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Numerical simulation of the roller cone drill bit lift-off phenomenon

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Rice University, 1994
RICE UNIVERSITY

Numerical Simulation of the Roller Cone Drill Bit Lift-Off Phenomenon

by

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Numerical Simulation of the Roller Cone Drill Bit Lift-Off Phenomenon

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Abstract

Rotary drilling with roller cone bit is often accompanied by wide fluctuations of the weight on bit (WOB). In certain cases the WOB periodically reduces to null. It has been postulated that in these cases the bit lifts off from the formation surface.

The present study models the interaction among the lift-off of the bit, the WOB variation, and the modulation of the amplitude of the lobes on the formation surface.

The analytical model is used to obtain numerical results for a specific drillstring. From these results it is inferred that the rotary speeds corresponding to the axial resonant frequencies of the system, determined from the functions of the driving point mobility and impedance at the bit, are critical regarding the wide fluctuations of the WOB. They may be associated with the sustenance of the lobes with large amplitude.
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Dedicated to my parents

Sri Achintya Kumar Sengupta
Smt. Dipa Sengupta
whose blessings and love
are always showered on me
from thousands of miles away.
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Chapter 1

Introduction

1.1 Observation

Downhole measurements of the forces acting on the drillstring and the motions it is subjected to, reveal some features which tend to remain undetected in the records obtained at the rig floor. One such feature is the wide and frequent fluctuations of the weight on bit (WOB) while drilling with roller cone bits.

The weight indicator suspended above the derrick floor displays the weight of the drillstring. After the downward push is applied the dial is reset so that the value displayed while drilling is equivalent to the WOB. Usually this value remains steady at a mean with some minor variations. This mean is close to the force of the downward push. Henceforth this force is referred to as the applied WOB to differentiate it from the fluctuating WOB which is the actual weight acting on the drill bit. Large bit displacement, corresponding to the fluctuating WOB, may of course be reflected in the higher kelly motion [41] and the whipping of the drilling lines [36].

But records from downhole measurement while drilling (MWD) instruments show wide fluctuations of the WOB about the mean, often the value even reducing to zero quite abruptly, as seen in Figs. 1.1 and 1.2 [34, 91, 97]. As a quantitative example, in a certain well of the experimental program of Esso Production Research Company [44], the mean of the WOB, measured at the surface at a rotary speed of 120 rpm, was 35,000 lb whereas the downhole recorder showed an average variation of ±10,000 lb, which is about ±28% off the mean. At times the load reached a peak value of 125,000 lb or 3.56 times the mean value. In other cases this phenomenon has occurred more or less periodically, where in every cycle the value has remained at zero for a considerable fraction of the time period. For such a clipped record it is predicted
Fig. 1. Weight versus time, Run 12 tests

Expanded as indicated

Weight fluctuations - 224
4 x Pcpm = 226
3 x rpm = 260
*Pump cycles per minute

Fig. 15 - Time histories of WOB and bending moment for Case 5 (linear coupling between axial force and bending vibration).

Figure 1.1: Records from refs. 34 and 91
Figure 1.2: Records from refs. 44 and 97
that the bit lifts off periodically from the formation surface corresponding to the null values of the WOB. The inference is supported by the fact that the simultaneous values of the torque on bit (TOB) are also zero. In the literature this lift-off is referred to as the “bit bounce”.

1.2 Postulation

In order to explain the variation of the WOB and the associated lift-off phenomenon, two primary causes were postulated by Cunningham (1968 [34]). They are the lobes in the formation surface, and the fluctuation in the drilling mud pressure.

1.2.1 Lobed Formation Profile

Samples of the core from hard rock formations drilled with roller cone bits reveal lobed surfaces with crests separated by troughs [36]. Usually the number of crests is equal to the number of cones in the bit, which is three for the commonly used tri-cone bit. In one reported case the number of lobes was six. As the bit rolls over the wavy surface under a high applied WOB, the displacement, velocity, and acceleration of the axial and torsional vibrations vary with frequencies which are multiples of the rotary speed (Ω). The spectral decompositions of these quantities produce peaks at values equal to three, six, nine or even twelve times the rotary speed. However, for most of the records of WOB the predominant frequency of fluctuation is found to be $3 \times \Omega$ [34, 91]. Thus the lobed formation surface is considered as a probable source of WOB fluctuation. The presence of the higher frequencies, as explained by Vandiver et al. (1990 [91]), is due to the nature of the cone movement which is periodic but not necessarily sinusoidal.
1.2.2 Mud Pressure Fluctuation

The pumps used in the circulation of the mud usually are of the reciprocating type. Commonly two cylinder pumps, each cylinder with double acting element, and three cylinder pumps, each cylinder with single acting element, are used. Depending on the type of the pump, the discharge flow rate fluctuates a certain number of times in each cycle of the rotating crank [63].

The pulsation in the discharge creates fluctuating forces at the locations where there is a change in the cross-sectional area of the conduit of the drillstring, as for example the pipe–collar junction. But perhaps the most important of all is the upward thrust created due to the ejection of the mud through the nozzles of the bit. There are many instances where the frequency of the WOB variation has been a multiple of the operating pump cycle, the factor being governed by the type of the pump [34]. Also the bit motion in turn influences the mud pressure in the part of the conduit that is close to the bit.

If there is tuning of the frequencies of the above excitations with the natural frequencies of axial vibration, then there can be considerable axial displacement resulting in the fluctuation of the WOB and the lifting off of the bit.

1.3 Aim

The development of the lobes in the formation surface, their influence on the axial vibrations of the drillstring and subsequently their sustenance or degeneration is an interactive process. It has been described as a feedback control mechanism [97]. The aim of the present study is to model this interaction and to identify the rotary speeds which cause the wide fluctuations of the WOB.

A preliminary study is made to model the effect of the mud pressure fluctuation by applying the principle of conservation of linear momentum to the mud ejecting from the nozzles of the bit. It is given in Appendix B.
Previous research efforts have focused on two of the three major problems in basic rotary drilling. They are stick-slip motion in torsional vibrations, and whirling in bending vibrations. The third one is the bit lift-off in axial vibrations. It still remains a hindrance against efficient drilling. The purpose of this study is to go beyond postulating and to present a qualitative account of the WOB fluctuation. Further, the study aims to predict the possibility of the bit lift-off, given the geometric properties of the drillstring, operating conditions, and the observed rate of penetration.
Chapter 2

Literature Review

2.1 Outline

The study of drillstring dynamics is a multifaceted problem. A broad review of the various aspects is necessary in order to investigate any particular phenomenon. The first section of this survey summarizes the experimental works conducted by the drilling companies. The next three address the principal kinds of vibration occurring in the drillstring. They are axial, torsional and bending. The stick-slip motion and whirling are grouped as special types of torsional and bending vibrations respectively. The work by Eronini (1978 [48]) encompasses all the above three vibrations. It is included as a separate section. In the dynamics of drillstring the bottom-hole assembly (BHA) plays the most important role. Models have been developed to find the static configuration or dynamic behavior of the BHA. Accordingly they are classified as static or dynamic models. The frequency response analysis is treated as a separate category. The rest of the literature is grouped under special topics. The basic components of rotary drilling are shown in Fig. 2.1.

2.2 Experimental Works

Finnie and Bailey (1960 [50]), reported random vibrations in the tests conducted by Shell Development Company. No well defined longitudinal modes were detected at the rotary table. Some torsional modes were observed. From the readings of the kelly acceleration at the rig of Socony Mobil Oil Co., it was found that the attenuation of axial waves generally increased with the frequency of vibration while using mud as the drilling fluid [5]. No such trend was found in case of water filled holes. As the
Basic Components of Rotary Drilling

Figure 2.1: Basic components of rotary drilling
attenuation was not large, a rotary drilling combined with induced vibratory motion was proposed.

The instrumented sub of the Esso Production Research Company [44, 64] detected the frequencies in the traces of WOB, TOB and bending forces, which are excited from the rotary speed, cone action, rock–bit-teeth interaction and pump pulsation. Normal variations of the WOB ranged within 25–50 % of the mean value. At times the WOB reached 3.5 times the average. But the most conspicuous feature was that it reduced to zero quite abruptly and almost at every cycle. The corresponding values of TOB were also zero. Cunningham (1968 [34]) inferred that the bit was lifting off from the formation during these null values. The coupling between the WOB and TOB was evident from the similarity of their traces. The bending forces also revealed the coupling with WOB. The rotation of the bit was found to be erratic. The torque applied at the rotary table was found to be much greater than the calculated torque which was necessary for the actual cutting of the rock. Shock subs were found to reduce the fluctuation levels. It was noted that the vibration records were influenced by the location of the instrumented sub.

The Advanced Drillstring Analysis and Measurement System (ADAMS) was developed by ARCO Oil and Gas Co. to detect the casing–drillstring interaction and general tubular failures along with the various vibrations [13, 14]. Two features detected are worth mentioning. First, the casing wear measured by calipers was significantly more severe than that had been theoretically predicted from the normal rubbing wear of tool joints in doglegs. The second is the side interaction between the drillpipe and the casing which caused the bit to stop intermittently. This is the stick-slip phenomenon, which is described in Sect. 2.4.1. Besaisow and Payne (1986 [15]) discussed the sources causing coupling between the lateral and axial vibrations. They are mass imbalance, bending of the pipe, misalignment in the drillstring elements, and the tilt in the bottom surface. Several of them were actually detected.
From the records of the Wire Telemetry System of NL Industries it was found that in the case of drilling with roller cutter bits, increase in the rate of penetration (ROP) for rotary speeds above a resonant frequency is not accompanied with a significant increase in the bending moment [97]. The better efficiency for rotary speeds beyond the resonance, as explained by Dareing (1983 [35]), is due to the degeneration of the lobes of the formation by the 180° out-of-phase axial vibrations of the bit. It was also noted that "the rotary speed that results in resonance is ... less than the drillstring's critical speed". Here of course the calculated critical speed corresponded to the first natural frequency of the drill collar. Thus, the detected resonance might have been due to the combined effect of the pipe and the collar acting as a system. Uncoupled fluctuations of the WOB and the bending moment were also noticed. Another interesting feature was that "once high bending occurred, it took a substantial decrease of the rotary speed in order to reduce the bending moment".

Field and laboratory drill-off tests were undertaken jointly by Sedco-Forex and Schlumberger Cambridge Research, where the string was left to drill with a certain initial WOB [23]. The rate of decay of the WOB in such a case is governed by the ROP and the drillstring compliance. It was found that the decay of the WOB was exponential with respect to time and that of the ROP was linear with WOB. In subsequent tests surface measurements of the WOB and torque were utilized to detect bit bearing failures and subsequent losses of cones.

The Down Hole Vibration Monitor system of Exploration Logging Inc. detected significant lateral vibrations during reaming and while drilling a casing shoe [30]. Vibrations were more severe when the bit was off-bottom than while it was drilling. In the tests, conducted by Shell Oil Co. and Anadrill/Schlumberger Inc., the magnitude of the axial acceleration, while drilling soft formation, was found to be much lower than that of the transverse acceleration. It was suspected that the casing had a detrimental effect during the transverse vibrations when the formation was of soft shale. The exposed formation would perhaps have a better cushioning effect.
2.3 Axial Vibration

Drillstring axial vibrations have been modeled by the one dimensional wave equation, subjected to suitable boundary conditions. Bailey and Finnie (1960 [7]) developed a trial-and-error method to calculate the natural frequencies. Starting from one end, it was based on satisfying the boundary condition at the other.

The mobility method was used by Paslay and Bogy (1963 [73]) to calculate the axial load at the bit due to a specified sinusoidal displacement at the same location. The values were consistent with the applied WOB.

In the work of Dareing and Livesay (1968 [41]) the amplitude of the periodic bit axial displacement was found by integrating the filtered record of the bit acceleration twice. The filtering was necessary to remove the high frequency components arising due to the bit-teeth-formation interaction. It was shown that due to the generation of standing waves, stress will be higher at the locations of the nodes of axial vibrations. As the natural frequencies become closely spaced with the increase in length of the drillstring, the tuning of the amplification factor of axial displacement becomes sensitive with respect to the frequency. The presence of shock subs lowers the natural frequencies and reduces the amplitudes of high frequency vibrations.

Ohanehi and Mitchell (1978 [71]) incorporated the drag forces acting on the outer and inner walls of the drillstring to study the energy loss due to the vibratory force in rotary-vibratory drilling. It was found that the net power delivered to the rock can vary widely according to the frequency of vibration. Vibration isolating subs reduce the loss and hence increase the efficiency regarding the transfer of the input power to the rock.

It was pointed out by Dareing (1984 [36]) that the length of the collar is an important parameter in determining the critical rotary speeds. The collars were described as “receivers and amplifiers of vibration energy from the drill bit”. For calculating the resonance frequencies, the collar was assumed to be fixed at the bit and free at the pipe–collar junction. The basis of the late argument is the low stiffness
offered by the pipe. In drilling with a roller cutter bit, the phase between the bit displacement and the fluctuating axial force acting on it, is an important criterion in the sustenance of the lobes. It was mentioned that rotary speeds greater than the critical values cause the displacement and the force to be out of phase, which subsequently leads to the degeneration of the lobes.

The importance of the inertia, damping and stiffness properties of the suspension system in determining the response at the top of the drillstring was noted by Aarrestad et al. (1987 [3]). A model of the coupling mechanism between the axial and torsional vibrations while drilling with roller cutter bits was developed using a polyharmonic formation profile [1]. The calculated axial acceleration and torque at the bit were compared with experimental results.

2.4 Torsional Vibration

Similar to the previous case, drillstring torsional vibrations can be modeled by the one dimensional wave equation. The excitations for the torsional oscillations are from the formation resistance at the bit, friction at the bit and the stabilizers, and friction during contact of the drillstring with the borehole wall due to bending. The large torsional flexibility of the pipe enhances the vibrations [85]. As large oscillations are observed while drilling vertical wells, even when the bit is off-bottom [43], it can be inferred that the friction at the bit and stabilizers is one of the primary excitations.

Halsey et al. (1987 [53]), included a correction factor in the wave number to take account of the tool joints. It was found experimentally that the torsional vibrations increase with increasing rotary speed.

Often the drillstring is idealized as a torsional pendulum, with the pipe as an inertialess spring and the BHA as a rigid mass. The use of this lumped parameter single degree of freedom (SDOF) model can be justified by the fact that the length of the BHA is much smaller than the total length of the pipes and hence the time taken by the torsional waves to travel through the BHA is much less compared to the time
taken to travel the overall length of the drillstring. Also, the mass moment of inertia and torsional rigidity of the BHA are quite high as compared to those of the pipe. One third of the mass moment of inertia of the pipe may be included in the effective mass moment of inertia of the BHA. The drawback of this simplification is that the downhole motion cannot be predicted accurately from the dynamic torque recorded at the top of the drillstring [59].

Narasimhan (1987 [70]) considered four friction models to study the stability of the torsional oscillations. The constants of the friction models were determined from the stability conditions in the state space. The perturbation method and the averaging technique with slowly varying parameter were also applied.

2.4.1 Stick-slip Motion

The typical stop-and-go motion arising in machine parts due to the friction–velocity relationship is referred to as the stick-slip motion. The torsional oscillations exhibit a similar kind of motion [13]. It is generated when the motive torque due to stiffness is less than the resisting torque due to static friction during a momentary stationarity of the BHA.

Dawson et al. (1987 [43]) found a cutoff value of the rotary speed above which the stick-slip motion ceases to occur. It was also suggested that the "oscillations could be reduced or eliminated by lowering the static friction coefficient between the pipe and the hole without necessarily changing the kinetic friction".

It was shown by Kyllingstad and Halsey (1987 [59]) that although the stick-slip period may be close to the natural period of the pendulum, the bit can be at rest for a considerable fraction of the vibration cycle. The stick-slip motion was found to become severe with increasing depth and inclination of the well. The value of the damping coefficient, above which the stick-slip motion does not arise, reduces with increasing rotary speed. The damping can be controlled by the use of lubricants in
the mud. The maximum angular velocity of the bit under stick-slip oscillations is always greater than twice the rotary speed. It can be even ten times as large [45].

Dufeyte and Henneuse (1991 [45]) described the stick-slip motion as a self-excited torsional vibration. The concept of the cutoff rotary speed was verified experimentally. The value was found to increase for higher WOB. Sometimes the stick-slip phenomenon was also observed in axial vibrations, “particularly during trips when the drillstring was pulled out in fits and starts”.

In the works of Lin and Wang (1991 [62]) an exponential friction–velocity model, similar to the one proposed by Bo and Pavelescu (1982 [19]) was used. It was suggested that the natural frequency of the pendulum can be altered, so that beyond a certain value the stick-slip motion will not occur. The beating phenomenon mentioned by Cunningham (1968 [34]), was explained through the stick-slip behavior.

2.5 Bending Vibration

Huang and Dareing (1968 [56]) developed a two dimensional model of the bending vibrations of the pipe, in which the pipe was considered simply supported at the top and guided at the junction with the collar. Curves were presented for determining the natural frequencies as a function of the specific weight of the pipe.

The fluid forces can be accounted for by considering a frequency dependent damping coefficient and an additional mass term, the later being commonly referred to as the added mass. Closed form expressions were given by Chen et al. (1976 [28]).

