( i = 29 \) gives

\[
\begin{align*}
\overline{T}_{30, j} &= 2 \overline{T}_{29, j} - \overline{T}_{28, j} \\
\overline{T}_{30}^R &= 2 \overline{T}_{29}^R - \overline{T}_{28}^R \\
\overline{T}_{30}^I &= 2 \overline{T}_{29}^I - \overline{T}_{28}^I
\end{align*}
\]

With these boundary conditions the difference equations can be solved for the functions \( F_k \).

This numerical treatment has involved several approximations such as (1) neglecting the truncation error which is \( O(\Delta \alpha^2 + \Delta \beta^2) \), (2) introducing finite boundaries where the fluid is assumed to become inviscid, and (3) assuming that conduction may be neglected in the \( \alpha \)-direction downstream from the cylinder. Since \( \Delta \alpha \) and \( \Delta \beta \) are not particularly small quantities it would be particularly desirable to estimate the error due to this truncation. However, it is a rather difficult job to estimate the truncation errors of the difference equations and almost impossible to determine their effect on the functions, judging from the experiences of Forsythe and Wasow (6). Even if it were known that large errors were caused by such truncations, very little could be done to improve the accuracy. A finer grid could not be used because of the limited memory size of the computer, and more accurate difference formulas would destroy the tri-diagonal nature of the matrices being inverted and thus increase computation time considerably.
As for the errors caused by the finite boundaries and by neglecting conduction, there seems to be no a priori method for their estimation. About the only way to determine the accuracy of the results is to compare them with experimental values in cases where the latter are available.
II. Calculations

The numerical calculations for this problem were performed on the Rice University Computer. This computer was built and is maintained by the staff of the Rice University Computer Project. It is a binary machine with random access, electrostatic storage. At present the memory is capable of storing 8192 words with 54 bits. Memory access time is about 10 microseconds, addition time is approximately 4 microseconds, and multiplication time is 120 microseconds. Input to the machine may be from paper or magnetic tape, and output may be on paper or magnetic tape, or on the line printer which is capable of printing a maximum of 600 lines per minute.

In addition to the electrostatic storage the machine has seven full length fast access registers, and seven short registers 15 bits long that are particularly useful as counters. Special provision was made for these registers in the instruction format and as a result extremely efficient coding is possible.

There are two basic methods of programming for this machine. The first is the symbolic machine language or assembly language method in which each instruction is represented by an alphabetic or alphanumeric symbol. For example, ADD represents the fixed point addition instruction and FAD represents floating point addition. Once a program composed of such symbols has been written it may then be "assembled" or converted into numeric instructions recognizable by the computer, and there is a one-to-one
correspondence between the symbolic and numeric instructions.

Programs may also be written in a compiler language which uses a pseudoalgebraic format. For example, one may write

\[ y = 3.0 z^2 - r_{ij} \]  \hspace{1cm} (91)

directly, provided certain basic rules are followed. Then, a program of such statements may be "compiled" or converted into numeric instructions. Of course, there is no one-to-one correspondence between compiler statements and numeric instructions.

The advantages of the compiler language are obvious. Since it is not necessary to break down the problem into a series of simple arithmetic and logical steps, a great savings can be made in programming time. Unfortunately, it has two major disadvantages: (1) it doesn't use the fast registers efficiently, and (2) the number of numeric instructions produced is more than necessary. The inefficient use of fast registers increases calculation time and the inefficient use of instructions causes storage problems. Some of the programs for this problem were written in both languages, and the compiled programs ran about four times longer and used two to three times as much memory space as the assembly language programs. Therefore, the major portion of this problem was programmed in assembly language.

Upon examination of Equations 29, 30, it is tempting to insert Equation 30 into Equation 29 and eliminate the
need to calculate vorticity. This was attempted, but it proved unsuccessful. The result of such a substitution is a fourth-order equation and its finite-difference representation is extremely complex. Furthermore, two boundary conditions on $\psi$ must be specified on each boundary, and many of them involve derivatives. Finite-difference equations with several derivative boundary conditions are notoriously slow converging; and, indeed, this case was no exception. For the method chosen to solve this equation it was estimated that it would require several million years to converge.

