BATTEN, Jr., George Washington, 1937–
ITERATIVE SOLUTION OF INTEGRAL EQUATIONS OF FIRST KIND WITH APPLICATIONS TO CONTINUATION PROBLEMS.

Rice University, Ph.D., 1963
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

RICE UNIVERSITY

Iterative Solution of Integral Equations of First Kind with Applications to Continuation Problems

by

George Washington Batten, Jr.

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Thesis Director's signature:

Houston, Texas

May, 1963

TO MY MOTHER,
FATHER, AND WIFE

ACKNOWLEDGEMENTS

Among those to whom I owe many thanks, I am most indebted to Professor Jim Douglas, Jr., my thesis director, who has been my friend and teacher since my undergraduate days, to my wife, who has been a constant source of encouragement during the development of this thesis, and to the two people, who, through their continual aid and encouragement of my scientific interests, have had the most influence on my entering the field of mathematics, my mother and father. I would like to express my appreciation to Humble Oil and Refining Company, through whose generosity I was introduced to applied mathematics, to Phillips Petroleum Company which has supported this research, and to Rice University.

George W. Batten, Jr.

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CHAPTER I

Iterative Procedure

Let L be a linear operator on a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let L have eigenvectors $\Phi_o, \Phi_1, \Phi_2, \ldots,$ with corresponding non zero eigen values $\lambda_o, \lambda_1, \lambda_2, \ldots.$ Assume that $\{\varphi_n\}$ forms a complete orthonormal sequence in H.

Let $g \in H$ and assume that there is a vector $f \in H$ such that

(1) $\quad Lf = g$

Consider the problem of finding f. If

(2) $\quad f = \Sigma_{k=o}^{\infty} \beta_k \Phi_k,$

where the β_k's are scalars, then

(3) $\quad g = Lf = \Sigma_{k=o}^{\infty} \lambda_k \beta_k \Phi_k.$

If the eigenvectors are known, then the solution is immediate: since the eigenvectors are orthogonal,

$$(g, \Phi_j) = \Sigma_{k=o}^{\infty} \lambda_k \beta_k (\Phi_k, \Phi_j) = \lambda_j \beta_j, \quad j = 0,1,\ldots$$

whence

$$f = \Sigma_{k=o}^{\infty} \lambda_k^{-1} (g, \Phi_k) \Phi_k.$$

There are many practical cases, however, for which the eigenvectors are not known, or for which the above procedure is computationally difficult or unstable.

In order to circumvent a priori knowledge of the eigenvectors, observe that the first term of

$$\lambda_o^{-1} g = \beta_o \Phi_o + \Sigma_{k=1}^{\infty} \lambda_o^{-1} \lambda_k \beta_k \Phi_k$$

is just the first term in the expansion of f. Thus, letting

$$e_1 \equiv f - \lambda_o^{-1} g,$$

we have

$$e_1 = \Sigma_{k=1}^{\infty} (1-\lambda_o^{-1} \lambda_k)\beta_k \Phi_k,$$

whence

$$Le_1 = \Sigma_{k=1}^{\infty} (1-\lambda_o^{-1} \lambda_k) \lambda_k \beta_k \Phi_k.$$

Similarly, the first term in

$$\lambda_1^{-1} Le_1 = (1-\lambda_o^{-1} \lambda_k) \beta_1 \Phi_1 + \Sigma_{k=2}^{\infty} (1-\lambda_o^{-1}\lambda_k)\lambda_1^{-1}\lambda_k\beta_k\Phi_k$$

is equal to the first term in the expansion of e_1, so that

$$e_1-\lambda_1^{-1}Le_1 = \Sigma_{k=2}^{\infty} (1-\lambda_o^{-1}\lambda_k)(1-\lambda_1^{-1}\lambda_k) \beta_k\Phi_k.$$

Let

$$e_2 \equiv e_1 - \lambda_1^{-1}Le_1.$$

We continue by an obvious induction step, thus obtaining

$$(4) \quad \begin{cases} e_o = f, \\ e_m = e_{m-1} - \lambda_{m-1}^{-1} Le_{m-1} = \Sigma_{k=m}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} (1-\lambda_\ell^{-1} \lambda_k)\right\} \beta_k \Phi_k, \\ Le_m = \Sigma_{k=m}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} (1-\lambda_\ell^{-1}\lambda_k) \right\} \lambda_k \beta_k \Phi_k, \qquad m = 1,2,\ldots. \end{cases}$$

Now let

$$f_m = \Sigma_{k=o}^{m-1} \lambda_k^{-1} Le_k, \qquad m = 1,2, \ldots.$$

Then

$$(5) \qquad f - f_m = e_m,$$

so that

$$(6) \qquad \|f-f_m\|^2 = \Sigma_{k=m}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} \left| 1-\lambda_\ell^{-1} \lambda_k \right|^2 \right\} \beta_k^{\ 2}, \ m=1,2,\ldots.$$

Clearly, if the λ_k are real, positive, and non-increasing, then

$$\Pi_{\ell=o}^{m-1} \left| 1 - \lambda_\ell^{-1} \lambda_k \right| \leq 1 \qquad \text{for } k \geq m.$$

Hence the right side of (6) tends to zero as m tends to infinity; that is,

$$(7) \qquad \|f-f_m\| \to 0 \quad \text{as} \quad m \to \infty .$$

The quantities $f_1, f_2, \ldots$ depend only on L, g, and $\lambda_o, \lambda_1, \ldots$: clearly

$$f_1 = \lambda_o^{-1} g,$$

and, by (4) and (5),

$$\begin{aligned} f_m &= f - (e_{m-1} - \lambda_{m-1}^{-1} Le_{m-1}) \\ (8) \qquad &= f - (f-f_{m-1}) - \lambda_{m-1}^{-1} L(f-f_{m-1}) \\ &= f_{m-1} + \lambda_{m-1}^{-1} (g-L f_{m-1}). \end{aligned}$$

Thus we have the

<u>Theorem</u>: If the eigenvalues $\lambda_o, \lambda_1, \ldots$ of L are real and satisfy $\lambda_k \geq \lambda_{k+1} > 0$, $k = 0,1,\ldots$, then $f_m \to f$ strongly.

Thus, we have a computational scheme for determining f if the eigenvalues are known. It will be shown that, in (8), the eigenvalues can be replaced by constants which are, to some extent, arbitrary.

For the purpose of generalizing the above results, let

(9) $$e_o = f, \quad e_m = e_{m-1} - c_{m-1} L^{p_{m-1}} e_{m-1}, \quad m=1,2,\ldots,$$

where p_m is a positive integer, L^p is defined recursively by

$$L^o h = h, \quad L^q h = L(L^{q-1} h), \quad q = 1,2,\ldots,$$

and c_m is a scalar, $m = 0,1,\ldots$. Let

(10) $$f_m = \Sigma_{k=o}^{m-1} c_k L^{p_k} e_k, \quad m = 1,2,\ldots.$$

It is easily seen that

(11) $$e_m = f - f_m, \quad m = 1,2,\ldots,$$

and that

(12) $$\begin{cases} f_1 = c_o L^{p_o} f = c_o L^{p_o - 1} g, \\ f_{m+1} = f - (e_m - c_m L^{p_m} e_m) \\ \qquad = f_m + c_m L^{p_m - 1} (g - L f_m) \end{cases}$$

so that $\{f_m\}$ can be obtained from L, g, and $\{c_m\}$.

Furthermore, by definition,

$$e_o = \Sigma_{k=o}^{\infty} \beta_k \Phi_k .$$

For the purpose of induction, assume that

(13) $$e_m = \Sigma_{k=o}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} \left(1 - c_\ell \lambda_k^{p_\ell}\right) \right\} \beta_k \Phi_k ,$$

then

$$\begin{aligned} e_{m+1} &= e_m - c_m L^{p_m} e_m \\ &= \Sigma_{k=o}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} (1 - c_\ell \lambda_k^{p_\ell}) \right\} (1 - c_m \lambda_k^{p_m}) \beta_k \Phi_k \\ &= \Sigma_{k=o}^{\infty} \left\{ \Pi_{\ell=o}^{m} (1 - c_\ell \lambda_k^{p_\ell}) \right\} \beta_k \Phi_k , \end{aligned}$$

so that (13) holds for all $m = 1,2,\ldots$ From (11) and (13) we have

(14) $$\|f-f_m\|^2 = \Sigma_{k=o}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} |1-c_\ell \lambda_k^{p_\ell}|^2 \right\} |\beta_k|^2 , \quad m=1,2,\ldots .$$

