INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Determination of the polytropic index of the free-streaming solar wind

Totten, Tracy L., M.S.
Rice University, 1994
RICE UNIVERSITY

DETERMINATION OF THE POLYTROPIC INDEX OF THE FREE-STRENGTHWIND

by

TRACY L. TOTTEN

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

MASTER OF SCIENCE

APPROVED, THESIS COMMITTEE:

John W. Freeman, Jr., Chair
Professor of Space Physics and Astronomy

Gerd-Hannes Voigt
Distinguished Faculty Fellow of Space Physics and Astronomy

Robert C. Haymes
Professor of Space Physics and Astronomy

Houston, Texas
August 1993
ABSTRACT

Determination of the Polytropic Index of the
Free-Streaming Solar Wind

by

Tracy L. Totten

Observations of solar wind temperatures near the Earth indicate that heating of the solar wind plasma exists. An alternate approach to finding explicit heating terms for the energy equation is to use a polytropic approximation. Using data from the Helios 1 spacecraft, an empirical value for the polytropic index of the solar wind is found to be independent of speed state, within statistical error, and has an average value of 1.47. Application of this empirically derived index to a solar wind computer model is examined by comparing the MHD energy equation and the polytropic relation. The result is obtained that the polytropic index can replace the adiabatic index in the MHD energy equation to simulate the effects of heat conduction if the assumptions are made that the heat conduction flux has a specific form and the particle pressure has no explicit time dependence. Justifications and limitations of this approach are discussed.
ACKNOWLEDGMENTS

I wish to thank my advisor, Dr. John Freeman, for his guidance and instruction these past few years. His encouragement and enthusiasm are an inspiration. I also thank Dr. Sharda Arya for all her assistance and many helpful discussions. Thanks go to my committee members, Dr. Hannes Voigt and Dr. Robert Haymes, who have been a pleasure to work with. I am also grateful for the many enlightening conversations with various members of this department and with Dr. Eric Priest and Dr. Vytenis Vasyliunas.

I thank my parents for their never-ending love and encouragement. I am also grateful for the support and patience of my fiancé and for the support of Marie, Maria, Umbe, and others who have continued to believe in me.

Thanks go to R. Schwenn and coworkers for making the Helios data available to the National Space Science Data Center and to Dr. Murray Dryer and Dr. Zdenka Smith of NOAA Space Environment Laboratory for providing the computer model used in this thesis. This work has been supported by NASA JOVE grants NAG8-146 and NAG8-818 and by Texas Advanced Technology Program grants 003604-01 and 003642-001.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>iii</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>iv</td>
</tr>
<tr>
<td>List of Tables and Figures</td>
<td>v</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>I. Physical Understanding of the Polytropic Equation</td>
<td>2</td>
</tr>
<tr>
<td>II. An Empirical Determination of the Polytropic Index for the Solar Wind</td>
<td>6</td>
</tr>
<tr>
<td>III. Application of the Polytropic Index to the MHD Energy Equation</td>
<td>25</td>
</tr>
<tr>
<td>IV. Justifications and Limitations of the Polytropic Approximation</td>
<td>35</td>
</tr>
<tr>
<td>V. Conclusions</td>
<td>40</td>
</tr>
<tr>
<td>References</td>
<td>44</td>
</tr>
</tbody>
</table>
TABLES AND FIGURES

Figure 1. Log-log plots of proton temperature versus radial distance by speed state. 9
Figure 2. Log-log plots of number density versus radial distance by speed state. 10
Table 1. Temperature power-law indices by speed state with various corrections. 11
Table 2. Velocity gradients for five velocity ranges in the solar wind. 13
Figure 3. Velocity probability distributions at perihelion and aphelion. 14
Table 3. Temperature and number density power-law indices and polytropic indices by speed state. 16
Figure 4. Scatterplots of log proton temperature versus velocity for uncorrected and corrected data sets. 17
Figure 5. Scatterplots of log number density versus velocity for uncorrected and corrected data sets. 18
Figure 6. Plot of temperature and density power-law indices and polytropic indices by speed range. 19
Table 4. Magnetic field power-law indices, plasma beta, and polytropic indices by speed state. 22
Table 5. Values for the constant C in the polytropic equation, by speed state, for two distances and for two forms of the pressure. 24
Figure 7. Cartoon of the region of interplanetary space considered by the computer model of Wu et al. [1983]. 26
Figure 8. Cartoon of the electron velocity distribution function in the solar wind. 37
Introduction

The continuous outflow of plasma from the Sun, known as the solar wind, has been a fascinating subject in space plasma physics for many years. Although countless observations have been performed and numerous fluid and magnetohydrodynamic models have been created, the many processes that occur in the solar wind are far from completely understood. Observations near the Sun and at Earth have indicated that the solar wind does not expand adiabatically in this region, implying that heating of the plasma occurs as it propagates through interplanetary space. The fact that the solar wind does not expand adiabatically has been known for quite some time, yet the heating mechanisms continue to elude space physicists.

An alternate approach to modeling a non-adiabatic fluid (or plasma) by using explicit heating terms in an energy equation is to utilize the polytropic approximation with a non-adiabatic exponent. This thesis focuses on such an approach. Chapter I presents a discussion on the assumptions and meaning of the polytropic approximation as well as the implications for the solar wind. Using data from the German spacecraft Helios 1, the polytropic index for the free-streaming solar wind is calculated in Chapter II. This empirically derived index provides a quantitative representation of the heating that exists in the solar wind. Next, Chapter III discusses the application of this polytropic index to a solar wind computer model. The computer model utilizes the MHD equations; thus, a comparison of the MHD energy equation and the polytropic approximation is performed. The value of the polytropic index is used to determine the amount of heat that is provided to the solar wind by a heat-conduction mechanism. Finally, Chapter IV provides the justifications and limitations of the polytropic approximation as applied to the solar wind. A summary of results and ideas for future work are presented in Chapter V.
Chapter I. Physical Understanding of the Polytropic Equation

The first step in applying a polytropic approximation to the expansion of a fluid is to understand the assumptions and thermodynamic properties of a polypeptide. This chapter lays the foundation needed to ensure that a meaningful discussion of the polytropic approximation as applied to solar wind propagation will exist. The derivation of the polytropic equation from the first law of thermodynamics along with the properties and assumptions of this result are presented. The application of the polytropic equation to the solar wind plasma is also discussed.

Chandrasekhar [1957] defines a quasi-static (reversible) process as one that occurs infinitely slowly so that at any given point in time, the system can be assumed to be in a state of equilibrium. For a quasi-static process, the first law of thermodynamics can be written as follows:

\[(I.1) \quad dQ = dU + pdV.\]

$Q$ represents the quantity of heat (per unit mass) added to or expelled from the system, $U$ is the internal energy per unit mass of the system, $p$ is pressure, and $V$ is the specific volume. For an ideal gas, $U$ is a function of only the temperature, $T$. Thus, \(dU = c_VdT\) where $c_V$ is defined to be the specific heat of the gas at constant volume. By definition, a polytropic process is a quasi-static change of state in which the specific heat, $c = c_VdT$, is held constant [Chandrasekhar, 1957]. Substituting $dU = c_VdT$ and $dQ = cdT$ into equation (I.1) and rearranging produces the following equation.

\[(I.2) \quad (c_V - c)dT + pdV = 0\]
For an ideal gas, \( pV = RT \) where \( R = (c_p - c_v) \) is the gas constant. Note that \( c_p \) is the specific heat at constant pressure. Applying these additional pieces of information to equation (I.2) and integrating, one obtains the polytropic equation.

\[
(I.3) \quad pV^\alpha = \text{constant}
\]

\( \alpha \) is called the polytropic index and can be written in terms of specific heats in the following manner:

\[
(I.4) \quad \alpha = \frac{c_p - c}{c_v - c}.
\]

Following the notation used by Parker [1963, 1965] and Priest [1982], the polytropic index is represented by the symbol \( \alpha \) rather than \( \gamma \) (as is more commonly seen) to emphasize the fact that a polytropic expansion need not be adiabatic. The symbol \( \gamma \) is specifically reserved for the ratio of specific heats: \( \gamma = c_p/c_v \) and is also related to the number of degrees of freedom, \( f \), of the fluid; viz., \( \gamma = \frac{f + 2}{f} \) [Farris et al., 1991]. Recall that \( V \) is the specific volume; i.e., \( V \) is the volume per unit mass. Consequently, \( V \) in equation (I.3) can be replaced by \( 1/\rho \) where \( \rho \) is the mass density of the fluid.

