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Numerical study of three dimensional incomprehensible thermal flows in complex geometries

Moreno, Rafael, M.S.
Rice University, 1994

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NUMERICAL STUDY OF THREE DIMENSIONAL INCOMPRESSIBLE THERMAL FLOWS IN COMPLEX GEOMETRIES

by

Rafael Moreno

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE Master of Science

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Rafael Moreno

Abstract

In our study, an iterative point successive over-relaxation (PSOR) finite difference scheme has been used to solve the coupled unsteady Navier-Stokes and energy equations for incompressible, viscous and laminar flows in their primitive variable form. Three problems have been studied in detail: (1) two-dimensional and three-dimensional natural convection in a cavity with differentially heated vertical walls; (2) two-dimensional and three-dimensional natural convection in cavity whose surface is cooled while two internal blocks are heated; (3) two-dimensional and three-dimensional natural convection in the region defined by two interconnected cavities of different sizes which are differentially heated. All computations have been performed for a Prandtl number of 1.0, and different values of the Rayleigh number ranging between $10^3$ and $10^7$ depending on the problem. The scheme has been found to be accurate even for large Rayleigh numbers.
Acknowledgments

I sincerely thank my thesis advisor Dr. Balasubramaniam Ramaswamy, for his guidance, advice, and encouragement during this research. I would also like to thank Dr. John E. Akin for serving on the thesis committee, Dr. Andrew J. Meade for his insight on where this work can lead me in the future, and my wife Evelmarie for her patience and support during the final stages of this work.

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Final thanks to Mr. Alvaro A. Fernandez and Mr. Gregory S. Lind with whom I had many fruitful discussions on the future path of this research.
# Nomenclature

<table>
<thead>
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<th>Symbols</th>
<th>Explanation</th>
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<tbody>
<tr>
<td>$x^<em>, y^</em>, z^*$</td>
<td>coordinate system with nodal origin</td>
</tr>
<tr>
<td>$u^*$</td>
<td>horizontal velocity in the $x^*$ direction</td>
</tr>
<tr>
<td>$v^*$</td>
<td>vertical velocity in the $y^*$ direction</td>
</tr>
<tr>
<td>$w^*$</td>
<td>velocity in the $z^*$ direction</td>
</tr>
<tr>
<td>$p^*$</td>
<td>pressure</td>
</tr>
<tr>
<td>$t^*$</td>
<td>time</td>
</tr>
<tr>
<td>$\theta^*$</td>
<td>temperature</td>
</tr>
<tr>
<td>$\theta_r^*$</td>
<td>reference temperature</td>
</tr>
<tr>
<td>$\theta_h^*$</td>
<td>maximum (hot) temperature</td>
</tr>
<tr>
<td>$\theta_c^*$</td>
<td>minimum (cold) temperature</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>non-dimensional coordinate system with nodal origin</td>
</tr>
<tr>
<td>$u$</td>
<td>non-dimensional velocity in the $x$ direction</td>
</tr>
<tr>
<td>$v$</td>
<td>non-dimensional vertical velocity in the $y$ direction</td>
</tr>
<tr>
<td>$w$</td>
<td>non-dimensional velocity in the $z$ direction</td>
</tr>
<tr>
<td>$p$</td>
<td>non-dimensional pressure</td>
</tr>
<tr>
<td>$t$</td>
<td>non-dimensional time</td>
</tr>
<tr>
<td>$\theta$</td>
<td>non-dimensional temperature</td>
</tr>
<tr>
<td>$\ddot{u}$</td>
<td>intermediate velocity in the $x$ direction</td>
</tr>
<tr>
<td>$\ddot{v}$</td>
<td>intermediate velocity in the $y$ direction</td>
</tr>
</tbody>
</table>
\( \tilde{\omega} \) intermediate velocity in the \( z \) direction

\( U \) reference velocity

\( g \) gravitational acceleration

\( L \) length of cavity in the \( x \) direction

\( H \) length of cavity in the \( y \) direction

\( W \) length of cavity in the \( z \) direction

\( \Delta x \) mesh spacing in the \( x \) direction

\( \Delta y \) mesh spacing in the \( y \) direction

\( \Delta z \) mesh spacing in the \( z \) direction

\( \Delta t \) time step

\( N_x \) number of grid points in the \( x \) direction

\( N_y \) number of grid points in the \( y \) direction

\( N_z \) number of grid points in the \( z \) direction

\( \bar{x}, \bar{y}, \bar{z} \) non-dimensional coordinate system with cell-centered origin

\( N_{\bar{x}} \) number of grid points in the \( \bar{x} \) direction

\( N_{\bar{y}} \) number of grid points in the \( \bar{y} \) direction

\( N_{\bar{z}} \) number of grid points in the \( \bar{z} \) direction

\( ppp \) non-dimensional pressure at the cell centers

\( X, M \) unknown quantities

\( a, b, c, d, e, f \) coefficients

\( A, B, C, D, E, F \) coefficients

\( U_1, U_2, U_3, U_4 \) coefficients

\( V_1, V_2, V_3, V_4 \) coefficients

\( W_1, W_2, W_3, W_4 \) coefficients

\( \theta_1, \theta_2, \theta_3, \theta_4 \) coefficients

\( c_1 \) to \( c_{20} \) coefficients
Greek Letters

\(\alpha\)  thermal diffusivity  \\
\(\bar{\beta}\)  thermal expansion coefficient  \\
\(\rho\)  density  \\
\(\mu\)  viscosity  \\
\(\nu\)  kinematic viscosity  \\
\(\epsilon\)  iterative error  \\
\(\omega\)  overrelaxation parameter  \\
\(\omega_u\)  overrelaxation parameter for \(\bar{u}\) calculation  \\
\(\omega_v\)  overrelaxation parameter for \(\bar{v}\) calculation  \\
\(\omega_w\)  overrelaxation parameter for \(\bar{w}\) calculation  \\
\(\omega_{ppp}\)  overrelaxation parameter for \(ppp\) calculation  \\
\(\omega_\theta\)  overrelaxation parameter for \(\theta\) calculation  \\
\(\Gamma\)  system boundary  \\
\(\theta\)  non-dimensional temperature

Subscripts

\(i\)  grid point counter in the \(x\) direction  \\
\(j\)  grid point counter in the \(y\) direction  \\
\(k\)  grid point counter in the \(z\) direction  \\
\(r\)  grid point counter in the \(\bar{x}\) direction  \\
\(s\)  grid point counter in the \(\bar{y}\) direction  \\
\(t\)  grid point counter in the \(\bar{z}\) direction
Superscripts

$m \quad$ dummy time index for counting iterations

$n \quad$ time index

Nondimensional Numbers

$Ra \quad$ Rayleigh number

$Re \quad$ Reynolds number

$Pr \quad$ Prandtl number

$Nu \quad$ Nusselt number

Note: The symbols defined above are subject to alternation on occasion.
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Chapter 1

Introduction

Three-dimensional buoyancy-driven flow phenomena are part of every day life. The flow of air in a building, the heating and cooling of electronic equipment by natural and forced convection, and the heating of bottled products for pasteurization are only a few examples of how three-dimensional flows are manipulated. Nevertheless, accurate simulation of these problems is desired in order to obtain quantitative and qualitative information which can then be used to improve and even maximize the effectiveness of the processes that these flows regulate. This is no simple task, due not only to the complexity of both the equations and the domains in question, but also to computational time limitations. However, the implementation of finite difference schemes for the solution of the pertaining partial differential equations has proven successful now and in the past.

