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Finite Element Reliability Analysis of Inelastic Dynamic Systems

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE Master of Science

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Abstract

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Due to the inherent uncertainties present in nature and given the imperfect state of our knowledge, it is impossible to guarantee the satisfactory performance of any system in an absolute sense. Therefore, an approach such as reliability based design, which offers a rational basis for taking into account in the design process the various sources of uncertainty and checking the computed probability of failure, is desirable. Structural reliability analysis also provides as a by-product various reliability sensitivity measures, which are very useful for rational decision making in structural design. In addition, the performance of large and complex structural systems can be predicted only through complicated numerical algorithms, such as the powerful finite element method. Hence, in order to evaluate the probability of failure of such systems for given limit-states or failure criteria, finite element analysis and reliability analysis must be linked together to produce the finite element reliability method.

In this study, the link between a general purpose, research oriented, finite element program (FEAP) and a reliability analysis program (CALREL) is established. In order to realistically model the inelastic behavior of structural systems, several inelastic element routines are developed and implemented in FEAP. The algorithms required for computing accurately and efficiently the structural response gradient (needed in reliability analysis) with respect to basic material properties are also formulated and implemented in FEAP. Finally, finite element sensitivity and reliability analyses of several realistic structural examples are performed.
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Table of Contents

Abstract ................................................................. i
Acknowledgments ......................................................... ii
Table of Contents .......................................................... iii
List of Figures ............................................................... vii
List of Tables ............................................................... xii

CHAPTER 1 INTRODUCTION ........................................... 1
  1.1 General Remarks ................................................... 1
  1.2 Objectives and Scope of Study ..................................... 2
  1.3 Organization of Thesis ............................................. 3

CHAPTER 2 STRUCTURAL RELIABILITY METHODS ................. 6
  2.1 Introduction ....................................................... 6
  2.2 Structural Component Reliability ................................ 6
  2.3 First- and Second-Order Reliability Methods ................. 8
    2.3.1 Transformation to the Standard Normal Space ........... 9
    2.3.2 Approximations of the Failure Surface in the U-Space ... 13
    2.3.3 Determination of the Design Point .......................... 16
  2.4 First-Order Reliability Sensitivity Measures .................. 18
  2.5 Characteristics of FORM and SORM ............................ 23
  2.6 Conclusions ..................................................... 24

CHAPTER 3 THE FINITE ELEMENT METHOD .......................... 29
  3.1 Introduction ..................................................... 29
  3.2 Differential Form of Equations of Motion ..................... 29
  3.3 Finite Element Discretization of Equations of Motion ........ 31
    3.3.1 Consistent versus Lumped Mass Matrices ................. 37
    3.3.2 Damping ................................................... 38
    3.3.3 Earthquake Loading ....................................... 41
  3.4 Step-by-Step Methods for the Response to General Dynamic Loading ... 43
    3.4.1 Newmark-beta Methods .................................. 44
    3.4.2 Explicit Formulation of the Newmark-beta Methods .... 45
3.4.3 Incremental Formulation for Nonlinear Analysis .................. 47
3.5 Numerical Algorithms for Nonlinear Static Analysis .................. 48
3.6 Plane Bilinear Isoparametric Element .......................... 51
   3.6.1 Displacement Interpolation Functions .................. 52
   3.6.2 Strain-Displacement Relations .................. 52
   3.6.3 Stress-Strain Relations .................. 55
   3.6.4 Element stiffness matrix .................. 56
3.7 Conclusions ........................................... 56

CHAPTER 4  CONSTITUTIVE EQUATIONS FOR PLASTICITY FORMULATIONS .................. 58
4.1 Introduction ........................................... 58
4.2 Basic Concepts of Plasticity Theory .......................... 58
4.3 Incremental Stress-Strain Relations .................. 63
4.4 Derivation of the “Continuum” Elastoplastic Constitutive Matrix .......... 64
4.5 Return Map Algorithm .................................. 66
4.6 Constitutive Equations for the Associative J2 Plasticity Model .......... 67
4.7 J2 Constitutive Equations in Discrete Form .................. 71
4.8 Return Map Algorithm for the J2 Plasticity Model .................. 73
   4.8.1 Elastic Predictor Step .......................... 73
   4.8.2 Plastic Corrector Step .......................... 74
   4.8.3 Consistent (Algorithmic) Tangent Elastoplastic Moduli .......... 77
4.9 Application of the Return Map Algorithm to the Plane-Strain Case .......... 81
   4.9.1 The Elastic Predictor Step .......................... 82
   4.9.2 Plastic Corrector Step .......................... 83
   4.9.3 Consistent Tangent Stiffness Matrix for the Plane-Strain Case .... 84
4.10 Application of the J2 Plasticity Model to the Truss Element ............ 87
   4.10.1 Constitutive Equations - Continuum (Infinitesimal) Form .......... 87
   4.10.2 Constitutive Equations in Discrete Form .............. 91
   4.10.3 Return Map Algorithm .................................. 92
      4.10.3.1 Elastic Predictor Step .......................... 92
      4.10.3.2 Plastic Corrector Step .......................... 92
      4.10.3.3 Consistent Tangent Stiffness Matrix for the Truss Element .......... 94
4.11 Constitutive Relations for the Cap Model .............................................. 98
  4.11.1 Return map Algorithm ............................................................. 102
  4.11.2 Loading in the Various Modes of the Cap Model ......................... 103
  4.11.3 Consistent Elastoplastic Tangent Moduli ...................................... 108
  4.12 Conclusions .................................................................................. 109

CHAPTER 5  THE FINITE ELEMENT RELIABILITY METHOD ............ 110
  5.1 Introduction .............................................................. 110
  5.2 Random Field Discretization .......................................................... 111
    5.2.1 Basic Theory of Random Fields ................................................. 112
    5.2.2 Methods of Random Field Discretization .................................... 120
    5.2.3 Selection of the Random Field Mesh ......................................... 126
  5.3 Response Sensitivity Analysis ......................................................... 128
    5.3.1 Conditional Derivative Method (CDM) .................................... 131
    5.3.2 Incremental Solution of the Equation of Motion ......................... 133
    5.3.3 Numerical Integration of the Gradient Equation ......................... 136
    5.3.4 Conditional Derivative of the Stress Tensor .............................. 138
    5.3.5 Application of the CDM to the Plasticity Model ......................... 139
    5.3.6 Application of the CDM to the Cap Model .................................. 143
    5.3.7 Summary of the CDM ............................................................. 151
  5.4 Definition of Limit-State or Performance Function ......................... 153

CHAPTER 6  IMPLEMENTATION IN FEAP ............................... 155
  6.1 Introduction .................................................................................. 155
  6.2 Link Between CALREL and FEAP ............................................... 155
  6.3 Definition of the Random Field Mesh in FEAP ............................... 158
  6.4 New Macro-Commands in FEAP .................................................... 158
    6.4.1 Input for Ground Motion Excitation (mac3) ............................. 158
    6.4.2 Double Updating of the State Variables During the Same
      Time Step (mac2) ........................................................................ 160
    6.4.3 Response Sensitivity Analysis (mac1) ..................................... 161
  6.5 Efficient Utilization of the Memory Space ...................................... 162
  6.6 Conclusions .................................................................................. 163

CHAPTER 7  APPLICATION EXAMPLES ....................................... 164
  7.1 Introduction .................................................................................. 164
7.2 Elasto-plastic Single-Degree-of-Freedom System .................................. 164
7.3 Inelastic Ten Member Truss ................................................................. 171
7.4 Elastic Dam Subjected to Static Load .................................................... 179
7.5 Perforated Inelastic Plate with J 2 Constitutive model ......................... 188
7.6 Single Plane Strain Element with Cap Plasticity Model ....................... 197
  7.6.1 Simple Calculations for Verification of the Implementation of the Cap Model .......................................................... 199
  7.6.2 Monotonic Uni-Axial Tension Load .................................................. 200
  7.6.3 Monotonic Uni-Axial Compression Load (Cap Surface Activated First) .......................................................... 202
  7.6.4 Monotonic Uni-Axial Compression Load (Failure Surface Activated First) .......................................................... 204
  7.6.5 Cyclic Loading ............................................................................. 206
7.7 Inelastic Plane Strain Element Subjected to Earthquake-Type Loading .... 208
7.8 Inelastic Dam Subjected to Earthquake Loading .................................... 213
7.9 Inelastic Dam Subjected to Scaled Up Earthquake Loading .................. 221
7.10 Reliability Analysis of Simple Dynamic Systems ................................. 230
7.11 Conclusions ..................................................................................... 237

CHAPTER 9  CONCLUSIONS ..................................................................... 238
  8.1 Summary of Work ............................................................................ 238
  8.2 Summary of Findings ........................................................................ 239
  8.3 Scope of Future Work ....................................................................... 240

REFERENCES .......................................................................................... 241
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 2.1</td>
<td>Outcome Space of Basic Random Variables</td>
<td>26</td>
</tr>
<tr>
<td>Figure 2.2</td>
<td>Transformation of Uniformly Distributed Random Variable</td>
<td>27</td>
</tr>
<tr>
<td>Figure 2.3</td>
<td>One-to-One Mapping from the X-Space to the U-Space</td>
<td>27</td>
</tr>
<tr>
<td>Figure 2.4</td>
<td>HL-RF Algorithm to Find the Design Point</td>
<td>28</td>
</tr>
<tr>
<td>Figure 3.1</td>
<td>Four-noded, plane, bilinear isoparametric element in (a) (x,y) space, and (b) (ξ,η) space.</td>
<td>52</td>
</tr>
<tr>
<td>Figure 4.1</td>
<td>Geometric interpretation of a general return mapping algorithm based on an elastoplastic split of the constitutive equations.</td>
<td>68</td>
</tr>
<tr>
<td>Figure 4.2</td>
<td>Geometrical Interpretation of the Radial return algorithm</td>
<td>76</td>
</tr>
<tr>
<td>Figure 4.3</td>
<td>Uniaxial J2 plasticity model with kinematic hardening only also called the bilinear inelastic model; (a) σ-ε diagram, (b) linear kinematic hardening model.</td>
<td>90</td>
</tr>
<tr>
<td>Figure 4.4</td>
<td>Return map algorithm in the uniaxial case with pure kinematic hardening (H_{iso}= 0).</td>
<td>97</td>
</tr>
<tr>
<td>Figure 4.5</td>
<td>The yield surface of the cap model</td>
<td>98</td>
</tr>
<tr>
<td>Figure 4.6</td>
<td>Hardening rule for the Cap model</td>
<td>100</td>
</tr>
<tr>
<td>Figure 4.7</td>
<td>Hardening / softening of the cap</td>
<td>101</td>
</tr>
<tr>
<td>Figure 4.8</td>
<td>Contraction behavior of the cap</td>
<td>101</td>
</tr>
<tr>
<td>Figure 4.9</td>
<td>The yield surface of the cap model</td>
<td>108</td>
</tr>
<tr>
<td>Figure 5.1</td>
<td>Newton-Raphson scheme for nonlinear dynamic equations</td>
<td>136</td>
</tr>
<tr>
<td>Figure 6.1</td>
<td>Flow chart diagram of the FEAP-CALREL merging strategy</td>
<td>157</td>
</tr>
<tr>
<td>Figure 7.1</td>
<td>SDOF system: elasto-plastic truss element</td>
<td>164</td>
</tr>
<tr>
<td>Figure 7.2</td>
<td>Imperial Valley earthquake, May 18, 1940, El Centro Site, Component S00E (N-S)</td>
<td>165</td>
</tr>
<tr>
<td>Figure 7.3</td>
<td>Verification of the inelastic truss element</td>
<td>166</td>
</tr>
<tr>
<td>Figure 7.4</td>
<td>Stress-strain response history of inelastic truss element</td>
<td>167</td>
</tr>
<tr>
<td>Figure 7.5</td>
<td>Displacement response sensitivity with respect to the material mass density, ρ.</td>
<td>167</td>
</tr>
<tr>
<td>Figure 7.6</td>
<td>Displacement response sensitivity with respect to the yield stress, σ_y</td>
<td>168</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Figure 7.7</td>
<td>Displacement response sensitivity with respect to the Young's modulus, $E$</td>
<td>168</td>
</tr>
<tr>
<td>Figure 7.8</td>
<td>Most probable &quot;failure&quot; modes of the SDOF inelastic truss system</td>
<td>170</td>
</tr>
<tr>
<td>Figure 7.9</td>
<td>Ten-member inelastic truss under consideration</td>
<td>171</td>
</tr>
<tr>
<td>Figure 7.10</td>
<td>Concentrated dynamic load applied at nodes 2 and 4 of the ten-member truss.</td>
<td>171</td>
</tr>
<tr>
<td>Figure 7.11</td>
<td>Vertical displacement time history at node 2 of the ten-member truss.</td>
<td>172</td>
</tr>
<tr>
<td>Figure 7.12</td>
<td>Displacement response sensitivity with respect to the Young's modulus, $E$</td>
<td>173</td>
</tr>
<tr>
<td>Figure 7.13</td>
<td>Displacement response sensitivity with respect to the material mass density, $\rho$</td>
<td>174</td>
</tr>
<tr>
<td>Figure 7.14</td>
<td>Displacement response sensitivity with respect to the yield stress, $\sigma_y$</td>
<td>174</td>
</tr>
<tr>
<td>Figure 7.15</td>
<td>Most probable &quot;failure&quot; modes for the inelastic truss</td>
<td>175</td>
</tr>
<tr>
<td>Figure 7.16</td>
<td>Stress-strain history of element 6</td>
<td>178</td>
</tr>
<tr>
<td>Figure 7.17</td>
<td>Stress-strain history of element 10</td>
<td>178</td>
</tr>
<tr>
<td>Figure 7.18</td>
<td>Finite element model of the Pine Flat Dam on King's River near Fresno, California</td>
<td>180</td>
</tr>
<tr>
<td>Figure 7.19</td>
<td>Finite element and random field meshes over the dam cross-section</td>
<td>182</td>
</tr>
<tr>
<td>Figure 7.19 (contd.)</td>
<td>Finite element and random field meshes over the dam cross-section</td>
<td>183</td>
</tr>
<tr>
<td>Figure 7.20</td>
<td>Effect of random field size on reliability index</td>
<td>184</td>
</tr>
<tr>
<td>Figure 7.21</td>
<td>Effect of correlation length on reliability index</td>
<td>184</td>
</tr>
<tr>
<td>Figure 7.22</td>
<td>Contour plot of the horizontal displacement field at the design point</td>
<td>185</td>
</tr>
<tr>
<td>Figure 7.23</td>
<td>Contour plot of the sensitivity of the reliability index with respect to the mean of the local Young's modulus</td>
<td>186</td>
</tr>
<tr>
<td>Figure 7.24</td>
<td>Contour plot of the sensitivity of the reliability index with respect to the standard deviation of the local Young's modulus</td>
<td>187</td>
</tr>
<tr>
<td>Figure 7.25</td>
<td>Inelastic plate with a circular hole</td>
<td>188</td>
</tr>
<tr>
<td>Figure 7.26</td>
<td>Finite element mesh of perforated plate</td>
<td>189</td>
</tr>
<tr>
<td>Figure 7.27</td>
<td>Applied loading on the perforated plate</td>
<td>190</td>
</tr>
<tr>
<td>Figure 7.28</td>
<td>Horizontal displacement response time-history at point A</td>
<td>190</td>
</tr>
<tr>
<td>Figure 7.29</td>
<td>Displacement response sensitivity with respect to the initial yield stress, $\sigma_{y0}$</td>
<td>191</td>
</tr>
</tbody>
</table>
Figure 7.30  Random field mesh for the inelastic perforated plate .......................... 192
Figure 7.31  Yield zone of the inelastic plate at t = 0.75 sec ................................. 193
Figure 7.32  Yield zone of the inelastic plate at t = 0.875 sec ............................. 194
Figure 7.33  Yield zone of the inelastic plate at t = 1.0 sec ................................. 194
Figure 7.34  Horizontal displacement response time-history at point A for the
               mean point and the design point ........................................... 195
Figure 7.35  Contour plot of the initial yield stress at the design point .................... 195
Figure 7.36  Contour plot of the normalized sensitivity of the first-order reliability
               index with respect to the mean of the local initial yield stress ............ 196
Figure 7.37  Contour plot of the normalized sensitivity of the first-order
               reliability index with respect to the standard deviation of the
               local initial yield stress ....................................................... 196
Figure 7.38  Single test element under plane strain condition ............................ 198
Figure 7.39  Load-displacement curve at node 3 for monotonic uni-axial
               tension load ..................................................................... 201
Figure 7.40  Stress path in the I₁-|ls| space for monotonic uni-axial tension
               load case ........................................................................ 201
Figure 7.41  Load-displacement curve at node 3 for monotonic uni-axial
               compression load (cap surface activated first) ............................. 203
Figure 7.42  Stress path in the I₁-|ls| space for monotonic uni-axial
               compression load (cap surface activated first) ............................. 203
Figure 7.43  Load-displacement curve at node 3 for monotonic uni-axial
               compression load (failure surface activated first) .......................... 205
Figure 7.44  Stress path in the I₁-|ls| space for monotonic uni-axial
               compression load (failure surface activated first) .......................... 205
Figure 7.45  Time history of the applied cyclic load .............................................. 206
Figure 7.46  Load-displacement curve at node 3 for cyclic load case ..................... 207
Figure 7.47  Stress path in the I₁-|ls| space for cyclic load case ........................... 207
Figure 7.48  Inelastic plane strain element subjected to earthquake ground
               excitation .................................................................. 208
Figure 7.49  Inelastic vs. elastic response time history of the single plane strain
               element ..................................................................... 209
Figure 7.50  Displacement response sensitivity with respect to the parameter α ...... 210
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.51</td>
<td>Displacement response sensitivity with respect to the parameter $\beta$</td>
<td>210</td>
</tr>
<tr>
<td>7.52</td>
<td>Displacement response sensitivity with respect to the parameter $R$</td>
<td>211</td>
</tr>
<tr>
<td>7.53</td>
<td>Displacement response sensitivity with respect to the parameter $T$</td>
<td>211</td>
</tr>
<tr>
<td>7.54</td>
<td>Displacement response sensitivity with respect to the parameter $W$</td>
<td>212</td>
</tr>
<tr>
<td>7.55</td>
<td>Displacement response time history of the top of the dam</td>
<td>215</td>
</tr>
<tr>
<td>7.56</td>
<td>Location of points A, B, C and D</td>
<td>215</td>
</tr>
<tr>
<td>7.57</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point A</td>
<td>216</td>
</tr>
<tr>
<td>7.58</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point B</td>
<td>216</td>
</tr>
<tr>
<td>7.59</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point C</td>
<td>217</td>
</tr>
<tr>
<td>7.60</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point D</td>
<td>217</td>
</tr>
<tr>
<td>7.61</td>
<td>Plot of mean stress vs. volumetric strain at point A</td>
<td>218</td>
</tr>
<tr>
<td>7.62</td>
<td>Plot of mean stress vs. volumetric strain at point B</td>
<td>218</td>
</tr>
<tr>
<td>7.63</td>
<td>Plot of mean stress vs. volumetric strain at point C</td>
<td>219</td>
</tr>
<tr>
<td>7.64</td>
<td>Plot of mean stress vs. volumetric strain at point D</td>
<td>219</td>
</tr>
<tr>
<td>7.65</td>
<td>Displacement response sensitivity with respect to the parameter $T$</td>
<td>220</td>
</tr>
<tr>
<td>7.66</td>
<td>Response of the dam subjected to scaled up El-Centro earthquake</td>
<td>222</td>
</tr>
<tr>
<td>7.67</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point A</td>
<td>222</td>
</tr>
<tr>
<td>7.68</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point B</td>
<td>223</td>
</tr>
<tr>
<td>7.69</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point C</td>
<td>223</td>
</tr>
<tr>
<td>7.70</td>
<td>Plot of the stress path in the $I_1$-I$_{lsll}$ space at point D</td>
<td>224</td>
</tr>
<tr>
<td>7.71</td>
<td>Plot of mean stress vs. volumetric strain at point A</td>
<td>224</td>
</tr>
<tr>
<td>7.72</td>
<td>Plot of mean stress vs. volumetric strain at point B</td>
<td>225</td>
</tr>
<tr>
<td>7.73</td>
<td>Plot of mean stress vs. volumetric strain at point C</td>
<td>225</td>
</tr>
<tr>
<td>7.74</td>
<td>Plot of mean stress vs. volumetric strain at point D</td>
<td>226</td>
</tr>
<tr>
<td>7.75</td>
<td>Displacement response sensitivity with respect to the parameter $\alpha$</td>
<td>226</td>
</tr>
<tr>
<td>7.76</td>
<td>Displacement response sensitivity with respect to the parameter $\lambda$</td>
<td>227</td>
</tr>
<tr>
<td>7.77</td>
<td>Displacement response sensitivity with respect to the parameter $R$</td>
<td>227</td>
</tr>
<tr>
<td>7.78</td>
<td>Displacement response sensitivity with respect to the parameter $T$</td>
<td>228</td>
</tr>
<tr>
<td>7.79</td>
<td>Displacement response sensitivity with respect to the parameter $W$</td>
<td>228</td>
</tr>
<tr>
<td>7.80</td>
<td>Displacement response sensitivity with respect to the parameter $\beta$</td>
<td>229</td>
</tr>
<tr>
<td>7.81</td>
<td>Displacement response sensitivity with respect to the parameter $\theta$</td>
<td>229</td>
</tr>
<tr>
<td>7.82</td>
<td>Uncertain SDOF system subjected to harmonic loading</td>
<td>230</td>
</tr>
<tr>
<td>7.83</td>
<td>Limit-state surface and failure domain of linear elastic SDOF truss</td>
<td></td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>7.84</td>
<td>Limit-state surface and failure domain of linear elastic SDOF truss structure (δc = 1.5 in)</td>
<td>231</td>
</tr>
<tr>
<td>7.85</td>
<td>Sketch of response spectrum for time-limited harmonic load</td>
<td>233</td>
</tr>
<tr>
<td>7.86</td>
<td>Stress-strain diagram of truss material</td>
<td>234</td>
</tr>
<tr>
<td>7.87</td>
<td>Limit-state surface and failure domain of elasto-perfectly plastic SDOF truss structure</td>
<td>235</td>
</tr>
<tr>
<td>7.88</td>
<td>Limit-state surface and failure domain of linear elastic 2-DOF truss structure</td>
<td>236</td>
</tr>
</tbody>
</table>
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2.1</td>
<td>Probability of Failure vs. Safety Index</td>
<td>26</td>
</tr>
<tr>
<td>Table 7.1</td>
<td>Basic random variables of the SDOF inelastic truss system</td>
<td>169</td>
</tr>
<tr>
<td>Table 7.2</td>
<td>Design points for the SDOF inelastic truss system</td>
<td>170</td>
</tr>
<tr>
<td>Table 7.3</td>
<td>Reliability results corresponding to the first design point</td>
<td>176</td>
</tr>
<tr>
<td>Table 7.4</td>
<td>Reliability results corresponding to the second design point</td>
<td>176</td>
</tr>
<tr>
<td>Table 7.5</td>
<td>Reliability results corresponding to the third design point</td>
<td>177</td>
</tr>
<tr>
<td>Table 7.6</td>
<td>Probability distributions of material parameters</td>
<td>179</td>
</tr>
<tr>
<td>Table 7.7</td>
<td>Parameters of the cap constitutive model</td>
<td>197</td>
</tr>
<tr>
<td>Table 7.8</td>
<td>Constitutive parameters for the concrete of the dam</td>
<td>214</td>
</tr>
</tbody>
</table>
CHAPTER 1 INTRODUCTION

1.1 General Remarks

The design of any structural system is performed in the face of numerous uncertainties. These uncertainties arise due to the inherent variability present in nature and due to the imperfect state of our knowledge. It may also arise due to improper modelling (simplified, idealized, incorrectly calibrated) of actual structural behavior. It is therefore impossible, under these conditions of uncertainty, to guarantee the satisfactory performance of any system in an absolute sense. It is essential to realize that any system has a finite probability of failure or of not performing as intended, and that one can only specify the maximum acceptable probability of failure. The probability that the system will perform satisfactorily is termed the reliability of the system, and the design methodology which takes into account the uncertainties of the system and its environment is called reliability based design.

Reliability analysis not only offers a rational basis for taking into account the basic sources of uncertainty in the design process, but also provides important sensitivity measures which can be very useful for rational decision making during the design.

For large and complex structural systems, the performance of the structure can be predicted only in the algorithmic sense, since no closed form analytical solution for the response of realistic structures is available. Hence, the designer is forced to use numerical methods such as the finite element method for assessing the performance of the structure. In order to predict the failure probability of such complicated structures, the reliability analysis must be performed in parallel with the finite element analysis. The merger of the two methods is termed the finite element reliability method.

Another issue which needs to be addressed is the accurate modelling of the inelastic behavior of structures. Most structural failures take place in the inelastic range. Therefore, in order to get a good estimate of the failure probability in structural safety stud-
ies, realistic models of nonlinear behavior of structures must be incorporated into the finite element methodology.

With the increased availability of computational power and the growing realization of the importance of probabilistic structural design, the finite element reliability method is emerging as a very important and useful tool for performing reliability analysis of real structures. The reliability of large, complex structural systems can be evaluated by linking the now well established methods of reliability analysis and the powerful finite element method.

The present study is motivated by the above considerations. The objectives and scope of this work are described in detail in the following section.

1.2 Objectives and Scope of Study

The goals of the present study are:

(a) to develop nonlinear finite element routines to accurately model the nonlinear material behavior of real structures;

(b) to compute the response sensitivities of an inelastic dynamic system in an efficient and accurate manner in order to establish the link between the finite element method and the reliability methods;

(c) to perform the finite element reliability analysis of inelastic static and dynamic structural systems.

As explained earlier, this work focuses on linking reliability methods with finite element methods. For this purpose, a general-purpose, research oriented finite element analysis program, FEAP, and a general-purpose structural reliability analysis program, CALREL, are used. In order to model realistically structural behavior, nonlinear inelastic element routines have been added to the element library of FEAP. Further, a key ingredient of the link between CALREL and FEAP is the efficient and accurate
computation of the structural response sensitivities with respect to the design parameters or the basic random variables of the system. This could be a very complicated problem for a general inelastic system. Hence, prior to performing a finite element reliability analysis, the subroutines required for the analytical computation of the inelastic response gradients are implemented in FEAP.

Finally, with the help of the tools developed, the finite element reliability analysis of various inelastic dynamic systems is performed and the corresponding results are presented.

1.3 Organization of Thesis

This thesis is organized as follows:

In chapter 2, the fundamental concepts of modern structural reliability analysis are presented. The now well established first- and second-order reliability methods (FORM and SORM, respectively) are introduced. The four important steps to be taken in a reliability analysis, namely the transformation to the standard normal space, the determination of the design point (or most likely failure point), the approximation of the limit state surface, and the computation of the failure probability are described. The reliability sensitivity measures obtained through FORM analysis are discussed and finally a brief review of the characteristics of FORM and SORM is given illustrating their comparative merits and demerits.

In chapter 3, the finite element method is briefly reviewed. The finite element discretization of the equations of motion, the development of the stiffness and mass matrices, and the condensation of the matrices into lumped and consistent matrices are discussed. The issue of damping is also addressed. Further, the iterative methods required for inelastic analysis, the time domain discretization for dynamic analysis and the direct integration methods for inelastic dynamic analysis are explained. Finally, the four-noded isoparametric plane strain element used in this study is formulated.
In chapter 4, the classical small strain theory of plasticity is reviewed. The essential concepts of yield criterion, flow rule and hardening rules are discussed. The return map algorithm using the notion of closest point projection is explained in detail. The continuum constitutive formulations of two different plasticity models, namely the single surface, rate independent $J_2$ or Von Mises plasticity model and the non smooth multi-surface, rate independent cap model are presented. Finally, the return map algorithm as applied to the two plasticity models in their discrete form is explained. The application of the return map algorithm is also discussed for the one-dimensional case of the truss element and for the two-dimensional plane strain case.

In chapter 5, the finite element reliability method (FERM) is introduced. The importance and usefulness of the FERM for large and complex systems involving complicated limit state functions are discussed. A brief review of the theory of random fields is also given. Then, two major issues involved in FERM, namely random field discretization and response sensitivity analysis are discussed. Various methods available for efficient and accurate representation of random fields are described and their relative merits are compared. Finally, the recently developed conditional derivative method (CDM) for the response sensitivity analysis is explained. The essential step in the CDM is the computation of the element level stress gradients with respect to the constitutive model parameters with strains fixed. The computation of these gradients is described in detail for both the $J_2$ plasticity model and the cap model.

In chapter 6, the details of the link between CALREL and FEAP are presented. The new macro commands added to enhance the performance of FEAP, when used in the context of reliability analysis, are described. These macro commands include the capabilities of introducing ground excitation input and conducting a response sensitivity analysis using the CDM. The computational aspects of updating the state variables and their gradients from one time step to the next are addressed. It is shown that by the efficient utilization of memory space available, it is possible to solve complex prob-
lems in structural reliability without encountering severe restrictions regarding memory space allocation.

In chapter 7, the results of the FERM as applied to the truss element and the plane strain element are presented and discussed. Application examples involving both inelastic and dynamic systems are considered. The results obtained through the CDM for the response sensitivities are compared with those derived using the finite difference method, to verify the implementation of the CDM in FEAP. It is shown that the reliability analysis of a dynamic system may be significantly different from the reliability analysis of a static or pseudo-static system. These differences arise because of the multiply-connected nature of the limit state surface for a dynamic system, which may pose serious problems regarding the approximation of the limit-state surface and the computation of the failure probability. The shape of the limit state surface is shown for some very simple dynamic systems. As application examples, a finite element reliability analysis is carried out for a ten member truss, for a concrete gravity dam, and for a perforated inelastic plate assuming plane strain conditions.

Finally, in chapter 8, the summary and conclusions of this study are presented.
CHAPTER 2  STRUCTURAL RELIABILITY METHODS

2.1 Introduction

The theory and methods of structural reliability have developed significantly during the last 15-20 years. The methods and application of structural and mechanical reliability theory have been documented in an increasing number of textbooks such as (Ditlevsen 1981a; Ang and Tang 1984; Augusti et al. 1984; Madsen et al. 1986; Melchers 1987). Mathematical reliability models can be classified according to the level of description used in the uncertainty modeling. These models can be categorized in three groups, namely (1) random variable models, (2) random process models, and (3) random field models. Each type of model has its own computational methods. The random variable model is the simplest, for which general, practical, and powerful computation methods exist. The most important of these methods are the analytically based probability integration methods referred to as first- and second-order reliability methods (FORM/SORM). Commercial software with the computational methods for random variable reliability problems implemented is available (PROBAN 1992; STRUREL; CALREL 1989). The present study will be concerned with time-invariant random variable and random field representations. However, the random field modeling will be reduced, through discretization, to a random variable formulation and random variable reliability methods will be used throughout this study. The first- and second-order reliability methods are briefly reviewed in the next sections of this chapter.

2.2 Structural Component Reliability

The structural reliability problems under consideration are based on two fundamental assumptions. First, the uncertainties characterizing the structure and its environment are assumed to be time-invariant and modeled as random variables. These uncertain basic variables may include material properties, loads, structural configuration, member dimensions, and model uncertainty parameters. The set of all the basic random
variables is represented by the random vector \( \mathbf{X} = [X_1, \ldots, X_n] \). Second, the structure may fail in any of a finite number of modes, and with respect to each mode it is either in a safe state or in a failure state. In the context of structural reliability, the term \textit{component} has a mathematical meaning. It describes a structure or structural element whose state is defined in terms of a single, continuous function, known as the \textit{limit-state function}, which reflects an underlying physical model. Also, the term \textit{failure} is used in a general sense. It might denote the physical failure of the structure or an element thereof (e.g., fracture or collapse) or the exceedence of a serviceability limit-state. Alternatively, a component reliability analysis can be defined as the analysis of the safety of a structure with respect to a given failure mode. The limit-state function, denoted by \( g(\mathbf{x}) \), subdivides the outcome space \( \mathbf{x} \) of the basic random variables \( \mathbf{X} \) into a \textit{safe set} (or domain) and a \textit{failure set} (or domain) defined such that \( \{ \mathbf{x} | g(\mathbf{x}) \leq 0 \} \) corresponds to the failure domain and \( \{ \mathbf{x} | g(\mathbf{x}) > 0 \} \) corresponds to the safe domain, see Figure 2.1. The boundary between the safe and the failure domains, defined by \( g(\mathbf{x}) = 0 \), is called the \textit{failure surface} or \textit{limit-state surface}.

The primary purpose of a component reliability analysis is to compute the probability of failure given by the n-fold integral

\[
p_f = \int_{\{g(\mathbf{x}) = 0\}} f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} \tag{2.1}
\]

where \( f_{\mathbf{X}}(\mathbf{x}) = f_{X_1 \ldots X_n}(x_1, \ldots, x_n) \) denotes the joint probability density function (PDF) of the basic variables. Hence, the probability of failure corresponds to the probability volume in the failure domain as indicated in Figure 2.1. An alternative to \( p_f \) as a measure of safety is provided by the (strict) reliability index obtained through the one-to-one transformation

\[
\beta = \Phi^{-1}(1 - p_f) = -\Phi^{-1}(p_f) \tag{2.2}
\]

in which \( \Phi^{-1}(\cdot) \) denotes the inverse of the standard normal cumulative distribution
function (CDF). The reliability index $\beta$ is a more convenient measure than $p_F$, since for most structural problems it varies from 1 to 5, while $p_F$ varies from $10^{-1}$ to $10^{-7}$, see Table 2.1. Furthermore, it is useful to use the reliability index since various estimates of $\beta$ have been defined and used in the past as measures of safety when the available statistical information is incomplete. The function $g(x)$ can be implicit in $x$ (e.g., computation of the value of $g(x)$ may require the solution of a linear system of equations) and its evaluation may be defined algorithmically only, such as in finite element analysis. Note that the dimension of $x$ can be large (e.g., tens, hundreds or even thousands), particularly when random fields of material properties or loads are discretized into random variables, as is done in stochastic finite element methods.

Only a few analytical and exact results (corresponding to very specific limit-state functions and joint PDF’s) are known for the multi-dimensional integral in Eq. (2.1). However, these results are extremely important for the analytical computational methods as will be seen below. Direct computation of the integral in Eq. (2.1) for a general limit-state function and large $n$ is a formidable task. Standard numerical integration techniques are generally not feasible because of the arbitrary nature of the integration domain and the typically high dimension of the problem. Therefore, either Monte Carlo simulation (MCS), the analytically based first- and second-order reliability methods (FORM/SORM), or hybrid methods combining simulation with FORM/SORM must be used. Since FORM/SORM are analytical probability integration methods, they require that the problem at hand satisfies, or is modeled to satisfy, the necessary analytical conditions. For a large class of engineering problems, where FORM/SORM do apply and $p_F$ is very small ($10^{-4}$ to $10^{-8}$), FORM/SORM are generally preferred to MCS since they are much more efficient computationally.

2.3 First- and Second-Order Reliability Methods

FORM and SORM apply to random variable reliability problems, where the basic variables are continuous, i.e., they do not apply in the case of discrete random vari-
ables. Also, the limit-state function $g(x)$ should be sufficiently smooth, that is generally the function must be differentiable in order to find the expansion point(s) through an efficient algorithm.

A probability computation by FORM/SORM consists of three steps: (1) transformation of the $x$-space (or physical space) into the $u$-space (or standard normal space), (2) approximation of the failure surface in the $u$-space, and (3) computation of the failure probability corresponding to the approximating failure surface(s). These three steps are described in detail in the following sections.

### 2.3.1 Transformation to the Standard Normal Space

One of the major breakthroughs within FORM/SORM was the identification of a generally applicable method for transformation of a random vector $X$ with strictly increasing continuous joint cumulative distribution function $F_X(x)$ into a standard normal vector $U$. The standard normal space is defined by a set of independent, standard normal variates $U$ having zero means and unit covariance matrix and the following joint PDF:

$$f_U(u) = \phi_n(u) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}u^\top u}$$

(2.3)

A one-to-one transformation or mapping between the $x$-space and the $u$-space

$$u = T(x)$$

(2.4)

always exist for basic random variables having strictly increasing continuous joint cumulative distribution functions. In fact, the number of possible transformations is infinite. If the variables $X$ are normal themselves (but not standard), the linear transformation

$$u = L^{-1}D^{-1}(x - \mu_X)$$

(2.5)

produces the desired result. In Eq. (2.5), $D = \text{diag} [\sigma_i]$ where $\sigma_i = (\text{Var}[X_i])^{1/2}$, $L$ is a
lower triangular matrix obtained from Cholesky decomposition of the correlation matrix \( \mathbf{R} = [\rho_{ij}] \) such that \( \mathbf{R} = \mathbf{L}\mathbf{L}^T \), and \( \mathbf{\mu}_X \) is the mean vector of \( \mathbf{X} \). The correlation matrix of the random vector \( \mathbf{X} \) is defined as \( \mathbf{R} = \mathbf{D}^{-1}\mathbf{\Sigma}_{XX}\mathbf{D}^{-1} \) where \( \mathbf{\Sigma}_{XX} = \mathbb{E}[(\mathbf{X}-\mathbf{\mu}_X)(\mathbf{X}-\mathbf{\mu}_X)^T] = [\rho_{ij}\sigma_i\sigma_j] \) denotes the covariance matrix of vector \( \mathbf{X} \).

More generally, when the variables are non-normal, a nonlinear transformation is required. For example, when the basic variables are mutually independent with cumulative (marginal) distribution functions \( F_{X_i}(x_i) \), a transformation that produces the desired result is

\[
u_i = \Phi^{-1}[F_{X_i}(x_i)] \quad i = 1, 2, \ldots, n
\]

Therefore, in the case non-normal, independent variables, each variable is transformed separately and the vector transformation \( \mathbf{u} = \mathbf{T}(\mathbf{x}) \) is diagonal. As an example, the transformation defined in Eq. (2.6) is illustrated in Figure 2.1 for a single uniformly distributed random variable.

For statistically dependent random variables, the required transformation is necessary coupled. In the particular case of incomplete statistical information where only the marginal distributions and the covariances are known, a joint distribution for \( \mathbf{X} \), called the \textit{Nataf model} (Nataf 1962), is formally assumed in the form

\[
f_{\mathbf{X}}(\mathbf{x}) = \phi_n(z, \mathbf{R}_0) \frac{f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n)}{\phi(z_1)\phi(z_2)\ldots\phi(z_n)}
\]

(2.7)

where \( f_{X_i}(x_i) \) are the known marginal PDFs of \( X_i \), \( \phi(.) \) stands for the standard normal PDF, \( \phi_n(., \mathbf{R}_0) \) denotes the \( n \)-dimensional standard normal PDF with correlation matrix \( \mathbf{R}_0 \), and \( z_i = \Phi^{-1}[F_{X_i}(x_i)] \) are transformed correlated standard variates. The elements \( \rho_{0,ij} \) of the correlation matrix \( \mathbf{R}_0 \) are obtained from the known correlation coefficients \( \rho_{ij} = \rho_{X_iX_j} \) through the integral relation
\[ \rho_{ij} = \int \int_{-\infty}^{\infty} \left( \frac{x_i - \mu_i}{\sigma_i} \right) \left( \frac{x_j - \mu_j}{\sigma_j} \right) \phi_2(z_i, z_j, \rho_{0,ij}) \, dz_i \, dz_j \]  

(2.8)

To avoid computing this integral, simple, semi-empirical formulas relating \( \rho_{0,ij} \) to \( \rho_{ij} \) for common marginal distributions have been developed by Liu and Der Kiureghian (1986). To map a set of basic variables distributed according to the Nataf model in Eq. (2.7) into a set of uncorrelated standard normal variates, a possible transformation is

\[ u = L_0^{-1} \begin{bmatrix} \Phi^{-1}[F_{X_1}(x_1)] \\ \vdots \\ \Phi^{-1}[F_{X_n}(x_n)] \end{bmatrix} \]  

(2.9)

in which \( L_0 \) is the lower triangular decomposition of \( R_0 \). Winterstein and Bjerager (1987) have suggested a similar approach for a more general case of statistical information. They use multivariate Hermite series to model the distribution of \( X \) in terms of any number of available marginal distributions and joint moments (of any order). It is noteworthy that in the case of incomplete statistical information (i.e., when the full joint distribution of \( X \) is unknown), some arbitrariness is introduced in the definition of the joint density distribution model. That is, the corresponding transformation from the \( x \)-space to the \( u \)-space is not exact (the exact transformation being unknown since the full joint density is unknown).

In the case of complete statistical information (i.e., the full joint distribution of \( X \) is known), a possible transformation, known in the structural reliability field as the \textit{Rosenblatt transformation} (Rosenblatt 1952; Hohenbichler and Rackwitz 1981), is of the form
\[ u_1 = \Phi^{-1} [F_{X_1}(x_1)] \]
\[ u_2 = \Phi^{-1} [F_{X_1|X_1}(x_2|x_1)] \]
\[ \vdots \]
\[ u_n = \Phi^{-1} [F_{X_1|X_1 \ldots X_{n-1}}(x_n|x_1 \ldots x_{n-1})] \]  \hspace{1cm} (2.10)

where \( F_{X_1|X_1 \ldots X_{n-1}}(x_n|x_1 \ldots x_{n-1}) \) is the conditional CDF of \( X_n \) for given \( X_1, \ldots, X_{n-1} \). The transformation first transforms \( X_1 \) into a standardized normal variable, \( U_1 \). Then the conditional variable \( X_2|X_1 \) is transformed into a standardized normal variable, and so forth. This transformation is convenient when the conditional distributions are known or can be easily obtained from the joint distribution. It is noted that for statistically independent basic random variables, both the transformations in Eqs. (2.9) and (2.10) reduce to that in Eq. (2.6). The fact that the transformation in Eq. (2.6) is diagonal, and those in Eqs. (2.9) and (2.10) triangular in form is important, since this facilitates numerical inversion of these transformations during FORM and SORM analyses.

The one-to-one transformation \( u = T(x) \) maps the limit-state surface in the \( x \)-space, \( \{x \mid g(x) = 0\} \), into the limit-state surface in the \( u \)-space, \( \{u \mid G(u) = g(T^{-1}(u)) = 0\} \), as illustrated in Figure 2.2. The transformation preserves the probability mass, \( p_f \), of the failure set and the failure probability integral becomes

\[ p_f = \int_{\{g(x) = 0\}} f_X(x) \, dx = \int_{\{G(u) = 0\}} \phi_n(u) \, du \]  \hspace{1cm} (2.11)

where \( \phi_n(u) \) denotes the standard normal joint density of \( U \).

The standard normal space has three important properties: (1) the probability density function is rotationally symmetric about the origin, see Figure 2.2, (2) the probability density decays exponentially with the square of the distance in the radial and tangential directions, and (3) the probability content of some simple sets in this space are
available for arbitrary dimensions. These properties form the bases for the efficient techniques of probability integration mentioned above, namely FORM and SORM.

2.3.2 Approximations of the Failure Surface in the U-Space
From the foregoing first property of the u-space, it follows that the point \( u^* \), which is the point on the failure surface \{ \( u \mid G(u) = 0 \) \} closest to the origin (see Figure 2.2), has the highest probability density among all failure points in the standard normal space. This point is known as the most likely failure point or design point or also \( \beta \)-point. In general, there may be more than one such point. From the foregoing second property of the \( u \)-space, it follows that the main contribution to the probability integral in Eq. (2.11) comes from the neighborhood of \( u^* \), provided the limit-state surface is not strongly nonlinear and there is only one significant design point. Therefore, the limit-state surface in the neighborhood of the design point is approximated by a first- or second-order surface for which the probability content is available (exactly or approximately). Therefore, using FORM/SORM essentially replaces a multi-dimensional integral by a mathematical programming problem for finding the most likely failure point(s). In FORM, the limit-state surface is approximated by the tangent hyperplane at \( u^* \) (Hasofer and Lind 1974; Rackwitz and Fiessler 1978; Ditlevsen 1981b; Hohenbichler and Rackwitz 1981), i.e., \( G(u) \) is replaced by its first-order Taylor expansion at \( u^* \) (see Figure 2.2). The corresponding first-order approximation of the failure probability, \( p_f \), is equal to the probability mass in the linearized failure domain, and is given by

\[
P_f = p_{f1} = \Phi (-\beta)
\]

In the above equation, \( \beta \), called the (first-order) reliability index, corresponds to the distance from the origin to the tangent hyperplane and is given by \( \beta = \hat{\alpha} \cdot u^* \) where

\[
\hat{\alpha} = \frac{\nabla_u G(u^*)}{\| \nabla_u G(u^*) \|} = \frac{1}{\beta} u^*
\]

(2.13)
is the unit normal vector at the design point directed towards the failure set as shown in Figure 2.2.