\[
(I.5) \quad \frac{p}{\rho^\alpha} = \text{constant}
\]

This is the form of the polytropic relation that will be used throughout this thesis. Consequently, a polytropic expansion or compression is defined as a process in which the pressure and density vary according to equation (I.5).
Now that the polytropic equation has been derived, it is beneficial to discuss the implications of such a relation. All polytropic processes are, in theory, reversible. Furthermore, the power index may have any non-negative value from zero to infinity [Van Nostrand's Scientific Encyclopedia, 1958]. An isobaric process is represented by \( \alpha = 0 \), and an isometric process has \( \alpha = \infty \). For an isothermal expansion, \( \alpha = 1 \) which implies that the heat capacity, \( c \), is infinite. Perhaps the most common polytropic approximation employed in models today is the adiabatic case: \( \alpha = \gamma = c_p/c_v \); i.e., \( c = 0 \). Since a reversible, adiabatic process has constant entropy, this is also referred to as the isentropic case. Entropy is a fundamental parameter that indicates the amount of "disorder" in the system. Specifically, the second law of thermodynamics for a reversible process can be expressed as \( dQ = TdS \) where \( S \) represents entropy [Chandrasekhar, 1957]. If \( \alpha \) is greater than 1, the temperature will decrease as the gas expands and increase as the gas is compressed. Also, if \( \alpha \) is less than \( \gamma \), heat must be supplied to the system in order for the fluid to expand [Van Nostrand's Scientific Encyclopedia, 1958]. As one can see, the value of the polytropic index of a given fluid can give an indication as to how the fluid will respond in certain situations.

The basic properties of polytropes can be applied to the propagation of the solar wind through interplanetary space. Observations of solar wind plasma have provided much information on some of the basic properties of this fluid as it propagates. First of all, the solar wind is observed to expand outward from the Sun with supersonic velocity beyond a few solar radii from the coronal base [e.g., Neugebauer and Snyder, 1966]. Temperature measurements near the Sun and at Earth indicate that the solar wind cools as it expands, but does not cool rapidly enough to be considered an adiabatic expansion [Neugebauer and Snyder, 1966]. From the discussion presented above, observations of the solar wind plasma imply a polytropic index with a value greater than one but less than
the adiabatic value. The solar wind plasma is regarded as having three degrees of freedom, implying $\gamma = 5/3$. Hence, $1 < \alpha < 5/3$ for the solar wind.

This range of values for the polytropic index of the solar wind indicates that although the wind is being heated, the temperature falls as it expands. The reason for this is that the rate at which heat is added to the plasma is less than the rate at which work is done by the plasma in the process of expanding [Van Nostrand's Scientific Encyclopedia, 1958]. So how much heat is being added to the solar wind? The amount by which $\alpha$ is less than the adiabatic value of $5/3$ can give an indication of the amount of heating that occurs [Parker, 1963; Belcher, 1971]. In fact, several researchers [Parker, 1965; Belcher, 1971; Siscoe and Finley, 1972; Goldstein and Jokipii, 1977; Priest, 1982; Habbal, 1985] have suggested that a non-adiabatic polytrope may roughly simulate the effects of heat conduction. A more detailed study of how a polytropic equation may represent heat conduction is presented in Chapter III.

One final point to be discussed concerning the polytropic index of the solar wind is the dependence of $\alpha$ on radial distance. Observations of the corona and near solar wind predict the polytropic index to be about 1.1 [Parker, 1963, 1965; Habbal, 1985]. Since the heating effects that exist in the solar wind are presumed to decrease with distance from the Sun, the polytropic index is expected to slowly increase and eventually reach the adiabatic value of $5/3$ [Parker, 1963]. The point at which the polytropic index reaches the adiabatic value is not yet known, but it is believed to be beyond the orbit of Earth. In fact, Whang et al. [1989], states that heating must be present out to at least 10 AU because the proton temperature there is observed to be greater than the value predicted by an adiabatic model. Furthermore, the assumption is made in this thesis that $\alpha$ varies with radius from the Sun slowly enough to consider it approximately constant for small separations in distance.
Chapter II. An Empirical Determination of the Polytropic Index for the Solar Wind

This chapter deals with determining an empirical value of the polytropic index for the solar wind using data from the Helios 1 spacecraft. Assuming the solar wind plasma parameters, namely temperature and number density, have power-law relations with radial distance, a simple equation for the polytropic index is derived from the polytropic equation. Next, a description of the data and the orbit of the spacecraft are presented. The assumption of power-law dependencies for proton temperature and number density are justified using the Helios 1 proton data, and the calculated power indices of these parameters for six solar wind states are provided. Corrections to the power indices are performed. These corrections address the effects of velocity gradients, non-uniformity in radial sampling, and heating due to stream-stream interactions. Using the corrected power indices for temperature and number density, the polytropic index for protons for the six solar wind states is calculated. Finally the results and implications of this calculation are discussed.

The polytropic relation is derived from the first law of thermodynamics in Chapter I, and the equation is rewritten below for reference.

\begin{equation}
\frac{P}{\rho^\alpha} = \text{constant} \tag{II.1}
\end{equation}

For an ideal, isotropic fluid, \( P = n k T \) and \( \rho = n m \) where \( n \) represents the number density. The polytropic equation now has the following form:

\begin{equation}
T n^{(1-\alpha)} = \text{constant} \tag{II.2}
\end{equation}
Taking the radial derivative of (II.2) and assuming $\alpha$ to be constant, the following expression is obtained: ($r$ is radial distance)

\[(II.3) \quad n \frac{dT}{dr} + (1 - \alpha)T \frac{dn}{dr} = 0.\]

The assumption is made that temperature and number density ($T$ and $n$) are power-law relations of the radial distance, $r$; viz.,

$$T \propto r^{-\delta}$$

$$n \propto r^{-\beta}$$

where $\delta$ is the power index for temperature and $\beta$ is the power index for number density. Substituting these forms for $T$ and $n$ in equation (II.3) and simplifying yields a simple equation for the polytropic index in terms of the power indices $\delta$ and $\beta$.

\[(II.4) \quad \alpha = 1 + \frac{\delta}{\beta}\]

To satisfy physical intuition of what is already known about the polytropic index, several limiting cases are addressed. An isothermal expansion implies that the power index for temperature, $\delta$, is zero. Note that $\alpha = 1$ in this case, conforming to the expected value discussed in Chapter I. For an isometric expansion, $\beta$ is set equal to zero, and equation (II.4) yields the predicted result of $\alpha = \infty$. Another limiting case to consider is spherically symmetric, adiabatic expansion. Here $\delta = 4/3$ and $\beta = 2$. Again, equation (II.4) gives the desired result of $\alpha = 5/3$. The fact that equation (II.4) produces the correct values in various limits lends confidence to the application of this equation to the solar wind.
Before calculating an empirical value of the polytropic index for the solar wind, a description of the data is required. The data for this calculation are for protons and are in the form of one-hour averages. These data were collected by the Helios 1 spacecraft [Rosenbauer et al., 1977] and were obtained from the National Space Science Data Center, made available by R. Schwenn and coworkers. Helios 1 has a highly eccentric orbit around the Sun with perihelion at 0.3 AU and aphelion at 1.0 AU. The data span the time from launch in December 1974 through the year 1980. This six-year period covers roughly one-half of a solar cycle. The data set, taken as a whole, may be used to predict properties of the "free-streaming" solar wind because the large amount of data causes an averaging effect. In other words, transient events that make up only a small portion of the data set will be "blended in" with the much more frequent events that represent the continuous, quiescent solar wind. With the assumption that the polytropic index is constant in the range from 0.3 to 1.0 AU, the statement can be made that the polytropic index of the free-streaming solar wind can be calculated from the Helios 1 data.

The state of the solar wind is most easily characterized by speed [Schwenn, 1983]. Therefore, the power indices for proton temperature and number density are determined by first sorting the data into 100-km/s speed bins and plotting against radial distance. Figure 1 [Freeman, 1988] shows a log-log plot of proton temperature versus radial distance for six of the seven speed ranges. The highest speed range (velocities greater than 800 km/s) contains only 79 of the 42,470 data points used in this analysis, and is found to be statistically unreliable throughout the calculations presented in this thesis. Figure 2 shows the corresponding log-log plots for number density. For each plasma parameter in each speed range, the points tend to lie along a straight line. A linear trend on a log-log plot indicates a power-law relation between the two variables, with the slope of the line corresponding to the power index. A linear regression analysis is performed for temperature and number density in each speed range. The slopes/power indices are
Figure 1. Scatterplots of log temperature versus log radial distance for six of the seven speed ranges considered. The slopes of the lines are determined by a linear regression analysis and correspond to the power indices shown in column three of Table 1. See Freeman [1988].
Figure 2. Scatter plots of log number density versus log radial distance for six of the seven speed ranges considered. The slopes of the lines correspond to the power indices for density and are calculated using a linear regression analysis.
calculated and displayed in the appropriate bins. Column three of Table 1 lists the power indices for the temperature data by speed state. Column two of this table lists the results calculated by _Schwenn_ [1983] for comparison. The remaining columns of Table 1 contain the temperature power indices that are calculated with various corrections to the data. The corrections made will now be discussed.