A program was also written to solve Equations 31, 32 explicitly. It also converged too slowly; but, in all fairness, it must be noted that the influence of the convergence parameter was not as thoroughly examined as it was for the alternating-direction method.

As has been noted the alternating-direction method was finally chosen for the calculations. These programs were divided into two parts; the first performed the iterative calculations for Equations 29, 30, 56 and the second examined the calculations to test for convergence. Convergence tests may be made by calculating

$$\sum_{i,j} \frac{|F_k^{(n+1)} - F_k^{(n)}|}{F_k^{(n+1)}}$$

or

$$\sum_{i,j} \frac{|R_k^{(n+1)}|}{R_k^{(n+1)}}$$

(92)

(93)
The first of these is obviously the easier to calculate but the second was chosen because it is a more exacting test. If there is an error in the program the numbers calculated may still converge to some value and Equation 92 would indicate convergence. Equation 93 will indicate convergence only if the numbers calculated satisfy the residue equations. Since the residue is never explicitly calculated in the iterative portion of the program there is little danger of making the same mistake in both parts of the program. This direct calculation of the residues proved to be quite helpful in finding such errors.

Equation 90 defined the convergence criterion. Obviously, the smaller the value of $\varepsilon$ the more accurate will be the results and the longer it will take to calculate them. At first a value of $10^{-8}$ was chosen but later it was found that $10^{-4}$ could be used without noticeably changing the results. For $\varepsilon = 10^{-4}$ the largest possible error would be $10^{-4}$, and if it is uniformly distributed the error at each point would be $1.6 \times 10^{-7}$.

The convergence parameter was introduced in Equation 85 and no restrictions were placed upon it. Peaceman and Rachford (11) proved for a problem they were considering that there was no restriction upon its value when the alternating-direction method was used. They were also able to calculate the set of values for $\gamma$ which gave the most rapid rate of convergence. Unfortunately, the method they used to prove convergence and calculate the optimum values of $\gamma$ is
inapplicable for the difference equations of this paper because of non-linearities and discontinuous boundary conditions. Therefore, it was necessary to choose $\gamma$ empirically. It was found that there was always an upper limit beyond which the iterative procedure diverged. This value was usually one, but some cases were found where it was as high as four and as low as 0.1. The sequence of values used was also found to be important. In some cases the solution would converge for $\gamma^{(n)} = 0.1, 0.2, \ldots, 1.0$, and would diverge for $\gamma^{(n)} = 0.6, 0.7, \ldots, 1.0$. In other cases, the second set would cause the solution to converge more rapidly. No obvious explanation could be found for such phenomena. No lower limit was found for $\gamma$ and the rate of convergence increased as $\gamma$ increased until divergence occurred. The rate of convergence could be changed considerably by changing $\gamma$ so considerable effort was expended in choosing it.

The time required to solve the various equations varied with Re, $\gamma$, and $\omega$. The steady-state equations were solved at various Reynolds numbers and it was found that the flow equations, Equations 31, 32, required about six hours and the energy equation, Equation 57, required about three hours. It usually took longer to solve them at low Reynolds numbers (0.1, say) and they tended to diverge at lower values of $\gamma$.

The equations for the fluctuating components were solved only for Re = 10. The value of $\gamma$ was not as
critical with these as with the steady-state equations. Usually, the set \( \gamma^{(n)} = 0.1, 0.2, \ldots, 1.0 \), was used. The flow equations required about eight hours and the energy equations required about four hours to converge. They converged more slowly at higher \( \omega \) because of the greater coupling between the equations.

During the calculations of \( \bar{f} \) an anomalous behavior was observed. Near the boundary \( i = 0 \) where \( \bar{f} \) tends to zero the value of \( \bar{f} \) fluctuated in sign anomalously. Checking these values showed that they all satisfied the difference equation. It was noted that the magnitude of these numbers was several orders less than the values a short distance downstream. Since \( \bar{f} \) would be expected to be quite small in this region, it was theorized that the anomalous behavior was due to the truncation error. If the solution of the differential equation increases rapidly from zero to a large value a polynomial approximation such as Equations 68, 69 could be expected to be in error.