Although the pipe may be subjected to lateral vibrations, it is the bending of the BHA which affects the trajectory of the bit. It was found that the lateral acceleration of the BHA was greater than the longitudinal one by one order of magnitude [30]. Stabilizers are placed in between the collars in order to design building, dropping or holding assemblies. In a series of papers by Millheim (1978–1979 [65]) the behavior of these assemblies were analyzed. In the finite element program of Amoco Production Co. the bit is assumed to be fixed, as regards the boundary condition for lateral
vibrations of the BHA. Contrary to the common assumption of treating the stabilizers as pinned supports, it was found that for soft formations the cross-section of the borehole may become elliptical due to the reaming by the stabilizers. Besaisow et al. (1985 [13]) noted that the bending vibrations depend significantly on the moments and clearances at the stabilizers and the effective lengths of the stabilizer blades.

From the results of the harmonic analysis using MARC, a general purpose finite element program, Mitchell and Allen in 1985 showed that the lateral vibrations could have been the cause behind the failures of the BHA in three wells [68]. Their conclusion was based on the matching of the frequencies of the model with those in the field. In order to determine the critical rotary speeds, parts of the BHA between two nodes were idealized as pipes with fixed ends [4]. The stabilizers, reamers, tools of larger diameter and the bit were considered as nodes. But the correlation with the actual operating speeds at failure was not satisfactory. The effect of the mud was found to be significant in lowering the critical rotary speeds.

In the finite element model described by Burgess et al. (1987 [27]) an initial two dimensional static analysis is carried out before the dynamic phase to identify the points of contact of the BHA with the borehole wall. The linear coupling between the WOB and the bending of an initially curved collar was identified as a principal source of lateral vibrations [91].

2.5.1 Whirling

Parts of the drillstring may be subjected to whirling like those observed in rotating shafts. In case of drilling with a polycrystalline diamond compact (PDC) bit, the bit itself whirls around the overgauge borehole. Different bottom hole patterns due to the whirling of the bit were noticed in the experiments conducted by Amoco Production Co. [26]. They were classified as ribbed, star and basket weave. There are two extreme cases. One is a pattern with concentric circles, when the bit rotates without whirling. The other is a chaotic pattern, when the bit whirls without any well defined
trajectory. Thus, these patterns are in sharp contrast to the three lobes formed by a tri-cone roller cutter bit. A PDC bit under backward whirl, without any slippage, was described as "a pinion in a hole that acts as gear". The whirling sustains due to the centrifugal force which occurs simultaneously. The instantaneous center of rotation can be at any cutter, located either at the face or the side of the bit.

For a whirling bit there exists an unbalanced radial force. In order to reduce it, several changes in the design were recommended [94]. First is the use of reamer-stabilizer assembly just behind the slightly undergauge bit. Second is the use of chamfered cutters in a bit with flat profile. Introduction of a concave cone, with cutters, at the center of the bit face is another. But the use of low friction gauge cutters, where the resultant of the radial forces point towards a pad with a lower contact friction, was suggested as the best remedy. In a subsequent paper, a three dimensional model of the performance of the PDC bit was presented [12].

A roller cone bit can also exhibit whirling [94]. Vandiver et al. (1990 [91]) developed an expression relating the rotary speed \( (\Omega) \), whirl rate \( (\Omega_w) \) and the tangential component of the velocity of the collar while slipping along the borehole wall \( (v) \). It is

\[
v = (r_h - r_{oc})\Omega_w + r_{oc}\Omega \tag{2.1}
\]

where \( r_h \) is the radius of the borehole and \( r_{oc} \) is the outer radius of the collar. Several case studies of whirling were done by a continuous beam model. It was found that whirling, especially in the collar just above the bit, effects the bit tilt, side forces and subsequently the direction of drilling.

The lateral vibrations and whirling are effected by fluid forces, stabilizer clearance and contact with the borehole. In order to analyze the motion, the BHA can be viewed as a mechanical rotor, with the stabilizers as disks which are constrained to move within the borehole. Based on the theory of rotor dynamics, equations were developed for a whirling collar by Jansen (1990 [57]). From the solutions it was found that the rotating motion of the fluid, subjected to a variable flow rate due to
the collar eccentricity, aggravates the whirling. This self-excited, flow induced whirl was referred to as the oil whirl or oil whip. The whirling reduces with the borehole inclination. The added mass and the stabilizer clearance lower the critical rotary speed. When the stabilizer clearance exceeds the eccentricity of the center of mass of the collar, chaotic motion develops. In such a motion, the solution trajectory is highly sensitive to the initial conditions, and the motion is studied with the concept of attractors in the state space.

An analytical technique was developed by Paslay et al. (1992 [75]) to detect the forward whirling of the BHA from the vibration signature of axial force and torque measured at the top of the drillstring. It was shown that the shortening of the BHA during whirling gives rise to axial acceleration with frequencies equal to or twice the rotary speed.

2.6 A Comprehensive Model

A comprehensive rig-to-bit model of the drilling with roller cone bits was developed by Eronini (1978 [48, 49]). In the section of vibration transmission from the bit to the rig, a continuous model was selected for the inclined drillstring. The fluid force was expressed as a sum of the inertial and hydrodynamic components. Equivalent toothed wheels were substituted for the cones in the section of the bit model. The rock fracturing was described to occur in two phases which are mentioned in the forthcoming section on rock fracture. Eronini's study also dealt with the bit-teeth and bearing wear, effect of jet bits and bottom hole cleaning. Expressions of the ROP and the specific energy for unit volume of rock fractured were derived.

The results revealed that the bending vibrations have considerable effect on the WOB, TOB, and bit rotational speed. The hydrodynamic damping was found to be significant. The signals detected downhole and at the kelly were correlated with the formation properties and the ROP. As the correlation with specific energy was low, it was mentioned that "the specific energy may not be an adequate drilling performance
indicator ...”. The ROP was found to decrease beyond a certain WOB, the reason being the increase in TOB. It was also pointed out that there exists a particular value of the differential pressure across the rock surface for an optimum ROP.

2.7 Bottom-Hole Assembly Behavior

The study of the BHA dynamics is necessary due to the influence of the BHA in the generation of axial and torsional harmonics and bending vibrations which subsequently lead to fatigue and failure. It effects the ROP and the measurements of MWD instruments. Moreover, with the introduction of directional drilling, predictions of the ROP and the direction to be drilled require the development of BHA vibration models along with that of the bit-formation interaction.

Most of the BHA models use the finite element method. The application of finite element methods and the efficient use of computers were encouraged by Williams and Apostal (1986 [95]). Each model can be classified as either static or dynamic, depending on whether the calculation is for the static configuration or the dynamic behavior.

2.7.1 Static Models

The static version of the program developed by Amoco Production Co. [65] calculates the side forces at the stabilizers and the bit.

MARC is a general purpose finite element program developed by Control Data Corporation. It was used to analyze the BHA with straight beam elements for the collars and truss elements as gapping members in between the nodes and the borehole wall [67]. The use of gapping elements permits a variable elastic modulus of the nodal constraint in response to the magnitude of deflection. In a second set of analyses, refinement was done by using curved beam elements with consistent nonlinear elastic foundation. This reduced the number of elements required for the same accuracy.
In the three dimensional model used by Total-CFP, France, the effect of the portion of the drillstring above the point of tangency was neglected [90]. It was concluded that “from the standpoint of trajectory deviation control, assemblies with two stabilizers are ideal”. Firpo (1986 [51]) used the two dimensional beam-column equation and a transfer matrix approach to find the deflected configuration of the BHA. Another two dimensional program (BHAP) with continuous beam elements was used in the parametric studies of packed and slick assemblies [96].

The static version of the program ORPHEE, developed by Elf Aquitaine, France, is used for the preliminary prediction of the BHA trajectory. The equilibrium curvature, for which the side forces cancel out, is calculated by an iterative process. From the ROP characteristics of the formation the increase in hole size can be found out. The three dimensional model incorporates quasi-dynamic friction at the contact points. The adjective quasi-dynamic has been used because the model considers the sliding of the BHA along the borehole wall without any rolling. It was found that the prediction of the inclination side force was more accurate than that of the azimuthal side force [16, 17].

2.7.2 Dynamic Models

The GEODYN program developed by Sandia National Laboratories is a “...finite element computer program, ..., capable of simulating the three dimensional transient dynamic response of a polycrystalline diamond compact bit interacting with a non-uniform formation” [8, 9, 10, 89]. It uses two types of elements. The first is the 8-noded brick elements for the bit and the second is the three dimensional linear and non-linear beam elements for the collars and the stabilizers. The cutters of the bit and the stabilizer blades have been modeled as massless elements through which the BHA interacts with the formation. The formation surface profile and its properties are incorporated through a ‘discrete point’ model. The program can also handle non-
circular borehole cross-sections. It finds out rock penetration depth, TOB, side forces and directional tendencies of the BHA.

In the dynamic version of ORPHEE, the discrete and continuous contacts are modeled as a shock of zero duration and a series of low intensity shocks, respectively. In a new release, the program can include elements with knee, as for example bent sub, bent housing etc. [18].

The model developed at University of Tulsa incorporates rock–bit interaction for both the PDC and the roller cutter bits [25]. It was found that the fluctuations of the WOB and the TOB are higher in case of drilling rocks with greater shear strength. The radial clearance at a stabilizer was not evident as a predominant variable effecting the directional behavior of the BHA. The holes drilled with a PDC bit were found to have stronger curvatures in the horizontal plane as compared to those drilled with a roller cone bit under similar conditions. The inclination tendencies for both the bits were same.

2.7.2.1 Frequency Response Analyses

Apostal et al. (1990 [6]), cited the importance of a frequency response analysis for the BHA. The procedure developed was “based on a quasi-static nonlinear finite element solution formulated to accommodate intermittent contact/friction, finite displacement, buoyancy, and other diverse effects which characterized the interaction of a rotating BHA with the formation”. Of course the incapability of the method to include the transient behavior was pointed out.

In the work of Payne (1991 [76, 88]) the importance of the added mass term was evident from the frequency response of the Mises stress. The excitation was by a lateral force applied at the bit. It was found that the response was not significantly effected when the modeling of the stabilizer was changed from a pinned support to a contact along a length of two feet. But the response was highly sensitive to the lateral boundary condition at the bit.
2.8 Special Topics

2.8.1 Damping

The sources of damping, as listed by Aarrestad et al. (1987 [3]) are the losses in the suspension, bit-rock interaction and those in the drillstring as for example viscous drag, friction during contact with the borehole, radiation of acoustic waves and dissipation in shock subs. Calculations by Squire and Whitehouse (1979) showed that for high frequency vibrations, the loss due to radiation was greater than that from the viscous drag. An expression of the radiation damping coefficient ($\gamma_r$), in US customary units, was given as

$$\gamma_r = 0.5\pi^2 \nu^2 d_o^2 \rho_f \omega$$

(2.2)

where, $\nu$ is the Poisson’s ratio, $d_o$ the outer diameter of the drillstring element, $\rho_f$ the density of the drilling fluid and $\omega$ the vibration frequency. If the damping coefficient of the suspension is comparable with the characteristic impedance of the pipe, axial waves with frequencies close to the natural frequency of the suspension are heavily damped during reflection.

An equation for the damping ratios ($\zeta$) for axial and torsional vibrations was developed by curve fitting of experimental data [76, 88]. It is

$$\zeta = a\omega^b$$

where, $a = (5.23 \times 10^{-9})\rho_f^{0.75}$ and $b = 0.15 - 0.123\rho_f$.

(2.3)

2.8.2 Shock Absorbers

Subs with shock absorbers are incorporated in the BHA, usually just above the bit, to dampen the dynamic bit forces and vibrations. They lower the natural frequencies of the BHA. The spring rate of any absorber can be a function of the frequency and the amplitude of vibration induced in it. If elastomers are used, the damping coefficient can depend on the loading rate and the temperature. There is additional internal friction in the absorber which is predominantly governed by the laws of Coulomb
friction. A linear model may not suffice in such a case. The Rogaland Research Institute, Norway, recommended the use of structural type of damping, as opposed to viscous, in order to model the damping characteristics of the shock absorbers [86]. The performances of the shock absorbers can be preferential regarding frequency of vibration; this fact was pointed out by Dunayevsky et al. [46].

2.8.3 Stability

Bogy and Paslay (1964 [22]) identified two modes of drillpipe buckling. The first one is the classical buckling for vertical stems, in which the pipe remains in contact with the hole. For this the critical load has a specific value. The second one is the large deflection buckling for inclined stems, where the pipe lifts off from the hole wall during buckling. Here the critical load is a function of the initial imperfections. Energy method was applied to evaluate the critical load for a two dimensional model of the pipe, simply supported at both ends. In a follow-up discussion, Dareing proposed the calculation of the critical angle of inclination for a given WOB. Beyond that inclination the pipe stays in contact with the borehole without getting buckled. In a subsequent paper the critical load was calculated for a circular rod, placed within a cylindrical chamber and subjected to end torques [74].

The dynamic buckling of a long vertical pipe, subjected to end torques and placed within a viscous medium was studied by Huang and Dareing (1966 [55]). The three dimensional model did not include axial loads and radial constraints. It was claimed that the rotary speed does not influence the stability, and that the damping coefficient acts as a bifurcation parameter. That is there exists a critical value above which the system is stable. The stable region reduces with increasing specific weight. For a pipe, simply supported at the top and guided at the junction with the collar, the magnitude of the critical load was found to become independent of the pipe length as the depth of drilling was increased [56]. Curves were produced to determine the critical load as a function of the specific weight.
Dunayevsky et al. (1984 [46]) showed that the occurrence of precession or backward whirl in a drillstring is a case of resonance under parametric excitation, where the parameters are the frequency of WOB variation, $3\Omega$ for drilling with a tri-cone bit, and the ratio of the amplitude of the same to the applied WOB ($\epsilon$). The governing partial differential equations were transformed to Mathieu ordinary differential equations by the variational approach. Zones of stability were plotted in the $\Omega-\epsilon$ plane. Even if the WOB is less than the static critical buckling load, precession can occur if $\Omega$ and $\epsilon$ fall in the unstable region. In a sequel paper, the variations of a severity factor with respect to several variables like rotary speed, hole inclination, WOB, length of heavy weight pipe, depth of drilling and presence of shock subs, were presented [47]. It is interesting that “...the shock-sub does produce a mitigating effect on the vibrations in one range of rotary speed but, contrary to intuitive expectations, can be inefficient or even detrimental in some other range of rotary speeds”.

### 2.8.4 Self-excited Oscillations

The drillstring exhibits self-excited oscillations, wherein the cause of the motion depends on the motion itself.

Saito and Someya (1980 [81]) studied the vibrations of a rotating hollow shaft, partially filled with a viscous liquid. The unbalanced force applied by the moving liquid and the whirling of the shaft interact to sustain each other. Such a phenomenon may occur in the drillpipe when the flow rate is insufficient to fill the conduit.

Self-excited vibrations can develop while drilling with a drag bit. Due to the axial vibrations of the drillstring, the cutting edges of the bit generate a wavy surface, which in turn increases the vibration. The stability issue is governed by the number and distribution of the cutting blades and the velocity of the vibration [42, 98].
2.8.5 Random Vibration

Bogdanoff and Goldberg (1958 [20]) pointed out that in drilling it is not quite realistic to have a deterministic characterization of the load due to the inherent randomness in the interaction with the formation. Indeed random components are detected during experimental measurements. A continuous model described by the one dimensional wave equation, with uniform viscous damping, was selected. Each of the axial load and the TOB was considered as a sum of a constant part, equal to the mean, and a weakly stationary random component of zero mean. The later components of the axial load and the TOB were characterized by a jointly Gaussian random process. Probabilities of exceedence and the mean upcrossing rates beyond certain levels of the normal and shear stresses, at any location and inclination, were formulated in terms of the power spectral densities of the input loads. The failure criteria for the levels were determined from the maximum shear stress theory. It was suggested that the coupling constants at the kelly can be adjusted to reduce the variance of the shear stress. In a follow-up paper the forces acting along the length of the pipe were also included [21].

Skauge (1987 [85]) considered the bit displacement \( Z \) to be caused by the lobed formation, with an additional effect from the irregularities on the lobes. The amplitudes of them were weighted with random functions. The expression was

\[
Z = R\left(\frac{1}{T}\right)S\sin(n\theta) + R(f)S_{ir}
\]

(2.4)

where \( R\left(\frac{1}{T}\right) \) and \( R(f) \) are random functions with cut-off frequencies \( \frac{1}{T} \) and \( f \) respectively. \( T \) is the approximate time in which a multilobe pattern can generate and break down, \( f \) is an arbitrary multiple of the rotary speed, \( S \) is the amplitude of each lobe, \( n \) is a multiple depending on the number of lobes, \( \theta \) is the angular position of the bit and \( S_{ir} \) is the amplitude of the irregularities.
2.8.6 Rock–Bit Interaction

Sheppard and Lesage (1988 [83]) calculated the forces acting on the teeth of a tri-cone bit by considering an ideal force–penetration relation and satisfying the equilibrium of the vertical and lateral forces. Expressions of the teeth displacement were worked out from the geometry of the bit.

