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ASYMPTOTIC VALUES
OF MEROMORPHIC FUNCTIONS

by

Karl Frederick Barth

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Thesis Director's signature:

G.R. Mac Lane

Houston, Texas
May, 1964
TO MY WIFE,

MY FATHER, AND MY MOTHER
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1. Introduction. The purpose of this thesis is to derive some results on asymptotic values of functions meromorphic in the unit circle. G. R. MacLane [12] considered the classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{L}$ of functions non-constant and holomorphic in $\{|z| < 1\}$. $\mathcal{A}$ is the class of functions having asymptotic values at a dense set on $\{|z| = 1\}$. $\mathcal{B}$ is the class of functions such that there is a set of Jordan arcs $\Gamma$ in $\{|z| = 1\}$, such that the end points are dense on $\{|z| = 1\}$ and such that on each $\Gamma$ either $f \to \infty$ or $f$ is bounded. The class $\mathcal{L}$ is defined as follows: $f \in \mathcal{L}$ if and only if each level set $\{z : |f(z)| = \lambda\}$ "ends at points" of $\{|z| = 1\}$. (The precise definition will be found early in section 3.) MacLane proved that $\mathcal{A} = \mathcal{B} = \mathcal{L}$. We shall consider the corresponding classes $\mathcal{A}_m$, $\mathcal{B}_m$, and $\mathcal{L}_m$ of meromorphic functions.

Definitions and some known results are stated in section 2.

The classes $\mathcal{A}_m$, $\mathcal{B}_m$, and $\mathcal{L}_m$ are defined in section 3. It is proved th- $\mathcal{A}_m \subset \mathcal{B}_m$, $\mathcal{L}_m \subset \mathcal{B}_m$, and examples are given showing that $\mathcal{B}_m \cap \mathcal{A}_m$, $\mathcal{B}_m \cap \mathcal{L}_m$, $\mathcal{A}_m \cap \mathcal{L}_m$, $\mathcal{L}_m \cap \mathcal{A}_m$.

Sections 4 and 5 are concerned with the existence of asymptotic values on sets of positive measure. There we prove (Theorem 5) that if $f \in \mathcal{A}_m$ and there exists a complex number $a$ (possibly $\infty$) such that $\sup \{N(r,a,f) ; 0 \leq r < 1\} < \infty$, ($N(r,a,f)$ is the Nevanlinna counting function of $f$), then on any subarc $\gamma$ of $\{|z| = 1\}$ on which $f$ does not have the
asymptotic value $a$ has asymptotic values on a set of positive measure. This generalizes a theorem of MacLane [12, Theorem 11]. Theorems 5 and 8 extend a theorem of Begemihl [2, Theorem 1] which is a generalization of [5, Theorem 3].

Section 6 is devoted to proving sufficient conditions for $f \in \mathcal{A}_m$. The fundamental condition, see Theorem 7, is as follows. If there exists a complex number $a$ (possibly $\infty$), and a set $\Theta$ dense on $[0, 2\pi]$ such that

$$(1) \quad \int_0^1 (1 - r)\log^+ \frac{1}{f(re^{i\theta}) - a} < \infty \quad (\theta \in \Theta, a \neq \infty)$$

and $\sup \{N(r, a, f); 0 \leq r < 1\} < \infty$

(If $a = \infty$, change $1/(f - a)$ to $f$), then $f \in \mathcal{A}_m$. A more restrictive condition is

$$(III) \quad \int_0^1 (1-r)T(r)dr < \infty \text{ and } \sup \{N(r, a, f); 0 \leq r < 1\} < \infty$$

where $T(r)$ is the Nevanlinna characteristic of $f$. These generalize conditions (I) and (III) of MacLane [12, section 7]. An example is given showing that the condition $\sup \{N(r, a), 0 \leq r < 1\} < \infty$ cannot be relaxed to $\delta(a) = 1$ where $\delta(a)$ is the Nevanlinna defect of $a$. Some condition other than just a restriction on the growth of $T(r)$ is necessary since MacLane [11] has constructed a function $f$ meromorphic in $\{|z| < 1\}$ such that $T(r, f)$ is of arbitrarily slow growth and $f$ has no asymptotic values.

Another important sufficient condition, Theorem 8, is that if $f$ is non-constant, meromorphic, normal in the sense of
Lehto and Virtanen, and there exists a complex number \( a \) (possibly \( \infty \)) such that \( \sup \{ N(r,a,f); 0 \leq r < 1 \} < \infty \), then \( f \in \mathcal{A}_m \). This generalizes a theorem which was proved independently by Bagemihl and Seidel [5, Corollary 1] and MacLane [12, Theorem 17]. It also extends [? , Corollary 1].

In section 7 the classes \( \mathcal{A}'_m, \mathcal{B}'_m, \) and \( \mathcal{L}'_m \) are defined. \( f \in \mathcal{A}'_m, \mathcal{B}'_m, \) or \( \mathcal{L}'_m \) if and only if \( f \in \mathcal{A}_m, \mathcal{B}_m, \) or \( \mathcal{L}_m \) (respectively) and \( \sup \{ N(r,\infty,f); 0 \leq r < 1 \} < \infty \). It is proved that \( \mathcal{A}'_m = \mathcal{B}'_m \supset \mathcal{L}'_m \) and that Koebe's Lemma holds for functions in \( \mathcal{A}'_m \). The extension of Koebe's Lemma generalizes a result of MacLane [12, Theorem 9] and overlaps a theorem of Bagemihl and Seidel [5, Theorem 1].

Section 8 is devoted to proving results about asymptotic tracts of functions in \( \mathcal{A}_m \). One of the most interesting results is that if \( f \in \mathcal{A}_m \) and there exist complex numbers \( a,b \) (one of which may be \( \infty \)) such that \( a \neq b \) and \( \sup \{ N(r,a); 0 \leq r < 1 \} < \infty \), \( \sup \{ N(r,b); 0 \leq r < 1 \} < \infty \), then \( f \) has no arc tracts.

The author wishes to express his deepest appreciation to Professor G. R. MacLane who suggested the problem considered here and gave much advice and encouragement during the writing of this thesis. Several of the examples are due to his suggestions.
2. Definitions and known results. If $f$ is meromorphic in $\{|z| < 1\}$, the symbols

$$N(r,a), m(r,a), T(r), \delta(a)$$

will have their usual meanings. See [15] for definitions.

It is convenient to make the following definition. Let

$\{\gamma_n\}$ be a sequence of continuous compact curves in $\{|z| < 1\}$

and let $\gamma$ be an arc $\{z : |z| = 1, \alpha \leq \arg z \leq \beta\}$.

DEFINITION A. $\gamma_n \to \gamma$ if and only if for each $\varepsilon > 0$

there exists an $n_0$ such that

$$\gamma_n \subset \{1 - \varepsilon < |z| < 1\}$$

$$\inf_{\gamma_n} \arg z - \alpha < \varepsilon, \quad \sup_{\gamma_n} \arg z - \beta < \varepsilon$$

Theorem 10 is a generalization of the following well

known lemma of Koebe [8].

THEOREM A. Let $f(z)$ be holomorphic and bounded in

$\{|z| < 1\}$. Let $\gamma_n \to \gamma$ where $\gamma_n \subset \{|z| < 1\}$ and $\gamma$ is an arc

of $\{|z| = 1\}$. If $\sup_{\gamma_n} |f(z) - a| \to 0$ as $n \to \infty$ then $f = a$.

We shall also need some basic facts about asymptotic

tracts. For a more detailed treatment of them see [12,

section 2].

DEFINITION B. Let $f$ be meromorphic in $\{|z| < 1\}$. A

tract $\{T(\varepsilon), a\}$ associated with the finite value $a$ is a

set of non-void domains $T(\varepsilon)$, one for each $\varepsilon > 0$, such that

$$T(\varepsilon) \text{ is a component of the open set}$$

$$\{z : |z| < 1, |f(z) - a| < \varepsilon\}$$

(2.1)

$$0 < \varepsilon_1 < \varepsilon_2 \Rightarrow T(\varepsilon_1) \subset T(\varepsilon_2)$$

(2.2)
\(|T(\varepsilon)\cap \varepsilon > 0 = \varepsilon .

If a is replaced by \(\sim\) the only change is to replace \(|f(z) - a| < \varepsilon\) in (2.1) by \(|f(z)| > 1/\varepsilon\).

One can easily show that
\[(2.4) \quad K = \bigcap_{\varepsilon > 0} T(\varepsilon)\]
is a non-void, connected, closed subset of \(|z| = 1\) . We shall call \(K\) the end of the tract. If \(K\) is a point (arc), then the tract will be called a point- (arc-) tract. The tract is a point tract if and only if the diameter of \(T(\varepsilon)\) tends to zero as \(\varepsilon \to 0\) . The tract will be called global if and only if \(K\) is the whole circumference \(|z| = 1\) and for each arc \(\gamma \subset \{|z| = 1\}\) there is a sequence of arcs \(\gamma_n \subset T(1/n)\) such that \(\gamma_n \to \gamma\).

**Definition C.** If \(\Gamma \subset \{|z| < 1\}\) is a continuous curve such that \(|z| = 1\) on \(\Gamma\), then we shall say that \(f\) has the asymptotic value \(a\) (possibly \(\sim\)) on \(\Gamma\) if and only if
\[
\lim_{|z| \to 1, z \in \Gamma} f(z) = a .
\]

Let \(\{T(\varepsilon), a\}\) be a tract and let \(\Gamma : z = \delta(t), 0 \leq t < 1\) be a continuous curve in \(|z| < 1\) such that \(\delta(t) \in T(\varepsilon)\) for \(1 - \delta(\varepsilon) < t < 1\) . In this case we shall say that \(\Gamma\) belongs to \(\{T(\varepsilon), a\}\) . Because of (2.3), \(|\delta(t)| \to 1\) as \(t \to 1\) .

Clearly \(f\) has the asymptotic value \(a\) on \(\Gamma\) . Given the tract, there exist curves \(\Gamma\) belonging to it. Conversely, if \(f\) has the asymptotic value \(a\) on some curve \(\Gamma\) on which \(|z| \to 1\) , then there is exactly one tract \(\{T(\varepsilon), a\}\) to which \(\Gamma\) belongs. A
curve belonging to \{T(\epsilon), a\} will always tend either to a single point of \(K\) or an arc of \(K\). There will always be some \(\Gamma\) which tends to all of \(K\); there may or may not be curves \(\Gamma\) belonging to the tract which tend to proper subsets of \(K\).

For convenience we make the following definition.

**DEFINITION D.** We shall say that \(f\) has the asymptotic value a at \(\zeta\), \(|\zeta| = 1\), if and only if there is an arc \(\Gamma \subset \{|z| < 1\}\) tending to \(\zeta\) such that \(f \to a\) on \(\Gamma\).

Recall that Lehto and Virtanen have defined the concept of normal function [9].

**DEFINITION E.** \(f\), meromorphic in \(\{|z| < 1\}\), is normal if and only if the set of functions \(f(T(z))\), where \(T\) ranges over the linear transformations preserving \(\{|z| < 1\}\), is a normal family.

We shall need the following two theorems about normal functions [9, Theorems 2 and 3].

**THEOREM B.** If \(f\) is a normal meromorphic function in \(\{|z| < 1\}\) and has the asymptotic value a along an arc ending at \(\zeta\), \(|\zeta| = 1\), then \(f\) has the angular limit a at \(\zeta\).

**THEOREM C.** If \(f\) is meromorphic in \(\{|z| < 1\}\), then it is normal if and only if

\[
|f'(z)| \leq \frac{C}{1 + |f(z)|^2} \quad (|z| < 1)
\]

where \(C\) is a constant.

Lehto and Virtanen remarked [9, p. 53] that the following theorem is true.
THEOREM D. If $f$ is a normal meromorphic function in $\{|z| < 1\}$ and $g$ is bounded and holomorphic in $\{|z| < 1\}$, then $f + g$ is normal and meromorphic in $\{|z| < 1\}$.

DEFINITION F. $f \in \mathcal{H}_m$ if and only if $f$ is a non-constant normal meromorphic function in $\{|z| < 1\}$.
3. The Classes $\mathcal{A}_m, \mathcal{B}_m, \mathcal{L}_m$. Let $f(z)$ be meromorphic and non-constant in $\{|z| < 1\}$. Let $a$ be any complex number (possibly $\infty$) and consider any $\zeta$ such that $|\zeta| = 1$.