Now assume that, for some set $\{p_\ell\}$ of positive integers, the set

$$\{\lambda_i^{p_k}\}$$

lies in a circle, say $\{|z-z_o| < |z_o|\}$, not containing the origin. If

$$c_\ell = \frac{1}{2} z_o^{-1} \rho_\ell ,$$

where $0 < \rho_\ell \leq 1$, $\ell = 0,1,\ldots,$ then

$$|1 - c_\ell \lambda_k^{p_\ell}| = |1 - \frac{1}{2} z_o^{-1} \rho_\ell \lambda_k^{p_\ell}| \leq$$

$$\leq |2z_o|^{-1} (|2z_o - z_o| + |z_o - \rho_\ell \lambda_k^{p_\ell}|)$$

$$< 1, \quad \text{for } k,\ell = 0,1,\ldots.$$

Hence, for each $k = 0,1,\ldots,$

$$\alpha_{km} = \Pi_{\ell=o}^{m-1} |1-c_\ell \lambda_k^{p_\ell}|$$

is a decreasing function of m. If, in addition, α_{km} tends to zero as $m \to \infty$, then the right side of (14) tends to zero. For this to happen, it is sufficient to assume that $\{c_\ell\}$ and $\{\rho_\ell\}$ take on only a finite number of values, for then

$$|1 - c_\ell \lambda_k^{p_\ell}| \leq M_k < 1, \ \ell = 0,1,\ldots,$$

so that

$$\alpha_{km} \leq M_k^m \to 0 \quad \text{as} \quad m \to \infty.$$

Of course, from the standpoint of making the process converge as rapidly as possible, it may be desirable to let the constants c_ℓ grow unboundly as $\ell \to \infty$; this happens, for example, in equation (8). In fact, the situation leading to (8) appears in a more general setting. Suppose that

$$|\lambda_o| \geq |\lambda_1| \geq \ldots,$$

and that $\{\lambda_k^{p_\ell}\}$ is contained in an angular sector $\theta_1 \leq \theta \leq \theta_2$, where $|\theta_1 - \theta_2| \leq \frac{1}{6}\pi$. Then it is sufficient to take

$$c_\ell = \lambda_\ell^{-p_\ell},$$

for then

$$c_\ell \quad \lambda_k^{p_\ell}$$

is contained in the region

$$D = \{z \mid |z| \le 1, \ |\arg z| \le \tfrac{1}{6}\pi\}$$

provided $k \ge \ell$. Thus

$$|1 - c_\ell \lambda_k|^{p_\ell} \le 1$$

for $k \ge \ell$. It follows that α_{km} vanishes for $k \le m-1$, and is bounded by 1 if $k \ge m$. Thus, by (14),

$$\|f-f_m\|^2 \le \Sigma_{k=m}^{\infty} |\beta_k|^2 \to 0 \text{ as } m \to \infty.$$

Particular examples of this situation are treated in later sections.

We summarize the results in

<u>Theorem 1.</u> Let L have a complete orthogonal set of eigenvectors with corresponding eigenvalues $\lambda_o, \lambda_1, \ldots$ Let $\{p_\ell\}$ be a sequence of positive integers. If $\{\lambda_k^{p_\ell}\}$ is contained in a circle not containing the origin, then there exists a sequence $\{c_\ell\}$ of numbers such that

$$\|f - f_m\| \to 0 \qquad \text{as} \quad m \to \infty,$$

where

$$Lf = g$$

and $f_1, f_2, \ldots$ are given by equations (12).

If, in addition

(14) $$|\lambda_o| \geq |\lambda_1| \geq \ldots,$$

and

$$\{\lambda_k^{p_\ell}\} \subset D,$$

then it is sufficient to take

$$c_\ell = \lambda_\ell^{-p_\ell}, \qquad \ell = 0,1,\ldots.$$

In this case,

$$\|f-f_m\| \leq \Sigma_{k=m}^{\infty} |\beta_k|^2.$$

In particular, for the applications to be presented later, the following theorem is sufficient.

Theorem 2: Suppose that the eigenvalues are bounded and non zero, and that there exist a positive integer j, and numbers α and ϵ, $0 \leq \alpha < 2\pi, 0 < \epsilon < \pi$, such that all of the eigenvalues lie in one of the sectors

$$\alpha + \frac{2\pi i}{j} - \frac{\pi-\epsilon}{2j} \leq \arg \lambda \leq \alpha + \frac{2\pi i}{j} + \frac{\pi-\epsilon}{2j}, \quad i=0,1,\ldots,j-1.$$

Then if $p_\ell \equiv j$, there exists a sequence $\{c_\ell\}$ such that

$$\|f - f_m\| \to 0.$$

If

(15) $$|\lambda_o| \geq |\lambda_1| \geq \ldots,$$

and if every λ_k lies in one of the sectors

$$\frac{2\pi i}{j} - \frac{\pi}{6j} \leq \arg \lambda \leq \frac{2\pi i}{j} + \frac{\pi}{6j},$$

then it is sufficient to take

$$p_\ell \equiv j, \quad c_\ell = \lambda_\ell^{-j}, \qquad \ell = 0,1,\ldots.$$

In this case

$$\|f-f_m\| \leq \Sigma_{k=m}^{\infty} |\beta_k|^2 .$$

<u>Proof</u>: The first statement follows from Theorem 1 and the inequalities

$$\alpha j + \frac{\pi+\varepsilon}{2} \leq \arg \lambda_k^j \leq \alpha j + \frac{\pi-\varepsilon}{2}, \qquad k = 0,1,\ldots.$$

The second statement follows from

$$-\frac{\pi}{6} \leq \arg \lambda_k^j \leq \frac{\pi}{6}, \qquad k = 0,1,\ldots.$$

In particular, if the eigenvalues are real, take $j = 2$, if they are positive, take $j = 1$.

Remark: Theorems 1 and 2 also hold if conditions (14) and (15) are replaced by

$$(16) \qquad |\lambda_m| \geq |\lambda_{m+1}| \geq \ldots$$

for some m sufficiently large.

These results can be extended to certain operators with a continuous spectrum. Suppose that L has the spectral representation

$$L = \int_{-\infty}^{\infty} \varphi(t)\, d\, E_t,$$

where E_t is a partition of unity, and $\varphi(t)$ is a convenient function. Further, assume that

$$L^q = \int_{-\infty}^{\infty} \varphi^q(t)\, d\, E_t, \qquad q = 2,3,\ldots.$$

Then, from equations (9),

$$e_o = f = \int_{-\infty}^{\infty} d\, E_t\, f,$$

$$e_1 = f - c_o\, L^p\, f$$

$$= \int_{-\infty}^{\infty} (1-c_1\, \varphi^p(t))\quad d\, E_t\, f.$$

For the purpose of induction, assume that

$$(16) \qquad e_m = \int_{-\infty}^{\infty} \Pi_{\ell=o}^{m-1}\ (1 - c_\ell\, \varphi^{p_\ell}(t))\, d\, E_t f.$$

From (9),

$$e_{m+1} = e_m - c_m\, L^{p_m}\, e_m$$

$$= \int_{-\infty}^{\infty} \left[1-c_m\, \varphi^{p_m}(t)\right] d\, E_t\, e_m.$$

By (16),

$$E_t e_m = \int_{-\infty}^{t} \Pi_{\ell=o}^{m-1}\ (1 - c_\ell\, \varphi^{p_\ell}(\tau))\ d\, E_\tau f.$$

Hence,

$$e_{m+1} = \int_{-\infty}^{\infty} \Pi_{\ell=o}^{m}(1-c_\ell\, \varphi^{p_\ell}(t))\quad d\, E_t f.$$

Therefore, (16) holds for $m = 1,2,\ldots.$ Thus,

$$\|e_m\|^2 = \int_{-\infty}^{\infty} \Pi_{\ell=o}^{m} \left| 1-c_\ell\, \varphi^{p_\ell}(t)\right|^2 d(E_t f,f).$$

It follows that $\|e_m\| \to 0$ if

$$\prod_{\ell=0}^{m-1} |1-c_\ell \, \varphi^{p_\ell}(t)| \downarrow 0$$

for all t as $m \uparrow \infty$. In particular, if φ is real and bounded, then it is possible to find $\{(c_\ell, p_\ell)\}$ such that $\|e_m\| \to 0$.

CHAPTER II

Stablity of the Iteration

Let $g^* = g + \gamma$, where g is defined in section I, and $\|\gamma\| < \delta$ for some $\delta > 0$. Consider applying iteration (I-12) to g^* instead of g. Denote the sequence of functions thus obtained by $\{f_m^*\}$. The function f_m^* can be thought of as an "approximation" to f_m. Indeed, if $\gamma = 0$, then $f_m^* = f_m$; in this case $f_m^* \to f$ if $f_m \to f$. On the other hand, if $\gamma \neq 0$, then it does not necessarily happen that $\{f_m^*\}$ converges to something close to f, or even that $\{f_m^*\}$ converges.