In SORM, the limit-state surface is approximated by a second-order surface at \( \mathbf{u}^* \) (see Figure 2.2) such as the second-order Taylor expansion of \( G(\mathbf{u}) \) at \( \mathbf{u}^* \), or by a curvature-fitted or point-fitted parabolic surface, the axis of which is in the direction of \( \mathbf{u}^* \) (Fiessler et al. 1979; Breitung 1984; Der Kiureghian et al. 1987). In the curvature-fitting method, the approximating parabola is determined by matching its principal curvatures (at the apex) to those of the limit-state surface at the design point. This fitting operation requires computing the \((n \times n)\) Hessian matrix of the limit-state surface at the design point,

\[
H = \frac{\partial^2 G}{\partial u_i \partial u_j} \bigg|_{n \times n},
\]

and solving for its eigenvalues. This is a formidable task, particularly when a large number of random variables is involved as is the case in reliability finite element applications. Usually, the finite difference scheme is used to compute the Hessian matrix directly in the standard normal space. The motivation behind the point-fitting method (Der Kiureghian et al. 1987) is to avoid the eigen solution by assuming that the rotated axes of the standard normal space (in which the \( n \)-th rotated axis corresponds to \( \mathbf{u}^* \)) coincide with the principal axes of the parabola regardless of the true orientation of the principal axes of the limit-state surface. This assumption eliminates not only the requirement of an eigen solution but also the Hessian matrix computation. In the point fitting method the approximating parabola is defined by fitting to a set of 2 \((n-1)\) discrete points selected on the limit-state surface at prescribed distances from the design point, \( \mathbf{u}^* \). These fitting points are chosen so as to account approximately for the effects of the higher-order terms of the limit-state surface in the "wide" neighborhood of \( \mathbf{u}^* \). Advantages of the point-fitting method are that it does not require differentiability of the limit-state surface and it is insensitive to numerical noise.
generated by the numerical algorithm used to evaluate the limit-state function (e.g.,
truncation errors, discretizations in time and space, tolerances allowed for convergence
of iterative schemes, etc.).

Recently, alternative approaches have been developed in which the principal curva-
tures of the limit-state surface at the design point are obtained directly without directly
computing the Hessian, but using only gradient computations (Der Kiureghian and De
Stefano 1991; Geyskens et al. 1991). An alternative SORM approach has also been
suggested by Breitung (1991), in which the failure probability is approximated by
maximizing the log-likelihood in the original \( x \)-space.

Once the limit-state surface is approximated by a parabola, a second-order approxima-
tion of the failure probability is obtained through the asymptotically exact result for
parabolas derived by Breitung (1984):

\[
 p_f \approx p_{f2} \approx p_{\text{asymptotic}}^{\text{SORM}} = \Phi(-\beta) \prod_{j=1}^{n-1} (1 + \beta \kappa_j)^{-1/2}
\]  

(2.15)

in which \( n \) denotes the number of basic random variables, \( \kappa_i \) are the \((n-1)\) principal
curvatures of the limit-state surface at the design point, taken positive for a convex
surface with respect to the origin. Breitung has shown that \( p_{f2} \) asymptotically
approaches \( p_f \) as \( \beta \) approaches infinity with \( \beta \kappa_i \) remaining fixed. This approximation
is slightly improved for small values of \( \beta \) if the term \( \beta \kappa_i \) inside the parenthesis is
replaced by \( [\Phi(\beta) / \Phi(-\beta)] \kappa_i \) (Hohenbichler and Rackwitz 1988). Other improved
approximation formulas have been proposed such as Tvedt's three-term formula,
Tvedt's single-integral formula, and Tvedt's double-integral formula (Tvedt 1983,
1985, 1988). It is only recently that Tvedt (1988) derived an exact result for the prob-
ability content of a parabolic set in the standard normal space in terms of the single-
fold integral.
\[ P_{f_1} = P_{f_2} = P_{\text{parabolic}} = \text{SORM} = \phi(\beta) \Re \left[ i \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty \frac{e^{(u+\beta)^2/2}}{u} \left( \prod_{i=1}^{n-1} (1 - \kappa_i u)^{-1/2} \right) du \right] \]  

(2.16)

which must be evaluated numerically for practical purposes. In the above equation, \( \Re[.] \) denotes the real part of an imaginary number and \( i = \sqrt{-1} \). This exact result for the parabolic set has also been extended to cover all quadratic forms of Gaussian variables (Tvedt 1990), rendering the second-order Taylor expansion SORM feasible. The second-order Taylor expansion of the failure function at the design point yields elliptic, hyperbolic, and parabolic approximations to the failure domains. It is worth mentioning that whereas the curvature-fitted parabola is invariant with respect to the choice of the limit-state function for a fixed limit-state surface, this is not the case of the second-order Taylor expansion. For this reason, the parabolic SORM may be preferred. Using the one-to-one transformation in Eq. (2.2), the SORM reliability index, \( \beta_{\text{SORM}} \), is defined to be

\[ \beta_{\text{SORM}} = -\Phi^{-1}(p_{f_2}) \]  

(2.17)

2.3.3 Determination of the Design Point

The main effort in the first-order reliability method is in finding the minimum-distance point from the limit-state surface to the origin in the standard normal space. This requires the solution of a constrained optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \|u\| \\
\text{subject to} & \quad G(u) = 0
\end{align*} 
\]  

(2.18)

Many algorithms are available to solve this problem. The HL-RF method, originally proposed by Hasofer and Lind (1974) for second-moment reliability analysis and later extended by Rackwitz and Fiessler (1978) to include distribution information, is currently the most widely used method for solving the optimization problem in structural reliability (Madsen et al. 1986). The HL-RF method is a simple gradient algorithm
based on the following recursive formula:

$$u_{k+1} = \frac{1}{|\nabla G(u_k)|^2} [\nabla G(u_k) \cdot u_k - G(u_k)] \nabla G(u_k)^T$$  \hspace{1cm} (2.19)

where $\nabla G = [\partial G/\partial u_1 \ldots \partial G/\partial u_n]$ denotes the gradient row vector of $G(u)$. The algorithm starts at an initial point $u_1$ (usually the image in the $u$-space of the mean vector in the $x$-space) where the limit-state function and its gradient vector are computed. Using these values in the recursion formula, a new iteration point $u_2$ is obtained. If convergence has not been achieved (i.e., if the norm $|u_2 - u_1|$ is larger than a pre-specified tolerance), a new iteration cycle is performed and the process is continued until convergence is obtained. The algorithm is illustrated in Figure 2.3. In the special case of a single random variable, the HL-RF algorithm to find the design point on the limit-state surface amounts to using the Newton-Raphson method to find the root of $G(u) = 0$. Usually it is more convenient to compute the limit-state function and its gradient in the original space rather than the standard space. For that purpose, each iteration point $u$ is first mapped back into the original space by inverting the one-to-one transformation in Eq. (2.4), i.e.,

$$x = T^{-1}(u)$$  \hspace{1cm} (2.20)

Using the chain rule of differentiation, the gradient of the limit-state function in the standard space is then obtained from

$$\nabla G(u) = \nabla g(x) J^{-1}$$  \hspace{1cm} (2.21)

where $J = [\partial u_i/\partial x_j]$ denotes the Jacobian matrix of the probability transformation in Eq. (2.4). Since the transformation $T$ is always triangular, the inversion of $J$ can easily be carried out numerically.

Experience has shown that for most problems the HL-RF algorithm converges in a small number (often less than 10) of iterations. However, there are cases in which this
algorithm behaves in a seesaw fashion and fails to converge. Therefore, modifications have been suggested to improve the robustness of this method. Some of these improvements in both robustness and efficiency consisted of adding a line search procedure (Liu and Der Kiureghian 1991a; Abdo and Rackwitz 1990). Liu and Der Kiureghian (1991a) performed a comparative evaluation of several standard optimization algorithms for their applicability in structural reliability calculations. The algorithms investigated included the gradient projection method, the augmented Lagrangian method, the original HL-RF algorithm, a modified HL-RF algorithm, and the sequential quadratic programming method which theoretically is now considered the most efficient method (Schittkowski 1983, 1985/6). Their study indicated that the sequential quadratic programming method, the gradient projection method, and the modified HL-RF method performed best both in terms of robustness and efficiency.

2.4 First-Order Reliability Sensitivity Measures

Analysis of the sensitivities of the reliability index or the failure probability with respect to the various deterministic parameters defining a problem is an important part of reliability analysis. A significant advantage of FORM is that it provides, at relatively little extra computational effort, measures of sensitivities of the reliability index and first-order estimate of the failure probability, $p_{f1}$, to the different variables and parameters involved. The first set of sensitivities is with respect to variations in the coordinates of the design point, $u^*$. It can be shown that the unit vector $\hat{\alpha}$, a by-product of the optimization algorithm to find the design point, provides these sensitivities, i.e.,

$$\nabla_{u^*} \beta = \hat{\alpha}$$

(2.22)

in which $\nabla_{u^*} \beta = [\frac{\partial \beta}{\partial u^*_1} \ldots \frac{\partial \beta}{\partial u^*_n}]$ denotes the row vector of partial derivatives. This vector gives an indication as to the relative importance of the standard variates $U_i$. The values $\alpha_i$ are often referred to as the sensitivity factors in the termi-
nology of structural reliability.

The sensitivities of the reliability index to variations in the coordinates of the design point in the original space, denoted $x^* = T^{-1}(u^*)$, are given by

$$
\nabla x^* \beta = \left( \nabla u \beta \right) \bigg|_{x^* = x'} = \hat{\alpha} J \bigg|_{x = x'}
$$

(2.23)

where $J \big|_{x = x'} = \frac{\partial u}{\partial x} \bigg|_{x = x'}$ is the Jacobian matrix of the transformation at the design point. However, it is noted, that the sensitivities in Eq. (2.23) depend on the units of the basic variables $X_i$. Therefore, to compare the relative importance of the basic random variables, Der Kiureghian and Ke (1985a) defined the unit dimensionless vector

$$
\gamma = \frac{\hat{\alpha} JD}{|\hat{\alpha} JD|}
$$

(2.24)

where $D$ is the diagonal matrix of the standard deviations. The scaling of the sensitivities in Eq. (2.23) by the corresponding standard deviations yields sensitivities of the reliability index with respect to equally likely (or statistically equivalent) changes in the basic random variables $x_i^*$. Thus, the unit vector $\gamma$ gives an indication on the relative importance of the basic variables $X_i$.

Madsen (1988) showed that, for independent basic variables, an estimate (asymptotically exact) of the first-order reliability index obtained by replacing the random variable $X_i$ by a deterministic parameter equal to the median of $X_i$, $m_i$, is given by

$$
\frac{\beta (X_i = m_i)}{\beta} \approx \frac{1}{\sqrt{1 - \alpha_i^2}}
$$

(2.25)

The ratio in Eq. (2.25), called omission sensitivity factor, gives the relative error in the first-order reliability index $\beta$ when a basic variable is replaced by its deterministic median value. Therefore, for independent basic variables, the components $\alpha_i$ of the
unit normal vector \( \hat{\alpha} \) at the design point can be taken as first-order measures of the relative importance of the uncertainty in the basic variables. If a basic variable associated with a small \( \alpha^2 \) value is substituted by its median, the first-order reliability result will remain almost unchanged. The \( \alpha^2 \) can thus be interpreted as the (first-order) contribution (or fraction) of the total uncertainty which can be attributed to the corresponding basic random variable. The quantity \( \alpha^2 \) is commonly referred to as an importance factor. The omission sensitivity factors or importance factors can be used to reduce the number of variables in the search for the most likely failure point (Madsen 1988). This is especially useful in probabilistic finite element analysis which is characterized by a large number of random variables (Der Kiureghian 1985a; Arnbjerg-Nielsen and Bjerager 1988).

Other important sets of sensitivity measures, called parametric sensitivity factors, consist of: (a) sensitivity measures with respect to distribution parameters, and (b) sensitivity measures with respect to limit-state function parameters. The joint PDF in Eq. (2.1) depends on distribution parameters such as mean values, standard deviations and correlation coefficients. The distribution parameters are grouped in the vector \( \theta_d \). The joint PDF of \( X \) can be explicitly expressed as \( f_X(x|\theta_d) \). Thus the probability transformation in Eq. (2.4) can be written as \( u = T(x, \theta_d) \). It can be shown (Hohenbichler and Rackwitz 1986; Madsen et al. 1986) that the sensitivities of \( \beta \) with respect to the distribution parameters are given by

\[
\nabla_{\theta_d} \beta = \hat{\alpha} \left. \frac{\partial T(x, \theta_d)}{\partial \theta_d} \right|_{x = x^*} \tag{2.26}
\]

where the last term represents the matrix of partial derivatives of the transformed variables with respect to \( \theta_d \) for fixed \( x^* \). These partial derivatives can be easily computed analytically or numerically and represent the only additional computations since \( \hat{\alpha} \) is already available from the algorithm used for finding the design point.
Let the limit-state function be defined in terms of a set of deterministic parameters denoted by $\theta_i$, i.e., $g(x) = g(x, \theta_i)$ and $G(u) = G(u, \theta_i)$. It can be shown (Madsen et al. 1986) that the sensitivities of $\beta$ with respect to the limit-state function parameters can be expressed as

$$\nabla_{\theta_i} \beta = \frac{1}{|\nabla_u G(u, \theta_i)|} \left. \frac{\partial g(x, \theta_i)}{\partial \theta_1} \right|_{x = x^*}$$  \hspace{1cm} (2.27)

where the last term represents the row vector of partial derivatives of the limit-state function with respect to the parameters $\theta_i$ for fixed $x^*$. These partial derivatives can be easily computed analytically or numerically and represent the only additional calculations since $\nabla_u G(u, \theta_i)$ is already available from the search algorithm for the design point.

Important among the sensitivities relating to distribution parameters are the sensitivity vectors with respect to the means, $\mu = \{\mu_1, \mu_2, \ldots\}$, and standard deviations, $\sigma = \{\sigma_1, \sigma_2, \ldots\}$, of the random variables $X_i$. When scaled by the diagonal matrix of standard deviations, these sensitivity vectors,

$$\delta = \text{diag}\{\sigma\} \nabla_\mu \beta \quad \text{and} \quad \eta = \text{diag}\{\sigma\} \nabla_\sigma \beta,$$ \hspace{1cm} (2.28)

represent dimensionless measures of variation in $\beta$ with respect to equally likely variations in the mean and standard deviation of each variable. The elements within each of these vectors can be compared regardless of the nature of each variable or the units used to describe it. The former vector gives the relative importance of the random variables in terms of their central values, whereas the latter gives the relative importance with respect to their variabilities. When the variability of a random variable has a relatively small influence on $\beta$, then that variable can be replaced with a deterministic quantity (e.g., mean value or median value), thus reducing the size of the reliability problem. It is noted that the information provided by the last sensitivity vector in Eq. (2.28) is similar to the information given by the importance safety factors and the
omission sensitivity factors defined previously.

For either set of parameters $\theta (= \theta_d$ or $\theta_l)$, the sensitivities of the first-order failure probability, $p_{fl}$, are obtained through applying the chain rule of differentiation in Eq. (2.12), i.e.,

$$\nabla_{\theta} p_{fl} = -\phi(\beta) (\nabla_{\theta} \beta)$$

where $\phi(.)$ denotes the standard normal PDF.

The sensitivity measures with respect to limit-state function parameters can be used to construct the distribution of a structural response quantity. Assuming that a certain response quantity $R$ is defined, explicitly or implicitly, in terms of the basic random variables $X$, the probability density function of $R = g(X)$ can be formulated as a parametric sensitivity factor by

$$f_R (r) = \frac{d}{dr} P [g(X) - r \leq 0]$$

where the target value $r$ which also acts as a limit-state function parameter can be systematically varied to cover the domain of realization of the random variable $R$.

In summary, sensitivity measures in reliability analysis are useful in a number of ways. They help identify the important sources of uncertainty. They are used by the optimization algorithm in reliability-based optimization and can be employed in reliability-based design to guide the design process. They are essential to reliability upgrading. They are also needed in nested reliability analysis and are used in reliability uncertainty analysis to estimate the variance of the reliability index when uncertainties exist due to parameter estimation errors and model uncertainty. They are valuable in gaining insight into the behavior of complex engineering systems or in making optimum decisions (e.g., optimum maintenance and inspection programs, optimum allocation of resources, design of experiments, etc.) in the face of uncertainty. In many applications, the sensitivity measures turn out to be even more useful than the
safety measure itself.

2.5 Characteristics of FORM and SORM

FORM and SORM are analytical and approximate methods to evaluate the probability of failure defined in Eq. (2.1). To be applicable, they require that the problem at hand satisfies a few analytical properties. First, the basic random variables must be continuous and second, each component limit-state function must be continuous and preferably smooth. When the limit-state surface is not sufficiently smooth, the design point cannot be identified by efficient mathematical programming methods making use of the gradient of the function. The analytical properties of FORM/SORM enables these methods to yield relatively inexpensive sensitivity factors.

These methods are approximate and, unfortunately, it is presently not possible to give a precise measure of the potential error. In the case of FORM, the approximation is clearly weakest when the limit-state surface in the standard normal space is highly nonlinear in the neighborhood of the design point. Such nonlinearity may arise from nonlinearity of the limit-state function in the original space, or from the probability transformation in Eq. (2.4). This probability transformation can introduce strong nonlinearity in the limit-state surface in the u-space when the random variables are highly non-normal. Usually, a good estimate of the potential error is obtained by comparison with a SORM estimate of the failure probability. Nevertheless, years of experience by numerous investigators have shown that for the vast majority of structural engineering problems the FORM and SORM approximations are well within acceptable bounds.

Using FORM/SORM, the probability integral is computed with a relatively small number of computations of the limit-state function and its gradient. Experience has shown that the number of iterations needed for convergence to the design point is independent of the number of random variables. Typically for FORM the number of evaluations of the limit-state function and its gradient is in the order of 10 to 100. In the case of SORM, depending on the method used, extra limit-state function evaluations
in the order of $10n$ or $n^2$ are needed to determine the principal curvatures ($n$ denotes the number of basic random variables). A major advantage of FORM/SORM is that these numbers of limit-state and gradient evaluations are independent of the magnitude of the failure probability. Therefore, for small order probabilities ($10^{-3} - 10^{-8}$), FORM and SORM are extremely efficient as compared to simulation methods, and without competition as regards CPU-time. The CPU-time for FORM (corresponding mainly to the effort involved in the computation of the gradient) is approximately linear in $n$, and the additional CPU-time for a SORM computation grows approximately with $n^2$. The absolute computation time depends on the time necessary to evaluate the limit-state function. This time may in fact depend on the actual values of the basic variables. Extreme values may take longer due to increased nonlinearities in the problem. The CPU-time is independent of the probability level, assuming a constant time for evaluation of the failure functions.

2.6 Conclusions

In this chapter, the fundamental concepts of probabilistic methods of analysis were discussed. The popular reliability methods like FORM/SORM are described in detail. The key ingredients of the reliability methods like the transformation of the basic random variables into a standard normal space, determination of the design point and the approximation of the failure surface at the design point are discussed. An important result of FORM/SORM analysis, namely the sensitivity measures are also explained. Finally, the important characteristics of FORM/SORM are discussed.

With the growing realization of the importance of the probabilistic approach to analysis of uncertain structures, FORM/SORM are becoming one of the most important tools for performing reliability based design of structures. With the help of these methods, the inherent uncertainty of a system can be taken into account in a rational manner. As shown in the subsequent chapters, FORM/SORM coupled with the finite element method enables the structural engineer to analyze extremely complex struc-
tures in the framework of probabilistic analysis.
Table 2.1  Probability of Failure vs. Safety Index

<table>
<thead>
<tr>
<th>$p_f$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>1.28</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>2.33</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>3.09</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>3.72</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>4.26</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>4.75</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>5.20</td>
</tr>
</tbody>
</table>

Figure 2.1  Outcome Space of Basic Random Variables
Figure 2.2 Transformation of Uniformly Distributed Random Variable

Figure 2.3 One-to-One Mapping from the X-Space to the U-Space
Figure 2.4 HL-RF Algorithm to Find the Design Point
CHAPTER 3  THE FINITE ELEMENT METHOD

3.1 Introduction

The field of finite element method (FEM) has grown extremely rapidly in the last two decades, and has become a very popular technique for the solution of complex engineering problems. The finite element method represents a very general analysis tool and has been used practically in all fields of engineering analysis. With the advent of powerful digital computers, the FEM has been used for the solution of increasingly complex engineering problems.

This chapter describes the finite element discretization of the equation of motion, development of the stiffness matrices and the issues involved in the representation of the damping and mass matrices. The problems of nonlinear analysis of structures and the discretization in the time domain are also addressed. Finally, the four-noded isoparametric linear elastic element used in FEAP is also described.

3.2 Differential Form of Equations of Motion

The equations of motion of an elastoplastic body occupying a region $\Omega$ in $\mathbb{R}^n$ and undergoing small (infinitesimal) deformations can be expressed as

$$
\rho \ddot{u} = \nabla \cdot \sigma + \rho b, \quad \sigma = D : \left( \nabla (\epsilon) \dot{u} - \epsilon^p \right),
$$

$$
u = \bar{u} \text{ on } \partial \Omega_u, \quad \sigma \cdot n = \mathbf{t} \text{ on } \partial \Omega_\sigma
$$

where $\partial \Omega_u \cup \partial \Omega_\sigma = \partial \Omega$ and $\partial \Omega_u \cap \partial \Omega_\sigma = \phi$. In Eq. (1), $\dot{u}$ ($u$) and $\sigma$ denote the velocity (displacement) and stress fields over $\Omega$, $\epsilon^p$ the plastic strain rates, $\rho b$ the body forces per unit volume, $\rho$ the density and $D$ the elastic modulus tensor, with the usual symmetries. The symbol $:$ implies the contraction $D : V^k = D_{ijkl} u^i u^j$. Finally, $\bar{u}$ and $\mathbf{t}$ denote the prescribed displacements and tractions over the kinematic and traction boundaries $\partial \Omega_u$ and $\partial \Omega_\sigma$, respectively. The symmetric part of the gradient of the
displacement vector defines the small (infinitesimal) strain tensor, i.e.,

\[
\varepsilon = \nabla^{(s)} \mathbf{u} = 1/2 \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right].
\]

(3.2)

The constitutive equation, expressed in a rate form in Eq. (3.1), is also commonly expressed in the infinitesimal incremental form

\[
d\mathbf{\sigma} = \frac{d\mathbf{\sigma}}{dt} dt = \mathbf{\dot{\sigma}} dt = D : \left( d\varepsilon - d\varepsilon^p \right).
\]

(3.3)

In order to have a complete set of equations one has to supplement Eq. (3.1) with some constitutive relations for \( \varepsilon^p \). At this point, it suffices to assume that \( \varepsilon^p \) can be expressed as a function of the stresses and some set of plastic internal variables \( \mathbf{q} \), i.e.,

\[
\varepsilon^p = T(\mathbf{\sigma}, \mathbf{q}).
\]

(3.4)

The internal variables \( \mathbf{q} \) may, for example, represent the yield stress for an isotropic hardening model or the translation of the elastic domain for a kinematic hardening model. Eq. (3.4) is general enough to accommodate perfect and hardening plasticity. It is finally assumed that the evolution of the internal variables is governed by kinetic equations of the form

\[
\dot{\mathbf{q}} = f(\mathbf{\sigma}, \mathbf{q}).
\]

(3.5)

Combining Eqs. (3.1), (3.4), and (3.5), the following set of equations of evolution is obtained

\[
\begin{align*}
\rho \ddot{\mathbf{u}} &= \nabla \cdot \mathbf{\sigma} + \rho \mathbf{b}, \\
\mathbf{\sigma} &= D : \nabla^s \mathbf{u} - D : T(\mathbf{\sigma}, \mathbf{q}), \\
\mathbf{u} &= \mathbf{u}_0 \text{ on } \partial \Omega_u, \\
\mathbf{\sigma} \cdot \mathbf{n} &= \mathbf{t} \text{ on } \partial \Omega_\sigma
\end{align*}
\]

which, together with the initial conditions

\[
\begin{align*}
\mathbf{u}(x, 0) &= \mathbf{u}_0, \\
\mathbf{u}(x, 0) &= \mathbf{u}_0, \\
\mathbf{\sigma}(x, 0) &= \mathbf{\sigma}_0, \\
\mathbf{q}(x, 0) &= \mathbf{q}_0.
\end{align*}
\]

(3.7)

define an initial boundary value problem for the unknowns \( \mathbf{u}, \mathbf{\sigma}, \mathbf{q} \).
3.3 Finite Element Discretization of Equations of Motion

In practice, approximate solutions of Eqs. (3.6) are obtained using the finite element method for the spatial discretization combined with a time stepping scheme for the integration over time. For the sake of completeness, a brief account of the derivation of the discretized equations of motion is given next.

By applying the principle of virtual work (displacement), a weak form of the linear momentum balance equation is obtained as

$$
\int_{\Omega} \left[ \rho \left( \ddot{u} - b \right) \cdot \delta u + \sigma : V^k \delta u \right] d\Omega = \int_{\partial\Omega_p} t \cdot \delta u \ d\Gamma
$$

(3.8)

in symbolic vector notation, or

$$
\int_{\Omega} \rho \ddot{u}_i \delta u_i d\Omega + \int_{\Omega} \sigma_{ij} \delta \epsilon_{ij} d\Omega = \int_{\Omega} \rho b_i \delta u_i d\Omega + \int_{\partial\Omega_p} \tau_i \delta u_i d\Gamma
$$

(3.9)

in indicial-tensor notation in which a repeated index implies summation over the range of the index, i.e., $\rho b_i \delta u_i = \rho b_1 \delta u_1 + \rho b_2 \delta u_2 + \rho b_3 \delta u_3$. In Eqs. (3.8) and (3.9), $\delta u_i$ and $\delta \epsilon_{ij}$ are virtual displacement increments and virtual (small) strain increments, respectively, and they form a compatible set of deformations; and the set $\{ \sigma_{ij}, \rho \ddot{u}_i, \rho b_i, \text{ and } \tau_i \}$ is in dynamic equilibrium.

In a matrix form, Eq. (3.8) or (3.9) becomes

$$
\int_{\Omega} \rho \delta u^T \cdot \ddot{u} d\Omega + \int_{\Omega} \delta \epsilon^T \cdot \sigma d\Omega = \int_{\Omega} \delta u^T \cdot \rho b d\Omega + \int_{\partial\Omega_p} \delta u^T \cdot t \ d\Gamma
$$

(3.10)

where the vectors for acceleration $\{ \ddot{u} \}$, displacement $\{ u \}$, strain $\{ \epsilon \}$, and stress $\{ \sigma \}$ are defined as

$$
\begin{align*}
\mathbf{u}^T &= \{ u_1, u_2, u_3 \}, & \delta \mathbf{u}^T &= \{ \delta u_1, \delta u_2, \delta u_3 \}
\end{align*}
$$

(3.11)
\[
\epsilon^T = \{ \epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31} \} \\
(3.12)
\]

\[
\delta\epsilon^T = \{ \delta\epsilon_{11}, \delta\epsilon_{22}, \delta\epsilon_{33}, 2\delta\epsilon_{12}, 2\delta\epsilon_{23}, 2\delta\epsilon_{31} \} \\
(3.13)
\]

\[
\sigma^T = \{ \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31} \} \\
(3.14)
\]

For a \textit{geometrically linear} analysis, or a small-deformation analysis, the strains are linearly related to the displacement field as

\[
\epsilon = Lu, \quad \delta\epsilon = L\delta u \\
(3.15)
\]

where \( L \) is the differential operator matrix defined as

\[
L = \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 \\
0 & \frac{\partial}{\partial x_2} & 0 \\
0 & 0 & \frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\
0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\
\end{bmatrix}_{(6 \times 3)} \\
(3.16)
\]

The \textit{finite element approximation} is introduced such that the displacement field \( u \) within the element \( e \) is defined as

\[
u^{(e)} = \sum_{\text{Nodes}} N_I(\xi) u^I = [N_I, N_J, \ldots] \begin{bmatrix} u^I \end{bmatrix}^{(e)} = NU^{(e)} \\
(3.17)
\]

in which the functions \( N_I \) are called the \textit{shape functions} at node \( I \), \( \xi \) are the natural
coordinates for the element, and \( \mathbf{u}^I \) are the values of the displacement vector at node I. The shape functions \( N_I, N_J, \ldots \) are chosen such that they give appropriate nodal displacements when the coordinates of the corresponding nodes are inserted in Eq. (3.17). In general, if all the components of the displacement field are interpolated in an identical manner, we have

\[
N_I = N_I(\xi) \mathbf{I}
\]  
(3.18)

where \( \mathbf{I} \) denotes the (3x3) identity matrix, and \( N_I \) is the shape function at node I chosen such that \( N_I = 1 \) at \( x_1^I, x_2^I, \) and \( x_3^I \) but zero at the other nodes of the element.

By combining Eqs. (3.15) and (3.17), the strain field within the element is directly related to the nodal displacement vectors,

\[
\nabla^{(s)} \mathbf{u} = \boldsymbol{\epsilon}(\mathbf{u}) = \sum_{\text{Nodes}} \mathbf{B}_I \mathbf{u}^I = \mathbf{B} \mathbf{U}^{(e)}
\]  
(3.19)

in which \( \mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \ldots] \), and \( \mathbf{B}_I \) is the strain-displacement matrix for node I of the element, expressed as

\[
\mathbf{B}_I = \begin{bmatrix}
N_{I,1} & 0 & 0 \\
0 & N_{I,2} & 0 \\
0 & 0 & N_{I,3} \\
N_{I,2} & N_{I,1} & 0 \\
0 & N_{I,3} & N_{I,2} \\
N_{I,3} & 0 & N_{I,1}
\end{bmatrix}_{(6 \times 3)}
\]  
(3.20)

where

\[
N_{I,i} = \frac{\partial N_I}{\partial x_i}.
\]

The approximated displacement field over the whole region \( \Omega \) is obtained by summing
the approximated displacement fields over all the elements $\Omega_e$, i.e.,

$$\mathbf{u} = \sum_{e=1}^{Nel} \mathbf{u}^{(e)}$$  \hfill (3.21)

Using the same shape or interpolation functions for both the real and virtual displacements and Eqs. (3.17) and (3.19) in Eq. (3.10), it is found that

$$\sum_{e=1}^{Nel} \left( \delta \mathbf{U}^{(e)} \right)^T \int_{\Omega_e} \rho \mathbf{N}^T \mathbf{d} \Omega_e \mathbf{U}^{(e)} + \sum_{e=1}^{Nel} \left( \delta \mathbf{U}^{(e)} \right)^T \int_{\Omega_e} \mathbf{B}^T \mathbf{\sigma} \mathbf{d} \Omega_e = $$

$$\sum_{e=1}^{Nel} \left( \delta \mathbf{U}^{(e)} \right)^T \int_{\Omega_e} \mathbf{N}^T \mathbf{b} \mathbf{d} \Omega_e + \sum_{e=1}^{Nel} \left( \delta \mathbf{U}^{(e)} \right)^T \int_{\partial \Omega_{e,c}} \mathbf{N}^T \mathbf{t} \mathbf{d} \Gamma_e$$  \hfill (3.22)

The above equation can be rewritten as

$$\sum_{e=1}^{Nel} \left( \delta \mathbf{U}^{(e)} \right)^T \left[ \int_{\Omega_e} \rho \mathbf{N}^T \mathbf{d} \Omega_e \mathbf{U}^{(e)} + \int_{\Omega_e} \mathbf{B}^T \mathbf{\sigma} \mathbf{d} \Omega_e - \int_{\partial \Omega_{e,c}} \mathbf{N}^T \mathbf{b} \mathbf{d} \Omega_e - \int_{\partial \Omega_{e,c}} \mathbf{N}^T \mathbf{t} \mathbf{d} \Gamma_e \right] = 0$$  \hfill (3.23)

Since the above equality is valid for any arbitrary set of virtual element nodal displacement vectors, $\delta \mathbf{U}^{(e)}$, the following equality must hold

$$\sum_{e=1}^{Nel} \left[ \int_{\Omega_e} \rho \mathbf{N}^T \mathbf{d} \Omega_e \mathbf{U}^{(e)} + \int_{\Omega_e} \mathbf{B}^T \mathbf{\sigma} \mathbf{d} \Omega_e - \int_{\Omega_e} \mathbf{N}^T \mathbf{b} \mathbf{d} \Omega_e - \int_{\partial \Omega_{e,c}} \mathbf{N}^T \mathbf{t} \mathbf{d} \Gamma_e \right] = 0$$  \hfill (3.24)

Eq. (3.24) represents the spatially discretized weak form of the linear momentum balance equations and expresses the dynamic equilibrium between the inertia forces (first term), the internal resisting forces (second term) and the external forces (third and fourth terms) of the whole system or structure.

The various terms in Eq. (3.24) define the following element matrices:
\[
\mathbf{m}^{(e)} = \int_{\Omega_e} \rho \mathbf{N}^T \mathbf{N} \, d\Omega_c ,
\]

(3.25)

\[
\mathbf{r}^{(e)} = \int_{\Omega_e} \mathbf{B}^T \mathbf{\sigma} \, d\Omega_c ,
\]

(3.26)

\[
\mathbf{f}^{(e)} = \int_{\Omega_e} \mathbf{N}^T \rho \mathbf{b} \, d\Omega_c + \int_{\partial \Omega_{\alpha,c}} \mathbf{N}^T \mathbf{t} \, d\Gamma_c ,
\]

(3.27)

where \(\mathbf{m}^{(e)}\) is the element mass matrix, \(\mathbf{r}^{(e)}\) is the element internal resisting force vector, and \(\mathbf{f}^{(e)}\) is the element external force vector. Using the relations in Eqs. (3.17) to (3.19), the above element matrices can be further expressed as an assembly of element submatrices:

\[
\mathbf{m}^{(e)} = \sum_I \sum_J \mathbf{m}_{IJ}^{(e)} \quad \text{where} \quad \mathbf{m}_{IJ}^{(e)} = \int_{\Omega_e} \rho N_I N_J \mathbf{I} \, d\Omega_c
\]

(3.28)

\[
\mathbf{r}^{(e)} = \sum_I \mathbf{r}_I^{(e)} \quad \text{where} \quad \mathbf{r}_I^{(e)} = \int_{\Omega_e} \mathbf{B}_I^T \mathbf{\sigma} \, d\Omega_c
\]

(3.29)

\[
\mathbf{f}^{(e)} = \sum_I \mathbf{f}_I^{(e)} \quad \text{where} \quad \mathbf{f}_I^{(e)} = \int_{\Omega_e} N_I \rho \mathbf{b} \, d\Omega_c + \int_{\partial \Omega_{\alpha,c}} N_I \mathbf{t} \, d\Gamma_c
\]

(3.30)

In the above equations, the subscripts \(I\) and \(J\) over which the sums are carried out denote the element node numbers.

By assembling the element matrices or submatrices into the corresponding system matrices, Eq. (3.24) reduces to the following matrix differential equation or spatially discretized equation of motion:

\[
\mathbf{M} \ddot{\mathbf{u}} (t) + \mathbf{R} (\mathbf{u}, t) = \mathbf{F} (t)
\]

(3.31)

where \(t\) denotes the time, \(\mathbf{u}(t)\) denotes the system vector of nodal displacements, \(\mathbf{M}\) is the system mass matrix, \(\mathbf{R}(\mathbf{u}, t)\) is the system internal resisting force vector, \(\mathbf{F}(t)\) is the system external load vector, and a superposed dot indicates differentiation with respect to time.
In an inelastic analysis, because of the nonlinear relationship between the stress, \( \sigma \), and the strain, \( \varepsilon \), the internal resisting force vector of both the element and the system is a nonlinear equation of strains, and therefore, is a nonlinear function of the nodal displacements. Due to this nonlinearity, iterative methods are usually employed to solve Eq. (3.31) for \( \mathbf{u}(t) \) corresponding to a given set of time-varying external forces \( \mathbf{F}(t) \). Moreover, for an elastic-plastic constitutive relation, the resisting force vector is also dependent on the deformation history, and an incremental analysis procedure should be used to trace the time-variation of the displacement, strain, and stresses.

In the particular case of a linear elastic material, the stresses are linearly related to the strains and can be expressed as

\[
\sigma = D\varepsilon = D\mathbf{B}U^{(e)}
\]  

(3.32)

using Eq. (3.19). Hence, the element internal resisting force vector in Eq. (3.26) becomes

\[
\mathbf{r}^{(e)} = \int_{\Omega_e} B^T DB d\Omega_e \mathbf{U}^{(e)} = \mathbf{k}^{(e)} \mathbf{U}^{(e)}
\]  

(3.33)
in which \( \mathbf{k}^{(e)} \) is the element stiffness matrix. Using Eq. (3.19), the element stiffness matrix can be further expressed as

\[
\mathbf{k}^{(e)} = \sum_i \sum_j \mathbf{k}_{ij}^{(e)} \quad \text{where} \quad \mathbf{k}_{ij}^{(e)} = \int_{\Omega_e} B_i^T DB_j d\Omega_e .
\]  

(3.34)

In the linear elastic case, the system spatially discretized matrix equation of motion is given by

\[
\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t)
\]  

(3.35)

where \( \mathbf{K} \) is the system stiffness matrix.
3.3.1 Consistent versus Lumped Mass Matrices

When the shape functions used in Eqs. (3.25) and (3.28) are the same as those used in deriving the internal resisting force vector or stiffness matrix, Eqs. (3.29) and (3.34), the resulting matrix is called consistent mass matrix. A simpler and historically earlier formulation is the lumped mass matrix, which is obtained by assuming that the distributed mass of an element can be lumped as point masses along the translational degrees of freedom at the nodes. Use of a lumped mass matrix in place of a consistent mass matrix in dynamic analysis results in considerable savings in computation costs, for two reasons: (1) the lumped mass matrix is diagonal (which simplifies considerably the eigenvalue analysis), while the consistent mass matrix has many off-diagonal terms leading to what is called mass coupling; (2) the rotational degrees of freedom can be eliminated from a lumped-mass analysis (by neglecting rotational inertia and using static condensation), whereas all rotational and translational degrees of freedom must be included in a consistent mass analysis. In terms of computational efficiency, lumped mass matrices are simpler to form, occupy less storage space, and require less computational effort. Indeed, some methods of dynamic analysis are practicable only with lumped mass matrices (Cook et al., 1989).

The procedure of mass lumping must satisfy the essential requirement of mass preservation, i.e.,

$$\sum_i M_{ii} = \int_\Omega \rho d\Omega$$

(3.36)

where $M_{ii}$ is the diagonal of the system mass matrix $M$. The earliest procedures performed mass diagonalization using physical intuition only. For example, the total mass of an element is equally distributed at the nodes of the element. Later, rational procedures were proposed, three of which are:

(1) The row sum method, which consists in making the diagonal the sum of all terms in the row of the consistent matrix, i.e.,
\[ M_{ii} \leftarrow \sum_j M_{ij}. \] (3.37)

(2) The diagonal scaling method, which consists of scaling the diagonals of the consistent matrix to satisfy the mass conservation requirement, i.e.,

\[ M_{ii} \leftarrow c M_{ii}. \] (3.38)

This method requires only the computation of the diagonal terms of the consistent mass matrix.

(3) Evaluation of \( M \) using a quadrature involving only the nodal points as sampling points, thus automatically yielding a diagonal mass matrix for standard finite element shape functions in which \( N_j = 0 \) for \( x = x_j, J \neq I \).

Methods (1) and (3) can lead to negative or zero masses (Zienkiewicz and Taylor, 1989). Such a loss of positive definiteness may have adverse effects in certain algorithms for dynamic analysis, and may cancel out the advantages of lumping. In general, the use of lumped mass matrices is recommended only as a convenient numerical device generally paid for by some loss of accuracy. Comparative studies have shown that mass lumping is appropriate if the continuum model is represented by differential equations of second order, but that the consistent mass formulation should be used for higher order systems such as beams and plates (Tong et al., 1971).

3.3.2 Damping

Damping is present in all structures. The primary effect of the damping is to remove energy from the structural system. This loss of energy from the damped system results in the decay of amplitude of free vibration. In steady-state forced vibration, the loss of energy is balanced by the energy that is supplied by the excitation. There are several mechanisms which can induce damping in a system: internal friction, material microcracking, imperfect elasticity or hysteresis in the material, air resistance, fluid resistance, dry friction between components, looseness of joints, and many other complex
causes. Usually, damping forces are small in magnitude compared to inertia forces or internal resisting forces. Despite this, damping forces may significantly affect structural response, especially for harmonic-type loading at a frequency near a natural frequency of the structure. Therefore, damping should be accounted for in a dynamic response analysis.

The mass and stiffness of a dynamic system can be determined from its physical characteristics. Mass can be calculated from the known geometry and the mass density, whereas stiffness is determined based on the geometry and material properties. However, it is difficult, impractical, if not impossible to relate damping resistance to known or measurable physical characteristics. Therefore, the actual damping mechanism is usually approximated by viscous damping or equivalent viscous damping at the system level. The concept of equivalent viscous damping is based on the equivalence of energy removed by a viscous damping mechanism and by the actual nonviscous damping mechanism (Craig, 1981). Linear viscous damping provides the simplest mathematical model of damping, namely a force directly proportional to the velocity. In general, viscous damping forces are known to provide analytical results for the response of a system that, for small amounts of damping, conform reasonably well to experimental observations (Humar, 1990).

Under the assumption of linear viscous damping, the matrix differential equation of motion, Eq. (3.31), becomes

\[ \mathbf{M} \ddot{\mathbf{u}} (t) + \mathbf{C} \dot{\mathbf{u}} (t) + \mathbf{R} (\mathbf{u}, t) = \mathbf{F} (t) \]  

(3.39)

where \( \mathbf{C} \) is the system damping matrix. Unlike the mass and stiffness matrices which could be obtained through assembling element mass and stiffness matrices derived from the internal mass and stiffness distributions of the system, respectively, the system damping matrix cannot be obtained from element damping matrices determined from the internal damping characteristics which are difficult or impossible to define. Furthermore, if the structure is linear elastic, the above equations of motion reduce to
\[ M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = F(t) \]  

(3.40)

When using the mode superposition method of dynamic analysis for linear elastic structures and assuming that the undamped mode shapes are $C$-orthogonal, there is no need to specify a physical damping matrix as the damping characteristics are represented more conveniently in terms of the modal damping ratios. The modal damping ratios are specified based on experiments on similar structures. However, there are dynamic analysis situations where an explicit system damping matrix is needed. These situations are: (1) analysis of a linear system using a direct step-by-step integration method, (2) analysis of a linear system with nonproportional linear viscous damping, and (3) analysis of a nonlinear system for which a viscous damping matrix is required to account for the energy dissipated by all mechanisms other than material yielding.

One procedure for defining a physical damping matrix is to employ a particular form of proportional damping called Rayleigh damping, in which the damping matrix is chosen to be a linear combination of the system mass and stiffness matrices, i.e.,

\[ C = \alpha M + \beta K \]  

(3.41)

where $\alpha$ and $\beta$ are constants. These constants can be selected to produce specified modal damping factors for two given modes. After uncoupling the equations of motion, Eq. (3.40), by using modal coordinates, it can be shown that modal damping ratios are given by

\[ \xi_n = \frac{1}{2} \left( \frac{\alpha}{\omega_n} + \beta \omega_n \right) \]  

(3.42)

† The undamped mode shapes are $C$-orthogonal if $\Phi^T C \Phi$ is diagonal where $\Phi$ is the matrix of undamped mode shapes. In this case, the transformation from geometric to modal coordinates uncouples the equations of motion. When the damping matrix possesses such a property, damping in the system is referred to in the literature as proportional, orthogonal, classical, modal (Craig, 1981). Proportional damping is suitable for modeling the behavior of most structural systems, in which the damping mechanism is distributed rather uniformly throughout the structure. However, for structures made up of more than a single type of material, where the different materials provide drastically differing energy-loss mechanisms in various parts of the structure, the resulting damping will be nonproportional (Clough and Penzien, 1993).
Thus, Rayleigh damping is easy to define by choosing $\xi_n$ for two modes and solving for $a$ and $\beta$. The damping in the remaining modes is then determined by Eq. (3.42). Eq. (3.42) shows that for mass proportional damping ($\beta=0$), the damping ratio is inversely proportional to the corresponding undamped modal frequency while for stiffness proportional damping ($\alpha=0$) it is directly in proportion with the modal frequency.

In performing a nonlinear analysis, it is appropriate to define the proportional damping matrix for the initial elastic state of the system (before nonlinear deformations have occurred) and to assume that this damping property remains constant during the response (Clough and Penzien, 1993).

The disadvantage of Rayleigh damping is that it does not permit realistic damping to be defined for all the modes of interest. An alternative method exists for evaluating an orthogonal damping matrix associated with any given set of modal damping ratios (Wilson and Penzien, 1972).

### 3.3.3 Earthquake Loading

The equation of motions of a system express the equilibrium between the inertia forces, $f_i(t)$, the damping forces, $f_D(t)$, the system restoring or resisting forces, $f_R(t)$, (not necessarily elastic), and the applied external dynamic loads, $f_E(t)$, i.e.,

$$f_i(t) + f_D(t) + f_R(t) = f_E(t) \quad (3.43)$$

The inertia forces depend on the total acceleration response of the system and are given by

$$f_i(t) = M\ddot{u}_i(t) \quad (3.44)$$

When dealing with earthquake excitation, it is convenient to express the total displacement vector, $u_i(t)$, as the superposition of the relative displacement vector, $u(t)$, and the quasi-static displacement, $u^s(t)$, that would result from a static-support displacement (Clough and Penzien, 1993), namely
\[ \mathbf{u}^1(t) = \mathbf{u}(t) + \mathbf{u}^s(t). \]  

(3.45)

The quasi-static displacements can be expressed conveniently by an influence coefficient vector \( \mathbf{r} \) which represents the displacements resulting from a unit support displacement; thus \( \mathbf{u}^s(t) = \mathbf{r} \mathbf{u}_g(t) \) and

\[ \mathbf{u}^1(t) = \mathbf{u}(t) + \mathbf{r} \mathbf{u}_g(t) \]  

(3.46)

where \( \mathbf{u}_g(t) \) represents the free-field input displacement at the base of the structure. It is important to emphasize that the influence coefficient vector depends on the type of rigid base displacement considered (e.g., translational or rotational ground excitation) and the structural configuration. For example, in a two-dimensional case with horizontal ground motion, the components of \( \mathbf{r} \) will be unity if they correspond to a horizontal degree of freedom or zero if not. In the present work, it is assumed that the same free-field ground motion acts simultaneously at all support points of the structure with its foundation, which is equivalent to considering the foundation soil or rock to be rigid. Moreover, only the case of a single horizontal ground motion component will be considered, i.e., \( \mathbf{u}_g(t) \) is a scalar-valued time function. However, the generalization to multiple-component rigid-soil excitation is straightforward. Substituting Eqs. (3.44) and (3.46) in Eq. (3.43) and noticing that in the case of ground excitation the applied external dynamic loads are zero, it is found that

\[ \mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{f}_D(t) + \mathbf{f}_R(t) = -\mathbf{M} \ddot{\mathbf{u}}_g(t) \]  

(3.47)

where \( \ddot{\mathbf{u}}_g(t) \) represents the free-field input rigid-base acceleration. Usually, the damping and resisting forces depend on the relative velocities, \( \dot{\mathbf{u}}(t) \), and the relative displacements, \( \mathbf{u}(t) \), respectively. The right-hand-side of Eq. (3.47) is referred to as the effective earthquake force or the effective-force vector generated by the earthquake motion.
3.4 Step-by-Step Methods for the Response to General Dynamic Loading

The *step-by-step procedure*, also called the *direct integration method*, is a general approach for the computation of the structural response to a dynamic load. It is well suited for nonlinear dynamic response analysis, since it avoids any use of the principle of superposition. In a step-by-step procedure, the loading and the response time histories are subdivided into a series of time intervals or time steps. The response during each step is determined by using the history of the loading during the time step and the displacement and velocity fields of the structure at the beginning of the time step as initial conditions. Nonlinear material behavior can be easily incorporated into this procedure by assuming that the properties of the structure remain constant during each time step. Hence, in this fashion, the nonlinear dynamic analysis of a structure is converted into a series of linear analyses. Any degree of refinement may be achieved by decreasing the length of the time step.