Buoyancy-driven flows, specially in two (2) dimensions have been the object of thorough study for over fifty years. Due to the nature of these partial differential equations, most studies in the past deal with simplified rectangular domains with different aspect ratios. Nevertheless, without a rigorous, accurate and well understood numerical solution, there was no real way of comparing different numerical schemes in terms of their accuracy, and now way of validating their solutions.

In 1983, de Vahl Davis [1] presented a study which became a bench mark solution for the basic problem of a square cavity (1:1 ratio of height and width) which is heated from the left, cooled on the right and insulated on its top and bottom boundaries. He used the stream-vorticity formulation of the governing equations. Today, there
are a few other benchmark solutions available, such as that by Saitoh and Hirose [2]. These solutions are now used as validation tools for both old and new schemes and solutions. Most computational fluid dynamic studies of buoyant-driven flows in the present use this benchmark problem as their test problem.

With the accelerated rate at which computer speed has increased in the past thirty years and the drop in the cost of performing computations [3], problems that could not be solved years ago are now in our grasp. One such area deals with the study of three-dimensional fluid dynamic problems, and in particular with Navier-Stokes flows in three (3) dimensions. We have been involved in the study of the confined flow of a laminar, incompressible and viscous fluid which is subjected to differential heating and is characterized by two dimensionless parameters: the Rayleigh number and the Prandtl number. In the study the Prandtl number is one (1) and the Rayleigh number is varied from $10^3$ to $10^7$. This involves the solution of the coupled momentum, energy and continuity equations for a fluid. Our goal is to show that the actually available computer resources are sufficient for the solution of simple and complex three-dimensional problems, that the use of iterative schemes in three-dimensions is easy to implement, and that with careful tracking of the problem variables these schemes can generate fast and accurate solutions. To accomplish these goals, we have duplicated and extended several two-dimensional studies in three-dimensions.

We have implemented the fractional step method or projection method proposed by Chorin, by Temam and by Fortin et al. [4]. Using such an approach we are able to solve the governing equations in their primitive variable form, obtain an explicit formulation for the pressure together with valid boundary conditions for it, and march accurately through time. The pressure calculations are initially done at the center of the mesh cells, but it is later distributed to the mesh nodes in order to correct the fluid velocities at those points.
In order to solve for the unknowns, we identified about a dozen possible solution algorithms [3] [5]. We chose to use the point succesive overrelaxation (PSOR) scheme because, using a first-order accurate finite difference in time as Fortin [4], the fractional step method transforms the hyperbolic governing equations over time into elliptic equations at every time step. The PSOR scheme is efficient in the solution of elliptic partial differential equations and it is relatively easy to implement in the computer. It is also very flexible in terms of the required accuracy, which can be controlled by changing the tolerance of the iterative solver.

In chapter 3 we solve the bench mark problem in two-dimmensions. Then we proceed to extend it to three-dimensions. We have found several studies of this form and we use the one by Pepper [6] for qualitative comparison. In chapter 4, we increase the complexity of the system by solving the problem where two heat sources are placed inside a square cavity whose walls are cooled. This problem was solved in two dimensions by Adlam [7]. We have extended his work from two to three dimensions. Finally, in chapter 5 we wanted to explore the flow of gases in a three-dimensional structure, so we decided to extend the work of Evren-Selamet, Arpacı and Borgnakke [8] from the two-dimensional realm to the three-dimensional one.
Chapter 2

Mathematical Model and Numerical Algorithms

2.1 Problem Formulation and Governing Equations

In the current study, the fluid in question is considered to be incompressible and viscous, the flow is laminar, and the domain is three-dimensional (3-D). To examine the fluid flow, we will solve the unsteady Navier-Stokes equations coupled with the energy equation, all in their primitive variable \((u-v-w-p-\theta)\) form. In this formulation, \(u, v, \) and \(w\) represent the fluid velocities in the \(x, y\) and \(z\) directions respectively, while \(p\) and \(\theta\) represent pressure and temperature in that order. This approach was selected over the streamline-vorticity approach due to the fact that the primitive variable introduces unknowns that are directly observable in real-life systems.

The three-dimensional unit volume of the problem in question is covered by a regular mesh with spacings of length \(\Delta x, \Delta y\) and \(\Delta z\) such that the number of grid points in the \(x, y\) and \(z\) directions is \(N_x, N_y\) and \(N_z\), respectively. The length of the cavity in the \(x\) direction is \(L\), the length of the cavity in the \(y\) direction is \(H\), and the length in the \(z\) direction is \(W\).

By introducing a reference velocity

\[
U = \frac{\alpha}{H} \tag{2.1}
\]

the set of non-dimensional variables

\[
u = \frac{u^*}{U}, \quad v = \frac{v^*}{U}, \quad w = \frac{w^*}{U}, \quad \phi = \frac{(\theta^* - \theta_e^*)}{(\theta_h^* - \theta_e^*)} \tag{2.2}
\]
\[ x = \frac{x^*}{H} \quad y = \frac{y^*}{H} \quad z = \frac{z^*}{H} \]  
\[ p = \frac{p^*}{\rho U^2} \]  
\[ t = \frac{t^* \alpha}{H^2} \]  
is generated, as well as the set of dimensionless parameters
\[ Ra = \frac{\rho \beta g (\theta_b^* - \theta_e^*) H^3}{\mu \alpha} \]  
\[ Re = \frac{U H}{\nu} \]  
\[ Pr = \frac{\nu}{\alpha} \]  
Given these, we can write the momentum, energy and conservation of mass equations as
\[ a \frac{\partial u}{\partial t} + b \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + c \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]  
\[ a \frac{\partial v}{\partial t} + b \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = - \frac{\partial p}{\partial y} + c \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + s \theta \]  
\[ a \frac{\partial w}{\partial t} + b \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} + c \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \]  
\[ f \frac{\partial \theta}{\partial t} + d \left( u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right) = e \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) \]  
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  
\[ \frac{\partial \theta}{\partial t} + \frac{\partial u \theta}{\partial x} + \frac{\partial v \theta}{\partial y} + \frac{\partial w \theta}{\partial z} = 0 \]
where, in our study,

\[ a = 1 \quad b = 1 \quad c = Pr \quad d = 1 \quad e = 1 \quad f = 1 \quad s = Ra Pr \quad (2.15) \]

As we can observe from the equations, there is no explicit formulation for the pressure. This is one of the reasons why the primitive variable formulation is sometimes avoided. In addition, there is no explicit information available about the initial pressure profile or the pressure boundary conditions. The non-dimensional form of these equations has been used extensively in the literature [9] [10].