In developing a rock–bit interaction model for the PDC bit [25], the resultant force \(F\) on a single cutter was expressed as

\[
F = \frac{2A_c \alpha_s}{[1 - \sin(\beta_t - \alpha_r)]} \tag{2.5}
\]

where \(A_c\) is the cut area, \(\alpha_s\) the shear angle of rock, \(\beta_t\) the friction angle and \(\alpha_r\) the rake angle. \(A_c\) is a function of the cutting depth and cutter placement on the bit body. The unbalanced forces of the interaction have a sinusoidal pattern. The model for the roller-cone bit utilized the concept of rolling resistance of a toothed wheel.

2.8.7 Rock Fracture

In an experimental study of the rock breakage under rotary drilling, Somerton (1958 [87]) pointed out two principal modes of failure. The first is the impact–compressive failure for true rolling hard-rock bits and the second one is the torsional–shear failure due to the scraping and gouging action of a non-true rolling soft-rock bit. Due to the complex nature of the rock breakage in rotary drilling and due to the variability of strength characteristics of the rock, it was mentioned that the ultimate compressive strength is not an adequate measure of rock drillability. Outmans (1960 [72]) presented a model of the rock fracture under a vertically penetrating bit tooth.

It has been concluded by Simon (1963 [84]) that most of the work required to break a unit volume of rock is consumed in overcoming the strain energy of the portion of the rock underlying the fractured part. Various modes of energy consumptions were tabulated. Possible methods of increasing efficiency of energy consumption through generation and breakage of special rock sections were discussed.
The rock fracturing by a cone tooth occurs in two phases [48, 49]. At first there is indentation due to the incident axial stress waves which causes a minor fracturing. Then there is fragmentation due to the rotation of the cones.

2.8.8 Rate of Penetration

The ROP usually increases with increasing rotary speed ($N$) and applied WOB ($\bar{W}$). But as the rock fracturing is effected by the flow rate, mud column pressure, formation pressure, rock permeability, state of the bit wear etc. it is difficult to have an explicit expression for the ROP. The experiments of Murray and Cunningham (1955 [69]) with a two-cone bit showed that the drilling rate decreases with the increase in mud column pressure. Based on experimental data, an expression of the average ROP ($\bar{R}$) was proposed by Gatlin (1957 [52])

$$\bar{R} = c + d\bar{W}N^e,$$

(2.6)

where $c$, $d$ and $e$ are constants depending on the rock type and the size and design of the bit. Somerton (1958 [87]) developed a relation for drilling with two-cone bits. Outmans (1960 [72]) charted the effects of hydraulic horsepower, WOB, rotary speed and mud pressure on ROP.

Separate expressions for the cases of perfect and imperfect cleaning were presented by Warren (1987 [93]). They are

$$\bar{R} = \begin{cases} 
(aS^2d^3 + \frac{\gamma_l}{N})^{-1} & \text{for perfect cleaning} \\
(aS^2d^3 + \frac{b}{N} + \frac{c\gamma_l}{F_{jm}})^{-1} & \text{for imperfect cleaning}
\end{cases}$$

(2.7)

where $a$, $b$, $c$ are dimensionless constants, $S$ is the rock strength, $d$ is the bit diameter, $\gamma_l$ and $\mu$ are the specific gravity and plastic viscosity of the drilling fluid respectively and $F_{jm}$ is a modified impact force of the jet. It was pointed out that the ROP is effected by both the generation and subsequent removal of the cuttings.
2.8.9 Directional Drilling

The control of deviation of the borehole is the most important criterion in directional drilling. Moreover problems like crookedness, casing wear, key seating, sticking of the pipe and need to fish out detached components appear during deviation. There are several factors which affect the deviation, as for example BHA configuration, borehole trajectory, bit, formation and operating conditions. Bradley (1975 [24]) propounded the importances of the flexibilities of the components of the active length of the BHA, which govern the bit forces, and the rock and bit anisotropies.

In a series of papers Millheim (1978-1979 [65]) gave a thorough account of the variables involved in directional drilling. The methods for controlled deviation were categorized under three heads: mechanical, hydraulic, and the rest.

DIDRIL, developed by NL Industries, is a three dimensional program to predict the build/drop and walk tendencies of the BHA [78]. It was pointed out that the curvature of the borehole should be considered in the classification of the BHA that is, whether it is building, dropping, or holding. An expression for determining the direction of drilling was given as

\[ r_n \vec{E}_r = I_b I_r \vec{E}_f + (1 - I_b) \cos A_{af} \vec{E}_a + (1 - I_r) r_n \cos A_{rd} \vec{E}_d \]  \hspace{1cm} (2.8)

where \( r_n \) is the normalized drilling efficiency, \( \vec{E}_r \) the unit vector along the drilling direction, \( I_b \) and \( I_r \) the bit and rock anisotropy indices respectively, \( \vec{E}_f \) the unit vector along the resultant bit force on the formation, \( A_{af} \) the angle between the bit axis direction and the resultant bit force direction, \( \vec{E}_a \) the unit vector along the bit axis, \( A_{rd} \) the angle between the drilling direction and the formation normal and \( \vec{E}_d \) the unit vector normal to the formation bedding. \( \vec{E}_f \) and \( \vec{E}_a \) are determined by the program. Ho (1987 [54]) pointed out that a "2D analysis not only completely ignores the walk tendency of the BHA, it also incorrectly predicts the build/drop tendency".
2.8.10 Torque on Bit

The popular method of calculating the resisting torque, that acts on the bit while drilling, is based on the energy balance approach. The difficulty hampering this method involves the determination of the specific energy of rock breakage and the efficiency factor [92]. Moreover it was found by Eronini (1978 [48, 49]) that "the specific energy may not be an adequate drilling performance indicator ...".

For a roller cone bit, since the resistance to rolling arises from the contact of the teeth with the formation and as each cone has rows of teeth which are at offset to the rows of the other cones, Warren considered only one cone with contiguous rows of teeth in the derivation based on a force balance approach [92]. Next, that cone was viewed as an extension of a toothed wheel. The torque was calculated from the rolling resistance of the wheel. It was found to be proportional to the WOB. The effects of the cone offset and the presence of more than one basic cone angle were considered by incorporating two constants, an additive and a multiplicative, in the factor of the WOB. The final expression of the TOB was

\[ T_{TOB} = [C_1 + C_2 \sqrt{\frac{R}{Nd}}] Wd \]  \hspace{1cm} (2.9)

where \( C_1, C_2 \) are constants determined by curve fitting of experimental data.

Sheppard and Lesage (1988 [83]) used the force balance approach to derive the following expression of the torque acting on the \( i \)th cone

\[ T_i = \left( \frac{\mu}{\sigma} a(R_j, r_j) + c(R_j, r_j) \sqrt{\frac{R}{N}} \right) W_i \]  \hspace{1cm} (2.10)

where \( \mu \) and \( \sigma \) are parameters of idealized force–displacement models, \( a(R_j, r_j) \) and \( c(R_j, r_j) \) are coefficients as functions of the radial distances of the tip of the \( j \)th tooth from the bit \( R_j \) and the cone axis \( r_j \) respectively.
2.8.11 Drag

During tripping operations, resisting forces develop due to the well bore friction, tight hole conditions, key seats, cuttings buildup, differential sticking and sloughing in the hole. This force is known as drag. The treatment of Johanscik et al. (1984 [58]) was analogous to a belt friction problem, in which the weight of the drillstring was included but not the bending stiffness. The drag on an element of the drillstring due to well bore friction was expressed as the product of the normal load applied by the drillstring on the formation and the coefficient of friction. For holes with high curvature the bending stiffness of the drillstring becomes important. Dareing and Ahlers (1991 [40]) included the stiffness and expressed the equations in polar co-ordinates. In a subsequent paper Rocheleau and Dareing (1992 [80]) calculated the “vertical force required within a drillstring at the kick-off point to develop a given drilling force at the drill bit”.
Chapter 3

Modeling

3.1 Preliminary Remarks

The objective of this chapter is to develop a model accounting for the lift-off phenomenon of the tri-cone bit, the time-wise variation of the WOB and the evolution of the lobes. Several simplifications are introduced in this process. First, only a straight and vertical bore well is considered. In this regard it is noted that the presence of the lobes and the subsequent WOB variation, along with the periodic clipping of the trace, have been detected in all types of bore well, even in those which are intended to be straight and vertical. Hence, the curvature of the well bore is neglected and a one dimensional vibration model is considered. Second, coupling of the axial and bending vibrations is neglected, although it can lead to additional axial shortening of the drillstring [91]. This simplification is done because the intent of this study is to present a description of the bit lift-off phenomenon which is more qualitative than quantitative in nature. Finally torsional oscillations of the drillstring are considered.

3.2 Continuous Model

3.2.1 Axial Vibration

For the idealized one dimensional continuous system, the equation of motion for axial vibrations is given by the second order hyperbolic equation with suitable boundary conditions

\[
\rho_s A \frac{\partial^2 u}{\partial t^2} + c_n(\omega_n) \frac{\partial u}{\partial t} - AE \frac{\partial^2 u}{\partial x^2} = f(x, t),
\]  

(3.1)
where \( \rho_s \) is the density of the steel, \( A \) the cross-sectional area of the member, which is either the pipe or collar, \( u(x, t) \) the axial displacement of a section of the member, \( E \) the modulus of elasticity of steel and \( f(x, t) \) the distributed axial load. The time and axial location variables are denoted by \( t \) and \( x \) respectively. The frequency dependence of the damping coefficient \( c_a(\omega) \) arises due to the viscous drag forces of the drilling mud. The motion of the mud is of course influenced by the oscillations of the drillstring. The expression of the damping coefficient is given in Sect. 3.2.3. Aarrestad et al. (1986 [3]) showed that a frequency dependent damping term is more appropriate than a constant value in fitting theoretical results with experimental ones. It was of course pointed out by Eronini et al. (1982 [49]) that “the damping of the longitudinal vibrations of the drillstring could be predominantly hydrodynamic as opposed to viscous”. Friction due to internal damping from the straining and hysteresis of the material is assumed to be negligible. Damping forces caused by wave radiation due to joint offset and diameter variation from the “breathing” effect have been found to be insignificant [85]. Stabilizers are not included, and the drillstring is considered to be undergoage throughout. Hence, there is no contact of the drillstring with the borehole wall and the corresponding rubbing. To justify the exclusion of the various forms of damping other than the viscous type, Dareing and Livesay (1968 [41]) mentioned that “…viscous friction can be substituted for other types of friction with reasonable accuracy, provided the decay per vibration cycle is nearly the same in both cases”.

When the equation of axial vibrations is applied to a composite system like a drillstring, the solution has to satisfy the displacement and force conditions at the boundaries, as well as at the junctions of the subsystems. The determination of the natural frequencies and mode shapes and hence the application of the modal superposition technique become tedious. Bailey and Finnie (1960 [7]) applied a trial and error procedure to determine the frequencies. Starting from one end, it sought to satisfy the boundary conditions at the other end.
The mobility analysis, applied by Paslay and Bogy (1963 [73]), is convenient for determining the steady state response. As the present study involves transient motions, this method is not found to be suitable for finding solutions in the time domain. Of course it is used to compare the natural frequencies, calculated from the eigenvalue analysis of the discretized model, with those determined by using the continuous system.

3.2.1.1 Boundary Conditions

Regarding the boundary conditions, which reflect the physics of the problem, there have been various opinions. Before downhole MWD instruments were introduced, the lifting off of the bit from the formation surface was not suspected. Hence, the bit was assumed to have no axial displacement [7]. Later on with the detection of the lobing pattern, a sinusoidal displacement of the bit was assumed [3, 29, 85, 86]. But many authors, based on the periodicity of the WOB variation, idealized a sinusoidal force to be acting on the bit [13, 20, 73]. The selection of either of them depends on the problem to be solved. The rock bit interface has been modeled by a spring and a damper to simulate the stiffness and dissipation effects of the rock [29, 71]. The characteristic values of the elements were calculated from the force-displacement hysteresis loop. They were found to depend primarily on the WOB. Clayer et al. (1990 [29]) introduced a displacement source in between the bit and the rock to simulate the axial movement of a tri-cone bit. They had concluded that the boundary condition, instead of remaining a particular kind, evolves with time depending on the frequency of vibration.

In the present study the boundary condition at the bit is considered to alternate between the two kinds; one in which the external force on the bit is specified, and the other in which the displacement of the bit is specified. When the bit is off-bottom the boundary condition is of the first kind where the specified value of the force is zero. The bit is then a free end. When the bit is moving in contact with the formation, it
is of the second kind, the displacement being dictated by the profile of the formation. Thus the mathematical representation of the formation is a critical element of the present analysis. It is addressed in Sect. 4.3.1. The alternating nature of the boundary condition at the bit couples the axial and torsional vibrations. Further, it necessitates a numerical solution scheme. It has been noted that the tri-lobe formation and the corresponding lift-off phenomenon occur mostly for hard formation. Hence, in the present study the surface is assumed to be infinitely stiff. For soft formations usually PDC bits are preferred to roller cone bits.

The boundary condition at the top has been taken as either fixed with zero displacement [13, 73] or mixed in which the force and displacement are governed by the inertia, damping and stiffness characteristics of the suspension. In the latter case the inertia of the travelling block, swivel and kelly and the elastic and damping effects of the drilling lines and derrick are modeled as a lumped SDOF system [3, 7, 41, 42, 49]. Clayer et al. (1990 [29]) separated the swivel and included it as a continuous member. Aarrestad and Kyllingstad (1989 [2]) noted that a single lumped mass–spring system idealization may not be adequate. They treated the effect of the suspension as a combination of two subsystems, the derrick being the first one and the rest in the second. The drilling lines act like a taut string subjected to parametric excitations. In case of matching of their natural frequencies with the frequency of drillstring axial vibrations they may undergo resonance and subsequently feed back energy to the drillstring. The derrick may also be subjected to resonant vibrations.

It should be noted, that the actual nature of the fixity may be preferential according to the frequency. Based on the damping characteristics of the waves reflected from the top of the drillstring, Aarrestad et al. (1986 [3]) stated that the "suspension acts as a nearly fixed point for frequencies far away from (its) natural frequency". In some cases the stiffness of the suspension has been neglected to consider the top of the drillstring to be a free end [71]. Paslay and Bogy (1963 [73])
pointed out that the influence of the nature of the boundary condition on the BHA motion is not important.

In the present study the suspension is idealized as a mass-spring-dashpot system as shown in Fig. 3.1. Due to lack of pertinent data, the characteristic values are selected from Ref. [29].

3.2.2 Torsional Vibration

The equation of motion for torsional vibrations is similar to the previous case. It can be written as

\[ \rho_s J \frac{\partial^2 \theta}{\partial t^2} + c_t(\omega_t) \frac{\partial \theta}{\partial t} - JG \frac{\partial^2 \theta}{\partial x^2} = \tau(x, t), \]

where, \( J \) is the polar moment of inertia of the cross-section of the member, \( \theta(x, t) \) denotes the absolute angular displacement of a section of the member, \( G \) is the shear modulus of steel and \( \tau(x, t) \) is the distributed torsional load. The damping considered is due to the viscous drag only. Note that in actual cases the friction losses at the bit, stabilizers, and other points of contact can be quite large [34]. The mobility method is used to compare the natural frequencies of the discretized model with those determined by using the continuous system.

3.2.2.1 Boundary Conditions

As far as the boundary conditions for torsional vibrations are concerned, the force on the bit and the displacement at the top are specified. The latter is governed by the base motion at the kelly bushing [50, 53, 59]. In the present study the boundary conditions are similar, as shown in Fig. 3.1. The force at the bottom, the TOB, is a function of the WOB, as given in Eq. 4.15.

In some cases the torque due to friction has been incorporated as a damping term [43, 62]. Clayer et al. (1990 [29]) included the inertia and stiffness effects of the suspension.
Axial vibration

\[ K \quad C \quad M \]

\[ x, \ u(x,t), \ f(x,t) \]

Torsional vibration

\[ \theta_k \]

\[ x, \ \theta(x,t), \ \tau(x,t) \]

Figure 3.1: Simplified models for axial and torsional vibrations
3.2.3 Generalized Formulation

The equations for both the axial and torsional vibrations are of the form

\[
C_M \frac{\partial^2 X}{\partial t^2} + C_C(\omega) \frac{\partial X}{\partial t} - C_K \frac{\partial^2 X}{\partial z^2} = p(x, t),
\]

(3.3)

where, \( C_M \) is the generalized inertia coefficient substituted for \( \rho_s A \) or \( \rho_s J \), \( C_C(\omega) \) the generalized damping coefficient for \( c_a(\omega_a) \) or \( c_t(\omega_t) \), \( C_K \) the generalized stiffness coefficient for \( AE \) or \( JG \), \( X(x, t) \) the generalized displacement for \( u(x, t) \) or \( \theta(x, t) \) and \( p(x, t) \) the generalized load for \( f(x, t) \) or \( \tau(x, t) \). Considering the mud to be an incompressible Newtonian viscous fluid and that the damping depends only on the dominant frequency of vibration \( \omega \), the coefficient can be expressed as

\[
C_C(\omega) = \mu \pi \left[ r_o \Re \left\{ (J_{n-1}(zr_o) - J_{n+1}(zr_o)) Y_n(zr_h) - (Y_{n-1}(zr_o) - Y_{n+1}(zr_o)) J_n(zr_h) \right\} \frac{zf_2}{\Delta} + r_i \Re \left\{ (J_{n-1}(zr_i) - J_{n+1}(zr_i)) \frac{zf_1}{J_n(zr_i)} \right\} \right],
\]

(3.4)

where \( \mu \) and \( \rho_f \) are the coefficient of viscosity and the density of the drilling mud respectively, \( r_i/o \) the inner/outer radius of the member, \( r_h \) the radius of the borehole and \( \Re \) denotes the real part. \( J_n(zr) \) and \( Y_n(zr) \) are Bessel functions of order \( n \) and of the first and second kind respectively with argument \( zr \),

\[
\Delta = J_n(zr_o)Y_n(zr_h) - J_n(zr_h)Y_n(zr_o) \quad \text{and} \quad z^2 = -\left( \frac{\omega \rho_f}{\mu} \right)i.
\]

For axial vibrations \( n = 0 \) and \( f_1, f_2 \) are equal to 1. For torsional vibrations \( n = 1 \), \( f_1 = r_i \) and \( f_2 = r_o \). To reduce the requisite computational time, the first term in the expression, that is the damping at the outer surface, is not included in the numerical implementation. The plots and the derivation of the coefficients are given in Fig. 3.2 and in Appendix A respectively.
Viscous damping coefficients

$\omega [\text{rad/s}]$

Axial vibration

$\omega [\text{rad/s}]$

Torsional vibration

Figure 3.2: Plot of viscous damping coefficients
Ohanehi and Mitchell (1978 [71]) adopted a simpler expression for the damping coefficient used for axial vibrations. It is applicable for a rod of negligible diameter vibrating in a fluid of infinite domain.