**Table 1.**

Temperature Power-law Index, $\delta$, With Various Corrections

<table>
<thead>
<tr>
<th>Speed Range (km/s)</th>
<th>Schwenn [1983]</th>
<th>Lopez and Freeman [1986]</th>
<th>With Distance Correction</th>
<th>With Sampling Density Correction</th>
<th>With Stream-Stream Interaction Correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 300</td>
<td>-1.33</td>
<td>-1.33 ±0.13</td>
<td>-0.93 ±0.14</td>
<td>-0.93 ±0.14</td>
<td>-0.94 ±0.23</td>
</tr>
<tr>
<td>300-400</td>
<td>-1.12</td>
<td>-1.22 ±0.087</td>
<td>-1.04 ±0.088</td>
<td>-1.03 ±0.090</td>
<td>-1.04 ±0.12</td>
</tr>
<tr>
<td>400-500</td>
<td>-0.8</td>
<td>-1.03 ±0.095</td>
<td>-0.97 ±0.090</td>
<td>-0.98 ±0.092</td>
<td>-0.98 ±0.12</td>
</tr>
<tr>
<td>500-600</td>
<td>-0.7</td>
<td>-0.83 ±0.099</td>
<td>-0.82 ±0.098</td>
<td>-0.82 ±0.094</td>
<td>-0.85 ±0.13</td>
</tr>
<tr>
<td>600-700</td>
<td>-0.6</td>
<td>-0.76 ±0.092</td>
<td>-0.77 ±0.093</td>
<td>-0.78 ±0.092</td>
<td>-0.83 ±0.12</td>
</tr>
<tr>
<td>700-800</td>
<td>-0.71</td>
<td>-0.81 ±0.17</td>
<td>-0.80 ±0.176</td>
<td>-0.84 ±0.15</td>
<td>-0.95 ±0.26</td>
</tr>
<tr>
<td>&gt; 800</td>
<td>--</td>
<td>0.89</td>
<td>0.76 ±0.76</td>
<td>0.87 ±0.65</td>
<td>0.30 ±1.03</td>
</tr>
</tbody>
</table>

As mentioned earlier, the three adjustments to the power indices for proton temperature and number density are made to account for the effects of non-zero velocity gradients, non-uniformity in radial sampling, and heating due to stream-stream...
interactions. The first of these refers to the fact that the velocity of the solar wind is not constant with radial distance but steadily increases [Arya and Freeman, 1991]. This property of the solar wind velocity introduces a bias when sorting the data by speed state. Therefore, it is prudent to normalize the velocity data to some common radial distance, chosen here to be at 1 AU, before sorting the temperature and number density data into speed bins.

Normalizing the velocity data to 1 AU requires the use of velocity gradients. Arya and Freeman [1991], determined the velocity gradients with the Helios 1 data by comparing the velocity probability distributions at perihelion (0.96-1.0 AU) and aphelion (0.3-0.4 AU). The distribution at each distance range is divided into five bins such that the same number of events exist in each bin. The basic assumption inherent in the following calculation is that the solar wind velocity maps from aphelion to perihelion in a proportional manner. In other words, the lowest 20% of velocity events at aphelion maps to the lowest 20% of events at perihelion. The second 20% at aphelion maps to the second 20% at perihelion, and so on. The median velocity for each bin at each radial distance is calculated, and the results are listed in columns two and three of Table 2 [Arya and Freeman, 1991]. The gradients are then determined by the shifts in the median velocities from aphelion to perihelion for each of the five ranges. See Figure 3. Columns four and five of Table 2 show the results of assuming 1) a radial power law of velocity on radius and 2) a linear slope relation, respectively.

Using the velocity gradients just described, the entire set of velocity data from Helios 1 is normalized to 1 AU. The process by which this is done is as follows. For a given data point, both the velocity and the radial distance are known. Using the velocity gradients, the boundaries of all five bins are calculated at the position of the data point. The velocity of this data point is compared with the values for the bin boundaries, and the appropriate bin for the given data point is determined. The velocity value of the data
point is extrapolated to 1 AU by using the velocity gradient corresponding to the appropriate velocity bin. This process is applied to every point in the data set. The proton temperature and number density data are then resorted by speed state, and linear regression analysis is repeated. The slopes/power indices for both plasma parameters for each speed state are again computed. Column four of Table 1 shows the results for temperature. One can see that this correction produces a significant difference in the values for the power indices in the three lowest speed ranges.

Table 2.

Velocity Gradients for Normalization of Velocities to 1 AU

<table>
<thead>
<tr>
<th>Velocity Range</th>
<th>Median Velocity at 0.3-0.4 AU (km/s)</th>
<th>Median Velocity at 0.96-1.0 AU (km/s)</th>
<th>Power-law Index, $\kappa$ ($V \propto R^\kappa$)</th>
<th>Linear Slope (km/s/AU)</th>
<th>% Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>274 ±7</td>
<td>316 ±4</td>
<td>0.139 ±0.028</td>
<td>66.7 ±12.8</td>
<td>14.2 %</td>
</tr>
<tr>
<td>2</td>
<td>326 ±10</td>
<td>360 ±5</td>
<td>0.096 ±0.033</td>
<td>54.0 ±17.7</td>
<td>9.9 %</td>
</tr>
<tr>
<td>3</td>
<td>375 ±12</td>
<td>399 ±5</td>
<td>0.063 ±0.034</td>
<td>38.1 ±20.6</td>
<td>6.2 %</td>
</tr>
<tr>
<td>4</td>
<td>436 ±16</td>
<td>455 ±7</td>
<td>0.041 ±0.038</td>
<td>30.2 ±27.7</td>
<td>4.3 %</td>
</tr>
<tr>
<td>5</td>
<td>558 ±16</td>
<td>555 ±110</td>
<td>-0.005 ±0.033</td>
<td>-4.8 ±29.9</td>
<td>-0.5 %</td>
</tr>
</tbody>
</table>

The next correction deals with the fact that, due to the elliptical orbit of the spacecraft, the number of points in the data set is different for different radial distances. At aphelion, Helios 1 moves slowly and orthogonal to the radial direction; consequently, the highest density of data points is near 1 AU. The next highest density of points is at
Figure 3. Probability distributions of velocity at perihelion and aphelion used to calculate velocity gradients from the Helios 1 data. Taken from Arya and Freeman [1991].
perihelion because the spacecraft is again moving orthogonal to the radial direction. In order to compensate for this effect, the data are divided into 0.1 AU bins, and the fraction of points in each bin (compared to the total number of points in the data set) is calculated. These fractions are used to weight each data point to eliminate any possible biasing by radial distance. With this correction to the temperature and number density data, the power indices for each speed state are recalculated. The results for temperature are given in column five of Table 1. As can easily be seen, this correction has a negligible effect on the power indices.

The final correction to the power indices concerns the possible heating effects of stream-stream interactions. Although the heating effects are predicted to be quite small [Burlaga and Ogilvie, 1973; Lopez and Freeman, 1986], this correction is performed to ensure that the values for the power indices are computed as accurately as possible. The method adopted is similar to that employed by Burlaga and Ogilvie [1973], and constitutes the removal of any data that is believed to lie near stream interaction regions. If two consecutive data points show a change in speed of 20 km/s or more (either increasing or decreasing), and the following four data points show a 10 km/s or more change (all with the same trend), then all five data points are discarded. Furthermore, the three previous and three subsequent data points are discarded from the data set. This process removes 1170 hourly averages from the data set. Again the power indices for the plasma parameters considered here are calculated. The results for temperature are shown in column six of Table 1 and in column two of Table 3. Column three of Table 3 shows the results for number density with all of these same corrections applied. This correction provides slight adjustment to the indices for the three highest speed ranges.
Table 3.