In order to solve the set of coupled PDE's we will use a finite difference approach. In addition, we will transform the equations using the fractional step method such that we obtain an explicit equation for the pressure. In the fractional step method, a set of intermediate velocities \( \bar{u} \), \( \bar{v} \) and \( \bar{w} \) are introduced such that they are independent from the pressure gradients. The fractional step method entails introducing these intermediate velocities into the governing equations, therefore decoupling them from the pressure. The selected finite difference approach involves central differencing in the advective and convective parts of the equations, and forward differencing in time. This process yields a set of finite difference equations as follows:

**step 1:**

**x-momentum without pressure term**:

\[ a \frac{\partial u}{\partial t} + b \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = c \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (2.16) \]

\[ a \frac{\bar{u}_{i,j,k}^{n+1} - u_{i,j,k}^n}{\Delta t} + b u \left( \frac{u_{i+1,j,k}^n - u_{i-1,j,k}^n}{2\Delta x} \right) + b v \left( \frac{u_{i,j+1,k}^n - u_{i,j-1,k}^n}{2\Delta y} \right) \]

\[ + b w \left( \frac{u_{i,j,k+1}^n - u_{i,j,k-1}^n}{2\Delta z} \right) = c \left( \frac{\bar{u}_{i,j,k}^{n+1} - 2\bar{u}_{i,j,k}^{n} + \bar{u}_{i,j,k}^{-1}}{\Delta x^2} \right) \quad (2.17) \]
\[ +c \left( \frac{\tilde{u}_{i,j-1,k}^{n+1} - 2\tilde{u}_{i,j,k}^{n+1} + \tilde{u}_{i,j+1,k}^{n+1}}{\Delta y^2} \right) + c \left( \frac{\tilde{u}_{i,j,k-1}^{n+1} - 2\tilde{u}_{i,j,k}^{n+1} + \tilde{u}_{i,j,k+1}^{n+1}}{\Delta z^2} \right) \]

\[ + \left[ \frac{a}{\Delta t} + \frac{2c}{\Delta x^2} + \frac{2c}{\Delta y^2} + \frac{2c}{\Delta z^2} \right] \tilde{u}_{i,j,k}^{n+1} + \left[ - \frac{c}{\Delta x^2} \right] \tilde{u}_{i-1,j,k}^{n+1} + \left[ - \frac{c}{\Delta z^2} \right] \tilde{u}_{i+1,j,k}^{n+1} \]

\[ + \left[ \frac{-c}{\Delta y^2} \right] \tilde{u}_{i,j-1,k}^{n+1} + \left[ - \frac{c}{\Delta y^2} \right] \tilde{u}_{i,j+1,k}^{n+1} + \left[ - \frac{c}{\Delta z^2} \right] \tilde{u}_{i,j,k-1}^{n+1} + \left[ - \frac{c}{\Delta z^2} \right] \tilde{u}_{i,j,k+1}^{n+1} \]

\[ = \left[ \frac{a}{\Delta t} \right] u_{i,j,k}^n + \left[ - \frac{b}{2\Delta x} \right] \left[ (u_{i,j,k}^n)(u_{i+1,j,k}^n) - (u_{i,j,k}^n)(u_{i-1,j,k}^n) \right] \]

\[ + \left[ - \frac{b}{2\Delta y} \right] \left[ (u_{i,j,k}^n)(u_{i,j+1,k}^n) - (u_{i,j,k}^n)(u_{i,j-1,k}^n) \right] \]

\[ + \left[ - \frac{b}{2\Delta z} \right] \left[ (u_{i,j,k}^n)(u_{i,j,k+1}^n) - (u_{i,j,k}^n)(u_{i,j,k-1}^n) \right] \]

\[ y\text{-momentum without pressure term:} \]

\[ a \frac{\partial v}{\partial t} + b \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = c \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + s \theta \]  

\[ a \frac{\tilde{v}_{i,j,k}^{n+1} - \tilde{v}_{i,j,k}^{n}}{\Delta t} + bu \left( \frac{v_{i+1,j,k}^{n+1} - v_{i-1,j,k}^{n}}{2\Delta x} \right) + bv \left( \frac{v_{i,j+1,k}^{n+1} - v_{i,j-1,k}^{n}}{2\Delta y} \right) \]

\[ +bw \left( \frac{v_{i,j,k+1}^{n} - v_{i,j,k-1}^{n}}{2\Delta z} \right) = c \left( \frac{\tilde{v}_{i-1,j,k}^{n+1} - 2\tilde{v}_{i,j,k}^{n+1} + \tilde{v}_{i+1,j,k}^{n+1}}{\Delta x^2} \right) \]

\[ +c \left( \frac{\tilde{v}_{i,j-1,k}^{n+1} - 2\tilde{v}_{i,j,k}^{n+1} + \tilde{v}_{i,j+1,k}^{n+1}}{\Delta y^2} \right) + c \left( \frac{\tilde{v}_{i,j,k-1}^{n+1} - 2\tilde{v}_{i,j,k}^{n+1} + \tilde{v}_{i,j,k+1}^{n+1}}{\Delta z^2} \right) + s \theta_{i,j,k}^n \]  

\[ = \left[ \frac{a}{\Delta t} + \frac{2c}{\Delta x^2} + \frac{2c}{\Delta y^2} + \frac{2c}{\Delta z^2} \right] \tilde{v}_{i,j,k}^{n+1} + \left[ - \frac{c}{\Delta x^2} \right] \tilde{v}_{i-1,j,k}^{n+1} + \left[ - \frac{c}{\Delta z^2} \right] \tilde{v}_{i+1,j,k}^{n+1} \]

\[ + \left[ - \frac{c}{\Delta y^2} \right] \tilde{v}_{i,j-1,k}^{n+1} + \left[ - \frac{c}{\Delta y^2} \right] \tilde{v}_{i,j+1,k}^{n+1} + \left[ - \frac{c}{\Delta z^2} \right] \tilde{v}_{i,j,k-1}^{n+1} + \left[ - \frac{c}{\Delta z^2} \right] \tilde{v}_{i,j,k+1}^{n+1} \]
\[
\begin{align*}
\frac{a}{\Delta t} w_{i,j,k}^{n+1} &+ \left[ -\frac{b}{2\Delta x} \right] \left[ (u_{i,j,k}^{n}) (w_{i+1,j,k}^{n}) - (u_{i,j,k}^{n}) (w_{i-1,j,k}^{n}) \right] \\
&+ \left[ -\frac{b}{2\Delta y} \right] \left[ (v_{i,j,k}^{n}) (w_{i,j+1,k}^{n}) - (v_{i,j,k}^{n}) (w_{i,j-1,k}^{n}) \right] \\
&+ \left[ -\frac{b}{2\Delta z} \right] \left[ (w_{i,j,k}^{n}) (w_{i,j,k+1}^{n}) - (w_{i,j,k}^{n}) (w_{i,j,k-1}^{n}) \right] + s\theta_{i,j,k}^{n} \\
\end{align*}
\]

z-momentum without pressure term:

\[
\frac{a}{\Delta t} \frac{\partial w}{\partial t} + b \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = c \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{2.22}
\]

\[
\frac{a}{\Delta t} \frac{\bar{w}_{i,j,k}^{n+1} - \bar{w}_{i,j,k}^{n}}{2\Delta t} + \frac{b}{2\Delta x} \left( \frac{w_{i+1,j,k}^{n} - w_{i-1,j,k}^{n}}{2\Delta x} \right) + \frac{b}{2\Delta y} \left( \frac{w_{i,j+1,k}^{n} - w_{i,j-1,k}^{n}}{2\Delta y} \right) \tag{2.23}
\]