### 3.3 Discrete Model

The pipes and collars can be discretized into a finite number of elements and the set of equations can then be developed by the dynamic matrix approach [77]. In matrix notation this set of equations of motion is

\[
M \ddot{\mathbf{X}} + C(\omega) \dot{\mathbf{X}} + K \mathbf{X} = \mathbf{P},
\]

where the overhead dots denote differentiation with respect to time. \(M\), \(C\) and \(K\) are the global mass, damping and stiffness matrices respectively. They are the generalized forms of the corresponding matrices for the axial and torsional vibrations. \(\mathbf{X}\) is the generalized displacement vector equal to \(\mathbf{U}\) or \(\mathbf{Q}\) and \(\mathbf{P}\) is the generalized load vector equal to \(\mathbf{F}\) or \(\mathbf{Q}\). The consistent mass, damping and stiffness matrices of the elements are given as

\[
M_e = C_M \frac{L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad C_e = C_C(\omega) \frac{L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad K_e = C_K \frac{1}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},
\]

(3.6)

where \(L_e\) is the length of the element and the subscript \(e\) denotes the quantities for the elements. The elements of the matrices are calculated from the relations

\[
m_{ij} = \int_0^{L_e} C_M N_i(x) N_j(x) \, dx
\]

(3.7)

\[
c_{ij} = \int_0^{L_e} C_C(\omega) N_i(x) N_j(x) \, dx
\]

(3.8)

\[
k_{ij} = \int_0^{L_e} C_K N_i'(x) N_j'(x) \, dx.
\]

(3.9)

The overhead prime denotes differentiation with respect to \(x\). The shape functions are

\[
N_1(x) = 1 - \frac{x}{L_e} \quad \text{and} \quad N_2(x) = \frac{x}{L_e}, \quad x \in [0, L_e].
\]

(3.10)
For the axial vibrations, the global $M$, $C$ and $K$ matrices include the lumped mass-spring-dashpot system at the top. The top of the suspension, that is the crown block, is assumed to be fixed and the force generated at that location is not within the domain of this study. Hence it has not been treated as another degree of freedom.

### 3.3.1 Treatment of Boundary Conditions

#### 3.3.1.1 Axial Vibration

Two cases are considered to treat the alternating boundary condition for the discrete model.

**Case of no contact**

If the bit loses contact with the formation due to lift-off, the vectors $\mathbf{X}$, $\dot{\mathbf{X}}$ and $\ddot{\mathbf{X}}$ or specifically $\mathbf{U}$, $\dot{\mathbf{U}}$ and $\ddot{\mathbf{U}}$ can be calculated from the free vibration equations, which is Eq. 3.5 with $P = F = 0$. The corresponding value of WOB is zero.

**Case of contact**

If the bit moves over the formation profile, the displacement, velocity and acceleration of the bit are known ($U_k = U_b$) and the corresponding element of $P$, which is the WOR, is unknown ($F_u = W$). For the rest of the nodes, the displacement, velocity and acceleration are unknown and the forces are known ($F_k = 0$). The set of equations can be partitioned as [11]

\[
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
\ddot{U}_u \\
\ddot{U}_k
\end{bmatrix}
+
\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{U}_u \\
\dot{U}_k
\end{bmatrix}
+
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
U_u \\
U_k
\end{bmatrix}
=
\begin{bmatrix}
0 \\
F_u
\end{bmatrix},
\]

(3.11)

where, $M_{ij}$, $C_{ij}$ and $K_{ij}$ are the submatrices of $M$, $C$ and $K$ respectively. The subscripts $u$ and $k$ stand for the unknown and known parts of the vectors respectively. The subscript $a$ denote the case of axial vibration. The unknown displacements and their derivatives are governed by the top equation.

\[
M_{11a} \ddot{U}_u + C_{11a} \dot{U}_u + K_{11a} U_u = 0 - M_{12a} \ddot{U}_k - C_{12a} \dot{U}_k - K_{12a} U_k = F_{\text{eff}}.
\]

(3.12)
The right hand side is known and it can be termed as the effective load vector $E_{\text{eff}}$. The WOB ($W'$) is governed by the bottom equation.

$$ W = F_u = M_{21} \ddot{u}_a + M_{22} \ddot{u}_k + C_{21} \dot{u}_a + C_{22} \dot{u}_k + K_{21} \ddot{u}_a + K_{22} \ddot{u}_k . $$ (3.13)

After the unknown quantities are determined from the top equation, the right hand side of this equation becomes known.

### 3.3.1.2 Torsional Vibration

The torsional vibrations on the other hand have only one case, where at the top, the displacement ($\Theta_k$) and its derivatives are always known and the driving torque ($T_{dr}$) is unknown. For the rest of the nodes, including the bit, the forces are known ($T_k = [0 \quad T_{TOB}]^T$) and the displacement, velocity and acceleration are unknown. When the bit is not in contact with the formation $T_{TOB} = 0$, otherwise the TOB is a function of the WOB, as given in Eq. 4.15. The equations can be partitioned as before

$$ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_k \\ \dot{\Theta}_u \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_k \\ \dot{\Theta}_u \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \Theta_k \\ \Theta_u \end{bmatrix} = \begin{bmatrix} T_u \\ T_k \end{bmatrix} . $$ (3.14)

The subscript $i$ denote the case of torsional vibration. The bottom equation is used to express the unknown displacements and their derivatives.

$$ M_{22} \ddot{\Theta}_u + C_{22} \dot{\Theta}_u + K_{22} \Theta_u = T_k - M_{21} \ddot{\Theta}_k - C_{21} \dot{\Theta}_k - K_{21} \Theta_k = T_{\text{eff}} . $$ (3.15)

The driving torque required at the rotary table is governed by the top equation.

$$ T_{dr} = T_u = M_{11} \ddot{\Theta}_k + M_{12} \ddot{\Theta}_u + C_{11} \dot{\Theta}_k + C_{12} \dot{\Theta}_u + K_{11} \Theta_k + K_{12} \Theta_u . $$ (3.16)

The above equations are solved numerically for each time step by the Newmark–Wilson integration scheme.
Chapter 4

Response Determination

4.1 Preliminary Remarks

In order to assess the reliability of the discrete model, its natural frequencies are compared with those of the continuous system. Regarding excitation there are three sources. First, for axial vibrations the excitation is due to the lobed formation surface. Second is the torque acting on the bit due to the resistance from the lobes of the formation surface. The third is the base motion applied on the drillstring by the rotation of the kelly. The last two excitations are for the torsional vibrations.

It has been mentioned at the end of the previous chapter, that the equations of motion are solved by the Newmark–Wilson integration scheme. There are several aspects of the solution in order to incorporate the bit lift-off phenomenon, the coupling of axial and torsional vibrations at the bit and the frequency dependence of damping matrices. These are described in the following sections.

4.2 Evaluation of Resonant Frequencies

To evaluate the resonant frequencies of the continuous system, the transfer function approach is used. A transfer function for a system can be defined as the ratio of the complex amplitudes of the output to the input. Here the output or the input refer to quantities involved in the dynamic behavior of the system. For a linear and constant parameter system, if the input is sinusoidal with a single frequency of variation, then the output is also sinusoidal with the same frequency of variation.

It has been mentioned in Sect. 3.2.1.1 that for axial vibrations the boundary condition at the bit alternates between two kinds. When the bit is not in contact
with the formation it is the force which is specified. On the contrary when the bit is in contact with the formation it is the displacement which is specified. Clay et al. (1990 [29]) pointed out that the relevant transfer functions for the two cases are entirely different.

### 4.2.1 Case of Specified Force

When the force at the boundary is specified, the input in the definition of the transfer function is the force. The output is the displacement of the boundary due to that force. The transfer function in this case is called the mobility. The following derivation is based on the mobility method developed by Paslay and Bogy (1963 [73]) for drillstring vibrations.

The individual mobilities of the elements of a mass–spring–dashpot system, and the resultant mobility are [33]

\[ M_M(\omega) = -\frac{1}{\omega^2 C_M}, \quad M_C(\omega) = \frac{1}{i\omega C_C}, \quad M_K = \frac{1}{C_K} \]

and

\[ M_{M - C}(\omega) = \frac{1}{-\omega^2 C_M + i\omega C_C + C_K}, \quad (4.1) \]

where, \(\omega\) is the frequency of the specified force and \(C_M, C_C\) and \(C_K\) are the mass, damping and stiffness parameters respectively. For an element of axial or torsional vibrations of length \(l\), if 1 and 2 represent the two ends, then the displacement at location \(r\) due to the force at \(s\) are related by the following equation of the mobility

\[ M_{rs}(\omega) = \sum_{k=0}^{\infty} c_{k,s} \psi_{k,r} \quad (4.2) \]

where

\[ c_{k,s} = \frac{\psi_{k,s}}{C_M(\nu_k^2 - \omega^2) + i\omega C_C}, \]

and \(\psi_{k,r}\) is the value of the normalized mode shape of the element at \(r\), corresponding to \(\nu_k\), the \(k\)th natural frequency. For a free–free element, \(r\) and \(s\) can be either 1 or
2 that is \(x_{r/s} = 0\) or \(l\), the load can be applied at either end, that is
\[
p(x,t) = p(t)\delta(x = x_r)
\]
and
\[
\psi_{k,r} = \begin{cases} 
\sqrt{\frac{3}{l}} \cos k\pi \frac{x_r}{l} & k = 1, 2, \ldots \\
\sqrt{\frac{1}{l}} & k = 0 
\end{cases}
\]
\[
\nu_k = k\pi \frac{c}{l},
\]
where \(c\) is the velocity of the vibrational wave through the material of the element.

For a fixed–free element, as the driving point mobility is the one which is of interest, \(r\) and \(s\) can be that corresponding to the free end only that is \(x_{r/s} = l\), the load can be applied at the same end and
\[
\psi_{k,r} = \sqrt{\frac{2}{l}} \sin \frac{1}{2} (2k - 1)\pi \frac{x_r}{l} \quad k = 1, 2, \ldots 
\]
\[
\nu_k = \frac{1}{2} (2k - 1)\pi \frac{c}{l}.
\]

In the formulation of the driving point mobility, the pipes and collars are considered as single elements. This is applicable when both of them have uniform cross-section. This is idealized in the present study and is mentioned in Sect. 5.1. The pipe element is a free–free element for the axial vibrations and a fixed–free one for the torsional vibrations. The collar element is a free–free element for both the vibrations.

Satisfying the junction conditions and the reciprocity theorem, the resultant driving point mobility of the bit for the case of axial vibrations is given by the equation
\[
M_{bb}(\omega) = M_{22c}(\omega) - \frac{M_{12c}(\omega)}{M_{22p}(\omega) + M_{11c}(\omega) - \left[M_{12p}(\omega)/\{M_{ss}(\omega) + M_{11p}(\omega)\}\right]}, \quad (4.5)
\]
where the subscripts \(c\) and \(p\) refer to the collar and pipe respectively.

\(M_{ss}(\omega) = M_{M-C-K}(\omega)\) with \(C_M = M\), \(C_C = C\) and \(C_K = K\) where, \(M\), \(C\) and \(K\) are the mass, damping and stiffness parameters of the idealized suspension respectively.

For the case of torsional vibrations the mobility of the bit is
\[
M_{bb}(\omega) = M_{22c}(\omega) - \frac{M_{12c}(\omega)}{M_{22p}(\omega) + M_{11c}(\omega)}.
\] 

The resonant frequencies correspond to the peaks of the mobility curves.
4.2.2 Case of Specified Displacement

When the displacement of the boundary is specified, the input is the displacement and the output is the force necessary to cause the displacement. The corresponding transfer function is called the impedance. An impedance of an element is the inverse of the corresponding mobility. Proceeding as previously, the resultant driving point impedance of the bit is given by the equation

$$Z_{bb}(\omega) = Z_{22c}(\omega) - \frac{Z^2_{12c}(\omega)\{Z_{11c}(\omega) + [Z_{12p}(\omega)/Z_{22p}(\omega)]\}}{Z_{22p}(\omega) + Z_{11c}(\omega) - [Z^2_{12p}(\omega)/Z_{22p}(\omega)]}. \quad (4.7)$$

The corresponding expression for the torsional vibrations is not necessary as the boundary condition at the bit is not of this kind. The resonant frequencies correspond to the troughs of the impedance curves.

4.3 Excitation

4.3.1 Formation Surface Profile

For axial vibrations the excitation is due to the lobes of the formation surface. To approximate the formation profile generated from the vibrations of a tri-cone bit, the angular variation of the surface elevation is usually idealized as sinusoidal. Artificial tri-lobed exciters have been designed with sinusoidal profiles [3, 86]. In the present study a profile with sinusoidal angular variation and constant radial elevation, as shown in Fig. 4.1, is selected. Since the lateral movement of the bit is not considered, the radial variation is neglected. To make the profile continuous at the center, a quarter sine radial variation is introduced within a small circular zone of radius $\Delta r_h$ which is concentric with the center of the borehole. The later of course is not relevant in the model. The equation of the surface is then given as

$$S(r, \phi) = \begin{cases} 
S_0 \sin \left( \frac{r - \Delta r_h}{2} \right) \sin (3\phi), & 0 \leq r \leq \Delta r_h \\
S_0 \sin (3\phi), & \Delta r_h \leq r \leq r_h
\end{cases} \quad \text{and} \quad 0 \leq \phi \leq 2\pi, \quad (4.8)$$
\[ \Delta n_h \leq r \leq r_h \]

Figure 4.1: Representative profile of the formation surface
where $S_0$ is the amplitude of the lobes, $r_h$ is the radius of the borehole, and $r$ and $\phi$ are the radial and angular co-ordinates respectively.

4.3.2 Torque on Bit

For torsional vibrations the excitations are induced through the TOB and the base motion at the kelly bushing. Like the WOB, the TOB also fluctuates and that too almost with the same frequency [34]. The common method of calculating the TOB is based on the energy balance approach. The difficulty in this method is the determination of the specific energy of rock breakage and the efficiency factor [92]. It has been mentioned by Eronini et al. (1982 [49]) that “the specific energy may not be an adequate drilling performance indicator . . .”.

The following derivation is based on a force balance approach developed by Warren (1984 [92]). Several simplifications are made, as shown in Figs. 4.2 and 4.3.

i. For a roller cutter bit the resistance to rolling arises from the contact of the teeth with the formation. As each cone has rows of teeth which are at offset to the rows of the other cones, only one cone with contiguous rows of teeth can be substituted for all the cones.

ii. As the study is for a hard formation surface, the cone has a single basic angle $\gamma$. The apex of the cone lies at the bit center line. Hence it is of the true rolling type without any slipping.

iii. The cone axis does not have any offset from the bit center line. To be noted that the cones of a soft formation bit have two or more basic angles and their axes are at considerable offset from the bit center line.

iv. The cone is an extension of an inclined toothed wheel. The TOB is calculated from the force needed to overcome the rolling resistance acting on the wheel.
Plan of location of the cone axes

Cones of a tri-cone bit

Typical cones at offset from the bit center line and with multiple cone angles

Idealized cones without the offset and each cone with one basic angle

Substituted idealized cone with contiguous rows of teeth

Figure 4.2: Idealization of the cones
Figure 4.3: Model of the cone
v. As the weight of the cone is negligible compared to the applied WOB, the difference between the vertical downward force acting on the cone through the journal bearing and the upward WOB from the formation, that arises due to the acceleration of the bit, is neglected.

vi. The vertical downward force is uniformly distributed along the journal bearing.

vii. The diameter of the hole is same as that of the bit, which is again twice the length of the cone side.

viii. The lowering of the mean elevation of the profile after a single passage of the idealized cone is constant and is equal to the average depth of cut per revolution. Its value is equal to the mean ROP divided by the applied rotary speed. It implies that the direct effects of WOB, rotary speed and other variables are not included each time the bit is in contact with the formation, rather their net result is incorporated as an ensemble through the value of the observed ROP. The assumption can be supported by the fact that the effect of any minor variation of the instantaneous depth of cut is reduced by a power of 0.5 in the first dominant term of the expression of the TOB. To be noted that a single passage of the idealized cone is equivalent to the combined effect of the passes of the three cones.

ix. The depth of cut is same throughout the profile, that is, there is no radial variation and it is independent of the inclinations of the profile. The latter is relaxed in the model of the profile amplitude modulation.