We say that $\zeta \in A_a$ if and only if $f$ has the asymptotic value at $\zeta$. If $S$ is any subset of the sphere we set

$$A(S) = \bigcup_{a \in S} A_a, \quad A(S) = \emptyset \text{ is } S = \emptyset.$$  

In particular if $b$ is any complex number we set

$$A_b^* = \bigcup_{a \neq b} A_a, \quad A = A_b^* \cup A_\infty.$$  

In order to avoid confusion $A_a$ will sometimes be denoted by $A_a(f)$.

**DEFINITION.** Let $f$ be meromorphic and non-constant in $\{|z| < 1\}$. Then $f \in \mathcal{A}_m$ if and only if $A(f)$ is dense on $\{|z| = 1\}$.

We now define the set $B^*$. A point $\zeta$ such that $|\zeta| = 1$ is said to belong to $B^*$ if and only if there exists a continuous arc $\Gamma \subset \{|z| < 1\}$ which ends at $\zeta$ such that $|f|$ is bounded by some finite constant $M$ on $\Gamma$.

We set

$$B(f) = B = A_\infty \cup B^*.$$  

**DEFINITION.** $f \in \mathcal{B}_m$ if and only if $f$ is meromorphic and non-constant in $\{|z| < 1\}$ and $B$ is dense on $\{|z| = 1\}$.

It is clear that $A^*_\infty \subset B^*$, $A \subset B$ and hence

$$\mathcal{A}_m \subset \mathcal{B}_m.$$  

For any $f(z)$ in $\{|z| < 1\}$ and any $\lambda > 0$ we will denote the level set, $\{z : |f| = \lambda\}$, by $L(\lambda)$. A component of $L(\lambda)$, a level curve, will be denoted by $C(\lambda)$.

Let $S$ be any subset of $\{|z| < 1\}$. For each $r$, $0 < r < 1$, let the components of
\[ S \cap \{ r < |z| < 1 \} \]
be \( S_i(r) \), \( i \in I \). Let \( \delta_i(r) = \text{diam } S_i(r) \) and set
\[ \delta(r) = \sup_{i \in I} \delta_i(r) \]
with \( \delta(r) = 0 \) if \( I \) is void. Clearly, \( \delta(r) \) is as \( r \to 1 \). We shall say that \( S \) **ends at points** of \( \{|z| = 1\} \) if and only if \( \delta(r) \downarrow 0 \) as \( r \to 1 \).

**Definition.** \( f(z) \) **belongs to the class** \( \mathcal{L}_m(\mathcal{L}_m^*) \) **if** and **only if** \( f(z) \) **is meromorphic and non-constant in** \( \{|z| < 1\} \) **and every level set** (level curve) \( L(\lambda)(C(\lambda)) \) **ends at points** of \( \{|z| = 1\} \).

It is clear that
\[ (3.5) \quad \mathcal{L}_m \subset \mathcal{L}_m^* . \]

G. R. MacLane has previously defined the classes \( \mathcal{A} \), \( \mathcal{B} \), and \( \mathcal{L} \) for holomorphic functions. See [12, p. 7]. The definitions are the same as for \( \mathcal{A}_m \), \( \mathcal{B}_m \), and \( \mathcal{L}_m \) except that the word "holomorphic" is substituted for "meromorphic" in each of the definitions. He proved that \( \mathcal{A} = \mathcal{B} = \mathcal{L} \) [12, p. 10]. The object of this section is to determine whether or not \( \mathcal{A}_m = \mathcal{B}_m = \mathcal{L}_m \). Specifically, we shall prove that \( \mathcal{A}_m \subset \mathcal{B}_m \), \( \mathcal{L}_m \subset \mathcal{B}_m \) and give examples showing that \( \mathcal{A}_m \), \( \mathcal{B}_m \not\subset \mathcal{L}_m \) and \( \mathcal{B}_m \), \( \mathcal{L}_m \not\subset \mathcal{A}_m \).

First we prove two preliminary theorems.

**Theorem 1.** Let \( f \in \mathcal{A}_m \), and let \( \gamma_n \) be a sequence of distinct simple arcs in \( \{|z| < 1\} \) which tend to the arc \( \gamma \) of \( \{|z| = 1\} \) with the property that there exists a complex number \( a \) such that
\[ (3.6) \quad \sup_{\gamma_n} |f(z) - a| = \mu_n \to 0 \quad (n \to \infty) \text{ if } a \neq \infty , \quad \text{or} \]
(3.7) \[ \inf_{\gamma_n} |f(z)| = \omega_n \to \infty (n \to \infty) \text{ if } a = \infty. \]

Then \( f(z) \) has an asymptotic tract \( \{T(\xi), a\} \) with end \( K \) such that \( \gamma \subset K \) and such that if \( \zeta \) is an arbitrary point of \( K \), then there exists a curve \( \Gamma \) belonging to \( \{T(\xi), a\} \) which ends at \( \zeta \). At any interior point \( \zeta \) of \( K \) the only asymptotic values come from this tract \( \{T(\xi), a\} \). If \( f \in \mathcal{L}_m \) (but not necessarily in \( A_m \)), the above conclusions are true for \( a = \infty \).

**Remark.** This theorem was proved for \( f \) holomorphic and in \( A, B \), or \( L \) by G. R. MacLane [12, Theorem 3].

**Proof.** For \( f \in \mathcal{L}_m \) and \( a = \infty \) the proof is exactly the same as that of [12, Theorem 3]. Thus it will not be given here.

Now suppose \( f \in A_m \) and \( a = \infty \). Let \( \gamma = \{e^{i\theta} : a \leq \theta \leq \delta\} \) and let \( S(\alpha, \beta) \) denote the sector \( \{z : \alpha \leq \arg z \leq \beta, |z| < 1\} \). Then for all \( \lambda > 0 \) \( L(\lambda) \cap S(\alpha, \beta) \) ends at points of \( \{|z| = 1\} \). Suppose not. Then there would exist a \( \lambda_1 > 0 \), a subarc, \( \Delta \), of \( \gamma \), and a sequence, \( \{\Delta_n\} \), of continuous compact (in \( \{|z| < 1\}\)) arcs such that \( \Delta_n \subset L(\lambda_1) \) for all \( n \) and \( \Delta_n \to \Delta \). Let \( \zeta \) be any interior point of \( \Delta \) at which \( f \) has an asymptotic value. However, any curve ending at \( \zeta \) must cross all but a finite number of the \( \Delta_n \) and \( \gamma_n \), and thus \( f \) cannot have an asymptotic value at \( \zeta \). This contradicts \( f \in A_m \). Hence \( L(\lambda) \cap S(\alpha, \beta) \) ends at points of \( \{|z|=1\} \) and again exactly the same proof as for [12, Theorem 3] works.

Finally, suppose \( a \) is finite. By applying Theorem 1
with $a = \infty$ to the function $1/(f - a)$ we have the desired result.

**THEOREM 2.** Let $f \in \mathcal{L}_m$. Suppose $\gamma = \{e^{i\theta} : \alpha \leq \theta \leq \beta, \alpha \neq \beta\}$ is a subarc of $\{|z| = 1\}$ such that no level curve of $f$ ends at any point of $\gamma$. Then exactly one of the following two statements is valid.

For each interior point, $e^{i\phi} (\alpha < \phi < \beta)$, of $\gamma$

- there exists a continuous curve $\Gamma(e^{i\phi}) \subset \{|z| < 1\}$
- which ends at $e^{i\phi}$ and is such that $|f|$ is bounded on $\bigcup_{\alpha < \phi < \beta} \Gamma(e^{i\phi})$. Moreover, $f$ does not have the asymptotic value $\infty$ at any interior point of $\gamma$.

(3.8)

There exists an arc tract for $\infty$ of $f$ with end $K$

such that $\gamma \subset K$.

(3.9)

**Proof.** It is easy to show that (3.8) and (3.9) cannot occur for the same $\gamma$; we shall prove that either (3.8) or (3.9) must be valid. Let

(3.10) $S(\alpha, \beta) = \{z : |z| < 1 \text{ and } \alpha < \arg z < \beta\}$.

Recall that for any positive real number $\lambda$, $L(\lambda)$ denotes the level set $\{|f| = \lambda\}$ and $C(\lambda)$ denotes a component (level curve) of $L(\lambda)$.

Pick $\{\lambda_n\}_{n=1}^{\infty}$ such that $0 < \lambda_n \uparrow \infty$ and $L(\lambda_n)$ has no multiple points. Then, since $f \in \mathcal{L}_m$, each $C(\lambda_n)$ is either a closed Jordan curve or a crosscut of $\{|z| = 1\}$. We may suppose that the origin, $0$, is not a pole of $f$. If it is, pick a point a near $0$ and repeat the following argument using $a$ in place of $0$. Let $N$ be such that $0 \in \{z : |f| < \lambda_N\}$.
For any $n \in \mathbb{N}$ let $\Delta(\lambda_n)$ denote the component of $\{z : |f| < \lambda_n\}$ which contains 0. Since $f \in \mathcal{L}_m$ and no level curve of $f$ ends at any point of \( \gamma \), at least one of the following statements must be valid for any $n \in \mathbb{N}$:

There exists a $\tau_n \subset \partial \Delta(\lambda_n)$ such that $\tau_n$ is a cross-cut of the sector $S(\alpha, \beta)$ which joins a point of $\arg z = \alpha$ to a point of $\arg z = \beta$.

$$\partial \Delta(\lambda_n) \supset \gamma.$$  \hfill (3.11)

If (3.11) is valid for all $n \in \mathbb{N}$, it is clear that $\tau_n \to \gamma$, and thus by Theorem 1 $f$ has an arc tract for $\omega$ with end $K \supset \gamma$. Hence (3.9) occurs.

Now suppose (3.12) is true for some $n = M$. Let \( \zeta = e^{i\phi}(\alpha < \phi < \beta) \) be any interior point of $\gamma$. By (3.12) $\zeta \in \partial \Delta(\lambda_M)$. Since $f \in \mathcal{L}_m$ and no level curves of $f$ end at points of $\gamma$, there exists a $\delta(\zeta) > 0$ such that all the components of $\partial \Delta(\lambda_M)$ which have non-empty intersection with $\gamma$ are closed Jordan curves which are contained in $S(\alpha, \beta)$.

This together with the hypothesis that ($f \in \mathcal{L}_m$) the diameter of $L(\lambda_M) \cap \{z : 1 - \epsilon < |z| < 1\}$ tends to zero as $\epsilon \searrow 0$ implies that 0 and $\zeta$ may be connected by a continuous curve $\Gamma(e^{i\phi}) \subset \Delta(\lambda_M) \cup \zeta$.

To prove the final statement in (3.8) note that the existence of the asymptotic value $\omega$ at $\zeta$ implies that $L(\lambda)$ for all $\lambda > \lambda_M$ ends at $\zeta$ which is contradictory. Thus (3.8) is valid and the proof is complete.
It is clear from the proof that Theorem 2 may be generalized as follows: \( f \in \mathcal{L}_m \) may be replaced by for each \( \zeta \in \gamma^0 \) there exists a \( \delta(\zeta) > 0 \) such that \( \{|f| = \lambda\} \cap U(\delta, \zeta) \) ends at points of \( \{|z| = 1\} \) for all \( \lambda > 0 \).

Now we shall prove the promised results.

**Theorem 3.** \( \mathcal{A}_m \subset \mathcal{B}_m, \mathcal{L}_m \subset \mathcal{B}_m, \) and no other inclusion relations between \( \mathcal{A}_m, \mathcal{B}_m, \) and \( \mathcal{L}_m \) are valid.

**Remarks.** Recall that for holomorphic functions \( \mathcal{A} = \mathcal{B} = \mathcal{L} \). The situation for meromorphic functions is illustrated by Figure 1.

**Proof.** We have already shown that \( \mathcal{A}_m \subset \mathcal{B}_m \) (see (3.4)). We shall now prove that \( \mathcal{L}_m \subset \mathcal{B}_m \). Suppose \( f \in \mathcal{L}_m \) and consider any subarc, \( \gamma = \{e^{i\theta} : \alpha \leq \theta \leq \beta\} \), of \( \{|z| = 1\} \). We shall show that either there is a continuous curve ending at some point of \( \gamma \) on which \( |f| \) is bounded or there is a continuous curve ending at some point of \( \gamma \) on which \( f \) has the asymptotic value \( \infty \). If a level curve of \( f \) ends at a point of \( \gamma \), we are done. If not, Theorem 2 applies and we have either for each interior point \( e^{i\theta}, \alpha < \theta < \beta \), of \( \gamma \) there exists a continuous curve \( \Gamma(\theta) \subset \{|z| < 1\} \) which ends at \( e^{i\theta} \) and on which \( |f| \) is bounded or there is an arc tract of \( f \) for \( \infty \) with end \( K \) such that \( \gamma \subset K \). If the first case occurs, we are through; in the second case, applying Theorem 1, we see that \( f \) has the asymptotic value \( \infty \) at each point of \( \gamma \). Hence \( f \in \mathcal{B}_m \). Examples 1 and 2 (see below) imply
that no other inclusion relations exist between $A_m$, $B_m$, and $L_m$.