The step-by-step methods of dynamic response analysis provide the most general approach to the analysis of the nonlinear response of a structure. However, they are also applicable for the analysis of linear elastic structures. In fact, due to their convenience and effectiveness, these methods are often used in time-history analysis.

The step-by-step methods may be classified as either explicit or implicit. An explicit method is defined as one in which the response values computed at the end of a time step depend only on the response quantities at the end of the previous time step. On the contrary, in an implicit method the response quantities computed at the end of a new time step involve other response quantities at the end of the new time step. Hence, trial values of the response must first be assumed and then these are refined by further iterations until convergence is achieved. Therefore, implicit procedures are computationally more expensive than explicit methods. However, implicit methods offer other advantages in terms of stability and convergence properties of the numerical scheme. For example, explicit methods are usually only conditionally stable,
whereas implicit methods are often unconditionally stable.

The choice of the length of the time increment is an important issue in any step-by-step procedure. The accuracy of the method can be improved by decreasing the time interval leading to a trade-off between accuracy and computational cost. In any case, the time interval must be short enough for an accurate description of the loading history. Special care must also be taken to define the time step in order to capture the high-frequency response in sufficient detail.

Apart from the accuracy considerations, the choice of the time interval also depends on the stability properties of the method at hand. Instability of a numerical scheme appears as an unbounded amplification of the errors with elapsing time. If the method is only conditionally stable, then the time step must be short enough to avoid any instability problem. Other factors that contribute to the numerical errors are the round-off errors which result from not using a sufficient number of significant digits and the truncation errors which arise from using too few terms in the Taylor series expansion of a response quantity.

Many classes of step-by-step methods are available in the literature. Among the most popular methods for dynamic response analysis of structures is the class of Newmark-beta methods which is briefly reviewed in the next section.

### 3.4.1 Newmark-beta Methods

In the Newmark-beta schemes, the velocities and displacements at the end of the time step are expressed in terms of the velocities and displacements at the beginning of the time step and the accelerations at the end of the time step as

\[
\begin{align*}
\ddot{u}_{n+1} &= \dot{u}_n + (1 - \gamma) h \ddot{u}_n + \gamma h \ddot{u}_{n+1} \\
\dot{u}_{n+1} &= u_n + h \dot{u}_n + \left( \frac{1}{2} - \beta \right) h^2 \ddot{u}_n + \beta h^2 \ddot{u}_{n+1}
\end{align*}
\]

(3.48)  (3.49)
In the above, h denotes the length of the time increment and the subscripts n and n+1 refer to the response values at the end of the n-th and (n+1)-th time step, respectively. The parameter γ provides a linearly varying weighting factor between the influence of the initial and final accelerations on the change of velocity during the time step, and the parameter β provides a weighting factor between the influence of the initial and final accelerations on the change of displacement during the time step.

The factor γ also controls the amount of artificial or numerical damping induced by the procedure. There is no numerical damping if a value of γ = 1/2 is used. Two particular cases of the Newmark-beta methods are the constant-average-acceleration method which corresponds to (γ = 1/2, β = 1/4) and the linear acceleration method corresponding to (γ = 1/2, β = 1/6). The constant-average-acceleration method can be derived by assuming that the acceleration remains constant during the time step, whereas the linear acceleration method assumes that the acceleration varies linearly during the time step. A significant advantage of the constant-average-acceleration method is that it is unconditionally stable. Hence, the length of the time increment may be selected considering only the accuracy requirement.

3.4.2 Explicit Formulation of the Newmark-beta Methods

In general, implicit formulations of the Newmark-beta methods are computationally expensive since several iterations are required at each time step to determine the response time history. Therefore, they are usually recast into an explicit form in which the response quantities at the end of a time step are expressed in terms of the response quantities obtained at the end of the previous time step. The derivation of the explicit formulation of the constant-average-acceleration method is presented below.

Denoting the accelerations at the beginning and at the end of the time step as \( \ddot{u}_n \) and \( \ddot{u}_{n+1} \), respectively, the average acceleration during the time step is
\[ \ddot{u}_{av} = \frac{1}{2} (\ddot{u}_n + \ddot{u}_{n+1}) . \]  

(3.50)

Assuming that the acceleration over the time step is constant and equal to the average acceleration, the velocity \( \dot{u}_{n+1} \) and the displacement \( u_{n+1} \) at the end of the time step are given by

\[ \dot{u}_{n+1} = \frac{h}{2} \left( \ddot{u}_n + \ddot{u}_{n+1} \right) + \dot{u}_n \]  

(3.51)

\[ u_{n+1} = u_n + \dot{u}_n h + \left( \ddot{u}_n + \ddot{u}_{n+1} \right) \frac{h^2}{4} \]  

(3.52)

The acceleration \( \ddot{u}_{n+1} \) and the velocity \( \dot{u}_{n+1} \) in Eqs. (3.51) and (3.52) can be solved in terms of the displacement at the end of the time step, \( u_{n+1} \), as

\[ \ddot{u}_{n+1} = \frac{4}{h^2} (u_{n+1} - u_n) - \frac{4}{h} \ddot{u}_n - \ddot{u}_n \]  

(3.53)

\[ \dot{u}_{n+1} = \frac{2}{h} (u_{n+1} - u_n) - \dot{u}_n \]  

(3.54)

In the linear elastic case, the equations of equilibrium at time \( t_{n+1} \) reduce to

\[ M \ddot{u}_{n+1} + C \dot{u}_{n+1} + K u_{n+1} = F_{n+1} \]  

(3.55)

Substituting the expressions for \( \ddot{u}_{n+1} \) and \( \dot{u}_{n+1} \) from Eqs. (3.52) and (3.53), we have

\[ \bar{K} u_{n+1} = \bar{F}_{n+1} \]  

(3.56)

where the effective dynamic stiffness matrix, \( \bar{K} \), and the effective load vector, \( \bar{F}_{n+1} \), are

\[ \bar{K} = K + \frac{2}{h} C + \frac{4}{h^2} M \]  

(3.57)

and

\[ \bar{F}_{n+1} = F_{n+1} + C \left( \frac{2}{h} u_n + \dot{u}_n \right) + M \left( \frac{4}{h^2} u_n + \frac{4}{h} \ddot{u}_n + \ddot{u}_n \right) . \]  

(3.58)
Hence, using the step-by-step procedure, the analysis of a dynamic system can be reduced to a series of static analyses with an effective stiffness matrix, $\tilde{K}$, and an effective load vector, $\tilde{F}_{n+1}$.

Using this formulation, the nodal displacements at the end of the time step can be computed by solving the linear system of equations (3.56) and using only the response data available at the beginning of the time step. Once the displacement vector is known, the nodal velocities at the end of the time step can be determined using Eq. (3.54). Finally, to preserve the conditions of dynamic equilibrium at the end of the time step, the acceleration vector, $\ddot{u}_{n+1}$, is derived from the dynamic equilibrium at $t_{n+1}$ as

$$\ddot{u}_{n+1} = M^{-1} (F_{n+1} - Ku_{n+1} - Cu_{n+1}).$$

(3.59)

### 3.4.3 Incremental Formulation for Nonlinear Analysis

For performing a nonlinear analysis, it is assumed that the system properties (i.e., stiffness, damping) remain constant only during short time increments. Hence, it is necessary to reformulate the step-by-step procedure in terms of the incremental equations of motion.

Denoting the inertial forces on the body as $f_I(t)$, the damping forces as $f_D(t)$ and the spring restoring forces as $f_S(t)$, the equations of dynamic equilibrium of the body at time $t = t_n$ are expressed as

$$f_{I,n} + f_{D,n} + f_{S,n} = F_n.$$  

(3.60)

At the end of the time step, the requirement for dynamic equilibrium is

$$f_{I,n+1} + f_{D,n+1} + f_{S,n+1} = F_{n+1}.$$  

(3.61)

Subtracting Eq. (3.61) from Eq. (3.60), we have the equation of equilibrium in the incremental form:
\[ \Delta f_{l, n+1} + \Delta f_{D, n+1} + \Delta f_{S, n+1} = \Delta F_{n+1} \]  
(3.62)

in which

\[ \Delta f_{l, n+1} = f_{l, n+1} - f_{l, n} = M \Delta \bar{u}_{n+1} \]  
(3.63)

\[ \Delta f_{D, n+1} = f_{D, n+1} - f_{D, n} = C \Delta u_{n+1} \]  
(3.64)

\[ \Delta f_{S, n+1} = f_{S, n+1} - f_{S, n} = K \Delta u_{n+1} \]  
(3.65)

\[ \Delta F_{n+1} = F_{n+1} - F_{n} \]  
(3.66)

For a general nonlinear system, the stiffness matrix \( K \) depends on the displacements \( u \), which are not known in advance. Hence, an iteration scheme such as the Newton-Raphson method must be used to integrate the equations of motion.

### 3.5 Numerical Algorithms for Nonlinear Static Analysis

Since the stiffness of a nonlinear system depends on its displacement field, at each time step iterations are required to solve for the nodal displacements which satisfy the equilibrium conditions. In the following discussion, \( u_{n+1}^{i} \) denotes the i-th iteration for the nodal displacement vector \( u \) at the (n+1)-th time step. The converged values for \( u \) at the (n+1)-th time step is denoted by \( u_{n+1} \). In the static case, the equilibrium of the internal resisting forces, \( R_{n+1} \), with the external forces, \( F_{n+1} \), can be written in the form of a nonlinear matrix equation with the nodal displacements, \( u_{n+1} \), as unknown:

\[ \Psi(u_{n+1}) = R_{n+1}(u_{n+1}) - F_{n+1} = 0 \]  
(3.67)

It is assumed that the i-th approximation to the displacement vector, \( u_{n+1}^{i} \), is available.

Expanding \( \Psi(u_{n+1}) \) using the Taylor series expansion about \( u_{n+1}^{i} \) and neglecting the higher order terms, we have
\[ \Psi(u_{n+1}) = \Psi(u^i_{n+1}) + \left( \frac{\partial \Psi}{\partial u} \right)^i_{n+1} (u_{n+1} - u^i_{n+1}) \]  

(3.68)

Hence,

\[ \left( \frac{\partial R}{\partial u} \right)^i_{n+1} \delta u^i = R^i_{n+1} - F_{n+1} = 0 \]  

(3.69)

in which

\[ \delta u^i = u_{n+1} - u^i_{n+1} \]  

(3.70)

and

\[ R^i_{n+1} = R(u^i_{n+1}) \]  

(3.71)

However,

\[ \left( \frac{\partial R}{\partial u} \right)^i_{n+1} = (K_{T})^i_{n+1} = \sum c \int_{\Omega_c} B^T D_T(e^i_{n+1})B \, d\Omega_c, \]  

(3.72)

where \((K_T)^i_{n+1}\) is the structure tangent stiffness matrix and \(D_T\) is the tangent constitutive matrix.

Therefore, the iterative scheme for the Newton-Raphson algorithm reduces to

\[ (K_T)^i_{n+1} \delta u^i = F_{n+1} - R^i_{n+1} \]  

(3.73)

\[ u^i_{n+1} = u^i_{n+1} + \delta u^i \]  

(3.74)

The initial values for starting the iteration scheme are obtained by setting

\[ u^i_{n+1} = u_n, \]  

(3.75)

\[ (K_T)^i_{n+1} = (K_T)_n, \]  

(3.76)

and

\[ R^i_{n+1} = R_n \]  

(3.77)

The Newton-Raphson method has an asymptotic rate of quadratic convergence. This is probably the most rapidly convergent process for solution of nonlinear problems pro-
vided that the initial step falls within the "zone of attraction". It should be noted that the jacobian or tangent stiffness matrix is evaluated and factorized at each iteration step. Several modifications of the Newton-Raphson method have been proposed (Chen and Han, 1988; Zienkiewicz and Taylor, 1991) which avoid the calculation and factorization of the tangent stiffness matrix at each iteration step, therefore improving the computational efficiency.

The iteration scheme continues until a proper convergence criterion is satisfied. A robust criterion for convergence is essential for the successful implementation of an efficient iterative scheme. At the end of each iteration, the solution obtained must be checked against a selected tolerance to see whether convergence has occurred. For a finite element displacement response analysis, the computed displacement should approach the true displacement. Since the true displacement is not known in advance, an approximation of the convergence criterion can be expressed as

$$
\|\delta u_i^i\| < \varepsilon_D \|u_{n+1}^i - u_n\| 
$$

(3.78)

where $\delta u_i^i$ is the incremental displacement obtained in the $i$-th iteration, the symbol $\| \|$ denotes the Euclidean norm of a vector, and $\varepsilon_D$ is a prescribed tolerance for the displacement $u$. This criterion is therefore called the displacement criterion.

Instead of the displacement criterion, a force criterion could also be used. It is of the form

$$
\|F_{n+1} - R_{n+1}^i\| < \varepsilon_F \|F_{n+1} - R_n\|
$$

(3.79)

where $\varepsilon_F$ is a prescribed tolerance for the norm of the out-of-balance force vector. A third approach is to use the internal energy criterion and compare the work done by the out-of-balance force on the last incremental displacement to the same work corresponding to the first iteration of the current step. This criterion is expressed as

$$
\left\|\begin{pmatrix} \delta u_i^i \end{pmatrix}^T \left( F_{n+1} - R_{n+1}^i \right) \right\| < \varepsilon_E \left\|\begin{pmatrix} \delta u^1 \end{pmatrix}^T \left( F_{n+1} - R_n \right) \right\|
$$

(3.80)
where $\varepsilon_E$ is a prescribed tolerance for the internal energy. This last criterion provides a measure of how close both displacements and forces are to their equilibrium values. In any of the above convergence criteria, the tolerance value must be carefully chosen. Indeed, a very strict tolerance will result in wasteful computations whereas a too loose tolerance will result in inaccurate results. It is the internal energy criterion which is implemented in FEAP, the finite element analysis program used in this study.

The above iterative/incremental algorithm can be readily extended for nonlinear dynamic analysis. This extension is presented in Section 5.3.2.

3.6 Plane Bilinear Isoparametric Element

Figure 5.1 shows the four-noded isoparametric element in the (x,y) plane and ($\xi, \eta$) plane, respectively, where (x,y) represent the geometrical coordinates while ($\xi, \eta$) denote the natural coordinates. For a four-noded element, the axes $\xi$ and $\eta$ pass through the midpoints of the two sides facing each other. These axes may not be orthogonal, and neither do they need to be parallel to the x or y axes. In natural coordinates, the sides of the element are described by $\xi = \pm 1$ and $\eta = \pm 1$.

The geometrical coordinates x and y within the element are defined by

$$x = \sum_{i=1}^{4} N_i(\xi, \eta) x_i \quad \text{and} \quad y = \sum_{i=1}^{4} N_i(\xi, \eta) y_i$$  \hspace{1cm} (3.81)

where $x_i$ and $y_i$ denote the geometrical coordinates of node i and $N_i$, i=1,...,4, denote the geometry shape functions given by

$$N_1(\xi, \eta) = \frac{1}{4} (1 - \xi) (1 - \eta)$$  \hspace{1cm} (3.82)

$$N_2(\xi, \eta) = \frac{1}{4} (1 + \xi) (1 - \eta)$$  \hspace{1cm} (3.83)

$$N_3(\xi, \eta) = \frac{1}{4} (1 + \xi) (1 + \eta)$$  \hspace{1cm} (3.84)
\[ N_4 (\xi, \eta) = \frac{1}{4} (1 - \xi) (1 + \eta) \] (3.85)

Figure 3.1 Four-noded, plane, bilinear isoparametric element in (a) \((x,y)\) space, and (b) \((\xi,\eta)\) space.

3.6.1 Displacement Interpolation Functions

Let vector \(\{u\} = [u_1 \ v_1 \ ... \ u_4 \ v_4]^T\) represent the vector of nodal displacements. The displacement field within the element is then given by

\[ u (x, y) = \sum_{i=1}^{4} N_i (\xi, \eta) u_i \quad \text{and} \quad v (x, y) = \sum_{i=1}^{4} N_i (\xi, \eta) v_i \] (3.86)

in which the displacement interpolation functions \(N_i\) are assumed to be identical to the geometry interpolation functions.

3.6.2 Strain-Displacement Relations

The strain components are related to the displacement field components as follows:

\[ \varepsilon_x = \frac{\partial u}{\partial x} \] (3.87)
\[ \varepsilon_y = \frac{\partial v}{\partial y} \]  

(3.88)

\[ \gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]  

(3.89)

Expressed in matrix form, this gives

\[
\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \end{bmatrix} \]  

(3.90)

Now, we need to express the partial derivatives of \( u \) and \( v \) with respect to the geometrical coordinates \( x \) and \( y \) in terms of the partial derivatives of \( u \) and \( v \) with respect to the natural coordinates, \( \xi \) and \( \eta \). This is done using the chain rule of differentiation:

\[ \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} \]  

(3.91)

and

\[ \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta} \]  

(3.92)

In matrix form, Eqs. (3.91) and (3.92) become

\[
\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} \]  

(3.93)

or the inverse relationship
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \left[ \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \right]^{-1} \left[ \frac{\partial u}{\partial \xi} \frac{\partial u}{\partial \eta} \right] \\
\frac{\partial u}{\partial y} &= \left[ \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \right]^{-1} \left[ \frac{\partial u}{\partial \eta} \frac{\partial u}{\partial \xi} \right] \tag{3.94}
\end{align*}
\]

Similarly, for the \( v \) displacement component,
\[
\begin{align*}
\frac{\partial v}{\partial x} &= \left[ \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \right]^{-1} \left[ \frac{\partial v}{\partial \xi} \frac{\partial v}{\partial \eta} \right] \\
\frac{\partial v}{\partial y} &= \left[ \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \right]^{-1} \left[ \frac{\partial v}{\partial \eta} \frac{\partial v}{\partial \xi} \right] \tag{3.95}
\end{align*}
\]

Hence,
\[
\begin{align*}
\frac{\partial u}{\partial x} &= J^{-1} \left[ \frac{\partial u}{\partial \xi} \frac{\partial u}{\partial \eta} \right] \quad \text{and} \quad \frac{\partial v}{\partial x} &= J^{-1} \left[ \frac{\partial v}{\partial \xi} \frac{\partial v}{\partial \eta} \right] \tag{3.96}
\end{align*}
\]

In Eq. (3.96), \( J \) denotes the jacobian transformation matrix defined as
\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}. \tag{3.97}
\]

Thus,
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & 0 & 0 \\
\Gamma_{21} & \Gamma_{22} & 0 & 0
\end{bmatrix} \frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial y} &= \begin{bmatrix}
0 & 0 & \Gamma_{11} & \Gamma_{12} \\
0 & 0 & \Gamma_{21} & \Gamma_{22}
\end{bmatrix} \frac{\partial u}{\partial \eta} \\
\frac{\partial v}{\partial x} &= \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & 0 & 0 \\
\Gamma_{21} & \Gamma_{22} & 0 & 0
\end{bmatrix} \frac{\partial v}{\partial \xi} \\
\frac{\partial v}{\partial y} &= \begin{bmatrix}
0 & 0 & \Gamma_{11} & \Gamma_{12} \\
0 & 0 & \Gamma_{21} & \Gamma_{22}
\end{bmatrix} \frac{\partial v}{\partial \eta} \tag{3.98}
\end{align*}
\]

where \( \Gamma_{11} \), \( \Gamma_{12} \), \( \Gamma_{21} \) and \( \Gamma_{22} \) are the elements of the matrix \( \Gamma \) which is given by
\[ \Gamma = J^{-1} \]  

(3.99)

Now, substituting the expressions for \( u \) and \( v \) given in Eq. (3.86), we obtain

\[
\begin{bmatrix}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial v}{\partial \xi} \\
\frac{\partial v}{\partial \eta}
\end{bmatrix}
= \begin{bmatrix}
N_1,\xi & 0 & N_2,\xi & 0 & N_3,\xi & 0 & N_4,\xi & 0 \\
N_1,\eta & 0 & N_2,\eta & 0 & N_3,\eta & 0 & N_4,\eta & 0 \\
0 & N_1,\xi & 0 & N_2,\xi & 0 & N_3,\xi & 0 & N_4,\xi \\
0 & N_1,\eta & 0 & N_2,\eta & 0 & N_3,\eta & 0 & N_4,\eta
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vdots \\
v_4 \\
\end{bmatrix}
\]  

(3.100)

where \( \mathbf{u} = [u_1 \; v_1 \; \ldots \; u_4 \; v_4]^T \).

Finally, by substituting Eq. (3.100) into Eq. (3.98) and the result into Eq. (3.90), it is found that

\[ \mathbf{e} = \mathbf{B}\mathbf{u} \]  

(3.101)

in which \( \mathbf{B} \) is the product of the rectangular matrices of equations Eqs. (3.90), (3.98) and (3.100) and is called the strain displacement matrix or the discrete strain operator.

3.6.3 Stress-Strain Relations

Assuming a linear elastic material, the stress vector is related to the strain vector through the elasticity matrix as

\[ \mathbf{\sigma} = \mathbf{D}\mathbf{e} \]  

(3.102)

For plane stress condition and linear elastic isotropic material, the elasticity matrix \( \mathbf{D} \) takes the form

\[
\mathbf{D} = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{(1-\nu)}{2}
\end{bmatrix},
\]  

(3.103)
whereas for plane strain condition and linear elastic isotropic material,

$$
D = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix}
1 & \frac{v}{(1-v)} & 0 \\
\frac{v}{(1-v)} & 1 & 0 \\
0 & 0 & (1-2v)/[2(1-v)]
\end{bmatrix}.
$$

(3.104)

### 3.6.4 Element stiffness matrix

The element stiffness matrix, $k^{(e)}$, of a plane stress/strain element of linear elastic material is defined as

$$
k^{(e)} = \int_{\Omega_e} \int \mathbf{B}^T \mathbf{DB} t \, dx dy = \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{DB} t \, (d\xi)(d\eta)
$$

(3.105)

where $t$ is the element thickness and $J$ is the determinant of the jacobian transformation defined by

$$
J(\xi, \eta) = \frac{\partial(x, y)}{\partial(\xi, \eta)}.
$$

(3.106)

thus,

$$
J(\xi, \eta) = \det J(\xi, \eta).
$$

(3.107)

In the case of elastoplastic material, the formulation of the element stiffness matrix given in Eq. (3.105) remains with the exception of the substitution of the elasticity matrix $D$ by the tangent constitutive matrix $D_T$.

### 3.7 Conclusions

The basic concepts of the finite element method for linear/nonlinear static/dynamic analysis of solids and structures are discussed in this chapter. The discretization in the time domain with the step-by-step direct integration methods used in structural dynamics are described. The incremental formulation required for the nonlinear static/dynamic analysis is also presented. The issues of damping matrices and lumping of mass matrices are also addressed. The formulation of the four-noded isoparametric plane strain element used in FEAP is presented.
With the availability of ever increasing computational power, the solutions of increasingly complicated problems of structural mechanics can be obtained through the finite element method. The finite element method used in conjunction with the modern methods of structural reliability analysis can be used to assess the reliability of complex structural systems. The link between finite element analysis and reliability analysis is explained in detail in Chapters 5 and 6.
CHAPTER 4  CONSTITUTIVE EQUATIONS FOR PLASTICITY FORMULATIONS

4.1 Introduction

The theory of plasticity represents a necessary extension to the theory of elasticity and it provides more realistic estimates of the load carrying capacities of engineering structures, which in general fail in the materially nonlinear range. The mathematical theory of plasticity models the irreversible straining exhibited by many materials. In this chapter, the fundamental concepts of plasticity theory are presented (Chen 1982; Chen and Han 1988). The basic ingredients of rate-independent plasticity theory, namely the yield criterion, the flow rule and the hardening rule are discussed. The constitutive formulations of two plasticity models, namely the \( J_2 \) plasticity model and the cap model, are described. The return map algorithm which is used to integrate the rate constitutive equations is carefully reviewed. The geometric interpretation of the return map algorithm is also shown. Finally, the numerical implementation of the return map algorithm is described in the context of the incremental formulation of nonlinear analysis used in the general-purpose finite element program FEAP.

4.2 Basic Concepts of Plasticity Theory

It is often convenient to decompose the symmetric stress tensor into two symmetric tensors, the hydrostatic or spherical stress tensor and the deviatoric stress tensor. The decomposition is given by

\[
\sigma_{ij} = s_{ij} + \frac{1}{3} \sigma_{kk} \delta_{ij} \tag{4.1}
\]

where \( s_{ij} \) is the deviatoric stress tensor, \( \frac{1}{3} \sigma_{kk} \) is the hydrostatic or spherical stress, and \( \delta_{ij} \) is the Kronecker delta. The tensor \( s_{ij} \) is that part of the stress state which represents the shear or the deviatoric state of stress and excludes the hydrostatic state of stress. The diagonal tensor \( \frac{1}{3} \sigma_{kk} \delta_{ij} \) refers strictly to the spherical or the hydrostatic stress. Each component on the diagonal of the hydrostatic stress tensor, that is \( \frac{1}{3} \sigma_{kk} \).
denotes the mean pressure:

$$p = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) .$$  \hspace{1cm} (4.2)$$

The deviatoric stress tensor can be expressed as follows:

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} = \sigma_{ij} - p \delta_{ij} .$$  \hspace{1cm} (4.3)$$

The invariants of the stress tensor can be defined as:

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \text{tr} (\sigma) ,$$  \hspace{1cm} (4.4)$$

$$I_2 = \frac{1}{2} \sigma_{ij} \sigma_{ji} = \frac{1}{2} \text{tr} [ (\sigma)^2 ] ,$$  \hspace{1cm} (4.5)$$

$$I_3 = \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki} = \frac{1}{3} \text{tr} [ (\sigma)^3 ] .$$  \hspace{1cm} (4.6)$$

In the above, the index notation is used and the repetition of an index within any term implies summation over the range of the index. Similarly, the invariants of the deviatoric stress tensor can be defined as:

$$J_1 = s_{ii} = \text{tr} (s) = 0 ,$$  \hspace{1cm} (4.7)$$

$$J_2 = \frac{1}{2} s_{ij} s_{ji} = \frac{1}{2} \text{tr} [ (s)^2 ] ,$$  \hspace{1cm} (4.8)$$

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} = \frac{1}{3} \text{tr} [ (s)^3 ] .$$  \hspace{1cm} (4.9)$$

The norm of a second order tensor \(a\) is defined as

$$||a|| = (a:a)^{1/2} = \{ \text{tr} [ (a)^2 ] \}^{1/2}$$  \hspace{1cm} (4.10)$$

in which the symbol \((::)\) signifies doubly contracted tensor product or dyadic product, e.g., \((D:\varepsilon)_{ij} = D_{ijkl} \varepsilon_{kl}\). From the definitions in Eqs. (4.8) and (4.10), it is noted that

$$||s|| = \sqrt{2 J_2}$$  \hspace{1cm} (4.11)$$
Any plasticity model is defined by the following three ingredients:

(1) An initial *yield criterion* represented by a yield surface in the stress space, defining the stress level at which plastic deformations begin.

(2) A *flow rule* which relates the increment of plastic strain tensor to the stress state on the yield surface (or plastic potential).

(3) A *hardening rule* which defines the change of configuration of the yield surface during the plastic deformation process.

Without loss of generality, the *yield function* can be expressed in scalar form as:

\[
F\left(\sigma_{ij}, \varepsilon_{ij}^p, \kappa\right) = f\left(\sigma_{ij}, \varepsilon_{ij}^p\right) - Y(\kappa)
\]  
(4.12)

where the function \( f \), which indicates the form of the yield criterion, depends on the stress tensor, \( \sigma_{ij} \), and the plastic strain tensor, \( \varepsilon_{ij}^p \), and the current yield stress, \( Y(\kappa) \), is related to the deformation history of the material via the hardening parameter \( \kappa \). Usually the hardening parameter is defined as either the effective plastic strain \( \varepsilon^p \) :

\[
\kappa = \varepsilon^p = \int_0^t \dot{\varepsilon} \, dt \quad \text{where} \quad \dot{\kappa} = \varepsilon^p = \left[\frac{2}{3} \left( \varepsilon_{ij}^p \varepsilon_{ij}^p \right) \right]^{1/2}
\]  
(4.13)

or the plastic work \( W_p \) :

\[
\kappa = W_p = \int_0^t \dot{W}_p \, dt \quad \text{where} \quad \dot{W}_p = \sigma_{ij} \varepsilon_{ij}^p.
\]  
(4.14)

Use of the effective plastic strain for \( \kappa \) implies a "strain-hardening" model, whereas a "work-hardening" model is obtained in the second case. The yield surface in the stress space is defined by

\[
F\left(\sigma_{ij}, \varepsilon_{ij}^p, \kappa\right) = 0
\]  
(4.15)

By convention, the yield function is defined in such a way that the elastic range forms the interior of the yield surface, that is, the material is elastic if
\[ f(\sigma_{ij}, \varepsilon_{ij}^p) - Y(\kappa) < 0. \]  

(4.16)

The yield function is physically constrained to be less than or equal to zero. Thus, the stress state cannot lie outside the yield surface. When plastic deformations occur, the yield surface expands (in the case of hardening material), but the stress state remains on the “expanding” yield surface. Thus, the material state is given by the value of the yield function as:

\[ F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) < 0 : \text{ elastic state} \]  

(4.17)

\[ F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) = 0 : \text{ yielding state} \]  

(4.18)

\[ F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) > 0 : \text{ not physically possible} \]  

(4.19)

Given the material state and the infinitesimal increment of the yield function, \( dF \), the material state event is given by:

\[ F < 0 \text{ and } dF > 0 : \text{ elastic loading} \]  

(4.20)

\[ F = 0 \text{ and } dF < 0 : \text{ elastic unloading from plastic state} \]  

(4.21)

\[ F = 0 \text{ and } dF = 0 : \text{ continued yielding} \]  

(4.22)

\[ F = 0 \text{ and } dF > 0 : \text{ not possible in the plastic regime} \]  

(4.23)

As expressed by (4.22), in case of continued yielding, the increment of yield function, \( dF \), is constrained to be zero. Thus,

\[
dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial F}{\partial \kappa} d\kappa = 0
\]  

(4.24)

The above relation is called the \textit{consistency condition}. The loading-unloading conditions can also be formulated in the standard \textit{Kuhn-Tucker conditions} of optimization theory.
\[ d\lambda \geq 0, \quad F d\lambda = 0, \quad F\left(\sigma, \epsilon^p, \kappa\right) \leq 0 \]  \hspace{1cm} (4.25)

If the material is assumed to be isotropic, it does not have any preferred directions and the state of stress is uniquely defined by the three principal stresses. Therefore, the yield criterion can be described only in terms of the three principal stresses rather than the six components of the stress tensor. More conveniently, the yield criterion can also be expressed in terms of the three stress invariants \( I_1, I_2 \) and \( I_3 \), where \( I_1 \) is the first invariant of the stress tensor \( \sigma_{ij} \), and \( I_2 \) and \( I_3 \) are the second and third invariants of the deviatoric stress tensor \( s_{ij} \). That is, for an isotropic material,

\[ f\left(\sigma_{ij}, \epsilon^p_{ij}\right) = f\left(I_1, I_2, I_3\right). \hspace{1cm} (4.26) \]

The flow rule relates the plastic strain increments to the stress state. It is generally defined by using the concept of a plastic potential function, \( Q\left(\sigma_{ij}, \epsilon^p_{ij}, \kappa\right) \), to which the incremental plastic strain tensor is orthogonal. In other words, the gradient of the plastic potential surface in the stress space, \( \frac{\partial Q}{\partial \sigma_{ij}} \), defines the direction of the incremental plastic strain tensor, i.e.,

\[ d\epsilon^p_{ij} = d\lambda \frac{\partial Q}{\partial \sigma_{ij}}. \hspace{1cm} (4.27) \]

The flow rule is said to be **associative** if the plastic potential function \( Q\left(\sigma_{ij}, \epsilon^p_{ij}, \kappa\right) \) is assumed to be identical to the yield function \( F\left(\sigma_{ij}, \epsilon^p_{ij}, \kappa\right) \). Therefore, for an associative flow rule,

\[ d\epsilon^p_{ij} = d\lambda \frac{\partial F}{\partial \sigma_{ij}} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}. \hspace{1cm} (4.28) \]

The associative flow rule implies that plastic strain increments are normal to the yield surface in the stress space. The scalar parameter \( d\lambda \) in Eqs. (4.27) and (4.28) is called the **plastic consistency parameter**. This parameter is zero for elastic behavior and positive for plastic behavior.
The hardening rule controls the change of configuration of the yield surface upon subsequent yielding. Two basic models of hardening exist: (a) *isotropic hardening*, and (b) *kinematic hardening*. In the case of isotropic hardening, the yield surface is assumed to expand uniformly about the origin in the stress space during the plastic deformation, while in the case of kinematic hardening, the yield surface simply translates in the stress space preserving its initial shape. More complex mixed hardening rules may be obtained by combining the isotropic and the kinematic hardening rules.

### 4.3 Incremental Stress-Strain Relations

In small-strain plasticity theory, the constitutive equations of a material for deformations beyond the elastic limit are expressed in the incremental form by assuming that the total increment of the strain tensor can be decomposed additively into elastic and plastic parts:

\[ \text{d} \varepsilon = \text{d} \varepsilon^e + \text{d} \varepsilon^p. \tag{4.29} \]

where \( \text{d} \varepsilon^e \) and \( \text{d} \varepsilon^p \) are the incremental elastic and plastic strain tensors, respectively. The incremental stress tensor, \( \text{d} \sigma \), is related to the incremental elastic strain tensor, \( \text{d} \varepsilon^e \) through the material elastic constitutive matrix \( \mathbf{D} \) as:

\[ \text{d} \sigma = \mathbf{D} \text{d} \varepsilon^e = \mathbf{D} \left( \text{d} \varepsilon - \text{d} \varepsilon^p \right) \tag{4.30} \]

The incremental plastic strains, \( \text{d} \varepsilon^p \), are given by the flow rule in Eq. (4.27) or (4.28). For a wide class of associative plasticity models, the yield function in (4.12) takes the form:

\[ F\left( \sigma, \alpha, \varepsilon^p \right) = f \left( \Sigma \right) - Y \left( \varepsilon^p \right) \tag{4.31} \]

where the "relative" stress tensor \( \Sigma \) is defined as:

\[ \Sigma = \sigma - \alpha \tag{4.32} \]

and
\[ Y\left( \dot{\varepsilon}^p \right) = Y_0 + H_{iso} \dot{\varepsilon}^p \]  
(4.33)

The stress tensor \( \alpha \) in Eqs. (4.31) and (4.32) is called the back stress tensor. It locates the center of the yield surface in the stress space as shown in Figure 4.2. The back stress tensor is related to the plastic strain rate through

\[ \dot{\alpha} = \frac{2}{3} H_{kin} \dot{\varepsilon}^p = \frac{2}{3} H_{kin} \hat{\lambda} \frac{\partial F}{\partial \sigma} \]  
(4.34)

where \( H_{kin} \) is a kinematic hardening modulus. (4.34) is referred to as a linear kinematic hardening law, since the back stress increment \( \dot{\alpha} \) is linearly related to the plastic strain increment \( \dot{\varepsilon}^p = \hat{\lambda} \frac{\partial F}{\partial \sigma} \). Initially, before any plastic strains appear, \( \alpha = 0 \). The kinematic hardening model of (4.34) indicates that the center of the yield surface moves in the direction of the plastic straining. (4.33) represents a linear isotropic hardening model in which the function \( Y\left( \varepsilon^p \right) \) measures the size of the current yield surface. Commonly, \( Y_0 \) is related to \( \sigma_y \), the yield stress in uniaxial tension. The parameter \( H_{iso} \) is an isotropic hardening modulus.

From the computational view point, the nonlinear behavior of a material is treated as a strain driven problem. In other words, the stress history is obtained from the strain history by means of an integration algorithm, such as the return map algorithm, which is described below.

### 4.4 Derivation of the “Continuum” Elastoplastic Constitutive Matrix

In this section, the derivation of the “continuum” elastoplastic constitutive matrix, which relates the infinitesimal increase in the stress tensor to an infinitesimal increase in the strain tensor is derived (Zienkiewicz and Taylor 1991). By definition of the elastic constitutive matrix we have

\[ d\varepsilon^e = \mathbf{D}^{-1} d\sigma \]  
(4.35)

The flow rule defined in Eq. (4.27) can be rewritten in matrix form as
\[ d\epsilon^p = d\lambda \frac{\partial Q}{\partial \sigma} \]  \hfill (4.36)

For an associative plasticity model, the flow rule takes the form

\[ d\epsilon^p = d\lambda \frac{\partial F}{\partial \sigma} \]  \hfill (4.37)

where \( F(\sigma, \alpha, \kappa) \) is the yield function.

Substituting Eqs. (4.35) and (4.37) into Eq. (4.29) yields

\[ d\epsilon = D^{-1}d\sigma + d\lambda \frac{\partial F}{\partial \sigma} \tag{4.38} \]

Using the chain rule of differentiation, we express an infinitesimal change in the yield function \( F(\sigma, \epsilon^p, \kappa) \) as

\[ dF(\sigma, \alpha, \kappa) = \frac{\partial F^T}{\partial \sigma} d\sigma + \frac{\partial F^T}{\partial \alpha} d\alpha + \frac{\partial F^T}{\partial \kappa} d\kappa d\lambda = 0 \tag{4.39} \]

Rewriting Eqs. (4.38) and (4.39) in matrix form gives

\[
\begin{bmatrix}
  d\epsilon \\
  0
\end{bmatrix} = \begin{bmatrix}
  \{ D^{-1} \} & \{ \frac{\partial F}{\partial \sigma} \} \\
  \{ \frac{\partial F^T}{\partial \sigma} \} & \{ \frac{2}{3} H_{\text{kin}} \frac{\partial F^T}{\partial \alpha} \frac{\partial F}{\partial \sigma} - A \}
\end{bmatrix}
\begin{bmatrix}
  d\sigma \\
  d\lambda
\end{bmatrix}
\tag{4.40}
\]

where use has been made of the relation \( d\alpha = \frac{2}{3} H_{\text{kin}} d\epsilon^p = \frac{2}{3} H_{\text{kin}} \frac{\partial F}{\partial \sigma} d\lambda \) and the definition \( A = \frac{\partial F^T}{\partial \kappa} d\kappa d\lambda \).

Using Eqs. (4.30) and (4.27) we can rewrite the term \( \frac{\partial F^T}{\partial \sigma} d\sigma \) as

\[ \frac{\partial F^T}{\partial \sigma} d\sigma = \frac{\partial F^T}{\partial \sigma} D d\epsilon - \frac{\partial F^T}{\partial \sigma} D d\epsilon - \frac{\partial F^T}{\partial \sigma} D d\epsilon \]  \hfill (4.41)

Substituting the above equation into the second row of Eq. (4.40) yields
\[
\frac{\partial F^T}{\partial \sigma} \text{D} \text{d} \varepsilon = \left[ \frac{\partial F^T}{\partial \sigma} \text{D} \frac{\partial F}{\partial \sigma} - \frac{2}{3} H_{\text{kin}} \frac{\partial F^T}{\partial \alpha} \frac{\partial F}{\partial \sigma} + A \right] \text{d} \lambda = 0
\]

(4.42)

Elimination of \( \text{d} \lambda \) from the first row of Eq. (4.40) gives

\[
\text{d} \varepsilon = \text{D}^{-1} \text{d} \sigma + \frac{\partial F}{\partial \sigma} \left[ \frac{\partial F^T}{\partial \sigma} \text{D} \frac{\partial F}{\partial \sigma} - \frac{2}{3} H_{\text{kin}} \frac{\partial F^T}{\partial \alpha} \frac{\partial F}{\partial \sigma} + A \right]^{-1} \frac{\partial F^T}{\partial \sigma} \text{D} \text{d} \varepsilon
\]

(4.43)

Rearranging terms gives

\[
\text{d} \sigma = \{ \text{D} - \text{D} \frac{\partial F}{\partial \sigma} \frac{\partial F^T}{\partial \sigma} \text{D} \left[ \frac{\partial F^T}{\partial \sigma} \text{D} \frac{\partial F}{\partial \sigma} - \frac{2}{3} H_{\text{kin}} \frac{\partial F^T}{\partial \alpha} \frac{\partial F}{\partial \sigma} + A \right]^{-1} \} \text{d} \varepsilon
\]

(4.44)

Hence,

\[
\text{d} \sigma = \text{D}_{\text{ep}} \text{d} \varepsilon
\]

(4.45)

where the elastoplastic constitutive matrix \( \text{D}_{\text{ep}} \) is given by

\[
\text{D}_{\text{ep}} = \text{D} - \text{D} \frac{\partial F}{\partial \sigma} \frac{\partial F^T}{\partial \sigma} \text{D} \left[ \frac{\partial F^T}{\partial \sigma} \text{D} \frac{\partial F}{\partial \sigma} - \frac{2}{3} H_{\text{kin}} \frac{\partial F^T}{\partial \alpha} \frac{\partial F}{\partial \sigma} + A \right]^{-1}
\]

(4.46)

### 4.5 Return Map Algorithm

In the context of nonlinear finite element analysis, it is necessary to update the stresses at the Gauss points of each element for a given incremental deformation. This phase of the calculation is called state determination. Knowing the state variables at the converged time step \( t_n \) and given an increment of the total strain tensor, \( \text{d} \varepsilon \), the problem consists of finding the values of the state variables at time \( t_{n+1} \) such that they satisfy the constitutive equations (yield criterion, flow rule and hardening rule). The return map algorithm provides an efficient and robust integration algorithm to obtain the update of the state variables. It belongs to the family of elastic-predictor plastic-corrector algorithms, and hence it is a two-step algorithm (Ortiz, Pinsky and Taylor 1983). In the first step, a purely elastic trial state is computed, and if the trial state violates the yield criterion, the plastic corrector is applied such that the final state
satisfies the constitutive model.

The return map algorithm is a natural consequence of the fact that the constitutive relations can be split into elastic and plastic parts (Simo and Ortiz 1985). In the elastic step, the plastic response of the material is frozen, so that all the incremental strain goes into the elastic strain. In the plastic corrector phase, however, the spatial configuration of the material remains fixed, and the constitutive laws are satisfied through a "plastic" relaxation of the elastic trial stresses towards a suitably updated yield surface. Based on the notion of the elastic-plastic operator split, a return map algorithm may be conveniently defined by first defining an elastic predictor state, and then applying the plastic corrector using the elastic predictor as an initial condition. A geometrical interpretation of the algorithm is shown in Figure 4.1.

It can be seen that the elastic predictor is returned to the updated yield surface in successive steps. Each one of these steps involves a projection of the stresses onto a linear approximation to the yield surface. For an associative flow rule, the computed return path is an approximation to the steepest descent path as defined by the tensor $C^e - C^{ep}$ (Ortiz and Simo 1986). In general, the return path is not known in advance, nor can it be determined analytically. It is therefore necessary to compute the return path numerically in an iterative fashion. At every iteration, the yield function is linearized at the current values of the state variables, onto which the stress point is projected to obtain the state variables for the next iteration. The iterative procedure is stopped as soon as the yield criterion is satisfied.

If the yield function is convex and the plastic flow is derived from a convex loading function, the return map algorithm can be shown to be unconditionally stable and quadratically convergent (Simo and Ortiz 1985).

**4.6 Constitutive Equations for the Associative $J_2$ Plasticity Model**

By definition, the yield function of the $J_2$ plasticity model, also called von Mises
model, depends only on the second invariant, $J_2$, of the deviatoric stress tensor and not on all the components of the stress tensor. An associative flow rule is assumed and only linear isotropic and kinematic hardening mechanisms are considered in this study.

The total strain is decomposed into the deviatoric part, $\mathbf{e}$, and the volumetric part, $\theta$, and is expressed, in matrix notation, as

$$\mathbf{e} = \mathbf{e} + (1/3) \theta \mathbf{1}.$$  \hfill (4.47)
where $\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is the volumetric strain and 1 denotes the unit second order tensor.