Consider, for example, the operator L on $\mathfrak{L}^2(0,2\pi)$ which is defined by the characteristic equations

$$\begin{cases} L \sin k\theta = a^k \sin k\theta, & k = 1,2,\ldots, \\ L \cos k\theta = a^k \cos k\theta, & k = 0,1,2,\ldots, \end{cases}$$

where $0 < a < 1$ (L is the operator which appears in the problem of harmonic continuation to be considered in Section III). Let $g = 0$, and let

$$\gamma(\theta) = \Sigma_{k=o}^{\infty} (\lambda_k \sin k\theta + \mu_k \cos k\theta),$$

$$0 \leq \theta < 2\pi.$$

In section III it will be shown that if $p_\ell \equiv c_\ell \equiv 1$, $\ell = 0,1,\ldots,$ then

$$\|f-f_m\| \to 0,$$

where $\|\cdot\|$ is the norm on $\mathfrak{L}^2(0,2\pi)$ (in the present situation, this statement is trivial since $f = f_m = 0$; however, the statement is also true for any g for which $Lf = g$ has a solution). However, we also have

$$\begin{cases} f^*_1 = \gamma \\ f^*_{m+1} = \gamma + f_m^* - L f_m^*. \end{cases}$$

Suppose that

$$f_m^* = \Sigma_{k=o}^{\infty} (\lambda_k \rho_{mk} \sin k\theta + \mu_k \sigma_{mk} \cos k\theta),$$

$$m = 1, 2, \ldots,$$

where the ρ_{mk} and the σ_{mk} are constants. Then

$$f^*_{m+1}(\theta) = \Sigma_{k=o}^{\infty} \Big\{\lambda_k(1 + \rho_{mk} - a^k \rho_{mk}) \sin k\theta +$$

$$+ \mu_k(1 + \sigma_{mk} - a^k \sigma_{mk}) \cos k\theta\Big\}$$

$$= \Sigma_{k=o}^{\infty} (\lambda_k \rho_{m+1,k} \sin k\theta + \mu_k \sigma_{m+1,k} \cos k\theta).$$

As this holds for all values of θ between 0 and 2π, we have $\rho_{mk} \equiv \sigma_{mk}$ satisfies

$$\begin{cases} \rho_{1k} = 1, \\ \rho_{m+1,k} = 1 + \rho_{mk}(1-a^k). \end{cases}$$

Consider

$$\rho_k = 1 + \rho_k(1-a^k),$$

or

$$\rho_k = a^{-k}.$$

We have

$$\rho_k - \rho_{m+1,k} = (\rho_k - \rho_{mk})(1-a^k),$$

whence

$$\rho_k - \rho_{mk} = (a^{-k}-1)(1-a^k)^{m-1} > 0, \quad m = 1,2,\ldots,$$

since $0 < a < 1$. Thus ρ_{mk} increases to a^{-k} as m increases to infinity.

Assume that γ is not analytic. Then

$$\Sigma_{k=o}^{\infty} (\lambda_k^2 + \mu_k^2)\, a^{-2k}$$

diverges, for otherwise, for sufficiently large k,

$$|\lambda_k|, |\mu_k| < a^k,$$

whence

$$g(z) = \Sigma_{k=o}^{\infty} (\mu_k - i\, \lambda_k) z^k$$

is harmonic in $\{|z| < a^{-1}\}$, and

$$\gamma(\theta) = \mathcal{R}\, g(ae^{i\theta})$$

is analytic, which is a contradiction. From this it follows that

$$\|f_m^*\|^2 = \Sigma_{k=o}^{\infty} \rho_{mk}^{\ 2} (\lambda_k^2 + \mu_k^2) \rightarrow \infty \text{ as } m \rightarrow \infty.$$

Thus $\{f_m^*\}$ is divergent.

Hence we see that the iteration is <u>not</u> stable in the sense that the error

$$\|f - f_m^*\|$$

can be made arbitrarily small by taking δ sufficiently small and m sufficiently large. The situation is not hopeless, however. Indeed, f_m^* is an asymptotic approximation to f as given by the following theorem.

Theorem 3: If L is bounded, if

$$\|f-f_m\| \to 0 \quad \text{as} \quad m \to \infty,$$

where f_m is given by (I-12), if f_m^* is given by (I-12) with g replaced by g^*, and if

$$\|g - g^*\| < \delta,$$

then there exists a number $M(\delta)$, which tends to infinity as $\delta \to 0$, such that

$$\|f - f_m^*\| \to 0 \qquad \text{as } \delta \to 0$$

if $m \le M(\delta)$ and $m \to \infty$ as $\delta \to 0$.

Proof: According to equations (I-12),

$$(1)\quad \begin{cases} f_1 - f_1^* = c_o L^{p_o-1}(g-g^*), \\ f_{m+1} - f_{m+1}^* = (f_m-f_m^*) + c_m L^{p_m-1}\left[(g-g^*) - L(f_m-f_m^*)\right], \end{cases}$$

or

$$(2)\quad \begin{cases} \|f_1-f_1^*\| \le |c_o|\, \|L\|^{p_o-1} \|g-g^*\| < |c_o|\, \|L\|^{p_o-1}\,\delta, \\ \|f_{m+1}-f_{m+1}^*\| \le \|f_m-f_m^*\| + |c_m|\, \|L\|^{p_m-1} \|g-g^*\| + \|L\|\, \|f_m-f_m^*\| \\ \qquad < |c_m|\, \|L\|^{p_m-1}\,\delta + (1+|c_m|\, \|L\|^{p_m})\, \|f_m-f_m^*\| \\ \qquad = K_{m+1}\,\delta, \end{cases}$$

Where K_{m+1} is some constant. Thus,

$$\|f-f_m^*\| \leq \|f-f_m\| + \|f_m-f_m^*\| < \|f-f_m\| + K_m \delta.$$

We therefore have an asymptotic approximation. Choose $M(\delta)$ as large as possible so that

$$K_m \delta \leq \|f-f_m\| \qquad \text{if } m \leq M(\delta).$$

Then $M(\delta) \to \infty$ as $\delta \to 0$. Thus, $\|f-f_m\|$, and <u>a fortiori</u> $\|f-f_m^*\|$, can be made arbitrarily small if δ is sufficiently small. QED.

Note that, in general, $\{K_m\}$ is not bounded. In fact, in the examples given in the following sections, $K_m \to \infty$ as $m \to \infty$.

Equation (2) does not give the best possible bound on $\|f_m-f_m^*\|$. Suppose that γ has the representation

$$(3) \qquad \gamma = \Sigma_{k=o}^{\infty} \lambda_k \gamma_k \Phi_k.$$

Then, by (1)

$$f_1-f_1^* = c_o \Sigma_{k=o}^{\infty} \lambda_k^{p_o-1} \lambda_k \gamma_k \Phi_k$$

$$= \Sigma_{k=o}^{\infty} \left[1 - (1-c_o \lambda_k^{p_o})\right] \gamma_k \Phi_k.$$

By indiction, it follows that

$$(4) \qquad f_m-f_m^* = \Sigma_{k=o}^{\infty} \left[1 - \Pi_{\ell=o}^{m-1} (1-c_\ell \lambda_k^{p_\ell})\right] \gamma_k \Phi_k,$$

whence

$$\|f_m-f_m^*\|^2 = \Sigma_{k=o}^{\infty} |\lambda_k^{-1}(1-\Pi_{\ell=o}^{m-1}(1-c_\ell\lambda_k^{p_\ell}))|^2 |\lambda_k|^2 |\gamma_k|^2.$$

Now

$$1 - \Pi_{\ell=o}^{m-1} (1 - c_\ell \lambda_k^{p_\ell}) = \lambda_k P_m(\lambda_k),$$

where P_m is a polynominal. Thus,

$$\lambda_k^{-1} \left[1 - \Pi_{\ell=o}^{m-1} (1 - c_\ell \lambda_k^{p_\ell})\right] = P_m(\lambda_k)$$

is bounded for fixed m. Let

(5) $$K_m = \max_k \left| \lambda_k^{-1} \left[1 - \Pi_{\ell=o}^{m-1} (1 - c_\ell \lambda_k^{p_\ell})\right] \right|.$$

Then

$$\|f_m - f_m^*\|^2 \leq K_m^2 \Sigma_{k=o}^{\infty} |\lambda_k \gamma_k|^2$$
$$= K_m^2 \|\gamma\|^2 ,$$

whence

(6) $$\|f_m - f_m^*\|^2 < K_m \delta,$$

and we have

(7) $$\|f - f_m^*\| \leq \|f - f_m\| + K_m \delta.$$

CHAPTER III

APPLICATION TO HARMONIC CONTINUATION

$\mathcal{L}^2$ Convergence

Let $u(r,\theta)$ be harmonic in the disk $S_1 = \{|z| < 1\}$. Let K_1 denote the boundary of S_1, and let K_a denote the circle $\{|z| = a\}$.