For a toothed wheel the rolling resistance can be calculated based on the concept that is applied to a roller. When an inclined roller revolves around a vertical axis, using the cylindrical polar co-ordinates, the angular velocity and moment can be decomposed into axial and radial components, where the radius refers to the horizontal line connecting the vertical axis and the center of mass of the roller. For a constant
rolling speed, neglecting any bearing friction, the resultant moment is zero. Applying the equilibrium condition for the radial component of the moment, the relation between the horizontal force $\Delta F_h$ from the applied torque and the vertical downward force $\Delta F_v$, due to the weight or any over bearing pressure, is

$$\Delta F_h = \left( \frac{b - c \tan \alpha}{c + b \tan \alpha} \right) \frac{1}{\cos \frac{\gamma}{2}} \Delta F_v,$$

where $b$ and $c$ are geometric variables of the wheel, $\tan \alpha$ is the slope of the surface and $\gamma$ is the basic cone angle, as shown in Fig. 4.3. Here the equilibrium is about the point at which the resultant of the pressure from the deformed surface acts on the circumference which is also the instantaneous body centrode. Substituting

$$b = \sqrt{2ar_w} - a^2 \quad \text{and} \quad c = r_w - a \quad \text{where} \quad a = \frac{\delta_c}{\cos \frac{\gamma}{2}},$$

$$\Delta F_h \approx \left( \frac{\sqrt{2a} - \sqrt{r_w} \tan \alpha}{\sqrt{r_w} + \sqrt{2a} \tan \alpha} \right) \frac{1}{\cos \frac{\gamma}{2}} \Delta F_v$$

$$= \left( \frac{\sqrt{2\delta_c} - \sqrt{r_w} \sqrt{\cos \frac{\gamma}{2}} \tan \alpha}{\sqrt{r_w} \sqrt{\cos \frac{\gamma}{2}} + \sqrt{2\delta_c} \tan \alpha} \right) \frac{1}{\cos \frac{\gamma}{2}} \Delta F_v.$$ 

Here $r_w$ is the radius of the wheel and $\delta_c$ is the average depth of cut per revolution of the bit. For the specified surface profile

$$\tan \alpha = \frac{1}{h} \cos \frac{\gamma}{2} \frac{\partial S}{\partial \phi},$$

where, $h$ is the distance of the wheel from the cone apex. For the cone the WOB is equal and opposite to the vertical downward force, which acts uniformly on the journal bearing of the cone. Then

$$\Delta F_v = -\frac{W}{H} \Delta h,$$

where $W$ is the WOB, $H = r_h \cos \frac{\gamma}{2}$ is the height of the cone, $r_h$ is the length of the side of the cone and $\Delta h$ is the thickness of the wheel. The TOB is also equal and
opposite to the torque required to generate the horizontal force. As $\Delta h \to 0$, the expression of the TOB is

$$T_{\text{TOB}} = -\int_{\delta H}^{H} h \cos \frac{\gamma}{2} dF_h$$

$$\simeq W \left[ \frac{2\sqrt{2}}{3} \frac{H \delta_c}{\sin \frac{\gamma}{2}} - \cos \frac{\gamma}{2} \frac{\partial S}{\partial \phi} \right]$$

$$= W \cos \frac{\gamma}{2} \left[ \frac{A}{3} \frac{r_h \delta_c}{\sin \gamma} - \frac{\partial S}{\partial \phi} \right],$$

(4.14)

where $\delta H$ is the distance of the tip of the body of the cone from the tip of the entire cone which includes the teeth. The second term inside the bracket represents an additional fluctuation generated due to the presence of the lobes. To take account of the possibility of the bit rotating backwards due to the torsional oscillations, the factor $\text{sign}(\dot{b}_b)$ has to be incorporated, where $\dot{b}_b$ is the torsional velocity of the bit.

The final expression is

$$T_{\text{TOB}} = W \cos \frac{\gamma}{2} \frac{A}{3} \frac{r_h \delta_c}{\sin \gamma} - \text{sign}(\dot{b}_b) \frac{\partial S}{\partial \phi}.$$  

(4.15)

Here,

$$\frac{\partial S}{\partial \phi} = 3S_0 \cos(3\phi) \quad \text{and} \quad \delta_c = \frac{\bar{R}}{N},$$

(4.16)

in which $\bar{R}$ is the average ROP and $N$ is the rotary speed.

Negative TOB has been recorded, which at times tends to unscrew the drill-string components [28, 97]. Warren had incorporated an additive constant and a multiplicative constant to consider the effects of the cone offset and the presence of more than one basic cone angle. The constants were determined by curve fitting of experimental data. It was stated that "the model is insensitive to moderate changes in factors such as bit hydraulics, fluid type, and formation type". The variation of these factors and others like bit geometry, bit wear and the differential pressure acting across the formation surface are taken into account through the value of the ROP.

This formulation of course does not include any phase difference between the WOB and the TOB which has been detected by MWD instruments [34].
4.3.3 Base Rotation

The base refers to the kelly bushing. The base rotation is given as

\[
\Theta_k = \begin{cases} 
\frac{1}{2} \dot{\Omega} t^2 + \Theta_0, & t \leq t_r \\
\frac{1}{2} \dot{\Omega} t^2 + \Omega_f (t - t_r) + \Theta_0 & t > t_r \\
= \Omega_r (t - \frac{1}{2} t_r) + \Theta_0 
\end{cases} \tag{4.17}
\]

\[
\dot{\Theta}_k = \begin{cases} 
\dot{\Omega} t, & t \leq t_r \\
\Omega_f & t > t_r 
\end{cases} \text{ and} \tag{4.18}
\]

\[
\ddot{\Theta}_k = \begin{cases} 
\ddot{\Omega}, & t \leq t_r \\
0 & t > t_r 
\end{cases} \tag{4.19}
\]

where \( \dot{\Omega} \) is the rotary speed ramping rate, \( \Theta_0 \) is the initial position of the drillstring and \( \Omega_f \) is the final value of the rotary speed. It is assumed that the rotary speed imparted by the rotary table is increased at a steady rate to the specified final value, after which it is kept constant. In the field of course a constant rotary speed may not be truly satisfied [59]. The total time of rotary speed ramping is known as the rise time and is given as \( t_r = \frac{\Omega_f}{\dot{\Omega}} \).

4.4 Lobe Amplitude Modulation

The generation of the lobes in the formation surface, their influence on the axial vibrations of the drillstring and subsequently their sustenance or degeneration is an interactive process. As a first study of this interaction, the modulation of the amplitude of the lobes is modeled as follows.

The change in the amplitude of the lobes is governed by the zones of contact of the formation with the bit-teeth. As the number of teeth or their parts, in contact with the formation, is more when the bit rolls in the troughs than that when it rolls over the crests, there is always a tendency to deepen the troughs. This increase in amplitude of the lobes occurs up to a certain limit which is governed by the size and
geometry of the cones. But if the contacts are uneven with respect to the crests and the troughs then there is the modulation. That is if the bit rolls in the troughs and jumps over the crests then the height of the later remains unaltered, which leads to an additional increase in the amplitude. On the contrary if the bit rolls over the crests and jumps across the troughs then the crests are eroded and there is no change in the depth of the troughs. Hence the amplitude decays.

Let $\Delta t_c$ and $\Delta t_t$ be the values of time per revolution during which the bit rolls over the crests and in the troughs respectively. Also let $\Delta S_{0c}$ and $\Delta S_{0t}$ be the corresponding change in the amplitudes, as shown in Fig. 4.4. The mean of the profile lowers by an amount equal to the depth of cut per revolution, $\delta_c$. Thus

$$\delta_c = \frac{1}{2}(\Delta S_{0c} + \Delta S_{0t}).$$

(4.20)

It is assumed that the change in amplitude of each part is proportional to the respective time of contact. Specifically

$$\Delta S_{0c/t} = k_{c/t} \Delta t_{c/t},$$

(4.21)

where, $k_t$ and $k_c$ are the proportionality constants and $k_t > k_c$. It has been mentioned earlier, that the number of teeth or their parts, in contact with the formation, is more when the bit rolls in the troughs than that when it rolls over the crests. In terms of $\delta_c$,

$$\Delta S_{0c/t} = 2\delta_c \frac{k_{c/t} \Delta t_{c/t}}{k_t \Delta t_t + k_c \Delta t_c}.$$  

(4.22)

The modified amplitude is

$$\dot{S}_0 = S_0 + \frac{1}{2}(\Delta S_{0t} - \Delta S_{0c}).$$

(4.23)

In the numerical implementation the two proportionality constants are related as $k_t = 1.25 \times k_c$. The change in amplitude is considered to occur at a uniform rate over the revolution. Note that in this model the cone geometry and the details of the bit-teeth-formation interaction are neglected.
Figure 4.4: Lobe amplitude modulation
It has been predicted by previous researchers that the lobes precess about the bore-hole axis as the impact from the bit is greater on the uphill sides of the lobes [39, 85]. A hypothesized precession rate was also presented [85]. In this study this precession is not considered. This simplification can be justified by the argument that although precession might occur, its predicted rate is much less than the rotary speed and hence it is not influential in the bit lift-off and the WOB variation.

4.5 Solution Aspects

4.5.1 Algorithm

The basic steps of the algorithm are as follows.

Part I

i. Input the geometric and material properties of the drillstring, that is the mass $M$, stiffness coefficient $K$ and damping coefficient $C$ of the suspension system; the inner and outer radii $r_{i/o}$ and the total length $L$ of the pipe and collar; density $\rho_s$, modulus of elasticity $E$ and Poisson's ratio $\nu$ of steel.

ii. Develop the mass and stiffness matrices for axial and torsional vibrations, $M_a$, $M_t$, $K_a$, and $K_t$.

iii. Calculate the natural frequencies.

Part II

i. Input the properties of the bit, that is the radius $r_b$ and the basic cone angle $\gamma$; density $\rho_l$ and coefficient of viscosity $\mu$ of the drilling mud; applied WOB $\tilde{W}$ and the rotary speed $N$; the initial value of the lobe amplitude $S_0$, and the parameters of the Newmark–Wilson scheme $\delta$, $\beta$ and $\theta$.

ii. Initialize the displacement, velocity and acceleration vectors for axial and torsional vibrations, that is $U$, $\dot{U}$, $\ddot{U}$ and $\Theta$, $\dot{\Theta}$, $\ddot{\Theta}$. Initialize the formation surface variable $S$. 
iii. Develop the frequency dependent damping matrices $C_a(\omega_a)$ and $C_t(\omega_t)$.

iv. For each time step, check the possibility of bit lift-off by solving the equations of free vibration, as given in Sect. 3.3.1.

v. If the bit has been in contact with the formation in the previous time step, and it remains so for the current step, solve the equations for the case of contact. Calculate $W$, $T_{TOB}$ and $\Delta S_0$. Otherwise if the bit undergoes lift-off, retain the solution of the free vibration equations. $W$, $T_{TOB}$ and $\Delta S_0$ are equal to zero.

vi. If the bit has not been in contact in the previous timestep, and remains so for the current step then retain the values of the solution of free vibration equations. $W$, $T_{TOB}$ and $\Delta S_0$ are null. Otherwise if the bit comes in contact with the formation, calculate the intermediate value of the time, at which the bit comes in contact, by an interpolation scheme. Solve the equations for the case of no contact for the interpolated time step to verify the resuming of contact.

vii. Repeat the steps iv to vi, for each time step, up to the final time of calculation.

The corresponding flowchart is given in Fig. 4.5. The equations of motion are integrated numerically in the time domain by the Newmark–Wilson scheme. The Newton–Raphson method is selected as the interpolation scheme.

4.5.2 Condition for Lift-off

When the bit moves in contact with the formation at a certain time, given the axial displacement, velocity and acceleration vectors, the same are calculated for the next time step, from the equations of free vibration, as given in Sect. 3.3.1. If the displacement of the bit is above the corresponding value of the profile elevation, then the bit is no more in contact with the formation. The steps are

i. Solve $M_{22} \ddot{\Theta}_u + C_{22} \dot{\Theta}_u + K_{22} \Theta_u = T_k - M_{21} \dot{\Theta}_k - C_{21} \dot{\Theta}_k - K_{21} \Theta_k$
where, $T_k = 0$. Find $\Theta_b$. 
Start

Geometric and material properties of drillstring

Develop $M_a$, $M_t$, $K_a$, $K_t$ and calculate frequencies

$\gamma$, $\mu$, $\rho_f$, $\bar{W}$, $N$, $S_{0p}$, $\delta$, $\beta$, $\bar{\theta}$

Initialize $\Theta$, $\dot{\Theta}$, $\ddot{\Theta}$, $U$, $\dot{U}$, $\ddot{U}$, $S$

Develop $C_a(\omega_a)$, $C_t(\omega_t)$

Solve $M_a \ddot{U} + C_a(\omega_a) \dot{U} + K_a U = 0$

$M_{22t} \ddot{\Theta}_u + C_{22t}(\omega_t) \dot{\Theta}_u + K_{22t} \Theta_u = T_{eff}$

$U_b \geq S$

Yes

Solve $M_{11} \ddot{U}_u + C_{11} \dot{U}_u + K_{11} U_u = E_{eff}$

$W = F_u$, $T_{TOB} = f(W)$

Calculate $\Delta S_0$

No

$w = 0$

$T_{TOB} = 0$

$\Delta S_0 = 0$

Stop

Figure 4.5: Flowchart
ii. Calculate \( S(\Theta_b) \).

iii. Solve \( M_a \ddot{U} + C_a(\omega) \dot{U} + K_a U = 0 \). Find \( U_b \).

iv. For lift-off \( U_b < S(\Theta_b) \).

4.5.3 Condition for Resuming Contact

When the bit is not in contact with the formation at a certain time and the displacement calculated from the free vibration equation for the next time step is below the corresponding value of the profile elevation, that is if \( U_b \geq S(\Theta_b) \), then the bit will be intercepted by the profile. To calculate the intermediate value of the time at which the bit comes in contact with the formation, the Newton–Raphson interpolation scheme is applied. The steps are

i. Interpolate as

\[
\frac{U_b - U_{bp}}{\Delta t} = \frac{S(\Delta t_{ip}) - U_{bp}}{\Delta t_{ip}}. \tag{4.24}
\]

ii. Solve the following nonlinear function of \( \Delta t_{ip} \) by the interpolation scheme

\[
\{S(\Delta t_{ip}) - U_{b_p}\} \Delta t - \{U_h - U_{b_p}\} \Delta t_{ip} = 0, \tag{4.25}
\]

where \( \Delta t \) is the default time step, \( \Delta t_{ip} \) is the interpolated time step and the subscript \( p \) denotes the quantity at the previous value of the time.

4.5.4 Criterion of Bit Motion

When the bit is in contact with the formation, its rotational motion is governed by the motive torque from stiffness of the drillstring above and the resisting TOB from the formation. The bit accelerates when the first is greater than the second and vice versa. But the TOB is a function of the WOB, which again depends on the position of the bit. Thus, the axial and torsional vibrations get coupled at the bit when the latter is in contact with the formation. Hence, an iterative scheme is employed to
the simultaneous sets of equations of axial and torsional vibrations. First the position of the bit is estimated. From that, the WOB and TOB are calculated, the later being used to find the bit position. The steps are subsequently repeated. They can be listed as

i. Assume $\Theta_b$.

ii. Calculate $U_k$, $\dot{U}_k$ and $\ddot{U}_k$.

iii. Calculate the WOB and the TOB.

iv. Solve $M_{22,\Theta} \ddot{\Theta}_u + C_{22,\Theta} \dot{\Theta}_u + K_{22,\Theta} = T_k - M_{21,\Theta} \dot{\Theta}_k - C_{21,\Theta} \dot{\Theta}_k - K_{21,\Theta} \Theta_k$

where, $T_k = [0 \ T_{TOB}]^T$. Find $\Theta_b$.

v. In case of disagreement, repeat the steps.