The deviatoric and volumetric strains can be further decomposed into elastic and plastic parts. Hence,

$$e = e^e + e^p$$ \hspace{1cm} (4.48)

and

$$\theta = \theta^e + \theta^p.$$ \hspace{1cm} (4.49)

In the case of the $J_2$ plasticity model, plasticity is restricted to the deviatoric part only. Thus, $\theta^p = 0$ at all times, and the yield function depends only on the norm of the deviatoric component of the stress tensor. The $J_2$ plasticity model is therefore pressure insensitive and the plastic deformations are isochoric.

The total stresses can be split into the deviatoric and volumetric parts also:

$$\sigma = s + p1$$ \hspace{1cm} (4.50)

in which $s$ is the deviatoric stress matrix and $p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ denotes the mean stress. The constitutive equations are now given by

$$s = 2G\left(e - e^p\right)$$ \hspace{1cm} (4.51)

and

$$p = K\theta$$ \hspace{1cm} (4.52)

where $G$ and $K$ represent the shear modulus and bulk modulus, respectively.

The yield function for the associative $J_2$ plasticity model is defined as

$$F\left(s, \alpha, e^p\right) = ||\Sigma|| - Y\left(e^p\right) \leq 0$$ \hspace{1cm} (4.53)

where $\Sigma$ is the relative deviatoric stress tensor defined as the difference between the
deviatoric stress tensor and the back stress tensor. Therefore,

$$\Sigma = s - \alpha.$$  \hspace{1cm} (4.54)

In (4.41), \(|\cdot|\) denotes the Euclidean norm of a second-order tensor (or matrix) previously defined in (4.10), and \(Y\) is the current radius of the yield surface which is represented by a cylinder centered on the hydrostatic or tridiagonal axis in the principal stress space. The current radius of the yield surface, \(Y(t)\), is directly related to the current uniaxial yield stress, \(\sigma_y(t)\), through

$$Y(t) = \sqrt{\frac{2}{3}} \sigma_y(t).$$  \hspace{1cm} (4.55)

The current uniaxial yield stress \(\sigma_y(t)\) is given by the linear isotropic hardening rule

$$\sigma_y(t) = \left( \sigma_{y0} + \text{H}_{\text{iso}} e^p \right).$$  \hspace{1cm} (4.56)

Thus, the material hardening parameter \(\kappa\) used in the linear isotropic hardening law assumed is the accumulated effective plastic strain, namely

$$\kappa(t) = \bar{e}^p(t) = \int_0^t \sqrt{\frac{2}{3}} \| e^p(t) \| dt = \int_0^t \sqrt{\frac{2}{3}} \| e^p(t) \| dt = e^p(t).$$  \hspace{1cm} (4.57)

The plastic component of the deviatoric strain rate is given by

$$\dot{e}^p = \dot{\gamma}(t) \frac{\partial F}{\partial s} = \dot{\gamma}(t) \frac{\partial F}{\partial \Sigma} \frac{\partial \Sigma}{\partial s} = \dot{\gamma}(t) \frac{\partial F}{\partial \Sigma}$$  \hspace{1cm} (4.58)

The linear kinematic hardening rule assumed is given by the following evolution of the back stress

$$\alpha = \frac{2}{3} \text{H}_{\text{kin}} e^p = \frac{2}{3} \text{H}_{\text{kin}} \dot{\gamma}(t) \frac{\partial F}{\partial \Sigma}.$$  \hspace{1cm} (4.59)

It is worth noting that since the present study deals only with rate independent plasticity models, \(t\) is a pseudo-time variable introduced to describe the plasticity as an evolutionary process.
It can be shown that for the case of von Mises yield criterion, the following relation holds (Simo and Taylor 1985):

\[ \frac{\partial F(t)}{\partial \Sigma(t)} = \frac{\partial \| \Sigma(t) \|}{\partial \Sigma(t)} = \frac{\Sigma(t)}{\| \Sigma(t) \|} = n(t) \]  \hspace{1cm} (4.60)

where \( n(t) \) is a tensor “normal” to the yield surface at \( \Sigma(t) \) and has a unit norm in the sense of Eq. (4.10), i.e.,

\[ \| n \| = \left[ \text{tr} \left( n^2 \right) \right]^{1/2} = 1 \]  \hspace{1cm} (4.61)

Using the previous relation, Eqs. (4.46) and (4.47) can be rewritten as

\[ \varepsilon^p(t) = \dot{\gamma}(t) \cdot n(t) \]  \hspace{1cm} (4.62)

and

\[ \alpha(t) = \frac{2}{3} H_{\text{kin}} \dot{\gamma}(t) n(t) \]  \hspace{1cm} (4.63)

From Eq. (4.34) and given that \( n \) is a “unit” tensor, it follows that:

\[ \| \varepsilon^p(t) \| = \dot{\gamma}(t) \]  \hspace{1cm} (4.64)

and the accumulated effective plastic strain in Eq. (4.45) becomes:

\[ \varepsilon^p(t) = \int_0^t \sqrt{\frac{2}{3}} \dot{\gamma} \, dt \]  \hspace{1cm} (4.65)

4.7 \( J_2 \) Constitutive Equations in Discrete Form

From the computational standpoint, the stress history is obtained from the strain history in an incremental fashion through a numerical integration algorithm, such as the return map algorithm used in this study (Auricchio, Taylor and Lubliner 1992). In this section, the discretized version of the constitutive equations of the \( J_2 \) plasticity model is presented.

In the discussion below, the following notation is introduced:
\[a_n = a(t_n) \quad \text{and} \quad a_{n+1} = a(t_{n+1})\]

in which "a" is any generic quantity (scalar, vector, tensor, ...). Thus in the discretized form, a quantity with a subscript \(n\) refers to this quantity evaluated at time \(t_n\) and a quantity with a subscript \(n+1\) refers to the quantity evaluated at time \(t_{n+1}\). Assuming that the quantities \((s_n, p_n, \varepsilon_n, \theta_n, e\_n^p, \alpha_n, \varepsilon\_n^p)\) at time \(t_n\) are known, the same quantities must be computed at time \(t_{n+1}\) given the total strain \(\varepsilon_{n+1}\), which is obtained from the incremental displacements corresponding to a Newton-Raphson iteration at the structure level.

The plastic rate of the deviatoric strains, Eqs. (4.46), and the back stress rate equation, (4.47), can be discretized using the *backward Euler implicit method*. Thus,

\[e\_n^{p} + \lambda\_n^{p} n\_n + 1 = e\_n^{p} + \lambda\_n^{p} n\_n + 1 (4.67)\]

\[\alpha\_n + 1 = \alpha\_n + \frac{2}{3} H\_kin \lambda\_n + 1 n\_n + 1 (4.68)\]

where

\[\lambda\_n + 1 = \int_{t_n}^{t\_n+1} \gamma dt = \gamma\_n + 1 - \gamma n (4.69)\]

The deviatoric stress tensor, \(s\_n+1\), is expressed in the incremental form as

\[s\_n + 1 = 2G(e\_n + 1 - e\_n^{p} + 1) = 2G(e\_n + 1 - e\_n^{p}) - 2G\lambda\_n + 1 n\_n + 1 (4.70)\]

The relative deviatoric stress tensor, \(\Sigma\_n + 1\), at time \(t\_n + 1\) is given by

\[\Sigma\_n + 1 = s\_n + 1 - \alpha\_n + 1\]

\[= 2G(e\_n + 1 - e\_n^{p}) - 2G\lambda\_n + 1 n\_n + 1 - \alpha\_n - \frac{2}{3} H\_kin \lambda\_n + 1 n\_n + 1 (4.71)\]

The uniaxial yield stress at time \(t\_n + 1\), \(\sigma\_y.n + 1\), is given by the linear isotropic hardening law of Eq. (4.44):
\[ \sigma_{y, n+1} = \sigma_{y, n} + \frac{2}{3} H_{iso} \lambda_{n+1} \cdot \]  

(4.72)

Hence, the new radius of the yield surface can be expressed as

\[ Y_{n+1} = Y_n + \frac{2}{3} H_{iso} \lambda_{n+1} \cdot \]  

(4.73)

### 4.8 Return Map Algorithm for the J2 Plasticity Model

As seen earlier, the return map algorithm is an elastic-predictor, plastic-corrector scheme. Its implementation is described below:

#### 4.8.1 Elastic Predictor Step

From the specified total strain at time \( t_{n+1} \), \( \varepsilon_{n+1} \), the volumetric state of the material can be readily obtained as:

\[ \theta_{n+1} = \text{trace}(\varepsilon_{n+1}) \]  

(4.74)

\[ p_{n+1} = K \theta_{n+1} \]

The deviatoric strains at \( t_{n+1} \) can also be obtained directly:

\[ \varepsilon_{n+1} = \varepsilon_{n+1} - \frac{1}{3} \theta_{n+1} I \]  

(4.75)

In the first iteration, a trial deviatoric state is computed by assuming linear elastic behavior between times \( t_n \) and \( t_{n+1} \). Hence the following relations are true for the elastic-predictor step:

\[ \lambda_{n+1}^{\text{Trial}} = 0 \]  

(4.76)

\[ \left( \varepsilon_{n+1}^p \right)^{\text{Trial}} = \varepsilon_n^p \]  

(4.77)

\[ \left( \varphi_{n+1}^p \right)^{\text{Trial}} = \varphi_n^p \]  

(4.78)

\[ \alpha_{n+1}^{\text{Trial}} = \alpha_n \]  

(4.79)
\[ s_{n+1}^{\text{Trial}} = 2G \left( e_{n+1} - e_n^p \right) \]  
(4.80)

\[ \Sigma_{n+1}^{\text{Trial}} = s_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}} = s_{n+1}^{\text{Trial}} - \alpha_n \]  
(4.81)

\[ Y_{n+1}^{\text{Trial}} = Y_n \]  
(4.82)

where the superscript "Trial" refers to the trial value for the elastic predictor step. If the trial state is found to satisfy the yield criterion (4.41), then it represents the true solution at time \( t_{n+1} \). Consequently, the plastic corrector part of the algorithm is skipped and the solution is updated with the trial state.

### 4.8.2 Plastic Corrector Step

If the trial state violates the yield condition, then the discrete plastic consistency parameter \( \lambda_{n+1} \) needs to be evaluated through the plastic corrector algorithm.

Rewriting Eqs. (4.58) and (4.59), the deviatoric stress tensor and the relative stress tensor can be expressed as:

\[ s_{n+1} = s_{n+1}^{\text{Trial}} - 2G \lambda_{n+1} n_{n+1} \]  
(4.83)

and

\[ \Sigma_{n+1} = \Sigma_{n+1}^{\text{Trial}} - \left( 2G + \frac{2}{3} H_{\text{kin}} \right) \lambda_{n+1} n_{n+1} \]  
(4.84)

However, it is clear from (4.48) that

\[ \Sigma_{n+1} = \| \Sigma_{n+1} \| n_{n+1} \]  
(4.85)

Therefore, from Eqs. (4.72) and (4.73), it can be deduced that \( \Sigma_{n+1}^{\text{Trial}} \) and \( \Sigma_{n+1}^{\text{Trial}} \) have the same "direction" \( n_{n+1} \). That is,

\[ \frac{\Sigma_{n+1}^{\text{Trial}}}{\| \Sigma_{n+1}^{\text{Trial}} \|} = \frac{\Sigma_{n+1}}{\| \Sigma_{n+1} \|} = n_{n+1} \]  
(4.86)

Hence, (4.72) represents the radial return map algorithm (see Fig. 4.2), and it is possi-
ble to write the following scalar equation:

\[ \| \mathbf{\Sigma}_{n+1} \| = \| \mathbf{\Sigma}_{n+1}^{\text{Trial}} \| - \left( 2G + \frac{2}{3} H_{\text{kin}} \right) \lambda_{n+1} \]  
(4.87)

Now, for the associative \( J_2 \) model, the yield criterion is

\[ \| \mathbf{\Sigma}_{n+1} \| - Y_{n+1} = 0 \]  
(4.88)

Use of Eqs. (4.75) and (4.61) yields

\[ \left\| \mathbf{\Sigma}_{n+1}^{\text{Tr}} \right\| - \left( 2G + \frac{2}{3} H_{\text{kin}} \right) \lambda_{n+1} - \left( Y_{n} + \frac{2}{3} H_{\text{iso}} \lambda_{n+1} \right) = 0. \]  
(4.89)

Finally, the above equation can be solved directly for the consistency parameter \( \lambda_{n+1} \):

\[ \lambda_{n+1} = \frac{\left\| \mathbf{\Sigma}_{n+1}^{\text{Tr}} \right\| - Y_{n}}{2G + \frac{2}{3} (H_{\text{iso}} + H_{\text{kin}})} . \]  
(4.90)

Note that in the case of a nonlinear hardening law, Eq. (4.77) is nonlinear in \( \lambda_{n+1} \) and an iterative scheme is required to find \( \lambda_{n+1} \). Once the value of \( \lambda_{n+1} \) is known, the solution at time \( t_{n+1} \) can be updated using the following relations:

\[ \mathbf{n}_{n+1} = \frac{\mathbf{\Sigma}_{n+1}^{\text{Trial}}}{\left\| \mathbf{\Sigma}_{n+1}^{\text{Trial}} \right\|} \]  
(4.91)

\[ \left\| \mathbf{\Sigma}_{n+1} \right\| = \left\| \mathbf{\Sigma}_{n+1}^{\text{Trial}} \right\| - \left( 2G + \frac{2}{3} H_{\text{kin}} \right) \lambda_{n+1} \]  
(4.92)

\[ \mathbf{\Sigma}_{n+1} = \left\| \mathbf{\Sigma}_{n+1} \right\| \mathbf{n}_{n+1} \]  
(4.93)

\[ \mathbf{\alpha}_{n+1} = \mathbf{\alpha}_{n} + \frac{2}{3} H_{\text{kin}} \lambda_{n+1} \mathbf{n}_{n+1} \]  
(4.94)

\[ \mathbf{\varepsilon}_{n+1} = \mathbf{\varepsilon}_{n} + \lambda_{n+1} \mathbf{n}_{n+1} \]  
(4.95)

\[ \mathbf{\varepsilon}^{p}_{n+1} = \mathbf{\varepsilon}^{p}_{n} + \frac{\sqrt{2}}{3} \lambda_{n+1} \]  
(4.96)
\[
\sigma_{y,n+1} = \left( \sigma_{y0} + H_{iso} \bar{\varepsilon}_n^p \right) \quad (4.97)
\]
\[
s_{n+1} = s_{n+1}^{\text{Trial}} - 2G\lambda_{n+1} n_{n+1} \quad (4.98)
\]
\[
p_{n+1} = K\theta_{n+1} \quad (4.99)
\]

The total stress tensor is now obtained from the relation:
\[
\sigma_{n+1} = s_{n+1} + \left( \frac{1}{3} \right)p_{n+1} I. \quad (4.100)
\]

Figure 4.2 Geometrical Interpretation of the Radial return algorithm
It should be noted that for the associated plasticity model, the points in the stress space corresponding to \( \alpha_n \), \( \alpha_{n+1} \), \( s_{n+1}^{\text{Trial}} \), and \( s_{n+1} \) are col-linear. In other words, \( n_{n+1} \) and \( n_{n+1} \) are identical.
4.8.3 Consistent (Algorithmic) Tangent Elastoplastic Moduli

The detailed derivation of the consistent algorithmic moduli for the $J_2$ plasticity model is available in the literature (Auricchio, Taylor and Lubliner 1992). This derivation is summarized below.

From the return map algorithm for the $J_2$ plasticity model, we have

$$
e^p_{n+1} = e^p_n + \lambda \ n_{n+1} \tag{4.101}$$

$$\alpha_{n+1} = \alpha_n + \frac{2}{3} H_{\text{kin}} \lambda_{n+1} n_{n+1} \tag{4.102}$$

$$s_{n+1} = 2G\left(e^p_{n+1} - e^p_n\right) - 2G\lambda_{n+1} n_{n+1} \tag{4.103}$$

$$\Sigma_{n+1} = s_{n+1} - \alpha_{n+1}. \tag{4.104}$$

Linearizing Eqs. (4.102) and (4.103) about the solution point gives

$$ds_{n+1} = 2Gde_{n+1} - 2Gd\lambda_{n+1} n_{n+1} - 2G\lambda_{n+1} d\Sigma_{n+1} \tag{4.105}$$

and

$$d\alpha_{n+1} = \frac{2}{3} H_{\text{kin}} d\lambda_{n+1} n_{n+1} + \frac{2}{3} H_{\text{kin}} \lambda_{n+1} d\Sigma_{n+1}. \tag{4.106}$$

Now, it can be shown that (Simo and Hughes, to appear)

$$d\Sigma_{n+1} = \hat{1} \frac{d}{d||\Sigma_{n+1}||} (\Sigma_{n+1}). \tag{4.107}$$

In the above relation,

$$\hat{1} = [I - (n_{n+1} \otimes n_{n+1})] \tag{4.108}$$

in which $I$ is the fourth-order identity tensor and $\otimes$ denotes the tensor cross product. Therefore, Eq. (4.105) can be rewritten as

$$ds_{n+1} + 2Gd\lambda_{n+1} n_{n+1} + 2G\lambda_{n+1} \hat{1} \frac{d}{d||\Sigma_{n+1}||} (\Sigma_{n+1}) = 2Gde_{n+1} \tag{4.109}$$
Using Eq. (4.104), we get

\[ ds_{n+1} + 2Gd\lambda_{n+1} n_{n+1} + 2G\lambda_{n+1} \hat{\mathbf{i}} (ds_{n+1} - d\alpha_{n+1}) = 2Gde_{n+1} \]  \hspace{1cm} (4.110)

which becomes, after rearranging the terms,

\[ ds_{n+1} \left\{ I - \frac{2G\lambda_{n+1} \hat{\mathbf{i}}}{\| \Sigma_{n+1} \|} \right\} + 2Gd\lambda_{n+1} n_{n+1} - \frac{2G\lambda_{n+1} }{\| \Sigma_{n+1} \|} d\alpha_{n+1} = 2Gde_{n+1} \]  \hspace{1cm} (4.111)

Similarly, Eq. (4.106) can be rewritten as

\[ d\alpha_{n+1} - \frac{2}{3} H_{\text{kin}} d\lambda_{n+1} n_{n+1} + \frac{2}{3} H_{\text{kin}} \lambda_{n+1} \hat{\mathbf{i}} \frac{d}{d\| \Sigma_{n+1} \|} (\Sigma_{n+1}) = 0 \]  \hspace{1cm} (4.112)

or

\[ d\alpha_{n+1} - \frac{2}{3} H_{\text{kin}} d\lambda_{n+1} n_{n+1} + \frac{2}{3} H_{\text{kin}} \lambda_{n+1} \hat{\mathbf{i}} \left( \frac{ds_{n+1} - d\alpha_{n+1}}{\| \Sigma_{n+1} \|} \right) = 0 \]  \hspace{1cm} (4.113)

which becomes, after rearranging the terms,

\[ -ds_{n+1} \left\{ \frac{2}{3} H_{\text{kin}} \lambda_{n+1} \right\} \hat{\mathbf{i}} + d\alpha_{n+1} \left\{ I - \frac{2}{3} H_{\text{kin}} \lambda_{n+1} \hat{\mathbf{i}} \right\} - \frac{2}{3} H_{\text{kin}} d\lambda_{n+1} n_{n+1} = 0 \]  \hspace{1cm} (4.114)

Rewriting the Eqs. (4.111) and (4.114) in matrix form yields

\[
\begin{bmatrix}
\{ I + a \hat{\mathbf{i}} \} & \{-a \hat{\mathbf{i}} \} & \{ 2Gn_{n+1} \} \\
\{-b \hat{\mathbf{i}} \} & \{ I - b \hat{\mathbf{i}} \} & \{- \frac{2}{3} H_{\text{kin}} n_{n+1} \}
\end{bmatrix}
\begin{bmatrix}
ds_{n+1} \\
d\alpha_{n+1} \\
d\lambda_{n+1}
\end{bmatrix}
= 
\begin{bmatrix}
2Gde_{n+1} \\
0
\end{bmatrix}
\]  \hspace{1cm} (4.115)

where \( a = \frac{2G\lambda_{n+1}}{\| \Sigma_{n+1} \|} \) and \( b = \frac{2}{3} H_{\text{kin}} \lambda_{n+1} \).

Now, from the consistency condition for the J_2 plasticity model,
\[ f_{n+1} = \| \Sigma_{n+1} \| - R_{n+1} = 0 \] (4.116)

Differentiating the above equation, we get \( df_{n+1} = 0 \). Hence,

\[ d\| \Sigma_{n+1} \| - dR_{n+1} = 0 \] (4.117)

Therefore,

\[ n_{n+1} : d\Sigma_{n+1} - \frac{2}{3} H_{iso} d\lambda_{n+1} = 0. \] (4.118)

This leads to

\[ n_{n+1} : d \{ ds_{n+1} - d\alpha_{n+1} \} - \frac{2}{3} H_{iso} d\lambda_{n+1} = 0. \] (4.119)

Using Eqs. (4.105) and (4.106), we obtain

\[ n_{n+1} \{ 2Gde_{n+1} - 2Gd\lambda_{n+1} n_{n+1} + 2G\lambda_{n+1} d\Sigma_{n+1} \} - \]

\[ n_{n+1} \{ \frac{2}{3} H_{kin} d\lambda_{n+1} n_{n+1} + \frac{2}{3} H_{kin} \lambda_{n+1} d\Sigma_{n+1} - \frac{2}{3} H_{iso} d\lambda_{n+1} \} = 0 \] (4.120)

On re-arrangement of terms, we have

\[ 2G (n_{n+1} : de_{n+1}) - \left( 2G + \frac{2}{3} H_{kin} + \frac{2}{3} H_{iso} \right) (n_{n+1} : n_{n+1}) d\lambda_{n+1} - \]

\[ 2G\lambda_{n+1} (n_{n+1} : d\Sigma_{n+1}) - \frac{2}{3} H_{kin} \lambda_{n+1} (n_{n+1} : d\Sigma_{n+1}) = 0 \] (4.121)

However, the following relations hold for the tensor \( n_{n+1} \):

\[ n_{n+1} : n_{n+1} = I \] (4.122)

\[ n_{n+1} : d\Sigma_{n+1} = n_{n+1} : \frac{\partial}{\partial \| \Sigma_{n+1} \|} (\Sigma_{n+1}). \] (4.123)

Making use of the above identities, we get
\[ 2G \left( n_{n+1} : \mathbf{d}e_{n+1} \right) - \left( 2G + \frac{2}{3} H_{\text{kin}} + \frac{2}{3} H_{\text{iso}} \right) d\lambda_{n+1} = 0. \] (4.124)

Hence,

\[ d\lambda_{n+1} = \left( \frac{2G}{2G + \frac{2}{3} H_{\text{kin}} + \frac{2}{3} H_{\text{iso}}} \right) \left( n_{n+1} : \mathbf{d}e_{n+1} \right). \] (4.125)

Substituting Eq. (4.125) into Eq. (4.115) yields

\[ \begin{bmatrix} I + a\hat{I} & -a\hat{I} \\ -b\hat{I} & I + b\hat{I} \end{bmatrix} \begin{bmatrix} \mathbf{d}e_{n+1} \\ \mathbf{d}\alpha_{n+1} \end{bmatrix} = \begin{bmatrix} 2G \left( \{1 + A\} I - A\hat{I} \right) \mathbf{d}e_{n+1} \\ \frac{2}{3} H_{\text{kin}} \left( n_{n+1} \otimes n_{n+1} \right) \mathbf{d}e_{n+1} \end{bmatrix}. \] (4.126)

The inverse of the above system of equations is

\[ \begin{bmatrix} \mathbf{d}s_{n+1} \\ \mathbf{d}\alpha_{n+1} \end{bmatrix} = \begin{bmatrix} I - C\hat{I} & -C\hat{I} \\ -D\hat{I} & I - D\hat{I} \end{bmatrix} \begin{bmatrix} 2G \left( \{1 + A\} I - A\hat{I} \right) \mathbf{d}e_{n+1} \\ \frac{2}{3} H_{\text{kin}} \left( n_{n+1} \otimes n_{n+1} \right) \mathbf{d}e_{n+1} \end{bmatrix}, \] (4.127)

where

\[ C = \frac{a}{1 + a + b} = \frac{2G\lambda_{n+1}}{\left\| \Sigma_{n+1}^{\text{Tr}} \right\|} \quad \text{and} \quad D = \frac{b}{1 + a + b} = \frac{2}{3} \frac{H_{\text{kin}}\lambda_{n+1}}{\left\| \Sigma_{n+1}^{\text{Tr}} \right\|} \]

Expansion of the first row of Eq. (4.127) yields

\[ \mathbf{d}s_{n+1} = 2G \left[ (1 - C) I + (C - A) \left( n_{n+1} \otimes n_{n+1} \right) \right] \mathbf{d}e_{n+1} \] (4.128)

In general, the total stress tensor can be expressed as

\[ \mathbf{\sigma}_{n+1} = K \left( \mathbf{I} \otimes \mathbf{1} \right) \mathbf{e}_{n+1} + s_{n+1}. \] (4.129)

where \( \mathbf{1} \) is the second-order identity tensor. Differentiating the above equation, we
have

\[ d\sigma_{n+1} = K (1 \otimes 1) d\epsilon_{n+1} + ds_{n+1}. \]  (4.130)

Substituting for \( ds_{n+1} \) from Eq. (4.128) we get

\[ d\sigma_{n+1} = K (1 \otimes 1) d\epsilon_{n+1} + 2G \left[ (1 - C) I + (C - A) (n_{n+1} \otimes n_{n+1}) \right] d\epsilon_{n+1}. \]  (4.131)

Defining the tensor \( I_{\text{dev}} \) such that \( e_{n+1} = I_{\text{dev}} \epsilon_{n+1} \), we obtain the algorithmic modulus for the \( J_2 \) plasticity model as

\[ D_{\text{ep}} = K (1 \otimes 1) + 2G (1 - C) I_{\text{dev}} + (C - A) (n_{n+1} \otimes n_{n+1}) I_{\text{dev}}. \]  (4.132)

4.9 Application of the Return Map Algorithm to the Plane-Strain Case

For ease of computations, the tensorial quantities are mapped into vectors and matrices. The following conventions are adopted for the plane strain case:

\[ \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} \]  (4.133)

\[ \sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \]  (4.134)

In tensor notation, the stresses and strains have the following form:

\[ \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \]  (4.135)
\[ \mathbf{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  
(4.136)

Hence, the volumetric strain \( \theta \) and the mean pressure \( p \) are:

\[ \theta = \text{trace} (\mathbf{\epsilon}) = (\epsilon_{11} + \epsilon_{22}) \]  
(4.137)

and

\[ p = \frac{1}{3} \text{trace} (\mathbf{\sigma}) = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \]  
(4.138)

Therefore, the deviatoric stress tensor, \( \mathbf{s} \), and the deviator strain tensor, \( \mathbf{e} \), become:

\[
\mathbf{s} = \begin{bmatrix}
\frac{(2\sigma_{11} - \sigma_{22} - \sigma_{33})}{3} & \sigma_{12} & 0 \\
\sigma_{12} & \frac{(-\sigma_{11} + 2\sigma_{22} - \sigma_{33})}{3} & 0 \\
0 & 0 & \frac{(-\sigma_{11} - \sigma_{22} + 2\sigma_{33})}{3}
\end{bmatrix}
\]  
(4.139)

and

\[
\mathbf{e} = \begin{bmatrix}
\frac{2\epsilon_{11} - \epsilon_{22}}{3} & \epsilon_{12} & 0 \\
\epsilon_{12} & -\frac{\epsilon_{11} + 2\epsilon_{22}}{3} & 0 \\
0 & 0 & \frac{-\epsilon_{11} - \epsilon_{22}}{3}
\end{bmatrix}
\]  
(4.140)

4.9.1 The Elastic Predictor Step

The trial deviatoric stress tensor is computed as

\[ s_{n+1}^{\text{Trial}} = 2G (\mathbf{e}_{n+1} - \mathbf{e}_n^p). \]  
(4.141)
The trial relative deviatoric stress tensor is now evaluated by using Eq. (4.69) as

\[ \Sigma_{n+1}^{\text{Trial}} = s_{n+1}^{\text{Trial}} - \alpha_n. \]  

(4.142)

It should be recognized that for the plane strain case, the \( \alpha_{13}, \alpha_{23}, \alpha_{31} \) and \( \alpha_{32} \) components of the back stress tensor are zero. The yield condition is now checked with the help of Eq. (4.41):

\[ \left\| \Sigma_{n+1}^{\text{Trial}} \right\| - Y_n \leq 0 \]

If the yield criterion is satisfied, then the trial state represents the true solution at time \( t_{n+1} \). The plastic corrector step is then skipped, and the solution is updated with the trial state. The total stress tensor is now given by:

\[ \sigma_{n+1} = s_{n+1}^{\text{Trial}} + K\theta_{n+1}^1I \]

(4.143)

### 4.9.2 Plastic Corrector Step

If the yield condition is violated, then the consistency parameter \( \lambda_{n+1} \) must be computed using the plastic corrector algorithm through the relation

\[ \lambda_{n+1} = \frac{\left\| \Sigma_{n+1}^{\text{Trial}} \right\| - Y_n}{2G + \frac{2}{3}(H_{\text{iso}} + H_{\text{kin}})}. \]

(4.144)

Once the value of \( \lambda_{n+1} \) has been determined, the solution at time \( t_{n+1} \) can be updated using Eqs. (4.79) through (4.87) as:

\[ n_{n+1} = \frac{\Sigma_{n+1}^{\text{Trial}}}{\left\| \Sigma_{n+1}^{\text{Trial}} \right\|} \]

\[ \Sigma_{n+1} = \Sigma_{n+1} - \left( 2G + \frac{2}{3}H_{\text{kin}} \right) \lambda_{n+1} n_{n+1} \]

(4.145)  

(4.146)
(4.147) \[ \alpha_{n+1} = \alpha_n + \frac{2}{3} H_{\text{kin}} \lambda_{n+1} n_{n+1} \]

(4.148) \[ e_{n+1}^p = e_n^p + \lambda_{n+1} n_{n+1} \]

(4.149) \[ \varepsilon_{n+1}^p = \varepsilon_n^p + \frac{\sqrt{2}}{\sqrt{3}} \lambda_{n+1} \]

(4.150) \[ \sigma_{y,n+1} = \left( \sigma_{y0} + H_{\text{iso}} \varepsilon_{n+1}^p \right) \]

The deviatoric stress tensor is calculated as:

(4.151) \[ s_{n+1} = \sum_{n+1} - \alpha_{n+1} \]

from which the total stress tensor is obtained:

(4.152) \[ \sigma_{n+1} = s_{n+1} + K\theta_{n+1} I \]

4.9.3 Consistent Tangent Stiffness Matrix for the Plane-Strain Case

From Section 4.8.3, the consistent tangent moduli for the general three dimensional case was found to be

\[ D^{ep} = K (I \otimes I) + 2G (1 - C) I_{\text{dev}} + (C - A) (n_{n+1} \otimes n_{n+1}) I_{\text{dev}} \]

In order to avoid errors in computing the norm of a tensor, the second-order stress and strain tensors are mapped into five-component vectors. Hence,

(4.153) \[ e = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{21} \end{bmatrix} \]
\[ s = \begin{bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \\ s_{21} \end{bmatrix} \] (4.154)

It should be noted that the above representation of a second-order tensor is redundant, since \( s_{12} = s_{21} \), \( e_{12} = e_{21} \) and \( e_{33} = -(e_{11} + e_{22})/3 \). However, this mapping simplifies a great deal the computation of the norm of a second-order tensor.

Using the above mapping of the tensorial quantities into vectors, the tensors \( \mathbf{1} \), \( \mathbf{I}_{\text{dev}} \), and \( \mathbf{n}_{n+1} \mathbf{\Sigma}_{n+1} \) can be represented in vectorial form as

\[ \mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^T \] (4.155)

\[ \mathbf{I}_{\text{dev}} = \begin{bmatrix} \left( \frac{2}{3} \right) & \left( \frac{-1}{3} \right) & \left( \frac{-1}{3} \right) & 0 & 0 \\ \left( \frac{-1}{3} \right) & \left( \frac{2}{3} \right) & \left( \frac{-1}{3} \right) & 0 & 0 \\ \left( \frac{-1}{3} \right) & \left( \frac{-1}{3} \right) & \left( \frac{2}{3} \right) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \] (4.156)

\[ \mathbf{n}_{n+1} = \left( \frac{1}{\| \mathbf{\Sigma}_{n+1} \|} \right) \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{22} & \mathbf{\Sigma}_{33} & \mathbf{\Sigma}_{12} & \mathbf{\Sigma}_{21} \end{bmatrix}^T \] (4.157)

\[ \mathbf{\Sigma}_{n+1} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{22} & \mathbf{\Sigma}_{33} & \mathbf{\Sigma}_{12} & \mathbf{\Sigma}_{21} \end{bmatrix}^T \] (4.158)

The norm of the tensor \( \mathbf{\Sigma}_{n+1} \) (represented in vector form) is

\[ \| \mathbf{\Sigma}_{n+1} \| = \sqrt{\begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{22} & \mathbf{\Sigma}_{33} & \mathbf{\Sigma}_{12} & \mathbf{\Sigma}_{21} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{22} & \mathbf{\Sigma}_{33} & \mathbf{\Sigma}_{12} & \mathbf{\Sigma}_{21} \end{bmatrix}^T} \] (4.159)

The term \( \mathbf{n}_{n+1} \otimes \mathbf{n}_{n+1} \) is then represented by the outer vector product...
\[ n_{n+1} \otimes n_{n+1} = \begin{bmatrix} n_{11} & n_{22} & n_{33} & n_{12} & n_{21} \end{bmatrix}^T \times \begin{bmatrix} n_{11} & n_{22} & n_{33} & n_{12} & n_{21} \end{bmatrix} \quad (4.160) \]

Using the above contraction of tensorial quantities into vectors, the constitutive tangent matrix can now be rewritten for the plane strain case as

\[ D_{(5x5)}^{ep} = K \left( I_{(5x1)} \otimes I_{(1x5)}^T \right) + 2G \left( 1 - C \right) I_{\text{dev}, (5x5)} + (C - A) \left( n_{(5x1)}^T \otimes n_{(1x5)} \right) I_{\text{dev}, (5x5)} \quad (4.161) \]
4.10 Application of the $J_2$ Plasticity Model to the Truss Element

4.10.1 Constitutive Equations - Continuum (Infinitesimal) Form

For the truss element, the stress and strain spaces are uni-dimensional. Therefore, the formulation of the constitutive equations is greatly simplified, since all equations are in scalar form. Let $\sigma$ and $\varepsilon$ denote the axial stress and axial strain, respectively, of the element. According to the basic assumption of small-strain plasticity, the axial strain $\varepsilon$ can be additively decomposed into an elastic part, $\varepsilon^e$, and a plastic part, $\varepsilon^p$, as:

$$\varepsilon = \varepsilon^e + \varepsilon^p$$ (4.162)

The uniaxial stress $\sigma$ is simply given by

$$\sigma = E \varepsilon^e = E \left( \varepsilon - \varepsilon^p \right)$$ (4.163)

where $E$ is the Young’s modulus of the material. The flow rule for the plastic component of the strain is given by:

$$\varepsilon^p = \dot{\gamma} \frac{\partial F}{\partial \sigma}$$ (4.164)

in which $F$ represents the yield function defined as:

$$F = ||\sigma - \alpha|| - \sigma_y$$ (4.165)

where $\alpha$ denotes the back stress which satisfies the linear kinematic hardening rule:

$$\alpha = H_{\text{kin}} \varepsilon^p$$ (4.166)

and $\sigma_y$ is the current yield stress given by the linear isotropic hardening rule:

$$\sigma_y = \sigma_{y0} + H_{\text{iso}} \varepsilon^p$$ (4.167)

In the above equation, the cumulative effective plastic strain $\varepsilon^p$ is defined as:

$$\varepsilon^p = \int |\varepsilon^p| \, dt$$ (4.168)
From the yield function defined in Eq. (4.112), it follows that

\[ \frac{\partial F}{\partial \sigma} = n = \frac{\sigma - \alpha}{|\sigma - \alpha|} = \begin{cases} +1 & \text{if yielding in tension} \\ -1 & \text{if yielding in compression} \end{cases} \]  
(4.169)

Using the above result, the plastic strain rate, \( \varepsilon^p \), and the rate of effective plastic strain, \( \dot{\varepsilon}^p \), can be expressed as:

\[ \varepsilon^p = \gamma n \]  
(4.170)

where

\[ \dot{\gamma} = \begin{cases} 0 & \text{during elastic behavior} \\ >0 & \text{during plastic behavior} \end{cases} \]  
(4.171)

\[ \dot{\varepsilon}^p = |\varepsilon^p| = \dot{\gamma} \]  
(4.172)

From the yield function in Eq. (4.112), the consistency condition can be expressed as:

\[ \dot{F} = \frac{\partial F}{\partial \sigma} \dot{\sigma} + \frac{\partial F}{\partial \alpha} \dot{\alpha} + \frac{\partial F}{\partial \varepsilon^p} \dot{\varepsilon}^p = 0 \]  
(4.173)

Using Eqs. (4.111), (4.113), (4.114), (4.116), and (4.119), the above equation can be further developed as:

\[ \dot{F} = n \dot{\sigma} - (H_{\text{kin}} + H_{\text{iso}}) \dot{\gamma} = 0 \]  
(4.174)

Using Eqs. (4.109), (4.110), (4.111), and (4.116), the total strain rate can be rewritten as:

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p = \frac{1}{E} \dot{\sigma} + n \dot{\gamma} \]  
(4.175)

Equations (4.121) and (4.122) can now be written in a single matrix form as:

\[ \begin{bmatrix} \dot{\varepsilon} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & n \\ n - (H_{\text{kin}} + H_{\text{iso}}) \end{bmatrix} \begin{bmatrix} \dot{\sigma} \\ \dot{\gamma} \end{bmatrix} \]  
(4.176)
The indeterminate constant $\dot{\gamma}$ can now be eliminated by multiplying the first set of Eq. (4.123) by $nE$, giving

$$n\sigma = nE\varepsilon - E\dot{\gamma}$$  \hspace{1cm} (4.177)

Substituting the above relation into the second set of Eq. (4.123) yields

$$nE\varepsilon - [E + H_{\text{kin}} + H_{\text{iso}}] \dot{\gamma} = 0$$  \hspace{1cm} (4.178)

Elimination of $\dot{\gamma}$ from the first equation of (4.123) now gives:

$$d\sigma = E^{ep} d\varepsilon$$  \hspace{1cm} (4.179)

where

$$E^{ep} = E \left[ 1 - \frac{E}{E + H_{\text{kin}} + H_{\text{iso}}} \right]$$  \hspace{1cm} (4.180)

The elastoplastic tangent modulus $E^{ep}$ takes the place of the elastic Young's modulus in infinitesimal incremental analysis. In the case of perfect plasticity, $H_{\text{kin}} = H_{\text{iso}} = 0$, and Eq. (4.127) becomes:

$$E^{ep} = 0$$  \hspace{1cm} (4.181)

For the popular case of pure kinematic hardening ($H_{\text{iso}} = 0$) in uniaxial plasticity, also called the bilinear inelastic model, the elastoplastic tangent modulus in Eq. (4.127) reduces to

$$E^{ep} = \frac{EH_{\text{kin}}}{E + H_{\text{kin}}}$$  \hspace{1cm} (4.182)

The uniaxial $J_2$ plasticity model with kinematic hardening only is represented in Fig. 4.3. Usually, the bilinear inelastic model is defined in terms of the post-yield stiffness, $E_1$, instead of the kinematic hardening modulus, $H_{\text{kin}}$, and the ratio $E_1/E$ is termed the “strain hardening ratio”. The relationship between $E_1$ and $H_{\text{kin}}$ can be obtained by expressing the back stress $\alpha$ in the two equivalent forms:
\[ \dot{\alpha} = H_{\text{kin}} \epsilon^p \]  

(4.183)

\[ \dot{\alpha} = E_1 \epsilon \]  

(4.184)

from which it follows that:

Figure 4.3 Uniaxial $J_2$ plasticity model with kinematic hardening only also called the bilinear inelastic model; (a) $\sigma$-$\epsilon$ diagram, (b) linear kinematic hardening model.
\[ E_1 = \frac{\varepsilon^p}{\varepsilon} H_{\text{kin}} \]  

(4.185)

By substituting Eqs. (4.117) and (4.120) into Eq. (4.132), it is found that

\[ E_1 = \frac{E}{E + H_{\text{kin}} + H_{\text{iso}}} H_{\text{kin}} \]  

(4.186)

In the case of pure kinematic hardening for which \( H_{\text{iso}} = 0 \),

\[ E_1 = \frac{E}{E + H_{\text{kin}}} H_{\text{kin}} \]  

(4.187)

**4.10.2 Constitutive Equations in Discrete Form**

A discrete solution at time \( t_n \) is defined in terms of the state \( \varepsilon_n, \sigma_n, \alpha_n, \varepsilon^p_n, \) and \( \varepsilon^p_n \). The solution is then advanced to time \( t_{n+1} \) by specifying the strain, \( \varepsilon_{n+1} \). Applying the implicit backward Euler rule to Eqs. (4.111), (4.113), and (4.119) and adopting the previously adopted convention, the following incremental relations are obtained:

\[ \varepsilon^p_{n+1} = \varepsilon^p_n + \lambda_n n_{n+1} \]  

(4.188)

\[ \alpha_{n+1} = \alpha_n + H_{\text{kin}} \lambda_{n+1} n_{n+1} \]  

(4.189)

\[ \varepsilon^p_{n+1} = \varepsilon^p_n + \lambda_{n+1} \]  

(4.190)

where \( \lambda_{n+1} \) and \( n_{n+1} \) are defined in Eqs. (4.57) and (4.116), respectively. The discrete consistency parameter \( \lambda_{n+1} \) is zero in case of purely elastic deformations and positive in case of elastoplastic deformations. The stress quantities are also incremented as follows:

\[ \sigma_{n+1} = E\left(\varepsilon_{n+1} - \varepsilon^p_{n+1}\right) = E\left(\varepsilon_{n+1} - \varepsilon^p_n\right) - E\lambda_{n+1} n_{n+1} \]  

(4.191)

\[ \sigma_{n+1} - \alpha_{n+1} = E\left(\varepsilon_{n+1} - \varepsilon^p_n\right) - E\lambda_{n+1} n_{n+1} - \alpha_n - H_{\text{kin}} \lambda_{n+1} n_{n+1} \]  

(4.192)

\[ \sigma_{y, n+1} = \sigma_{y, n} + H_{\text{iso}} \lambda_{n+1} \]  

(4.193)
The yield criterion is now given by:

\[ F_{n+1} = |\sigma_{n+1} - \alpha_{n+1}| - \sigma_{y, n+1} \leq 0 \]  \hspace{1cm} (4.194)

### 4.10.3 Return Map Algorithm

The return map algorithm for the uni-dimensional version of the \(J_2\) plasticity model in discrete form is described below.

#### 4.10.3.1 Elastic Predictor Step

In the elastic predictor step, it is assumed that no plastic deformation occurs during the current time step. Therefore,

\[ \lambda_{n+1}^{\text{Trial}} = 0 \]  \hspace{1cm} (4.195)

\[ \left( \epsilon_n^p \right)^{\text{Trial}} = \epsilon_n^p \]  \hspace{1cm} (4.196)

\[ \alpha_{n+1}^{\text{Trial}} = \alpha_n \]  \hspace{1cm} (4.197)

\[ \left( \varepsilon_n^p \right)^{\text{Trial}} = \varepsilon_n^p \]  \hspace{1cm} (4.198)

\[ \sigma_{n+1}^{\text{Trial}} = E\left( \epsilon_{n+1}^p - \epsilon_n^p \right) \]  \hspace{1cm} (4.199)

\[ \sigma_{y, n+1}^{\text{Trial}} = \sigma_{y, n} \]  \hspace{1cm} (4.200)

where the superscript "Trial" denotes the trial values for the elastic predictor step.

Now, the trial current state of stress is checked against the yield criterion, as given in Eq. (4.141). If the yield criterion is satisfied, then the trial state of stress represents the true state of stress at time \(t_{n+1}\). Otherwise, a plastic corrector step must be applied.