**EXAMPLE 1.** We shall construct a function $f$ meromorphic and non-constant in $\{ |z| < 1 \}$ such that $f \in B_m$ and $f \in L_m$ but $f \notin A_m$. Thus $B_m \not\subset A_m$ and $L_m \not\subset A_m$. This example is due to Lehto and Virtanen; see [9, p. 58].

According to [10, Theorem 6], if $E \subset \{ |z| = 1 \}$, is of type $F_\sigma$, and is of measure zero, then there exists a function $g(z)$, holomorphic and bounded in $\{ |z| < 1 \}$, such that the radial limit $g(e^{i\theta})$ exists for each $e^{i\theta}$ in $\{ |z| = 1 \} - E$ and for no $e^{i\theta}$ in $E$.

Now let $h(z)$ be a "modular function" omitting the values 0, 1, and 5 i.e. $h$ maps the unit disc one-one and conformally onto the universal covering surface of the complex sphere with the points 0, 1, and 5 removed. We know that $h(z) \in \mathcal{H}_m$ (see Definition F). Also $h(e^{i\theta})$ exists for $e^{i\theta} \in M$, and $h(e^{i\theta})$ does not exist for any $e^{i\theta}$ in $\{ |z| = 1 \} - M$ where $M$ is a countable dense subset of $\{ |z| = 1 \}$. Choose $E = M$.

Then $f(z) = g(z) + h(z)$ is normal since $g$ is bounded and $h \in \mathcal{H}_m$ (Theorem D), and $f(z)$ cannot have any radial limits. Since $f \in \mathcal{H}_m$, it can have no asymptotic values either (Theorem B). If $e^{i\theta} \in M$, then there exists a $t_\circ(e^{i\theta})$ such that $|h(te^{i\theta})|$ is bounded for all $t$ satisfying $t_\circ(e^{i\theta}) < t < 1$. Thus $|f(te^{i\theta})|$ is bounded for all $t$ such that $t_\circ(e^{i\theta}) < t < 1$, $e^{i\theta} \in M$. Hence there is a curve ending at each point of $M$ on which $|f|$ is bounded. Therefore $f \in B_m$, but $f \notin A_m$. 
Next we want to show that \( f \in \mathcal{L}_m \). We know that 
\( \mathcal{A}_o(h), \mathcal{A}_1(h), \) and \( \mathcal{A}_5(h) \) are all dense, and we may suppose that \( |g| \equiv 1 \). Now suppose that \( f \notin \mathcal{L}_m \); then for some \( \lambda > 0 \) there exists a sequence of arcs, \( \{ \gamma_n \} \), such that \( \gamma_n \neq \gamma \), a subarc of \( \{|z| = 1\} \), and \( |f| = \lambda \) on \( \gamma_n \). Pick \( e^{i\theta_1} \in \gamma \) such that \( h(e^{i\theta_1}) = 0 \) and \( e^{i\theta_2} \in \gamma \) such that \( h(e^{i\theta_2}) = 5 \). Let \( z(1, n) (z(2, n)) \) be a point of intersection of the radius to \( e^{i\theta_1}(e^{i\theta_2}) \) with \( \gamma_n \). (If necessary, pick a subsequence of the \( \gamma_n \) such that \( z(1, n) \) and \( z(2, n) \) exist for all \( n \).) Now there exists an \( N \) such that for \( n > N \) we have

(3.14)

\[
\lambda = |f(z(1,n))| \equiv |g(z(1,n))| + |h(z(1,n))| \equiv 1 + 1/2 = 3/2
\]

and

(3.15)

\[
\lambda = |f(z(2,n))| \equiv |h(z(2,n))| - |g(z(2,n))| \equiv 4 - 1 = 3.
\]

Since (3.14) and (3.15) are contradictory, we find that \( f \in \mathcal{L}_m \). Thus \( f \in \mathcal{L}_m \), but \( f \notin \mathcal{A}_m \). Note that Example 1 also shows that \( \mathcal{A}_m \) is not a linear space.

EXAMPLE 2. Using a theorem of Mergelyan we shall construct a function, \( f \), meromorphic in \( \{|z| < 1\} \), such that \( f \in \mathcal{A}_m \), \( f \in \mathcal{B}_m \), but \( \{|f| = 1\} \) contains a sequence of arcs which approaches \( \{|z| = 1\} \). i.e. \( f \notin \mathcal{L}_m \). The construction is similar to that used by Bagemihl and Seidel in [3].

Let \( \{r_n\}_{n=1}^\infty \) be a sequence of real numbers such that \( 0 < r_1 < r_2 < \cdots < r_n < \cdots \rightarrow 1 \). We make the following
definitions for \( n \geq 1 \):

\begin{align*}
(3.16) \quad C_n &= \{|z| = r_n\} \\
(3.17) \quad D_n &= \{|z| < r_n\}
\end{align*}

and

\begin{equation*}
(3.18) \quad E_n = \{z : r_n \leq |z| \leq r_{n+1} ; \arg z = (k/2^n)2\pi ; \quad k = 0, 1, 2, \ldots, 2^n - 1\}.
\end{equation*}

Also for \( n > 1 \) let

\begin{equation*}
(3.19) \quad F_n = D_{n-1}^- \cup E_{n-1} \cup C_n.
\end{equation*}

Now we shall inductively define two sequences of functions, \( \{f_n(z)\}_{n=1}^\infty \) and \( \{R_n(z)\}_{n=1}^\infty \).

First let \( f_1(z) \) and \( R_1(z) \) be defined on \( D_1^- \) such that

\begin{equation*}
f_1(z) = R_1(z) = 1/2.
\end{equation*}

Next construct \( f_2(z) \) to be continuous on \( F_2 \) so that

\begin{align*}
(3.20a) \quad f_2(z) &= f_1(z) \text{ on } D_1^- , \\
(3.20b) \quad f_2(z) &= 5/4 \text{ on } C_2 ,
\end{align*}

and

\begin{equation*}
(3.20c) \quad f_2(z) \text{ is linear on each component of } E_1.
\end{equation*}

It is clear that \( F_2 \) is closed and that it divides the plane into a finite number of regions. Also \( f_2(z) \) is continuous on \( F_2 \) and analytic in the interior of \( F_2 \). Thus by a remark of Mergelyan [14, p. 24] there exists a rational function \( R_2(z) \) such that

\begin{equation*}
(3.21) \quad \max_{z \in F_2} |f_2(z) - R_2(z)| < 1/2^4.
\end{equation*}

Let \( \{a(2, k)\}_{k=1}^N \) denote the poles of \( R_2(z) \) which are contained in \( D_2^- \) and let \( P(2, k, z) \) denote the principal part
of $R_2(z)$ at $a(2, k)$. Now construct $f_3(z)$ to be continuous (in the spherical metric) on $F_3$ such that

\[(3.22a) \quad f_3(z) = R_2(z) \text{ on } D_2^- , \]
\[(3.22b) \quad f_3(z) = 1 - 1/2^3 \text{ on } C_3 , \]
and
\[(3.22c) \quad f_3(z) \text{ is linear on each component of } E_2 . \]

Then

\[(3.23) \quad g_3(z) = f_3(z) - \sum_{k=1}^{N_2} P(2, k, z) \]

is continuous on $F_3$ and analytic at interior points of $F_3$. As before, there is a rational function, $S_3(z)$, such that

$$\max_{z \in F_3} |g_3(z) - S_3(z)| < 1/2^5$$

or

$$\max_{z \in F_3} |f_3(z) - [S_3(z) + \sum_{k=1}^{N_2} P(2, k, z)]| < 1/2^5 .$$

Letting

$$R_3(z) = S_3(z) + \sum_{k=1}^{N_2} P(2, k, z)$$

we have

\[(3.24) \quad \max_{z \in F_3} |f_3(z) - R_3(z)| < 1/2^5 . \]

Denote the poles of $R_3(z)$ by $\{a(3, k)\}_{k=1}^{N_3}$ and the principal part of $R_3(z)$ at $a(3, k)$ by $P(3, k, z)$.

In general let $f_n(z)$ be continuous (spherically) on $F_n$ such that

\[(3.25a) \quad f_n(z) = R_{n-1}(z) \text{ on } D_{n-1}^- , \]
\[(3.25b) \quad f_n(z) = 1 + (-1)^n/2^n \text{ on } C_n , \]
and
\[(3.25c) \quad f_n(z) \text{ is linear on each component of } E_{n-1} . \]
We can find an $R_n(z)$ such that
\[
(3.26) \quad \max_{z \in F_n} |f_n(z) - R_n(z)| < 1/2^{n+2}.
\]

Now we will show that $\{R_n(z)\}$ converges to a meromorphic function, $R(z)$, for $\{|z| < 1\}$. First consider any $n > 1$.

Using (3.25a), we have for $z \in D_n^-$
\[
|\overline{R}_{n+1}(z) - R_n(z)| = |\overline{R}_{n+1}(z) - f_{n+1}(z)|;
\]
thus (3.26) implies
\[
(3.27) \quad |\overline{R}_{n+1}(z) - R_n(z)| < 1/2^{n+3} (z \in D_n^-).
\]

Let $k$ be any positive integer and $z \in D_n^-$. Then
\[
|\overline{R}_{n+k}(z) - R_n(z)| \leq \sum_{i=n+1}^{n+k} |\overline{R}_i(z) - R_{i-1}(z)|,
\]
and applying (3.27) to each term of the right hand member we have
\[
(3.28) \quad |\overline{R}_{n+k}(z) - R_n(z)| \leq \sum_{i=n+1}^{n+k} 1/2^{i+2} < 1/2^{n+2} (z \in D_n^-).
\]

Now for any compact set $K \subset \{|z| < 1\}$ pick $n_0$ such that $K \subset D_n^{n_0}$. For any $z \in K$, any $n > n_0$, and any positive integer $k$ we have by (3.28)
\[
(3.29) \quad |\overline{R}_{n+k}(z) - R_n(z)| < 1/2^{n+2} (z \in K).
\]

Thus $\{R_n(z)\}$ converges to a meromorphic function in $\{|z| < 1\}$.

In order to show that $R(z) \in \mathcal{L}_m$ we shall show that there exists a component of $\{|R| = 1\}$ which separates $C_n$ from $C_{n+1}$ for any $n$. If we show that
\[
(3.30) \quad |R(z) - (1 + (-1)^n/2^n)| < (1/2)(1/2^n) (z \in C_n),
\]
it is clear that a component of $\{|R| = 1\}$ must separate $C_n$ and $C_{n+1}$. Thus it suffices to prove (3.30).
Now
\[ R(z) = R_n(z) + \sum_{k=n}^{\infty} (R_{k+1}(z) - R_k(z)). \]

Thus
\[
|R(z) - (1 + (-1)^n/2^n)| \leq |R_n(z) - (1 + (-1)^n/2^n)| \\
+ \sum_{k=n}^{\infty} |R_{k+1}(z) - R_k(z)| .
\]

We chose \( R_n(z) \) such that
(3.31) \[ |R_n(z) - (1 + (-1)^n/2^n)| < 1/2^{n+2} \quad (z \in C_n) . \]

Since \( C_n \subseteq C_k^- \) for \( k = n, n + 1, \ldots \), using (3.27) we have
(3.32) \[ |R_{k+1}(z) - R_k(z)| < 1/2^{k+3} \quad (z \in C_n) . \]

Hence
\[
|R(z) - (1 + (-1)^n/2^n)| < 1/2^{n+2} + \sum_{k=n}^{\infty} 1/2^{k+3} \\
= (1/2)(1/2^n)
\]

which completes the proof that \( f \notin L_m \).

It remains to be shown that \( f \in \mathcal{A}_m \). We will prove that \( f \) has the asymptotic value one on any radius of the form
(3.33) \[ \{z : 0 \leq |z| < 1, \arg z = k/2^n \text{ where} \]
\[ n = 1, 2 \ldots \text{ and } k = 0, 1, 2, \ldots, 2^n - 1 \} . \]

It suffices to consider only the radius corresponding to \( \arg z = 0 \); call it \( \Delta \). The argument for any other is similar.