Consider the problem of harmonic continuation: given the values $u(a,\theta) = g(\theta)$ of u on K_a, where $a < 1$, determine u throughout S. It is well known that this problem is not well posed in the sense of Hadamard; that is, given $\varepsilon > 0$ and $M > 0$, there is a function u_o such that $|u-u_o| \le \varepsilon$ on K_a, but $|u-u_o| > M$ at points near K_1, as can be seen by considering the function $u_o = u+v_{\varepsilon,n}$, where

$$v_{\varepsilon,n} = \varepsilon \left(\frac{r}{a}\right)^n \sin n\theta.$$

This improperly posed problem can be changed into a well posed problem by suitably restricting the class of harmonic functions under consideration. For this purpose, assume that u is continuous in $\{|z| \le 1\}$, and that

(1) $$\|u\|_r \le M$$

for $0 \le r \le 1$, where M is a constant, and

(2) $$\|u\|_r = \left\{\int_o^{2\pi} [u,(r,\theta)]^2 d\theta\right\}^{\frac{1}{2}}.$$

John [3] has shown that under these conditions the solution of the harmonic continuation problem is stable. We now apply the results of the preceding sections to this problem.

Let L denote the linear operator which maps the $\mathcal{L}^2(0,2\pi)$ function $u(1,\theta)$ into $u(a,\theta)$. Since $u(r,\theta)=r^n \sin n\theta$ and $u(r,\theta) = r^n \cos n\theta$ are harmonic functions, it is seen that the eigenfunctions and the corresponding eigenvalues for L are

$$(3) \qquad \begin{cases} (2\pi)^{-\frac{1}{2}}, & 1, \\ \pi^{-\frac{1}{2}} \sin k\theta, & a^k, \qquad k = 1,2,\ldots, \\ \pi^{-\frac{1}{2}} \cos k\theta, & a^k, \qquad k = 1,2,\ldots, \end{cases}$$

Moreover, the eigenfunctions form a complete orthornormal family in $\mathcal{L}^2(0,2\pi)$, and the eigenvalues are real, positive, and bounded. Therefore, the results of sections I and II apply with $\rho_m = 1$. In particular, from Theorem 1 it follows that the iterates

$$(4) \qquad \begin{cases} u_1(1,\theta) = u(a,\theta), \\ u_m(1,\theta) = u_{m-1}(1,\theta) + c_{m-1}[u(a,\theta)-L\, u_{m-1}(1,\theta)], \end{cases}$$

converge to $u(a,\theta)$ for some $\{c_\ell\}$. Indeed, it is sufficient to take

$$(5) \qquad 0 < c_\ell = c = \text{const.} < 2, \qquad \ell = 0,1,\ldots,$$

for in this case,

$$\Pi_{\ell=1}^{m-1} |1 - c_\ell\, a^k| = |1 - c\, a^k|^{m-1} \downarrow 0 \quad \text{as} \quad m \uparrow \infty,$$

and, by equation (I-14),

$$(6) \quad \|u-u_m\|_1^2 = \Sigma_{k=o}^{\infty} \left\{ \Pi_{\ell=o}^{m-1} |1-c_\ell a^k| \right\} (\alpha_k^2 + \beta_k^2) \downarrow 0 \text{ as } m\uparrow\infty,$$

where

$$(7)\quad u(1,\theta) = (2\pi)^{-\frac{1}{2}}(\alpha_o+\beta_o) + \Sigma_{k=1}^{\infty}\ \pi^{-\frac{1}{2}}(\alpha_k \sin k\theta + \beta_k \cos k\theta).$$

According to Theorem 1, another choice of $\{c_k\}$ is

$$(8)\qquad c_\ell = a^{-\ell}, \qquad \ell = 0,1,\ldots.$$

In this case,

$$(9)\qquad \|u-u_m\|_1^2 \le \Sigma_{k=m}^{\infty} (\alpha_k^2 + \beta_k^2).$$

Now assume, as in section II, that the function $u(a,\theta)$ is not known exactly, but that the computations are done using

$$u(a,\theta) + \gamma(\theta),$$

where

$$(10)\qquad \|\gamma\| \equiv \left(\int_o^{2\pi} \gamma^2(\theta)\, d\theta\right)^{\frac{1}{2}} < \delta.$$

According to equation (II-6), for $\{c_\ell\}$ given by (5) we have

$$(11)\qquad \|u-u_m^*\| \le \left[\Sigma_{k=o}^{\infty}(1-c\ a^k)^{2m}(\alpha_k^2 + \beta_k^2)\right]^{\frac{1}{2}} + K_m\ \delta,$$

where

$$(12)\qquad K_m = \max_k \left\{a^{-k}\left(1-(1-c\ a^k)^m\right)\right\}.$$

If $\{c_\ell\}$ is given by (8), then

$$(13)\qquad \|u-u_m^*\| \le \left[\Sigma_{k=m}^{\infty} (\alpha_k^2 + \beta_k^2)\right]^{\frac{1}{2}} + K'_m\ \delta,$$

where

$$(14) \qquad K'_m = \max_k \left\{ a^{-k} \left[1 - \Pi_{\ell=0}^{m-1} (1 - c_\ell \, a^k) \right] \right\};$$

here

$$a^{-k} \left[1 - \Pi_{\ell=0}^{m-1} (1 - c_\ell \, a^k) \right] = \begin{cases} a^{-k} & \text{if} \quad k \le m-1, \\ a^{-k} \left[1 - \Pi_{\ell=0}^{m-1} (1 - a^{k-\ell}) \right], & \text{if } k \ge m. \end{cases}$$

By Taylor's Theorem, if p and q are positive integers, then, for any constant c_*,

$$1 - (1 - c_* a^p)^q = q \, c_* \, a^p - \tfrac{1}{2} q(q-1) \, c_*^2 \, a_o^{2p},$$

where $0 < a_o < a$, whence

$$a^{-p} \left[1 - (1 - c_* \, a^p)^q \right] \le q \, c_*.$$

Thus,

$$K_m \le m \, c,$$

and

$$K'_m \le \max (a^{1-m}, \, m a^{1-m}) < m a^{-m},$$

since

$$a^{-k} \left[1 - \Pi_{\ell=0}^{m-1} (1 - a^{k-\ell}) \right] \le a^{-k} \left[1 - (1 - a^{1+k-m})^m \right]$$

$$\le a^{-k} \, m \, a^{1-m+k}$$

$$= m \, a^{1-m}.$$

Thus

$$(15) \qquad \|u-u_m^*\| \le \left[\Sigma_{k=o}^{\infty} |1-ca^k|^{2m} (\alpha_k^2 + \beta_k^2)\right]^{\frac{1}{2}} + m\, c\, \delta$$

if $\{c_\ell\}$ is given by (5), and

$$(16) \qquad \|u-u_m^*\| \le \Sigma_{k=m}^{\infty} (\alpha_k^2 + \beta_k^2)^{\frac{1}{2}} + m\, a^{-m}\, \delta$$

if $\{c_\ell\}$ is given by (8).

In order to obtain some estimates for the rate of convergence of the iteration, it is necessary to make some additional assumptions about the rate of convergence of

$$\Sigma_{k=1}^{\infty} (\alpha_k^2 + \beta_k^2).$$

Such assumptions are essentially assumptions about the behavior of the high order derivatives of u near $r = 1$. Thus, we will restrict our attention to functions which are quite smooth near the boundary.

Assume, for example, that

$$(17) \qquad \alpha_k^2 + \beta_k^2 \le M\, k^{-1-\lambda} \;,$$

where M and λ are positive constants. Then the first term on the right side of (16) satisfies

$$(18) \qquad \begin{aligned} \Sigma_{k=m}^{\infty} (\alpha_k^2 + \beta_k^2) &\le \Sigma_{k=m}^{\infty} M\, k^{-1-\lambda} \\ &\le M \int_{m-1}^{\infty} x^{-1-\lambda}\, dx \\ &= M\, \lambda^{-1}(m-1)^{-\lambda} \\ &\le M\, m^{-\lambda}, \end{aligned}$$

where M is a generic for a positive constant. Moreover, for any positive integer k_o, the first term on the right side of (15) satisfies

$$\sum_{k=o}^{\infty} |1-ca^k|^{2m} (\alpha_k^2+\beta_k^2) \le \sum_{k=o}^{k_o} |1-ca^k|^{2m} (\alpha_k^2+\beta_k^2) + \\ + \sum_{k=k_o+1}^{\infty} |1-ca^k|^{2m}(\alpha_k^2+\beta_k^2) \\ \le M \sum_{k=o}^{k_o} |1-ca^k|^{2m} + \sum_{k=k_o+1}^{\infty} (\alpha_k^2 + \beta_k^2).$$

For sufficiently large k_o,

(19) $$\sum_{k=o}^{k_o} |1-ca^k|^{2m} < (k_o+1) |1-ca^{k_o}|^{2m} ,$$

so that

(20) $$\sum_{k=o}^{\infty} |1-ca^k|^{2m} (\alpha_k^2 + \beta_k^2) \le M\left[(k_o+1) |1-ca^{k_o}|^{2m} + \\ + k_o^{-\lambda}\right].$$