Temperature and Density Indices and the Polytropic Index
for Several Solar Wind Speed Ranges

<table>
<thead>
<tr>
<th>Speed Range (km/s)</th>
<th>Temperature Index, $\delta$</th>
<th>Density Index, $\beta$</th>
<th>Polytropic Index, $\alpha$ ($\alpha = 1 + \delta/\beta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 300</td>
<td>-0.94±0.23</td>
<td>-2.12±0.30</td>
<td>1.44±0.12</td>
</tr>
<tr>
<td>300-400</td>
<td>-1.04±0.12</td>
<td>-2.14±0.11</td>
<td>1.49±0.06</td>
</tr>
<tr>
<td>400-500</td>
<td>-0.98±0.12</td>
<td>-2.02±0.11</td>
<td>1.48±0.07</td>
</tr>
<tr>
<td>500-600</td>
<td>-0.85±0.13</td>
<td>-1.86±0.13</td>
<td>1.46±0.08</td>
</tr>
<tr>
<td>600-700</td>
<td>-0.83±0.12</td>
<td>-1.90±0.11</td>
<td>1.44±0.07</td>
</tr>
<tr>
<td>700-800</td>
<td>-0.95±0.26</td>
<td>-1.92±0.22</td>
<td>1.50±0.15</td>
</tr>
<tr>
<td>&gt; 800</td>
<td>0.30±1.03</td>
<td>-2.02±1.0</td>
<td>0.85±0.51</td>
</tr>
</tbody>
</table>

To lend support to the idea that the corrections described above actually improve the results, scatter plots of temperature and number density versus velocity are presented in Figures 4 and 5. The top panel of Figure 4 plots log temperature against velocity before any corrections to the data are performed. The lower panel plots the same parameters after all three corrections are made. The number of data points lying in the lowest velocity range (< 300 km/s) is greatly reduced, and the overall scatter is significantly diminished. Figure 5 shows the uncorrected data (top panel) for number density versus velocity, and the bottom figure shows the corresponding plot with all corrections applied. Again there is a marked reduction in the number of points in the lowest velocity range, and the scatter is somewhat curtailed. Less scatter in a data set
Figure 4. Scatter plots of log temperature versus velocity (a) with no corrections made to the data, and (b) with all corrections applied.
Figure 5. Scatter plots of log number density versus velocity (a) with no corrections made to the data, and (b) with all corrections applied.
implies a more accurate representation of Mother Nature's intentions; therefore, these scatter plots lend confidence to the use of the corrections in determining the power indices for temperature and number density.

With a fair amount of certainty in the values for the indices $\delta$ and $\beta$, a value for the polytropic index of the solar wind can now be determined. Inserting the values for the temperature indices, $\delta$, and the number density indices, $\beta$, into equation (II.4), the value for the polytropic index, $\alpha$, in each speed range is obtained. The results are presented in column four of Table 3. Figure 6 shows a plot of the temperature and number density indices, as well as the polytropic index, for all of the seven speed states.

![Graph of Power Indices for Various Speed Ranges](image)

**Figure 6.** Power Indices for Various Speed Ranges
As mentioned earlier, the highest speed state (> 800 km/s) has statistically unreliable results for all parameters due to the small number of data points in this range. Except for the highest speed range, the value for \( \alpha \) is the same, within statistical error. The average value is found to be \( 1.47 \pm 0.04 \). This result suggests that the heat flux for both the high- and low-speed streams is the same. Note that as the temperature index changes from one solar wind state to the next, the power index for density adjusts in such a manner as to keep \( \alpha \) roughly constant. This suggests that the difference in velocity of the high- and low-speed streams is due to a difference in expansion geometry, rather than a difference in heat addition.

Suppose that the magnetic pressure is included in the polytropic equation (II.1) so that \( p = \text{particle pressure} + \text{magnetic pressure} = nkT + \frac{B^2}{2\mu_0} \). The polytropic equation now has the form shown below. Recall \( \rho = nm \).

\[
(II.5) \quad \frac{nkT + \frac{B^2}{2\mu_0}}{n^\alpha} = \text{constant}
\]

Taking the radial derivative and simplifying, equation (II.5) becomes:

\[
(II.6) \quad \left( nk \frac{dT}{dr} + kT \frac{dn}{dr} + \frac{B}{\mu_0} \frac{dB}{dr} \right) - \left( nkT + \frac{B^2}{2\mu_0} \right) \frac{\alpha \ dn}{n \ dr} = 0.
\]

As with number density and temperature, let the magnetic field magnitude, \( B \), have a power-law relation with radial distance; viz.,

\[ B \propto r^{-\lambda} \]
Applying these forms for n, T, and B to equation (II.6) and rearranging, the following expression is obtained.

\[
\frac{\alpha \beta - 2\lambda}{\delta + \beta - \alpha \beta} = \frac{n k T}{B^2/2\mu_o} \equiv \text{plasma beta}
\]  

(II.7)

Recall that \(\alpha\) is the polytropic index, and \(\delta\), \(\beta\), and \(\lambda\) are the power indices for temperature, number density, and magnetic field magnitude, respectively. The values for \(\delta\) and \(\beta\) have already been determined and are shown in Table 3. The Helios magnetometer data are used to calculate the power index for the magnetic field \((\lambda)\) in the same manner, including all the same corrections, that the indices for the density and temperature are obtained from the plasma data. The results, by speed state, are shown in column two of Table 4. Column three of the same table shows the calculated values of the plasma beta, also determined using the data from the Helios spacecraft. These values are substituted into equation (II.7), and the new values for the polytropic index are determined for each speed range. The results are shown in column four of Table 4. As in the case discussed previously, except for the highest speed range, the polytropic index is independent of speed state, within statistical error. The average value for \(\alpha\) is \(1.60 \pm 0.06\). It should be pointed out that the data set used in this calculation is not identical to the data set used to calculate the polytropic index considering particle pressure only because the magnetometer and plasma experiments were not always operational during the same time periods.
Table 4.

Parameters Involving the Polytropic Index that Includes Magnetic Pressure

<table>
<thead>
<tr>
<th>Speed Range (km/sec)</th>
<th>Magnetic Field Index $\lambda$</th>
<th>Plasma Beta $nkT/(B^2/2\mu_0)$</th>
<th>Polytropic Index $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 300</td>
<td>-1.63 ± 0.20</td>
<td>0.28 ± 0.12</td>
<td>1.52 ± 0.24</td>
</tr>
<tr>
<td>300-400</td>
<td>-1.56 ± 0.10</td>
<td>0.36 ± 0.05</td>
<td>1.47 ± 0.09</td>
</tr>
<tr>
<td>400-500</td>
<td>-1.73 ± 0.09</td>
<td>0.51 ± 0.08</td>
<td>1.63 ± 0.09</td>
</tr>
<tr>
<td>500-600</td>
<td>-1.76 ± 0.09</td>
<td>0.71 ± 0.11</td>
<td>1.71 ± 0.11</td>
</tr>
<tr>
<td>600-700</td>
<td>-1.62 ± 0.09</td>
<td>0.73 ± 0.11</td>
<td>1.59 ± 0.09</td>
</tr>
<tr>
<td>700-800</td>
<td>-1.50 ± 0.17</td>
<td>0.70 ± 0.18</td>
<td>1.53 ± 0.18</td>
</tr>
<tr>
<td>&gt; 800</td>
<td>-0.82 ± 0.36</td>
<td>0.22 ± 0.24</td>
<td>0.82 ± 0.44</td>
</tr>
</tbody>
</table>

An attempt has been made to determine the polytropic index directly from log-log plots of pressure versus number density. The scatter is great, and the values of the polytropic index for the several speed ranges vary widely. After normalizing the temperature and density data to 1 AU using the corrected power indices, $\delta$ and $\beta$ (see Table 3), the data points cluster in all directions about a central point and an accurate determination of the polytropic index is greatly inhibited. This method is thought to be less accurate than the process described earlier. In fact, Zhu [1990] discusses this approach as applied to calculating the polytropic index for the Earth's plasma sheet. Zhu [1990] states that the scatter in log-log plots of pressure (or temperature) against number density is caused by variations of the specific entropy from one flux tube to another and that this scatter will affect the determination of the polytropic index. Consequently, the
process for calculating the polytropic index described earlier in this chapter is presumed to be more accurate than the "log p versus log n" approach for the case of the solar wind.

As already stated, the average value of the polytropic index for the solar wind is 1.47, neglecting the (unreliable) highest speed range. If the magnetic pressure is included, the average value of $\alpha$ is 1.60. As predicted in Chapter I, these values are less than the adiabatic value of 5/3 (1.66) but greater than the isothermal value of 1. This result conforms to the physical understanding of the polytropic equation and to in situ observations made of the solar wind. Specifically, heat is added to the plasma as it expands, yet the temperature declines with distance from the Sun. However, the empirical calculation of $\alpha$ moves one step beyond what is already known by providing a quantitative representation of the heating that exists in the solar wind. Furthermore, the heating is found to be independent of solar wind state, to within statistical accuracy.