For any \( n > 2 \) consider any \( z_0 \in \Delta \) such that \( r_n < |z_0| \leq 1 \).

There exists an \( N \) such that \( r_N < |z_0| \leq r_{N+1} \). If we recall that \( f_{N+1}(z) \) is linear on each segment of \( E_n \), it is clear that
(3.34) \[ \sup_{z \in \Delta \cap A_n} |f_{N+1}(z) - 1| = \max \{ |R_N(r_N) - 1|, 1/2^{N+1} \} \]
where $A_N = \{r_N < |z| \leq r_{N+1}\}$. Using (3.25b) and (3.26) we have
\begin{equation}
\max \{ |R_N(r_N) - 1|, 1/2^{N+1} \} \leq \{ 1/2^{N+2} + 1/2^N, 1/2^{N+1} \}
= 1/2^{N+2} + 1/2^N
< 1/2^{N-1}.
\end{equation}

Now by (3.26)
\begin{equation}
|R_{N+1}(z_0) - f_{N+1}(z_0)| < 1/2^{N+3},
\end{equation}
and using (3.27) we have
\begin{equation}
|R(z_0) - R_{N+1}(z_0)| \leq \sum_{k=N+2}^{\infty} |R_k(z_0) - R_{k-1}(z_0)|
< \sum_{k=N+2}^{\infty} 1/2^{k+2}
= 1/2^{N+3}.
\end{equation}

Also
\begin{equation}
|R(z_0) - 1| \leq |R(z_0) - R_{N+1}(z_0)| + |R_{N+1}(z_0) - f_{N+1}(z_0)|
+ |f_{N+1}(z_0) - 1|
\end{equation}

Using (3.34), (3.35), (3.36), and (3.37) we have
\begin{equation}
|R(z_0) - 1| < 1/2^{N+3} + 1/2^{N+3} + 1/2^{N-1}
< 1/2^{N-2}
\leq 1/2^{n-2}
\end{equation}

Thus $R(z)$ has the asymptotic value one on $\Delta$. Since (3.33) is dense, we see that $f \in A_m$. Thus $A_m \not\subset L_m$; also note that $f \in B_m$ which implies that $B_m \not\subset L_m$.

MacLane [12, p. 18] has shown that if $f$ is holomorphic and $f \in L$, then $f + a \in L$ for any finite complex number $a$. We shall show that this is not the case for meromorphic functions. Consider the function $f$, constructed in
Example 2. We know that $f \notin \mathcal{L}_m$ and that $A_1(f)$ is dense. Thus $A_0(f - 1)$ is dense which implies that $f - 1 \in \mathcal{L}_m$. Hence, $f \notin \mathcal{L}_m$, but $f - 1 \in \mathcal{L}_m$.

One naturally raises the following question: Can $\mathcal{B}_m$ be redefined in some way so that it will be contained in $\mathcal{L}_m$? A natural choice of a new definition is as follows: For any complex number $a$ (possibly $\infty$) and $\zeta \in \{|\zeta| = 1\}$ let $\zeta \in B_a^* \text{ if and only if there exists a continuous curve ending at } \zeta \text{ on which } |f| \text{ is bounded away from } a$. Let

$B_a = B_a^* \cup A_a$.

(Note that our old $B$ is equal to $B_\infty$.) If $f$ is meromorphic and non-constant in $\{|z| < 1\}$, we say that $f \in \tilde{\mathcal{B}}_m$ if and only if $B_a$ is dense for each complex number $a$. It is clear that $\mathcal{A}_m \subset \tilde{\mathcal{B}}_m$, but the function constructed in Example 2 shows that $\tilde{\mathcal{B}}_m \not\subset \mathcal{L}_m$. Hence, it is of no use to redefine $\mathcal{B}_m$ in this way.
4. **Measurability of \( A(S) \).** In the next section we shall need the measurability of the set \( A(S) \) defined in Section 3. For this we prove Theorem 4.

**THEOREM 4.** Let \( f \in \mathcal{A}_m \) and let \( S \) be a Borel set on the sphere. Then \( A(S) \) is measurable.

**Remarks.** Measurable here means Lebesgue measurable as a set in \([0, 2\pi]\). Theorem 4 was proved by MacLane [12, p.22] for \( f \in \mathcal{A} \) (see section 3). It was later proved by McMillan [13] for any function which is holomorphic in \(|z| < 1\). The following proof is very similar to that used by MacLane.

**Proof.** The following relations are obvious for any finite or countable collection of sets \( S_n \) on the sphere.

\[
\begin{align*}
(4.1) \quad & A(\cup S_n) = \cup A(S_n) . \\
(4.2) \quad & A(\cap S_n) \subseteq \cap A(S_n) .
\end{align*}
\]

The reason for inclusion only in (4.2) is that \( f \) may have more than one asymptotic value at a given point. This difficulty can be obviated by means of the following result [1].

**THEOREM OF BAGEMIHL.** Let \( f(z) \) be meromorphic in \(|z| < 1\). The set of points \( E \subset \{|z| = 1\} \) at which \( f \) has more than one asymptotic value is at most countable.

It follows then that (4.2) may be improved to

\[
(4.3) \quad \cap A(S_n) = A(S_n) \cup E_1, \quad E_1 \text{ countable.}
\]
Then from (4.1) and (4.3) it follows that we need only prove the measurability of \( A(S) \) for the two cases: \( S \) closed and bounded, \( S = \{ \sigma \} \). For then, if \( S_0 \) is the whole sphere, \( A = A(S_0) \) is measurable. If \( A(S) \) is measurable and \( S' = S - S_0 \), then

\[
A(S') = [A - A(S)] \cup \{ \text{countable set} \}
\]

is measurable, etc.

Let now \( S \) be closed and bounded. For each \( n \geq 1 \) let

\[
\Delta(n,1), \ldots, \Delta(n,\nu_n)
\]

be a finite set of open discs of radius \( 4^{-n} \) which cover \( S \) and is not redundant in the sense that

\[
(4.4) \quad \Delta(n,k) \cap S \neq \emptyset, \quad 1 \leq k \leq \nu_n.
\]

Let \( \Delta^*(n,k) \) be the disc of radius \( 2 \cdot 4^{-n} \) with the same center as \( \Delta(n,k) \). The discs \( \Delta \) and \( \Delta^* \) may be chosen so that their circumferences contain no projections of the branch points of \( \mathcal{S} \). Here \( \mathcal{S} \) is the Riemann surface onto which \( w = f(z) \) maps \( \{|z| < 1\} \). Then each component of \( \mathcal{S} \) over \( \Delta(n,k) \) will correspond to a domain \( D(n,k,p) \) in \( \{|z| < 1\} \) which is bounded by level curves, without multiple points, of \( f(z) - a(n,k) \), where \( w = a(n,k) \) is the center of \( \Delta(n,k) \), and possibly some points on \( \{|z| = 1\} \). We will consider only those \( D \)'s with boundary points on \( \{|z| = 1\} \); for a given \( n, k \) let these be \( D(n,k,p), p \in P(n,k) \). The set \( P(n,k) \) may be void, finite, or countably infinite. We will let \( E^*(n,k,p) \) denote \( D^-(n,k,p) \cap \{|z| = 1\} \). Now each \( E^*(n,k,p) \) is closed and hence measurable.

Consider any point \( \zeta \in E^*(n,k,p) \). We shall say that \( \zeta \in E_0(n,k,p) \) if and only if there exists a sequence of
subarcs, \( \{ \gamma_n \} \), of \( \mathbb{D}(n,k,p) \) such that \( \gamma_n \rightarrow \gamma \) where \( \gamma \) is a subarc of \( \{ |z| = 1 \} \) and \( \zeta \in \gamma^0 \). It is clear that \( E_0(n,k,p) \) is open, and thus, if it is non-empty, it is the union of a finite or countable number of disjoint open subarcs of \( \{ |z| = 1 \} \). Let \( E(n,k,p) = E^*(n,k,p) - E_0(n,k,p) \). From now on we will consider only those \( E \)'s for which \( E \) is non-void. We shall call the new index set \( P'(n,k) \). It is clear that \( E(n,k,p) \) is measurable. Now \( E(n,k,p) \) will contain a finite or countable set \( F(n,k,p) \) which consists of the end points of the level curves on the boundary of \( D(n,k,p) \) which end at points of \( \{ |z| = 1 \} \) and of the end points of the components of \( E_0(n,k,p) \). The set

\[
H(n,k,p) = E(n,k,p) - F(n,k,p)
\]

is then measurable.

Set

\[
E(n) = \bigcup_{(k,p)} E(n,k,p), \quad H(n) = \bigcup_{(k,p)} H(n,k,p)
\]

\[
E = \bigcap_{n=1}^{\infty} E(n), \quad H = \bigcap_{n=1}^{\infty} H(n).
\]

These sets are all measurable. Also,

\[
E(n) = H(n) \cup \{ \text{countable set} \}, \quad E = H \cup \{ \text{countable set} \}.
\]

Now we prove that

\[
H \subseteq A(S) \subseteq E
\]

from which it follows that \( A(S) \) is measurable. If \( \zeta \in A(S) \), then \( z \rightarrow a \) as \( z \rightarrow \zeta \) along a curve \( \Gamma \) ending at \( \zeta \), and also \( a \in S \). For each \( n \), \( a \in \Delta(n,k) \) for some \( k \) and hence a tag end of \( \Gamma \) lies in some \( D(n,k,p) \). Then \( \zeta \) is on the boundary
of \( D(n,k,p) \); that is, \( \zeta \in E(n,k,p) \) for some pair \( k,p \), since if \( \zeta \) were in \( E_o(n,k,p) \), \( \Gamma \) would intersect the boundary of \( D(n,k,p) \) an infinite number of times contradicting the fact that \( \Gamma \subset D(n,k,p) \). (Note that this also justifies considering only those \( D \)'s for which \( E \) is non-void.) Hence \( \zeta \in E(n) \) for each \( n \), \( \zeta \in E \), and thus \( A(S) \subset E \).

Now assume \( \zeta \in H \). Then for each \( n \) there is a pair \( (k_n,p_n) \) such that \( \zeta \in H(n,k_n,p_n) \). The corresponding domains \( D(n,k_n,p_n) \) are such that any two of them must intersect; for if two \( D \)'s are disjoint then the only boundary points on \( \{|z|=1\} \) which they have in common will be points in \( E_o(n,k_n,p_n) \) and \( F(n,k_n,p_n) \) both of which are disjoint from \( H(n,k_n,p_n) \). Since

\[
(4.9) \quad D(m,k_m,p_m) \cap D(n,k_n,p_n) \neq \emptyset \quad (m,n \geq 1),
\]

it follows that

\[
(4.10) \quad \Delta(m,k_m) \cap \Delta(n,k_n) \neq \emptyset \quad (m,n \geq 1).
\]

Since the radii of these discs tend to zero, these discs \( \Delta(n,k_n) \) tend to a unique point \( a \). Since the covering \( \Delta(n,k) \) of \( S \) was non-redundant and \( S \) is closed, it follows that \( a \in S \). It follows from (4.10) and the choice of the radii of the \( \Delta \) and \( \Delta^* \) that

\[
(4.11) \quad \Delta^*(n+1,k_{n+1}) \subset \Delta^*(n,k_n).
\]

One component \( L_n \) of \( \mathcal{L} \) over \( \Delta^*(n,k_n) \) will contain the component over \( \Delta(n,k_n) \) which corresponds to \( D(n,k_n,p_n) \). Let \( L_n \) correspond to the domain \( D_n^* \) in \( \{|z|<1\} \). Then
(4.12) \[ D(n,k_n,p_n) \subset D_n^* \]

and because of (4.9) and (4.11)

(4.13) \[ D_{n+1}^* \subset D_n^* \]

From (4.12) it follows that \( \zeta \) is a boundary point of each \( D_n^* \). Now let \( \Gamma : z = \varphi(t), \ 0 \leq t < 1 \), be a continuous curve such that \( \varphi(t) \in D_n^* \) for \( \tau(n) \leq t < 1 \). The domains \( D_n^* \) are such that \( D_n^* \cap \{ |z| \leq r \} = \emptyset \) for \( r < 1, \ n > n_0(r) \), as otherwise we would have \( f = a \) which is impossible. Hence \( \Gamma \) tends to \( \{ |z| = 1 \} \) and \( f \) has the asymptotic value \( a \) on \( \Gamma \). We must show that there exists a curve tending to \( \zeta \) on which \( f \) has the asymptotic value \( a \).