The integer k_o can be chosen to minimize the last expression, but the effort does not seem to be justified since the estimates are rather crude. Instead, we will choose k_o to make the two terms approximately equal. Given $\varepsilon > 0$, which is so small that (19) is satisfied by k_o given by

$$M\, k_o^{-\lambda} \le \tfrac{1}{8}\,\varepsilon^2 < M(k_o-1)^{-\lambda} ,$$

which is equivalent to

(21) $$k_o-1 < \left(\frac{\varepsilon^2}{8M}\right)^{-\frac{1}{\lambda}} \le k_o.$$

If m is such that

(22) $$M(k_o+1)\ |1-c\ a^{k_o}|^{2m} \le \frac{1}{8}\varepsilon^2 < M(k_o+1)\ |1-c\ a^{k_o}|^{2(m-1)},$$

then the right side of (20) is no greater than $\frac{1}{4}\varepsilon^2$. Thus, for this choice of m,

(23) $$\Sigma_{k=o}^{\infty} |1-c\ a^k|^{2m} \left(\alpha_k{}^2 + \beta_k{}^2\right) \le \frac{1}{4}\varepsilon^2 .$$

Hence,

$$\|u-u_m^*\|_1 \le \frac{1}{2}\varepsilon + m\ c\ \delta.$$

Thus, for

(24) $$\delta < \frac{\varepsilon}{2m\delta},$$

we have

(25) $$\|u-u_m^*\|_1 < \varepsilon.$$

From (21) and (22), it is easily seen that, as $\varepsilon \to 0$, the number of iterations, m, required to reduce the error to less than ε tends to infinity like

(26) $$a^{-M}\ \varepsilon^{-\frac{2}{\lambda}}.$$

As m increases with decreasing c, it is desirable to choose c close to but less than 2 (the process does not necessarily converge for $c \ge 2$).

If $\{c_\ell\}$ is given by (8), then from (16) and (18) we have

$$\|u-u_m^*\|_1 \le M\ m^{-\frac{1}{2}\lambda} + m\ a^{2-m}\ \delta,$$

where, as before, M is a generic for a constant. Now given δ, choose m so that the two terms are approximately equal. For example, choose m such that

$$(27) \qquad M(m+1)^{-\frac{1}{2}\lambda} < m\, a^{2-m}\, \delta \le M\, m^{-\frac{1}{2}\lambda} .$$

Then

$$(28) \qquad \|u-u_m^*\|_1 \le M\, m^{-\frac{1}{2}\lambda} .$$

Thus, in order to make $\|u-u_m^*\| < \epsilon$, it is sufficient to take m and δ such that (17) is satisfied and

$$M\, m^{-\frac{1}{2}\lambda} < \epsilon ;$$

i.e.,

$$(29) \qquad m > (\tfrac{\epsilon}{M})^{-\frac{2}{\lambda}} .$$

Note that the rate of convergence indicated by (29) is much greater than that indicated by (26).

In many cases, the bound (17) can be replaced by one which tends to zero much faster. For example, if u is harmonic in the disk $S_R = \{|z| \leqslant R\}$, where $R > 1$, then the coefficients satisfy

$$(30) \qquad \alpha_k^{\,2} + \beta_k^{\,2} \le M\, R^{-2k} .$$

In this case,

$$(31) \qquad \Sigma_{k=m}^{\infty} (\alpha_k^{\,2} + \beta_k^{\,2}) \le M\, R^{-2m} ,$$

so that in place of (20) we have

$$(32)\quad \Sigma_{k=o}^{\infty} |1-c\ a^k|^{2m} (\alpha_k^2 + \beta_k^2) \le M \left[(k_o+1)|1-c\ a^{k_o}|^{2m} + \right.$$
$$\left. + R^{-2k_o}\right],$$

Again we choose k_o small enough so that the terms in brackets are approximately equal. Suppose $\epsilon > 0$ is so small that (19) is satisfied by k_o given by

$$M\ R^{-2k_o} \le \frac{1}{8}\epsilon^2 < M\ R^{-2(k_o-1)} ,$$

or

$$(33)\qquad k_o-1 < \frac{\log \frac{\epsilon^2}{8M}}{2 \log R} \le k_o .$$

If m satisfies (22), then the right side of (20) is less that $\frac{1}{4}\epsilon^2$, so that

$$(34)\qquad \Sigma_{k=o}^{\infty} |1-c\ a^k|^{2m} \left(\alpha_k^2 + \beta_k^2\right) \le \frac{1}{4}\epsilon^2.$$

Hence,

$$(35)\qquad \|u-u_m^*\|_1 < \epsilon$$

if δ satisfies (24). In this case, m behaves like

$$(36)\qquad a^{\log \epsilon} = \epsilon^{\log a} .$$

Similarly, for $\{c_\ell\}$ given by (8), from (16) we have

$$\|u-u_m^*\| \le M\ R^{-m} + m\ a^{2-m}\ \delta.$$

If m is chosen so that

$$M\,R^{-(m+1)} < m\,a^{2-m}\,\delta \le M\,R^{-m},$$

then

$$\|u-u_m^*\| \le M\,R^{-m},$$

whence

$$\|u - u_m^*\| < \varepsilon$$

if

(37) $$m > \log\left(\frac{\varepsilon}{R}\right),$$

so that the convergence is quite rapid in this case.

A situation somewhat intermediate to (17) and (30) arises if the inner normal derivative u_n of u exists on K_1 and satisfies

$$\|u_n(1,\theta)\|^2 \le M,$$

or, equivalently,

(38) $$\Sigma_{k=1}^{\infty}\; k^2(\alpha_k^2 + \beta_k^2) \le M.$$

In this case,

$$\Sigma_{k=m}^{\infty}\;(\alpha_k^2 + \beta_k^2) \le m^{-2}\,M,$$

so that in place of (20) we have

$$\Sigma_{k=o}^{\infty}\;|1-c\,a^k|^{2m}\,(\alpha_k^2 + \beta_k^2) \le M\left[(k_o+1)\,|1-c\,a^{k_o}|^{2m} + k_o^{-2}\right].$$

Given $\varepsilon > 0$, we choose k_o satisfying

(39) $$k_o - 1 < 2\sqrt{2m}\ \varepsilon^{-1} \le k_o.$$

If m satisfies (22), and if δ satisfies (24), then

$$\|u - u_m^*\|_1 < \varepsilon.$$

In this case, m behaves like

(40) $$a^{-1/\varepsilon}.$$

For $\{c_\ell\}$ given by (8), it is easily seen that in order to make

$$\|u - u_m^*\|_1 < \varepsilon,$$

it is sufficient to take

(41) $$m > M\,\varepsilon^{-1}.$$

Uniform Convergence

Until now we have been concerned with $\mathcal{L}^2$ estimates on the error. For numerical purposes, however, uniform estimates, which we will now derive, are more useful. First consider the error

$$e_m(\theta) = u(1,\theta) - u_m(1,\theta).$$

According to equations (7) and (I-13),

$$(42) \qquad e_m(\theta) = (2\pi)^{-\frac{1}{2}} \Pi_{\ell=o}^{m-1} (1-c_\ell)\ (\alpha_o + \beta_o) +$$

$$+ \pi^{-\frac{1}{2}} \Sigma_{k=1}^{\infty} \Pi_{\ell=o}^{m-1} (1-c_\ell\ a^k)(\alpha_k \sin k\theta + \beta_k \cos k\theta).$$

Also,

$$(43) \qquad \Sigma_{k=o}^{\infty} (\alpha_k^2 + \beta_k^2) \leq M.$$

This seems to be a good place to apply (see Miller [6],p.2)

Lemma I: Let a_k and b_k be real numbers for $k = 0,1,\ldots.$ For every sequence $\{x_k\}$ of real numbers, the maximum of

$$\Sigma_{k=o}^{\infty} x_k\ a_k$$

under the condition

$$\Sigma_{k=o}^{\infty} x_k^2\ b_k^2 \leq M$$

is

$$(44) \qquad \left[M\ \Sigma_{k=o}^{\infty} \left(\frac{a_k}{b_k}\right)^2\right]^{\frac{1}{2}}.$$

Proof: According to the inequality of Schwarz,

$$\Sigma_{k=o}^{\infty} x_k a_k = \Sigma_{k=o}^{\infty} (x_k b_k) \left(\frac{a_k}{b_k}\right)$$

$$\leq \left[\Sigma_{k=o}^{\infty} x_k^2 b_k^2 \cdot \Sigma_{k=o}^{\infty} \left(\frac{a_k}{b_k}\right)^2\right]^{\frac{1}{2}}$$

$$\leq \left[M \, \Sigma_{k=o}^{\infty} \left(\frac{a_k}{b_k}\right)^2\right]^{\frac{1}{2}}.$$

On the other hand, if

$$x_k = m^{\frac{1}{2}} \left[\Sigma_{\ell=o}^{\infty} \left(\frac{a_\ell}{b_\ell}\right)^2\right]^{\frac{1}{2}} \left(\frac{a_k}{b_k^2}\right),$$

then

$$\Sigma_{k=o}^{\infty} x_k a_k = \left[M \, \Sigma_{k=o}^{\infty} \left(\frac{a_k}{b_k}\right)^2\right]^{\frac{1}{2}}.$$

QED

In this case, however, Lemma I is not sufficient because the series in (44) does not converge. Therefore, we make some additional assumptions.