Now that the polytropic index has been determined, only the "constant" in equation (II.1) remains unknown. Setting $p = mn$ where $m$ is the proton mass, equation (II.1) can be written as follows:

\[
\frac{D}{n^{\alpha}} = \text{constant} \equiv C. \tag{II.8}
\]

The constant $C$ can be determined at any specified point in the solar wind using the values of the pressure and number density corresponding to that point. Column two of Table 5 shows the quantities obtained for $C$, by speed state, at 0.3 AU. The values for the thermal pressure ($p = nkT$) and number density are determined using Helios 1 data obtained at 0.3 AU. The values for the pressure, number density, and polytropic index appropriate to each speed range are used to calculate $C$. Because the polytropic index for the highest speed range ($> 800 \text{ km/s}$) is so unreliable, the constant for this range cannot be accurately determined. Column three of the same table shows a similar calculation at
1.0 AU. Comparing these two columns reveals that, for a given speed range, C is constant with radial distance. However, the constant varies considerably from one speed state to another. Columns four and five of Table 5 display the values for C (at 0.3 and 1.0 AU, respectively) when magnetic pressure is included. \((p = nkT + B^2/2\mu_0)\). Again C is roughly constant with radial distance, as expected, yet the values change significantly with speed range.

**Table 5.**

Values for the Constant C in Equation (II.8) by Speed State

<table>
<thead>
<tr>
<th>Speed Range (km/s)</th>
<th>C at 0.3 AU ((p = nkT)) ((10^{-22} \text{ N-m}^3\alpha^{-2}))</th>
<th>C at 1.0 AU ((p = nkT)) ((10^{-22} \text{ N-m}^3\alpha^{-2}))</th>
<th>C at 0.3 AU ((p = nkT + B^2/2\mu_0)) ((10^{-22} \text{ N-m}^3\alpha^{-2}))</th>
<th>C at 1.0 AU ((p = nkT + B^2/2\mu_0)) ((10^{-22} \text{ N-m}^3\alpha^{-2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 300</td>
<td>1.95 ± 1.41 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>1.95 ± 0.72 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>2.52 ± 3.63 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>2.69 ± 1.88 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
</tr>
<tr>
<td>300-400</td>
<td>2.38 ± 0.79 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>2.38 ± 0.35 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>12.2 ± 5.99 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>12.2 ± 2.74 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
</tr>
<tr>
<td>400-500</td>
<td>6.17 ± 2.04 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>6.17 ± 0.87 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>1.15 ± 0.60 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>1.76 ± 0.35 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
</tr>
<tr>
<td>500-600</td>
<td>18.3 ± 6.14 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>18.3 ± 2.48 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>0.21 ± 0.13 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>0.90 ± 0.18 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
</tr>
<tr>
<td>600-700</td>
<td>38.7 ± 11.6 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>38.7 ± 4.38 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>5.28 ± 2.43 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>9.44 ± 1.41 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
</tr>
<tr>
<td>700-800</td>
<td>19.0 ± 11.8 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>19.0 ± 4.16 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>25.9 ± 20.0 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
<td>26.8 ± 7.00 ((10^{-22} \text{ N-m}^3\alpha^{-2}))</td>
</tr>
</tbody>
</table>

With the empirical determination of both constants \((\alpha \text{ and } C)\) in equation (II.8), the polytropic relation for the free-streaming solar wind is completely defined. The polytropic equation may be used to close a set of equations describing the solar wind plasma, rather than employing a more complicated energy equation.
Chapter III. Application of the Polytropic Index to the MHD Energy Equation

The next logical step is to apply the empirically derived polytropic index to a solar wind model. A computer model of solar wind propagation has been provided by Dr. Murray Dryer and Dr. Zdenka Smith of NOAA Space Environment Laboratory. The original intention was to substitute the empirical value for $\alpha$ into the code to replace the currently used adiabatic value of 5/3. However, since the computer model employs the magnetohydrodynamic (MHD) conservation relations, a comparison of the MHD energy equation and the polytropic approximation is warranted. This chapter is devoted to such a comparison. First, a brief description of the computer model is presented. Then the question is addressed as to whether or not it is possible, or even physically reasonable, to replace the ratio of specific heats, $\gamma$, in the MHD energy equation with the empirically determined polytropic index, $\alpha$.

The computer model supplied by Drs. Dryer and Smith is designed to study the effects of solar flares on the propagation of the solar wind, but the model can be used, with some adjustments, to simulate the steady-state solar wind. The computer model is a 2-1/2 dimensional code [Wu et al., 1983; Karlicky et al., 1991] that encompasses 180° of the solar equatorial plane from 18RS (0.08 AU) out to 1 AU. (See Figure 7.) The model is considered to be 2-1/2 dimensional because the calculations are performed in the equatorial plane, and all partial derivatives with respect to the polar angle (in spherical coordinates) are set equal to zero. However, the vector quantities, such as the solar wind velocity and magnetic field, have three components: radial, azimuthal, and polar. The region of space covered by this model is divided into a grid 108-radial steps by 61-azimuthal steps. Each radial grid step is 2RS and each azimuthal step is 3°. The equations used are the conservation of mass, momentum (three vector components), and energy as well as the induction equation (also three vector components) [Wu et al., 1983].
The solar wind is assumed to have infinite electrical conductivity and to flow adiabatically. The effects of the solar gravitational field are included in the momentum and energy equations. These equations are solved by a computer using the two-step Lax-Wendroff scheme [Karlicky et al., 1991] which is a second-order in time and space, finite difference method [Press et al., 1990; Nakagawa and Wellick, 1973]. The ambient, steady-state version of the code produces a one-dimensional flow and a magnetic field that is an Archimedean spiral, as predicted by Parker [1958]. Furthermore, the parameters at the inner boundary are adjusted to produce the desired results at 1 AU [Wu et al., 1983].

**Figure 7.** Cartoon of the grid set up in the computer model of Wu et al. [1983]. The region of space covered by the model is from $18R_s$ (0.08 AU) to $232R_s$ (1 AU) through $180^\circ$ of the ecliptic plane ($\theta = 0^\circ$). There are 108 grid spaces in the radial direction and 61 grid spaces in the azimuthal direction. This figure is not to scale.
As mentioned earlier, the computer model uses the MHD energy equation with adiabatic expansion assumed and the effects of the solar gravitational field included. Specifically, the following equation is employed:

\[
\frac{\partial}{\partial t} \left( \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left[ \frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \left( \frac{\vec{E} \times \vec{B}}{\mu_0} \right) \right] = \nabla \cdot \vec{F}_g.
\]

The usual symbols are used, namely: \( p \) is pressure; \( \rho \) is the mass density; \( v \) is the magnitude of the solar wind bulk flow velocity; \( B \) and \( E \) are the solar wind magnetic and electric fields, respectively; and \( F_g \) represents the gravitational force of the Sun. Note that \( \vec{E} = -\vec{v} \times \vec{B} \) because the solar wind is assumed to have infinite conductivity. Furthermore, no heating terms are included in equation (III.1), and \( \gamma = c_p/c_v \) is assumed to have a value of 5/3.

Some authors [Priest, 1982; Habbal, 1985] have suggested that, in a steady state, a non-adiabatic polytropic equation may roughly simulate the effects of heat conduction. The question presented here is as follows: Can \( \gamma \) in equation (III.1) be replaced by our empirically derived index, \( \alpha \) (Chapter II), to account for the effects of heat conduction? In other words, is the following relation true?

\[
\frac{\partial}{\partial t} \left( \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left[ \frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \left( \frac{\vec{E} \times \vec{B}}{\mu_0} \right) \right] = \nabla \cdot \vec{F} + \vec{\nabla} \cdot \vec{q}
\]

\[
= \frac{\partial}{\partial t} \left( \frac{p}{\alpha - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left[ \frac{\alpha p}{\alpha - 1} + \frac{\rho v^2}{2} \left( \frac{\vec{E} \times \vec{B}}{\mu_0} \right) \right] - \nabla \cdot \vec{F}
\]

Note that \( \vec{q} \) represents heat conduction. The first step in answering this question is to determine what assumptions need to be made in order for a polytropic equation to exactly represent an energy equation that includes the effects of heat conduction. To do this, a
"convenient form" of the energy equation will be derived from the first law of thermodynamics. The next step is to compare equation (III.1) with this "convenient form" of the energy equation so that the question presented in equation (III.2) can be easily addressed. Finally, this question is answered by using the assumptions made about the relation between the polytropic equation and the energy equation with heat conduction.