To study this behavior the boundary \( i = 0 \) was moved from \( \alpha = -2 \) to \( \alpha = -2.5 \). The same anomalous behavior occurred in the region where \( \bar{f} \) was small, but downstream the values were not changed. It was concluded that this behavior did not affect the values near the cylinder and no further effort was made to eliminate or reduce it.

One improvement in the calculation could have been made by changing the dependent variables so that their values were constant along the boundaries. For example, replacing \( \bar{f} \)
with $\varphi - \beta$ in the differential equation would eliminate the need to determine $\beta$ on the boundary. This type of substitution would simplify the calculations and reduce programming requirements. If a transformation for $T$ could be found that would eliminate the derivative boundary conditions it would simplify programming and probably increase the rate of convergence.

A further improvement could be made by expanding $\xi'$, $\psi'$, and $T'$ in a power series in $1/\omega$. Thus,

$$
\begin{align*}
\xi' &= \xi^{(0)} + \frac{k}{\omega} \xi^{(1)} + \frac{k^2}{\omega^2} \xi^{(2)} + \cdots + \frac{k^m}{\omega^m} \xi^{(m)} \\
\psi' &= \psi^{(0)} + \frac{k}{\omega} \psi^{(1)} + \frac{k^2}{\omega^2} \psi^{(2)} + \cdots + \frac{k^m}{\omega^m} \psi^{(m)} \\
T' &= T^{(0)} + \frac{k}{\omega} T^{(1)} + \frac{k^2}{\omega^2} T^{(2)} + \cdots + \frac{k^m}{\omega^m} T^{(m)} 
\end{align*}
$$

(94)

Substituting these into Equations 46, 47, 60, gives

$$
\begin{align*}
\xi^{(0)} &= 0 \\
L \psi^{(0)} &= 0 \\
T^{(0)} &= 0 \\
\end{align*}
$$

(95)

for $m = 0$,

$$
\begin{align*}
\frac{i \xi^{(1)}}{h^2} &= -M^{(0)} \xi \\
L \psi^{(1)} &= -\xi^{(1)} \\
\frac{i \, T^{(1)}}{h^2} &= -M^{(0)} T 
\end{align*}
$$

(96)
for $m = 1$,

$$
\frac{i \xi^{(2)}}{\hbar^2} = (\mathcal{L} - \mathcal{M}) \xi^{(1)} - \mathcal{M}^{(1)} \mathcal{F}
$$

$$
\mathcal{L} \psi^{(2)} = -\frac{\xi^{(2)}}{\hbar^2}
$$

$$
\frac{i \mathcal{T}^{(2)}}{\hbar^2} = \left( \frac{1}{P_r} \mathcal{L} - \mathcal{M} \right) \mathcal{T}^{(1)} - \mathcal{M}^{(1)} \mathcal{T}
$$

for $m = 2$, etc. These equations are easy to solve since only the one involving $\psi^{(m)}$ requires an iterative solution. Once these solutions have been obtained the temperature distribution at any frequency greater than one can be computed. For temperatures less than one, an expansion of $\xi'$, $\psi'$, and $\mathcal{T}'$ in $\omega$ can be made, but the solution is not as easy because all of the equations require an iterative solution. Nevertheless, once the solution is obtained all the temperature distributions for frequencies less than one can be constructed.

The difficulty with this series expansion method is that neither series converges for $\omega = 1$. In this problem this was the most important region. For other problems, or if one is interested in a higher range of frequencies, this technique would appear to be the most desirable since the method used in this paper becomes more difficult as the frequency increases while the technique described here requires less effort at higher frequencies.
III. Results

The steady-state flow equations, Equations 31, 32, and the steady-state energy equation, Equation 57, were solved for Reynolds numbers ranging from 0.1 to 40 for a Prandtl number of 0.7. The stream-function is plotted in Figure 2, Figure 3 shows the vorticity, and Figure 4 shows the temperature distribution around the cylinder for a Reynolds number of ten. The Nusselt number has been calculated from the temperature distribution and is plotted versus the Reynolds number in Figure 5. It is also recorded in Table 1.