\[
+ b \frac{w_{i,j,k+1}^{n} - \bar{w}_{i,j,k}^{n}}{2\Delta z} = c \left( \frac{\bar{w}_{i,j,k+1}^{n+1} - 2\bar{w}_{i,j,k}^{n+1} + \bar{w}_{i,j,k}^{n+1}}{\Delta x^2} \right) \\
+ c \left( \frac{\bar{w}_{i-1,j,k}^{n+1} - 2\bar{w}_{i,j,k}^{n+1} + \bar{w}_{i+1,j,k}^{n+1}}{\Delta y^2} \right) + c \left( \frac{\bar{w}_{i,j,k+1}^{n+1} - 2\bar{w}_{i,j,k+1}^{n+1} + \bar{w}_{i,j,k}^{n+1}}{\Delta z^2} \right) \\
\left[ \frac{a}{\Delta t} + \frac{2c}{\Delta x^2} + \frac{2c}{\Delta y^2} + \frac{2c}{\Delta z^2} \right] \bar{w}_{i,j,k}^{n+1} + \left[ -\frac{c}{\Delta x^2} \right] \bar{w}_{i-1,j,k}^{n+1} + \left[ -\frac{c}{\Delta y^2} \right] \bar{w}_{i,j,k+1}^{n+1} \tag{2.24}
\]

\[
\left[ -\frac{c}{\Delta y^2} \right] \bar{w}_{i,j,k-1}^{n+1} + \left[ -\frac{c}{\Delta z^2} \right] \bar{w}_{i,j,k+1}^{n+1} + \left[ -\frac{c}{\Delta z^2} \right] \bar{w}_{i,j,k-1}^{n+1} + \left[ -\frac{c}{\Delta z^2} \right] \bar{w}_{i,j,k+1}^{n+1}
\]

\[
= \left[ \frac{a}{\Delta t} \right] w_{i,j,k}^{n} + \left[ -\frac{b}{2\Delta x} \right] \left[ (u_{i,j,k}^{n}) (w_{i+1,j,k}^{n}) - (u_{i,j,k}^{n}) (w_{i-1,j,k}^{n}) \right]
\]

\[
+ \left[ -\frac{b}{2\Delta y} \right] \left[ (v_{i,j,k}^{n}) (w_{i,j+1,k}^{n}) - (v_{i,j,k}^{n}) (w_{i,j-1,k}^{n}) \right]
\]

\[
+ \left[ -\frac{b}{2\Delta z} \right] \left[ (w_{i,j,k}^{n}) (w_{i,j,k+1}^{n}) - (w_{i,j,k}^{n}) (w_{i,j,k-1}^{n}) \right]
\]
step 2:

pressure equation and velocity correctors:

u velocity corrector:

\[
a \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}
\]  \hspace{1cm} (2.25)

\[
a \frac{u_{i,j,k}^{n+1} - \bar{u}_{i,j,k}^{n+1}}{\Delta t} = -(\frac{\partial p}{\partial x})^{n+1}
\]  \hspace{1cm} (2.26)

\[
u_{i,j,k}^{n+1} = \bar{u}_{i,j,k}^{n+1} - \left[ \frac{\Delta t}{2a\Delta x} \right] (p_{i+1,j,k}^{n+1} - p_{i-1,j,k}^{n+1})
\]  \hspace{1cm} (2.27)

partial derivative of equation 2.25 with respect to x:

\[
\frac{\partial u_{i,j,k}^{n+1}}{\partial x} = \frac{\partial \bar{u}_{i,j,k}^{n+1}}{\partial x} - \left[ \frac{\Delta t}{a} \right] \left( \frac{\partial^2 p}{\partial x^2} \right)^{n+1}
\]  \hspace{1cm} (2.28)

v velocity corrector:

\[
a \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}
\]  \hspace{1cm} (2.29)

\[
a \frac{v_{i,j,k}^{n+1} - \bar{v}_{i,j,k}^{n+1}}{\Delta t} = -(\frac{\partial p}{\partial y})^{n+1}
\]  \hspace{1cm} (2.30)

\[
v_{i,j,k}^{n+1} = \bar{v}_{i,j,k}^{n+1} - \left[ \frac{\Delta t}{2a\Delta y} \right] (p_{i,j+1,k}^{n+1} - p_{i,j-1,k}^{n+1})
\]  \hspace{1cm} (2.31)

partial derivative of equation 2.29 with respect to y:

\[
\frac{\partial v_{i,j,k}^{n+1}}{\partial y} = \frac{\partial \bar{v}_{i,j,k}^{n+1}}{\partial y} - \left[ \frac{\Delta t}{a} \right] \left( \frac{\partial^2 p}{\partial y^2} \right)^{n+1}
\]  \hspace{1cm} (2.32)

w velocity corrector:

\[
a \frac{\partial w}{\partial l} = -\frac{\partial p}{\partial z}
\]  \hspace{1cm} (2.33)

\[
a \frac{w_{i,j,k}^{n+1} - \bar{w}_{i,j,k}^{n+1}}{\Delta t} = -(\frac{\partial p}{\partial z})^{n+1}
\]  \hspace{1cm} (2.34)

\[
w_{i,j,k}^{n+1} = \bar{w}_{i,j,k}^{n+1} - \left[ \frac{\Delta t}{2a\Delta z} \right] (p_{i,j,k+1}^{n+1} - p_{i,j,k-1}^{n+1})
\]  \hspace{1cm} (2.35)
partial derivative of equation 2.33 with respect to z:

\[
\frac{\partial u_{i,j,k}^{n+1}}{\partial z} = \frac{\partial \bar{u}_{i,j,k}^{n+1}}{\partial z} - \left[ \frac{\Delta t}{a} \right] \left( \frac{\partial^2 p}{\partial z^2} \right)^{n+1}
\]  

(2.36)

If we add up equations 2.28, 2.32 and 2.36 we get

\[
\left( \frac{\partial u_{i,j,k}^{n+1}}{\partial x} + \frac{\partial v_{i,j,k}^{n+1}}{\partial y} + \frac{\partial w_{i,j,k}^{n+1}}{\partial z} \right) = \left( \frac{\partial \bar{v}_{i,j,k}^{n+1}}{\partial x} + \frac{\partial \bar{v}_{i,j,k}^{n+1}}{\partial y} + \frac{\partial \bar{w}_{i,j,k}^{n+1}}{\partial z} \right)
\]  

\[- \left[ \frac{\Delta t}{a} \right] \left( \frac{\partial^2 p_{i,j,k}^{n+1}}{\partial x^2} + \frac{\partial^2 p_{i,j,k}^{n+1}}{\partial y^2} + \frac{\partial^2 p_{i,j,k}^{n+1}}{\partial z^2} \right)
\]  

(2.37)

As one can see, the left hand side of equation 2.37 is the same as equation 2.14, i.e. a statement of conservation of mass, and is therefore equal to zero, such that we are left with

\[
\left( \frac{\partial^2 p_{i,j,k}^{n+1}}{\partial x^2} + \frac{\partial^2 p_{i,j,k}^{n+1}}{\partial y^2} + \frac{\partial^2 p_{i,j,k}^{n+1}}{\partial z^2} \right) = \left[ \frac{a}{\Delta t} \right] \left( \frac{\partial \bar{u}_{i,j,k}^{n+1}}{\partial x} + \frac{\partial \bar{v}_{i,j,k}^{n+1}}{\partial y} + \frac{\partial \bar{w}_{i,j,k}^{n+1}}{\partial z} \right)
\]  

(2.38)

because, in general, the intermediate velocities do not satisfy the conservation of mass equation.