As mentioned in Chapter I, the first law of thermodynamics for a quasi-static (reversible) process can be written as follows:

(III.3) \[ dU + pdV = dQ. \]

Substituting \( 1/\rho \) for \( V \) and defining \( L = -\rho \frac{D(Q)}{Dt} \), equation (III.3) simplifies to the equation below.

(III.4) \[ \frac{\rho}{T} \frac{D(U)}{Dt} - \frac{p}{\rho} \frac{D(p)}{Dt} = -L \]

\( D/Dt \) is the convective derivative, and \( L \) represents all of the energy gained or lost by the system [Priest, 1982]. Chapter I also states that \( dU = c_v dT \). With the assumption that \( c_v \) is independent of temperature, \( U = c_v T \) [Chandrasekhar, 1957]. For an ideal fluid \( pV = RT \), or \( p = \rho RT \). Recall that \( R = c_p - c_v \), and \( \gamma = c_p/c_v \). Consequently, the internal energy per unit mass can be written as follows:

(III.5) \[ U = \frac{p}{(\gamma - 1)p}. \]
Substituting equation (III.5) into (III.4) and simplifying, the desired energy equation is obtained [Priest, 1982].

\[
\frac{p' T}{\gamma - 1} \frac{D}{Dt} \left( \frac{p}{p'} \right) = -L
\]

(III.6)

If \( L = 0 \), indicating that no heat is added to or expelled from the system, then the adiabatic polytropic equation is recovered.

The current goal is to determine under what circumstances the polytropic equation can simulate the effects of heat conduction. As already stated, \( L \) is the net effect of all the energy gained or lost by the system. Specifically,

\[
L = \vec{V} \cdot \vec{q} + L_r - \frac{j^2}{\sigma} - H
\]

(III.7)

where \( \vec{q} \) is the heat conduction flux, \( L_r \) represents radiation, \( j^2/\sigma \) is the ohmic dissipation, and \( H \) is all other forms of energy addition [Priest, 1982]. Suppose the only contribution to \( L \) in equation (III.6) is from the heat conduction flux, viz., \( L = \vec{V} \cdot \vec{q} \). Following suggestions made by Siscoe and Finley [1972], Priest [1982], and Habbal [1985], let \( \vec{q} = \beta p \vec{V} \) where \( \beta \) is the proportionality constant distinct from the density power index discussed earlier. This constant of proportionality can be related to the polytropic index [Siscoe and Finley, 1972]. Using the "chain rule" to separate the convective derivative on the left-hand side of equation (III.6) and the divergence of \( \vec{q} \) on the right-hand side and simplifying, the following expression is obtained.

\[
\frac{1}{\gamma - 1} \frac{D(p)}{Dt} - \frac{\gamma p}{\gamma - 1} \frac{D(p)}{Dt} = -\beta p \vec{V} \cdot \vec{v} - \beta \vec{v} \cdot \vec{V} p
\]

(III.8)
Note that \( \vec{v} \cdot \vec{v}_p = \frac{Dp}{Dt} - \frac{\partial p}{\partial t} \) and \(-\vec{v} \cdot \vec{v} = \frac{1}{\rho} \frac{Dp}{Dt} \) (continuity equation). Substituting these expressions into (III.8) and combining like terms yields

\[
\left[ \frac{1}{\gamma - 1} + \beta \right] \frac{Dp}{Dt} - p \left[ \frac{\gamma}{\gamma - 1} + \beta \right] \frac{1}{\rho} \frac{D\rho}{Dt} = \beta \frac{\partial p}{\partial t}.
\]

Recall that the polytropic index, \( \alpha = \frac{c_p - c}{c_v - c} \). If \( c = -\beta(c_p - c_v) \), then \( \frac{1}{\gamma - 1} + \beta = \frac{1}{\alpha - 1} \) and \( \frac{\gamma}{\gamma - 1} + \beta = \frac{\alpha}{\alpha - 1} \). With these relations and the fact that \( \frac{-\alpha}{\alpha - 1} \frac{p Dp}{Dt} = \frac{1}{\alpha - 1} \rho^\alpha \frac{D}{Dt} \left( \frac{1}{\rho^\alpha} \right) \), equation (III.9) becomes

\[
\frac{1}{\alpha - 1} \frac{Dp}{Dt} + \frac{1}{\alpha - 1} \rho^\alpha \frac{D}{Dt} \left( \frac{1}{\rho^\alpha} \right) = \beta \frac{\partial p}{\partial t}.
\]

Again using the "chain rule" to combine the terms on the left-hand side of (III.10), the desired form is obtained.

\[
\frac{\rho^\alpha \frac{D}{Dt} \left( \frac{p}{\rho^\alpha} \right)}{\alpha - 1} = \beta \frac{\partial p}{\partial t}.
\]

If \( \partial p/\partial t = 0 \), then the polytropic expression is exactly recovered \([Habbar, 1985]\). Therefore, with the assumptions \( \partial p/\partial t = 0 \) and \( \tilde{q} \propto \rho \vec{v} \), a polytropic equation is equivalent to an energy equation with heat conduction as the only external source of energy.

Before applying these assumptions to the question posed in equation (III.2), it is beneficial to compare the MHD energy equation to the energy equation presented in
(III.6). Specifically, the MHD energy equation will be derived from equation (III.6), which is shown below for reference.

\[ \frac{\rho^\gamma}{\gamma-1} \frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = -L = -\vec{\nabla} \cdot \vec{q} - L_v + \frac{j^2}{\sigma} + H \]

Suppose, again, that the only contribution to \( L \) is from heat conduction and that the fluid has infinite conductivity. Taking the scalar product of the current density, \( \vec{j} \), with Ohm's law for an ideal conducting fluid yields the expression below.

\[ \vec{j} \cdot \vec{E} = \vec{j} \cdot \left( -\vec{\nabla} \times \vec{B} \right) = \vec{v} \cdot \left( \vec{j} \times \vec{B} \right). \]

The mechanical energy equation is acquired by taking the dot product of the solar wind bulk flow velocity, \( \vec{v} \), with the momentum equation [Priest, 1982], viz.,

\[ \rho \frac{D}{Dt} \left( \frac{v^2}{2} \right) = -\vec{v} \cdot \vec{\nabla} p + \vec{v} \cdot \left( \vec{j} \times \vec{B} \right) + \vec{v} \cdot \vec{F}. \]

The assumption is made in equation (III.13) that the plasma is electrically neutral, and \( \vec{F} \) represents any forces acting on the fluid; i.e., gravity or viscosity. Combining equations (III.12) and (III.13), substituting the result into (III.6a), and rearranging the convective derivative leads to the equation presented below.

\[ \rho \left( \frac{D}{Dt} \left( \frac{p}{\rho} \right) - \frac{Dp}{Dt} \frac{\rho}{D} \right) = -\vec{v} \cdot \vec{q} + \vec{j} \cdot \vec{E} - \rho \frac{D}{Dt} \left( \frac{v^2}{2} \right) - \vec{v} \cdot \vec{\nabla} p + \vec{v} \cdot \vec{F} \]
Using the continuity equation, the second term on the left-hand side of the above equation is combined with the fourth term on the right-hand side to give \( \vec{\nabla} \cdot (\rho \vec{v}) \). In addition, the following relation for any scalar quantity, A, is derived from the continuity equation and a vector identity.

\[
(\text{III.15}) \quad \rho \frac{D}{Dt} \left( \frac{A}{\rho} \right) = \frac{\partial A}{\partial t} + \vec{\nabla} \cdot (\vec{v} A)
\]

Applying this relation to equation (\text{III.14}) and combining like terms yields the equation below.

\[
(\text{III.16}) \quad \frac{\partial}{\partial t} \left( \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} \right) + \vec{\nabla} \cdot \left[ \vec{v} \left( \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} \right) \right] = -\vec{\nabla} \cdot q - \vec{\nabla} \cdot (\rho \vec{v}) + \vec{\nabla} \cdot \vec{F} + \vec{J} \cdot \vec{E}
\]

The following expression is derived from Maxwell's equations:

\[
(\text{III.17}) \quad \vec{J} \cdot \vec{E} = -\frac{\partial}{\partial t} \left( \frac{B^2}{2 \mu_o} \right) - \vec{\nabla} \cdot \left( \frac{\vec{E} \times \vec{B}}{\mu_o} \right)
\]