<table>
<thead>
<tr>
<th>Reynolds Number</th>
<th>Nusselt Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.862</td>
</tr>
<tr>
<td>0.9</td>
<td>1.07</td>
</tr>
<tr>
<td>1.0</td>
<td>1.08</td>
</tr>
<tr>
<td>3.0</td>
<td>1.35</td>
</tr>
<tr>
<td>10.0</td>
<td>2.09</td>
</tr>
<tr>
<td>20.0</td>
<td>2.77</td>
</tr>
<tr>
<td>40.0</td>
<td>3.64</td>
</tr>
</tbody>
</table>
McAdams (10) has plotted the available experimental data for air flowing normal to a cylinder. These data were corrected for radiation to the surroundings. A portion of his recommended curve is shown in Figure 5 for the range of Reynolds numbers being considered in this paper. In addition, he gives several empirical formulae that fit certain portions of his recommended curve. He gives

\[ Nu = 0.891 \, Re^{0.330} \]  \hspace{1cm} (98)

for \( 1 \leq Re \leq 4 \), and

\[ Nu = 0.821 \, Re^{0.385} \]  \hspace{1cm} (99)

for \( 4 \leq Re \leq 40 \), or

\[ Nu = 0.32 + 0.43 \, Re^{0.52} \]  \hspace{1cm} (100)

for \( 0.1 \leq Re \leq 1000 \). The equation developed by Kramers (8) is

\[ Nu = 0.42 \, Pr^{0.2} + 0.57 \, Pr^{0.33} \, Re^{0.50} \]  \hspace{1cm} (101)

This reduces to

\[ Nu = 0.39 + 0.506 \, Re^{0.52} \]  \hspace{1cm} (102)

for \( Pr = 0.7 \). Collis and Williams (3) have developed an equation which reduces to

\[ Nu = 0.24 + 0.56 \, Re^{0.45} \]  \hspace{1cm} (103)
for small temperature differences between the surface of the cylinder and the oncoming air. This equation applies for $0.02 \leq \text{Re} \leq 44$. Equations 102, 103 are also shown in Figure 5.

McAdams (10) claims his recommended curve correlated the data with an average deviation of $\pm 20$ per cent. The data of Collis and Williams (3) seem to correlate much better and deviate by perhaps two per cent from Equation (103). Both of these equations are empirical fits of the data. On the other hand, Equation 101 might be considered semi-empirical since the exponent of the Reynolds number seems to have been obtained from theoretical considerations while the Prandtl number dependence was determined empirically. At least Kramer (8) does give some theoretical justification for his choice of the exponent and he claims that his equation correlates the data "well".

Figure 5 shows that the values calculated in this paper agree within the range of accuracy given my McAdams (10) except at the lowest Reynolds number considered (0.1). An examination of the calculated data indicated that the deviation was due to the choice of finite boundaries. At this low Reynolds number a significant portion of the heat transfer is by conduction and since the boundary at $i = 0$ is so close to the leading stagnation point of the cylinder, an unrealistic amount of heat flux is calculated to flow upstream. Thus, at such a low Reynolds number the assumption that the dimensionless temperature at $i = 0$ is zero breaks down. At
higher Reynolds numbers, the convective forces increase and force the heat flux downstream, and the boundary condition proves valid. Figure 6 shows the percentage of the total heat transfer which crosses the upstream boundary.

Since the data agree with the experimental except at this lowest Reynolds number the assumptions made in the theoretical development of the problem seem to be justified. Further arguments on this point will, of course, be made.

For the unsteady-state the Reynolds number was held constant at ten and the Prandtl number considered was 0.7. It proved to be impractical to consider other values of these parameters because the time required for such calculations would be too great. The only parameter varied in these calculations was the frequency of fluctuation.

The results of these calculations are perhaps best presented in the form of the fluctuating Nusselt number. Since the velocity was considered to vary as

$$V = \nabla + V' e^{i\omega t}$$

(104)

the Nusselt number will be

$$Nu = \overline{Nu} + Nu' e^{i\omega t}$$

(105)

where $Nu'$ is complex. That is,

$$Nu' = Nu^R + iNu^I$$

(106)

In polar form this becomes

$$Nu' = A e^{i\phi}$$

(107)
where
\[ A = \sqrt{(N_u^R)^2 + (N_u^I)^2} \]  
and
\[ \phi = \arctan\left(\frac{N_u^I}{N_u^R}\right) \]

Now, Equation 105 becomes
\[ N_u = \overline{N_u} + A e^{i(\omega t + \phi)} \]