#### 4.10.3.2 Plastic Corrector Step

The plastic corrector step is based upon satisfying the consistency condition in discrete form:

\[ |\sigma_{n+1} - \alpha_{n+1}| - \sigma_{y, n+1} = 0 \]  \hspace{1cm} (4.201)
where

\[
\sigma_{n+1} = E\left( e_{n+1} - e_{n+1}^p \right)
\]
\[
= E\left( e_{n+1} - e_n - \lambda_{n+1} n_{n+1} \right)
\]
\[
= \sigma_{n+1}^{\text{Trial}} - E\lambda_{n+1} n_{n+1}
\]

\[
\alpha_{n+1} = \alpha_n + H_{\text{kin}} \lambda_{n+1} n_{n+1}
\]
\[
= \alpha_{n+1}^{\text{Trial}} + H_{\text{kin}} \lambda_{n+1} n_{n+1}
\]

\[
\sigma_{y,n+1} = \sigma_{y,n} + H_{\text{iso}} \lambda_{n+1}
\]

In Eqs. (4.149) and (4.150), the “unit” directional vector \( n_{n+1} \) is given by:

\[
n_{n+1} = \frac{\sigma_{n+1} - \alpha_{n+1}}{\sqrt{\sigma_{n+1} - \alpha_{n+1}^2}}
\]

By substituting Eqs. (4.149) and (4.150) into Eq. (4.148) and using Eq. (4.151), it follows that:

\[
|\sigma_{n+1} - \alpha_{n+1}| n_{n+1} = \left( \sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}} \right) - (E + H_{\text{kin}}) \lambda_{n+1} n_{n+1}
\]

Since two terms out of three in Eq. (4.153) are in the direction \( n_{n+1} \), it follows that the third term is also in this direction. Thus,

\[
\frac{\sigma_{n+1} - \alpha_{n+1}}{\sqrt{\sigma_{n+1} - \alpha_{n+1}^2}} = \frac{\sigma_{n+1}^{\text{Trial}}}{\sqrt{\sigma_{n+1} - \alpha_{n+1}^{\text{Trial}}}} = n_{n+1}
\]

Using the above relation, Eq. (4.153) reduces further to:

\[
|\sigma_{n+1} - \alpha_{n+1}| = |\sigma_{n+1} - \alpha_{n+1}^{\text{Trial}} - (E + H_{\text{kin}}) \lambda_{n+1}|
\]

By substituting Eqs. (4.151) and (4.154) into Eq. (4.148), the discrete consistency condition becomes:

\[
|\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}} - (E + H_{\text{kin}}) \lambda_{n+1} - \sigma_{y,n} - H_{\text{iso}} \lambda_{n+1} = 0
\]
The discrete consistency parameter, \( \lambda_{n+1} \), is obtained from the above equation:

\[
\lambda_{n+1} = \frac{\left| \sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}} \right| - \sigma_{y,n}}{E + H_{\text{iso}} + H_{\text{kin}}}
\]  

(4.210)

Once \( \lambda_{n+1} \) has been determined, the complete solution at time \( t_{n+1} \) can be updated as follows:

\[
\eta_{n+1}^{\text{Trial}} = \frac{\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trail}}}{\left| \sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trail}} \right|}
\]  

(4.211)

\[
\alpha_{n+1} = \alpha_{n} + H_{\text{kin}} \lambda_{n+1} \eta_{n+1}^{n+1}
\]  

(4.212)

\[
\epsilon_{n+1}^{p} = \epsilon_{n}^{p} + \lambda_{n+1} \eta_{n+1}^{n+1}
\]  

(4.213)

\[
\bar{\epsilon}_{n+1}^{p} = \bar{\epsilon}_{n}^{p} + \lambda_{n+1}
\]  

(4.214)

\[
\sigma_{y,n+1} = \sigma_{y,n} + H_{\text{iso}} \lambda_{n+1}
\]  

(4.215)

\[
\sigma_{n+1} = E \left( \epsilon_{n+1} - \epsilon_{n+1}^{p} \right)
\]  

(4.216)

The return map algorithm, in the uniaxial stress case, is illustrated in Fig. 4.4 for the particular case of pure kinematic hardening (i.e., \( H_{\text{iso}} = 0 \)).

### 4.10.3.3 Consistent Tangent Stiffness Matrix for the Truss Element

The derivation of the consistent elastoplastic moduli for the truss element is particularly simple since the stresses, strains and the elastic moduli can be represented by scalar variables.

From the elastic stress-strain relationship and the flow rule, we have

\[
\sigma_{n+1} = E \left( \epsilon_{n+1} - \epsilon_{n+1}^{p} \right)
\]  

(4.217)

\[
\epsilon_{n+1}^{p} = \epsilon_{n}^{p} + \Delta \epsilon_{n+1}^{p} = \lambda_{n+1} \eta_{n+1}^{n+1}
\]  

(4.218)
Differentiating Eqs. (4.217) and (4.218), we obtain
\[ d\sigma_{n+1} = E \left( d\epsilon_{n+1} - d\epsilon^p_{n+1} \right) \] (4.219)
and
\[ d\epsilon^p_{n+1} = d\lambda_{n+1} n_{n+1} \] (4.220)
Substituting Eq. (4.220) into Eq. (4.219), we get
\[ d\sigma_{n+1} = E \left( d\epsilon_{n+1} - d\lambda_{n+1} n_{n+1} \right) \] (4.221)
From the consistency condition we have
\[ f_{n+1} = \|\Sigma_{n+1}\| - R_{n+1} = 0 \] (4.222)
Differentiating the above, we have
\[ df_{n+1} = d\|\Sigma_{n+1}\| - dR_{n+1} = 0. \] (4.223)
Recognizing that
\[ \|\Sigma_{n+1}\| = (\sigma_{n+1} - \alpha_{n+1}) n_{n+1} = (\sigma_{n+1} - H_{\text{kin}} \lambda_{n+1} n_{n+1}) n_{n+1} \] (4.224)
and
\[ R_{n+1} = H_{\text{iso}} \lambda_{n+1}, \] (4.225)
we have
\[ d\|\Sigma_{n+1}\| = (d\sigma_{n+1} - H_{\text{kin}} d\lambda_{n+1} n_{n+1}) n_{n+1} \] (4.226)
and
\[ dR_{n+1} = H_{\text{iso}} d\lambda_{n+1} \] (4.227)
Upon substituting Eqs. (4.227) and (4.226) into Eq. (4.223) we get
\[ (d\sigma_{n+1} - H_{\text{kin}} d\lambda_{n+1} n_{n+1}) n_{n+1} - H_{\text{iso}} d\lambda_{n+1} = 0 \] (4.228)
Substituting the value of \( d\sigma_{n+1} \) from Eq. (4.221) into Eq. (4.228) and rearranging the
terms, we obtain the expression for \( d\lambda_{n+1} \):

\[
d\lambda_{n+1} = \left( \frac{E}{E + H_{\text{kin}} + H_{\text{iso}}} \right) d\epsilon_{n+1} n_{n+1}. \tag{4.229}
\]

In the above re-arrangement use has been made of the identity

\[
n_{n+1} \cdot n_{n+1} = 1 \tag{4.230}
\]

Substitution of Eq. (4.229) into Eq. (4.221) leads to

\[
d\sigma_{n+1} = E \left( d\epsilon_{n+1} - \left\{ \frac{E}{E + H_{\text{kin}} + H_{\text{iso}}} \right\} d\epsilon_{n+1} \right). \tag{4.231}
\]

Hence,

\[
\frac{d\sigma_{n+1}}{d\epsilon_{n+1}} = E_{\text{alg}}^{\text{ep}} = E \left( 1 - \frac{E}{E + H_{\text{kin}} + H_{\text{iso}}} \right) \tag{4.232}
\]

where \( E_{\text{alg}}^{\text{ep}} \) is the algorithmic elasto-plastic tangent moduli for the J2 truss element. By comparing Eq. (4.232) with Eq. (4.127), it is observed that, unlike in the multi-dimensional case, the "continuum" and algorithmic tangent moduli are identical in the uni-axial case for the J2 plasticity model.
Figure 4.4 Return map algorithm in the uniaxial case with pure kinematic hardening ($H_{iso}=0$)
4.11 Constitutive Relations for the Cap Model

The cap model used in this study is a non-smooth, multisurface, rate independent, associative plasticity model (DiMaggio and Sandler 1971; Sandler et al. 1976; Simo et al. 1988a). As shown in Fig. 4.5, it is defined by a convex yield surface, which consists of a failure surface or envelope, \( f_1 (\mathbf{\sigma}) \), a hardening elliptical cap, \( f_2 (\mathbf{\sigma}, \kappa) \), and a tension cut-off region, \( f_3 (\mathbf{\sigma}) \), where \( \mathbf{\sigma} \) denotes the stress tensor and \( \kappa \) is the hardening parameter. Thus, the failure envelope and the tension cut-off region are modeled as ideal plasticity surfaces while the cap is modeled as a strain hardening surface.

![Diagram](image)

Figure 4.5 The yield surface of the cap model

The functional forms of \( f_1 \), \( f_2 \) and \( f_3 \) are:

\[
f_1 (\mathbf{\sigma}) = \|s\| - F_e (I_1), \quad \text{for } T \leq I_1 \leq \kappa, \quad (4.233)
\]

where

\[
F_e (I_1) = \alpha - \lambda e^{-\beta I_1} + \theta I_1, \quad (4.234)
\]

\[
f_2 (\|s\|, I_1, \kappa) = F_e (\|s\|, I_1, \kappa) - F_e (\kappa), \quad \text{for } \kappa \leq I_1 \leq X (\kappa) \quad (4.235)
\]

where
\[
F_c (|s|, I_1, \kappa) = \sqrt{||s||^2 + \left( \frac{I_1 - L(\kappa)}{R} \right)^2}, \quad (4.236)
\]

and

\[
f_3 (\sigma) = T - I_1, \quad \text{for } I_1 = T. \quad (4.237)
\]

In the above definitions, \(\alpha, \beta, \lambda, \theta, \text{and } R\) are material parameters for the cap and failure envelope, and \(T\) is the tension cut-off or maximum allowable hydrostatic tension, which is a material constant.

\(I_1\) and \(|s|\) are the first invariant of the stress tensor \(\sigma\) and the norm of the deviatoric stress tensor as defined in Eqs. (4.4) and (4.11), respectively. The function \(L(\kappa)\) is defined as:

\[
L(\kappa) = \begin{cases} 
\kappa & \text{if } \kappa > 0 \\
0 & \text{if } \kappa \leq 0
\end{cases} \quad (4.238)
\]

In the following analysis, compressive stresses and compressive strains (i.e., compaction) are assumed to be positive. The point of intersection of the cap with the \(I_1\)-axis is defined as

\[
X(\kappa) = \kappa + R F_c(\kappa) \quad (4.239)
\]

in which \(R\) represents the major to minor axis ratio. The deviatoric strain tensor is defined as

\[
e = \epsilon - \frac{1}{3} \tilde{I}_1 I \quad (4.240)
\]

where \(\epsilon\) denotes the strain tensor and \(\tilde{I}_1 = \text{trace}(\epsilon)\). The hardening parameter \(\kappa\) is implicitly defined (Sandler and Rubin, 1979) in terms of the effective plastic volumetric strain, \(\varepsilon^p_v\):}

\[
\varepsilon^p_v = W \left( 1 - e^{-D X(\kappa)} \right) \quad (4.241)
\]

in which \(W\) and \(D\) are material parameters. The effective plastic volumetric strain, \(\varepsilon^p_v\),
is a history dependent functional of the volumetric plastic strain and is defined in rate form as:

\[
\varepsilon^p_v = \begin{cases} 
\dot{\varepsilon}^p_v = \varepsilon^p_v & \text{if } \dot{I}_1 > 0, \text{ or } \kappa > 0 \text{ and } \kappa > I_1 \\
0, & \text{otherwise}
\end{cases}
\]  

(4.242)

The hardening relationship between \(\varepsilon^p_v\) and \(X(\kappa)\) in Eq. (4.172) is represented graphically in Fig. 4.6. The hardening parameter \(\kappa\) is thus obtained by substituting Eq. (4.173) into Eq. (4.172) and solving for \(X(\kappa)\), after which Eq. (4.170) is solved for \(\kappa\) given \(X(\kappa)\) using a Newton iterative procedure. Therefore, according to the hardening rule defined in Eqs. (4.169), (4.170), (4.172), and (4.173), the cap moves out or in as \(\varepsilon^p_v\) increases or decreases, respectively, as shown in Fig. 4.7. If the stress point reaches the failure envelope, the plastic strain rate vector has a negative volumetric component (dilatancy), see Fig. 4.8(a). This causes the cap to move back until it reaches the stress point, thus limiting further plastic volume increases, see Fig. 4.8(b). At the compressive corner point, movement of the cap is prevented and the model behaves as in perfect plasticity, thus avoiding a softening response.
The elastic domain is defined in terms of the three yield surfaces as

\[ f_i(\sigma) < 0, \quad i = 1, 2, 3 \]  \hspace{1cm} (4.243)

The flow rule is assumed to be associative and, in the case of multisurface plasticity, takes the generalized form:

\[ \varepsilon^p = \sum_{i=1}^{3} \gamma_i \frac{\partial f}{\partial \sigma}, \]  \hspace{1cm} (4.244)
where $\gamma_i$, $i=1,2,3$, are the plastic consistency parameters. Plastic loading or elastic loading/unloading can be formulated in the standard Kuhn-Tucker form as:

$$\gamma_i \geq 0, \quad f_i \leq 0, \quad \text{and} \quad \gamma_i f_i = 0, \quad i=1,2,3$$

(4.245)

4.11.1 Return map Algorithm

In order to compute the internal resisting force increments for a given incremental strain tensor, the corresponding incremental stress tensor must be determined at each integration point of the element. As for the $J_2$ single-surface plasticity model, the return map algorithm can be used to find the true internal stresses. It consists of two major steps, the elastic-predictor step and the return mapping to the yield surface, which can be geometrically interpreted as a closest point projection of the elastic trial stress onto the yield surface, also referred to as a plastic corrector.

In the following discussion, $\epsilon^p$, $\epsilon^e$ and $\epsilon$ denote the plastic strain tensor, the elastic strain tensor and the total strain tensor, respectively. Application of an implicit backward Euler integration scheme to the constitutive equations of the cap model results into the following incremental equations (Hofstetter et al. 1993):

$$\epsilon^p_{n+1} = \epsilon^p_n + \sum_{i=1}^{3} \Delta \gamma_{i,n+1} \left( \frac{\partial f_i}{\partial \sigma} \right)_{n+1}$$

(4.246)

$$\sigma^\text{Trial}_{n+1} = C : \left( \epsilon_{n+1} - \epsilon^p_n \right)$$

(4.247)

$$\sigma_{n+1} = C : \left( \epsilon_{n+1} - \epsilon^p_{n+1} \right) = \sigma^\text{Trial}_{n+1} - C : \Delta \epsilon^p_{n+1}$$

(4.248)

where $\Delta \gamma_{i,n+1} = \int_{t_n}^{t_{n+1}} \gamma_i \, dt$, $C$ is the elastic constitutive fourth-order tensor and the symbol $(\cdot)$ denotes the doubly contracted tensor product, e.g., $(C:\epsilon)_{ij} = C_{ijkl} \epsilon_{kl}$. The Kuhn-Tucker conditions in discrete form can be expressed as:

$$\Delta \gamma_{i,n+1} \geq 0, \quad f_{i,n+1} \leq 0 \quad \text{and} \quad \Delta \gamma_{i,n+1} f_{i,n+1} = 0, \quad i=1,2,3.$$ (4.249)

The trial elastic deviatoric and spherical stresses are defined as:
\[ s_{n+1}^{\text{Trial}} = 2G\left( e_{n+1}^p - e_n^p \right) \quad (4.250) \]

\[ I_{1,n+1}^{\text{Trial}} = 3K\left( \tau_{1,n+1} - \tau_{1,n}^p \right) \quad (4.251) \]

It can be shown (Simo and Hughes, to appear) that the convexity condition of the yield surface implies that \( f_{i,n+1} \leq f_{i,n+1}^{\text{Trial}} \) for \( i=1,2,3 \). Hence, if \( f_{i,n+1}^{\text{Trial}} \leq 0 \) for \( i=1,2,3 \), then the process is elastic. Otherwise, the current step is inelastic and it is necessary to determine \( \Delta e_{n+1}^p \) and \( \kappa_{n+1} \). Then, the true stresses can be obtained from the trial elastic stresses as:

\[ s_{n+1} = s_{n+1}^{\text{Trial}} - 2G \Delta e_{n+1}^p \quad (4.252) \]

\[ I_{1,n+1} = I_{1,n+1}^{\text{Trial}} - 3K \Delta I_{1,n+1}^p \quad (4.253) \]

The difficulty associated with multisurface plasticity is that the active set of yield surfaces is not known in advance, since \( f_{i,n+1}^{\text{Trial}} > 0 \) does not guarantee that \( f_{i,n+1} = 0 \) if more than one yield criterion is active (Simo, Kennedy, and Govindjee, 1988b).

**4.11.2 Loading in the Various Modes of the Cap Model**

The following sections summarize the return mapping algorithm for the various modes of the cap model. A detailed derivation of the algorithm is available in the literature (Hofstetter, Simo and Taylor, 1993).

(1) **Loading in the failure envelope mode:**

The loading in the failure envelope mode is characterized by \( f_{1,n+1}^{\text{Trial}} > 0 \), \( \Delta \gamma_{1,n+1} > 0 \), \( \Delta \gamma_{2,n+1} = \Delta \gamma_{3,n+1} = 0 \). To determine the discrete consistency parameter, \( \Delta \gamma_{1,n+1} \), associated with the failure surface \( f_1 = 0 \), the following nonlinear scalar equation must be solved for \( I_{1,n+1} \):

\[ I_{1,n+1}^{\text{Trial}} + 9K \Delta \gamma_{1,n+1} \left( \frac{dF_e}{dI_1} \right)_{n+1} - I_{1,n+1} = 0 \quad (4.254) \]

where
\[
\Delta \gamma_{I_{n+1}} = \frac{\|s_{n+1}^{\text{Trial}}\| - F_e (I_{I_{n+1}})}{2G}
\]

The above equation can be solved using a Newton iterative technique. Once \(I_{I_{n+1}}\) is known, \(\Delta \gamma_{I_{n+1}}\) is obtained from Eq. (4.186) and the true state at time \(t_{n+1}\) is determined through the following relations:

\[
\Delta T_{I_{n+1}}^p = -3 \Delta \gamma_{I_{n+1}} \frac{dF_e (I_{I_{n+1}})}{dI_{I_{n+1}}}
\]

\[
\Delta e_{n+1}^p = \Delta \gamma_{I_{n+1}} n_{n+1}
\]

\[
s_{n+1} = \left\| s_{n+1} \right\| n_{n+1}
\]

where

\[
\left\| s_{n+1} \right\| = \left\| s_{n+1}^{\text{Trial}} \right\| - 2G \Delta \gamma_{I_{n+1}} \quad \text{and} \quad n_{n+1} = \frac{s_{n+1}^{\text{Trial}}}{\left\| s_{n+1} \right\|}
\]

\[
\sigma_{n+1} = s_{n+1} + 1/3 I_{I_{n+1}} \quad (4.260)
\]

(2) Loading in the tensile corner region:

The loading in the tensile corner region is characterized by \(\Delta \gamma_{I_{n+1}} > 0\), \(\Delta \gamma_{2_{n+1}} = 0\) and \(\Delta \gamma_{3_{n+1}} > 0\). Since both the failure surface and the tension cut-off surface behave as in ideal plasticity, the return point, \(\sigma_{n+1}\), is the point of intersection of these two surfaces. Thus,

\[
I_{I_{n+1}} = T, \quad (4.261)
\]

\[
\left\| s_{n+1} \right\| = F_e (T). \quad (4.262)
\]

Solving for the discrete consistency parameters \(\Delta \gamma_{I_{n+1}}\) and \(\Delta \gamma_{3_{n+1}}\) results in:

\[
\Delta \gamma_{I_{n+1}} = \frac{\left\| s_{n+1}^{\text{Trial}} \right\| - F_e (T)}{2G}, \quad (4.263)
\]
\[
\Delta \gamma_{3,n+1} = \frac{T - I_{1,n+1}^{\text{Trial}}}{9K} - \Delta \gamma_{1,n+1} \frac{dF_e(T)}{dI_1} .
\]
(4.264)

Once \( \Delta \gamma_{1,n+1} \) and \( \Delta \gamma_{3,n+1} \) are known, the true state at time \( t_{n+1} \) is obtained according to the following relations:

\[
\Delta I_{1,n+1}^p = -3 \left( \Delta \gamma_{1,n+1} \frac{dF_e(T)}{dI_1} + \Delta \gamma_{3,n+1} \right) = -3 \left( \frac{T - I_{1,n+1}^{\text{Trial}}}{9K} \right)
\]
(4.265)

\[
\Delta e_{n+1}^p = \Delta \gamma_{1,n+1} \mathbf{n}_{n+1}
\]
(4.266)

\[
s_{n+1} = \|s_{n+1}\| \mathbf{n}_{n+1}
\]
(4.267)

\[
\sigma_{n+1} = s_{n+1} + 1/3 I_{1,n+1} I
\]
(4.268)

In Eq. (4.198), \( \|s_{n+1}\| \) and \( \mathbf{n}_{n+1} \) are given by Eqs. (4.193) and (4.190)_2, respectively.

(3) Loading in the cap mode:

Loading within the cap region is characterized by \( I_{2,n+1}^{\text{Trial}} > 0, \Delta \gamma_{2,n+1} > 0, \) and \( \Delta \gamma_{1,n+1} = \Delta \gamma_{3,n+1} = 0. \) The discrete consistency parameter \( \Delta \gamma_{2,n+1} \) is found to be the solution of the following set of nonlinear equations:

\[
\sqrt{\left( \frac{\left\| s_{n+1} \right\|}{F_e(\kappa_{n+1})} \right)^2 + \left( \frac{I_{1,n+1}^{\text{Trial}} - \kappa_{n+1}}{R + (9K\Delta \gamma_{2,n+1}) / (RF_e(\kappa_{n+1}))} \right)^2} = 0
\]
(4.269)

\[
-\frac{R^2H(\kappa_{n+1})F_e(\kappa_{n+1})}{3(I_{1,n+1} - \kappa_{n+1})} = 0
\]
(4.270)

and

\[
I_{1,n+1} = I_{1,n+1}^{\text{Trial}} - 3KH(\kappa_{n+1})
\]
(4.271)

where
\[ H(\kappa_{n+1}) = W\left( e^{-DX(\kappa_n)} - e^{-DX(\kappa_{n+1})} \right). \] (4.272)

Substituting Eq. (4.202) into Eq. (4.201) and inserting the result into Eq. (4.200) gives a scalar nonlinear equation in terms of \( \kappa_{n+1} \), which can be solved by a Newton iteration technique. Once \( \kappa_{n+1} \) is known, \( \Delta \gamma_{2,n+1} \) and \( I_{1,n+1} \) are determined by Eqs. (4.201) and (4.202), respectively.

In the case where \( I_{1,n+1} = \kappa_{n+1} \), the right-hand side of Eq. (4.201) is an indeterminate expression. However, in this case \( I_{1,n+1} = \kappa_{n+1} = I_{1,n+1}^{\text{Trial}} = \kappa_n \) and \( \Delta \gamma_{2,n+1} \) can be obtained from Eq. (4.200) as

\[ \Delta \gamma_{2,n+1} = \frac{\| s_{n+1} \| - F_c(\kappa_n)}{2G} \] (4.273)

The true state at time \( t_{n+1} \) is then obtained through the following relations:

\[ \Delta I_{1,n+1}^p = 3 \Delta \gamma_{2,n+1} \frac{\partial f_{2,n+1}}{\partial I_{1,n+1}} = W\left( e^{-DX(\kappa_n)} - e^{-DX(\kappa_{n+1})} \right) = H(\kappa_{n+1}) \] (4.274)

\[ \Delta e_{n+1}^p = \Delta \gamma_{2,n+1} \frac{\partial f_{2,n+1}}{\partial s_{n+1}} = \Delta \gamma_{2,n+1} \frac{s_{n+1}}{F_c(\| s_{n+1} \|, I_{1,n+1}, \kappa_{n+1})} \] (4.275)

where

\[ \| s_{n+1} \| = \frac{\| s_{n+1} \|}{1 + \frac{2G \Delta \gamma_{2,n+1}}{F_c(\kappa_{n+1})}} \] (4.276)

The deviatoric stresses, \( s_{n+1} \), and total stresses, \( \sigma_{n+1} \), are obtained using Eqs. (4.189), (4.190), and (4.191).

(4) Loading in the compressive corner region:

This mode is characterized by \( \Delta \gamma_{1,n+1} > 0 \), \( \Delta \gamma_{2,n+1} > 0 \) and \( \Delta \gamma_{3,n+1} = 0 \). From the hardening law, the movement of the cap is prevented if the stress point is at the com-
pressive corner point. Thus, the model behaves as in perfect plasticity. Hence, the final stress point must lie at the intersection of the cap and failure surfaces. In this case, the discrete consistency parameters are found to be:

\[
\Delta \gamma_{1,n+1} = \frac{\kappa_n - I_{1,n+1}^{\text{Trial}}}{9 K \frac{dF_c(\kappa_n)}{dI_1}}
\]

\[
\Delta \gamma_{2,n+1} = \frac{\|s_{n+1}\| - F_c(\kappa_n)}{2G} - \Delta \gamma_{1,n+1}
\]  

(4.277)  

(4.278)

Once \(\Delta \gamma_{1,n+1}\) and \(\Delta \gamma_{2,n+1}\) are known, the complete state at time \(t_{n+1}\) can be obtained as follows:

\[
\Delta I_{1,n+1}^p = -3 \Delta \gamma_{1,n+1} \frac{dF_c(\kappa_n)}{dI_1}
\]

\[
\Delta \epsilon_{n+1}^p = (\Delta \gamma_{1,n+1} + \Delta \gamma_{2,n+1}) n_{n+1}
\]

\[
I_{1,n+1} = \kappa_n
\]

\[
\|s_{n+1}\| = F_c(\kappa_n)
\]

(4.279)  

(4.280)  

(4.281)  

(4.282)

In Eq. (4.211), \(n_{n+1}\) is defined as in Eq. (4.190). The deviatoric stresses, \(s_{n+1}\), and total stresses, \(\sigma_{n+1}\), are obtained by substituting Eqs. (4.212) and (4.213) into Eqs. (4.189) and (4.191).

(5) Loading in the tension cut-off region:

Loading in the tension cut-off mode is characterized by \(f_{3,n+1}^{\text{Trial}} > 0\), \(\Delta \gamma_{3,n+1} > 0\) and \(\Delta \gamma_{1,n+1} = \Delta \gamma_{2,n+1} = 0\). Using the geometric interpretation of the return mapping algorithm as a closest-point projection of the elastic trial state onto the yield surface, it follows that:

\[
I_{1,n+1} = T
\]

(4.283)
and
\[ s_{n+1}^{\text{Trial}} = s_{n+1}^{\text{Trial}}. \]  
\(4.284\)

The plastic deviatoric and volumetric strains are given by:
\[ \Delta e_p^{n+1} = 0 \]  
\(4.285\)

and
\[ \Delta I_{1,n+1}^p = -3 \Delta \gamma_{3,n+1} \]  
\(4.286\)

where
\[ \Delta \gamma_{3,n+1} = \frac{T - I_{1,n+1}^{\text{Trial}}}{9K}. \]  
\(4.287\)

Finally, the return map algorithm is illustrated in Fig. 4.9 for the various regions of the cap model.

![Figure 4.9 The yield surface of the cap model](image)

### 4.11.3 Consistent Elastoplastic Tangent Moduli

The consistent elastoplastic tangent moduli for the cap model have been derived by
Hofstetter et al. (1993). The interested reader is referred to their referenced paper.

4.12 Conclusions

The basic principles of plasticity theory are reviewed in this chapter. In particular, two plasticity models, the $J_2$ or von Mises model and the cap model, are discussed. The return map algorithm using the notions of elastic-plastic operator split and closest-point projection in the stress space is explained. The formulation of the return map algorithm used to integrate the constitutive equations of the $J_2$ and cap models is presented in detail. The notion of consistent or algorithmic elastoplastic tangent moduli in contrast to continuum elastoplastic tangent moduli is discussed.

The $J_2$ plasticity model enables to model realistically the inelastic behavior of metals; while the predictive ability of the cap model is very good for soil and concrete materials. These inelastic constitutive models have the advantage to predict realistically the behavior of structures near their failure region, since most structural failures occur in the nonlinear range. These inelastic models can be incorporated in the framework of the finite element reliability methods and can be used to make realistic assessments of structural safety.
CHAPTER 5  THE FINITE ELEMENT RELIABILITY METHOD

5.1 Introduction

Over the last decade, the finite element methods for structural analysis of complex systems have become a widely used and powerful tool at the service of the structural engineering community. However, a vast majority of these methods are strictly deterministic. Unfortunately, uncertainties exist in all aspects of structural engineering. The loads applied to a structure, the material properties and the geometric properties of the structural members and overall structural configuration are intrinsically random. Moreover, our knowledge of the behavior of complex systems is not complete and the models used to describe the behavior of real structures are imperfect. Hence, the inherent uncertainties in the system must be taken into account when designing new structures or when assessing existing structures for strength and serviceability limit-states. The mathematical theory of probability and random processes provides an ideal framework for performing structural reliability and safety studies.

The term probabilistic finite element method is used to describe a finite element method which takes into account the inherent uncertainties in the geometry and material properties of a structure, as well as in the applied loading. In recent years the probabilistic finite element method has emerged as a popular tool for the analysis of complex structural systems because of the realization that even low levels of uncertainty in the material properties or geometry of the structure may affect the reliability or safety of the system to a large extent.

The probabilistic finite element methods can be divided into two broad classes: (1) the methods aimed at performing a second moment analysis of the structure, in which the mean vector and covariance matrix of the response quantities are computed given the first and second statistical moments of the basic parameters, and (2) the methods aimed at analyzing the reliability of the structure, where the objective is to compute the probability of exceeding a specified limit-state of the structure under consideration.
The second moment analysis may be performed by using a Taylor series expansion of the response expression (Hisada and Nakagiri, 1985; Liu et al., 1986) or using a direct simulation method (Yamazaki, 1988). The probabilistic finite element methods of the second class comprise the finite element reliability methods which consist of merging the finite element method and the reliability method (FORM/SORM) into a general framework for the reliability assessment of structures (Der Kiureghian, 1985; Der Kiureghian and Ke, 1988; Mahadevan and Haldar, 1988; and Liu and Der Kiureghian, 1991b). Finite element reliability analysis has also been performed using simulation in conjunction with the response surface method (Veneziano et al., 1983; Favarelli, 1989). This study deals with combining nonlinear static and dynamic finite element analysis with the first-order reliability analysis.

Several issues need to be addressed before the probabilistic fine element method can be applied in conjunction with the first-order reliability method. The main ingredients of the finite element reliability method are: (1) the efficient representation of the random spatial distribution of applied loads or material parameters into a set of correlated random variables (i.e., random field discretization), (2) the development and computer implementation of a robust nonlinear model of the structural system considered, (3) the accurate and efficient computation of the structural response gradients, and (4) the formulation of the limit-state or performance function for the structural reliability or safety analysis under consideration. The above ingredients, except for ingredient (2) which has already been treated extensively in Chapter 4, will be considered successively in the following sections.

5.2 Random Field Discretization

The mathematical reliability models presented in Chapter 2 are random variable models. The basic ingredient of these reliability models is the vector of basic random variables, \( \mathbf{X} \). However, system parameters encountered in engineering applications are usually not lumped but distributed in space. The moment of inertia of a beam, the
thickness of a plate or the yield strength of its material, and the compressive strength
of the concrete of a gravity dam are examples of spatially distributed parameters in
one, two and three dimensions, respectively. Such uncertain distributed quantities must
be formally modeled as random fields.

The powerful methods of reliability analysis described in Chapter 2 cannot handle ran-
dom fields directly. Thus, in order to perform a first- or a second-order reliability anal-
ysis of a structure, the random field models must first be discretized into a set of
correlated random variables or random vector. These random variables must jointly
represent the random field accurately and efficiently. The next sections are devoted to
a review of basic random field theory and the various methods available for random
field discretization.

5.2.1 Basic Theory of Random Fields
By definition, a random field is a parametered family of random variables with the
parameter(s) belonging to an indexing set(s). In general, the indexing sets are defined
by the space (s) or time (t) coordinates. Thus, the random field at a given set of
parameter values is a random variable. A random field can also be viewed as the
ensemble of all the possible realizations or outcomes of a random function. Usually, a
singly parametered (or uni-dimensional) random field is called a random process. A
practical example of a random process is given by the variability of the yield moment
of a reinforced concrete beam over its length. In this case, the single parameter set is
the abscissa along the length of the beam. Thus, the yield moment, X(s), at the beam
cross-section of abscissa s is a single random variable defined by its probability cumu-
lative distribution function (CDF):

\[ F(x, s) = P \{ X(s) \leq x \} \]  \hspace{1cm} (5.1)

or by its probability density function (PDF):

\[ f(x, s) = \frac{dF(x, s)}{dx} \]  \hspace{1cm} (5.2)
The joint statistical behavior of the yield moments at locations $s_1$ and $s_2$, is defined by the second-order (joint) CDF:

$$F(x_1, s_1; x_2, s_2) = P[X(s_1) \leq x_1 \cap X(s_2) \leq x_2]$$  \hspace{1cm} (5.3)

or second-order (joint) PDF:

$$f(x_1, s_1; x_2, s_2) = \frac{dF(x_1, s_1; x_2, s_2)}{dx_1 dx_2}$$  \hspace{1cm} (5.4)

Similarly, the joint statistical behavior of the yield moments at locations $s_1$, $s_2$, $s_3$ is described by the third-order (joint) CDF or PDF, and so on. Notice that in general, for a complete probabilistic specification of a random process, one needs the joint PDF’s or CDF’s of all orders up to infinity. A renowned exception is the Gaussian process which is completely defined by its first- and second-order CDF or PDF’s.

The ensemble average of any function $g(\cdot)$ of the random variables $X(s_1), X(s_2), \ldots, X(s_n)$ is termed the expected value of $g(\cdot)$ and defined as

$$E[g(X(s_1), X(s_2), \ldots, X(s_n))] =$$

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots \, dx_n$$  \hspace{1cm} (5.5)

As a special case of Eq. (5.5), the $m^{\text{th}}$ moment of the random process $X(s)$ at a given location $s$ is given by

$$\alpha_m(t) = E[X^m(t)] = \int_{-\infty}^{\infty} x^m(t) f(x, t) \, dx$$  \hspace{1cm} (5.6)

Thus the first moment, more commonly called the mean function of the process $X(s)$ and sometimes denoted by $m_X(s)$ or simply $m(s)$ is defined as

$$\alpha_1(s) = E[X(s)] = m(s)$$  \hspace{1cm} (5.7)

The mean square value of the random process $X(s)$ at location $s$ corresponds to $\alpha_2(s)$. 
The central moments are also useful in describing the statistics of a random process \( X(s) \) at an arbitrary point in time. The \( m \)th central moment of \( X(t) \) at location \( s \) is given by

\[
\mu_m(s) = E\left[ (X(s) - m(s))^m \right]
\]

(5.8)

The second central moment, \( \mu_2(s) \), is the variance of \( X(s) \) at location \( s \), which is often denoted by \( \sigma_X^2(s) \).

Another important special case of Eq. (5.5) is the family of joint moments of the random process \( X(s) \) at two different locations, \( s_1 \) and \( s_2 \). The \( mn \)th joint moment of \( X(s) \) at locations \( s_1 \) and \( s_2 \) is defined by

\[
\alpha_{mn}(s_1, s_2) = E\left[ X^m(s_1) X^n(s_2) \right] = \int \int s_1^m s_2^n f(x_1, x_2) \, dx_1 \, dx_2
\]

(5.9)

A special case of Eq. (5.9) is \( \alpha_{11}(s_1, s_2) \), called the auto-correlation function and often denoted by \( \phi_{XX}(s_1, s_2) \), which is an important measure of linear dependence between \( X(s_1) \) and \( X(s_2) \) and plays a central role in random process theory. Also important in the theory of random processes are the joint central moments at locations \( s_1 \) and \( s_2 \) given by

\[
\mu_{mn}(s_1, s_2) = E\left[ (X(s_1) - m(s_1))^m (X(s_2) - m(s_2))^n \right]
\]

(5.10)

The special case \( \mu_{11}(s_1, s_2) \) is called the auto-covariance function of the random process \( X(s) \) and is often denoted by \( \kappa_{XX}(s_1, s_2) \). It can be readily shown that the autocovariance function is related to the autocovariance function and the mean function through the relation:

\[
\kappa_{XX}(s_1, s_2) = \phi_{XX}(s_1, s_2) - m(s_1) m(s_2)
\]

(5.11)

By comparing Eqs. (5.8) and (5.10), it is observed that the variance of the process \( X(s) \) is a particular case of its autocovariance function:
\[ \sigma_X^2(s) = \kappa_{XX}(s, s) \]  

(5.12)

A normalized version of the auto-covariance function of the random process \( X(s) \) is the auto-correlation coefficient function \( \rho_{XX}(s_1, s_2) \) defined as

\[ \rho_{XX}(s_1, s_2) = \frac{\kappa_{XX}(s_1, s_2)}{\sigma_X(s_1) \sigma_X(s_2)} \]  

(5.13)

which is a measure of the linear dependence between the random variables \( X(s_1) \) and \( X(s_2) \).

The above definitions have been presented in the context of a single (or uni-variate) random field. In many structural applications, it is common to have several random fields, which may be correlated to each other. In this case, the \( m^{th} \) joint moment and joint central moment of \( X(s) \) at location \( s_1 \) and \( Y(s) \) at location \( s_2 \) are defined by

\[ \alpha_{mn}^{XY}(s_1, s_2) = E\left[X^m(s_1)Y^n(s_2)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^m y^n f(x, s_1; y, s_2) \, dx \, dy \]  

(5.14)

and

\[ \mu_{mn}^{XY}(s_1, s_2) = E\left[[X(s_1) - m_X(s_1)]^m [Y(s_2) - m_Y(s_2)]^n\right], \]  

(5.15)

respectively, where \( f(x, s_1; y, s_2) \) is the second-order joint PDF between \( X(s) \) at location \( s_1 \) and \( Y(s) \) at location \( s_2 \).

An important case of Eq. (5.14) is \( \alpha_{11}^{XY}(s_1, s_2) \), called the cross-correlation function between \( X(s) \) at location \( s_1 \) and \( Y(s) \) at location \( s_2 \), and which is often denoted by \( \phi_{XY}(s_1, s_2) \). The special case \( \mu_{11}^{XY}(s_1, s_2) \) of Eq. (5.15) is called the cross-covariance function between \( X(s) \) at \( s_1 \) and \( Y(s) \) at \( s_2 \) and is often denoted by \( \kappa_{XY}(s_1, s_2) \).

A random field is said to be homogeneous (or stationary) in the strict sense if its complete probability structure (described by either the joint PDFs of all orders or the joint moments of all orders) is invariant to a shift of the parametric origin. A random
field is said to be \textit{homogeneous (or stationary) in the wide sense} (or weakly stationary) if its first and second-order statistics are invariant to a shift of the parametric origin. Hence, in the case of a weakly stationary process:

\[ f(x, s) = f(x) \]  \hspace{1cm} (5.16)

and

\[ f(x_1, s_1; x_2, s_2) = f(x_1, x_2, \nu) \]  \hspace{1cm} (5.17)

where \( \nu = x_2 - x_1 \) is the distance separation. From Eqs. (5.16) and (5.17), it follows that:

\[ m_X(s) = E[X(s)] = \eta = \text{constant} \]  \hspace{1cm} (5.18)

and

\[ \phi_{XX}(s, s + \nu) = E[X(s)X(s + \nu)] = R_{XX}(\nu) \]  \hspace{1cm} (5.19)

In other words, the mean function of a weakly stationary process is constant, while its auto-correlation function depends only on the distance separation between the two locations under consideration.

The frequency domain (for \( X(t) \)) or wave number domain (for \( X(s) \)) version of the auto-correlation function of a weakly stationary, uni-variate random process is called the \textit{power spectral density (PSD) function} and is denoted by \( \Phi_{XX}(\omega) \) for \( X(t) \) and \( \Phi_{XX}(k) \) for \( X(s) \) where \( \omega \) and \( k \) refer to the frequency (radians per second) and wave number (radians per unit length) parameters, respectively. The auto-correlation function and power spectral density function form a Fourier transform pair known as the \textit{Wiener-Khinchine relations} (Vanmarcke, 1983), which is as follows in the case of a uni-variate, uni-dimensional random field, \( X(s) \):

\[ \Phi_{XX}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\nu) e^{-i\nu k} d\nu \]  \hspace{1cm} (5.20)

\[ R_{XX}(\nu) = \int_{-\infty}^{\infty} \Phi_{XX}(k) e^{i\nu k} dk \]  \hspace{1cm} (5.21)
The PSD plays a very important role in random vibration and random field theory, since it describes the distribution of the mean square value of the process over the frequency or wave number domain as shown by Eq. (5.21) for \( \nu=0 \):

\[
\alpha_2(s) = E[X^2(s)] = R_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(k) \, dk = \text{constant} \tag{5.22}
\]

Stationary random processes can be classified according to the bandwidth of their power spectral density. A random process is wide band (or broad band) if its power spectral density has significant values over a wide range of frequencies. On the other hand, a random process is narrow band if its power spectral density has significant values only over a narrow frequency range centered around a central or predominant frequency. The autocorrelation function of a wide-band process decays rapidly. This feature is characteristic of most spatially distributed parameters encountered in engineering applications, such as distributed loads and material properties. Thus the values of the random field at closely separated points are strongly correlated random variables and their correlation drops off rapidly as the separation distance increases.

The above concepts and definitions related to a unidimensional random field, \( X(s) \), or random process, \( X(t) \), can be generalized to a multidimensional random field. In the present study, time-invariant, two-dimensional random fields, \( W(x) \) where \( x = [x_1, x_2]^T \) denotes the cartesian coordinates of the domain of interest \( \Omega \), are used to represent the uncertain spatial variability of material constitutive parameters. If the random field \( W(x_1, x_2) \) is Gaussian, then it is completely described by its mean function \( m_W(x) \), variance function \( \sigma^2_W(x) \), and auto-correlation coefficient function \( \rho_{WW}(x,x') \) defined as

\[
\rho_{WW}(x,x') = \frac{E[ (W(x) - m_W(x))(W(x') - m_W(x')) ]}{\sigma_W(x) \sigma_W(x')} \tag{5.23}
\]

Many engineering quantities needing random field modeling are non-Gaussian. A non-
Gaussian random field model that is particularly useful in the context of FORM/SORM analysis is the Nataf model (Liu and Der Kiureghian 1986; Der Kiureghian 1987) already presented in Chapter 2 in the case of multiple random variables. The Nataf random field model is a particular member of the family of translation processes which defines a non-Gaussian random field as a nonlinear function of a Gaussian field (Grigoriu 1984). In the Nataf model, the random field is defined in terms of a first-order cumulative distribution function $F_w(w(x))$ and an auto-correlation coefficient function $\rho_{WW}(x,x')$. The field is completely defined probabilistically by assuming that the transformed random field

$$Z(x) = \Phi^{-1}\{F_w(w(x))\}$$

is Gaussian with zero mean, unit variance, and auto-correlation coefficient function $\rho_{ZZ}(x,x')$. In Eq. (5.24), $\Phi^{-1}(.)$ denotes the inverse of the standard normal cumulative distribution function, $\Phi(.)$. As in Eq. (2.8) for the case of multiple random variables, the autocorrelation coefficient functions of the original field $W(x)$ and transformed field $Z(x)$, are implicitly related through (Liu and Der Kiureghian 1986)

$$\rho_{WW}(x,x') = \int \int \left( \frac{w(x) - m_w(x)}{\sigma_w(x)} \right) \left( \frac{w(x') - m_w(x')}{{\sigma_w(x')}} \right) \phi_2[z, z', \rho_{ZZ}(x, x')] dz \, dz'$$

in which $z = \Phi^{-1}\{F[w(x)]\}$, $z' = \Phi^{-1}\{F[w(x')]\}$, and $\phi_2(.,.,\rho)$ is the bivariate normal density with zero means, unit variances, and correlation coefficient $\rho$. Liu and Der Kiureghian (1986) have shown that, in general, $|\rho_{ZZ}(x,x')| \leq |\rho_{WW}(x,x')|$, and, for most cases, $\rho_{ZZ}(x,x') = \rho_{WW}(x,x')$. It is therefore more convenient, in practice, to define the random field by prescribing $\rho_{ZZ}(x,x')$ rather than $\rho_{WW}(x,x')$. However, empirical formulae relating $\rho_{WW}(x,x')$ to $\rho_{ZZ}(x,x)$ for common marginal distributions (i.e., first-order probability density functions) can be found in the preceding reference.
The random field $W(\mathbf{x})$ is called homogeneous, in the strict sense, if the joint probability density functions of all orders (i.e., $f_w(w(x_1))$, $f_{ww}(w(x_1), w(x_2))$, $f_{www}(w(x_1), w(x_2), w(x_3))$, ...) are independent of a translation (but not rotation) of the set of locations $x_1, x_2, x_3, ...$, in the parameter space. The homogeneous filed is isotropic if the joint probability density functions of all orders are also independent of a rotation of the location points $x_1, x_2, x_3, ...$, in the parameter space. A random field is called weakly homogeneous if only the first- and second-order joint probability density functions (i.e., $f_w(w(x_1))$ and $f_{ww}(w(x_1), w(x_2))$) are independent of a translation, and weakly homogeneous isotropic if these first- and second-order joint probability density functions are also independent of a rotation in the parameter space. Therefore, the auto-correlation coefficient function, of a (weakly) homogeneous isotropic field depends only on the distance separation between the points under consideration, i.e., $\rho_{WW}(\mathbf{x}, \mathbf{x}') = \rho_{WW}(|\mathbf{x}-\mathbf{x}'|)$. An important measure of the rate of fluctuation of a homogeneous isotropic random field is the correlation length, $\lambda$, defined such that

$$
\rho_{WW}(|\mathbf{x}-\mathbf{x}'| = \lambda) = e^{-1} \approx 0.368
$$

(5.26)

Typical examples of auto-correlation coefficient functions of homogeneous isotropic random fields are:

$$
\rho_{WW}(\mathbf{x}, \mathbf{x}') = e^{\frac{|\mathbf{x}-\mathbf{x}'|}{\lambda}}
$$

(5.27)

$$
\rho_{WW}(\mathbf{x}, \mathbf{x}') = e^{\frac{|\mathbf{x}-\mathbf{x}'|^2}{\lambda}}
$$

(5.28)

$$
\rho_{WW}(\mathbf{x}, \mathbf{x}') = \frac{\sin\left(\frac{2.2|\mathbf{x}-\mathbf{x}'|}{\lambda}\right)}{2.2|\mathbf{x}-\mathbf{x}'|}
$$

(5.29)

By scaling the axes of a homogeneous isotropic random field, one obtains a homoge-
neous random field with an *ellipsoidal correlation structure* defined as

\[ \rho_{WW}(x, x') = \rho_{WW} \left( \frac{a_1 \Delta x_1}{\lambda_{x_1}} \right)^2 \left( \frac{a_2 \Delta x_2}{\lambda_{x_2}} \right)^2 \]  

(5.30)

where \( \Delta x_1 \) and \( \Delta x_2 \) are the cartesian components of the separation vector between the points \( x \) and \( x' \), and \( a_1 \) and \( a_2 \) are the scaling factors. If \( a_1 = a_2 \), the correlation structure becomes isotropic. The standard form of the auto-correlation coefficient function of a *homogeneous orthotropic random field* is:

\[ \rho_{WW}(x, x') = e^{-\left( \frac{\Delta x_1}{\lambda_{x_1}} \right)^2 \left( \frac{\Delta x_2}{\lambda_{x_2}} \right)^2} \]  

(5.31)

where \( \lambda_{x_1} \) and \( \lambda_{x_2} \) are the correlation lengths in the \( x_1 \)- and \( x_2 \)-directions, respectively.