Using equation (\text{III.17}) and again combining like terms, the MHD energy equation is finally obtained.

\[
(\text{III.18}) \quad \frac{\partial}{\partial t} \left( \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2 \mu_o} \right) + \vec{\nabla} \cdot \left[ \vec{v} \left( \frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \right) + \left( \frac{\vec{E} \times \vec{B}}{\mu_o} \right) \right] = -\vec{\nabla} \cdot q + \vec{\nabla} \cdot \vec{F}_s
\]

The above equation is identical to equation (\text{III.1}) with the exception of the heat conduction term, \( \vec{\nabla} \cdot \vec{q} \).
The computer model [Wu et al., 1983] described at the beginning of this chapter does not include the effects of heat conduction. But perhaps heat conduction can be accounted for by replacing $\gamma = 5/3$ in equation (III.1) with the empirically derived index, $\alpha$. In order to provide validation for this alteration of the energy equation, equation (III.2) must be shown to be a true relation.

\[
\begin{align*}
(III.2) \quad \frac{\partial}{\partial t} \left( \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_o} \right) + \bar{\nabla} \cdot \left[ \bar{\nabla} \left( \frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \right) + \frac{\bar{E} \times \bar{B}}{\mu_o} \right] &= -\bar{v} \cdot \bar{F} + \bar{v} \cdot \bar{q} \\
&= \frac{\partial}{\partial t} \left( \frac{p}{\alpha - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_o} \right) + \bar{\nabla} \cdot \left[ \bar{\nabla} \left( \frac{\alpha p}{\alpha - 1} + \frac{\rho v^2}{2} \right) + \frac{\bar{E} \times \bar{B}}{\mu_o} \right] - \bar{v} \cdot \bar{F}
\end{align*}
\]

With the assumptions necessary for a polytropic relation to simulate the effects of heat conduction, namely $\partial p/\partial t = 0$ and $\bar{q} = \beta p \bar{v}$, equation (III.2) reduces to the following form:

\[
(III.19) \quad \frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + \frac{B^2}{2\mu_o} \right) + \bar{\nabla} \cdot \left[ \bar{\nabla} \left( \frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \right) + \frac{\bar{E} \times \bar{B}}{\mu_o} \right] - \bar{v} \cdot \bar{F} + \bar{v} \cdot (\beta p \bar{v})
\]

\[
= \frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + \frac{B^2}{2\mu_o} \right) + \bar{\nabla} \cdot \left[ \bar{\nabla} \left( \frac{\alpha p}{\alpha - 1} + \frac{\rho v^2}{2} \right) + \frac{\bar{E} \times \bar{B}}{\mu_o} \right] - \bar{v} \cdot \bar{F}
\]

Cancellation of like terms leads to the simple relation shown below.

\[
(III.20) \quad \bar{\nabla} \cdot \left( \bar{v} \frac{\gamma p}{\gamma - 1} \right) + \bar{\nabla} \cdot (\beta p \bar{v}) = \bar{\nabla} \cdot \left( \bar{v} \frac{\alpha p}{\alpha - 1} \right)
\]

Integrating both sides of this equation and dividing through by $p \bar{v}$ yields a simple expression of constants.
(III.21) \[ \frac{\gamma}{\gamma - 1} + \beta = \frac{\alpha}{\alpha - 1} \]

The definitions of \( \beta \) presented below equation (III.9) verifies that (III.21) is a true statement. Consequently, the answer to the question posed in equation (III.2) is: Yes, heat conduction can be accounted for in the MHD energy equation by replacing the adiabatic index, \( \gamma \), with a non-adiabatic index, \( \alpha \), under the assumptions that \( \partial p / \partial t = 0 \) and \( \bar{q} \propto \rho \bar{v} \). Physical justifications for this form of the heat conduction flux and limitations to the result obtained here are discussed in the next chapter.
Chapter IV. Justifications and Limitations of the Polytropic Approximation

The assumption is made in the previous chapter that $\dot{q} \propto p\dot{v}$. Although this form was initially chosen for mathematical convenience, a physical justification is now presented for the case of the solar wind. Unfortunately, however, the reasoning behind this form of the heat flux applies to solar wind electrons. Furthermore, the polytropic index determined in Chapter II concerns only the protons. In addition, the MHD equations employed by the computer model [Wu et al., 1983], and consequently the derivations presented in Chapter III, are for a single fluid. The purpose of this chapter is to present the physical justification of the assumed form for the heat flux and to discuss the possibilities of applying this heat flux and the empirically derived polytropic index to the computer model.

When creating a mathematical model to describe the propagation of the solar wind through interplanetary space, some researchers [e.g., Noble and Scarf, 1963; Whang and Chang, 1965; Hartle and Sturrock, 1968; Leer and Axford, 1972] use the MHD energy equation with the standard Spitzer-Härm [Spitzer and Härm, 1953] heat conduction formula. However, this approach predicts electron temperatures and heat conduction fluxes at 1 AU that are considerably larger than the observed values [Hollweg, 1974, 1976]. In addition, the Spitzer-Härm conductivity formula is derived under the assumption that the plasma is collision-dominated [Hollweg, 1976]. The solar wind, on the other hand, is almost completely collisionless in the region from a few solar radii out to the orbit of Earth and beyond. Clearly a different form for the heat conduction flux, one that is applicable to collisionless fluids, needs to be considered.

assumes a radial solar magnetic field and radial flow of solar wind plasma, and considers
the motion of collisionless electrons. Since the electron thermal velocity is much greater
than the solar wind bulk-flow speed, an electrostatic potential is created far from the Sun.
Thus, many of the solar wind electrons are trapped between this potential barrier and a
magnetic mirror point close to the Sun. A few electrons have energies sufficient to
overcome the electrostatic barrier and escape. In situ observations made by Feldman et
al. [1974] at 1 AU show that the electron distribution can be divided into two
components: a core component that consists of trapped electrons, and a halo component.
The escaping electrons may correspond to those halo electrons with energies sufficient to
overcome the electrostatic barrier [Feldman et al., 1975]. These observations lend
support to the ideas presented by Perkins [1973] and Hollweg [1974, 1976].

Figure 8 shows a radial slice of the electron distribution function as described by
Perkins. Hollweg generalizes to the case of a spiral magnetic field due to solar rotation.
In this instance, Figure 8 represents a field-aligned slice through the distribution function.
Note that the trapped (core) electrons form a symmetric, truncated Maxwellian
distribution. The escaping (high-energy halo) electrons flow away from the Sun at very
high velocities and cause the average electron velocity to be offset from zero. In fact, the
electrostatic potential has just the right value so that the escaping electron flux is equal to
assumes that this offset of the peak of the electron distribution function from the mean
velocity is a "measure of the skewness" of the distribution function, and hence a measure
of the heat conduction flux. This assumption is based on the idea that the high-energy
halo electrons are responsible for both the average velocity offset and the heat conduction
flux. Consequently, the following relation for the collisionless heat conduction flux is
proposed [Hollweg 1976]:
(IV.1) \[ \tilde{q}_e = \frac{3}{2} n_e k T_e \bar{v}_e \eta \propto p_e \bar{v} . \]

The usual symbols are used with the subscript "e" referring to electrons, and \( \eta \) is an arbitrary parameter introduced to account for deviation of the core electron distribution from a Maxwellian due to Coulomb collisions or heat-conduction-driven instabilities [Hollweg, 1976]. Thus, equation (IV.1) is appropriate even if the core electrons are not Maxwellian or are not completely collisionless; the only requirement is that the escaping electrons be collisionless.

\[ f_e(v) \]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure8.png}
\caption{Cartoon of a radial (or magnetically-aligned) slice through the electron velocity distribution in the solar wind. Taken from Hollweg [1976].}
\end{figure}

One additional point to be made concerning equation (IV.1) is that this equation is obtained in a frame that is rotating with the Sun. In a non-rotating frame, the heat flux has the form shown below.
\[ \bar{q} = \eta \frac{3}{2} n_e k T_e \bar{v}_{\infty} \cos \psi \]

The symbol \( \psi \) represents the angle between the radial direction and the interplanetary magnetic field. Near the Sun, \( \psi = 0 \) and \( \cos \psi = 1 \). At the orbit of Earth, \( \psi = 45^\circ \) which implies that \( \cos \psi = 0.707 \). For the purposes of this analysis, \( \cos \psi \) is assumed to be equal to unity in the region of space under consideration (i.e., from 0.08 AU to 1.0 AU). With this assumption and the fact that the solar wind is collisionless in this region, the form of the heat flux in equation (IV.1) is applicable [Hollweg, 1976]. This point is addressed because the computer model of Wu et al. [1983] solves the MHD equations in a non-rotating frame.