It is a simple matter to calculate \( N_u^R \) and \( N_u^I \) from the fluctuating temperature distribution but their polar representation gives greater insight into their physical significance. Both forms are presented in Table 2. Both \( N_u^R \) and \( N_u^I \) are plotted versus frequency in Figure 7, \( A(\omega)/A(0) \) is plotted in Figure 8, and \( \phi \) versus \( \omega \) is shown in Figure 9.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( A )</th>
<th>( \phi ) (degrees)</th>
<th>( N_u^R )</th>
<th>( N_u^I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0756</td>
<td>0</td>
<td>0.0756</td>
<td>0</td>
</tr>
<tr>
<td>10(^{-5})</td>
<td>0.0756</td>
<td>-2.31 \times 10^{-4}</td>
<td>0.0756</td>
<td>-3.05 \times 10^{-7}</td>
</tr>
<tr>
<td>10(^{-4})</td>
<td>0.0756</td>
<td>-1.78 \times 10^{-3}</td>
<td>0.0756</td>
<td>-2.36 \times 10^{-6}</td>
</tr>
<tr>
<td>10(^{-3})</td>
<td>0.0756</td>
<td>-2.76 \times 10^{-2}</td>
<td>0.0756</td>
<td>-3.65 \times 10^{-5}</td>
</tr>
<tr>
<td>10(^{-2})</td>
<td>0.0756</td>
<td>-3.05 \times 10^{-1}</td>
<td>0.0756</td>
<td>-4.02 \times 10^{-4}</td>
</tr>
<tr>
<td>10(^{-1})</td>
<td>0.0755</td>
<td>-3.04</td>
<td>0.0754</td>
<td>-4.00 \times 10^{-3}</td>
</tr>
<tr>
<td>1</td>
<td>0.0660</td>
<td>-26.8</td>
<td>0.0589</td>
<td>-2.98 \times 10^{-2}</td>
</tr>
</tbody>
</table>
A negative value of $\phi$ indicates that the heat transfer fluctuation lags behind the velocity fluctuation. Since all values of $\phi$ in this paper are negative, $\phi$ shall be referred to as the phase lag.

As has been noted earlier Equation 4 predicts no phase lag. Therefore, it should correspond with the numerical solution when the frequency is zero. From Equation 7

$$\overline{Nu} + Nu' = A^* + B \overline{Re}^{\frac{1}{2}} (1 + \frac{1}{2} \frac{Re'}{Re} e^{i\omega t})$$  \hspace{1cm} (111)$$

and Equation 1

$$\overline{Nu} = A^* + B \overline{Re}^{\frac{1}{2}}$$  \hspace{1cm} (112)$$

Subtracting (112) from (111) gives

$$Nu' = \frac{B}{2} \overline{Re}^{\frac{1}{2}} \left( \frac{Re'}{Re} \right) e^{i\omega t}$$  \hspace{1cm} (113)$$

or

$$A = \frac{B}{2} \overline{Re}^{\frac{1}{2}} \left( \frac{Re'}{Re} \right)$$  \hspace{1cm} (114)$$

From Equation 102, $B = 0.506$, so for

$$Re = 10, \quad Re'/Re = 0.1$$

$$A = 0.0799$$

This value agrees quite well with the value of $A (\omega = 0)$ in Table 2 which again tends to confirm the assumptions made in the theoretical development of the problem.

Figures 8, 9, show that there exists a critical
frequency, $\omega_{cr}$, beyond which the phase lag and amplitude reduction may not be neglected. The value of $\omega_{cr}$ lies between 0.1 and 1.0. Although this value is for $Re = 10$ and $Pr = 0.7$, it would certainly be desirable to determine $\omega_{cr}$ for other values of these parameters as well. It was not practical to make these calculations, as has been noted previously, because of the time requirements.

A rough check on the influence of $Pr$ on $\omega_{cr}$ was found by Davis (4) using a boundary layer approximation with steady state conditions for a flat plate and no pressure variation outside the boundary layer. Two factors prevent the use of these boundary layer equations from giving quantitative results for a periodic disturbance: (a) one term must be neglected in the equations restricting validity to large distances from the leading edge, and (b) the periodic functions grow exponentially for large distances from the solid-fluid interface. In spite of these factors and the neglect of pressure variations in the direction of flow, good agreement was found between these restrictively approximate solutions and the solutions reported in this thesis.