For example, suppose that (17) holds. In this case,

$$(\alpha_k^2 + \beta_k^2)\, k^{h\lambda} \leq M\, k^{-1-(1-h)\lambda},$$

where h is a constant, $0 < h < 1$. Hence,

$$(\alpha_o^2 + \beta_o^2) + \Sigma_{k=1}^{\infty} (\alpha_k^2 + \beta_k^2) k^{h\lambda} \leq M,$$

so that if $h\lambda > 1$, Lemma I gives

$$|e_m(\theta)|^2 \leq M \left[\Pi_{\ell=o}^{m-1} (1-c_\ell)^2 + \Sigma_{k=1}^{\infty} k^{-h\lambda} \Pi_{\ell=o}^{m-1} (1-c_\ell a^k)^2\right].$$

For $\{c_\ell\}$ given by (8) (which, in view of the preceding results, is more reasonable than (5)),

$$(45) \qquad |e_m(\theta)| \le M \sum_{k=m}^{\infty} k^{-h\lambda} = M\, m^{1-h\lambda} .$$

Thus, the error tends to zero uniformly as $m \to \infty$. Clearly, in order that $h\lambda > 1$ for some h, it is sufficient to require that $\lambda > 1$.

If (30) holds, then

$$(\alpha_k^{\,2} + \beta_k^{\,2})\, R^{hk} \le R^{(h-2)k} ,$$

where $0 < h < 2$. Hence,

$$\sum_{k=o}^{\infty} (\alpha_k^{\,2} + \beta_k^{\,2}) R^{hk} \le M(1-R^{h-2})^{-1} ,$$

so that, by Lemma I,

$$|e_m(\theta)|^2 \le M \left[\prod_{\ell=o}^{m-1} (1-c_\ell)^2 + \sum_{\ell=o}^{\infty} \prod_{\ell=o}^{m-1} (1-c_\ell a^k) R^{-2hk}\right]$$

$$(46) \qquad \le M \sum_{k=m}^{\infty} R^{-2hk}$$

$$\le M\, R^{-2hm}$$

for $\{c_\ell\}$ given by (8), and the error tends to zero uniformly.

If (38) holds, then

$$\alpha_o^{\,2} + \beta_o^{\,2} + \sum_{k=1}^{\infty} k^2 (\alpha_k^{\,2} + \beta_k^{\,2}) \le M,$$

whence, by Lemma I,

$$(47) \qquad |e_m(\theta)|^2 \le M \frac{1}{m}$$

if $\{c_\ell\}$ is given by (8), and again the error tends to zero uniformly.

The bounds derived above apply only to the error $u-u_m$. For practical applications, it is more important to have a bound for $u - u_m^*$. We will now obtain uniform bounds for

$$\epsilon_m = u_m - u_m^* ,$$

which, when combined with (45), (46), and (47) provide uniform estimates for $u - u_m^*$.

Assume (cf. equation (II-3)) that

$$\text{(48)} \qquad \gamma(\theta) = (2\pi)^{-\frac{1}{2}} \cdot (\mu_o+v_o) + \pi^{-\frac{1}{2}} \Sigma_{k=1}^{\infty} a^k(\mu_k \sin k\theta + v_k \cos k\theta).$$

Then, according to equation (II-4),

$$\text{(49)} \qquad \epsilon_m = (2\pi)^{-\frac{1}{2}} \left[1 - \pi_{\ell=1}^{m-1}(1-c_\ell)\right] (\mu_o+v_o) + $$
$$+ \pi^{-\frac{1}{2}} \Sigma_{k=1}^{\infty} \left[1-\pi_{\ell=1}^{m-1} (1-c_\ell a^k)\right] \cdot (\mu_k \sin k\theta + v_k \cos k\theta).$$

Assume that γ satisfies inequality (10). We suppose further that

$$\text{(50)} \qquad \left\| \frac{d\gamma}{d\theta} \right\| \leq \delta.$$

Indeed, if γ does not satisfy condition (50), then let $\gamma(r,\theta)$ and $g(r,\theta)$ be functions which are harmonic in $\{|z| < a\}$, and which satisfy the boundary conditions

$$\lim_{r \uparrow a} \gamma(r,\theta) = \gamma(\theta) \text{ a.e.}, \quad g(a,\theta) = g(\theta).$$

Then, for any $r < a$, it follows from (10) that

$$\left\| \frac{d\gamma}{d\theta} \right\|_r \leq \text{const. } \delta,$$

and from the definition of u that

$$g(r,\theta) = u(r,\theta).$$

Hence the continuation problem can be considered as the continuation from a circle of radius $r < a$; this problem satisfies condition (50).

Condition (50) is equivalent to

$$\Sigma_{k=1}^{\infty} k^2 a^{2k}(\mu_k{}^2 + \nu_k{}^2) \le \delta^2,$$

so that, because of (10), we have

$$(\mu_o{}^2 + \nu_o{}^2) + \Sigma_{k=1}^{\infty} k^2 a^{2k} (\mu_k{}^2 + \nu_k{}^2) \le 2\delta^2.$$

Therefore, by Lemma 1,

$$(51) \qquad |\epsilon_m|^2 \le \frac{2\delta^2}{\pi} \Big[\{1 - \pi_{\ell=1}^{m-1} (1-c_\ell)\}^2 + \\ + \Sigma_{k=1}^{\infty} \{a^{-k}[1-\pi_{\ell=1}^{m-1} (1-c_\ell a^k)]\}^2 k^{-2}\Big].$$

According to inequality (II-5),

$$|a^{-k}[1-\pi_{\ell=1}^{m-1}(1 - c_\ell a^k)]| \le M_m,$$

where M_m depends only on $m, c_1, \ldots, c_{m-1}$. Thus the series on the right of (51) converges, and

$$|\epsilon_m| \le \text{const. } \delta,$$

where the constant tends to infinity with m. Thus, for fixed m, the error ϵ_m tends to zero uniformly as $\delta \to 0$. It follows that under conditions (10) and (50), if any one of (17), (30), (38) holds, then the total error

$$|u(1,\theta) - u_m^*(1,\theta)| \leq |e_m| + |\varepsilon_m|$$

tends to zero uniformly as $m \to \infty$ and $\delta \to 0$, provided that $m \to \infty$ sufficiently slowly as $\delta \to 0$.

Choice of $\{c_\ell\}$

Now consider the problem of optimizing the choice of $\{c_\ell\}$. The problem can be formulated as follows: given m, δ, what choice of $c_o \ldots, c_{m-1}$ minimizes $\|u-u_m^*\|_1$ (or $\sup |u(1,\theta) - u_m^*(1,\theta)|$). A somewhat simpler problem which will be treated below, is to choose $c_o, \ldots, c_{m-1}$ to minimize $\|u-u_m\|_1$.

By equation (6) we have

$$D_m \equiv \|u-u_m\|_1^2 = \Sigma_{k=o}^{\infty} \left[\Pi_{\ell=o}^{m-1} (1 - c_\ell a^k)^2\right](\alpha_k^2 + \beta_k^2).$$

Since

$$\Sigma_{k=o}^{\infty} (\alpha_k^2 + \beta_k^2)$$

converges, D_m can be differentiated term by term with respect to c_j:

$$-\tfrac{1}{2} \frac{\partial D_m}{\partial c_j} = \Sigma_{k=o}^{\infty} a^k \left[\Pi_{\substack{\ell=o \\ \ell\neq j}}^{m-1} (1 - c_\ell a^k)^2\right](\alpha_k^2 + \beta_k^2) -$$

$$- c_j \Sigma_{k=o}^{\infty} a^{2k} \left[\Pi_{\substack{\ell=o \\ \ell\neq j}}^{m-1} (1 - c_\ell a^k)^2\right](\alpha_k^2 + \beta_k^2), \; j=0,1,\ldots,m-1.$$