### 5.2.2 Methods of Random Field Discretization

As mentioned earlier, in order to be used in the finite element reliability method, a random field needs to be discretized into a set of correlated random variables. Therefore, a collection of locations has to be selected such that the values of the random field at these locations (i.e., random variables) together represent the random field adequately. These random variables will be correlated since they are all derived from the same parent random field which is characterized by its auto-correlation coefficient function as defined in Eq. (5.23). Although random field discretization is similar to finite element discretization, there is a significant difference since the finite element discretization is subjected to the requirement of an accurate representation of the deterministic mechanical behavior of the structure, while the random field discretization must represent accurately the probabilistic structure of a spatially varying parameter. Hence, the finite element reliability analysis of a continuum structure involves two different kinds of meshes:

(a) a finite element mesh for the deterministic mechanical analysis of the structure;  

(b) several *random field element meshes*, one for each spatially distributed random
parameter, for the reliability or stochastic analysis of the structure.

The set of random variables representing the random field may be obtained using several methods, such as the *midpoint method* (Hisada and Nakagiri 1985; Der Kiureghian and Ke 1988; Yamazaki et al. 1988), the *spatial averaging method* (Vanmarcke and Grigoriu 1983), the *interpolation (or shape function) method* (Liu et al. 1986), and the *optimum linear estimation method* (Li and Der Kiureghian 1993). An alternative approach, which does not require any random field mesh, for representing a random field in terms of random variables is the *series expansion method* (Lawrence 1987; Spanos and Ghanem, 1989).

The spatial averaging method describes the field within each element \( \Omega_e \) (of the random field mesh) in terms of the average value of the field over the element, namely

\[
W(x) = \hat{W}(x) = \frac{\int_{\Omega_e} W(x) \, d\Omega}{\int_{\Omega_e} d\Omega} ; \quad x \in \Omega_e 
\]

(5.32)

Thus, a realization of the approximating (or discretized) field, \( \hat{W}(x) \), is uniform over the elements and has discontinuities along the element boundaries. In general, due to the averaging process the variance of the spatial-average variable over the element is smaller than the local variance of the random field. The mean and correlation matrix of the vector of spatial-average random variables are given in terms of integrals of the moment functions of the parent random field (Vanmarcke 1983).

The interpolation or shape function method describes the random field within an element in terms of a set of nodal values and corresponding interpolation (shape) functions as

\[
W(x) \approx \hat{W}(x) = \sum_{i=1}^{n} N_i(x) W(x_i) , \quad x \in \Omega_e 
\]

(5.33)

where \( n \) = number of nodes of the element, \( x_i \) = coordinate vector of the \( i \)th node, and
\( N_i(x) \) is the \( i \)-th shape function as used in approximating the displacement field in finite element analysis. A realization of the approximating field, \( \hat{W}(x) \), is a continuous function, which is a clear advantage over the midpoint method (defined below) and the spatial averaging method.

The optimum linear estimation method approximates the random field within the whole domain under consideration, \( \Omega \), by a linear function of the nodal values \( W(x_i) \) in the form

\[
W(x) = \hat{W}(x) = a(x) + \sum_{i=1}^{N} b_i(x)^T W(x_i), \quad x \in \Omega
\]

where \( N \) = number of nodal points in the domain \( \Omega \); \( a(x) \) = scalar function of \( x \); and \( b(x) = [b_i(x)] = \) vector function of \( x \) with elements \( b_i(x) \); \( W = [W(x_i)] = \) vector of random variables describing the random field. The optimum functions \( a(x) \) and \( b_i(x) \) are determined by minimizing the variance of the error \( W(x) - \hat{W}(x) \), subject to the condition that \( \hat{W}(x) \) be an unbiased estimator of \( W(x) \) in the mean, i.e.,

\[
E[W(x) - \hat{W}(x)] = 0 \quad \text{for any} \quad x \in \Omega.
\]

The final results of this problem of optimal linear-estimation theory are:

\[
a(x) = m_W(x) - b_i^T(x) \mu_W
\]

\[
b(x) = \Sigma_{WW}^{-1} \Sigma_{W(x)W}
\]

in which \( \mu_W = E[W] \); \( \Sigma_{WW} = E[(W - \mu_W)(W - \mu_W)^T] \); and \( \Sigma_{W(x)W} = N \times 1 \) vector containing the covariances of \( W(x) \) with the elements of the random vector \( W \).

It can be shown (Li and Der Kiureghian 1993) that the estimate \( \hat{W}(x) \) always underestimates the variance of the actual random field \( W(x) \). Among all linear representations of \( W(x) \) in terms of the nodal random variables \( W \) that match the mean function, the preceding representation is optimal in the sense that it minimizes the error in the
variance at any given point.

The series expansion (or spectral) method makes use of the Karhunen-Loève theorem to express the field in terms of its spectral decomposition. In a normalized form, it can be expressed as:

$$W(x) = \mu_W(x) + \sigma_W(x) \sum_{i=1}^{\infty} \zeta_i \sqrt{\lambda_i} f_i(x), \quad x \in \Omega$$  \hspace{1cm} (5.36)

in which \( \zeta_i \) = independent standard variates (zero mean, unit variance, zero correlation); and \( \lambda_i, f_i(x) \) are the eigenvalues and eigenfunctions of the correlation kernel obtained from the integral equation

$$\int_{\Omega} \rho_{WW}(x, x') f_i(x') \, dx' = \lambda_i f_i(x)$$  \hspace{1cm} (5.37)

in which the eigenfunctions are normalized such that

$$\int_{\Omega} f_i(x) f_j(x) \, dx = \delta_{ij}$$  \hspace{1cm} (5.38)

where \( \delta_{ij} = \) Kronecker delta. Notice that the \( \zeta_i \)'s are normal if the actual field \( W(x) \) is Gaussian, otherwise they are non-normal. The field is thus represented by the infinite set of random variables \( \zeta_i, i = 1, 2, \ldots \). However, usually the infinite series in Eq. (5.36) is truncated and only a subset of terms with the largest eigenvalues is retained. Theoretically, this spectral method of random field representation does not require a discretization of the domain. If the exact eigenfunctions of the autocorrelation coefficient kernel can be obtained, then the series expansion method is the most efficient method of discretizing a random field in the sense that it requires the smallest number of random variables to capture the probabilistic structure of the random field within a given level of accuracy. However, in practice the integral eigenvalue problem rarely has an exact solution, and an approximation method must be used which eventually converts the integral eigenvalue problem into a matrix eigenvalue problem. In both the cases where a discrete integration rule or a Galerkin-type approximation (con-
sisting of expressing the eigenfunctions \( f_i(x) \) in terms of a set of prescribed basis functions) is used, the domain \( \Omega \) must be discretized and the approximate series expansion solution becomes equivalent to the shape function method.

The simplest of the random field discretization methods is the midpoint method which has the significant advantage of easily allowing for non-Gaussian random field representation. According to the midpoint method, the domain of a random field is discretized into a mesh of random field elements and the value of the random field over each element is represented by a single random variable, namely the value of the random field at the centroid (or any other characteristic central point) \( x_c \) of the element. That is,

\[
W(x) = \hat{W}(x) = W(x_c), \quad x \in \Omega_e
\]  

(5.39)

A realization of the approximating field \( \hat{W}(x) \) varies as a stepwise function with discontinuities along the element boundaries. The second order statistics, \( \mu_W \) and \( \Sigma_{WW} \), of the random vector \( W = [W(x_j)] \) are derived directly from the mean, variance, and autocorrelation coefficient functions of the actual field evaluated at the centroidal points \( x_{ci} \). Notice that the first-order probability density function of the discretized random field \( \hat{W}(x) \) remains the same as that of the actual random field \( W(x) \). In particular, the variance of \( \hat{W}(x) \) as defined by the midpoint rule is the same as the local or point variance of the actual process \( W(x) \).

The efficiency of the midpoint, spatial averaging, shape function, and optimum linear estimation methods described above can be increased by making use of the spectral decomposition of the covariance matrix, \( \Sigma_{WW} \), of the random vector \( W \) representing the random field \( W(x) \). Assuming that \( \Sigma_{WW} \) is nonsingular, the random vector \( W \) can be expressed in terms of its spectral decomposition

\[
W = \mu_W + \sum_{i=1}^{N} \zeta_i \sqrt{\theta_i} \phi_i
\]  

(5.40)
where \( N \) = size of \( W \) = number of random variables used to describe the random field \( W(x) \), \( \zeta_i \) = independent standard normal variates, and \( \theta_i \) and \( \phi_i \) are the eigenvalues and eigenvectors of the covariance matrix, i.e., they are the solutions of the following matrix eigenvalue problem

\[
\Sigma_{WW} \phi_i = \theta_i \phi_i
\]  

(5.41)

in which the eigenvectors are normalized such that \( \phi_i^T \phi_j = \delta_{ij} \). In the case where the actual random field \( W(x) \) is Gaussian, \( W = [W(x_i)] \), while in the case where \( W(x) \) is a non-Gaussian field defined according to the Nataf model in Eq. (5.24), \( W = [Z(x_i)] = \left[ \Phi^{-1} \{ F_W(W(x_i)) \} \right] \) where \( F_W(.) \) is the first-order cumulative distribution function of the field \( W(x) \). In each case, the random vector \( W \) is Gaussian. Retaining only a subset \( r < N \) of the terms in Eq. (5.40) which correspond to the largest eigenvalues, the number of random variables needed to describe the field \( W(x) \) can be reduced.

For all the above methods of random field discretization, \( W(x) \) is related to \( W \) through a linear transformation. Therefore, for a Gaussian random field \( W(x) \), \( W \) is also Gaussian and, hence, completely defined by its mean vector, \( \mu_W \), and covariance matrix, \( \Sigma_{WW} \). Since non-Gaussian distributions are not closed under linear transformations, it follows that, with the exception of the midpoint method, all the discretization methods described above are strictly applicable to Gaussian random fields only. For this reason and also for mathematical simplicity in deriving the covariance matrix of the random vector \( W \), the midpoint method of random field discretization has been adopted in this study for the implementation of the finite element reliability method.

When analyzing structures characterized by several cross-correlated random fields, the correlation coefficients between the random variables representing the various fields can be derived directly from the cross-correlation coefficient functions between the various fields.
5.2.3 Selection of the Random Field Mesh

The random field mesh size in stochastic finite element or finite element reliability analysis must be carefully selected from the viewpoints of accuracy, stability, and computational efficiency. Adopting a fine random field mesh results in the following important effects:

(1) The number of random variables representing the random field(s) is significantly increased, which results in higher computational cost during the reliability analysis part. Hence, one has to determine whether the increase in computational cost is justified by the improvement in the final results. Since for realistic problems of stochastic mechanics, the exact solution is usually not available, it is intuitively assumed that the accuracy of the computed solution increases with the refinement of the random field mesh.

(2) The use of a very fine random field mesh results in a set of highly correlated random variables representing the random field(s), which causes numerical instability in the probability transformation in Eq. (2.4) due to a near singularity of the correlation coefficient matrix. This difficulty should be taken into account while refining the mesh.

On the other hand, the size of the random field mesh should be selected small enough to capture, with sufficient accuracy, the uncertain spatial variability of the random field. In this regard, the size of the random field mesh is controlled by the correlation length of the field as defined in Eq. (5.26). Past investigations (Hisada and Nakagiri 1985; Der Kiureghian and Ke 1988) have shown that a mesh size of one half to one quarter of the field correlation length is appropriate to satisfy the accuracy requirement. Furthermore, a recent investigation by Liu and Liu (1993) on adaptive random field mesh refinement resulted in a mesh selection criterion which consists of adopting a coarse mesh in the area where the gradient of the limit-state function with respect to the discretized spatial variables is small, and a fine mesh in the area where this gradi-
ent is large.

The finite element and random field meshes are therefore subjected to different requirements: the finite element mesh is controlled by the gradient of the stress field, while the random field mesh is controlled by the field correlation length and the gradient of the limit-state function with respect to the spatially distributed parameter. For this reason, combined with the desire to use as few random variables as possible, Der Kiureghian and Ke (1988) have suggested using separate finite element and random field meshes. Their experience and others’ (Liu and Der Kiureghian 1991b) have shown that a satisfactory compromise is achieved by using a finite element mesh that satisfies the requirements for the stress gradient and the correlation lengths of all random fields considered, and then choosing for each random field a mesh which coincides or is coarser than the finite element mesh to ensure a well-conditioned correlation coefficient matrix. Therefore, a random field element typically consists of one or more finite element(s). Examples of random field meshes will be given in Chapter 7 in the case of a stochastic perforated plate and a stochastic concrete gravity dam.

For the sake of computational efficiency, all the parameters having an uncertain spatial distribution should not necessarily be modelled as random field and subsequently discretized into a set of random variables. Indeed, if the reliability index for a certain limit-state is insensitive to the uncertainty of a particular parameter, then it is inefficient to model this parameter as a random field. Hence, for computational efficiency, a trial or preliminary reliability analysis should be first performed considering all the uncertain parameters as single random variables, i.e., assuming no spatial variability. Only those parameters with high sensitivity indices should then be remodelled as random fields and discretized into sets of random variables.
5.3 Response Sensitivity Analysis

Beside its important role in probabilistic and reliability analysis of structures, structural response sensitivity analysis (Haftka and Adelman 1989) finds also important uses in other applications, such as structural optimization (Haftka and Gürdal 1992), design of experiments, optimum allocation of resources, and in determining the relative importance of the parameters in a system. Response sensitivities of any system with respect to a given parameter are expressed mathematically in terms of the partial derivative of the response with respect to the parameter under consideration. The analytical first- and second-order reliability methods (FORM and SORM, respectively) require the response sensitivities (or response gradient vector) to determine a new search direction in an iterative scheme for finding an optimum point, called the design point, at which the limit-state or performance function is approximated (linearized or quadratized). Accuracy and efficiency of the gradient computation are of prime importance for the successful implementation of these analytical reliability methods.

In nonlinear dynamic analysis of structures, the limit-state function is usually defined in terms of the maximum response during the entire time history for first-exursion type failure, and in terms of cumulative plastic deformations for fatigue type failures. Thus, there is no closed-form expression for the limit-state function which is available only in an algorithmic sense, i.e., from a finite element solution algorithm. Since in the case of nonlinear dynamic analysis of large complex structures, the evaluation of the limit-state function is very expensive computationally, the analytical reliability methods can be applied to these cases only if accurate and efficient algorithms are available to compute the response gradi-
ents. Such gradient computation capabilities are typically not available in existing finite element programs.

Several general methods are available for calculating the sensitivity of structural response. These methods have been the object of a comprehensive review by Haftka and Adelman (1989). The simplest technique for calculating the response gradient with respect to a set of parameters is the finite difference approximation. Several finite difference schemes are available, such as the first-order forward-difference approximation (also called Euler method), the second-order central-difference approximation, and other higher-order approximations. However, the finite difference methods are computationally expensive, since they require repeated solutions of the problem at least one time greater (for the forward-difference scheme) than the number of sensitivity parameters which is equal to the number of random variables in a reliability analysis. The finite difference method to approximate derivatives of a function has two source of errors: truncation and condition errors. The truncation error is a result of the neglected terms in the Taylor series expansion of the perturbed function and depends on the step size used in perturbing the sensitivity parameter and on the order of the finite difference scheme. The condition error is the difference between the numerical evaluation of the function to be differentiated and its exact value. A contribution to the condition error is the computer round-off error in evaluating the function at the original and perturbed values of the sensitivity parameters. Other conditional errors may come from the numerical noise stemming from lengthy or ill-conditioned numerical processes involving various tolerance parameters to control convergence of iterative schemes, typical of nonlinear finite element analysis. Unfortunately, the
finite difference methods suffer from the so-called "step-size dilemma": if a small step size is selected to reduce the truncation error, the condition error may become excessive. In some cases, there may not be any step size which yields an acceptable error (Haftka and Güralp 1992).

Perturbation methods (Nayfeh 1973) form another popular class of methods for computing the structural response sensitivities. In these methods, the governing equation of motion of the response for a perturbed value of the sensitivity parameter is expanded in a Taylor series, and after truncating the higher order terms, an equation for the corresponding perturbation in the response is obtained which is similar to the equation for the response itself. The perturbed equation can be solved at small extra computational cost by using the available solution of the response (e.g., factorization of the coefficient matrix for the system of equations). The accuracy of the response sensitivities computed according to the perturbation method depends on the size of the perturbation used and on the importance of the truncated higher order terms. In general, the perturbation methods are more efficient but less accurate than the finite difference methods.

A third approach for calculating the structural response gradient consists of directly differentiating the governing matrix equation of motion with respect to the sensitivity parameter (Tsay and Arora 1989, 1990) to obtain a matrix equation for the gradient which has the same left-hand-side as the equation for the response itself. The gradient equation can then be solved very efficiently by taking advantage of the available solution for the response. This so-called direct differentiation method (DDM) is computationally as efficient as the perturbation methods, but more accurate since it makes no approximations beside those
involved in the numerical solution of the equations for the response and the gradient. A new formulation using the direct differentiation approach, referred to as the conditional derivative method (CDM), was proposed recently by Zhang and Der Kiureghian (1993) in the context of the dynamic response of structures with the $J_2$ plasticity model. The CDM computes the partial derivative of the internal resisting force vector with respect to the sensitivity parameter in terms of the derivatives of the constitutive equations of the material for fixed values of the displacements (i.e., conditional derivatives). In the CDM, the gradient equation naturally involves the tangent stiffness matrix, which is available from the solution of the response if a Newton-type algorithm is used. Even though the equation of motion for the response is highly nonlinear, the equation for the gradient is linear once the displacement vector is known. The CDM is formulated in a manner consistent with the conventional finite element method for inelastic materials.

In the present study, the CDM is applied to the $J_2$ plasticity model and the cap model defined in Sections 4.7 and 4.11, respectively.

5.3.1 Conditional Derivative Method (CDM)

Let $x$ denote the parameter of interest, with respect to which response sensitivities have to be computed. Typical parameters of interest include material constitutive parameters (e.g., Young’s modulus, initial yield stress, hardening modulus), geometry parameters or loading parameters. The governing equation of motion for an inelastic dynamic system takes the form:

$$
M(x) \ddot{u}(x, t) + C(x) u(x, t) + R(u(x, t), x) = F(x, t)
$$

(5.42)

in which $t$ denotes the time, $u(x, t)$ denotes the displacement vector, $M(x)$ is the mass
matrix, \( C(x) \) is the damping matrix, \( R(u(x, t), x) \) denotes the history dependent internal resisting force vector, and \( F(x, t) \) denotes the external load vector. In the above equation, a superposed dot indicates differentiation with respect to time.

The response sensitivity or gradient equation is obtained by directly differentiating the above equation of motion with respect to \( x \), the sensitivity parameter. By defining \( v = \frac{\partial u}{\partial x} \) and reversing the order of differentiations with respect to \( t \) and \( x \), one finds

\[
M \ddot{v} + C \dot{v} + K(u) v = \frac{\partial F}{\partial x} - \frac{\partial M}{\partial x} \ddot{u} - \frac{\partial C}{\partial x} \dot{u} - \frac{\partial R}{\partial x} \bigg|_{u}
\]

in which \( K(u) \) represents the current tangent stiffness matrix. The above reversal in the order of differentiations with respect to \( t \) and \( x \) is possible provided that \( u(x, t) \) is continuous in \( x \) and \( t \). The quantity \( \frac{\partial R}{\partial x} \bigg|_{u} \) represents the derivative of the internal resisting force vector with respect to the sensitivity parameter conditioned on the displacement vector \( u \) being fixed. It is history dependent since the material is inelastic and it depends on the particular constitutive law of the material. It is important to note that once the nodal displacements \( u(t) \) are known, the gradient equation, Eq. (5.43), is linear in \( v \), which results in a very efficient computational algorithm for the gradient computation.

In a displacement-based finite element formulation, the resisting force vector takes the form

\[
R(u(x, t), x) = \sum_{e=1}^{\text{Nel}} \int_{\Omega_e} B^T \sigma(\epsilon(x, t), x) \, d\Omega_e
\]
where \( e \) denotes the element number and \( \text{Nel} \) is the total number of elements, \( B \) is the strain-displacement matrix, and \( \sigma (\epsilon (x, t), x) \) is the stress vector which is history dependent. Differentiating Eq. (5.44) with respect to the parameter \( x \) with \( u \) fixed gives

\[
\frac{\partial}{\partial x} \mathbf{R} \left( \mathbf{u} (x, t), x \right) \bigg|_u = \sum_{e=1}^{\text{Nel}} \int_{\Omega_e} \mathbf{B}^T \frac{\partial}{\partial x} \sigma (\epsilon (x, t), x) \bigg|_\epsilon \, d\Omega_e
\]  

(5.45)

where \( \frac{\partial}{\partial x} \sigma (\epsilon (x, t), x) \bigg|_\epsilon \) denotes the derivative of the stress vector with respect to \( x \) with the strains \( \epsilon \) fixed. The permutation of the differentiation and integration operators applied in deriving Eq. (5.45) assumes that the boundary of the element is independent of the sensitivity parameter \( x \).

5.3.2 Incremental Solution of the Equation of Motion

In any conventional finite element algorithm, a step-by-step numerical integration method is used to solve Eq. (5.42) at discrete time steps \( t = t_n \) for \( n = 0, 1, 2, \ldots \). The class of one-step implicit integration methods approximates the acceleration and velocity at \( t_{n+1} \) by

\[
\ddot{u}_{n+1} = a_0 \, \mathbf{u}_{n+1} - a_2 \, \mathbf{u}_n - a_4 \, \dot{u}_n - a_6 \, \ddot{u}_n
\]  

(5.46)

\[
\dot{u}_{n+1} = a_1 \, \mathbf{u}_{n+1} - a_3 \, \mathbf{u}_n - a_5 \, \dot{u}_n - a_7 \, \ddot{u}_n
\]  

(5.47)

where \( a_0 - a_7 \) are the integration coefficients, which depend on the specific time stepping algorithm used. The Newmark-beta method presented in Sections 3.4.1 and 3.4.2 is a subclass of the class of integration methods defined in Eqs. (5.46) and (5.47). The equation of motion in Eq. (5.42) is rewritten at time \( t_{n+1} \) in the following residual form:

\[
\Psi_{n+1} = M \ddot{u}_{n+1} + C \dot{u}_{n+1} + R_{n+1} (\mathbf{u}_{n+1}) - \mathbf{F}_{n+1} = 0
\]  

(5.48)
Substituting the finite difference approximations in Eqs. (5.46) and (5.47) into Eq. (5.48) results in

\[ \Psi_{n+1}(u_{n+1}) = a_0 M u_{n+1} + a_1 C u_{n+1} + R(u_{n+1}) - \tilde{F}_{n+1} = 0 \]  \hspace{1cm} (5.49)

where

\[ \tilde{F}_{n+1} = F_{n+1} + M \left( a_2 u_n + a_4 \dot{u}_n + a_6 \ddot{u}_n \right) + C \left( a_3 u_n + a_5 \dot{u}_n + a_7 \ddot{u}_n \right) \]  \hspace{1cm} (5.50)

is the effective loading at step n+1. The problem consists of solving Eq. (5.49) which is a set of nonlinear algebraic equations. The most commonly used method to solve this kind of problems is the Newton-Raphson iterative scheme which is summarized below.

Using the Newton-Raphson iterative scheme, a series of successive approximations, \( u^i_{n+1}, \ i = 1, 2, ..., \) will be obtained which finally converges to the desired result \( u_{n+1} \). The subscript "i" is the iteration counter starting from

\[ u^1_{n+1} = u_n \]  \hspace{1cm} (5.51)

At each iteration \( i+1 \), the residual function \( \Psi_{n+1}(u_{n+1}) \) is linearized around the known displacements at the previous iteration, \( u^i_{n+1} \), as

\[ \Psi(u_{n+1}) = \Psi(u^i_{n+1}) + \frac{\partial}{\partial u^i_{n+1}} \Psi(u^i_{n+1}) \delta u^i_n = 0 \]  \hspace{1cm} (5.52)

Using Eq. (5.49), it is found that the jacobian matrix \( \frac{\partial}{\partial u^i_{n+1}} \Psi(u^i_{n+1}) \), takes the form

\[ \frac{\partial}{\partial u^i_{n+1}} \Psi(u^i_{n+1}) = a_0 M + a_1 C + \frac{\partial}{\partial u^i_{n+1}} R(u^i_{n+1}) \]  \hspace{1cm} (5.53)

By definition, the gradient of the internal resisting force vector \( R(u^i_{n+1}) \) with respect to the current displacements \( u^i_{n+1} \) is the tangent stiffness matrix of the structure at
displacements \( u_{n+1}^i \):

\[
\frac{\partial}{\partial u_{n+1}^i} \mathbf{R} \left( u_{n+1}^i \right) = K_T \left( u_{n+1}^i \right)
\]  \( (5.54) \)

The jacobian matrix in Eq. (5.53) is also called the effective dynamic tangent stiffness matrix:

\[
\tilde{K}_T \left( u_{n+1}^i \right) = a_0 M + a_1 C + K_T \left( u_{n+1}^i \right)
\]  \( (5.55) \)

The current increment of displacements, \( \delta u_n^i \), is then obtained as

\[
\delta u_n^i = -\tilde{K}_T^{-1} \left( u_{n+1}^i \right) \Psi \left( u_{n+1}^i \right)
\]  \( (5.56) \)

A series of successive iterations for the time increment between \( t_n \) and \( t_{n+1} \) gives:

\[
u_{n+1}^{i+1} = u_n + \Delta u_n^i = u_{n+1}^i + \delta u_n^i
\]  \( (5.57) \)

where

\[
\Delta u_n^i = \sum_{j=1}^{i} \delta u_n^j
\]  \( (5.58) \)

The above iteration process is illustrated in Fig. 5.1. After convergence is reached, the displacement vector at \( t_{n+1} \), \( u_{n+1} \), is obtained. The velocity vector at \( t_{n+1} \), \( u_{n+1} \), is then obtained from Eq. (5.47). Finally the acceleration vector at \( t_{n+1} \), \( \dot{u}_{n+1} \), is derived by solving the equation of motion at time \( t_{n+1} \):

\[
\dot{u}_{n+1} = M^{-1} \left[ F_{n+1} - C \dot{u}_{n+1} - R \left( u_{n+1} \right) \right]
\]  \( (5.59) \)

rather than from Eq. (5.46) thus preserving the dynamic equilibrium condition.
5.3.3 Numerical Integration of the Gradient Equation

The differential equation governing the response sensitivity, Eq. (5.43), is solved at time $t_{n+1}$ after convergence of the solution of the equation of motion is reached at $t_{n+1}$. The gradient equation in Eq. (5.43) is rewritten at time $t_{n+1}$ as:

$$
M \ddot{\psi}_{n+1} + C \dot{\psi}_{n+1} + K_T(u_{n+1}) \psi_{n+1} = \frac{\partial F_{n+1}}{\partial x} - \frac{\partial M_{n+1}}{\partial u} \dot{u}_{n+1} \quad (5.60)
$$

$$
= - \frac{\partial C}{\partial x} \dot{u}_{n+1} + \frac{\partial R(u_{n+1})}{\partial u} \big|_{u_{n+1}}.
$$

The same finite difference scheme as in Eqs. (5.46) and (5.47) is used to integrate the gradient equation, namely

$$
\psi_{n+1} = a_0 \psi_{n+1} - a_2 \psi_n - a_4 \dot{\psi}_n - a_6 \ddot{\psi}_n
$$

$$
\dot{\psi}_{n+1} = a_1 \dot{\psi}_{n+1} - a_3 \dot{\psi}_n - a_5 \ddot{\psi}_n - a_7 \dddot{\psi}_n
$$
Substituting the above finite difference approximations into Eq. (5.60) results in:

\[ \tilde{K}_T(u_{n+1}) v_{n+1} = \frac{\partial \tilde{F}_{n+1}}{\partial x} - \left[ a_0 \frac{\partial M}{\partial x} u_{n+1} + a_1 \frac{\partial C}{\partial x} u_{n+1} + \frac{\partial R(u_{n+1})}{\partial x} \right]_{u_{n+1}} \]  

(5.61)

where the effective dynamic tangent stiffness matrix \( \tilde{K}_T(u_{n+1}) \) is identical to that of Eq. (5.55) at the solution point \( u_{n+1} \) and

\[ \frac{\partial \tilde{F}_{n+1}}{\partial x} = \frac{\partial F_{n+1}}{\partial x} + M \left[ a_2 v_n + a_4 \dot{v}_n + a_6 \ddot{v}_n \right] + C \left[ a_3 v_n + a_5 \dot{v}_n + a_7 \ddot{v}_n \right] 
+ \frac{\partial M}{\partial x} \left[ a_2 u_n + a_4 \dot{u}_n + a_6 \ddot{u}_n \right] + \frac{\partial C}{\partial x} \left[ a_3 u_n + a_5 \dot{u}_n + a_7 \ddot{u}_n \right] \]  

(5.62)

Since the matrix on the left-hand-side of Eq. (5.61) is identical to that of Eq. (5.56) expressed as

\[ \tilde{K}_T \delta u_n = -\Psi_{n+1} \left( u_{n+1} \right), \]  

(5.63)

only the vectors on the right-hand-side of Eq. (5.61) need to be computed. Since the factorization of the effective tangent dynamic stiffness matrix, \( \tilde{K}_T \), is already available at the converged time step \( t_{n+1} \), solution of Eq. (5.63) is computationally very inexpensive. The vector \( \frac{\partial F}{\partial x} \) and matrices \( \frac{\partial M}{\partial x} \) and \( \frac{\partial C}{\partial x} \) can be computed easily at the element level and then assembled at the structural level using the classical direct stiffness assembly procedure. Computation of the vector \( \frac{\partial R(u_{n+1})}{\partial x} \) however, is much more involved, since it is strongly history-dependent. First at the element level, according to Eq. (5.45), this vector is computed from the stress gradient \( \frac{\partial \sigma}{\partial x} \) which depends on the constitutive law of the material. Then the vector \( \frac{\partial R(u_{n+1})}{\partial x} \) at the structural level is computed by assembling its counterpart at the element level using the direct stiffness assembly procedure. The analytical expres-
sions to compute the stress gradient will be described in the following sections for the
J_2 and cap plasticity models.

5.3.4 Conditional Derivative of the Stress Tensor
In this section, the formulations required to compute the partial derivatives of the stress
tensor with respect to the sensitivity parameter x at a fixed strain are developed.
By virtue of the definition of the deviatoric and hydrostatic stress tensors, we have at time
t = t_{n+1}:

\[ s_{n+1}^{\text{Trial}} = 2G\left(e_{n+1}^p - e_n^p\right) \]  
(5.64)

\[ I_{1,n+1}^{\text{Trial}} = 3K\left(I_{1,n+1} - I_{1,n}^p\right) \]  
(5.65)

\[ s_{n+1} = s_{n+1}^{\text{Trial}} - 2G\Delta e_{n+1}^p \]  
(5.66)

\[ I_{1,n+1} = 3p_{n+1} = 3Ke_v e_n^p = I_{1,n+1}^{\text{Trial}} - 3K\Delta I_{1,n+1}^p \]  
(5.67)

\[ \sigma_{n+1} = s_{n+1} + \frac{1}{3} I_{1,n+1} \]  
(5.68)

Differentiating both sides of the above equation with respect to x gives

\[ \frac{\partial \sigma_{n+1}}{\partial x} = \frac{\partial s_{n+1}}{\partial x} + \frac{1}{3} \frac{\partial I_{1,n+1}}{\partial x} \]  
(5.69)

where, from the return map algorithm,

\[ \frac{\partial s_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} - 2G \frac{\partial}{\partial x} \Delta e_{n+1}^p - 2G \frac{\partial}{\partial x} \left(\Delta e_{n+1}^p\right) \]  
(5.70)

\[ \frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} - 3K \frac{\partial}{\partial x} \left(I_{1,n+1}^p - 3K \frac{\partial}{\partial x} \left(I_{1,n+1}^p\right) \right) \]  
(5.71)

The quantities \( \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \) and \( \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} \) can be evaluated from
\[
\frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} = 2 \frac{\partial G}{\partial x} \left( e_{n+1} - e_n^p \right) + 2G \left( \frac{\partial e_{n+1}}{\partial x} - \frac{\partial e_n^p}{\partial x} \right) \tag{5.72}
\]

and
\[
\frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} = 3 \frac{\partial K}{\partial x} \left( I_{1,n+1} - I_{1,n}^p \right) + 3K \left( \frac{\partial I_{1,n+1}^p}{\partial x} - \frac{\partial I_{1,n}^p}{\partial x} \right). \tag{5.73}
\]

The conditional derivative with strains fixed can be evaluated by substituting \( \frac{\partial e_n}{\partial x} = 0 \) and \( \frac{\partial I_{1,n}}{\partial x} = 0 \) in the equations above. Thus,
\[
\frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \bigg|_{e_{n+1}} = 2 \frac{\partial G}{\partial x} \left( e_{n+1} - e_n^p \right) - 2G \frac{\partial e_n^p}{\partial x} \tag{5.74}
\]
\[
\frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} \bigg|_{e_{n+1}} = 3 \frac{\partial K}{\partial x} \left( I_{1,n+1} - I_{1,n}^p \right) - 3K \frac{\partial I_{1,n}^p}{\partial x} \tag{5.75}
\]

In the above, \( \frac{\partial}{\partial x} \Delta e_{n+1}^p \) and \( \frac{\partial}{\partial x} \Delta I_{1,n+1}^p \) represent the derivative of the incremental plastic deviatoric strain tensor and the derivative of the first invariant of the incremental plastic strain tensor with respect to the sensitivity parameter \( x \). These incremental plastic, deviatoric and volumetric strain quantities depend on the past history of the response and the plasticity model assumed. In the next two sections, the gradient of these incremental plastic strains, \( \frac{\partial}{\partial x} \Delta e_{n+1}^p \) and \( \frac{\partial}{\partial x} \Delta I_{1,n+1}^p \), will be derived for both the \( J_2 \) and the cap plasticity models.

### 5.3.5 Application of the CDM to the \( J_2 \) Plasticity Model

For the \( J_2 \) plasticity model, the spherical part of the stress and strain tensors are assumed to remain purely elastic. Hence, the inelastic component of the first invariant of the stress and strain tensors is zero. Thus,
\[ I_{1,n+1}^p = I_{1,n+1}^p = \Delta I_{1,n+1}^p = 0 \]  

(5.76)

and Eqs. (5.56)-(5.59) become

\[ \frac{\partial s_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} - 2 \frac{\partial G}{\partial x} \Delta \epsilon_{n+1}^p - 2G \frac{\partial}{\partial x} \left( \Delta \epsilon_{n+1}^p \right) \]  

(5.77)

\[ \frac{\partial}{\partial x}(I_{1,n+1}) = \frac{\partial}{\partial x}(I_{1,n+1}^{\text{Trial}}) \]  

(5.78)

and

\[ \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} = 2 \frac{\partial G}{\partial x} \left( \epsilon_{n+1}^p - \epsilon_n^p \right) + 2G \left( \frac{\partial \epsilon_{n+1}^p}{\partial x} - \frac{\partial \epsilon_n^p}{\partial x} \right) \]  

(5.79)

\[ \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} = 3 \frac{\partial K}{\partial x} I_{1,n+1} + 3K \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} - 3 \frac{\partial K}{\partial x} \Delta I_{1,n+1}^p - 3K \frac{\partial}{\partial x} \left( \Delta I_{1,n+1}^p \right) \]  

(5.80)

The deviatoric component of the plastic strain tensor at time \( t = t_{n+1} \) is obtained from the discrete form of the flow rule in Eq. (4.55):

\[ \epsilon_{n+1}^p = \epsilon_n^p + \lambda_{n+1} \mathbf{n}_{n+1} \]  

(5.81)

where \( \lambda_{n+1} \) is the discrete plastic consistency parameter, and according to Eq (4.74):

\[ \mathbf{n}_{n+1} = \frac{\Sigma_{n+1}^{\text{Trial}}}{\left\| \Sigma_{n+1}^{\text{Trial}} \right\|} \]  

(5.82)

in which \( \Sigma_{n+1}^{\text{Trial}} \) is the trial back stress tensor given by

\[ \Sigma_{n+1}^{\text{Trial}} = s_{n+1}^{\text{Trial}} - \alpha_n. \]  

(5.83)

Therefore, given the incremental plastic deviatoric strain tensor,

\[ \Delta \epsilon_{n+1}^p = \lambda_{n+1} \mathbf{n}_{n+1}, \]  

(5.84)

its derivative with respect to the sensitivity parameter \( x \) can be expressed as
\[
\frac{\partial}{\partial x} \Delta e_{n+1}^p = \frac{\partial}{\partial x} \lambda_{n+1} + \lambda_{n+1} \frac{\partial}{\partial x} (n_{n+1})
\] (5.85)

where, from Eq. (5.82), the derivative of the unit normal tensor \( n_{n+1} \) with respect to \( x \) is

\[
\frac{\partial n_{n+1}}{\partial x} = \frac{\partial \Sigma_{n+1}^{\text{Trial}}}{\partial x} \frac{\| \Sigma_{n+1}^{\text{Trial}} \| - \frac{\partial}{\partial x} \| \Sigma_{n+1}^{\text{Trial}} \| \Sigma_{n+1}^{\text{Trial}}}{\| \Sigma_{n+1}^{\text{Trial}} \|^2}.
\] (5.86)

It can also be found that

\[
\frac{\partial}{\partial x} \| \Sigma_{n+1}^{\text{Trial}} \| = \frac{\Sigma_{n+1}^{\text{Trial}}}{\| \Sigma_{n+1}^{\text{Trial}} \|} \cdot \frac{\partial \Sigma_{n+1}^{\text{Trial}}}{\partial x}
\] (5.87)

in which the doubly contracted tensor product (\( \cdot \)) was defined previously in Eq. (4.10). In Section 4.8.2, it was shown that for the \( J_2 \) model with linear hardening, the discrete consistency parameter, \( \lambda_{n+1} \), can be obtained directly from:

\[
\lambda_{n+1} = \frac{\| \Sigma_{n+1}^{\text{Trial}} \| - Y_n}{2 G_1}
\] (5.88)

in which

\[
G_1 = 2G + \frac{2}{3} (H_{iso} + H_{kin})
\] (5.89)

Differentiating Eq. (5.88) with respect to the parameter \( x \) yields

\[
\frac{\partial \lambda_{n+1}}{\partial x} = \frac{2G_1 \left( \frac{\partial}{\partial x} \| \Sigma_{n+1}^{\text{Trial}} \| - \frac{\partial Y_n}{\partial x} \right) - 2 \frac{\partial G_1}{\partial x} \left( \| \Sigma_{n+1}^{\text{Trial}} \| - Y_n \right)}{(2G_1)^2}
\] (5.90)

Differentiating Eq. (5.83) with respect to the parameter \( x \) gives

\[
\frac{\partial \Sigma_{n+1}^{\text{Trial}}}{\partial x} = \frac{\partial \Sigma_{n+1}^{\text{Trial}}}{\partial x} - \frac{\partial \alpha_n}{\partial x}.
\] (5.91)
The derivative of the size of the yield surface, $Y_{n+1}$, with respect to the parameter $x$ is obtained by differentiating Eq. (4.61) with respect to $x$:

$$\frac{\partial Y_{n+1}}{\partial x} = \frac{\partial Y_n}{\partial x} + \frac{2}{3} H_{iso} \frac{\partial \lambda_{n+1}}{\partial x} + \frac{2}{3} \frac{\partial H_{iso}}{\partial x} \lambda_{n+1}.\quad (5.92)$$

in which $\frac{\partial \lambda_{n+1}}{\partial x}$ is obtained from Eq. (5.90).

The derivatives of the deviatoric strain tensor, $e_{n+1}$, and the volumetric strain, $I_{1,n+1}$, with respect to the parameter $x$ are simply obtained from the result of the gradient equation, Eq. (5.60), at time $t_{n+1}$. Once $v_{n+1} = \frac{\partial u_{n+1}}{\partial x}$ is known at the global structural level, the gradient of the total strain tensor at the element level is determined from:

$$\frac{\partial e_{n+1}^{(e)}}{\partial x} = \frac{\partial}{\partial x} \left[ B \, u_{n+1}^{(e)} \right] = B \left[ \frac{\partial u_{n+1}^{(e)}}{\partial x} \right] = B v_{n+1}^{(e)}\quad (5.93)$$

where the superscript $(e)$ denotes element "$e$" and $B$ is the strain-displacement matrix which depends on the local coordinates of the element. From the decomposition of the total strain tensor into the deviatoric and volumetric components, i.e.,

$$e_{n+1}^{(e)} = e_{n+1}^{(e)} + \frac{1}{3} I_{1,n+1}^{(e)} I$$

in which

$$I_{1,n+1}^{(e)} = \text{trace} \left[ e_{n+1}^{(e)} \right],\quad (5.95)$$

it follows that

$$\frac{\partial e_{n+1}^{(e)}}{\partial x} = \frac{\partial e_{n+1}^{(e)}}{\partial x} + \frac{1}{3} \frac{\partial I_{1,n+1}^{(e)}}{\partial x} I$$

and
\[
\frac{\partial I_{1,n+1}^{(e)}}{\partial x} = \text{trace} \left[ \frac{\partial \mathbf{e}_{n+1}^{(e)}}{\partial x} \right],
\]
(5.97)

\[
\frac{\partial \mathbf{e}_{n+1}^{(e)}}{\partial x} = \frac{\partial \mathbf{e}_{n+1}^{(e)}}{\partial x} - \frac{1}{3} \frac{\partial I_{1,n+1}^{(e)}}{\partial x} \mathbf{I}
\]
(5.98)

As mentioned earlier, the gradient of any quantity with \( u_{n+1} \) or \( e_{n+1} \) fixed (conditional derivative) can be obtained simply by setting the terms \( \frac{\partial I_{1,n+1}^{(e)}}{\partial x} \) and \( \frac{\partial \mathbf{e}_{n+1}^{(e)}}{\partial x} \) to be zero in the corresponding unconditional gradient expressions. Note that under elastic deformations, (i.e., when \( F_{n+1} < 0 \)), the discrete consistency parameter \( \lambda_{n+1} \) and its gradient are both zero and some of the above equations can be simplified. Also for \( F_{n+1} < 0 \), obviously \( \frac{\partial n_{n+1}}{\partial x} = 0 \) since no plastic deformation occurs. For the \( J_2 \) plasticity model, the parameter \( x \) is assumed to be one of the constants \( K, G, H_{\text{kin}}, H_{\text{iso}}, \) or \( \sigma_0 \). Thus, the gradient terms \( \frac{\partial K}{\partial x}, \frac{\partial G}{\partial x}, \frac{\partial H_{\text{kin}}}{\partial x}, \frac{\partial H_{\text{iso}}}{\partial x}, \) and \( \frac{\partial \sigma_0}{\partial x} \) are either one or zero, depending on the choice of \( x \).

5.3.6 Application of the CDM to the Cap Model

(1) Loading in failure envelope mode:

Differentiating Eqs. (4.185) and (4.186) with respect to \( x \) yields:

\[
\frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} + 9 \frac{\partial K}{\partial x} \Delta \gamma_{1,n+1} \left( \frac{dF_e}{dI_1} \right)_{n+1} + 9K \frac{\partial}{\partial x} (\Delta \gamma_{1,n+1}) \left( \frac{dF_e}{dI_1} \right)_{n+1} +
\]

\[
9K \Delta \gamma_{1,n+1} \left( \frac{\partial}{\partial x} \left[ \frac{dF_e}{dI_1} \right]_{I_{1,n+1}} \right) + \frac{\partial}{\partial x} \left[ \frac{dF_e}{dI_1} (I_{1,n+1}) \right] \frac{dI_{1,n+1}}{dx} - \frac{\partial I_{1,n+1}}{\partial x} = 0
\]
(5.99)

and
\[
\frac{\partial}{\partial x} \gamma_{1, n+1} = \frac{2G \left( \frac{\partial}{\partial x} \left[ S_{n+1} \right] - \frac{\partial}{\partial x} F_e \left( I_{1, n+1} \right) \right] + \frac{\partial F_e}{\partial I_1} \left( \frac{\partial I_{1, n+1}}{\partial x} \right)}{(2G)^2} 
\]

\[
= \frac{-2 \frac{\partial G}{\partial x} \left( S_{n+1} \right) - F_e \left( I_{1, n+1} \right)}{(2G)^2}
\]

in which the gradients \( \frac{\partial I_{1, n+1}}{\partial x} \) and \( \frac{\partial G}{\partial x} \) are given in Eqs. (5.80) and (5.87).