Equations 2.27, 2.31 and 2.35 are used to correct the intermediate velocities $\bar{u}$, $\bar{v}$ and $\bar{w}$ into $u$, $v$ and $w$ respectively, after the pressure has been updated by solving equation 2.38.

step 3:

energy equation:

\[
f \frac{\partial \theta}{\partial t} + d \left( u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right) = e \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right)
\]  

(2.39)

\[
f \frac{\bar{\theta}_{i,j,k}^{n+1} - \theta_{i,j,k}^{n}}{\Delta t} + du \left( \frac{\theta_{i+1,j,k}^{n} - \theta_{i-1,j,k}^{n}}{2\Delta x} \right) + dv \left( \frac{\theta_{i,j+1,k}^{n} - \theta_{i,j-1,k}^{n}}{2\Delta y} \right) + dw \left( \frac{\theta_{i,j,k+1}^{n} - \theta_{i,j,k-1}^{n}}{2\Delta z} \right) = e \left( \frac{\bar{\theta}_{i-1,j,k}^{n+1} - 2\bar{\theta}_{i,j,k}^{n+1} + \bar{\theta}_{i+1,j,k}^{n+1}}{\Delta x^2} \right)
\]  

(2.40)
\[ + e \left( \frac{\bar{\theta}^{n+1}_{i,j-1,k} - 2\bar{\theta}^{n+1}_{i,j,k} + \bar{\theta}^{n+1}_{i,j+1,k}}{\Delta y^2} \right) + e \left( \frac{\bar{\theta}^{n+1}_{i,j,k-1} - 2\bar{\theta}^{n+1}_{i,j,k} + \bar{\theta}^{n+1}_{i,j,k+1}}{\Delta z^2} \right) \]

\[ + \left[ \frac{f}{\Delta t} + \frac{2e}{\Delta x^2} + \frac{2e}{\Delta y^2} + \frac{2e}{\Delta z^2} \right] \bar{\theta}^{n+1}_{i,j,k} + \left[ -\frac{e}{\Delta x^2} \right] \bar{\theta}^{n+1}_{i-1,j,k} + \left[ -\frac{e}{\Delta z^2} \right] \bar{\theta}^{n+1}_{i+1,j,k} \]

\[ = \left[ \frac{f}{\Delta t} \right] \bar{\theta}^{n+1}_{i,j,k} + \left[ -\frac{d}{2\Delta x} \right] \left( u^n_{i,j,k} \left( \theta^n_{i+1,j,k} \right) - \left( u^n_{i,j,k} \right) \left( \theta^n_{i-1,j,k} \right) \right) \]

\[ + \left[ -\frac{d}{2\Delta y} \right] \left( v^n_{i,j,k} \left( \theta^n_{i,j+1,k} \right) - \left( v^n_{i,j,k} \right) \left( \theta^n_{i,j-1,k} \right) \right) \]

\[ + \left[ -\frac{d}{2\Delta z} \right] \left( w^n_{i,j,k} \left( \theta^n_{i,j,k+1} \right) - \left( w^n_{i,j,k} \right) \left( \theta^n_{i,j,k-1} \right) \right) \]

So, equations 2.18, 2.21, 2.24, 2.38 and 2.41 together with the correction equations mentioned previously are the finite difference representation of the governing equations. Because the finite difference formulation used is second-order in x, y and z, and first-order in time, the system behaves well for a large range of Rayleigh numbers. We have solved benchmark problems as well as a series of problems in 3-D that include both Dirichlet and Neumann boundary conditions, as well as problems with complex, multi-body geometries. These will be discussed in detail in later chapters. Nevertheless, because the problems do not give direct information about the pressure, such information has to be deduced in other ways. Based on the projection method proposed by Chorin, by Ternam and by Fortin et al., which is in fact the fractional step method with a first-order formulation in time we use, a Neumann condition for the pressure is obtained and is of the form

\[ \left( \frac{\partial p}{\partial N} \right)^{n+1}_\Gamma = -\frac{1}{\Delta t} \left( U^{n+1}_\Gamma - \bar{U}_\Gamma \right) \cdot N \]

(2.42)
where \( N \) is the vector of directions \([x \ y \ z]\) and \( U \) is the vector of velocities \([u \ v \ w]\). In all the problems presented in this study, the velocities in question are specified at the boundaries, leading to the conclusion that the pressure flux in any direction at the boundaries is zero, which is the necessary information we were looking for in the first place. Now, because we limit our work to regular meshes with parallelepiped cells, there are nodes, such as the corners of the cavities studied, at which the given pressure information is not helpful. Our solution is to transform equation 2.38 such that the solution scheme is applied to the pressure at the center of the mesh cells, yielding a value \( ppp \). Then, \( ppp \) is distributed back to the mesh nodes in order to obtain \( p \) at all points, including the cavity corners. Our solution of the benchmark problem suggests that our approach is not only valid but also accurate. We start by creating a new mesh system based on the center of the cells of the original grid. Therefore, the number of grid points in the \( \bar{x}, \bar{y} \) and \( \bar{z} \) directions is \( N_x, N_y \) and \( N_z \), where \( N_x = N_x - 1 \), \( N_y = N_y - 1 \) and \( N_z = N_z - 1 \). We want to solve equation 2.38 in the new grid system. Figure 2.1 shows the nodes of the original system as well as the nodes of the new one. In the new grid space \( \bar{x} \bar{y} \bar{z} \) we count the nodes in \( \bar{x} \) with \( r \), the nodes in \( \bar{y} \) with \( s \) and the nodes in \( \bar{z} \) with \( t \). Remember, in the original space \( x \ y \ z \) we count the nodes in \( x \) with \( i \), the nodes in \( y \) with \( j \) and the nodes in \( z \) with \( k \). So if we apply a finite difference formulation to equation 2.38 in the new grid system we obtain

\[
\left( \frac{pppr_{r+1,s,t} - 2pppr_{r,s,t} + pppr_{r-1,s,t}}{\Delta x^2} \right) + \left( \frac{pppr_{r,s+1,t} - 2pppr_{r,s,t} + pppr_{r,s-1,t}}{\Delta y^2} \right) + \left( \frac{pppr_{r,s,t+1} - 2pppr_{r,s,t} + pppr_{r,s,t-1}}{\Delta z^2} \right) = \left[ \frac{a}{\Delta t} \right] \left( \frac{\bar{u}_{r+\frac{1}{2},s,t} - \bar{u}_{r-\frac{1}{2},s,t}}{\Delta x} \right)
\]  

(2.43)
Figure 2.1: Two-dimensional projection of the three-dimensional grid system used. Notice that the thick line represents a system boundary.