At a minimum,

$$\frac{\partial D_m}{\partial c_j} = 0,$$

so that,

$$(52) \quad c_j = \frac{\Sigma_{k=o}^{\infty} a^k \left[\Pi_{\substack{\ell=o \\ \ell\neq j}}^{m-1} (1 - c_\ell a^k)^2\right](\alpha_k^2 + \beta_k^2)}{\Sigma_{k=o}^{\infty} a^{2k} \left[\Pi_{\substack{\ell=o \\ \ell\neq j}}^{m-1} (1 - c_\ell a^k)^2\right](\alpha_k^2 + \beta_k^2)},$$

$$j = 0, \ldots, \quad m-1.$$

Equations (52) do not have an obvious solution, but a number of observations can be made. In the first place, $\{c_\ell\}$ depends strongly on the coefficients α_k and β_k. For example, if $\alpha_k = \beta_k = 0$ for $k \neq n$, and if $\alpha_n^2 + \beta_n^2 = 1$,

$$c_j = a^{-n} \qquad \text{for} \qquad j = 0, \ldots, m-1,$$

is a solution of (46). Secondly, if $c_o, \ldots, c_{m-1}$ is any solution of (46), then any rearrangement of $c_o, \ldots, c_{m-1}$ is also a solution. If $\alpha_k^2 + \beta_k^2 \neq 0$ for some $k \geq 1$, then $c_j \geq 1$, for $j = 0, \ldots, m-1$. If $\alpha_k^2 + \beta_k^2 = 0$ for $k \geq k_o$, then, for $m = k_o+1$, the optimum choice is

$$c_j = a^{-j}, \quad j = 0, \ldots, \quad m-1.$$

Note that this sequence was treated earlier (cf. equation (8)). This is a minimizing sequence since $D_m = 0$.

In the case $c_1 = \ldots = c_{m-1} =$ constant, we have

$$D_m = \Sigma_{k=o}^{\infty} (1-c\, a^k)^{2m} \quad (\alpha_k^2 + \beta_k^2),$$

$$\frac{dD_m}{dc} = - \Sigma_{k=o}^{\infty} 2m(1 - c\, a^k)^{2m-1} a^k(\alpha_k^2 + \beta_k^2),$$

$$\frac{d^2D_m}{dc^2} = \Sigma_{k=o}^{\infty} 2m(2m-1)(1-c\ a^k)^{2m-2}\ a^{2k}(\alpha_k{}^2 + \beta_k{}^2) > 0,$$

so that any c satisfying

$$(53) \qquad c = \frac{\Sigma_{k=o}^{\infty} a^k(1 - c\ a^k)^{2m-2}\ (\alpha_k{}^2 + \beta_k{}^2)}{\Sigma_{k=o}^{\infty} a^{2k}\ (1 - c\ a^k)^{2m-2}(\alpha_k{}^2 + \beta_k{}^2)}$$

is a minimum of D_m. The right side of (53) is positive for $c = 0$, and it is bounded for $c > 0$; hence (53) is satisfied for some $c > 0$. Therefore D_m has a minimum.

CHAPTER IV

Backward Continuation of the Heat Equation.

Let $u(x,t)$ satisfy the heat equation

$$u_{xx} = u_t \tag{1}$$

in the region $R = \{0 < x < \pi,\ \ 0 < t < T\}$. Assume that u is continuous in the closure of R. Consider the following continuation problem: given the values

$$\begin{cases} u(x,T) = g(x), & 0 < x < \pi, \\ u(0,t) = h_o(t), & 0 \le t \le T, \\ u(\pi,t) = h_1(t), & 0 \le t \le T, \end{cases} \tag{2}$$

determine the initial values $u(x,0) = f(x)$, $0 < x < \pi$. By subtracting from u the function $v(x,t)$ satisfying (1) and

$$\begin{cases} v(x,0) = 0, & 0 < x < 1, \\ v(0,t) = h_o(t), & 0 \le t \le T, \\ v(\pi,t) = h_1(t), & 0 \le t \le T, \end{cases}$$

it is sufficient to consider the problem (1), (2) with $h_o \equiv h_1 \equiv 0$.

This problem like the problem of harmonic continuation is unstable in the sense of Hadamard. It can be stabilized by assuming that

$$\|u\|_t \le M, \qquad 0 \le t \le T, \tag{3}$$

where

$$(4) \qquad \|u\|_t = \left\{\int_0^\pi [u(x,t)]^2 \, dx\right\}^{\frac{1}{2}}.$$

Let L denote the linear operator which maps the $\mathcal{L}^2(0,\pi)$ function $u(x,0)$ into $u(x,T)$. Then, the eigenfunctions and corresponding eigenvalues of L are

$$(5) \qquad \begin{cases} \Pi^{-\frac{1}{2}}, & 1, \\ (2\Pi)^{-\frac{1}{2}} \sin kx, & e^{-k^2 T}, \quad k = 1,2,\ldots. \end{cases}$$

As the eigenfunctions form a complete orthonormal family in $\mathcal{L}^2(0,\pi)$, and as the eigenvalues are real, positive, and bounded, the results of sections I and II apply with $p_m = 1$. According to Theorem 1, the iterates

$$(6) \qquad \begin{cases} u_1(x,0) = u(x,T), \\ u_m(x,0) = u_{m-1}(x,0) + c_{m-1}\left[u(x,T) - L\, u_{m-1}(x,0)\right], \end{cases}$$

where

$$(7) \qquad c_\ell = e^{\ell^2 T}, \qquad \ell = 0,1,\ldots,$$

converge to $u(x,0)$, and

$$(8) \qquad \|u-u_m\|_0^2 \le \Sigma_{k=m}^{\infty} \alpha_k^2,$$

if

$$(9) \qquad u(x,0) = \Pi^{-\frac{1}{2}} \alpha_0 + (2\Pi)^{-\frac{1}{2}} \Sigma_{k=1}^{\infty} \alpha_k \sin kx.$$

If the computations are done using

$$u(x,T) + \gamma(x),$$

where

$$\|\gamma\| = \left[\int_0^{2\pi} \gamma^2(\theta)\, d\theta\right]^{\frac{1}{2}} < \delta,$$

instead of $u(x,T)$, then, by equation (II-6),

$$\|u-u_m^*\| \le \left[\Sigma_{k=m}^{\infty} \alpha_k^2\right]^{\frac{1}{2}} + K_m\, \delta,$$

where

$$K_m = \max_k \left\{ e^{k^2T}\left[1 - \Pi_{\ell=0}^{m-1}(1 - c_\ell\, a^{-k^2T})\right]\right\}.$$

Clearly, the quantity in braces is given by

$$\begin{cases} e^{k^2T}, & \text{if } k \le m-1, \\ e^{k^2T}\left[1 - \Pi_{\ell=0}^{m-1}\left(1-e^{(\ell^2-k^2)T}\right)\right], & \text{if } k \ge m. \end{cases}$$

By the argument following (II-14),

$$K_m \le \max\left(e^{(m-1)^2T},\; m\, e^{(m-1)^2T}\right) < m\, e^{m^2T}.$$

Thus,

$$\|u-u_m^*\| \le \left[\Sigma_{k=m}^{\infty} \alpha_k^2\right]^{\frac{1}{2}} + m\, e^{m^2T}\, \delta.$$

We now make some assumptions on the behavior of $\{\alpha_k\}$. If

$$\alpha_k^2 \le M\, k^{-1-\lambda},$$

then, the number m of iterations required to reduce the error to less than ε tends to infinity like

$$\varepsilon^{-\frac{2}{\lambda}}$$

as ε tends to zero. This rate is the same as that given by (III-29) for harmonic continuation. However, δ must tend to zero faster than indicated by (III-27). Indeed, the corresponding relation is

$$\delta \leq M\, m^{-\frac{1}{2}}\, \lambda - 1 \quad e^{-m^2 T}.$$

This fact is a consequence of the fact that the operator involved here has a much greater smoothing effect than the one involved in harmonic continuation. Other assumptions lead to similar results.

CHAPTER V

Determination of the Charge Density on a Ring

In this section we consider the following problem. Given that a potential $u(\gamma,\theta)$ is the potential of a charge of linear density $\rho(\theta)$ on the circle $\{|z| = 1\}$, determine ρ from the values of u on the circle $\{|z| = a\}$, $a > 1$.

More precisely, let

$$L_r\ \rho(\theta) = \int_{-\pi}^{\pi} \rho(\varphi)\ \log\left\{\left[1+r^2-2r\ \cos(\theta-\varphi)\right]^{-\frac{1}{2}}\right\} d\varphi$$

for any $\rho \in \mathcal{L}^2(-\pi,\pi)$. Then

$$u(r,\theta) = L_r\ \rho(\theta)$$

is the logrithmic potential produced by a change of density $\rho(\theta)$ on the circle $\{|z| = 1\}$. The problem is to find $\rho(\theta)$ given that

$$u(a,\theta) = g(\theta), \qquad a > 1;$$

i.e., to solve

$$L_a\ \rho = g$$

for ρ.