Davis' solution for $Pr = 0.7$ exhibited a curvature similar to Figures 8 and 9, but the curve was shifted to a lower frequency. His critical frequency was about 1.6 times lower than the one for this paper. Additional calculations by Davis (4) showed that the critical frequency decreased by a factor of about two when the Prandtl number was increased.
from 0.00391 to 128. These results clearly indicate that the Prandtl number does not strongly influence the response of the heat transfer process.

The influence of Reynolds number on the critical frequency is not known. However, the range of Reynolds numbers over which the hot-wire anemometer is operated is not extremely large. For Reynolds numbers above about 40, vortex separation occurs which disrupts the normal flow pattern, and at Reynolds numbers much below 0.1, conduction and natural convection become serious problems. Within this range the critical frequency is not believed to vary too greatly.

As has been noted the critical frequency is on the order of one. That is,

\[
\omega_{cr} = \frac{\omega'' D}{U_\infty} = O(1) \tag{115}
\]

or

\[
\omega'' \approx \frac{U_\infty}{D} \tag{116}
\]

Such a result seems reasonable since \(1/\omega''_{cr}\) is a measure of the time required for velocity fluctuation to complete one cycle and \(D/U_\infty\) is a measure of the time required for the fluid to flow past the cylinder. When these two quantities are of the same magnitude the fluid on the downstream side of the cylinder will have had a quite different history from that which the fluid on the upstream side will have. This
is because the velocity, \( U_\infty + u'_\infty \), has changed considerably during the time required for the fluid to flow over the wire.

On the other hand, if

\[ \omega'' \ll \omega''_c \]

so that

\[ \frac{1}{\omega''_c} \ll \frac{1}{\omega''} \]

then the velocity will not change appreciably during the time required for the fluid to pass the cylinder. In this case all of the fluid surrounding the wire will have nearly the same history. Therefore, it is permissible to assume that the instantaneous heat transfer coefficient is related to the instantaneous velocity by the same equation that relates their steady-state values. That is,

\[ \overline{Nu} + Nu' = A + B(Re + Re')^{1/2} \]  \hspace{1cm} (117)

Hinze (7) states that the maximum frequency that may be encountered in a real fluid is about \( 10^4 \) cycles per second. For the case considered on page 7

\[ \omega''_c \approx \frac{U_\infty}{D} = \frac{4000 \text{ cm/sec}}{5 \times 10^{-4} \text{ cm}} = 8 \times 10^6 \text{ sec}^{-1} \]

which is much greater than the frequencies which will be encountered in practice. The minimum velocity for this size wire would be

\[ U_\infty \approx \omega''_c D = \left(10^4 \text{ sec}^{-1}\right)\left(5 \times 10^{-4} \text{ cm}\right) = 5 \text{ cm/sec}. \]
It would be desirable to compare the thermal lag in the fluid with that in the wire, but such a comparison is difficult because the functional dependence is not the same. Hinze (7) has developed equations which show that the critical frequency is a function of the physical properties of the wire, the temperature difference, the diameter of the wire, the characteristics of the associated electronic measuring devices, etc. Fortunately, the smaller the wire diameter the greater the critical frequency so that the thermal lag of the wire and the fluid is decreased by decreasing the wire diameter. From the examples given by Hinze (7) it would appear that the thermal lag of the wire would be a more serious problem than that of the fluid, but, of course, this depends upon the values of parameters which do not influence the thermal lag of the fluid.
FIG. 3 - STEADY-STATE VORTICITY
FIG. 4 - STEADY-STATE TEMPERATURE
FIG. 5—STEADY-STATE NUSSELT NUMBER VS. REYNOLDS NUMBER
FIG. 7—FLUCTUATING NUSSELT NUMBER VS. FREQUENCY
Fig. 8 - Amplitude Reduction

\[ \frac{A(\omega)}{A(0)} \]

\[ A(0) = 0.0756 \]

\[ \bar{Re} = 10 \]
FIG. 9 - PHASE LAG
BIBLIOGRAPHY


4. Davis, S. H., unpublished calculations, Rice University, Houston, Texas.