4.5.5 Frequency Dependence of Damping Matrices

While deriving the expression of the damping matrix, it is assumed that the matrix depends only on the dominant frequency of vibration. In evaluating the expression at a certain step, the dominant mode is assumed and the system of equations are solved. Modal expansion of the displacement vector is carried out to check the dominant mode. In case of disagreement the steps are repeated.

i. Assume dominant mode of vibration and hence $\omega$.

ii. Calculate $C(\omega)$.

iii. Solve $M \ddot{X} + C(\omega) \dot{X} + K X = P$.

iv. $X = \sum_n C_n \phi_n$. Calculate the greatest $C_n$ and verify $n$.

v. In case of disagreement, repeat the steps.
Chapter 5

Application

5.1 Problem

An idealized drillstring is selected to study the applicability of the model. The suspension is modeled by a SDOF mass-spring-damper system. Due to lack of pertinent data, the characteristic values are selected from Ref. [29]. The pipes are considered to be circular and of uniform cross-section. The idealization of uniform cross-section of the pipes and thus neglecting the effects of the joints was justified by Bradbury and Wilhoit [41]. Of course it is applicable mostly for straight drillstrings, with idealized tool joints [49]. The BHA constitutes of collars of uniform circular cross-section and the bit. Stabilizers, heavy weight drillpipes, MWD subs and tool joints are neglected. It is assumed that the lobes in the profile exist at the beginning of the runs. The initial amplitude of the lobes is taken to be small as compared to the values measured in the core samples. They can be created during previous runs or due to compaction during the initial application of the WOB or from any heterogeneity of the formation. The numerical values of the various quantities are given in Table 5.1.

The values selected for the Newmark–Wilson integration scheme are

$$\delta = 0.600, \quad \beta = 0.303 \quad \text{and} \quad \vartheta = 1.000.$$  \hspace{1cm} (5.1)

5.1.1 Rate of Penetration Model

A modified form of the model proposed by Gatlin [52] and mentioned in Sect. 2.8.8, which relates the mean ROP $\bar{R}$ with the applied WOB $\bar{W}$ and the rotary speed $N$, is used. It is

$$\bar{R} = 1.57 \times 10^{-5} \times \bar{W} \sqrt{N} - 0.685,$$  \hspace{1cm} (5.2)
<table>
<thead>
<tr>
<th>Quantities</th>
<th>SI units</th>
<th>US customary units</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Suspension system</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mass</td>
<td>$2.376 \times 10^4$ Kg</td>
<td>$5.240 \times 10^4$ lb</td>
</tr>
<tr>
<td>Stiffness coefficient</td>
<td>$1.000 \times 10^7$ N m$^{-1}$</td>
<td>$6.852 \times 10^5$ lbf ft$^{-1}$</td>
</tr>
<tr>
<td>Damping coefficient</td>
<td>$4.000 \times 10^4$ N s m$^{-1}$</td>
<td>$2.741 \times 10^5$ lbf ft$^{-1}$</td>
</tr>
<tr>
<td><strong>Drillpipe</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner diameter</td>
<td>$1.050 \times 10^{-1}$ m</td>
<td>$3.445 \times 10^{-1}$ ft</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>$1.300 \times 10^{-1}$ m</td>
<td>$4.265 \times 10^{-1}$ ft</td>
</tr>
<tr>
<td>Total length</td>
<td>$3.000 \times 10^3$ m</td>
<td>$9.842 \times 10^3$ ft</td>
</tr>
<tr>
<td><strong>Drill collar</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner diameter</td>
<td>$1.050 \times 10^{-1}$ m</td>
<td>$3.445 \times 10^{-1}$ ft</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>$2.000 \times 10^{-1}$ m</td>
<td>$6.562 \times 10^{-1}$ ft</td>
</tr>
<tr>
<td>Total length</td>
<td>$2.000 \times 10^2$ m</td>
<td>$6.562 \times 10^2$ ft</td>
</tr>
<tr>
<td><strong>Drill bit</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diameter</td>
<td>$3.000 \times 10^{-1}$ m</td>
<td>$9.842 \times 10^{-1}$ ft</td>
</tr>
<tr>
<td>Cone angle</td>
<td>$\frac{\pi}{2}$ rad</td>
<td></td>
</tr>
<tr>
<td><strong>Steel</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>$7.830 \times 10^3$ Kg m$^{-3}$</td>
<td>$4.887 \times 10^2$ lbf ft$^{-3}$</td>
</tr>
<tr>
<td>Modulus of Elasticity</td>
<td>$2.000 \times 10^{11}$ N m$^{-2}$</td>
<td>$2.000 \times 10^4$ ksi</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$3.300 \times 10^{-1}$</td>
<td></td>
</tr>
<tr>
<td><strong>Drilling mud</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>$1.200 \times 10^3$ Kg m$^{-3}$</td>
<td>$7.490 \times 10^1$ lbf ft$^{-3}$</td>
</tr>
<tr>
<td>Coefficient of viscosity</td>
<td>$1.500 \times 10^{-2}$ Kg m$^{-1}$ s$^{-1}$</td>
<td>$3.131 \times 10^{-4}$ lbf s ft$^{-2}$</td>
</tr>
<tr>
<td><strong>Drilling variables</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Applied WOB</td>
<td>$7.000 \times 10^4$ N</td>
<td>$1.574 \times 10^5$ lbf</td>
</tr>
<tr>
<td>Rotary speed ramping</td>
<td>-</td>
<td>$10.000 \text{ rpm s}^{-1}$</td>
</tr>
<tr>
<td><strong>Lobe amplitude</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial value</td>
<td>$1.500 \times 10^{-3}$ m</td>
<td>$4.920 \times 10^{-3}$ ft</td>
</tr>
<tr>
<td>Upper limit</td>
<td>$5.000 \times 10^{-3}$ m</td>
<td>$1.640 \times 10^{-2}$ ft</td>
</tr>
</tbody>
</table>

Table 5.1: Numerical values of quantities
where the units of $\dot{R}$, $\dot{W}$ and $N$ are meters per hour, Newtons and revolutions per minute respectively. The function is plotted in Fig. 5.1. The exponent 0.5 for $N$ gave the best fit to the experimental data of Eckel. The constants are selected to satisfy a reasonable value of the ROP for the case of drilling hard rocks. In the field, the measured ROP can substitute for this theoretical expression.

![Rate of penetration model graph](image)

**Figure 5.1: Rate of penetration model**

### 5.2 Results

#### 5.2.1 Resonant Frequencies

The mobilities for axial vibrations are plotted in Figs. 5.2, 5.3 and 5.4. Similar plots can be done for mobilities for torsional vibrations and impedances for axial vibrations.
Mobilities for axial vibration

Figure 5.2: Mobilities of suspension and drillpipe
For axial vibration
No. of pipe elements = 1

N.B. Frequencies of discretized system shown as vertical lines
Mobilities for axial vibration

\[ |M_{22c}(\omega)| [\text{m/N}] \]

\[ |M_{bb}(\omega)| [\text{m/N}] \]

\[ \omega [\text{rad/s}] \]

Drill collar

System

N.B. Frequencies of discretized system shown as vertical lines

Figure 5.3: Mobilities of drill collar and system
For axial vibration
No. of pipe elements = 1
Mobilities for axial vibration

N.B. Frequencies of discretized system shown as vertical lines

Figure 5.4: Mobilities of drillpipe and system
For axial vibration
No. of pipe elements = 11
For the given geometric properties, the templates of the driving point mobility and impedance at the bit is primarily dictated by the mobility and the impedance of the collar respectively. But there are peaks at the location of the natural frequencies of the pipe.

The importance of the pipes and collars in the mobility or impedance diagrams depends on their relative lengths. Considering the total length of the collars to be fixed, the peaks or troughs corresponding to the frequencies of the pipe are significant when the length of the collars is comparable with the length of the pipes. But as the depth of drilling increases, the above mentioned peaks or troughs get insignificant and as was pointed out by Dareing (1984 [36]), the length of the collars becomes the important parameter in determining the critical rotary speeds. But it is true that the higher frequencies of the pipe, which are close to those peaks or troughs governed by the length of the collars, are also influential in generating resonance.

5.2.2 Calculation Aspects

5.2.2.1 Discretization Schemes

Two schemes of discretization are employed.

i. No. of pipe elements = 1, no. of collar elements = 1.
   In this scheme the pipes and the collars are considered as single elements. This coarse discretization is selected for the purpose of demonstration.

ii. No. of pipe elements = 11, no. of collar elements = 1.
   This scheme is the optimum discretization to capture all the frequencies of the system for axial vibration within the domain of interest. Considering that the maximum rotary speed to be 200 rpm, the value of the upper limit of the domain is $\omega = 3\Omega = 62.8$ rad $s^{-1}$. Here the system refers to the combination of the suspension and the drillstring.
In the plots of the mobilities, the frequencies calculated by the eigenvalue analysis of the discretized system are shown as vertical lines.

5.2.2.2 Time Parameters

First, the time step for calculation is one-fourth of the least of the lowest axial time period, the lowest torsional time period, the rise time for rotary speed and one-ninth of the period of base rotation.

Second, the calculation is terminated when the time crosses the final value and the number of base rotations is at least fifteen. The final value of the time is twice the greatest of the period of the base rotation, the largest axial time period, the largest torsional time period and the rise time for rotary speed.

5.2.3 Presentation of Results

The results from the output of the program are presented through graphs. Apart from the surface elevation record and the rotational displacement record, all other graphs are typical sections of the time records after the vibrations attain a steady pattern. The various plots are as follows.

i. Surface elevation record

It represents the surface profile that the bit is traversing. The reason behind selecting the rotational position of the base as the ordinate of the horizontal axis is that, it is a monotonically increasing function in contrast to the rotational position of the bit. Here the base refers to the kelly bushing. Also it is preferred to the time variable because the surface elevation is primarily a function of the rotational position of the drillstring. The relationship between the time and the rotational position of the base is linear only after the rotary speed reaches the designated limit. Before that, it is quadratic.
ii. Bit trajectory
   It is the trajectory of the node at the bit over the formation surface.

iii. Weight on bit record
   It is the trace of the WOB with respect to time.

iv. Rotational displacement record
   It plots the rotational positions of the bit and the base with respect to time.

v. Torque on bit record
   It is the trace of the TOB with respect to time.

vi. Base torque record
   It is the trace of the driving torque at the kelly with respect to time.

5.2.4 Dynamic Behavior

5.2.4.1 Scheme No. 1

The circular natural frequencies of the system and the rotary speeds necessary to generate vibrations of these frequencies are given in Table 5.2. The relation between them is

\[ N = \frac{\omega}{3} \times \frac{60}{2\pi}. \]  \hspace{1cm} (5.3)

It can be seen from the mobility diagrams, Figs. 5.2 and 5.3, that the calculated frequencies of the pipe, collar and the system are higher than those corresponding to the peaks or troughs, the offset being higher for higher frequencies. The selected rotary speeds out of the several runs are 75, 120, 150, 180 and 200 rpm. Out of them the values 75, 120 and 180 are close to some of those tabulated. The remaining values of the table are rather low from field considerations. The dynamic behavior under the selected rotary speeds is as follows.
| Quantity specified at the bit | Axial vibration |  | Torsional vibration |  |
|-----------------------------|-----------------|---------------------|---------------------|
|                             | $\omega$ | $N$ | $\omega$ | $N$ |
|                             | rad s$^{-1}$ | rpm | rad s$^{-1}$ | rpm |
| Scheme no. 1                |          |     |          |     |
| Force                       |          |     |          |     |
| 2.0                         | 6.4     | 1.1 | 3.4     |     |
| 13.7                        | 43.6    | 37.3| 118.7   |     |
| 56.8                        | 180.8   |     |          |     |
| Displacement                |          |     |          |     |
| 12.7                        | 40.4    |     |          |     |
| 24.1                        | 76.7    |     |          |     |
| Scheme no. 2                |          |     |          |     |
| Force                       |          |     |          |     |
| 1.97                        | 6.3     | 1.1 | 3.3     |     |
| 6.3                         | 19.9    | 3.7 | 11.8    |     |
| 11.0                        | 35.0    | 6.8 | 21.7    |     |
| 15.8                        | 50.3    | 10.2| 32.5    |     |
| 20.0                        | 63.7    | 13.8| 43.9    |     |
| 24.2                        | 77.0    | 17.7| 56.3    |     |
| 29.8                        | 94.8    | 21.9| 69.7    |     |
| 36.3                        | 115.5   | 26.3| 83.7    |     |
| 43.3                        | 137.8   | 30.8| 98.0    |     |
| 50.5                        | 160.7   | 35.1| 111.7   |     |
| 57.3                        | 182.4   | 38.2| 121.6   |     |
| 62.3                        | 198.3   | 51.6| 164.2   |     |
| Displacement                |          |     |          |     |
| 81.9                        | 260.7   |     |          |     |
| 5.07                        | 16.1    |     |          |     |
| 10.2                        | 32.5    |     |          |     |
| 15.1                        | 48.1    |     |          |     |
| 19.5                        | 62.1    |     |          |     |
| 23.6                        | 75.1    |     |          |     |
| 28.9                        | 92.0    |     |          |     |
| 34.9                        | 111.1   |     |          |     |
| 40.8                        | 129.9   |     |          |     |
| 46.0                        | 146.4   |     |          |     |
| 51.8                        | 164.9   |     |          |     |
| 57.9                        | 184.3   |     |          |     |
| 62.5                        | 198.9   |     |          |     |

Table 5.2: Circular natural frequencies and generating rotary speeds
75 rpm

The value is close to that for an axial resonance with specified bit displacement. Although the lobe amplitude is restricted to 0.002 m, the bit lifts off almost periodically. Correspondingly the WOB and TOB reduce to zero. The fluctuation of the base torque $T_{dr}$ is also high. The plots are given in Figs. 5.5, 5.6, 5.7 and 5.8.

Figure 5.5: Representative variation of surface profile
For 75 rpm
Time = 0 to 9.6 s
No. of pipe elements = 1
Rotary speed = 75 rpm

Figure 5.6: Rotational displacement and surface elevation records
For 75 rpm
No. of pipe elements = 1
Rotary speed = 75 rpm

Figure 5.7: Bit trajectory and WOB record
For 75 rpm
No. of pipe elements = 1
Rotary speed = 75 rpm

Figure 5.8: TOB and base torque records
For 75 rpm
No. of pipe elements = 1
120 rpm
The value is close to one of those which generate torsional resonance. But the lobe amplitude is restricted to 0.002 m. There is no lifting off of the bit and the fluctuation of the WOB is significantly less. The plots are in Figs. 5.9 and 5.10.

150 rpm
This value is not close to any one of those tabulated. The profile amplitude remains restricted at 0.0025 m. There is no lifting off of the bit and the fluctuation of the WOB is not wide, as seen in Figs. 5.11 and 5.12.

180 rpm
This is almost equal to one of those which generate axial resonance with specified force on the bit. The amplitude of the profile steadily increases till it reaches the upper limit where it remains almost constant. The bit lifts off at almost every cycle and correspondingly the WOB and TOB reduce to null values. The values remain at zero for almost half the cycle. The fluctuations in the records of WOB, TOB and base torque are large. The plots are in Figs. 5.13, 5.14 and 5.15.

200 rpm
The value is greater than those tabulated. Here the lobe amplitude increases up to the upper limit. But it does not remain constant and modulates slowly. Due to the high rotational speed the bit jumps over a side of the trough and the WOB reduces to zero for a small fraction of the fluctuation cycle, as seen in Figs. 5.16 and 5.17.
Figure 5.9: Surface elevation record
For 120 rpm
No. of pipe elements = 1
Rotary speed = 120 rpm

Figure 5.10: Bit trajectory and WOB record
For 120 rpm
No. of pipe elements = 1
Rotary speed = 150 rpm

Figure 5.11: Surface elevation record
For 150 rpm
No. of pipe elements = 1
Figure 5.12: Bit trajectory and WOB record
For 150 rpm
No. of pipe elements = 1
Surface elevation record for rotary speed = 180 rpm

Figure 5.13: Surface elevation record
For 180 rpm
No. of pipe elements = 1
Rotary speed = 180 rpm

Figure 5.14: Bit trajectory and WOB record
For 180 rpm
No. of pipe elements = 1
Rotary speed = 180 rpm

Figure 5.15: TOB and base torque records for 180 rpm.
No. of pipe elements = 1
Figure 5.16: Surface elevation record for rotary speed = 200 rpm (a)

Surface elevation (m)

Base rotational displacement (rad)

[ m ]

[ rad ]

No. of pipe elements = 1
Rotary speed = 200 rpm (a)

Figure 5.17: Bit trajectory and WOB record
For 200 rpm
No. of pipe elements = 1
5.2.4.2 Scheme No. 2

The circular natural frequencies of the system and the rotary speeds necessary to
generate vibrations of these frequencies are given in Table 5.2. The offsets of the cal-
culated frequencies of the pipe and the system from the actual resonant frequencies
are reduced, as seen in Fig. 5.4. The selected rotary speeds are 115, 145, 175 and
200 rpm. The first two and the last are close to some of those generating resonant
frequencies. The dynamic behavior for the four cases is as follows.

115 rpm

This is almost equal to one of those which generate axial resonance with specified
force on the bit. The lobe amplitude increases during the rotary speed ramping, but
subsequently reduces and remains restricted within 0.0015 m. The bit lifts off from
the downhill side of the profile almost periodically and correspondingly the WOB
reduces to zero. The plots are in Figs. 5.18 and 5.19.

145 rpm

The value is close to one of those which generate axial resonance with specified dis-
placement of the bit. After the initial increase the lobe amplitude decreases and is
limited to about 0.002 m. But the fluctuation of the WOB is quite high and periodi-
cally reduces to zero, as seen in Figs. 5.20 and 5.21.

175 rpm

This value is not close to any one of those tabulated. After the rotary speed
becomes constant, the amplitude of the lobes does not increase beyond 0.001 m. The
fluctuation of the WOB is limited. The plots are in Figs. 5.22 and 5.23.