If \( \Gamma \) ends at \( \zeta \), we're done. If not, suppose first that \( \Gamma \) tends to an arc of \( \{ |z| = 1 \} \) which contains \( \zeta \). By Theorem 1 there exists a curve ending at \( \zeta \) on which \( f \) has the asymptotic value \( a \). Next suppose \( \Gamma \) ends at \( \zeta_1 \neq \zeta \).

Since \( \zeta \) and \( \zeta_1 \) are on the boundary of each \( D_n^* \), we can find a curve \( \Gamma_1 \) running through the \( D_n^* \) which comes arbitrarily close to both \( \zeta \) and \( \zeta_1 \) and such that \( f \) has the asymptotic value \( a \) on \( \Gamma_1 \). Thus we are back to the previous situation. Finally suppose \( \Gamma \) tends to an arc of \( \{ |z| = 1 \} \) which does not contain \( \zeta \). Similar reasoning yields a curve which ends at \( \zeta \) on which \( f \) has the asymptotic value \( a \).

Hence \( \zeta \in A(S) \) which completes the proof of (4.8).

Finally we must prove that \( A_{\infty} \) is measurable. A simplified version of the above argument works (we begin with the discs \( \Delta_n = \{ |w| > n \} \), and we don't have to fool with the \( \Delta^* \)'s).
5. Asymptotic Values on Sets of Positive Measure.

We can now prove a generalization of [12, Theorem 11]. Theorem 5 and Theorem 8 extend [5, Theorem 3] and [2, Theorem 1]. It will be recalled that \( A_a \) denotes the set of points at which \( f \) has the asymptotic value \( a \), and \( A_a^\ast \) denotes the set of points at which \( f \) has an asymptotic value not equal to \( a \).

**Theorem 5.** Let \( f \in A_m \). Suppose \( a \) is a complex number (possibly \( \infty \)) such that \( \sup \{N(r,a,f) ; 0 \leq r < 1\} < \infty \) and let \( \gamma \) be any subarc of \( \{|z|=1\} \) such that \( A_a \cap \gamma = \emptyset \). Then \( \text{meas} \ (A_a^\ast \cap \gamma) > 0 \).

**Remark.** The inequality \( \text{meas} (A_a^\ast \cap \gamma) < \text{meas} (\gamma) \) is possible. See [12, p. 75].

**Proof.** By Theorem 4 \( A_a^\ast \) and hence \( A_a^\ast \cap \gamma \) is measurable. We may suppose that \( a = \infty \) since if \( \sup \{N(r,a); 0 \leq r < 1\} < \infty \) for some finite \( a \), we may apply Theorem 5 with \( a = \infty \) to the function \( 1/(f-a) \) and obtain the desired conclusion. Suppose also that \( f \) has a pole of order \( \lambda \) at \( z = 0 \) (where \( \lambda = 0 \) if \( f \) is holomorphic at \( z = 0 \)). Let \( \{b_k = |b_k|e^{i\theta_k}\} \) denote the poles of \( f \) where a pole of order \( \mu \) appears \( \mu \) times among the \( b_k \). It is known that \( N(r, \infty) \) bounded for \( 0 \leq r < 1 \) (see [15, p. 188]) implies

\[
B(z) = z^\lambda \prod_{k=1}^{\infty} \frac{|b_k| - ze^{i\theta_k}}{1 - \overline{b_k}z}
\]

(5.1)

converges subuniformly in \( \{|z| < 1\} \) to a holomorphic function, \( |B(z)| \leq 1 \), and \( B(b_k) = 0 \) for \( k = 1, 2, \ldots \). Then

\[
F(z) = f(z)B(z)
\]

(5.2)
is holomorphic in \(|z| < 1\).

There are two possibilities. Either

(5.3a) \(F\) is bounded in some neighborhood of some point on \(\gamma\)

or

(5.3b) \(\limsup_{z \to \zeta} |F(z)| = \infty\) \(\text{ (all } \zeta \in \gamma\)}.

First suppose that (5.3a) occurs i.e. that \(F\) is bounded
in some neighborhood of some point, \(\zeta_0\), of \(\gamma\). Let
\(U = \{|z - \zeta_0| < \delta\} \cap \{|z| < 1\}\) where \(\delta\) is chosen so that
\(F\) is bounded in \(U\). Also let \(\gamma_1 = U^- \cap \{|z| = 1\}\). Choose
an elementary function, \(g(Z)\), so that \(g(Z)\) maps \(|Z| < 1\)
one-to-one conformally onto \(U\). Now \(\gamma_1\) corresponds to a subarc,
\(\gamma_2\), of \(|Z| = 1\). Set

(5.4) \(\psi(Z) = F(g(Z)) = F(z)\)
and

(5.5) \(\psi(Z) = B(g(Z)) = B(z)\).

Then \(\psi(Z)\) and \(\psi(Z)\) are bounded in \(|Z| < 1\) and thus

\[\psi(Z)/\psi(Z) = F(g(Z))/B(g(Z))\]

is a function of bounded characteristic in \(|Z| < 1\).

Hence \(\psi/\psi\) has finite radial limits on a set \(E* \subset \gamma_2\) such that

(5.6) \(m(E*) = m(\gamma_2) > 0\).

Since \(U\) is a Jordan domain each radial limit \(\psi(e^{i\theta})/\psi(e^{i\theta})\)
\((e^{i\theta} \in E*)\) corresponds to a point asymptotic value of \(F\) in \(U\).

Also \(E*\) corresponds to a set \(E \subset \gamma_1\) such that \(m_e(E) > 0\) \((m_e(E)\)
is the exterior measure of \(E)\) since \(g\) is an elementary function.

Now \(E \subset A* \cap \gamma\) so we have

(5.7) \(m(A* \cap \gamma) \geq m_e(E) > 0\).
Now suppose that (5.3b) occurs. Pick two distinct points, \( \zeta_1 \) and \( \zeta_2 \), of \( \gamma \) and two curves \( \Delta(\zeta_1) \) and \( \Delta(\zeta_2) \) such that \( \Delta(\zeta_1) \) and \( \Delta(\zeta_2) \) end at \( \zeta_1 \) and \( \zeta_2 \) respectively and that \( f(z) \) tends to a finite limit on \( \Delta(\zeta_1) \) and \( \Delta(\zeta_2) \) as \( |z| \to 1 \). Since \( |B(z)| \leq 1 \), \( |F| \) is bounded on \( \Delta(\zeta_1) \) and \( \Delta(\zeta_2) \) for \( |z| \) sufficiently near one. Hence we can find a cross-cut \( \gamma_1 \) of \( \{|z| < 1\} \), joining \( \zeta_1 \) and \( \zeta_2 \), on which \( |F| \leq M \).

Now the cross-cut \( \gamma_1 \) and the arc \( \gamma' = (\zeta_1, \zeta_2) \subset \gamma \) bound a domain \( H \). Let \( \chi(s) \) map \( \{|s| \leq 1\} \) one-one onto \( H^- \) such that \( \chi(s) \) is conformal in \( \{|s| < 1\} \) and continuous in \( \{|s| \leq 1\} \). Consider the function, \( F_0(s) \), in \( \{|s| < 1\} \) given by

\[
F_0(s) = F(\chi(s));
\]

We shall show that \( F_0 \in \mathcal{B} \). Let \( \chi^{-1}[\gamma_1] \) and \( \chi^{-1}[\gamma'] \) denote the image by \( \chi^{-1}(s) \) of \( \gamma_1 \) and \( \gamma' \) respectively. It is obvious that \( F_0 \) is bounded on any curve which approaches a point of \( \chi^{-1}[\gamma_1] \). Since \( f \in \mathcal{A}_m \) and \( A_\infty(f) \cap \gamma = \emptyset \), there exists a dense subset of \( \gamma' \) such that there is a curve ending at each point of this subset on which \( f \) has a finite limit as \( |z| \to 1 \). Since \( |B(z)| \leq 1 \), \( F \) is bounded on each of these curves for \( T < |z| < 1 \) where \( T \) depends on the particular curve. Hence we can find a curve ending at each point of the above dense subset of \( \chi^{-1}[\gamma'] \) on which \( F_0 \) is bounded. Thus \( F_0 \in \mathcal{B} \) and by [12, Theorem 1] \( F_0 \in \mathcal{A} \).

Hence there is a dense subset of \( \gamma' \) such that \( F \) has an asymptotic value at each point of this subset. Recall that for any \( \lambda > 0 \) \( L(\lambda) \) denotes \( \{|F| = \lambda\} \). In view of the above it is clear that the following is true:
For any \( \lambda > 0 \) \( H \cap L(\lambda) \) ends at points of \( \gamma' \). \( (L(\lambda) = \{ \lambda = F \}). \)

\( F \) is unbounded in \( H \) by (5.3b). If \( n > M \), \( H \) will contain at least one component of \( \{ z : |F| > n \} \); let these components in \( H \) be \( D_{n,1} \), \( D_{n,2} \), \ldots. Now suppose \( F \) were bounded in every single set \( D_{n,k} \) for \( n > M \) and all \( k \) involved. Then \( D_{n,1} \) would contain at least one \( D_{n+1,1} \), say \( D_{n+1,1} \) which in turn would contain \( D_{n+2,1} \), etc. The domains \( D_{n,1} \supset D_{n+1,1} \supset \ldots \) determine an asymptotic tract of \( F \) with asymptotic value \( \alpha \). The end \( K \) of this tract is part of \( \gamma \). Because of (5.9) (\( F \) is "in \( L_m \) near \( \gamma' \)) it is clear from the proof of Theorem 1 that \( F \) has the asymptotic value \( \alpha \) at each point of \( K \). Since \( |F| = |f \cdot B| \leq |f| \), \( f \) has the asymptotic value \( \alpha \) at each point of \( K \), which contradicts the hypothesis that \( A_\infty(f) \cap \gamma = \emptyset \).

Hence there exists some \( D_{n,k} \) on which \( F \) is bounded; let us denote this domain simply by \( D_0 \). It is not necessary that \( n \) be an integer, and we may assume that \( n \) is such that the level set \( L(n) \) contains no multiple points. Then \( D_0 \) is bounded by various Jordan curves and cross-cuts, \( \Gamma_0 \), on which \( |F| = n \), and by a set \( E_1 \subset \gamma \). Also

(5.10) \[ n < |F(z)| < N \quad (z \in D_0) \]

The set \( E_1 \) is non-void, for otherwise we would have

\[ \lim \sup |F(z)| \leq n \] at every boundary point of \( D_0 \) implying \( |F| \leq n \) in \( D_0 \), which won't fit with (5.10). Now the effect of each Jordan curve in \( \Gamma_0 \) is to punch a hole in \( D_0 \), which makes the connectivity of \( D_0 \) nasty. Add all such holes
to $D_0$ to obtain a domain $D \subset \mathbb{H}$ with the properties: $D$ is simply connected and is bounded by cross-cuts $\Gamma$, on which $|F| = n$, and by $E_1$; also

$$|F(z)| < N \quad (z \in D).$$

Now if $\Gamma$ contains infinitely many cross-cuts, the diameters of these must approach zero by (5.9). It follows readily that the total boundary of $D$ is a Jordan curve. The set $E_1$ contains no arcs because of (5.3b) and (5.11), but we shall prove that a subset of $E_1$ has positive measure.

It is convenient to assume (use a linear transformation on the unit disc if necessary) that $z = 0 \in D$. Let $z = g_1(Z)$ map $\{|Z| < 1\}$ one-one conformally onto $D$ with $g(0) = 0$ and set

$$\psi_1(Z) = F(g_1(Z)) = F(z)$$

and

$$\psi_1(Z) = B(g_1(Z)) = B(z).$$

Then $\psi_1(Z)$ and $\psi_1(Z)$ are bounded in $\{|Z| < 1\}$ and by Fatou's Theorem have radial limits $\psi_1(e^{i\theta})$ and $\psi_1(e^{i\theta})$ (respectively) almost everywhere. Note also that $\psi_1(e^{i\theta}) \neq 0$ for almost all $\theta$. Now $\psi_1(Z)$ is harmonic and may be expressed by the Poisson integral with boundary function $\psi_1(e^{i\theta})$. Since $|\psi_1| > n$ in part of $\{|Z| < 1\}$ (corresponding to $D_0$) it follows that $|\psi_1(e^{i\theta})| > n$ on a set $E^*_{1}$ of positive measure. Thus

$$\frac{\psi_1(Z)}{\psi_1(Z)} = F(g_1(Z))/B(g_1(Z))$$

has finite radial limits on a set $E^*_{2} \subset E^*_{1}$ such that

$$m(E^*_{2}) = m(E^*_{1}) > 0.$$
Since $D$ is a Jordan domain each radial limit corresponds to a point asymptotic value of $f$ in $D$. The set $E^*_2$ maps onto a set $E_2 \subset E_1$ since $|F| = n$ on $\Gamma$.