\[
\begin{align*}
\frac{\Delta y}{\Delta z} & \left( \begin{array}{c}
\frac{\partial}{\partial x} \left( \bar{u}_{i+\frac{1}{2},j} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \bar{v}_{i+\frac{1}{2},j} \right) \\
\frac{\partial}{\partial z} \left( \bar{w}_{i+\frac{1}{2},j} \right) - \frac{1}{2} \frac{\partial}{\partial z} \left( \bar{w}_{i+1,j} \right) \\
\frac{\partial}{\partial z} \left( \bar{w}_{i+\frac{1}{2},j} \right) - \frac{1}{2} \frac{\partial}{\partial z} \left( \bar{w}_{i+\frac{1}{2},j} \right)
\end{array} \right) \right) \\
+ \left[ \frac{a}{\Delta t} \right] \left( \bar{v}_{i,\frac{1}{2},j} \right) \\
+ \left[ \frac{a}{\Delta t} \right] \left( \bar{w}_{i,\frac{1}{2},j} \right)
\end{align*}
\]

which, in terms of the intermediate velocity values at the nodes of the original grid, becomes

\[
\begin{align*}
\left( \frac{ppp_{\text{r},i+1,j,t}}{\Delta x^2} - \frac{2ppp_{\text{r},i,j,t} + ppp_{\text{r},i-1,j,t}}{\Delta y^2} \right) \\
+ \left( \frac{ppp_{\text{r},i,j,t+1} - 2ppp_{\text{r},i,j,t} + ppp_{\text{r},i,j,t-1}}{\Delta z^2} \right) = \\
\left[ \frac{a}{\Delta t} \right] \left( \frac{A - B}{\Delta x} \right) \\
+ \left[ \frac{a}{\Delta t} \right] \left( \frac{C - D}{\Delta y} \right) \\
+ \left[ \frac{a}{\Delta t} \right] \left( \frac{E - F}{\Delta z} \right)
\end{align*}
\]

where

\[
A = \bar{u}_{i+1,j} + \bar{u}_{i+1,j+1} + \bar{u}_{i+1,j+1} = \left( \frac{1}{2} \right) \left( \bar{u}_{i+1,j} + \bar{u}_{i+1,j+1} \right) = \left( \frac{1}{2} \right) \left( \bar{u}_{i+1,j+1} + \bar{u}_{i+1,j} \right) = \left( \frac{1}{2} \right) \left( \bar{u}_{i+1,j} + \bar{u}_{i+1,j+1} \right) = \left( \frac{1}{2} \right) \left( \bar{u}_{i+1,j} + \bar{u}_{i+1,j+1} \right)
\]

(2.44)
\begin{align*}
\left(\frac{1}{4}\right) (\bar{u}_{i+1,j,k} + \bar{u}_{i+1,j,k+1}) + \left(\frac{1}{4}\right) (\bar{u}_{i+1,j+1,k} + \bar{u}_{i+1,j+1,k+1})
\quad B = \bar{u}_{i,j+\frac{1}{2},k+\frac{1}{2}} = \left(\frac{1}{2}\right) (\bar{u}_{i,j,k+\frac{1}{2}} + \bar{u}_{i,j+1,k+\frac{1}{2}}) = \\
\left(\frac{1}{4}\right) (\bar{v}_{i,j,k} + \bar{v}_{i,j,k+1}) + \left(\frac{1}{4}\right) (\bar{u}_{i,j+1,k} + \bar{u}_{i,j+1,k+1})
\quad C = \bar{v}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \left(\frac{1}{2}\right) (\bar{v}_{i,j+1,k+\frac{1}{2}} + \bar{v}_{i+1,j+1,k+\frac{1}{2}}) = \\
\left(\frac{1}{4}\right) (\bar{v}_{i,j+1,k} + \bar{v}_{i,j+1,k+1}) + \left(\frac{1}{4}\right) (\bar{v}_{i+1,j+1,k} + \bar{v}_{i+1,j+1,k+1})
\quad D = \bar{v}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \left(\frac{1}{2}\right) (\bar{v}_{i,j,k+\frac{1}{2}} + \bar{v}_{i+1,j,k+\frac{1}{2}}) = \\
\left(\frac{1}{4}\right) (\bar{v}_{i,j,k} + \bar{v}_{i,j,k+1}) + \left(\frac{1}{4}\right) (\bar{v}_{i+1,j,k} + \bar{v}_{i+1,j,k+1})
\quad E = \bar{w}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \left(\frac{1}{2}\right) (\bar{w}_{i,j+\frac{1}{2},k+1} + \bar{w}_{i+1,j+\frac{1}{2},k+1}) = \\
\left(\frac{1}{4}\right) (\bar{w}_{i,j,k+1} + \bar{w}_{i,j,k+1}) + \left(\frac{1}{4}\right) (\bar{w}_{i+1,j,k+1} + \bar{w}_{i+1,j+1,k+1})
\quad F = \bar{w}_{i+\frac{1}{2},j+\frac{1}{2},k} = \left(\frac{1}{2}\right) (\bar{w}_{i,j+\frac{1}{2},k} + \bar{w}_{i+1,j+\frac{1}{2},k}) = \\
\left(\frac{1}{4}\right) (\bar{w}_{i,j,k} + \bar{w}_{i,j+1,k}) + \left(\frac{1}{4}\right) (\bar{w}_{i+1,j,k} + \bar{w}_{i+1,j+1,k})
\end{align*}

Given equation 2.44, we can solve for the pressure at the centers of the grid cells. Redistributing \( ppp \) to the nodes of the original mesh is a simple operation governed by the way the mesh cells surround the mesh node in question and the pressure boundary conditions.

Now that the finite difference formulation has been described, let us describe the solution method.
Elliptic partial differential equations, when reduced to finite difference equations by using central differencing, generate a set of algebraic equations that can be solved by two major classes of methods. Direct methods rely on solving the matrix problem at hand by Gaussian elimination, Cramer's rule, matrix inversion, LU decomposition or any other similar scheme. Direct solvers are versatile, but run into trouble as the number of unknowns of the system increases, because the number of operations required to solve it increases very rapidly, and so does the computational time. Nevertheless, there are direct solver algorithms that exploit characteristics of the system of equations to speed up the solution process, such as the Conjugate-Gradient method, but there is a tradeoff in terms of the complexity of the computer algorithms and their implementation. The other general class of solution schemes for elliptic equations has been labeled iterative schemes. Iterative schemes include schemes such as the Jacobi method, the point Gauss-Seidel method, the line Gauss-Seidel method, the point successive over-relaxation method (PSOR), the line successive over-relaxation method (LSOR), the alternating direction explicit method (ADI) and others. All of these schemes rely on the same basic strategy, with few changes designed to increase the rate of convergence to the actual solution. The tactic can be described as follows. To start the algorithms, a solution is guessed. The equation is rearranged such that the desired nodal value or values become the unknowns, and the other nodes are moved to the opposite side of the difference equation. A dummy time index \( m \) is introduced to represent the current iteration, such that \( m + 1 \) represents the next one. Then, the unknowns are indexed at the \( m + 1 \)th iteration level, and the knowns are indexed at the \( m \)th iteration level. The algorithm scans through the mesh in such a way that, at the end, both a value at the \( m + 1 \)th iteration and at the \( m \)th iteration exists for every point in the mesh. Then we define a parameter \( \epsilon \) such
that

$$
\epsilon = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} \text{ABS} \left[ X_{i,j,k}^{m+1} - X_{i,j,k}^{m} \right]
$$

(2.51)

that is, $\epsilon$ is the sum of all the differences between the value of $X$ at the $i,j,k$th mesh points at the $m$th time level, and the value of $X$ at the same mesh point but at the $m+1$th time level, where $X$ is the unknown quantity in question. The value of $\epsilon$ can be used as a measure of how different two consecutive iterations are. The process of solution is repeated until the value of $\epsilon$ goes below the desired tolerance.

At that moment, the latest value of $X$ is the solution. The most general version of the iterative schemes is the Jacobi method, which corresponds to the previous description.