200 rpm

It is close to two out of those tabulated. One of them generates an axial resonance
with specified displacement of the bit. The other generates an axial resonance with
specified force on the bit. The lobe amplitude remains restricted within 0.0015 m,
Fig. 5.24. The bit lift-off and the clipping of the WOB record are periodic, Fig. 5.25.
Figure 5.18: Surface elevation record
For 115 rpm
No. of pipe elements = 11

Rotary speed = 115 rpm
Rotary speed = 115 rpm

Figure 5.19: Bit trajectory and WOB record
For 115 rpm
No. of pipe elements = 11
Figure 5.20: Surface elevation record
For 145 rpm
No. of pipe elements = 11
Rotary speed = 145 rpm

Figure 5.21: Bit trajectory and WOB record
For 145 rpm
No. of pipe elements = 11
Surface elevation record for rotary speed = 175 rpm

Figure 5.22: Surface elevation record
For 175 rpm
No. of pipe elements = 11
Rotary speed = 175 rpm

Figure 5.23: Bit trajectory and WOB record
For 175 rpm
No. of pipe elements = 11
Figure 5.24: Surface elevation record
For 200 rpm
No. of pipe elements = 11
Figure 5.25: Bit trajectory and WOB record
For 200 rpm
No. of pipe elements = 11
<table>
<thead>
<tr>
<th>RPM</th>
<th>Description</th>
<th>Profile variation</th>
<th>WOB fluctuation</th>
<th>Nature of lift-off</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>Close to axial resonance with specified bit displacement</td>
<td>Restricted within 0.0020 m</td>
<td>Wide with periodic clipping at zero</td>
<td>Periodic</td>
<td>Critical</td>
</tr>
<tr>
<td>120</td>
<td>Close to torsional resonance</td>
<td>Restricted within 0.0020 m</td>
<td>Limited</td>
<td>Rare</td>
<td>Not critical</td>
</tr>
<tr>
<td>150</td>
<td>Not close to any resonance</td>
<td>Restricted within 0.0025 m</td>
<td>Limited</td>
<td>Rare</td>
<td>Not critical</td>
</tr>
<tr>
<td>180</td>
<td>Close to axial resonance with specified force on the bit</td>
<td>Increases upto the maximum limit and remains constant</td>
<td>Wide with periodic clipping at zero</td>
<td>Periodic</td>
<td>Critical</td>
</tr>
<tr>
<td>200</td>
<td>Not close to any resonance</td>
<td>Increases upto the maximum limit but modulates slowly</td>
<td>Limited with periodic clipping</td>
<td>Periodic</td>
<td>Not critical</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Scheme No. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>RPM</td>
</tr>
<tr>
<td>115</td>
</tr>
<tr>
<td>145</td>
</tr>
<tr>
<td>175</td>
</tr>
<tr>
<td>200</td>
</tr>
</tbody>
</table>

Table 5.3: Summary of results
5.3 Inference

From the preceding results the rotary speeds causing wide fluctuation of the WOB can be denoted as the critical rotary speeds. They may be associated with the increase of the lobe amplitude and its sustenance at high values. From the consideration of frequency bandwidth at resonance, it was pointed out by Dareing that "critical rotary speed should be expressed in terms of a critical rotary speed range instead of a single rotary speed ..." [35].

For scheme no. 1 rotary speeds corresponding to axial natural frequencies for either of the two kinds of boundary conditions, that is 75 and 180 rpm, are found to be critical. That corresponding to a torsional natural frequency, 120 rpm, is not found to be so. A speed which corresponds to a torsional natural frequency may cause torsional resonance. But this is not a part of the present investigation. A rotary speed which do not correspond to any natural frequency, as for example 150 rpm, is not critical. A very high rotary speed, like 200 rpm, may not be critical as the fluctuation of the WOB is limited. But the bit jumps over a side of the trough and the WOB drops down to zero for a small fraction of every cycle.

For scheme no. 2 it is verified that the rotary speeds corresponding to axial natural frequencies of both the kinds of boundary conditions are critical, as for example 115, 145 and 200 rpm. It should be mentioned that with the increase in the number of pipe elements, particularly for longer pipes, the resonant frequencies get closely spaced. Hence the vibration is more susceptible to change from one mode to another and it is difficult to maintain a resonant condition under a particular rotary speed [41]. A rotary speed which does not correspond to any natural frequency, such as 175 rpm is not critical. Thus, for the continuous system the rotary speeds corresponding to the axial resonant frequencies, determined from the functions of the driving point mobility and impedance at the bit, are critical regarding the wide fluctuations of the WOB.
Previously natural frequencies of the collar had been considered as approximate estimates for determining the critical rotary speeds. But from this study it is found that a rotary speed, for which the corresponding axial natural frequency is not sufficiently close to a dominant peak of the mobility diagram, may also be critical, as for example 115 rpm. Similar comment applies for a frequency which is not sufficiently close to a dominant trough of the impedance diagram. Hence, if the approximate estimates are applied, several critical speeds may remain undetected. The severity of the effect of course depends on the relative lengths of the collar and the pipe.

On the other hand if the BHA is treated as a lumped mass for axial vibrations, the dominant peaks are not taken into account. Such a treatment is applicable when the length of the collar is quite small as compared to that of the pipe, and the operating rotary speed is low, that is when the operating frequencies are far from those dominating peaks. Severe vibrations occur during the ramping of the rotary speed. But a detailed investigation of this aspect is outside the objective of the present study. The driving torque at the rotary table is found to be considerably large. This has been noticed in the field experiments as well [34]. Of course there are additional resisting torsional forces in the field which are not considered here.
Chapter 6

Concluding Remarks

Rotary drilling with roller cone bits is accompanied with wide and frequent fluctuations of the WOB, which tend to remain undetected in the indicator at the rig floor. It was recorded by downhole MWD instruments that the WOB may even drop down to null quite abruptly and that too periodically. Based on the traces it was postulated that the bit lifts off from the formation surface, corresponding to which the WOB values are zero. On the other hand, samples of cores from hard rock formation show typical lobes on the surface. The lobed formation surface and the mud pressure fluctuation were identified as the causes of the WOB variation and the simultaneous lift-off of the bit.

The generation and sustenance of these lobes were attributed to the hammering effect from axial vibrations of the drillstring. The present study is an initial approach to model the interaction among the lift-off of the bit, the WOB variation and the modulation of the amplitude of the lobes. It has been pointed out that for axial vibrations, a single kind of boundary condition at the bit where either the force or the displacement is specified, is inadequate to capture the intermittent lift-off phenomenon. The boundary condition should be considered to alternate between the above two kinds. This necessitates a numerical solution scheme. For torsional vibrations a boundary condition with specified force has been found to be adequate.

Instead of selecting arbitrary damping coefficients, the present study has modeled the viscous damping based on the drag generated on a body oscillating in a fluid. This generates damping coefficients as functions of the frequency of vibration. To have a standard basis for comparing the calculated natural frequencies, the transfer function method has been found to be suitable. The expression of the TOB has been
developed based on a force balance approach. A method, based on the relative values of time of contact of the bit with the crests and the troughs, has been developed to model the variation of the amplitude of the lobes.

From the numerical results it has been inferred that for the continuous model, the rotary speeds corresponding to the axial resonant frequencies of the system, determined from the functions of the driving point mobility and impedance at the bit, are critical regarding the wide fluctuations of the WOB. They may be associated with the sustenance of the lobes with large amplitude. Approximate estimates from the natural frequencies of the collar may not be sufficient in determining all the critical speeds within that range. Vibrations have been found to be more severe when the rotary speed is ramped than after it becomes steady.

The analytical model thus affords the option of a meaningful investigation of the complex phenomenon of bit lift-off. It is of course evident that the detection of the critical rotary speeds depends on the discretization scheme. Priority should be given to the scheme which can capture, at least approximately, all the resonant frequencies falling within the operating range.

In the present study the formation surface has only one spatial frequency, that corresponding to a tri-lobed formation. This represents the most widely accepted formation surface profile in case of drilling with tri-cone bits. The necessity of having a polyharmonic profile is not felt in modeling the interaction among the axial vibrations of the drillstring, the bit lift-off and the lobe amplitude modulation. Thus, the frequency of excitation is only three times the frequency of the rotary speed and higher multiples are neglected. A polyharmonic profile can be easily incorporated, in which case of course the optimum discretization scheme has to be changed accordingly.

Clearly the present method can be enhanced to reduce the assumptions and idealizations. First the mud pressure fluctuation and the bending vibrations can be incorporated. Next the drillstring model can be elaborated by introducing the formation impedance. Third is the inclusion of the friction from side wall. Note that for tor-
sional vibrations particularly, dry friction is the larger source of damping. Since the solution is based on numerical integration, it can accommodate nonlinear friction models. The development of the lobe amplitude modulation model by considering the cone geometry and the details of the bit-teeth-formation interaction is another possibility. The action of the individual teeth causes a superimposed high frequency fluctuation of the WOB.

For an adequate qualification, the model has to be tested for drillstrings with more complicated geometric properties and drilling under several operating conditions and formations. It can be then used for real-time feedback to the driller.
Bibliography


[75] Paslay, P.R., Jan, Y-M., Kingman, J.E.E and Macpherson, J., Detection of BHA Lateral Resonances While Drilling With Surface Longitudinal and Torsional Sensors, SPE 24583.


Appendix A

Coefficients of Viscous Damping

The objective is to calculate the coefficients of viscous damping for the equations of motion of axial and torsional vibrations of the drillstring.

The total drag force is treated as the algebraic sum of that acting on the outer surface, due to the fluid in the annulus between the drillstring and the bore-hole wall, and that acting on the inner surface due to the fluid inside the drillstring. The drilling muds are non-newtonian in nature. They are pseudo-plastic in their shear rate versus shear stress characteristics and thixotropic in the timewise variation of viscosity. Also, the flow around the drillstring tend to be turbulent. Mathematical modeling of turbulent flow and thixotropic fluids are not yet developed adequately. To simplify the computation, the following idealizations are made.

i. The fluid is newtonian.

ii. The fluid is incompressible, which implies constant density.

iii. The coefficient of viscosity is constant.

iv. The flow of the fluid is isothermal.

v. The flow rate is constant. Specifically, fluctuation of the discharge from the reciprocating pump is not considered.

vi. The flow is laminar.

vii. The net flow is in the direction parallel to the borehole axis.

viii. The flow characteristics are axi-symmetric.

ix. The end-effects are insignificant.
The hydrodynamic equations are written in cylindrical polar co-ordinates, $r, \phi, x$ [60].

Equation of continuity

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_x}{\partial x} = 0.$$  \hspace{1cm} \text{(A.1)}

Equations of motion

$$\rho_t \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} + v_x \frac{\partial v_r}{\partial x} - \frac{v_r^2}{r} \right) = \rho_t f_r - \frac{\partial p}{\partial r} + \mu (\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi}) \hspace{1cm} \text{(A.2)}$$

$$\rho_t \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_x \frac{\partial v_\phi}{\partial x} \right) = \rho_t f_\phi - \frac{1}{r} \frac{\partial p}{\partial \phi} + \mu (\nabla^2 v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi}) \hspace{1cm} \text{(A.3)}$$

$$\rho_t \left( \frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_x}{\partial \phi} + v_x \frac{\partial v_x}{\partial x} \right) = \rho_t f_x - \frac{\partial p}{\partial x} + \mu \nabla^2 v_x \hspace{1cm} \text{(A.4)}$$

where,

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial x^2},$$

$p$ is the hydrostatic pressure, $v_r, v_\phi, v_x$ and $f_r, f_\phi, f_x$ are the radial, transverse and axial components of the velocity and body force respectively. The equation of energy balance is not necessary. The conditions to be appended with the equations are as follows.

i. For the case of straight and vertical bore well, the only body force is due to the gravity which acts along the $x$ direction. Therefore,

$$f_r = 0, \quad f_\phi = 0 \quad \text{and} \quad f_x = g.$$  \hspace{1cm} \text{(A.5)}

ii. As the flow is axi-symmetric, there is no $\phi$ dependency and all the terms with $\frac{\partial}{\partial \phi}$ are zero. Also, the terms with $\frac{\partial}{\partial r}$ evaluated at $r = 0$ are zero.

iii. Due to the constant flow rate, velocity components are independent of $x$. 
From the above conditions and invoking the no-slip condition for the layers of fluid in contact with the drillstring and borehole wall, the boundary conditions for the two regions are as follows.

**Region I** The inner conduit

\[
\begin{align*}
\text{At } r &= 0, \quad \frac{\partial v_r}{\partial r} = 0 \quad \frac{\partial v_\phi}{\partial r} = 0 \quad \frac{\partial v_z}{\partial r} = 0 \\
\text{At } r &= r_1, \quad v_r(r, t) = 0 \quad v_\phi(r, t) = r_1 \frac{\partial \theta}{\partial t} \quad v_z(r, t) = \frac{\partial u}{\partial t} \quad (A.6)
\end{align*}
\]

**Region II** The outer annular conduit

\[
\begin{align*}
\text{At } r &= r_o, \quad v_r(r, t) = 0 \quad v_\phi(r, t) = r_o \frac{\partial \theta}{\partial t} \quad v_z(r, t) = \frac{\partial u}{\partial t} \\
\text{At } r &= r_h, \quad v_r(r, t) = 0 \quad v_\phi(r, t) = 0 \quad v_z(r, t) = 0 \quad (A.7)
\end{align*}
\]

The following three cases are considered separately for the simplification of the solution of the equations.

(A) The fluid is flowing without any vibration of the drillstring.

(B) The drillstring is vibrating axially when the tangential and the net axial flows are zero.

(C) The drillstring is rotating and vibrating torsionally, when the axial flow is zero.

As the effects of the three cases are not being superposed, the separate treatment is justified. For all the three cases, it can be derived from the continuity equation that \(v_r(r, t)\) is zero identically.

**Case A**

As the flow rate is constant the variables are independent of time. From Eq. A.3 it can be shown that \(v_\phi(r) = 0\) identically. Solving Eq. A.4 the velocity profile is given as

\[
v_z(r) = \frac{\dot{\rho}}{4\mu} \left[(r_o^2 - r^2) - (r_2^2 - r_1^2) \ln\left(\frac{r}{r_1}\right)\right] \quad (A.8)
\]
where, $\ddot{p} = \rho t g - \frac{\partial p}{\partial z}$. For region I, $r_1 = 0$, $r_2 = r$. This is the case of Poiseuille flow. For region II, $r_1 = r_o$, $r_2 = r_h$. The value of $\frac{\partial p}{\partial z}$ can be calculated in terms of the flow rate. As the drag generated is constant with respect to time, it is equivalent to a static force and hence it is not included in the damping coefficient.

**Case B**

Here also it can be proved from Eq. A.3 that $v_x(r, t) = 0$ identically. The pressure consists of two parts: $p = p_1(x) + p_2(t)$, where $p_1(x)$ is the one present in the fluid under static equilibrium. It satisfies $\frac{\partial p_1}{\partial x} = \rho t g$. Eq. A.4 then reduces to

$$\rho t \frac{\partial v_x}{\partial t} = \mu \left( \frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} \right)$$

Substituting $V_z(r, t) = F_1(r)e^{i\omega t}$ in the equation, such that $v_x(r, t) = \Re\{V_z(r, t)\}$, and considering $\frac{\partial u}{\partial t} = \tilde{U}e^{i\omega t}$ at any particular location of the drillstring, the equation transforms to

$$\frac{d^2 F_1}{dr^2} + \frac{1}{r} \frac{dF_1}{dr} + \frac{i\omega \rho t}{\mu} F_1 = 0 \quad (A.9)$$

with boundary conditions as follows.

i. For region I: $\frac{dF_1}{dr}|_{r=0} = 0$ and $F_1(r)|_{r=r_1} = \tilde{U}$

ii. For region II: $F_1(r)|_{r=r_o} = \tilde{U}$ and $F_1(r)|_{r=r_h} = 0$.

In the substitution $\frac{\partial u}{\partial t} = \tilde{U}e^{i\omega t}$, vibration of a single frequency is considered. This is the dominant frequency.

**Case C**

Here it can be proved from Eq. A.4 that $v_x(r, t) = 0$ identically. To satisfy Eq. A.2 the pressure has to satisfy the relation $\frac{\partial p}{\partial r} = \rho t \frac{\partial v_x}{\partial r}$, which is the pressure required to develop the centripetal forces. Eq. A.3 reduces to

$$\rho t \frac{\partial v_x}{\partial t} = \mu \left( \frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} - \frac{v_x}{r^2} \right).$$
As before, substituting \( V_\phi(r, t) = F_2(r)e^{i\omega t} \) in the equation and considering \( \frac{\partial \phi}{\partial t} = \ddot{\Theta} e^{i\omega t} \), the equation reduces to

\[
\frac{d^2 F_2}{dr^2} + \frac{1}{r} \frac{dF_2}{dr} + \left( \frac{i^3 \omega \mu}{\mu} - \frac{1}{r^2} \right) F_2 = 0
\] (A.10)

with boundary conditions as follows.

i. For region I : \( \frac{dF_2}{dr} |_{r=0} = 0 \) and \( F_2(r) |_{r=r_1} = r_1 \ddot{\Theta} \)

ii. For region II : \( F_2(r) |_{r=r_o} = r_o \ddot{\Theta} \) and \( F_2(r) |_{r=r_h} = 0 \).