The same argument as used by MacLane [12, p. 27] shows that $m_0(E_2) \geq m(E^*_2) > 0$. However, since $E_2 \subset A^*$, this implies that $m(A^*_\infty \cap \gamma) > 0$ which completes the proof of Theorem 5.

**Corollary 1.** Let $f$ and $\gamma$ satisfy the hypotheses of Theorem 5, and let $V$ be the set of asymptotic values which occur on $\gamma$. Then $V$ contains a closed set $V_1$ of positive harmonic measure.

**Proof.** This is immediate by applying Privalow's theorem [16, p. 210] either to $\hat{\psi}(Z)/\psi(Z)$ and its angular limits on the set $E^*$ or to $\hat{\psi}_1(Z)/\psi_1(Z)$ and its angular limits on the set $E^*_2$ (depending on whether (5.3a) or (5.3b) occurs).

**Corollary 2.** Let $f \in \mathcal{B}_m$. Suppose $\sup \{N(r, \infty, f); 0 \leq r < 1\} < \infty$, and let $\gamma$ be any subarc of $\{|z| = 1\}$ such that $A^*_\infty \cap \gamma = \emptyset$. Then $m_0(A^*_\infty \cap \gamma) > 0$ ($m_0(A^*_\infty \cap \gamma)$ denotes the exterior measure of $A^*_\infty \cap \gamma$).

**Proof.** The proof is essentially the same as that of Theorem 5.

Corollary 2 will be needed in the proof of Theorem 9.
6. Some Sufficient Conditions for $f \in A_m$. The most important sufficient condition we shall prove in this section is that if $\sup \{N(r,a,f); 0 \leq r < 1\} < \infty$ for some complex number $a$ (possibly $\infty$) and the growth of $T(r)$ is suitably restricted (see Theorem 7 for a precise statement), then $f \in A_m$. We shall first prove the following theorem.

**THEOREM 6.** Let $g(z)$ and $h(z)$ be holomorphic in $\{|z| < 1\}$, and let $g/h$ be non-constant. Suppose $g \in A$ and $h$ is bounded and let $f(z) = g(z)/h(z)$. Then $f \in A_m$ and $1/f \in A_m$.

**Remark.** If we assume Theorem 6, then $f$ satisfies the hypotheses of Theorem 5 ($\sup \{N(r,\infty,f); 0 \leq r < 1\} < \infty$).

Thus for any subarc, $\gamma$, of $\{|z| = 1\}$ such that $A_\infty(f) \cap \gamma = \emptyset$, we have $\text{meas.}(A_\infty(f) \cap \gamma) > 0$.

**Proof.** Consider any subarc, $\gamma$, of $\{|z| = 1\}$. We shall show that there exists a point $\zeta \in \gamma$ and a curve ending at $\zeta$ on which $f$ tends to a limit as $|z| \to 1$. First suppose $A_\infty(g) \cap \gamma \neq \emptyset$. Then there is a $\zeta \in \gamma$ and a curve $\Delta$ ending at $\zeta$ on which $g \to \infty$ as $|z| \to 1$ ($z \in \Delta$). Now,

$$\lim_{|z| \to 1} |f(z)| = \lim_{z \in \Delta} \lim_{|z| \to 1} \frac{|g(z)|}{|h(z)|}$$

$$\approx \lim_{|z| \to 1} \frac{|g(z)|}{M}$$

Since $|h(z)| \neq M$ in $\{|z| < 1\}$. Thus $f$ has the asymptotic value $\infty$ at $\zeta$.
Next suppose \( A_\infty(g) \cap \gamma = \emptyset \). If \( g \) is bounded in some neighborhood of some point of \( \gamma \), the conclusion is a trivial consequence of the Fatou-Nevanlinna Theorem. Thus we may suppose

\[
\limsup_{z \to \zeta} |g(z)| = +\infty \quad (\text{all } \zeta \in \gamma)
\]

Under these hypotheses MacLane has shown [12, p. 26] that there exists a \( D \subset \{|z| < 1\} \) with the properties: \( D \) is a simply-connected Jordan domain bounded by cross-cuts, \( \Gamma \), of \( \{|z| < 1\} \), on which \( |g| = \lambda \) for some \( \lambda > 0 \), and by a non-empty subset, \( F \), of \( \gamma \); also

\[
|g(z)| < N \quad (z \in D)
\]

Moreover,

\[
\lambda < |g(z)| < N \quad (z \in D_0)
\]

where \( D_0 \) is a non-empty subdomain of \( D \).

Now let \( z = G(Z) \) map \( \{|Z| < 1\} \) one-one conformally onto \( D \) and set

\[
\psi(Z) = g(G(Z)) = g(z)
\]

and

\[
\varphi(Z) = h(G(Z)) = h(z)
\]

then \( \psi(Z) \) and \( \varphi(Z) \) are bounded in \( \{|Z| < 1\} \) and by Fatou's Theorem have radial limits \( \psi(e^{i\theta}) \) and \( \varphi(e^{i\theta}) \) (respectively) almost everywhere. Now \( \psi(Z) \) is harmonic and may be expressed by the Poisson integral with boundary function \( \psi(e^{i\theta}) \). Since \( |\psi| > \lambda \) in part of \( \{|Z| < 1\} \) (corresponding to \( D_0 \)), it follows that \( |\psi(e^{i\theta})| > \lambda \) on a set \( E_1 \) of positive measure. Furthermore \( \psi(Z) \) has non-zero radial limits on a
set $E^* = E^*$ such that $m(E^*) = m(E^*)$. Thus $\frac{i}{y}(Z)/\frac{i}{y}(Z)$ has finite radial limits at all points of $E^*$. Since $D$ is a Jordan domain, each radial limit $\frac{i}{y}(e^{i\theta})/\frac{i}{y}(e^{i\theta})$ corresponds to a point asymptotic value of $f$ in $D$. The set $E^*$ maps onto $E \subset F$ since $|y| = 1$ on $\Gamma$, and thus $f$ has asymptotic values at some points of $\Gamma$. Hence, $f \in A_m$, and it is now obvious that $1/f \in A_m$.

The following three sufficient conditions generalize conditions (I), (II), and (III) of MacLane [12, p. 35-37] to meromorphic functions. We shall say that $f$, meromorphic in $\{|z| < 1\}$, satisfies condition (I) if and only if there exists a complex number $a$ (possibly $\infty$) and a set $\delta$ which is dense on $[0, 2\pi)$ such that

\[
\begin{align*}
\sup \{N(r,a); 0 \leq r < 1\} &< \infty \text{ and } \\
\frac{1}{\delta} \int_0^1 (1 - r) \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| \, dr < \infty \quad (\delta \in \mathbb{C})
\end{align*}
\]

(1)

If $a \neq \infty$. If $a = \infty$, the integral condition is

\[
\frac{1}{\delta} \int_0^1 (1 - r) \log^+ |f(re^{i\theta})| \, dr < \infty \quad (\delta \in \mathbb{C}).
\]

No uniformity is implied by the above; we just require that each individual integral converge.

We shall eventually prove (Theorem 7) that if $f$ satisfies (I), then $f \in A_m$. However, let us first look at two other sufficient conditions. The form of (I) suggests a connection between it and the Schmiegungsfunktion of Nevanlinna. In order to find it let
\[(6.3) \quad \sigma(a, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} (1-r) \log^+ \left| \frac{1}{f(re^{i\theta})-a} \right| dr \quad (a \neq \infty) . \]

Then
\[(6.4) \quad \frac{1}{2\pi} \int_0^{2\pi} \sigma(a, \varphi) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 (1-r) \log^+ \left| \frac{1}{f(re^{i\theta})-a} \right| dr d\theta \]
\[= \int_0^1 (1-r)m(r,a)dr \quad (a \neq \infty) \]

for any meromorphic function \(f\). If \(a = \infty\), make the obvious modifications in (6.3) and (6.4).

We shall say that \(f\), meromorphic in \(|z| < 1\), satisfies condition (II) if and only if there exists a complex number \(a\) (possibly \(\infty\)) such that
\[(II) \quad \sup \{N(r, a); 0 \leq r < 1\} < \infty, \text{ and} \]
\[\int_0^1 (1-r)m(r, a)dr < \infty . \]

Since (II) implies that \(\sigma(a, \varphi)\) is finite for almost all \(\varphi\) (see (6.4)), we have
\[(6.5) \quad (II) \Rightarrow (I) . \]

Finally, we shall say that \(f\), meromorphic in \(|z| < 1\), satisfies condition (III) if and only if
\[(III) \quad \sup \{N(r, a); 0 \leq r < 1\} < \infty \text{ and} \]
\[\int_0^1 (1-r)T(r)dr < \infty \]

for some complex number \(a\) (possibly \(\infty\)). If \(a = \infty\), it is clear that (III) \(\Rightarrow\) (II). If \(a\) is finite, using Nevanlinna's First Main Theorem [15, p. 168], we have
\[\int_0^1 (1-r)m(r, a)dr < \infty , \]
and thus

\[(6.6) \quad \text{III} = \text{II}. \]

**THEOREM 7.** Let \( f \) be meromorphic and non-constant in \(|z| < 1\). Suppose that \( f \) satisfies one of the conditions (I), (II), or (III). Then \( f \in \mathbb{A}_m \).

**Proof.** Because of (6.5) and (6.6) it suffices to prove that (I) implies \( f \in \mathbb{A}_m \). First suppose that \( a = \infty \).

As in the proof of Theorem 5, let \( B(z) \) be the Blaschke Product with zeros at the poles of \( f \). Then

\[(6.7) \quad g(z) = B(z)f(z) \]

is holomorphic in \(|z| < 1\), and

\[
\int_0^1 (1-r) \log^+|g(re^{i\theta})|dr = \int_0^1 (1-r) \log^+|B(re^{i\theta})f(re^{i\theta})|dr
\]

\[
\leq \int_0^1 (1-r) \log^+|B(re^{i\theta})|dr + \int_0^1 (1-r) \log^+|f(re^{i\theta})|dr \quad (\theta \in \Theta).
\]

Thus

\[(6.8) \quad \int_0^1 (1-r) \log^+|g(re^{i\theta})|dr \leq \int_0^1 (1-r) \log^+|f(re^{i\theta})|dr \quad (\theta \in \Theta) \]

since \(|B| \leq 1\). Using (6.8) and (I), we have

\[(6.9) \quad \int_0^1 (1-r) \log^+|g(re^{i\theta})|dr < \infty \quad (\theta \in \Theta). \]

Then by [12, Theorem 14], \( g \in \mathbb{A} \), and

\[ f = g/B \]

where \( g \in \mathbb{A} \) and \(|B| \leq 1\). Then by Theorem 6 we see that \( f \in \mathbb{A}_m \).
If \( a \neq \infty \), the above implies that \( 1/(f - a) \in \mathbb{H}_m \), and thus \( f \in \mathbb{H}_m \). Hence, the proof of Theorem 7 is complete.

In conditions (I), (II), or (III) the "global" restrictions on \( f \) may be replaced by the following "local" restrictions. Let \( \zeta \) be such that \( |\zeta| = 1 \), and let \( \delta > 0 \). Define
\[
U(\delta, \zeta) = \{ z : |z - \zeta| < \delta \text{ and } |z| < 1 \},
\]
and define
\[
U^*(\delta, \zeta) = \{ z : |z - \zeta| < \delta \text{ and } |z| \leq 1 \}.
\]
Suppose that a covering,
\[
\{ U^*(\delta(i), \zeta_i) \}_{i \in I}
\]
(where \( I \) is some index set), of \( \{ |\zeta| = 1 \} \) is given, and let
\[
F_i(Z) = f(G_i(Z)) = f(z) \quad (z \in U(\delta(i), \zeta_i))
\]
where \( G_i(Z) \) maps \( \{|Z| < 1\} \) one-one conformally onto \( U(\delta(i), \zeta_i) \). If for each \( i \in I \) \( F_i(Z) \) satisfies one of the conditions (I), (II), and (III), it is easily proved that \( f(z) \in \mathbb{H}_m \). Note in particular that "a" can be different for each \( i \).

[12, Examples 6, 7, 8, 9] show that the implications (6.5) and (6.6) cannot be reversed.

The following example shows that the hypothesis sup \( \{ N(r, a) ; 0 \leq r < 1 \} < \infty \) in (I), (II), and (III) is both essential and best possible.