For the cap model, the sensitivity parameter \( x \) is assumed to be one of the constants \( K, G, \alpha, \beta, \lambda, \theta, R, T, W \) or \( D \). Thus, the gradient terms \( \frac{\partial K}{\partial x}, \frac{\partial G}{\partial x}, \frac{\partial \alpha}{\partial x}, \frac{\partial \beta}{\partial x}, \frac{\partial \lambda}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial R}{\partial x}, \frac{\partial T}{\partial x}, \frac{\partial W}{\partial x} \) and \( \frac{\partial D}{\partial x} \) are either one or zero, depending on the choice of \( x \).

Substituting Eq. (5.100) into Eq. (5.99) results into a linear scalar equation in \( \frac{\partial I_{1, n+1}}{\partial x} \).

Once \( \frac{\partial I_{1, n+1}}{\partial x} \) is known, \( \frac{\partial}{\partial x} \Delta \gamma_{1, n+1} \) can be easily determined by evaluating Eq. (5.100).

The gradient of the deviatoric stress tensor can be obtained from

\[
\frac{\partial S_{n+1}}{\partial x} = \frac{\partial S_{n+1}^{\text{Trial}}}{\partial x} - 2\frac{\partial G}{\partial x} \Delta e_{n+1}^p - 2G \frac{\partial}{\partial x} \left( \Delta e_{n+1}^p \right) 
\]

where

\[
\frac{\partial}{\partial x} \left( \Delta e_{n+1}^p \right) = \frac{\partial}{\partial x} \left( \Delta \gamma_{1, n+1} \right) n_{n+1} + \Delta \gamma_{1, n+1} \frac{\partial n_{n+1}}{\partial x}
\]

and
\[ n_{n+1} = \frac{s_{n+1}}{\| s_{n+1} \|} = \frac{s_{n+1}}{\| s_{n+1} \|} \]  

\[ (5.103) \]

As in Eq. (5.86), \( \frac{\partial n_{n+1}}{\partial x} \) takes the form

\[ \frac{\partial n_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \frac{|| s_{n+1}^{\text{Trial}} || - s_{n+1}^{\text{Trial}} \frac{\partial || s_{n+1}^{\text{Trial}} ||}{\partial x}}{|| s_{n+1}^{\text{Trial}} ||^2} \]

\[ (5.104) \]

in which

\[ \frac{\partial}{\partial x} \left( \frac{s_{n+1}^{\text{Trial}}}{|| s_{n+1}^{\text{Trial}} ||} \right) = \frac{s_{n+1}^{\text{Trial}}}{|| s_{n+1}^{\text{Trial}} ||} \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \]

\[ (5.105) \]

The gradient of the trial deviatoric stress, \( \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \), is given by Eq. (5.72).

\[ \frac{\partial}{\partial x} \left( \Delta I_{1, n+1}^{p} \right) = -3 \left( \frac{\partial}{\partial x} (\Delta \gamma_{1, n+1}) \frac{d}{d I_{1}} F_{e} (I_{1}) + \right. \]

\[ \left. \Delta \gamma_{1, n+1} \left\{ \frac{\partial}{\partial x} \left[ \frac{d}{d I_{1}} F_{e} (I_{1}) \right] \left|_{I_{1}=T} \right] + \frac{\partial}{\partial I_{1}} \left[ \frac{d}{d I_{1}} F_{e} (I_{1}) \right] \left|_{x} \right] \frac{\partial I_{1}}{\partial x} \right\} \]

\[ (5.106) \]

It is worth recalling that immediately following convergence of the response at time \( t_{n+1} \), when computing the gradient of the internal resisting force vector with \( u_{n+1} \) fixed, \( \frac{\partial}{\partial x} R (u_{n+1}) \bigg|_{u_{n+1}} \) in Eq. (5.60), the conditional gradients \( \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \bigg|_{\epsilon_{n+1}} \) and \( \frac{\partial I_{1, n+1}}{\partial x} \bigg|_{\epsilon_{n+1}} \) are required. They are obtained simply by substituting

\[ \frac{\partial \epsilon_{n+1}}{\partial x} \bigg|_{\epsilon_{n+1}} = \frac{\partial I_{1, n+1}}{\partial x} \bigg|_{\epsilon_{n+1}} = 0 \]

in the above gradient equations. On the other hand, after the gradient equation at the
structural level, Eq. (5.60), is solved for $\mathbf{v}_{n+1} = \frac{\partial \mathbf{u}_{n+1}}{\partial x}$, the unconditional gradients $\frac{\partial s_{n+1}}{\partial x}$ and $\frac{\partial I_{1,n+1}}{\partial x}$ can be obtained by using the deviatoric and volumetric strain gradients, $\frac{\partial e_{n+1}}{\partial x}$ and $\frac{\partial I_{1,n+1}}{\partial x}$. These unconditional gradients will be needed to compute the conditional gradients at the next time step.

(2) *Loading in the tensile corner region:*

Differentiating Eq. (4.192) with respect to the sensitivity parameter $x$ yields

$$\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial T}{\partial x}$$

(5.107)

The gradient of the deviatoric stress vector is obtained from Eq. (5.101) in which

$$\frac{\partial}{\partial x} \left( \Delta \mathbf{e}^p_{n+1} \right) = \frac{\partial}{\partial x} (\Delta \mathbf{\gamma}_{I,n+1}) \mathbf{n}_{n+1} + \Delta \mathbf{\gamma}_{I,n+1} \frac{\partial \mathbf{n}_{n+1}}{\partial x}$$

(5.108)

The gradient $\frac{\partial}{\partial x} (\Delta \mathbf{\gamma}_{I,n+1})$ is obtained by differentiating Eq. (4.194) with respect to $x$:

$$\frac{\partial}{\partial x} (\Delta \mathbf{\gamma}_{I,n+1}) = \frac{2G \left( \frac{\partial}{\partial x} \| \mathbf{s}_{n+1} \| - \frac{\partial}{\partial x} \mathbf{F}_e (I_1 = T) \right) - 2 \frac{\partial}{\partial x} \left[ \| \mathbf{s}_{n+1} \| - \mathbf{F}_e (T) \right] }{(2G)^2}$$

(5.109)

where

$$\frac{\partial}{\partial x} \mathbf{F}_e (I_1 = T) = \frac{\partial \mathbf{F}_e}{\partial x} \bigg|_{l_1} + \frac{\partial \mathbf{F}_e}{\partial l_1} \frac{\partial l_1}{\partial x} =$$

$$\frac{\partial \mathbf{F}_e}{\partial \alpha} \bigg|_{l_1 = T} \frac{\partial \alpha}{\partial x} + \frac{\partial \mathbf{F}_e}{\partial \lambda} \frac{\partial \lambda}{\partial x} + \frac{\partial \mathbf{F}_e}{\partial \beta} \frac{\partial \beta}{\partial x} + \frac{\partial \mathbf{F}_e}{\partial \theta} \frac{\partial \theta}{\partial x} + \left( \frac{\partial \mathbf{F}_e}{\partial l_1} \right) \bigg|_{l_1 = T}$$

(5.110)

Finally, the gradient of the increment of volumetric plastic strain is derived by differentiating Eq. (4.196) with respect to $x$:
\[
\frac{\partial}{\partial x} \left( \Delta I_{1,n+1}^p \right) = -3 \left( \frac{\partial}{\partial x} \left( \Delta \gamma_{1,n+1} \right) \frac{d}{dI_1} F_e(T) \right) + \\
\Delta \gamma_{1,n+1} \left\{ \frac{\partial}{\partial x} \left[ \frac{d}{dI_1} F_e(T) \right] \bigg|_{x_{I_1}} + \frac{\partial}{\partial I_1} \left[ F_e(T) \right] \bigg|_{x_{I_1}} \frac{\partial I_1}{\partial x} \right\} + \frac{\partial (\Delta \gamma_{3,n+1})}{\partial x}
\]

\text{(5.111)}

(3) \text{Loading in the cap mode:}

Differentiating Eqs. (4.201) and (4.202) with respect to the parameter \( x \) gives

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial I_{1,n+1}}{\partial x}^{\text{Trial}} - 3 \frac{\partial K}{\partial x} H(\kappa_{n+1}) - 3 \frac{\partial K}{\partial x} H(\kappa_{n+1}) \]

\text{(5.112)}

and

\[
\frac{\partial}{\partial x} \Delta \gamma_{2,n+1} = \frac{3 \left( I_{1,n+1} - \kappa_{n+1} \right) \frac{\partial}{\partial x} \left[ R^2 H(\kappa_{n+1}) F_e(\kappa_{n+1}) \right]}{\left[ 3 \left( I_{1,n+1} - \kappa_{n+1} \right) \right]^2} + \frac{3 R^2 H(\kappa_{n+1}) F_e(\kappa_{n+1}) \left( \frac{\partial I_{1,n+1}}{\partial x} - \frac{\partial \kappa_{n+1}}{\partial x} \right)}{\left[ 3 \left( I_{1,n+1} - \kappa_{n+1} \right) \right]^2}
\]

\text{(5.113)}

in which

\[
H(\kappa_{n+1}) = W \left( e^{-D X(\kappa_n)} - e^{-D X(\kappa_{n+1})} \right)
\]

\text{(5.114)}

\[
\frac{\partial}{\partial x} H(\kappa_{n+1}) = \frac{\partial W}{\partial x} \left( e^{-D X(\kappa_n)} - e^{-D X(\kappa_{n+1})} \right) + \\
W \left[ -X(\kappa_n) e^{-D X(\kappa_n)} + X(\kappa_{n+1}) e^{-D X(\kappa_{n+1})} \right] \frac{\partial D}{\partial x} - \\
De^{-D X(\kappa_n)} \frac{\partial}{\partial x} X(\kappa_n) + De^{-D X(\kappa_{n+1})} \frac{\partial}{\partial x} X(\kappa_{n+1})
\]

\text{(5.115)}

The gradients \( \frac{\partial}{\partial x} X(\kappa_n) \) and \( X(\kappa_{n+1}) \) can be obtained from Eq. (4.170) as:
\[
\frac{\partial}{\partial x} X (\kappa_{n+1}) = \frac{\partial \kappa_{n+1}}{\partial x} + \frac{\partial R}{\partial x} F_e (\kappa_{n+1}) + R \frac{\partial}{\partial x} F_e (\kappa_{n+1})
\] (5.116)

in which, from Eq. (4.165),
\[
\frac{\partial}{\partial x} F_e (\kappa_{n+1}) = \frac{\partial}{\partial x} F_e (I_{1,n+1}) \bigg|_{I_{1,n+1} = \kappa_{n+1}} + \left[ \frac{\partial}{\partial x} F_e (I_{1,n+1} = \kappa_{n+1}) \right] \frac{\partial I_{1,n+1}}{\partial x}
\] (5.117)

The gradients \( \frac{\partial}{\partial x} F_e (\kappa_{n+1}) \) in Eq. (5.113) is obtained by differentiating Eq. (4.200) with respect to \( x \).

By substituting Eqs. (5.112), (5.113) and (5.117) into the expression for \( \frac{\partial}{\partial x} F_e (\kappa_{n+1}) \), a single nonlinear scalar equation is obtained, which can be solved for \( \frac{\partial \kappa_{n+1}}{\partial x} \) using the Newton-Raphson iteration scheme. Once \( \frac{\partial \kappa_{n+1}}{\partial x} \) is known, the gradients \( \frac{\partial I_{1,n+1}}{\partial x} \) and \( \frac{\partial}{\partial x} \Delta \gamma_{2,n+1} \) can be obtained by back-substitution. Then the gradient of the deviatoric and spherical stress tensors can be computed by differentiating the following relations with respect to \( x \)

\[
s_{n+1} = 2G \left[ e_{n+1} - \left( e^p_{n+1} + \Delta e^p_{n+1} \right) \right]
\]

\[
I_{1,n+1} = 3K \left[ I_{1,n+1} - \left( I_{1,n+1} + \Delta I^p_{1,n+1} \right) \right]
\] (5.118)

and

\[
\Delta e^p_{n+1} = \Delta \gamma_{2,n+1} \frac{s_{n+1}}{F_c (\| s_{n+1} \|, I_{1,n+1}, \kappa_{n+1})}
\]

\[
\Delta I^p_{1,n+1} = \Delta \gamma_{2,n+1} \frac{3 (I_{1,n+1} - \kappa_{n+1})}{R^2 (F_c (\| s_{n+1} \|, I_{1,n+1}, \kappa_{n+1}))}
\] (5.119)

In the case where \( I_{1,n+1} = \kappa_{n+1} \), the gradient \( \frac{\partial}{\partial x} \Delta \gamma_{2,n+1} \) in Eq. (5.113) must be substi-
tuted by the following expression obtained by differentiating Eq. (4.204) with respect to \( x \):

\[
\frac{\partial}{\partial x} (\Delta \gamma_{2,n+1}) = \frac{2G \left( \frac{\partial}{\partial x} \left\| \mathcal{S}_{n+1} \right\| - \frac{\partial}{\partial x} F_e (\kappa_{n+1}) \right) - 2\frac{\partial G}{\partial x} \left[ \left\| \mathcal{S}_{n+1} \right\| - F_e (\kappa_{n+1}) \right]}{(2G)^2}
\]

(5.120)

(4) Loading in the compressive corner region:

From Eq. (4.212), the gradient of the first invariant of the total stress tensor is given by:

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial \kappa_n}{\partial x},
\]

(5.121)

while the gradient of the deviatoric stress tensor is given as in Eq. (5.101). The gradient of the incremental plastic deviatoric strain tensor is obtained by differentiating Eq. (4.211) with respect to the parameter \( x \):

\[
\frac{\partial}{\partial x} \left( \Delta \mathbf{e}_n^{p,n+1} \right) = \left( \frac{\partial}{\partial x} (\Delta \gamma_{1,n+1}) + \frac{\partial}{\partial x} (\Delta \gamma_{2,n+1}) \right) \mathbf{n}_{n+1} + \left( \Delta \gamma_{1,n+1} + \Delta \gamma_{2,n+1} \right) \frac{\partial \mathbf{n}_{n+1}}{\partial x}
\]

(5.122)

in which \( \frac{\partial \mathbf{n}_{n+1}}{\partial x} \) is as in Eq. (5.104) and the gradients of the discrete consistency parameters \( \Delta \gamma_{1,n+1} \) and \( \Delta \gamma_{2,n+1} \) are given by differentiating Eqs. (4.208) and (4.209) with respect to \( x \) as:
\[
\frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) = \frac{9K \frac{d}{dI_{1,n+1}} F_e(\kappa_n) \left[ \frac{\partial \kappa_n}{\partial x} + \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} \right]}{\left( 9K \frac{d}{dI_{1,n+1}} F_e(\kappa_n) \right)^2} - \\
\left[ \kappa_n - I_{1,n+1}^{\text{Trial}} \right] \frac{9}{\left( 9K \frac{d}{dI_{1,n+1}} F_e(\kappa_n) \right)^2} \left[ \frac{\partial K}{\partial x} \frac{d}{dI_{1,n+1}} F_e(\kappa_n) + 9K \frac{\partial}{\partial x} \left( \frac{d}{dI_{1,n+1}} F_e(\kappa_n) \right) \right]_{\text{I}_{1,n+1}} + (5.123)
\]

\[
\frac{\partial}{\partial x}(\Delta \gamma_{2,n+1}) = \frac{2G \left( \frac{\partial}{\partial x} \left| s_{n+1}^{\text{Trial}} \right| - \frac{\partial}{\partial x} F_e(\kappa_n) \right)_{\text{I}_n} + \frac{\partial}{\partial I_{1}} F_e(\kappa_n) \frac{\partial I_{1,n+1}}{\partial x} }{(2G)^2} - (5.124)
\]

The partial derivatives in the right-hand-side of the above equations are readily available.

(5) Loading in the tension cut-off region:

Differentiating Eqs. (4.214), (4.216), and (4.217) with respect to the parameter x, we find:

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial T}{\partial x} \quad (5.125)
\]

\[
\frac{\partial}{\partial x}(\Delta e_{n+1}^p) = 0 \quad (5.126)
\]
\[
\frac{\partial}{\partial x} \Delta \Gamma_{1,n+1}^p = -3 \frac{\partial}{\partial x} (\Delta \gamma_{3,n+1})
\]

in which, from Eq. (4.218),

\[
\frac{\partial}{\partial x} (\Delta \gamma_{3,n+1}) = \frac{9K}{(9K)^2} \frac{\partial}{\partial x} \left( T - I_{1,n+1}^{\text{Trial}} \right) - 9 \frac{\partial K}{\partial x} \left( T - I_{1,n+1}^{\text{Trial}} \right)
\]

(5.128)

The gradient of the total stress tensor is then given by

\[
\frac{\partial \sigma_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} + \frac{1}{3} \frac{\partial I_{1,n+1}}{\partial x} \mathbf{I}
\]

(5.129)

5.3.7 Summary of the CDM

The above completes the set of equations required to compute the gradient of the internal resisting force vector with the displacements fixed, \( \frac{\partial}{\partial x} \mathbf{R}(\mathbf{u}_{n+1}) \bigg|_{u_{n+1}} \). Once this gradient is known, the gradient equation, Eq. (5.60), can be integrated over one time step to give the gradient of the inelastic displacement response, \( \mathbf{v}_{n+1} = \frac{\partial \mathbf{u}_{n+1}}{\partial x} \).

To summarize, the following procedure is used to compute the response sensitivity of dynamic inelastic systems:

1. Solve the incremental equations of motion, Eq. (5.49), in conjunction with the discrete constitutive relations of the assumed plasticity model by means of the return map algorithm, and compute quantities such as \( I_{1,n+1}, e_{n+1}, I_{1,n+1}^p, e_{n+1}^p, e_{n+1}^p, \kappa_{n+1}, s_{n+1}, I_{1,n+1}, \sigma_{n+1} \).
(2) At each integration point, compute the derivative of the stress vector with respect to the sensitivity parameter \( x \) with the strains fixed at step \( n+1 \), i.e., \( \frac{\partial \sigma_{n+1}}{\partial x} \bigg|_{\varepsilon_{n+1}} \). All the conditional derivatives are obtained simply by setting the terms \( \frac{\partial I_{1,n+1}}{\partial x} \) and \( \frac{\partial \varepsilon_{n+1}}{\partial x} \) to zero in the unconditional gradient equations.

(3) Compute (assemble) the partial derivative of the internal resisting force vector with the displacements fixed at step \( n+1 \) according to Eq. (5.45).

(4) Compute the right-hand-side vector of the gradient equation, Eq. (5.60).

(5) Solve the gradient equation, Eq. (5.60), for \( \mathbf{v}_{n+1} = \frac{\partial \mathbf{u}_{n+1}}{\partial x} \) using the factorized form of the effective dynamic stiffness matrix, \( \tilde{\mathbf{K}}_T (\mathbf{u}_{n+1}) \), available from step (1).

(6) Compute the “unconditional” gradient of the strain and its spherical and deviatoric components from the gradient of the displacement vector using the finite element strain-displacement relation:

\[
\frac{\partial \varepsilon_{n+1}}{\partial x} = \mathbf{B} \frac{\partial (\mathbf{u}_{n+1})}{\partial x} = \mathbf{B} \mathbf{v}_{n+1} \quad (5.130)
\]

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial \varepsilon_{11,n+1}}{\partial x} + \frac{\partial \varepsilon_{22,n+1}}{\partial x} + \frac{\partial \varepsilon_{33,n+1}}{\partial x} \quad (5.131)
\]

\[
\frac{\partial \varepsilon_{n+1}}{\partial x} = \frac{\partial \varepsilon_{n+1}}{\partial x} - \frac{1}{3} \frac{\partial (\mathbf{I}_{1,n+1})}{\partial x} \mathbf{I} \quad (5.132)
\]

(7) Using the results of step 6, update all the “conditional” derivatives at step \( n+1 \) to the “unconditional” derivatives at step \( n+1 \) such as \( \frac{\partial \varepsilon_{n+1}^p}{\partial x} \), \( \frac{\partial I_{1,n+1}^p}{\partial x} \), \( \frac{\partial \varepsilon_{n+1}}{\partial x} \), \( \frac{\partial I_{1,n+1}}{\partial x} \), \( \frac{\partial \varepsilon_{n+1}^p}{\partial x} \) for use in the
next time step.

The formulations presented here can easily be incorporated into any existing finite element code. The return map algorithm for the bilinear isoparametric plane strain element and subroutines needed to perform an analytical response sensitivity analysis have been implemented in FEAP for both the $J_2$ and the cap plasticity models.

5.4 Definition of Limit-State or Performance Function

As discussed in Chapter 2, the definition of a performance function or limit state-function is required which expresses analytically the failure condition of the structure under consideration. The term failure is used in a general sense. It might denote physical failure of the structure or an element thereof (e.g., failure or collapse criteria) or it may denote the exceedence of a serviceability criterion.

Definitions of structure limit-states can be classified into two categories, namely (1) first-excursion type limit-states, and (2) cumulative-damage type limit-states. In reality, structural failures usually correspond to a combination of the two aforementioned limit-states. In the case of a first-excursion type limit-state, the limit-state function takes the following form:

$$ g(x) = r_0(x) - \max_{0 \leq t \leq T(x)} [r(x, t)] $$

(5.133)

in which $r(x,t)$ represents the time history of a critical response quantity and $r_0(x)$ denotes the corresponding safe threshold. The critical response quantity can be either a single or combination of kinematic quantities such as nodal displacement and strain component, or a single or combination of stress or integrated stress quantities such as major/minor principal stress, base shear, and base overturning moment. Not that $r_0$, $r$, and $T$, the duration of loading, are expressed as functions of the basic random variables $x$. It is worth noting that the limit-state surface corresponding to Eq. (5.1) may not be continuously differentiable. Indeed the function $\max_{0 \leq t \leq T(x)} [r(x, t)]$ in general is
not a continuously differentiable function of \( x \). In other words, the time \( t_{\text{max}}(x) \) at which \( r(x,t) \) reaches its maximum value may make a finite jump for an infinitesimal change in \( x \). In such a case, the maximum shifts to another peak. Der Kiureghian and Zhang (1991) and Der Kiureghian (1992) found that because of this effect, the limit-state surface for the first-excursion problem usually exhibits multiple design points which require a system reliability approach to solve a component reliability problem. This feature will also be illustrated in the application examples considered in this study and presented in Chapter 7.

In the case of a cumulative-damage type limit-state, the limit-state function is expressed as:

\[
g(x) = D_0(x) - \int_0^{T(x)} \dot{D}(x,t) \, dt \tag{5.134}
\]

in which \( \dot{D}(x,t) \) and \( D_0(x) \) denote the damage rate and the tolerable damage threshold, respectively. Because of the integration involved in Eq. (5.2), the corresponding limit-state surface is smooth and continuously differentiable, which warrants a simple FORM or SORM analysis.
CHAPTER 6 IMPLEMENTATION IN FEAP

6.1 Introduction

For the successful computation of the analytical response gradients for inelastic structures responding statically or dynamically, the conditional derivative method presented in the previous chapter must be implemented in a nonlinear finite element program. This chapter discusses the details of such an implementation in the general purpose, research oriented finite element analysis program FEAP and of the link between the reliability analysis code CALREL and FEAP. The new macro-commands added to FEAP to enhance its capabilities for analytical sensitivity analysis are explained.

Memory requirements could become a major problem arising while performing the finite element reliability analysis of complex structural systems. In this chapter, efficient means of utilizing the memory space are described.

6.2 Link Between CALREL and FEAP

A schematic representation of the link between CALREL and FEAP is shown in Fig. 6.1. As discussed in Chapter 2, the reliability analysis of any system involves the search for the design point which is the failure point closest to the origin in the standard normal space. Several algorithms exist in the literature to perform this search, such as the gradient projection method, the HL-RF algorithm and the modified HL-RF algorithm (Liu and Der Kiureghian, 1991). Most of these procedures are iterative algorithms which require at each iteration the value of the limit-state function and its derivatives with respect to the basic system parameters modelled as random variables. These gradients can be obtained through a deterministic finite element algorithm. Hence, in the quest for the design point, the reliability analysis program CALREL calls the finite element program FEAP several times, iteratively. The purpose of each call is to compute the value of the limit-state function and its gradient for a specific realization of the set of basic random variables, in order to proceed to the next itera-
It should be recognized that while some of the system parameters in the reliability analysis are deterministic, others are random in nature. However, each run of FEAP is a deterministic finite element analysis for a specific set of realizations of the basic random variables. Hence, provision has been made in FEAP to distinguish between the deterministic parameters of the system and the uncertain parameters modelled as random variables.

Any random variable in the FEAP input file is represented as an integer, whereas a deterministic parameter is represented by a real number. When FEAP encounters an integer while reading the input file, it substitutes it with the current realization of the random variable corresponding to that integer. Hence, while numbering the random variables, the user must ensure compatibility between the CALREL input file and the FEAP input file.

A typical CALREL input file for performing a finite element reliability analysis is:

```
... stat 1 2 E 1,1,1000.,100.,0.,0. A 2,1,1.,0.1,0.,0. ...
```

The corresponding input file for FEAP is:

```
... mate 1,2,.1 ........
```

In the above, the Young's modulus E, and the cross-section area A of a truss element are treated as the first and second random variables, respectively.
Figure 6.1 Flow Chart Diagram of the FEAP-CALREL Merging Strategy

CALREL

Pass the vector of random variables for the current iteration

USER.F

If any parameter is an integer, then replace it by the appropriate random variable

FEAP (Nonlinear Finite Element Analysis Program)

Evaluate the limit-state function using an incremental/iterative solution technique
Store displacement vector and factorized effective dynamic stiffness matrix

Loop over each basic random variable

Compute gradients
\[ \frac{\partial g}{\partial x_i} \]

Loop over each finite element

Assemble the right-hand side of the gradient equation

Solve the response sensitivity equation
6.3 Definition of the Random Field Mesh in FEAP

The random field mesh in FEAP is defined by the creation of a boolean array called the random field mesh connectivity matrix. It is similar to the element connectivity matrix in a conventional finite element procedure. This two-dimensional boolean array is created by the user in the subroutine user.f of CALREL; the two arguments of the array are the element number and the random variable number. An entry in the connectivity matrix, conn(i, j), is one if the random variable j belongs to the element i and zero otherwise. At the element level, while performing the response sensitivity analysis for the random variable j in the element i, the value of conn(i, j) is checked. The sensitivity analysis is performed only if the corresponding boolean entry is one otherwise it is skipped. In this fashion, different random field meshes can be created for different parameters with uncertain spatial distribution.

6.4 New Macro-Commands in FEAP

Several new macro-commands have been introduced in FEAP to enhance its capabilities. These macro commands are described in the following sections.

6.4.1 Input for Ground Motion Excitation (mac3)

Standard finite element codes usually do not provide directly for dynamic response analysis to ground motion excitation. Hence, if the matrix equation of dynamic equilibrium is written as

\[ M \ddot{u} (t) + C \dot{u} (t) + K u (t) = P(t), \quad (6.135) \]

then \( P(t) \) can be an arbitrary time-varying loading function. However, it is usually not possible to directly specify the input ground motion excitation, \( \ddot{x}_g (t) \), and assemble the effective earthquake force vector, \( P(t) = -ML\ddot{x}_g (t) \), where \( L \) is an influence coefficient vector which represents the nodal static displacements resulting from a unit support displacement and which depends on the type of support motion (horizontal, vertical or rotational). Using this equivalent earthquake load vector, the time varying
vector of nodal displacements relative to the ground can be solved for.

In the enhanced version of FEAP, a ground motion acceleration time history is specified with the macro-command \textit{prop}. For example, the FEAP input file for the dynamic analysis of a structural system which is excited by a ground motion varying linearly from time zero to one second is specified as:

\begin{verbatim}
macro
mac3
prop
dt,.,01
beta
loop, time, 100
loop, newton, 10
tang., 1
next, newton
time
next, time
end
2
0.0 0.0
.01 .01
.........
1.00 1.00
0.0 0.0
stop
end
\end{verbatim}

The macro command \textit{mac3} in the second line of the input file indicates that it is a
ground excited system. Hence, FEAP will interpret the specified load history as the
ground acceleration, compute the corresponding influence coefficient vector \( \mathbf{L} \), and
compute the equivalent earthquake load vector.

A complete description of the other FEAP macro-commands is available in Zienkiewicz and Taylor (1989, 1991).

In order to improve the computational efficiency of the numerical integration scheme,
the equivalent earthquake load vector is computed only at the element level and added
to the negative sum of the element inertia, damping and resisting force vectors. This
strategy avoids creation of large matrices at the global structural level. In other words,
the response to a ground excited system is analyzed by computing the following residual
force vector at the element level:

\[
P^{(e)}(t) = -M^{(e)}L^{(e)}\dot{x}_g(t) - \left[ M^{(e)}\ddot{u}(t) + C^{(e)}\dot{u}(t) + R^{(e)}(t) \right]. \tag{6.136}
\]

If the ground motion input is specified in the prop command, then it is available in
the parameter dm of the element subroutine. The core of FEAP then assembles the
residual element force vectors into the residual structural force vector.

6.4.2 Double Updating of the State Variables During the Same Time Step (mac2)
As explained earlier in Chapter 5, in order to compute the response sensitivities, derivatives of the state variables at the element level must be computed twice during the
same time step. First, before integrating the gradient equation over the time step, the
conditional gradients of the state variables are computed with the strains fixed. Sec-
ond, after integrating the gradient equation over the time step, the unconditional gradi-
ents of the state variables must be computed for their use in the next time step, since
the state variables as well as their gradients are strongly history dependent for inelastic
systems. However, in order to compute the unconditional derivatives, some of the his-
tory variables need to be updated at the same time step using the quantities computed
in the conditional derivative analysis.
FEAP offers the macro command *time* to update the state or history variables. However, it also updates the time step at the same time. Hence a new-macro command, *mac2*, was incorporated in FEAP which allows the user to update the state variables without updating the time step.

**6.4.3 Response Sensitivity Analysis (mac1)**

The conditional derivative method (CDM) used to compute the response sensitivities is described in detail in Chapter 5. The CDM can be invoked in FEAP by means of the macro command *mac1*. This macro command causes FEAP to compute the conditional derivatives at the element level, and to form the right-hand-side of the gradient equation. Then the response gradient equation is solved using the factored effective dynamic stiffness matrix (available from the converged response calculation at the same time step) to yield the response sensitivities, \( v_{n+1} = \frac{\partial u^n}{\partial x} \). The macro command *mac1* must always be followed by the macro command *mac2* to update the conditional derivatives at the current time step to the unconditional derivatives, which are required to compute the conditional derivatives at the next time step.

The FEAP input file for performing a response sensitivity analysis for the example problem in Section 6.4.1 would be:

```
....
macro
prop
beta
dt,.01
loop, time, 100
loop, newton, 10
tang,, 1
next, newton
```
6.5 Efficient Utilization of the Memory Space

Several authors (Mahadevan and Haldar 1989) have reported that the major constraint in the implementation of the stochastic finite element method is the limitation imposed by the availability of the memory space. However, the solution strategy shown in Fig. 6.1 results in a very efficient utilization of the memory space available, since the derivatives of the limit state function with respect to the basic random variables are computed sequentially. Moreover, matrices like \( \frac{\partial M}{\partial X_i} \), \( \frac{\partial C}{\partial X_i} \) and \( \frac{\partial K}{\partial X_i} \) are never computed at the global structural level, but are determined at the element level and are directly assembled into the global vector of the right-hand-side of the gradient equation. Therefore, although the size of the stiffness, damping, and mass matrix can be very large for actual structural systems, large amounts of memory space are not occupied by the numerous partial derivatives of these matrices. Further, for solving the response gradient equation, use is made of the factored effective dynamic stiffness matrix available from the analysis of the response at the same time step. It should be emphasized that
the gradient governing equation is a linear matrix equation even though the equations governing the response may be highly nonlinear and inelastic.

Due to the foregoing advantages, the algorithm for the response gradient computation used in CALREL-FEAP is highly efficient, which enables the user to apply the finite element reliability methods to large and realistic structural systems.

6.6 Conclusions

In this chapter, the details of the implementation of FEAP-CALREL are presented. New macro-commands introduced in FEAP to enhance its capabilities are described. Samples of typical input files invoking these macro-commands are also shown.

Finally, the issue of efficient utilization of the memory space available is discussed for the implementation of the CDM used in this study.
CHAPTER 7  APPLICATION EXAMPLES

7.1 Introduction

This chapter provides application examples for the response gradient computation algorithms and the finite element reliability methods described in the previous chapters. The examples considered consist of an elastoplastic single-degree-of-freedom system and a ten member elastoplastic truss structure subjected to earthquake loading, a linear elastic gravity dam under hydrostatic load, an elastoplastic perforated strip with the $J_2$ constitutive model subjected to pseudo-static cyclic loading, and a single plane strain element with the cap plasticity model under ground shaking. The case of a dam subjected to earthquake loading is also presented for both the linear elasticity and cap plasticity models.

The problems associated with the reliability analysis of dynamic systems are also discussed. The complicated nature of the limit-state surface of dynamic systems in general is illustrated using some simple elastic and inelastic dynamic systems.

7.2 Elastoplastic Single-Degree-of-Freedom System

The ground excited single-degree-of-freedom system considered in this example is shown in Fig. 7.1 with all its defining parameters.

\[
A = 1 \text{ in}^2 \\
L = 20 \text{ in} \\
E = 1233.7 \text{ ksi} \\
\rho = 0.1 \text{ k-sec}^2/\text{in}^4 \\
\sigma_y = 61.84 \text{ ksi} \\
H_{\text{kin}} = 500 \text{ ksi} \\
H_{\text{iso}} = 0
\]

\[
g(x) = 5.6 - \max_{t} |u(x, t)|
\]

Figure 7.1  SDOF system: elasto-plastic truss element
The ground excitation consists of the first fifteen seconds of the S00E (North-South) component of the 1940 Imperial Valley earthquake recorded at the El Centro station and displayed in Fig. 7.2.

![Graph of ground excitation](image)

**Figure 7.2** Imperial Valley earthquake, May 18, 1940, El Centro Site, Component S00E (N-S)

In order to verify the implementation of the elastoplastic truss element and the FEAP macro-command to input the earthquake ground excitation, the response time history of the SDOF truss element is compared to that obtained from a specialized and widely tested in-house computer program for the dynamic response of inelastic SDOF systems. The time step used in integrating the equation of motion and gradient equations is $\Delta t = 0.02$ sec. The two response time histories agree very closely as shown in Fig. 7.3. The small differences existing between the two response time-histories are due to the fact that the specialized SDOF program determines exactly (to any pre-specified tolerance) the corner points of the hysteretic loops, while the FEAP implementation of the inelastic truss element does not, see Fig. 7.4. The hysteretic behavior (sequence of
yielding and elastic unloading) of the SDOF truss system is represented by the stress-strain response history shown in Fig. 7.4.

To verify the implementation of the Conditional Derivative Method (CDM) for the inelastic truss element in FEAP, the displacement response sensitivity of the truss element obtained via CDM is compared to that obtained through the finite difference method. The sensitivities of the displacement response, \( u(t) \), with respect to the material mass density, \( \rho \), the yield stress, \( \sigma_y \), and the Young's modulus, \( E \), of the element are shown in Figs. 7.5, 7.6, and 7.7, respectively. Each of these sensitivity response time histories is scaled by the parameter itself (\( \rho \), \( \sigma_y \), or \( E \)) and can be interpreted as a change in the displacement response time history, \( u(t) \), per percentage change in the parameter in question. The sensitivity results in Figs. 7.5, 7.6, and 7.7 indicate that as the parameter increment is decreased, the finite difference results converge to the analytical result obtained using CDM, thus verifying the accuracy of the implementation of the gradient computation algorithm in FEAP.
Figure 7.4 Stress-strain response history of inelastic truss element

Figure 7.5 Displacement response sensitivity with respect to the material mass density, $\rho$
Figure 7.6 Displacement response sensitivity with respect to the yield stress, $\sigma_y$

Figure 7.7 Displacement response sensitivity with respect to the Young's modulus, $E$
A reliability analysis of the SDOF inelastic truss element is also carried out using CALREL-FEAP by assuming that the Young’s modulus, $E$, the material mass density, $\rho$, and the yield stress, $\sigma_y$, are statistically independent random variables. The type and distribution parameters of these random variables are summarized in Table 7.1. The limit-state function is given in Fig. 7.1. According to this limit-state function, the system “fails” if the absolute value of the displacement response, $u(t)$, crosses the threshold of 5.6 in.

**Table 7.1 Basic random variables of the SDOF inelastic truss system**

<table>
<thead>
<tr>
<th>Random variable</th>
<th>Distribution</th>
<th>Mean [ksi]</th>
<th>Coefficient of variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus, $E$</td>
<td>lognormal</td>
<td>30000.0 [ksi]</td>
<td>10%</td>
</tr>
<tr>
<td>Material mass density, $\rho$</td>
<td>lognormal</td>
<td>$7.61 \times 10^{-5}$ [k-sec$^2$/in$^4$]</td>
<td>10%</td>
</tr>
<tr>
<td>Yield stress, $\sigma_y$</td>
<td>lognormal</td>
<td>12.0 [ksi]</td>
<td>10%</td>
</tr>
</tbody>
</table>

By running CALREL-FEAP several times using different initial conditions for the optimization algorithm to find the design point, two design points are obtained corresponding to two most likely “failure” modes. The two design points with their corresponding reliability index and first-order modal probability of failure are reported in Table 7.2. The displacement response time histories corresponding to these two design points are displayed in Fig. 7.8. It is observed that the first design point, $x_1^*$, corresponds to a stiffer system than the second design point, $x_2^*$. According to the above reliability analysis, the displacement response time history corresponding to $x_1^*$ represents the most likely “failure” mode (with $p_{F1} = 0.1377$), while that corresponding to $x_2^*$ represents the second most likely “failure” mode (with $p_{F1} = 0.006877$).
Table 7.2 Design points for the SDOF inelastic truss system

<table>
<thead>
<tr>
<th></th>
<th>(x_1^*)</th>
<th>(x_2^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_1) = 1.0905</td>
<td>(\beta_2 = 2.4636)</td>
<td></td>
</tr>
<tr>
<td>(p_{F1} = 0.1377)</td>
<td>(p_{F1} = 0.006877)</td>
<td></td>
</tr>
<tr>
<td>(E_1^* = 1318) [ksi]</td>
<td>(E_2^* = 1043) [ksi]</td>
<td></td>
</tr>
<tr>
<td>(\rho_1^* = 0.0917) [k-sec^2/in^4]</td>
<td>(\rho_2^* = 0.1195) [k-sec^2/in^4]</td>
<td></td>
</tr>
<tr>
<td>(\sigma_{y,1}^* = 62.00) [ksi]</td>
<td>(\sigma_{y,2}^* = 60.12) [ksi]</td>
<td></td>
</tr>
</tbody>
</table>

Figure 7.8 Most probable “failure” modes of the SDOF inelastic truss system
7.3 Inelastic Ten Member Truss

The ten member inelastic truss under consideration and its defining parameters are shown in Fig. 7.9, while Fig. 7.10 displays the dynamic concentrated load applied at

For all members:

\[ A = 3.0 \text{ cm}^2 \]
\[ E = 30,000 \text{ kN/cm}^2 \]
\[ \rho = 0.761 \times 10^{-4} \text{ kN/cm}^3 \]
\[ \sigma_y = 12 \text{ kN/cm}^2 \]
\[ H_{\text{kin}} = 300 \text{ kN/cm}^2 \]
\[ H_{\text{iso}} = 0. \]

Figure 7.9 Ten member inelastic truss under consideration

Figure 7.10 Concentrated dynamic load applied at nodes 2 and 4 of the ten member truss

nodes 2 and 4. The time step used in integrating the equations of motion and gradient
equations is $\Delta t = 0.002$ sec. The vertical displacement response time history at node 2, $v_2(t)$, is presented in Fig. 7.11.

![Graph of $v_2(t)$ vs. Time (sec)]

Figure 7.11 Vertical displacement time history at node 2 of the ten-member truss

To verify the implementation of the CDM in FEAP, the response sensitivities to the material parameters obtained using the CDM are compared to those computed through the finite difference method. Figures 7.12, 7.13 and 7.14 show the sensitivity of the vertical displacement time history, $v_2(t)$, with respect to the Young's elastic modulus, $E$, the yield stress, $\sigma_y$, and the material mass density, $\rho$, respectively. It can be seen that the finite difference results converge asymptotically to the CDM result as the parameter increments are decreased, thus validating the implementation of the gradient computation scheme in FEAP.

Finally, a reliability analysis of the ten member truss subjected to the deterministic dynamic load in Fig. 7.10 is carried out using CALREL-FEAP by assuming that the yield stresses of the different bars of the truss are lognormal random variables all correlated with 10% of correlation coefficient. These basic random variables share the same mean ($\mu_{\sigma_y} = 12$ kN/cm$^2$) and standard deviation ($\sigma_{\sigma_y} = 1.2$ kN/cm$^2$). The
other variables are considered deterministic and are as defined in Fig. 7.9. The truss is assumed to "fail" if the absolute value of the vertical displacement response at node 2 exceeds 0.7 cm. Therefore the limit-state function can be written as: the value of the vertical displacement response at node 2 exceeds 0.7 cm. Therefore the limit-state function can be written as:

\[ g(x) = 0.7 - \max_t \left| v_2(x, t) \right| \]

By repeating the reliability analysis several times using different initial conditions for the optimization algorithm to find the design point, three different design points are obtained, \( x_1^* \), \( x_2^* \), and \( x_3^* \), corresponding to three different "failure" modes. The results of the reliability analysis including the standardized sensitivities with respect to the means and standard deviations of the basic random variables (see Eq. (2.28)) are reported in Tables 7.3 to 7.5 for the first, second, and third design points, respectively. The vertical displacement response time histories corresponding to the three design points are shown in Fig. 7.15.
Figure 7.13 Displacement response sensitivity with respect to the material mass density, $\rho$

Figure 7.14 Displacement response sensitivity with respect to the yield stress, $\sigma_y$
From Table 7.3, it can be seen that the reliability against the first failure mode is much more sensitive to the statistics (mean and standard deviation) of the yield stress of member 6 than to those of member 10. To get a physical insight into the reasons behind these differences in sensitivity, the stress-strain histories of the two members are plotted in Figs. 7.16 and 7.17. From these two figures, it appears that member 6 undergoes more inelastic (hysteretic) behavior than member 10, and hence its yield stress is relatively more important than that of member 10 to the reliability of the truss.
Table 7.3 Reliability results corresponding to the first design point

<table>
<thead>
<tr>
<th>Member</th>
<th>$\sigma_y^*$ [kN/cm$^2$]</th>
<th>$\frac{\partial \beta_1}{\partial \mu} \sigma$</th>
<th>$\frac{\partial \beta_1}{\partial \sigma} \sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.53</td>
<td>.7791</td>
<td>-.9244</td>
</tr>
<tr>
<td>2</td>
<td>11.00</td>
<td>.1246</td>
<td>-.1040</td>
</tr>
<tr>
<td>3</td>
<td>10.82</td>
<td>.3773</td>
<td>-.3686</td>
</tr>
<tr>
<td>4</td>
<td>10.82</td>
<td>.3683</td>
<td>-.3581</td>
</tr>
<tr>
<td>5</td>
<td>11.21</td>
<td>-.1343</td>
<td>.0913</td>
</tr>
<tr>
<td>6</td>
<td>11.46</td>
<td>-.4331</td>
<td>.2109</td>
</tr>
<tr>
<td>7</td>
<td>11.03</td>
<td>.0922</td>
<td>-.0753</td>
</tr>
<tr>
<td>8</td>
<td>11.00</td>
<td>.1369</td>
<td>-.1153</td>
</tr>
<tr>
<td>9</td>
<td>10.94</td>
<td>.2104</td>
<td>-.1861</td>
</tr>
<tr>
<td>10</td>
<td>11.43</td>
<td>-.4082</td>
<td>.2056</td>
</tr>
</tbody>
</table>

$\beta_1 = 1.53675, \ p_{F1}^{(1)} = 0.06218$

Table 7.4 Reliability results corresponding to the second design point

<table>
<thead>
<tr>
<th>Member</th>
<th>$\sigma_y^*$ [kN/cm$^2$]</th>
<th>$\frac{\partial \beta_1}{\partial \mu} \sigma$</th>
<th>$\frac{\partial \beta_1}{\partial \sigma} \sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.28</td>
<td>.6839</td>
<td>-.4271</td>
</tr>
<tr>
<td>2</td>
<td>12.11</td>
<td>-.0219</td>
<td>-.0010</td>
</tr>
<tr>
<td>3</td>
<td>12.08</td>
<td>.0071</td>
<td>.0001</td>
</tr>
<tr>
<td>4</td>
<td>11.29</td>
<td>.6776</td>
<td>-.4195</td>
</tr>
<tr>
<td>5</td>
<td>12.84</td>
<td>-.5244</td>
<td>-.3472</td>
</tr>
<tr>
<td>6</td>
<td>13.36</td>
<td>-.8333</td>
<td>-.9423</td>
</tr>
<tr>
<td>7</td>
<td>12.09</td>
<td>.0001</td>
<td>.0010</td>
</tr>
<tr>
<td>8</td>
<td>12.09</td>
<td>.0006</td>
<td>.0007</td>
</tr>
</tbody>
</table>

$\beta_2 = 2.1712, \ p_{F1}^{(2)} = 0.01496$
### Table 7.4 Reliability results corresponding to the second design point

<table>
<thead>
<tr>
<th>Member</th>
<th>$\sigma_y$ [kN/cm$^2$]</th>
<th>$\frac{\partial \beta_1}{\partial \mu} \sigma$</th>
<th>$\frac{\partial \beta_1}{\partial \sigma} \sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>11.93</td>
<td>-.1183</td>
<td>-.0122</td>
</tr>
<tr>
<td>10</td>
<td>12.09</td>
<td>-.0050</td>
<td>-.0001</td>
</tr>
</tbody>
</table>

$\beta_2 = 2.1712$, $p_{F1}^{(2)} = 0.01496$

### Table 7.5 Reliability results corresponding to the third design point

<table>
<thead>
<tr>
<th>Member</th>
<th>$\sigma_y$ [kN/cm$^2$]</th>
<th>$\frac{\partial \beta_1}{\partial \mu} \sigma$</th>
<th>$\frac{\partial \beta_1}{\partial \sigma} \sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.51</td>
<td>.5411</td>
<td>-.2415</td>
</tr>
<tr>
<td>2</td>
<td>12.77</td>
<td>.0241</td>
<td>-.0146</td>
</tr>
<tr>
<td>3</td>
<td>12.70</td>
<td>.0050</td>
<td>-.0027</td>
</tr>
<tr>
<td>4</td>
<td>11.34</td>
<td>.6280</td>
<td>-.3615</td>
</tr>
<tr>
<td>5</td>
<td>13.91</td>
<td>.4025</td>
<td>-.6660</td>
</tr>
<tr>
<td>6</td>
<td>15.51</td>
<td>.7751</td>
<td>-.5851</td>
</tr>
<tr>
<td>7</td>
<td>12.71</td>
<td>.0010</td>
<td>-.0010</td>
</tr>
<tr>
<td>8</td>
<td>12.71</td>
<td>.0008</td>
<td>-.0010</td>
</tr>
<tr>
<td>9</td>
<td>12.56</td>
<td>.0568</td>
<td>-.0242</td>
</tr>
<tr>
<td>10</td>
<td>12.63</td>
<td>.0310</td>
<td>-.0149</td>
</tr>
</tbody>
</table>

$\beta_3 = 3.877$, $p_{F1}^{(3)} = 0.00005$
Figure 7.16  Stress-strain history of element 6

Figure 7.17  Stress-strain history of element 10
7.4 Elastic Dam Subjected to Static Load

In this application example, a reliability analysis of the Pine Flat Dam on King's River near Fresno, California, is presented. The finite element idealization of the dam is shown in Fig. 7.18. The dam is subjected to hydrostatic forces only and the limit-state function is defined by a maximum horizontal displacement of 0.5 inch at the crest of the dam (node 91). The material (plain concrete) is assumed to be linear elastic, homogeneous and isotropic, and the material constitutive parameters, namely the Young's modulus and the Poisson's ratio are considered to be uncertain in nature. The triangular hydrostatic loading on the dam is taken to be deterministic, and the cross-section of the dam is assumed to be under plane strain condition.