In this study, the PSOR scheme was selected as the solution scheme. In the PSOR, a parameter $\omega$ is introduced to accelerate the convergence of the scheme. The optimum value of $\omega$ can be calculated analytically for a very limited number of simple problems and boundary conditions. In general, the optimum value is found experimentally. In addition to this enhancement, the PSOR scheme utilizes the latest updated values as it scans the mesh, that is, as values are updated they are introduced into the formulation, increasing the scheme's performance even more. In the current study, the PSOR scheme is applied to all the governing equations. There are various reasons for this. First of all, it can be observed that after applying the fractional step method, the equation obtained for the pressure is in fact elliptic, and suited for solution by the PSOR scheme. But what about the other equations? When a forward difference is introduced to deal with the time derivatives, the time dependent equations become elliptic for any individual time step. Therefore, at every time step, $p$, $\theta$, $u$, $v$ and $\bar{w}$ can be solved for using the PSOR method. This can reduce the complexity of the computer program immensely. We will describe the elements of the computer code in detail in the following sections. The difference equations can be rewritten in PSOR form as follows:
\( \dot{u} \) equation:

\[
\dot{u}_{i,j,k}^{m+1} = (1 - \omega_0) \dot{u}_{i,j,k}^{m} + \left( \frac{\omega_0}{c_{ij}} \right) \left( U_1 + U_2 + U_3 + U_4 - a_{ij} \dot{u}_{i-1,j,k}^{m} - b_{ij} \dot{u}_{i+1,j,k}^{m} \right) (2.52)
\]

\[-d_{ij} \dot{u}_{i,j-1,k}^{m} - e_{ij} \dot{u}_{i,j+1,k}^{m} - f_{ij} \dot{u}_{i,j,k-1}^{m} - g_{ij} \dot{u}_{i,j,k+1}^{m} \]

where

\[
U_1 = - \left( \frac{b}{2 \Delta x} \right) (u_{i,j,k} u_{i+1,j,k} - u_{i,j,k} u_{i-1,j,k})
\]

\[
U_2 = - \left( \frac{b}{2 \Delta y} \right) (u_{i,j,k} u_{i,j+1,k} - u_{i,j,k} u_{i,j-1,k})
\]

\[
U_3 = - \left( \frac{b}{2 \Delta z} \right) (u_{i,j,k} u_{i,j,k+1} - w_{i,j,k} u_{i,j,k-1})
\]

\[
U_4 = \left( \frac{a}{\Delta t} \right) (u_{i,j,k})
\]

and

\[
a_{ij} = b_{ij} = - \left( \frac{c}{\Delta x^2} \right) \quad d_{ij} = e_{ij} = - \left( \frac{c}{\Delta y^2} \right) \quad f_{ij} = g_{ij} = - \left( \frac{c}{\Delta z^2} \right)
\]

\[
c_{ij} = \left( \frac{a}{\Delta t} \right) + \left( \frac{2c}{\Delta x^2} \right) + \left( \frac{2c}{\Delta y^2} \right) + \left( \frac{2c}{\Delta z^2} \right)
\]

\( \ddot{v} \) equation:

\[
\ddot{v}_{i,j,k}^{m+1} = (1 - \omega_0) \ddot{v}_{i,j,k}^{m} + \left( \frac{\omega_0}{c_{ij}} \right) \left( V_1 + V_2 + V_3 + V_4 - a_{ij} \ddot{v}_{i-1,j,k}^{m} - b_{ij} \ddot{v}_{i+1,j,k}^{m} \right) (2.53)
\]

\[-d_{ij} \ddot{v}_{i,j-1,k}^{m} - e_{ij} \ddot{v}_{i,j+1,k}^{m} - f_{ij} \ddot{v}_{i,j,k-1}^{m} - g_{ij} \ddot{v}_{i,j,k+1}^{m} + s_{ij} \theta_{i,j,k}^{m} \]

where

\[
V_1 = - \left( \frac{b}{2 \Delta x} \right) (u_{i,j,k} v_{i+1,j,k} - u_{i,j,k} v_{i-1,j,k})
\]
\[ V_2 = -\left( \frac{b}{2\Delta y} \right) (v_{i,j,k}v_{i,j+1,k} - v_{i,j,k}v_{i,j-1,k}) \]

\[ V_3 = -\left( \frac{b}{2\Delta z} \right) (w_{i,j,k}v_{i,j,k+1} - w_{i,j,k}v_{i,j,k-1}) \]

\[ V_4 = \left( \frac{a}{\Delta t} \right) (v_{i,j,k}) \]

and

\[ a_{ij} = b_{ij} = -\left( \frac{c}{\Delta x^2} \right) \quad d_{ij} = e_{ij} = -\left( \frac{c}{\Delta y^2} \right) \quad f_{ij} = g_{ij} = -\left( \frac{c}{\Delta z^2} \right) \]

\[ c_{ij} = \left( \frac{a}{\Delta t} \right) + \left( \frac{2c}{\Delta x^2} \right) + \left( \frac{2c}{\Delta y^2} \right) + \left( \frac{2c}{\Delta z^2} \right) \]

\( \tilde{w} \) equation:

\[ \tilde{w}_{i,j,k}^{m+1} = (1 - \omega) \tilde{w}_{i,j,k}^m + \left( \frac{\omega}{c_{ij}} \right) \left( W_1 + W_2 + W_3 + W_4 - a_{ij} \tilde{w}_{i-1,j,k}^m \right) \]

\[ -b_{ij} \tilde{w}_{i+1,j,k}^m - d_{ij} \tilde{w}_{i,j-1,k}^m - e_{ij} \tilde{w}_{i,j+1,k}^m - f_{ij} \tilde{w}_{i,j,k-1}^m - g_{ij} \tilde{w}_{i,j,k+1}^m \quad (2.54) \]

where

\[ W_1 = -\left( \frac{b}{2\Delta x} \right) (u_{i,j,k}w_{i+1,j,k} - u_{i,j,k}w_{i-1,j,k}) \]

\[ W_2 = -\left( \frac{b}{2\Delta y} \right) (v_{i,j,k}w_{i,j+1,k} - v_{i,j,k}w_{i,j-1,k}) \]

\[ W_3 = -\left( \frac{b}{2\Delta z} \right) (w_{i,j,k}w_{i,j,k+1} - w_{i,j,k}w_{i,j,k-1}) \]

\[ W_4 = \left( \frac{a}{\Delta t} \right) (w_{i,j,k}) \]

and

\[ a_{ij} = b_{ij} = -\left( \frac{c}{\Delta x^2} \right) \quad d_{ij} = e_{ij} = -\left( \frac{c}{\Delta y^2} \right) \quad f_{ij} = g_{ij} = -\left( \frac{c}{\Delta z^2} \right) \]
\[ c_{ij} = (\frac{a}{\Delta t}) + (\frac{2c}{\Delta x^2}) + (\frac{2c}{\Delta y^2}) + (\frac{2c}{\Delta z^2}) \]

**ppp equation:**

\[ ppp_{i,j,k}^{m+1} = (1 - \omega_{ppp}) ppp_{i,j,k}^{m} + \left( \frac{\omega_{ppp}}{c_{ij}} \right) \left( -a_{ij}ppp_{i-1,j,k}^{m} - b_{ij}ppp_{i+1,j,k}^{m} \right) - d_{ij}ppp_{i,j-1,k}^{m} - e_{ij}ppp_{i,j+1,k}^{m} - f_{ij}ppp_{i,j,k-1}^{m} - g_{ij}ppp_{i,j,k+1}^{m} + A \]  