The physically deduced form of the heat flux discussed above is applied in the analysis presented in Chapter III. As mentioned previously, however, this form for the heat conduction flux is only justified for electrons. It is doubtful that a similar argument can be made for protons. However, Habbal [1985] uses a single-fluid approach and defines the heat conduction flux to be \( \bar{q} = \beta p \bar{v} \) where \( p = n k (T_e + T_p) = 2nkT; n = n_e = n_p \). (See also Holzer and Leer [1980].) Although the physical justification for \( \bar{q} \propto p \bar{v} \) concerns only the electrons, perhaps the result can be applied to the case where electrons and protons are not treated as separate components of the plasma; i.e., in a single-fluid regime.

Another assumption that is made in the previous chapter is \( \partial p/\partial t = 0 \). This restriction is much more difficult to address because no physical reasoning exists to support it. Perhaps the contribution from this term in the MHD energy equation is small compared to all other terms. Certainly if one assumes the fluid to be in a steady-state, then \( \partial p/\partial t = 0 \) is a true statement. Although the assumption of steady-state reduces the
equations to a specific situation, this case may be adequate when describing the free-streaming solar wind.

The final point to be discussed concerns the empirically derived polytropic index for the solar wind. This index is derived for solar wind protons only, using Helios 1 proton data. Furthermore, conductive heating is assumed to be the dominant source for the non-adiabaticity of the empirical polytropic index; i.e., heating due to waves and other sources are assumed to be negligible. In order to apply our derived index to the results obtained in Chapter III, the assumption must be made that the index also applies to the solar wind electrons. An alternative approach would be to obtain electron data from the Helios spacecraft and perform one of two analyses. First, the data could be used to calculate an electron polytropic index in the same manner that the index for protons is determined, and interpolation between the two indices can be performed to find a value that represents the fluid as a single species. On the other hand, use of the electron temperature data, in addition to the proton data already obtained, can be employed to define the total particle pressure $p = nk(T_e + T_p)$ where $n = n_e = n_p$. Using this pressure, a polytropic index can be calculated for the fluid as a whole. Whatever the method, the end result must be a unique polytropic index representative of the plasma as a single fluid in order to apply the index to the MHD energy equation as described in Chapter III.
Chapter V. Conclusion

The purpose of this chapter is to reiterate the main ideas presented in this thesis. This work focuses on the properties of the polytropic relation and its application to the free-streaming solar wind in the region from 0.3 AU to the orbit of Earth. In addition to reviewing many of the concepts described above, a few ideas for applications of the results to the understanding of solar wind propagation are discussed.

The polytropic equation is derived from the first law of thermodynamics with the following two assumptions: 1) the expansion (or contraction) of the polytropic fluid is a reversible process, and 2) the specific heat, \( c = \frac{dQ}{dT} \), remains constant throughout the expansion/compression. The polytropic equation has the form shown below.

\[
(V.1) \quad \frac{P}{\rho^\alpha} = \text{constant}
\]

\[
(V.2) \quad \alpha = \frac{c_p - c}{c_v - c}
\]

The polytropic index, \( \alpha \), can have any non-negative value form 0 to \( \infty \). An index that is greater than one but less that the adiabatic value implies that heat is added to the fluid, yet the temperature declines during expansion. This is the situation observed in the solar wind. Consequently, the polytropic index for the free-streaming solar wind should have a value between one and the adiabatic value of 5/3. The amount by which \( \alpha \) is less than 5/3 gives an indication of the amount of heating that exists in the solar wind.

Next, an empirical value for the polytropic index of the free-streaming solar wind is calculated using Helios 1 proton data. The data are collected over a six-year period and cover the region from 0.3 AU to 1 AU. Assuming power-law relations for temperature
and number density on radial distance, a simple equation for the polytropic index, \( \alpha \), is derived.

\[
\alpha = 1 + \frac{\delta}{\beta}
\]

Recall that \( \delta \) and \( \beta \) are the temperature and density power indices, respectively. Log-log plots of temperature and number density versus radial distance are shown for various speed states. A linear regression analysis for each parameter in each speed range is performed, and the slope of the line corresponds to the power index. Various corrections to the data are applied to yield more accurate values for the temperature and density power indices. These corrections include effects of velocity gradients, non-uniformity in radial sampling by the spacecraft, and possible heating effects due to stream-stream interactions. Substituting the improved power indices, \( \delta \) and \( \beta \), into equation (V.3), the polytropic index for each speed range of the solar wind is calculated. Except for the anomalous speed range (\( v > 800 \) km/s), the polytropic index is found to be the same for all speed states, within statistical error, and has an average value of \( 1.47 \pm 0.04 \). However, if the effects of magnetic pressure are included, the polytropic index has an average value of \( 1.60 \pm 0.06 \), neglecting the highest speed range. As predicted, \( \alpha \) is between one and \( 5/3 \). What is surprising, however, is the fact that the polytropic index is roughly the same for all speed states, indicating that the heat addition is the same for both the high- and low-speed solar wind, within statistical accuracy. This determination of the polytropic index gives a quantitative representation of the heating that exists in the solar wind.

A computer model of solar wind propagation [Wu et al., 1983] has been provided by Dr. M. Dryer and Dr. Z. Smith of NOAA Space Environment Laboratory. This computer model employs the MHD equations, but no form of heat addition is
incorporated into the energy equation. Under certain conditions, the polytropic equation may represent energy conservation with the effects of particle heat conduction included. Consequently, the suggestion is made to replace the adiabatic index \( \gamma = 5/3 \) in the MHD energy equation in the computer model with the empirically derived polytropic index, \( \alpha \), to account for heat conduction. Through various mathematical manipulations and two assumptions, the result is obtained that such an alteration to the MHD energy equation is practicable. The two assumptions necessary are \( \partial p / \partial t = 0 \) and \( \bar{q} \propto \bar{p} \).

Although the mathematics involved in the analysis mentioned above is sound, the physical implications of the assumptions are also important. Physical justification for \( \bar{q} \propto \bar{p} \) is described for solar wind electrons, and this reasoning is expanded to the entire plasma for a single-fluid approach [Holzer and Leer, 1980; Habbal, 1985]. The assumption of steady-state of the fluid may be adequate to describe the free-streaming solar wind, in which case \( \partial p / \partial t = 0 \) is automatically satisfied. However, the polytropic index obtained in Chapter II concerns only the solar wind protons. In order to apply the polytropic index to the MHD energy equation in the computer model of Wu et al. [1983], a value for \( \alpha \) must be obtained that characterizes the entire plasma from a single-fluid viewpoint.

Many suggestions exist for future work. Some possibilities are discussed below.

- The empirically derived value of the polytropic index for the free-streaming solar wind can be applied to the computer model provided by Drs. Dryer and Smith. Predictions of shock arrival times for solar flares can be compared to the times predicted by the model before this adjustment is made. Furthermore, the "new" predictions can be compared to observational data.

- Obtaining electron data from the Helios spacecraft during the same time periods used in the above analysis will allow the determination of a polytropic index for
electrons. Consequently, a polytropic index that applies to the solar wind as a single fluid can be extrapolated and inserted in the computer model.

- Yet another possibility is to find a way to include the effects of Alfvén waves on the solar wind in the computer model. This can be done by adding a pressure-force term to the momentum equation and the corresponding energy term to the energy equation. However, if this effect is to be included in the computer model of Drs. Dryer and Smith, then a method must be found to supply and modify the wave quantities (specifically, $\delta B$) explicitly in the computer code.

These three approaches have an advantage in that the computer code needed to solve the equations already exists. Only minor alterations are required.

- A completely different approach is to create a new solar wind model. This model could use a two-fluid approach where the energy equation for protons can be represented by the polytropic equation with the empirically derived polytropic index. A polytropic equation for electrons may also be employed if data were available to calculate an empirical value for the polytropic index. If electron data are not available, then an alternative form of the energy equation for electrons, such as the equation proposed by Hollweg [1974, 1976], will need to be considered. This option has the advantage of treating the solar wind in a slightly more realistic manner by using a two-fluid approach. The disadvantage to this approach, however, is that the computer code necessary to solve the equations of the model will need to be created.

These are only a few of the many possibilities that exist. The solar wind contains many mysteries, and much work is yet to be done before all of the physical processes that exist in this plasma can be understood.
References