EXAMPLE 3. We shall show that the hypothesis
\[
\sup \{ N(r, a) ; 0 \leq r < 1 \} < \infty
\]
cannot be relaxed to $\delta(a) = 1$ where $\delta(a)$ is the Nevanlinna defect of $a$ [15, p. 269]. Specifically, we shall construct a function $f$, meromorphic in $\{|z| < 1\}$, such that $T(r, f) \leq \log (1 - r)^{-1}$ ($0 \leq r < 1$), $\delta(\infty) = 1$, and $f \notin A_m$.

By a theorem of MacLane [11] there exists a function $g$, meromorphic in $\{|z| < 1\}$, such that
\begin{equation}
T(r, g) \leq \log \log (1 - r)^{-1} \quad (0 \leq r < 1)
\end{equation}
and such that $g$ has no asymptotic values. Let
\[ h(z) = (1 - z)^{-1} \quad (|z| < 1) \]
and let
\begin{equation}
(6.11) \quad f(z) = g(z) + h(z) \quad (|z| < 1).
\end{equation}
Since $g(z)$ has no asymptotic values, the only point of $\{|z| = 1\}$ at which $f$ can have an asymptotic value is $z = 1$. Thus $f \notin A_m$. Also,
\[ T(r, h) \sim \log (1 - r)^{-1} \quad (r \downarrow 1). \]
Thus
\begin{equation}
(6.12) \quad T(r, f) \sim \log (1 - r)^{-1} \quad (r \downarrow 1),
\end{equation}
and
\begin{equation}
(6.13) \quad N(r, \infty, f) = N(r, \infty, g)
\end{equation}
\[ \equiv T(r, g) \equiv \log \log (1 - r)^{-1} \]
by (6.10). Using (6.12) and (6.13) we have
\[ \delta(\infty, f) = 1 - \limsup_{r \downarrow 1} \frac{N(r, \infty, f)}{T(r, \infty, f)} \]
\[ \equiv 1 - \limsup_{r \downarrow 1} \frac{\log \log (1-r)^{-1}}{\log (1-r)^{-1}} \]
\[ = 1. \]
Thus
\[ \delta(\infty, f) = 1, \]
and Example 3 is complete.

Bagemihl and Seidel [5, Corollary 1] proved that if f is holomorphic and normal (see Definition E) in \( \{|z| < 1\} \), then the set of points at which f has an angular limit is dense on \( \{|z| = 1\} \) i.e. if f is holomorphic, non-constant, and normal in \( \{|z| < 1\} \), then \( f \in A \). This result was proved independently by MacLane [12, p. 43]. Bagemihl [2, Corollary 1] has also proved that if f is meromorphic, normal, and omits at least one value in \( \{|z| < 1\} \), then the set of points at which f has an angular limit is dense on \( \{|z| = 1\} \). We can do somewhat better than that, as follows.

**THEOREM 8.** Let \( f \in H_m \) (see Definition F) and let there exist a complex number a (possibly \( \infty \)) such that
\[ \sup \{N(r, a); 0 \leq r < 1\} < \infty. \]
Then:

1° \( f \in A_m \).

2° If \( |\zeta| = 1 \), then f has at most one asymptotic value at \( \zeta \). Moreover, if f has the asymptotic value b at \( \zeta \), then f has the angular limit b at \( \zeta \).

**Remark.** The hypothesis \( \sup \{N(r, a); 0 \leq r < 1\} < \infty \) is essential because Lehto and Virtanen [9, p. 58] have constructed a normal meromorphic function without any asymptotic values.

**Proof.** Let \( T_s(r) \) denote the spherical characteristic of Schmizu and Ahlfors [15, p. 177]. We have
\[ T_s(r) = \int_{0}^{r} \frac{A(t)}{t} \, dt \quad (0 \leq r \leq 1) \]

where
\[ A(t) = \frac{1}{\pi} \int_{r}^{t} \frac{|f'(re^{i\theta})|^2 r \, dr \, d\theta}{(1 + |f(re^{i\theta})|^2)^2} \quad (0 \leq t < 1). \]

Using Theorem C, we have
\[ A(t) \leq \frac{1}{\pi} \int_{r}^{t} \frac{C^2 r \, dr \, d\theta}{(1 - r^2)^2} \quad (0 \leq r < t < 1), \]

and after an elementary computation we obtain
\[ (6.14) \quad T_s(r) \geq \frac{C^2}{2} \log \frac{1}{1 - r^2} \quad (0 \leq r < 1). \]

Also,
\[ (6.15) \quad T(r) = T_s(r) + O(1), \]

and by (6.14) and (6.15)
\[ (6.16) \quad T(r) \geq \frac{C^2}{2} \log \frac{1}{1 - r^2} + O(1) \]

where \( T(r) \) is the original Nevanlinna characteristic of \( f \).

It is immediate from (6.16) that \( f \) satisfies (III) so \( f \in A_m \). \( 2^0 \) is an obvious consequence of Theorem B.

A Theorem of Tsuji [17] yields another sufficient condition. Only part of the theorem is stated here.

**THEOREM OF TSUJI.** Let \( f(z) \) be meromorphic in \( \{ |z| < 1 \} \) and satisfy
\[ (6.17) \quad \int_{0}^{2\pi} \frac{f'(re^{i\theta}) d\theta}{1 + |f(re^{i\theta})|^2} \leq C \quad (0 \leq r < 1) \]

where \( C \) is a finite constant, so that the circle \( \{ |z| = r \} \) is mapped on a curve of length \( \leq C \) on the sphere.

Then for almost all \( \theta \) the radius \( [0, e^{i\theta}] \) is mapped
onto a rectifiable curve on the sphere, and hence the radial limit of \( f(z) \) exists almost everywhere.

Briefly, if \( f \), meromorphic and non-constant in \( \{ |z| < 1 \} \), satisfies (6.17), then \( f \in \mathbb{A}_m \). The proof of Tsuji's theorem comes from the fact that (6.17) implies

\[
(6.18) \quad \int_0^1 \frac{|f'(re^{i\theta})|dr}{1 + |f(re^{i\theta})|^2} < \infty
\]

for almost all \( \theta \). The condition (6.17) can be weakened to \( f \) satisfies (6.18) for a dense set of \( \theta \). It can be further weakened, as follows.

THEOREM 9. Let \( f(z) \) be meromorphic and non-constant in \( \{ |z| < 1 \} \) and let \( \mu(t) \) be a real positive increasing function on \([0, \infty)\). If

\[
(6.19) \quad \int_0^1 \frac{|f'(re^{i\theta})|dr}{\mu(|f(re^{i\theta})|)} < \infty \quad (\theta \in \Theta)
\]

then the radial limit

\[
\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})
\]

exists for each \( \theta \in \Theta \). If \( \Theta \) is dense in \([0, 2\pi]\), then \( f \in \mathbb{A}_m \).

Remark. This result may be generalized by using a different \( \mu(t) \) for each \( \theta \in \Theta \).

Proof. This theorem with the hypothesis "\( f \) meromorphic" replaced by "\( f \) holomorphic" was proved by MacLane [12, p. 45]. Exactly the same proof works for meromorphic functions so it will not be repeated here.
7. The Classes $A'_m$, $B'_m$, and $L'_m$. The object of this section is to find conditions on the functions in $A'_m$, $B'_m$, and $L'_m$ so that the conclusion of Theorem 3 may be strengthened i.e. under what conditions does $f \in B'_m$ imply $f \in A'_m$, etc.? Also, we shall prove a generalization of Koebe's Lemma (see Theorem A).

We shall say that $f \in A'_m$, $B'_m$, or $L'_m$ if and only if $f \in A'_m$, $B'_m$, or $L'_m$ (respectively) and $\sup \{N(r, \infty, f); 0 \leq r < 1\} < \infty$. The conclusion of Theorem 3 can be improved, as follows.

THEOREM 9. $A'_m = B'_m = L'_m$.

Remark. It is an open question whether or not $B'_m \subset L'_m$. If this were true it would be interesting because it would show that [12, Theorem 1] ($A = B = L$) can be generalized from holomorphic functions to meromorphic functions with $\sup \{N(r, \infty, f); 0 \leq r < 1\} < \infty$.

Proof. It has already been shown that $A'_m \subset B'_m$ and $L'_m \subset B'_m$; see (3.4) and Theorem 3. Thus, in particular, $A'_m \subset B'_m$, $L'_m \subset B'_m$, and we need only show $B'_m \subset A'_m$.

Suppose $f \in B'_m$, and let $\gamma$ be any subarc of $\{|z| = 1\}$. We shall show that $f$ has an asymptotic value at some point of $\gamma$. This is obvious if $A_\infty(f) \cap \gamma = \emptyset$, so we may suppose that $A_\infty(f) \cap \gamma = \emptyset$. Then by Corollary 2 of Theorem 5 we have $m_e(A_\infty^* \cap \gamma) > 0$ (where $m_e(A_\infty^* \cap \gamma)$ denotes the exterior measure of $A_\infty^* \cap \gamma$). Thus $f$ has asymptotic values at lots of points of $\gamma$. Hence $f \in A'_m$, and $B'_m \subset A'_m$. 
Koebe's Lemma was generalized from bounded holomorphic functions to meromorphic functions which omit three values by Gross [7]. Bagemihl and Seidel [5] then proved it for normal meromorphic functions, and MacLane [12, Theorem 9] later generalized it to functions in \( \mathcal{A} \). MacLane's and Bagemihl and Seidel's results overlap, but neither contains the other. We can improve MacLane's result, as follows.

**THEOREM 10.** Let \( f \in \mathcal{A} ' \) and let \( \{ \gamma_n \} \) be a sequence of simple arcs in \( \{ |z| = 1 \} \) which tend to an arc \( \gamma \subset \{ |z| = 1 \} \). Let \( a \) be any finite complex number, and let

\[
(7.1) \quad \gamma_n = \max_{z \in \gamma_n} |f - a|.
\]

Then

\[
(7.2) \quad \mu = \liminf_{n \to \infty} \gamma_n > 0.
\]

**Remarks.** Example 2 shows that some hypothesis other than just \( f \in \mathcal{A} ' \) is necessary for the conclusion of Theorem 10 to be true. However, it will be clear from the proof that the hypotheses of Theorem 10 may be weakened in the following manner: \( f \in \mathcal{A} ' \) may be replaced by \( f \in \mathcal{A} \) and there exists a point \( \zeta \in \gamma \) and a neighborhood, \( U(\delta, \zeta) = \{ z : |z - \zeta| < \delta \text{ and } |z| < 1 \} \), such that

\[
\sup \{ N(R, \infty, f(G(Z)) ; 0 \leq R < 1 \} < \infty.
\]

where \( G(Z) \) maps \( \{ |Z| < 1 \} \) one-one conformally onto \( U(\delta, \zeta) \).