The Poisson's ratio over the dam cross-section is modelled as a uniformly distributed single random variable (i.e., the spatial variability of the Poisson's ratio is neglected), since a preliminary reliability analysis indicated a much smaller sensitivity of the reliability index to the Poisson's ratio than to the Young's modulus. On the other hand, the Young's modulus is modelled as a lognormal random field, $E(x)$, with an exponentially decaying correlation coefficient function given by:

$$
p_{EE}(x_1, x_2) = e^{-\frac{\Delta}{a}}
$$

(7.1)

where $\Delta$ denotes the distance separation between points $x_1$ and $x_2$, and "a" is the correlation length of the random field. The distribution parameters for the Young's modulus and Poisson's ratio are presented in Table 7.6.

<table>
<thead>
<tr>
<th>Material property</th>
<th>Distribution</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus</td>
<td>lognormal</td>
<td>$4 \times 10^6$ psi</td>
<td>$4 \times 10^5$ psi</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>uniform</td>
<td>0.20</td>
<td>0.0289</td>
</tr>
</tbody>
</table>
Figure 7.18  Finite element model of the Pine Flat Dam on King’s River near Fresno, California

g = 0.5 in - d_{91}(horizontal)
The dam cross-section is sub-divided into a number of random field elements and the mid-point rule is used to discretize the random field of the Young's modulus into a set of correlated random variables. The effect of random field discretization on the first-order reliability index ($\beta_{\text{FORM}}$) is investigated by increasingly refining the random field mesh over the dam cross-section, see Fig. 7.19. The effect of the correlation length on the reliability index is also analyzed by repeating the reliability analysis for various values of the correlation length and for a fixed random field mesh. The results of these parametric reliability analyses are shown in Figs. 7.20 and 7.21. In Fig. 7.20, it is observed that the reliability index increases (i.e., the probability of failure decreases) with the refinement of the random field mesh and converges asymptotically to a constant value. Fig. 7.21 shows that, as the correlation length of the random field $E(\mathbf{x})$ increases, the reliability index decreases (i.e., the probability of failure increases) asymptotically to the reliability index obtained for a single random field element (point A of Fig. 7.20). This asymptotic relationship serves as a checking point of the present implementation of the finite element reliability method, since a random field with an infinite correlation length is a single random variable. Thus, the reliability response of the dam to correlation between elements Young's modulus is similar to that of a parallel system.

The contour plot of the horizontal displacement field at the design point is displayed in Fig. 7.22, while Figs. 7.23 and 7.24 are showing the contour plots of the sensitivities of the reliability index with respect to the mean and standard deviation of the local Young's modulus, respectively. These sensitivity plots are extremely instructive, since they indicate that the band (or compressive strut) of concrete which lies along the load transfer path of the hydrostatic load is the most important region of the dam cross-section. In other words, the horizontal displacement at the top of the dam is very sensitive to fluctuations in the mean and standard deviation of the local Young's modulus in this banded region. From the viewpoint of the deformation limit-state considered here, it would therefore be very important to maintain a high quality control during concreting of this region of the dam cross-section.
Figure 7.19 Finite element and random field meshes over the dam cross-section
15 random field elements

26 random field elements

48 random field elements

Finite element mesh
Random field mesh

Figure 7.19 (contd.) Finite element and random field meshes over the dam cross-section
Figure 7.20  Effect of random field size on reliability index

Figure 7.21  Effect of correlation length on reliability index
Plane strain condition - linear elastic concrete material

Figure 7.22  Contour plot of the horizontal displacement field at the design point
Plane strain condition - linear elastic concrete material

\[ \frac{\partial \beta_{\text{FORM}}}{\partial \mu_{E_i}} \]

Figure 7.23 Contour plot of the sensitivity of the reliability index with respect to the mean of the local Young's modulus
Plane strain condition - linear elastic concrete material

Figure 7.24 Contour plot of the sensitivity of the reliability index with respect to the standard deviation of the local Young's modulus
7.5 Perforated Inelastic Plate with $J_2$ Constitutive model

In this application example, an inelastic plate with a circular hole, shown in Fig. 7.25, and subjected to the time-varying pseudostatic loading plotted in Fig. 7.27 is analyzed by neglecting the inertia and damping effects. The plate is assumed to have an infinite thickness and to be in a state of plane strain. The material constitutive behavior is modeled with the $J_2$ plasticity model with linear isotropic and kinematic hardening. The constitutive parameters are chosen as follows: $E = 1000 \text{ kN/cm}^2$, $\nu = 0.30$, $H_{iso} = 50.0 \text{ kN/cm}^2$, $H_{kin} = 50.0 \text{ kN/cm}^2$, and $\sigma_{y0} = 1.0 \text{ kN/cm}^2$. This example has already been used in the literature on computational plasticity (Simo and Taylor 1986) and structural sensitivity analysis (Zhang and Der Kiureghian 1993) and therefore can be used as a benchmark problem to check the present implementation of the sensitivity analysis. Taking advantage of the symmetry of the problem, only one-quarter of the plate is analyzed as shown in Fig. 7.26. A total of 360 4-node isoparametric quadrilateral elements with bilinear interpolation of the displacement field are employed in the calculation. The time increment $\Delta t = 0.0625 \text{ sec}$ is used in the iterative/incremental procedure for solving the equations of motion and the gradient.

The displacement response time-history at point A (see Fig. 7.26) is given in Fig. 7.28 and agrees with the corresponding result obtained by Zhang and Der Kiureghian (1993), which validates the present implementation of the return map algorithm for the $J_2$ plasticity model in FEAP.

![Figure 7.25 Inelastic plate with a circular hole](image)
The present implementation of the conditional derivative method (CDM) for the $J_2$ plasticity model in FEAP is verified by computing the sensitivity of the displacement response time-history with respect to the initial yield stress, $\sigma_{y0}$, via the CDM and comparing it with results obtained from finite-difference analysis. Although not a limitation of the present implementation of sensitivity analysis, the initial yield stress parameter is taken to be constant over the whole plate region. Any constitutive parameter can be declared to vary from one finite element to the next, and the response sensitivity computed with respect to any local value of the constitutive parameter. The displacement response sensitivities computed using the CDM and finite-difference with decreasing value of the initial yield stress increment, $\Delta \sigma_{y0}$, are compared in Fig. 7.29. It can be seen that the finite-difference sensitivities converge asymptotically to the sensitivities computed using the CDM, thus verifying the implementation of the CDM for the $J_2$ plasticity model in FEAP.
Figure 7.27  Applied loading on the perforated plate

Figure 7.28  Horizontal displacement response time-history at point A
A reliability analysis is performed by modeling the initial yield stress of the material as a homogeneous and isotropic lognormal random field, with an exponentially decaying correlation coefficient function as defined in Eq. (7.1). The mean value and coefficient-of-variation of the initial yield stress are taken to be 1.0 kN/cm² and 10%, respectively. A correlation length of 0.50 cm is assumed for the random field. As shown in Fig. 7.30, the plate is subdivided into 40 random field elements, and the midpoint method defined in Section 5.2.2 is used to discretize the random field of the initial yield stress into a set of correlated random variables. The initial yield stress over an entire random field element is modeled as a single random variable equal to the value of the random field at the midpoint of the element. A first-order reliability analysis of the plate is conducted by assuming that “failure” occurs when the horizontal displacement at point A exceeds 0.1 cm.

As the finite element program (FEAP) evaluates the limit-state function and its gradient, it loops over all the finite elements and within each finite element, it loops over
the four Gauss points in order to evaluate the element stiffness matrix and nodal load vector. The value of the initial yield stress at the Gauss points of a given finite element is given by the particular realization of the random field at the mid-point of the random field element to which the current finite element belongs. The connectivity between finite elements and random field elements is established using a boolean connectivity matrix as explained in Section 6.3.

To gain more insight into the inelastic behavior of the plate, the spread of plasticity in the plate at the mean point (i.e., the initial yield stress is constant over the plate and equal to the mean value = 1 kN/cm$^2$) as the load increases is displayed in Figs. 7.31, 7.32, and 7.33. These yield patterns are consistent with those obtained by Simo and Taylor (1986).

The first-order reliability analysis produces the reliability index $\beta_{\text{FORM}} = 6.60$. The time-histories of the horizontal displacement response at point A are plotted in Fig. 7.34 for both the mean point and the design point. Fig. 7.35 shows the contour plot of the initial yield stress at the design point. By comparing Figs. 7.33 and 7.35, it is
observed that the spatial distribution of the initial yield stress at the design point has its lowest values in the region most plastified under the mean value condition. The contour plots of the normalized sensitivities of the first-order reliability index with respect to the mean and standard deviation of the local initial yield stress are presented in Figs. 7.36 and 7.37, respectively. Such reliability sensitivity results provide keen insight into the behavior of the plate, they indicate the relative importance of the uncertain local constitutive parameters in terms of the reliability against a given limit-state, and suggest design alternatives to improve the reliability of the plate.

Figure 7.31 Yield zone of the inelastic plate at $t = 0.75$ sec
Figure 7.32  Yield zone of the inelastic plate at $t = 0.875$ sec

Figure 7.33  Yield zone of the inelastic plate at $t = 1.0$ sec
Figure 7.34  Horizontal displacement response time-history at point A for the mean point and the design point.

Figure 7.35  Contour plot of the initial yield stress at the design point.
Figure 7.36 Contour plot of the normalized sensitivity of the first-order reliability index with respect to the mean of the local initial yield stress

Figure 7.37 Contour plot of the normalized sensitivity of the first-order reliability index with respect to the standard deviation of the local initial yield stress
7.6 Single Plane Strain Element with Cap Plasticity Model

In order to verify the implementation of the return map algorithm and the conditional derivative method for the cap model in FEAP, a simple test example consisting of a single four-noded plane strain element of unit thickness and with four Gauss points, see Fig. 7.38, is investigated in detail. The material constitutive law of the single element is assumed to follow the cap model defined in Section 4.11 and the constitutive parameters are reported in Table 7.7. The implementation of the return map algorithm for the cap model is verified by subjecting the test element to several basic load cases, such as monotonically increasing uni-axial tension and compression, and cyclic loading. Two cases are considered for the uni-axial compressive loading. In the first case, the stress path in the $I_1$-lssl space is allowed to hit the cap surface first, and in the other case, the failure surface is activated first by modifying the initial hardening parameter $\kappa_0$. The load-displacement relationship at node 3 and the stress path in the $I_1$-lssl space for these basic load cases are presented in Figs. 7.39 to 7.44. In each case, the updated (hardened) yield surface is shown for three different load levels. As part of the verification process, computation of the theoretical load at first yield of the test element is also presented for each load case. In order to simplify these calculations, the parameter $\beta$ is set equal to zero, which results in the use of a straight line for the failure surface instead of a more general exponential curve. In each case, it can be seen from the load-displacement curve produced by FEAP that the magnitude of the load initiating yielding of the material agrees well with the theoretical value.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>0.3 ksi</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>3.86 ksi</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.16 ksi</td>
</tr>
</tbody>
</table>
Table 7.7 Parameters of the cap constitutive model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>0.42</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0 ksi$^{-1}$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.11</td>
</tr>
<tr>
<td>R</td>
<td>4.43</td>
</tr>
<tr>
<td>D</td>
<td>0.0032 ksi$^{-1}$</td>
</tr>
<tr>
<td>$\kappa_0$</td>
<td>1.25 ksi</td>
</tr>
<tr>
<td>K</td>
<td>2100 ksi</td>
</tr>
<tr>
<td>G</td>
<td>1700 ksi</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.150 k/ft$^3$</td>
</tr>
</tbody>
</table>

Figure 7.38 Single test element under plane strain condition
7.6.1 Simple Calculations for Verification of the Implementation of the Cap Model

This section is concerned with the computation of the theoretical load at first yield of the simple test specimen shown in Fig. 7.38. Before yielding occurs, the single element is under linear elastic plane strain condition. Moreover, since the element is free to displace in the horizontal direction, it is stress free in that direction. Therefore, the stress state at each of the four Gauss points is defined by

\[
\sigma_x(t) = \frac{2P(t)}{A} \quad (7.2)
\]

\[
\sigma_y(t) = 0 \quad (7.3)
\]

\[
\sigma_z(t) = \nu \sigma_x(t) \quad (7.4)
\]

\[
\sigma_{xy}(t) = \sigma_{xz}(t) = \sigma_{yz}(t) = 0 \quad (7.5)
\]

in which \(P(t)\) is the pseudo-static load applied at time \(t\), \(A\) is the cross-section area of the element, and \(\nu\) denotes the Poisson’s ratio.

In matrix form, one has

\[
\mathbf{\sigma} = \begin{bmatrix}
\sigma_x & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \sigma_z
\end{bmatrix} \quad (7.6)
\]

For the above state of stress, the first invariant of the stress tensor, \(I_1\), and the deviatoric stress tensor, \(s\), become

\[
I_1 = \sigma_x + \sigma_y + \sigma_z = (1 + \nu) \sigma_x \quad (7.7)
\]

\[
s = \begin{bmatrix}
\left(\frac{2}{3} \sigma_x - \frac{\nu}{3} \sigma_z\right) & 0 & 0 \\
0 & \left(-\frac{1}{3} \sigma_x - \frac{\nu}{3} \sigma_z\right) & 0 \\
0 & 0 & \left(-\frac{1}{3} \sigma_x + \frac{2}{3} \nu \sigma_x\right)
\end{bmatrix} \quad (7.8)
\]

Hence,
\[ ||s|| = \sqrt{\left(\frac{2}{3} \sigma_x - \frac{\nu}{3} \sigma_x\right)^2 + \left(\frac{1}{3} \sigma_y + \frac{\nu}{3} \sigma_x\right)^2 + \left(-\frac{1}{3} \sigma_x + \frac{2}{3} \nu \sigma_x\right)^2} \]  
\[ (7.9) \]

where \( ||s|| \) is the norm of the deviatoric stress tensor as defined in Eq. (4.10). After simplification, Eq. (7.9) reduces to

\[ ||s|| = \sqrt{\frac{2}{3} \left(1 + \nu^2 - \nu\right) \sigma_x} = \frac{2}{\sqrt{3}} \left(1 + \nu^2 - \nu\right) \frac{I_1}{(1 + \nu)} \]  
\[ (7.10) \]

which for \( \nu = 0.18 \) gives

\[ ||s|| = 0.7538 \sigma_x \]  
\[ (7.11) \]

The following sections examine in detail the monotonic uni-axial tension and compression load cases.

### 7.6.2 Monotonic Uni-Axial Tension Load

By definition of the tension cut-off region, the first yield in tension occurs at

\[ I_1 = T \]  
\[ (7.12) \]

which from Eq. (7.7) gives

\[ \sigma_x^y = \frac{T}{(1 + \nu)} \]  
\[ (7.13) \]

where \( \sigma_x^y \) represents the yield stress under uni-axial tension. The corresponding tensile load \( P_y \) which causes the element to first yield is

\[ P_y = \frac{\sigma_x^y A}{2} = \frac{T}{2(1 + \nu)} A \]  
\[ (7.14) \]

Substituting the values \( A = 1 \ in^2 \), \( T = 0.3 \ ksi \) and \( \nu = 0.18 \), one finds

\[ P_y = 0.127 \ kips \]  
\[ (7.15) \]

which agrees well with the yield force obtained by FEAP as shown in Fig. 7.39. The stress path in the monotonic uni-axial tension load case is reported in Fig. 7.40.
Figure 7.39  Load-displacement curve at node 3 for monotonic uni-axial tension load

Figure 7.40  Stress path in the $I_1$-$||\sigma||$ space for monotonic uni-axial tension load case
7.6.3 Monotonic Uni-Axial Compression Load (Cap Surface Activated First)

The initial cap surface is characterized by, see Eq. (4.235),

\[
f_2(\sigma_x, \kappa_0) = F_c(\sigma_x, \kappa_0) - F_e(\kappa_0) = 0,
\]

(7.16)

where

\[
F_c(\sigma_x, \kappa_0) = \sqrt{||s||^2 + \frac{[I_1 - L(\kappa_0)]^2}{R^2}} = \sqrt{||s||^2 + \frac{[I_1 - 1.25]^2}{4.43^2}}
\]

(7.17)

and

\[
F_e(\kappa_0) = \alpha - \lambda e^{-\beta \kappa_0} + \theta \kappa_0
\]

(7.18)

Using the values of \( \alpha = 3.86 \text{ ksi}, \beta = 0., \lambda = 1.16 \text{ ksi}, \kappa_0 = 1.25 \text{ ksi} \) and \( \theta = 0.11, \)

\[
F_e(\kappa_0) = 2.8375 \text{ ksi}
\]

(7.19)

From Eqs. (7.7) and (7.11), and using \( R = 4.43 \) and \( \nu = 0.18, \) one finds

\[
F_c(\sigma_x, \kappa_0) = \sqrt{(0.7538\sigma_x)^2 + \frac{[1.18\sigma_x - 1.25]^2}{4.43^2}}
\]

(7.20)

The first time the cap mode is activated, \( F_e(\kappa_0) = F_c(\sigma_x^y, \kappa_0) \) which, after using Eqs. (7.19) and (7.20) and simplifying, yields

\[
12.543 \left(\sigma_x^y\right)^2 - 2.95 \sigma_x^y - 156.4455 = 0
\]

(7.21)

the solution of which is \( \sigma_x^y = 3.651 \text{ ksi}. \) Hence, the uni-axial compressive force at first yield, \( P_y, \) when the cap mode is activated is given by

\[
P_y = \frac{\sigma_x^y A}{2} = 1.825 \text{ kips}
\]

(7.22)

The load-displacement curve in Fig. 7.41 shows that the magnitude of the applied compressive load at first yield obtained by FEAP agrees with the above theoretical
value. The agreement between the computed and theoretical values is not as good as in Fig. 7.39 due to load increments (time steps) which are too large in the present case. The stress path followed during the monotonically applied uni-axial compressive load is displayed in Fig. 7.42 with three different positions of the hardened cap corre-

Figure 7.41 Load-displacement curve at node 3 for monotonic uni-axial compression load

Figure 7.42 Stress path in the $I_1$-lsl space for monotonic uni-axial compression load
sponding to three different load levels of increasing magnitude.

7.6.4 Monotonic Uni-Axial Compression Load (Failure Surface Activated First)

To enforce the single plane strain element to first yield on the failure surface, the initial yield surface is modified by changing the value of the initial hardening parameter to \( \kappa_0 = 10.0 \) ksi. When the failure surface is activated, one has

\[
f_1 (\sigma_x) = \|s\| - F_c (I_1) = 0
\]  

(7.23)

which, by using Eqs. (7.7) and (7.11), reduces to

\[
0.7538 \sigma_x^y = \alpha - \lambda e^{-\beta (1 + \nu) \sigma_x^y} + \theta (1 + \nu) \sigma_x^y
\]  

(7.24)

where \( \sigma_x^y \) denotes the yield stress under uni-axial compression when first yield is controlled by the failure surface. Substituting the parameter values \( \alpha = 3.86 \) ksi, \( \theta = 0.11 \), \( \lambda = 1.16 \) ksi, \( \beta = 0 \) and \( \nu = 0.18 \) into Eq. (7.24) and solving for \( \sigma_x^y \) yields

\[
\sigma_x^y = 4.327 \text{ ksi}
\]  

(7.25)

Hence, the uni-axial compressive force at first yield is

\[
P_y = \frac{\sigma_x^y A}{2} = 2.163 \text{ kips}
\]  

(7.26)

which agrees closely with the finite element solution shown in Fig. 7.43. The stress path followed during the monotonically applied uni-axial compressive load is displayed in Fig. 7.44 with three different positions of the hardened cap corresponding to three different load levels of increasing magnitude.
Figure 7.43 Load-displacement curve at node 3 for monotonic uni-axial compression load (failure surface activated first)

Figure 7.44 Stress path in the $I_1$-$\|s\|$ space for monotonic uni-axial compression load (failure surface activated first)
7.6.5 Cyclic Loading

Next, the inelastic single plane strain element is subjected to pseudo-static cyclic loading. The time history of the applied load, $P(t)$, is depicted in Fig. 7.45. The resulting load-displacement relationship is given in Fig. 7.46 and the path followed by the stress state in the $I_1$-Ilsl space is shown in Fig. 7.47. It is noted that a kink appears at the onset of every yield event in the load-displacement curve. This is due to the fact that the return map algorithm used here does not determine the exact occurrence time of the yield events, thus “truncating the corners” of the load-displacement curves. However, no error is introduced in the computed stresses and strains, since the return map algorithm ensures that the yield criteria, the flow rule, and the hardening law of the cap plasticity model are satisfied exactly at every time step. The size of the kinks mentioned above can be reduced by decreasing the size of the load increments (time steps) used in solving the nonlinear static equations.

![Graph showing time history of the applied cyclic load](image.png)

Figure 7.45 Time history of the applied cyclic load
Figure 7.46 Load-displacement curve at node 3 for cyclic load case

Figure 7.47 Stress path in the $I_1$-$I_{sl}$ space for cyclic load case
7.7 Inelastic Plane Strain Element Subjected to Earthquake-Type Loading

In this section, the response and response gradient of the single element with the cap model are analyzed when the element is subjected to an earthquake-type loading consisting of the first fifteen seconds of the S00E (North-South) component of the El Centro earthquake record scaled by a factor of 0.015 and applied as concentrated force at nodes 3 and 4 in the vertical direction, see Fig. 7.48. The El Centro record was shown earlier in Fig. 7.2. The inelastic displacement response time history, \( u_3(t) \), is presented in Fig. 7.49 where it is compared with the response of the system in case it were to behave linear elastically. In order to use a cap constitutive model calibrated against actual concrete data, the constitutive parameter \( \beta \) is changed back to \( \beta = 0.44 \) ksi\(^{-1} \). It is observed in Fig. 7.49 that the element undergoes a large number of positive and negative yield excursions.

The implementation of the CDM is verified by computing the displacement response sensitivities of the element with respect to the cap model parameters and comparing
them with those obtained using the finite difference method. The time step \( \Delta t = 0.02 \) sec is used to integrate the equation of motion and the gradient equations. The comparison between the CDM and the finite difference method for the displacement response sensitivities with respect to the parameters T, W, R, \( \alpha \) and \( \beta \) is shown in Figs. 7.50 to 7.54, respectively. Each of the sensitivity results is scaled by the nominal value of the constitutive parameter considered and can be interpreted as a change in the vertical displacement response time history per percentage change in the constitutive parameter. In each case, it can be seen that the two methods give practically the same sensitivity results, thus validating the CDM for the cap plasticity model and the correctness of its implementation in FEAP.
Figure 7.50  Displacement response sensitivity with respect to the parameter $\alpha$

Figure 7.51  Displacement response sensitivity with respect to the parameter $\beta$
Figure 7.52 Displacement response sensitivity with respect to the parameter $R$

Figure 7.53 Displacement response sensitivity with respect to the parameter $T$
Figure 7.54  Displacement response sensitivity with respect to the parameter W
7.8 Inelastic Dam Subjected to Earthquake Loading

The Pine Flat Dam on Kings River near Fresno, California, which was analyzed in Section 7.4 for hydrostatic load, is now subjected to the 1940 N-S component of the El Centro earthquake record, assuming fixed base and empty reservoir conditions. The finite element model shown in Fig. 7.18 is also used for the dynamic analysis considered here. The constitutive behavior of the plain concrete of the dam is modelled by the cap plasticity model described in Section 4.11. The parameters of the cap model are reported in Table 7.8. These cap constitutive parameter values were obtained by Simo et al. (1988b) through a least-square estimation procedure performed on the Colorado concrete data (Scavuzzo et al. 1983). The cap parameters are assumed to be constant over the whole cross-section of the dam. The gravity load is applied prior to the seismic load. The time step $\Delta t = 0.02$ sec was used in integrating the equation of motion and gradient equations.

The inelastic displacement response time history of the top of the dam (node 91 in Fig. 7.18) is displayed in Fig. 7.55 which also shows the displacement response in the case that the dam were linear elastic and undamped. It is observed that the inelastic top displacement response is slightly larger than the undamped linear elastic response.

To gain more insight into the inelastic behavior of the dam, the stress path is examined at the four different locations (A, B, C, and D) of the dam cross-section shown in Fig. 7.56. The path followed by the stress state in the $I_1$-lsll space at points A, B, C, and D is plotted in Figs. 7.57 to 7.60, respectively. It is observed that while the stress state remains within the elastic domain at points B and D (downstream face of the dam), the tension cut-off region is activated at points A and C (upstream face of the dam). This is confirmed by the plots of the mean stress $p = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3} I_1$ versus the volumetric strain $\epsilon_v = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \frac{1}{3} I_1$ displayed in Figs. 7.61 to 7.64 for points A, B, C, and D, respectively. These $p-\epsilon_v$ plots show that only points A and C (upstream face of the dam) undergo yielding in the tension cut-
off region, while points B and D remain linear elastic.

Since only the tension cut-off region of the cap model is activated, it follows that the only cap model parameter affecting the response of the dam is the tension cut-off $T$. The displacement response sensitivity with respect to the other cap constitutive parameters ($\alpha$, $\theta$, $\beta$, $R$, $W$, $D$, $\lambda$, and $\kappa_0$) is zero. The normalized sensitivity time history of the horizontal displacement at the top of the dam (i.e., $u_{91}(t)$) with respect to the parameter $T$ computed using both the CDM and the finite-difference method is shown in Fig. 7.65. The results of the CDM and finite difference method coincide, thus validating the implementation of the CDM for the tensile cut-off region of the cap model.

Table 7.8 Constitutive parameters for the concrete of the dam

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>2100 ksi</td>
</tr>
<tr>
<td>$G$</td>
<td>1700 ksi</td>
</tr>
<tr>
<td>$T$</td>
<td>0.3 ksi</td>
</tr>
<tr>
<td>$R$</td>
<td>4.43</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>3.86 ksi</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.11</td>
</tr>
<tr>
<td>$\beta$</td>
<td>.44 ksi$^{-1}$</td>
</tr>
<tr>
<td>$W$</td>
<td>0.42</td>
</tr>
<tr>
<td>$D$</td>
<td>0.0032 ksi$^{-1}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.16 ksi</td>
</tr>
<tr>
<td>$\kappa_0$</td>
<td>1.25 ksi</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.15 k/ft$^3$</td>
</tr>
</tbody>
</table>
Figure 7.55  Displacement response time history of the top of the dam

Figure 7.56  Location of points A, B, C and D
Figure 7.57  Plot of the stress-path in the $I_1$-$I_{	ext{sl}}$ space at point A

Figure 7.58  Plot of the stress-path in the $I_1$-$I_{	ext{sl}}$ space at point B
Figure 7.59  Plot of the stress-path in the $I_1$-$\sigma_{\text{yield}}$ space at point C

Figure 7.60  Plot of the stress path in the $I_1$-$\sigma_{\text{yield}}$ space at point D
Figure 7.61  Plot of mean stress vs. volumetric strain at point A

Figure 7.62  Plot of mean stress vs. volumetric strain at point B
Figure 7.63 Plot of mean stress vs. volumetric strain at point C

Figure 7.64 Plot of mean stress vs. volumetric strain at point D
Figure 7.65  Displacement response sensitivity with respect to the parameter $T$
7.9 Inelastic Dam Subjected to Scaled Up Earthquake Loading

In order to activate all regions of the cap model, the empty dam is subjected to the N-S component of the 1940 El Centro earthquake record (Fig. 7.2) scaled by a factor 3, such that the peak ground acceleration reaches 1.045 g. The time step $\Delta t = 0.02$ sec is used to integrate the equation of motion and the gradient equations. The inelastic displacement response at the top of the dam is shown in Fig. 7.66 with the undamped linear elastic response. It is observed that the undamped linear elastic response is larger than the inelastic response which is characterized by hysteretic energy dissipation. As in Section 7.8 for the unscaled El Centro record, the path of the stress state in the $I_1-I_{sl}$ space is displayed in Figs. 7.67 to 7.70 for points A, B, C, and D, respectively. In Fig. 7.67, the hardened yield surface is given at two intermediate times. It is observed that at point A (mid-height upstream face) all the regions of the cap plasticity model are activated during the inelastic response of the dam. The plots of the mean stress versus the volumetric strain at each of the control points A, B, C, and D are given in Figs. 7.71 to 7.74, respectively. In Figs. 7.71 and 7.73, notice that some of the corners of the hysteresis loops are truncated due to the discrete nature of the time when integrating the equation of motion.

The top displacement response sensitivities with respect to the cap constitutive parameters $\alpha$, $\lambda$, $R$, $T$, $W$, $\beta$, and $\theta$ are shown in Figs. 7.75 to 7.81, respectively. Each displacement response sensitivity has been computed using the CDM and the finite difference method. It is noted that for each sensitivity parameter, there is a very close agreement between the results of the CDM and finite difference method, which validates the correctness of the formulation of the CDM for the cap plasticity model and its implementation in FEAP. It is also noticed that the inelastic displacement response at the top of the dam has a higher sensitivity to $\alpha$, $\lambda$, and $T$ than it does to the remaining parameters. Parameters $\alpha$, $\lambda$, and $T$ characterize the failure envelope and tension cut-off region.
Figure 7.66  response of the dam subjected to scaled up El-Centro earthquake

Figure 7.67  Plot of the stress path in the $I_1$-I_{sl} space at point A
Figure 7.68  Plot of the stress path in the $I_1-\|s\|$ space at point B

Figure 7.69  Plot of the stress path in the $I_1-\|s\|$ space at point C
Figure 7.70 Plot of the stress path in the $I_1$-$\|\sigma\|$ space at point D

Figure 7.71 Plot of mean stress vs. volumetric strain at point A
Figure 7.72  Plot of mean stress vs. volumetric strain at point B

Figure 7.73  Plot of mean stress vs. volumetric strain at point C
Figure 7.74  Plot of mean stress vs. volumetric strain at point D

Figure 7.75  Displacement response sensitivity with respect to the parameter $\alpha$
Figure 7.76  Displacement response sensitivity with respect to the parameter $\lambda$

Figure 7.77  Displacement response sensitivity with respect to the parameter $R$
Figure 7.78 Displacement response sensitivity with respect to the parameter $T$

Figure 7.79 Displacement response sensitivity with respect to the parameter $W$
Figure 7.80  Displacement response sensitivity with respect to the parameter $\beta$

Figure 7.81  Displacement response sensitivity with respect to the parameter $\theta$
7.10 Reliability Analysis of Simple Dynamic Systems

To gain insight into the behavior of the limit-state surface of dynamic systems, simple one- and two-degree-of-freedom structures are considered in this section.

The first example consists of a linear elastic, undamped, single-degree-of-freedom (SDOF) truss structure subjected to harmonic loading as shown in Fig. 7.82. The truss structure is composed of two truss elements of Young’s moduli $E_1$ and $E_2$, respectively, cross-section area $A$, length $L$, and volumetric mass density $\rho$. The Young’s moduli $E_1$ and $E_2$ are considered to be the basic random variables, and the remaining variables are assumed to be deterministic and take the following the values: $A = 1.0 \text{ [in}^2\text{]}, \rho = 6.5 \text{ [k-sec}^2\text{/in}^4\text{]}, L_1 = 30 \text{ [in]}, \text{and } L_2 = 40 \text{ in.}$ The truss is excited at point $B$ by a harmonic load defined as

$$P(t) = 1000 \sin(\pi t) \quad (0 \leq t \leq 10 \text{ sec})$$

The structure is assumed to have “failed” if the horizontal displacement response at point $B$ exceeds a predetermined critical value, $\delta_c$, during the time of application of the load, i.e., $(0 \leq t \leq 10 \text{ s})$. Hence the limit-state function can be expressed as:

$$g(X) = \delta_c - \text{Max} \{|\delta(t)|, \; (0 \leq t \leq 10 \text{ sec})\}$$
where $X = [E_1, E_2]^T$ is the basic random vector. For the purpose of this illustration, the two critical values $\delta_c = 1.5 \text{ [in]}$ and $\delta_c = 4.0 \text{ [in]}$ are considered.

The limit-state function of the above reliability problem is constructed for each of the critical values $\delta_c$. For this purpose, different combinations of $E_1$ and $E_2$ are considered along the nodal points of a grid in the $(E_1, E_2)$ space, and for each combination the displacement response time history $\delta(t)$ at point B is computed and $\max_{0 \leq t \leq 10 \text{sec}} |\delta(t)|$ is determined. The limit-state surface then coincides with the contour line of level zero of the limit-state function $g(X)$ defined above. The results are shown in Figs. 7.83 and 7.84 for $\delta_c = 1.5 \text{ [in]}$ and $\delta_c = 4.0 \text{ [in]}$, respectively.

![Diagram](image)

**Figure 7.83** Limit-state surface and failure domain of linear elastic SDOF truss structure ($\delta_c = 1.5 \text{ in}$)

It is noted that the safe domain is not connected, i.e., the random parameter space $(E_1$
and $E_2$ is not subdivided into two semi-infinite regions (the safe and unsafe domains), as is usually assumed when applying the first- and second-order reliability methods. Instead, the unsafe domain lies sandwiched between two safe domains. Clearly, the basic assumption upon which the FORM/SORM methods are based is violated even in the case of this simple dynamic system.

In this first example, the maximum absolute value of the dynamic response is controlled by a single independent parameter: the natural frequency $\omega$ of the system. To understand the reasons behind the banded shape of the failure domain, the response spectrum of a SDOF oscillator subjected to a time-limited harmonic loading is sketched in Fig. 7.85. The response spectrum reaches its maximum in the neighborhood of the loading frequency, $\pi \text{ [rad/sec]}$. It is also obvious that the relationship
between the maximum displacement and the natural frequency of the system is not one-to-one. Typically, for any given maximum displacement \( \delta_c \) there exist two critical natural frequencies of the structure, \( \omega_1 \) and \( \omega_2 \), see Fig. 7.85. For any realization of the basic random variables \( E_1 \) and \( E_2 \) such that the natural frequency of the structure lies between \( \omega_1 \) and \( \omega_2 \), \( \text{Max} \left| \delta(t) \right| \) is greater than \( \delta_c \) and such realization belongs to the failure or unsafe domain. On the other hand, any realization of \( E_1 \) and \( E_2 \) such that \( \omega(E_1, E_2) \leq \omega_1 \) or \( \omega(E_1, E_2) \geq \omega_2 \) belongs to the safe domain. Thus the unsafe domain is sandwiched between two surfaces, one corresponding to \( \omega(E_1, E_2) = \omega_1(\delta_c) \) and the other corresponding to \( \omega(E_1, E_2) = \omega_2(\delta_c) \). In the present example, the function \( \omega^2(X) \) is linear in the basic random variables \( E_1 \) and \( E_2 \). Hence, in the \( X \)-space, the unsafe region is separated from the safe region by a pair of straight lines. The upper separating line in Fig. 7.83 is not straight due to an interpolation error of the contouring procedure (i.e., not enough grid points in the \( (E_1, E_2) \) space).

As the probability distribution of the basic random variables is undefined, the shape of
the limit-state surface in the standard normal space $U$ remains undefined. However, if the random variables $E_1$ and $E_2$ are assumed to be jointly normal, then the transformation from the $X$-space to the $U$-space is linear and the banded shape of the unsafe domain is conserved in the $U$-space. For non-normal probability distributions, the unsafe domain in the $U$-space is a distorted banded region.

In a more general problem than the one considered above, the limit-state function will be more complex. For example, if the loading function has more than one dominant frequency as is the case with earthquake excitation, then there will be several critical natural frequencies, $\omega_1$, $\omega_2$, ..., at which \[ \max_{0 \leq t \leq 10 \text{sec}} |\delta(t)| \] equals the critical displacement value $\delta_c$. In such a case, the unsafe region is “multiply-banded”, i.e., safe banded regions alternate with unsafe ones. Moreover, depending on the nature of the basic random variables $X$, the hypersurface described by $\omega(X) = \omega_i = \text{constant}$ may be highly nonlinear and may even be closed.

In the second example, the SDOF system considered in the first example is assumed to be elastic-perfectly plastic as shown by the stress-strain diagram represented in Fig. 7.86. The same geometric and material parameters (for the linear elastic part) as in

![Stress-strain diagram of truss material](image)

**Figure 7.86** Stress-strain diagram of truss material
the first example are considered with the additional constraint that $E_1 = E_2 = E$. The limit-state surface in the $(\varepsilon_y, E)$ space has been determined for $\delta_c = 1.5$ [in] and $\delta_c = 4.0$ [in] by running a set of dynamic response analyses along the grid points in the $(\varepsilon_y, E)$ space. The two limit-state surfaces are displayed in Fig. 7.87. As can be seen, the failure domain has a truncated banded shape.

As a third and final example, the linear elastic SDOF truss structure considered in the first example, see Fig. 7.82, is analyzed as a two-degree-of-freedom system by removing the vertical support at point B. The geometric and material parameters and the loading function are kept the same. The failure surface is constructed in the $(E_1, E_2)$ space for $\delta_c = 6.5$ [in] (= critical value of largest absolute horizontal displacement at point B) by performing a set of dynamic response analyses. It is displayed in Fig.
7.88. In this case, the failure domain is closed and of elliptic shape.

\[ g(X) = 6.5 - \delta(X) \]

Figure 7.88  Limit-state surface and failure domain of linear elastic 2-DOF truss structure

To summarize, even for simple one- or two-degree-of-freedom dynamic systems, the failure region may not be simple. In other words, the limit-state surface may not subdivide the random parameter space into two infinite half regions. The failure region may be banded, closed, multiply-banded, or composed of disconnected subregions, depending on the parameters modelled as random and the properties of the dynamic loading. Since a basic assumption upon which the first- and second-order reliability methods are based is violated, it is expected that in some cases the computed reliability index or probability of failure might be significantly erroneous. It seems that in studying dynamic systems in a reliability framework, it is necessary to have some prior understanding/knowledge of the failure region.
7.11 Conclusions

This chapter presents several application examples. First, a ground excited inelastic single-degree-of-freedom truss structure is analyzed to validate the formulation and implementation of the $J_2$ inelastic truss element and the corresponding "exact" differentiation method called conditional derivative method (CDM). The analytical sensitivity results are verified by finite difference. Then, a $J_2$ inelastic ten-member truss subjected to cyclic loading is considered. Its inelastic displacement response and response sensitivities with respect to various material parameters are computed. The application of the CDM to the case of a linear elastic, empty dam subjected to hydrostatic loading is also presented. A first-order reliability analysis for maximum displacement limit-state is performed for each of these three examples. In each dynamic case, more than a single design point is obtained. For the linear elastic dam example under hydrostatic condition, the Young’s modulus is modelled as a lognormal random field and the effects of its correlation length and discretization on the reliability index is investigated.

Next, a perforated plate with the $J_2$ constitutive model is analyzed under plane strain condition and for pseudo-static cyclic loading. A first-order reliability analysis is carried out by modelling the initial yield stress as a lognormal random field. Emphasis is placed on the sensitivity results of the reliability analysis.

Finally, the formulation and implementation of the cap plasticity model is verified based on two examples: a single plane strain element under various pseudo-static load cases and an earthquake-type dynamic load case and an empty dam subjected to earthquake excitation. In each case, the formulation and implementation of the CDM adapted for the cap model is verified using the finite difference method.

Some important issues in the reliability analysis of dynamic systems are also addressed. The limit-state surface of simple elastic and inelastic dynamic systems is investigated and found to be significantly different than in the static case.
CHAPTER 9 CONCLUSIONS

8.1 Summary of Work

In this project, some important tools required for the finite element reliability analysis of inelastic dynamic systems were developed and implemented. A finite element reliability analysis program was built by merging a general purpose, research oriented finite element analysis program, FEAP, with a reliability analysis program, CALRE. The finite element code FEAP was first enhanced by the addition of inelastic element routines for the truss and plane strain elements. More specifically, the truss element with the $J_2$ plasticity model and the four-noded quadrilateral plane strain element with the $J_2$ and cap plasticity models were implemented in FEAP. In order to accurately and efficiently compute the structural response sensitivities needed in first- and second-order reliability analysis, a subset of the “exact” differentiation method, called the conditional derivative method (CDM), was formulated and implemented in FEAP for the $J_2$ and cap plasticity models. The implementation of the CDM in FEAP was cross-checked by the finite difference method for several application examples and the sensitivity results produced by the two methods were found to agree closely in each case.

Several application examples were analyzed using the FEAP-CALREL software. These examples included the dynamic sensitivity analysis and reliability analysis of a ten-member inelastic truss. The earthquake response sensitivity of an empty concrete gravity dam with the cap constitutive model was also performed.

In order to gain more insight into the nature and complexity of the limit-state surface of dynamic systems, simple elastic and inelastic one- and two-degree-of-freedom systems were investigated.

To conclude, this work lays a foundation for the development of a comprehensive software package for the reliability and safety analysis of large and complex inelastic dynamic systems using the powerful finite element method.
8.2 Summary of Findings

In this study, the Conditional Derivative Method (CDM) was established as an efficient and accurate procedure for computing the response sensitivities of inelastic dynamic systems. The sensitivity results produced by the CDM were cross-checked with those obtained by finite-difference for various application examples and in each case the two methods were found to be in close agreement.

Using the CDM for computing the response gradient, several examples of first-order reliability analysis were performed. Multiple design points corresponding to different failure modes were typically found for dynamic systems. Two types of instructive contour plots were presented: (i) the contour plots of the distribution of the material properties at the design point (most likely failure point); and (ii) the contour plots of the sensitivities of the reliability index with respect to the distribution parameters of the material properties. These contour plots offer significant insight into the load transfer mechanisms of structural systems and serve to identify the regions in a structure which are critical from a reliability or safety standpoint. The dependence of the reliability index on the random field discretization and the correlation length of the random field was also investigated. In the example considered, it was found that the reliability index converges asymptotically from below as the random field mesh is refined, and that it converges asymptotically from above as the correlation length of the random field increases.

It was also discovered that the reliability analysis of dynamic systems is much more complicated than that of static systems due to the complex nature of their limit-state surface and failure region.

Use of an efficient memory management strategy in FEAP-CALREL resulted in a superior utilization of the memory space, and it was found that reliability analysis of large and complex structural systems involving a large number of degrees of freedom and random variables can be tackled using this strategy.
8.3 Scope of Future Work

The main thrust of this work encompasses three different areas of structural engineering, namely finite element analysis, plasticity theory, and reliability analysis. Several powerful innovations can be made in the existing FEAP-CALREL code with minimal effort in each of the three areas. A better estimate of the structural reliability of a dynamic system can be achieved by using more realistic and sophisticated failure criteria based on a combination of maximum response and cumulative damage as opposed to the maximum displacement based criterion used in this study. The code can be easily modified to incorporate different failure criteria. As already mentioned, the failure regions of dynamic structural systems are very complex and there is a need to investigate the nature of these failure regions in order to establish a reliable procedure for the determination of the failure probability (or probability content of the failure region).

Use can be made of a variety of other plasticity and rheological models to represent different material behaviors. By adding different inelastic element routines and retaining the rest of the software package, a library of material models can be incorporated into FEAP-CALREL with relatively small additional effort. Such an addition would contribute to a more realistic modeling and analysis of large and complex structural systems, especially in their failure region.

Accurate and efficient algorithms for determining the dynamic response sensitivities of inelastic structural systems have been implemented in FEAP-CALREL. Therefore, other applications requiring the use of response sensitivities, such as structural optimization, system identification, and damage detection, can be readily targeted by modifying the FEAP-CALREL package.
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