Observe that $\rho \equiv 1$ is an eigenfunction of L_a, and that the corresponding eigenvalue is

$$\lambda_o = \int_{-\pi}^{\pi} \log\left\{\left[1+r^2 - 2r\ \cos\ \varphi\right]^{\frac{1}{2}}\right\} d\varphi$$
$$= -2\pi\ \log\ r,$$

the integral being easily evaluated by contour integration. Moreover, for any $k = 1,2,\ldots,$ an integration by parts gives

$$L_a \sin k\theta = \frac{r}{k} \int_{-\pi}^{\pi} \frac{\cos k\varphi \sin(\theta-\varphi)}{[1+r^2 - 2r\cos(\theta-\varphi)]} d\varphi,$$

$$L_a \cos k\theta = -\frac{r}{k} \int_{-\pi}^{\pi} \frac{\sin k\varphi \sin(\theta-\varphi)}{[1+r^2 - 2r\cos(\theta-\varphi)]} d\varphi.$$

Now

$$\cos k\varphi = \sin k(\theta-\varphi) \sin k\theta + \cos k(\theta-\varphi) \cos k\theta,$$
$$\sin k\varphi = -\sin k(\theta-\varphi) \cos k\theta + \cos k(\theta-\varphi) \sin k\theta.$$

Also,

$$\int_{-\pi}^{\pi} \frac{\cos k(\theta-\varphi) \sin(\theta-\varphi)}{1+r^2 - 2r\cos(\theta-\varphi)} d\varphi = 0,$$

since the integrand is odd, and

$$\int_{-\pi}^{\pi} \frac{\sin k\varphi \sin\varphi}{1+r^2 - 2r\cos\varphi} d\varphi = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos(k-1)\varphi - \cos(k+1)\varphi}{1+r^2 - 2r\cos\varphi} d\varphi$$

$$= \frac{1}{2r^2} \int_{-\pi}^{\pi} \frac{\cos(k-1)\varphi - \cos(k+1)\varphi}{r^{-2} + 1 - 2r^{-1}\cos\varphi} d\varphi,$$

which is the Poisson integral for the harmonic function

$$\frac{\pi}{r^2 - 1} \left[r^{1-k} \cos(k-1)\theta - r^{-1-k} \cos(k+1)\theta \right],$$

so that

$$\int_{-\pi}^{\pi} \frac{\sin k\varphi \sin\varphi}{1+r^2 - 2r\cos\varphi} d\varphi = \frac{\pi}{r^2 - 1} \left[r^{1-k} - r^{-1-k} \right]$$

$$= \pi r^{-1-k}.$$

Thus,

$$L_a \sin k\theta = \pi k^{-1} r^{-k} \sin k\theta,$$

$$L_a \cos k\theta = \pi k^{-1} r^{-k} \cos k\theta, \qquad k = 1,2,\ldots,$$

whence the eigenfunctions and corresponding eigenvalues, respectively, of L_a are

$$\begin{cases} (2\pi)^{-\frac{1}{2}}, & \lambda_o = -2\pi \log a, \\ \pi^{-\frac{1}{2}} \sin k\theta, & \lambda_k = \pi k^{-1} a^{-k}, \\ \pi^{-\frac{1}{2}} \cos k\theta, & \lambda_k = \pi k^{-1} a^{-k}, \qquad k = 1,2,\ldots. \end{cases}$$

Note that the eigenvalues, with the exception of the first, are positive and bounded. Therefore, by Theorem 1 of section I, there exists $\{c_\ell\}$ such that the functions ρ_m defined by

$$\begin{cases} \rho_1 = c_o L g, \\ \rho_m = \rho_{m-1} + c_{m-1} (g - L \rho_{m-1}), \qquad m = 2,3,4,\ldots, \end{cases}$$

converge to ρ in the $\mathcal{L}^2(-\pi,\pi)$ norm:

$$\|\rho - \rho_m\| \to 0 \qquad \text{as} \qquad m \to 0$$

(note that $p_o = 2$; this is because the first eigenvalue can be negative).

By Theorem 1, take

$$\begin{cases} c_o = (2\pi \log a)^{-2} \\ c_\ell = \pi^{-1} \ell a^\ell, \qquad \ell = 1,2,\ldots. \end{cases}$$

If

$$\rho(\theta) = (2\pi)^{-\frac{1}{2}} (\alpha_o + \beta_o) + \Sigma_{k=1}^{\infty} \pi^{-\frac{1}{2}} (\alpha_k \sin k\theta + \beta_k \cos k\theta),$$

then, by (I-14),

$$\|\rho-\rho_m\|^2 = (\alpha_o{}^2 + \beta_o{}^2) \Pi_{\ell=o}^{m-1} \left[1-c_\ell(2\pi \log a)^{-2}\right]^2 +$$

$$+ \Sigma_{k=1}^{\infty} \Pi_{\ell=o}^{m-1} \left[1-c_\ell \pi k^{-1}a^{-k}\right]^2 (\alpha_k{}^2 + \beta_k{}^2)$$

$$= \Sigma_{k=m}^{\infty} \Pi_{\ell=o}^{m-1} \left[1-c_\ell \pi k^{-1}a^{-k}\right]^2 (\alpha_k{}^2 + \beta_k{}^2),$$

since

$$\Pi_{\ell=o}^{m-1} \left[1 - c_\ell \pi k^{-1}a^{-k}\right] = 0 \quad \text{if} \quad m-1 \geq k.$$

Now if ℓ is sufficiently large, $c_\ell \geq c_{\ell+1} \geq \ldots .$

Thus, by the remark following Theorem 2,

$$\|\rho-\rho_m\|^2 \leq \Sigma_{k=m}^{\infty} (\alpha_k{}^2 + \beta_k{}^2) \to 0 \qquad \text{as } m \to \infty.$$

Hence, by (II-7),

$$\|\rho-\rho_m{}^*\| \leq \left[\Sigma_{k=m}^{\infty} (\alpha_k{}^2 + \beta_k{}^2)\right]^{\frac{1}{2}} + K_m \delta,$$

where K_m is a bound for

$$|\lambda_k{}^{-1} \left[1 - (1-c_o \lambda_k{}^2) \Pi_{\ell=1}^{m-1} (1-c_\ell \lambda_k)\right]| =$$

$$= \begin{cases} |\lambda_k{}^{-1}| & \text{for} \quad k \leq m-1, \\ \pi^{-1}k\ a^k \left[1-\left(1-\{2k\ a^k \log a\}^{-2}\right) \Pi_{\ell=1}^{m-1}\left(1-\frac{\ell}{k}\ a^{\ell-k}\right)\right] & \text{for } k \geq m, \end{cases}$$

if $m \geq 2$. Now,

$$1 - \left(1-\{2k\ a^k \log a\}^{-2}\right) \Pi_{\ell=1}^{m-1} \left(1 - \frac{\ell}{k}\ a^{\ell-k}\right) \leq$$

$$\leq 1- \left(1-\frac{m-1}{k}\ a^{m-1-k}\right)^{m-1} + (2k\ a^k \log a)^{-2}(1 - \frac{1}{k}a^{1-k})^{m-1}.$$

Thus, by the argument leading to (III-15),

$$\pi^{-1}k\ a^k \left[1 - \left(1-\{2k\ a^k \log a\}^{-2} \Pi_{\ell=1}^{m-1} \left(1 - \frac{\ell}{k}\ a^{\ell-k}\right)\right] \leq$$

$$\leq \pi^{-1}k\ a^k \left[1 - \left(1-\frac{m-1}{k}a^{m-1-k}\right)^{m-1}\right] +$$

$$+ \pi^{-1}\ k^{-1}\ a^{-k} \log^{-2} a \left(1 - \frac{1}{k}\ a^{1-k}\right)^{m-1} \leq$$

$$\leq \pi^{-1}(m-1) \left[(m-1)\ a^{m-1} + k^{-2}\ a \log^{-2} a\right]$$

$$\leq \pi^{-1}\ m^2\ a^m$$

for sufficiently large k. Hence,

$$K_m \leq \max\ (\pi^{-1}\ m\ a^m,\ \pi^{\ 1}\ m^2\ a^m) = \pi^{-1}\ m^2\ a^m.$$

Thus,

$$\|\rho - \rho_m^*\| \leq \left[\Sigma_{k=m}^{\infty} (\alpha_k^{\ 2} + \beta_k^{\ 2})\right]^{\frac{1}{2}} + \pi^{-1}\ m^2\ a^m\ \delta.$$

Now if $a_o > a$, then, for sufficiently large m,

$$\pi^{-1}\ m^2\ a^m < m\ a_o^{\ m},$$

whence

$$\|\rho-\rho_m{}^*\| \leq \left[\Sigma_{k=m}^{\infty} (\alpha_k{}^2 + \beta_k{}^2)\right]^{\frac{1}{2}} + m\, a_o^m\, \delta,$$

which is just inequality (III-16). Hence, the corresponding estimates of section III apply to $\|\rho-\rho_m{}^*\|$.

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