**Proof.** Suppose \( \mu = 0 \). As in the proof of Theorem 5, let \( B(z) \) be the Blaschke Product with zeros at the poles
of \( f \). Then

\begin{equation}
(7.3) \quad g(z) = f(z)B(z) \quad (|z| < 1)
\end{equation}

is holomorphic in \(|z| < 1\). The assumption \( \mu = 0 \) implies that \( A_a(f) \cap Y^0 \) is dense in \( Y^0 \). Let \( \zeta_1, \zeta_2 \in A_a(f) \cap Y^0 \), and let \( \zeta_1 = e^{i\theta_1}, \zeta_2 = e^{i\theta_2} \) with \( \theta_1 < \theta_2 \). Since \( |B(z)| < 1 \), we have

\begin{equation}
(7.4) \quad |g(z)| \leq |f(z)| \quad (|z| < 1).
\end{equation}

Thus we may construct two curves \( \Gamma_1 \) and \( \Gamma_2 \) such that

\( \Gamma_1, \Gamma_2 \subseteq \{|z| < 1\}, \Gamma_1 \) begins at \( z = 0 \) and ends at \( \zeta_1(i = 1, 2), \Gamma_1 \cap \Gamma_2 = 0, \) \( |g| \) is bounded on \( \Gamma_1 \cup \Gamma_2 \), and \( f \) has the asymptotic value \( a \) on \( \Gamma_1(i = 1, 2) \). Let \( \gamma_0 = \{e^{i\theta}: \theta_1 \leq \theta \leq \theta_2 \}, \) and let \( \mathcal{H} \) be the domain bounded by \( \Gamma_1 \cup \Gamma_2 \setminus \gamma_0 \). Now \( \Gamma_1 \) and \( \Gamma_2 \) intersect all but a finite number of the \( \gamma_n \); thus we can find a sequence, \( \{\tau_k\}_{k=1}^\infty \), of crosscuts of \( \mathcal{H} \) such that \( \tau_k \) joins a point of \( \Gamma_1 \) to a point of \( \Gamma_2 \), \( \tau_{k+1} \) separates \( \tau_k \) from \( \gamma_0 \) in \( \mathcal{H} \), for each \( k \) there exists an \( n \) such that \( \tau_k \subset \gamma_n \), and \( \tau_k \setminus \gamma_0 \). (see Fig. 2)

Let \( D_k \) be the subdomain of \( \mathcal{H} \) bounded by \( \tau_k, \tau_{k+1} \), and subarcs of \( \Gamma_1 \) and \( \Gamma_2 \). Using (7.4), applying the maximum principle to \( D_k \), and letting \( k \to \infty \), we have

\[ \lim_{z \to \gamma_0} \sup_{z \in \mathcal{H}^0} |g(z)| \leq |a|. \]
Thus,

\[(7.5) \quad g \text{ is bounded in } H.\]

Now map $H$ one-one and conformally onto $\{|Z| < 1\}$ by $z = G(Z)$. Let

\[\psi(Z) = g(G(Z)) = g(z) \quad (z \in H)\]
\[\gamma(Z) = B(G(Z)) = B(z) \quad (z \in H).\]

By (7.5) $\psi(Z)$ and $\gamma(Z)$ are bounded in $\{|Z| < 1\}$. Hence we have that

\[F(Z) = \frac{\psi(Z)}{\gamma(Z)} = f(z) \quad (z \in H)\]

is the quotient of two bounded holomorphic functions in $\{|Z| < 1\}$ and thus is of bounded characteristic. $\gamma_0$ corresponds to a subarc $\gamma_0'$ of $\{|Z| = 1\}$. Thus by (7.1) and the assumption that $\omega = 0$ (see (7.2)) $F$ has the angular limit $a$ at almost every point of $\gamma_0'$. Then by [15, p. 209] $F \equiv a$ which implies $f \equiv a$. This contradicts $f \in A_m'$ so we must have $\omega > 0$.

Some results on asymptotic tracts of functions in $A_m'$ are stated in section 8. Note that if $f$ satisfies at least one of the conditions (I), (II), or (III) with $a = \infty$ of section 6, then $f \in A_m'$. Also if $f$ satisfies the hypotheses of Theorem 8, then $f \in A_m'$. As a special case of Theorem 5, we see that if $f \in A_m'$, $\gamma$ is any sub-arc of $\{|z| = 1\}$, and $A_m \cap \gamma = \emptyset$, then $\text{meas}(A_m \cap \gamma) > 0$. 
8. Results on asymptotic tracts. Theorems 1 and 2 give information about asymptotic tracts of functions in \( \mathbf{A}_m \). The next few theorems we shall prove give conditions for the existence or non-existence of arc tracts for functions in \( \mathbf{A}_m \). Before we prove them it is interesting to notice that MacLane [12, p. 61] has pointed out that if \( f \in \mathbf{A} \) is unbounded, then the growth of \( M(r) \) has nothing to do with the existence of arc tracts. He constructed a function in \( \mathbf{A} \) with no arc tracts and such that \( M(r) \) can be either arbitrarily small or arbitrarily large. He also constructed a function in \( \mathbf{A} \) with an arc tract and such that \( M(r) \) is arbitrarily large. Another of his examples is of a function in \( \mathbf{A} \) with an arc tract and such that \( M(r) \) is arbitrarily small. However, we shall see (Theorem 12) that conditions on the growth of \( N(r, a, f) \) do influence the non-existence of arc tracts. We first prove a generalization of [12, Theorem 4].

**THEOREM 11.** Let \( f \in \mathbf{A}_m \) and let \( a \) be any complex number (possibly \( \infty \)) such that \( \sup \{N(r, a, f); 0 \leq r < 1\} < \infty \). If \( \{T(\varepsilon), b\} \), \( b \neq a \), is a tract of \( f \), then \( \{T(\varepsilon), b\} \) is a point tract.

**Proof.** Suppose that \( \{T(\varepsilon), b\} \) is an arc tract. First consider the case where \( a = \infty \). Then \( f \in \mathbf{A}_m' \) and we can find an arc \( \gamma \subset \{|z| = 1\} \) and a sequence of continuous compact (in \( \{|z| < 1\} \)) arcs \( \gamma_n \) such that \( \gamma_n \to \gamma \) and

\[
(8.1) \quad \liminf_{n \to \infty} \omega_n = 0 \quad (\omega_n = \max_{z \in \gamma_n} |f(z) - b|).
\]
But (8.1) contradicts Theorem 10, so \( \{T(\varepsilon), b\} \) must be a point tract.

Now consider the case where \( a = \infty \). Then \( g = 1/(f-a) \in \mathbb{A}'_m \) since \( \sup \{N(r, \infty, g); 0 \leq r < 1\} = \sup \{N(r, a, f); 0 \leq r < 1\} < \infty \). Using the same argument as above, we obtain a contradiction. Hence in all cases we see that \( \{T(\varepsilon), b\} \) is a point tract.

Some results about the non-existence of arc tracts are an easy consequence of Theorem 11. The following theorem extends [6, Theorem 4].

**THEOREM 12.** Let \( f \in \mathbb{A}_m \) and suppose there exist two complex numbers \( a, b \) (one of which may be \( \infty \)) such that \( a \neq b \) and \( \sup \{N(r, a, f); 0 \leq r < 1\} < \infty \), \( \sup \{N(r, b, f); 0 \leq r < 1\} < \infty \). Then \( f \) has no arc tracts.

**Proof.** By Theorem 11 all tracts \( \{T(\varepsilon), c\} \) for \( c \neq a \) must be point tracts, and again by Theorem 11 all tracts \( \{T(\varepsilon), d\} \) for \( d \neq b \) must be point tracts. Hence all tracts of \( f \) are point tracts.

**COROLLARY.** Let \( f \in \mathbb{A} \) and suppose \( \sup \{N(r, a, f); 0 \leq r < 1\} < \infty \) for some finite complex number \( a \). Then \( f \) has no arc tracts.

Bagemihl and Seidel [4, Corollary 1 to Theorem 3] proved that a function which is non-constant, meromorphic in \( \{|z| < 1\} \), and normal (see Definition E) has no arc tracts, so in particular a function in \( \mathbb{A}_m \) which is normal has no arc tracts.
The next theorem concerns conditions for the existence of global tracts. It is a generalization of [12, Theorem 6B].

**Theorem 13.** Let \( f \in \mathcal{A}_m \). Then \( f \) has a global tract for \( a \) if and only if on every curve \( \Gamma \) in \( \{ |z| < 1 \} \) on which \( |z|-1 \) is not bounded away from \( a \).

**Proof.** Suppose first that \( a = \infty \). i.e. we shall prove that \( f \) has a global tract for \( \infty \) if and only if \( f \) is unbounded on every curve \( \Gamma \) in \( \{ |z| < 1 \} \) on which \( |z|-1 \).

Suppose that \( f \) has a global tract for \( \infty \). Since \( f \in \mathcal{A}_m \), we know that \( A_\infty \) is dense in \( \{ |z| = 1 \} \). It follows easily that \( f \in \mathcal{L}_m \). Now we shall show that all level curves of \( f \) are compact. Suppose not, then a level curve, \( C(\lambda) \), ends at a point \( \zeta \) of \( \{ |z| = 1 \} \). There exists a subarc, \( \gamma \), of \( \{ |z| = 1 \} \) and a sequence, \( \{ \gamma_n \} \subseteq \gamma \), of continuous compact (in \( \{ |z| < 1 \} \)) arcs such that \( \zeta \in \gamma^{o}, \gamma_n \to \gamma \), and

\[
\lim_{n \to \infty} \inf_{z \in \gamma_n} |f(z)| = \infty.
\]

Then \( C(\lambda) \) must intersect all but a finite number of the \( \gamma_n \), and we have a contradiction \( (|f| = \lambda \text{ on } C(\lambda)) \). Thus all level curves of \( f \) are compact. Hence \( f \) satisfies the hypotheses of Theorem 2, and one can easily prove that (3.8) can't happen. From the proof of (3.9) it is clear that there exists a sequence of closed Jordan curves \( J(\lambda_n) \) such that

\[
J(\lambda_n) = \{ |z| = 1 \} \quad (n \to \infty)
\]

and

\[
\lim_{n \to \infty} \inf_{z \in J(\lambda_n)} |f(z)| = \infty.
\]
Any curve $\Gamma$ on which $|z| - 1$ must cross all but a finite number of the $J(\lambda_n)$ and thus by (8.4) $f$ is unbounded on $\Gamma$.

Now suppose $f$ is unbounded on every curve $\Gamma$ on which $|z| - 1$. This means that $A_\infty$ must be dense in $\{|z| = 1\}$. Again we see that $f \in L_m^\infty$. Also all level curves of $f$ must be compact since $f$ is unbounded on every curve $\Gamma$ on which $|z| - 1$. Therefore $f$ satisfies the hypotheses of Theorem 2. Again (3.8) cannot happen, so (3.9) must happen. It follows from the proof of (3.9) that the arc tract of (3.9) is actually a global tract.

If $a$ is finite, the result follows from applying the above to $1/(f - a)$.

If we recall the generalization of Theorem 2 mentioned at the end of the proof of it, then it is clear that the proof of Theorem 13 may be generalized to yield a corresponding result for arc tracts.

THEOREM 14. Let $f \in A_m^\infty$ and let $\gamma$ be a subarc of $\{|z| = 1\}$. Then $f$ has an arc tract for the value $a$ with end $K \supset \gamma$ if and only if on any curve $\Gamma$ on which $|z| - 1$ and such that $\Gamma^- \cap \gamma^0 \neq \emptyset$ $f$ is not bounded away from $a$.

The following theorem was proved by MacLane [12, Theorem 7] for $f \in A$. Exactly the same proof works for $f \in A_m^\infty$ so it will not be repeated here.

THEOREM 15. Let $f \in A_m^\infty$ and let $[T(\varepsilon), \infty]$ be an arc-tract of $f$ with end $K$. Let $\zeta$ be any point of $K$, let $\delta > 0$, and let
\[ U(\delta, \zeta) = \{ |z| < 1 \} \cap \{ |z - \zeta| < \delta \}. \]

Then:

(A) \( f(z) \) assumes every finite value infinitely many times in \( U(\delta, \zeta) \).

(B) Let \( w = f(z) \) map \( \{ |z| < 1 \} \) onto the Riemann surface \( \mathcal{S} \), a covering of the \( w \)-sphere. For any \( r > 0 \) let the components of \( \mathcal{S} \) over \( \{ |w| < 1 \} \) be \( \Delta(r,1), \Delta(r,2), \ldots \). Let \( G(r,n) \) be the domain in \( \{ |z| < 1 \} \) corresponding to \( \Delta(r,n) \). Then for any given \( r > 0 \) there are infinitely many of the components \( \Delta(r,n) \), say \( \Delta(r,n_k) \), \( k = 1, 2, \ldots \), such that each \( \Delta(r,n_k) \) is relatively compact and \( G(r,n_k) \subseteq U(\delta, \zeta) \) for \( k \leq 1 \).

(C) Each \( T(\varepsilon) \) has infinite connectivity.

Remark. Information about an arc tract \( \{ T(\varepsilon), a \} \) where \( a \) is finite may be obtained by applying Theorem 15 to \( 1/(f - a) \).

We end this section with an extension of [12, Theorem 5].

THEOREM 16. Let \( f \in A_m \) and let \( B^* \) be defined as at the beginning of section 3. Let the open set \( B^{*-1} \) consist of the open arcs \( J_n \). Then each \( J_n^- \) is the end of a single tract \( \{ T(\varepsilon,n), \omega \} \) of the type of Theorem 1. In particular

\[ B^{*-1} \subseteq A_\infty. \]

(8.5)

Here closure (bar) and complement (prime) are relative to \( \{ |z| = 1 \} \).

Proof. \( A_\infty \) must be dense on each \( J_n \), and it is easy to show that \( f \) and each \( J_n \) satisfy the hypotheses of the
of the generalization of Theorem 2 stated at the end of the proof of it. Clearly, (3.8) can't happen on $J_n$, so (3.9) is valid. It is obvious from the definition of the $J_n$ that the end of the tract $\{T(\epsilon,n), \omega\}$ must be precisely $J_n^-$. The analog of (8.5) with $B^*$ replaced by $A^*$ is false, as is seen from [12, Example 15].
References