Solution

Both Eqs. A.9 and A.10 are of the form

\[
F'' + \frac{1}{r} F' + \left( z^2 - \frac{n^2}{r^2} \right) F = 0
\] (A.11)

where,

\[ z^2 = -\left( \frac{\omega \mu}{\mu} \right) i. \]

Eq. A.11 is a transformed Bessel equation of order \( n \). The generalized solution is

\[
F(r) = c_1 J_n(zr) + c_2 Y_n(zr)
\] (A.12)

where, \( J_n(zr) \) and \( Y_n(zr) \) are Bessel functions of the first and second kind respectively [79]. Satisfying the boundary conditions, the solutions are

i. for region I :

\[
F(r) = \frac{J_n(zr)}{J_n(zr_1)} X_1
\] (A.13)

ii. for region II :

\[
F(r) = \frac{X_2}{\Delta} [J_n(zr)Y_n(zr_h) - Y_n(zr)J_n(zr_h)]
\] (A.14)

where,

\[ \Delta = J_n(zr_o)Y_n(zr_h) - J_n(zr_h)Y_n(zr_o), \]

\[ X_1 = X_2 = \ddot{U} \text{ for case B and } X_1 = r_1 \ddot{\Theta}, X_2 = r_o \ddot{\Theta} \text{ for case C.} \]
Damping Coefficient

Applying the constitutive law for Newtonian fluids $\tau = \mu \dot{\gamma}$, the viscous damping force $F_{vd}$ per unit length of the drillstring is given as

$$F_{vd} = \sum_{j=1,2} \tau_j A_j = \sum_{j=1,2} \mu \frac{\partial v}{\partial r} 2\pi r_j$$

$$= \mu \pi \left[ \rho_o \Re \{ (J_{n-1}(zr_o) - J_{n+1}(zr_o))Y_n(zr_h) ight. $$

$$\left. - (Y_{n-1}(zr_o) - Y_{n+1}(zr_o))J_n(zr_h) \right] \frac{zX_2}{\Delta} e^{i\omega t}$$

$$+ r_i \Re \{ (J_{n-1}(zr_i) - J_{n+1}(zr_i)) \frac{zX_1}{J_n(zr_i)} e^{i\omega t} \} \right]$$

$$= C_C(\omega) \tilde{X} e^{i\omega t}. \quad (A.15)$$

Here $\sum_{j=1,2}$ represents the algebraic summation for the outer and inner surfaces and $\tilde{X} = \tilde{U}$ for case B and $\tilde{X} = \tilde{G}$ for case C. Thus the damping coefficient is

$$C_C(\omega) = \mu \pi \left[ \rho_o \Re \{ (J_{n-1}(zr_o) - J_{n+1}(zr_o))Y_n(zr_h) ight. $$

$$\left. - (Y_{n-1}(zr_o) - Y_{n+1}(zr_o))J_n(zr_h) \right] \frac{z f_1}{\Delta}$$

$$+ r_i \Re \{ (J_{n-1}(zr_i) - J_{n+1}(zr_i)) \frac{z f_1}{J_n(zr_i)} \} \right] \quad (A.16)$$

where, $f_1 = f_2 = 1$ for case B and $f_1 = r_i$, $f_2 = r_o$ for case C.

If all the frequencies of vibration are to be included then the expression of the damping force per unit length is

$$F_{vd} = \mu \pi \left[ \rho_o \Re \left\{ \int_{-\infty}^{\infty} \frac{(J_{n-1}(zr_o) - J_{n+1}(zr_o))Y_n(zr_h)}{\Delta} $$

$$\frac{zX_2}{e^{i\omega t}} \, d\omega \right\}$$

$$+ r_i \Re \left\{ \int_{-\infty}^{\infty} \frac{(J_{n-1}(zr_i) - J_{n+1}(zr_i))}{J_n(zr_i)} \frac{zX_1}{e^{i\omega t}} \, d\omega \right\} \right] \quad (A.17)$$
Appendix B

Mud Pressure Fluctuation

The objective is to model the force generated on the bit due to the ejection of the drilling mud from the bit nozzle.

For a double acting duplex pump the flow rate \( q(t) \) can be modeled as

\[
q(t) = q_0(|\sin \Psi t| + |\cos \Psi t|),
\]

where each trigonometric term represents one cylinder and the absolute value is required to consider the unidirectional flow of the fluid from the piston strokes. \( \Psi \) is \( 2\pi \times \) the pump cycle. The average and extreme values of the flow rate are then given as

\[
\bar{q} = \frac{1}{T} \int_0^T q(t) \, dt = \frac{4}{\pi} q_0 \\
q_{\text{min}} = q(t = n \frac{T}{8}) = \frac{\pi}{4} \bar{q} \\
q_{\text{max}} = q(t = (2n - 1) \frac{T}{8}) = \frac{\pi}{2\sqrt{2}} \bar{q}.
\]

Thus in terms of the average value, the flow rate variation is

\[
q(t) = \frac{\pi}{4} \bar{q} (|\sin \Psi t| + |\cos \Psi t|).
\]

The flow of the mud through the nozzles of the bit creates an upward thrust, the timewise variation of which is governed by that of the flow rate. Applying the principle of conservation of linear momentum to a mass of the fluid for a certain time interval \( \Delta t \),

\[
I_{|t+\Delta t} = I_{|t} \\
M_b[\dot{U}_b(t) + \Delta \dot{U}_b(t)] + q(t)\rho_f \Delta t[\dot{U}_b(t) + \Delta \dot{U}_b(t) + v_0] = M_b \dot{U}_b(t) + q(t)\rho_f \Delta t[\dot{U}_b(t) + v_0],
\]

\[
= M_b \dot{U}_b(t) + q(t)\rho_f \Delta t[\dot{U}_b(t) + v_0],
\]

\[
= M_b \dot{U}_b(t) + q(t)\rho_f \Delta t[\dot{U}_b(t) + v_0].
\]
as shown in Fig. B.1. Here $I$ is the linear momentum, $M_b$ is the mass of the bit, $v_{ej}$ is the velocity of ejection of the mud from the bit and $v_{cd}$ is the velocity of the mud in the conduit of the bit. Cancelling the common terms and as $\Delta t \to 0$, the differential equation for the thrust is

$$p(t) = M_b \ddot{U}_b(t) = -q(t) \rho_t (v_{ej} - v_{cd})$$
$$= -q^2(t) \rho_t \left( \frac{1}{A_{nz}} - \frac{1}{A_{cd}} \right)$$
$$= p_0(1 + 2|\sin \Psi t||\cos \Psi t|) \quad (B.5)$$

where,

$$p_0 = -\frac{\pi^2}{16} q^2 \rho_t \left( \frac{1}{A_{nz}} - \frac{1}{A_{cd}} \right), \quad (B.6)$$

$A_{nz}$ is the total area of the nozzles and $A_{cd}$ is the area of the conduit in the bit. The drillstring vibration alters the mud pulsations, sometimes even creating pressure surges [49]. This coupling has not been included in this preliminary formulation. Cook et al. (1989, [31]) pointed out that “…estimates of the pressure drop across the bit, using traditional calculation methods based on pump strokes, are likely to be inaccurate”. Hence the above formulation needs further development.
At time 't'

\[ q(t) \rho_f \Delta t [\dot{U}_b(t) + v_{cd}] \]

\[ M_b \dot{U}_b(t) \]

At time 't + \Delta t'

\[ M_b [\dot{U}_b(t) + \Delta \dot{U}_b(t)] \]

\[ q(t) \rho_f \Delta t [\dot{U}_b(t) + \Delta \dot{U}_b(t) + v_{ej}] \]

Figure B.1: Model for the mud pressure fluctuation
Appendix C

Notation and Abbreviations

C.1 Notation

- $a$: variable related with the depth of cut
- $A$: cross-sectional area of the pipe or collar
- $A_{cd}$: area of the conduit within the bit
- $A_{nz}$: total area of the nozzles
- $b, c$: geometric variables of the toothed wheel
- $c$: velocity of the linear wave through the material of the drillstring
- $c_1, c_2$: constants of integration
- $c_{\omega}(\omega)$: frequency dependent damping coefficient
- $c_{ij}$: element of $C_e$
- $c_{k,s}$: parameter in the expression of $M_{ks}(\omega)$
- $C$: damping coefficient of the mass–spring–dashpot system
- $C_C(\omega)$: generalized damping coefficient = $c_\alpha(\omega_k)$ or, $c_1(\omega_k)$ or damping parameter
- $C_K$: generalized stiffness coefficient = $AE$ or, $JG$ or stiffness parameter
- $C_M$: generalized inertia coefficient = $\rho_A A$ or, $\rho_J J$ or mass parameter
- $C$: generalized global damping matrix
- $C_e$: generalized elemental damping matrix
- $C_{ij}$: submatrix of $C$
- $E$: modulus of elasticity of steel
- $f(x, t)$: axial load
- $f_1, f_2$: parameters in the expression of the viscous damping coefficient
- $f_r, f_\phi, f_x$: radial, transverse and axial components of body force
$F(r)$ generalized variable for $F_1(r), F_2(r)$

$F_1(r), F_2(r)$ component functions of velocity after separation of variables

$F_{vd}$ viscous damping force per unit length

$E$ axial load vector

$G$ shear modulus of steel

$h$ distance of the toothed wheel from the cone apex

$H$ height of the cone

$i$ $\sqrt{-1}$

$I$ linear momentum

$J$ polar moment of inertia of the cross-section of the pipe or collar

$J_n(\bullet)$ Bessel function of the first kind of order $n$

$k_\omega$ proportionality constant

$k_{ij}$ element of $K_e$

$K$ stiffness of the mass-spring-dashpot system

$K$ generalized global stiffness matrix

$K_e$ generalized elemental stiffness matrix

$K_{ij}$ submatrix of $K$

$l, L_e$ length of an element

$L$ length of the pipe or collar

$m_{ij}$ element of $M_e$

$M$ mass of the mass-spring-dashpot system

$M_b$ mass of the bit

$M_{bb}(\omega)$ driving point mobility of the bit

$M_{C/K/M}$ mobilities of the dashpot / spring / mass elements

$M_{M-C-K}(\omega)$ resultant mobility of the mass-spring-dashpot system

$M_{ss}(\omega)$ mobility of an element, displacement at $r$ and force at $s$

$M_{ss}(\omega)$ mobility of the suspension
M generalized global mass matrix
\( M_e \) generalized elemental mass matrix
\( M_{ij} \) submatrix of \( M \)
\( N \) number of rotations of the drillstring per unit time
\( N_i(x) \) shape function
\( p \) hydrostatic pressure
\( \bar{p} \) substituted pressure variable
\( p_0 \) amplitude of axial thrust variation
\( p_1, p_2 \) pressure variables
\( p(t) \) axial thrust
\( p(x, t) \) generalized distributed load = \( f(x, t) \) or, \( \tau(x, t) \)
\( \mathcal{P} \) generalized load vector = \( F \) or, \( T \)
\( \bar{q} \) average value of the flow rate
\( q_0 \) amplitude of the flow rate variation
\( q_{\text{max}} \) maximum value of the flow rate
\( q_{\text{min}} \) minimum value of the flow rate
\( q(t) \) flow rate of the mud
\( r \) radial co-ordinate
\( r_h \) radius of the borehole
\( r_{i/o} \) inner/outer radius of the pipe or collar
\( r_w \) radius of the toothed wheel
\( r_1, r_2 \) radius variables
\( R \) average rate of penetration
\( \Re(\sim) \) real part of \( \sim \)
\( S(r, \phi) \) elevation variable for the formation surface profile
\( S_0 \) amplitude of sinusoidal variation of the formation surface profile
\( \hat{S}_0 \) modified amplitude of the formation surface profile
\( t \)  
\( t_r \)  
\( T \)  
\( T \)  
\( T_{dr} \)  
\( T_{TOB} \)  
\( u \)  
\( \tilde{U} \)  
\( U \)  
\( v_{cd} \)  
\( v_{ej} \)  
\( v_r, v_\phi, v_x \)  
\( W \)  
\( \tilde{W} \)  
\( x \)  
\( x_r \)  
\( X \)  
\( X_1, X_2 \)  
\( \tilde{X} \)  
\( \dot{X} \)  
\( Y_n(\cdot) \)  
\( z \)  
\( Z_{bb}(\omega) \)  
\( Z_{rs}(\omega) \)  
\( Z_{ss}(\omega) \)  

- time variable
- rise time
- time period of one pump cycle
- torsional load vector
- driving torque at the rotary table
- torque on bit
- axial displacement of a section of the pipe or collar
- Fourier transform of \( \frac{\partial u}{\partial t} \)
- axial displacement vector
- velocity of drilling fluid in the conduit of the bit
- velocity of drilling fluid during ejection from the bit
- radial, transverse and axial components of velocity
- weight on bit
- applied weight on bit
- axial co-ordinate
- axial co-ordinate of the end designated by \( r \)
- generalized displacement = \( u \) or \( \theta \)
- variables related with the Fourier transforms \( \tilde{U} \) and \( \dot{\theta} \)
- generalized variable for \( \tilde{U} \) and \( \dot{\theta} \)
- generalized displacement vector = \( U \) or \( \Phi \)
- Bessel function of the second kind of order \( n \)
- parameter in the expression of the viscous damping coefficient
- driving point impedance of the bit
- impedance of an element, displacement at \( r \) and force at \( s \)
- impedance of the suspension
\( \sim_a \) for axial vibrations

\( \sim_b \) of the bit

\( \sim_c \) for crest or collar or cut

\( \sim_C \) for damping element

\( \sim_{cd} \) for the conduit within the bit

\( \sim_e \) for the element

\( \sim_{eff} \) effective value of \( \sim \)

\( \sim_{ej} \) for ejection

\( \sim_f \) for the drilling fluid or final value

\( \sim_h \) for the borehole or horizontal component

\( \sim_i \) inner

\( \sim_{ip} \) interpolated value

\( \sim_k \) known value of \( \sim \)

\( \sim_K \) for stiffness element

\( \sim_M \) for mass element

\( \sim_{nz} \) for the nozzles

\( \sim_o \) outer

\( \sim_p \) for pipe or previous time step

\( \sim_r \) for radial component or rise

\( \sim_s \) for steel or the suspension

\( \sim_t \) for torsional vibrations or trough

\( \sim_u \) unknown value of \( \sim \)

\( \sim_v \) for vertical component

\( \sim_{vd} \) viscous damping

\( \sim_w \) for the toothed wheel

\( \sim_x \) for axial component
\( \alpha \) argument of the slope of the surface profile

\( \beta \) parameter of the Newmark–Wilson method

\( \gamma \) basic cone angle

\( \dot{\gamma} \) rate of shear strain

\( \delta \) parameter of the Newmark–Wilson method

\( \delta(x) \) delta function

\( \delta_c \) depth of cut per revolution of the bit

\( \delta H \) distance of the tip of the body of the cone from the tip of the entire cone which includes the teeth

\( \Delta \) parameter in the expression of the viscous damping coefficient

\( \Delta F_h \) horizontal component of the force acting at the center of the toothed wheel

\( \Delta F_v \) vertical component of the force acting at the center of the toothed wheel

\( \Delta h \) thickness of the toothed wheel

\( \Delta r_h \) radius of the small concentric circular zone in the formation surface

\( \Delta S_{0*} \) change in the amplitude of the lobes for the crests / troughs

\( \Delta t \) time step

\( \Delta t_{ip} \) interpolated time step

\( \Delta t_* \) time per revolution during which the bit rolls over the crests / troughs

\( \mu \) coefficient of viscosity of the mud

\( \nu_k \) \( k \)th natural frequency of an element

\( \pi \) ratio of the circumference to the diameter of a circle

\( \rho_t \) density of the mud

\( \rho_s \) density of the steel

\( \theta \) absolute angular displacement of a section of the pipe or collar

\( \vartheta \) parameter of the Newmark–Wilson method

\( \Theta_0 \) initial angular position of the drillstring

\( \tilde{\Theta} \) Fourier transform of \( \frac{\partial \theta}{\partial t} \)
\( \Theta \)  angular displacement vector
\( \tau \)  shear stress
\( \tau(x,t) \)  torsional load
\( \phi \)  angular co-ordinate
\( \psi_{k,r} \)  value of the normalized mode shape of an element at the location \( r \), corresponding to the frequency \( \nu_k \)
\( \Psi \)  \( 2\pi \times \) pump cycle
\( \omega_\omega \)  frequency of vibration
\( \omega \)  generalized frequency of vibration = \( \omega_h \) or, \( \omega_l \)
\( \Omega \)  applied rotary speed
\( \dot{\Omega} \)  ramping rate of the rotary speed
\( \Omega_f \)  final value of \( \Omega \)

\~\phi \quad \text{for transverse component}

C.2  Abbreviations

BHA  bottom-hole assembly
bit  drill bit
collar  drill collar
mud  drilling mud
MWD  measurement while drilling
PDC  polycrystalline diamond compact
pipe  drillpipe
ROP  rate of penetration
SDOF  single degree of freedom
suspension  suspension system
TOB  torque on bit
WOB  weight on bit