(2.55)

where

\[ a_{ij} = b_{ij} = \left( \frac{1}{\Delta x^2} \right) \quad d_{ij} = e_{ij} = \left( \frac{1}{\Delta y^2} \right) \quad f_{ij} = g_{ij} = \left( \frac{1}{\Delta z^2} \right) \]

\[ c_{ij} = \left( \frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2} + \frac{-2}{\Delta z^2} \right) \]

**\(\theta\) equation:**

\[ \theta_{i,j,k}^{m+1} = (1 - \omega_{\theta}) \theta_{i,j,k}^{m} + \left( \frac{\omega_{\theta}}{c_{ij}} \right) \left( \theta_{1} + \theta_{2} + \theta_{3} + \theta_{4} - a_{ij}\theta_{i-1,j,k}^{m} - b_{ij}\theta_{i+1,j,k}^{m} \right) - d_{ij}\theta_{i,j-1,k}^{m} - e_{ij}\theta_{i,j+1,k}^{m} - f_{ij}\theta_{i,j,k-1}^{m} - g_{ij}\theta_{i,j,k+1}^{m} \]  

(2.56)

where

\[ \theta_{1} = -\left( \frac{d}{2\Delta x} \right) (u_{i,j,k}\theta_{i+1,j,k} - u_{i,j,k}\theta_{i-1,j,k}) \]

\[ \theta_{2} = -\left( \frac{d}{2\Delta y} \right) (v_{i,j,k}\theta_{i,j+1,k} - v_{i,j,k}\theta_{i,j-1,k}) \]

\[ \theta_{3} = -\left( \frac{d}{2\Delta z} \right) (w_{i,j,k}\theta_{i,j,k+1} - w_{i,j,k}\theta_{i,j,k-1}) \]

\[ \theta_{4} = \left( \frac{f}{\Delta t} \right) (\theta_{i,j,k}) \]
\[ a_{ij} = b_{ij} = -\left(\frac{e}{\Delta x^2}\right) \quad d_{ij} = e_{ij} = -\left(\frac{e}{\Delta y^2}\right) \quad f_{ij} = g_{ij} = -\left(\frac{e}{\Delta z^2}\right) \]

\[ c_{ij} = \left(\frac{f}{\Delta t}\right) + \left(\frac{2e}{\Delta x^2}\right) + \left(\frac{2e}{\Delta y^2}\right) + \left(\frac{2e}{\Delta z^2}\right) \]

2.2 Computer Algorithm and Code Structure

The computer code used for the solution of the equations described in section 2.1 is written in C. In this section, we describe how the computer code works, how it is structured, and what kinds of problems it is able to solve. We also describe the different pre-processing and post-processing tools we have developed in order to calculate the data present in the result and discussion chapters.

2.2.1 The PSOR Function

The computational engine of the computer code is the PSOR function. This external function is called from the main program to solve for the different unknowns of the problem, and is also used by some of the post-processing functions to solve for other values wanted. In order to allow the function to deal with all the different types of nodes it has to solve unknowns for, we write a generalized PSOR equation capable of handling multiple type of nodal geometries as well as multiple types of boundary conditions. The generalized PSOR function then receives information in the form of flags, in our case, zeroes (0) and ones (1) which indicate which terms of the generalized formulation are needed for the current node, and also provide information on the initial condition and boundary condition of the nodal point in question. The specific function of the flags is discussed in detail in section 2.2.2.
The generalized PSOR formula is of the form

\[ M_{i,j,k} = c_{14} (1 - \omega) M_{i,j,k} + \left( \frac{\omega}{c_{ij}} \right) [c_{15} F - a_{ij} c_1 M_{i+1,j,k}] \] (2.57)

\[-a_{ij} c_2 [c_{16} \Delta x + M_{i,j,k}] - b_{ij} c_3 M_{i-1,j,k} - b_{ij} c_4 [-c_{17} \Delta x + M_{i,j,k}] \]

\[-d_{ij} c_6 M_{i,j+1,k} - d_{ij} c_6 [c_{18} \Delta y + M_{i,j,k}] - e_{ij} c_7 M_{i,j-1,k} \]

\[-e_{ij} c_8 [-c_{19} \Delta y + M_{i,j,k}] - f_{ij} c_9 M_{i,j,k+1} - f_{ij} c_{10} [c_{20} \Delta z + M_{i,j,k}] \]

\[-g_{ij} c_{11} M_{i,j,k-1} - g_{ij} c_{12} [-c_{21} \Delta z + M_{i,j,k}] + c_{13} M_{i,j,k} \]

where \( M \) is the unknown value to be calculated at the \( i, j, k \)th node, \( F \) is a collection of extra terms that is dependent on the unknown in question, \( \omega \) is the corresponding over-relaxation parameter which is also dependent on the unknown and the constants \( c_1 \) to \( c_{21} \) are the flags corresponding to the \( i, j, k \)th node. The other terms are also problem dependent. For example, to solve equation 2.52, \( M = \bar{u}, F = U_1 + U_2 + U_3 + U_4 \) and \( \omega = \omega_0 \).

### 2.2.2 The Pre-Processor

The function assigned to the pre-processor in our code is that of creating a data file in which all the nodes in the defined grid system are described by a series of constants that serve as flags, boundary conditions, initial conditions and grid location. In our study, every problem requires an individual pre-processor. Our pre-processors create six (6) individual data files that are later appended to each other to create a final data file. The individual data files correspond to the six (6) different unknowns of the problems at hand, which again are \( \bar{u}, \bar{v}, \bar{w}, p_{ppp}, p \) and \( \theta \). For each node we define the following values and flags:
1. Flags 1 to 3 - ith, jth and kth location of the node in question.

2. Flag 4 - initial value of the unknown at this node.

3. Flags 5 to 19 - zeroes (0) and ones (1) that indicate which of the terms in equation 2.57 are to be used and which are not. This series of flags is dependent on the location of the current node and the location of its closest neighbors in the x, y and z directions. Flag 17, for example, holds the value of constant $c_{13}$ in equation 2.57. This flag is one (1) only if the node in question is a node with a Dirichlet boundary condition. In that case, all the other flags, from 5 to 19, which correspond to the values of the constants $c_1$ to $c_{15}$, are zero (0) to ensure that the value of the unknown at the node in question does not change with time.

4. Flags 20 to 25 - Magnitude of the Neumann boundary conditions in the x, y and z directions for the node in question and the unknown in question. If the flux of the unknown is specified, these flags store its magnitude.

**2.2.3 The Main Program**

Our main program is a straightforward implementation of the mathematical arguments described in section 2.1. We can better describe the program structure by listing the program operation as follows:

1. User input - consists of a series of prompts to the user requesting geometrical information as well as some necessary parameters for the solution of the problem, which include the $Pr$ and $Ra$ numbers. In addition, output filenames as well as input data filenames are requested.

2. Processing of user input - the information provided by the user is processed and used to calculate the necessary parameters for the program, such as:
(a) Number of grid points.

(b) Mesh spacings.

(c) Constants for the governing equations.

3. Load data file created by the pre-processor - the input data filename provided by the user is that of the data file created by the pre-processor. It is loaded into memory and stored in a multi-dimensional array for faster access.

4. Start the time loop - the time loop for the time dependent problem is started. Within this loop all the unknowns are solved for at all the mesh nodes, until steady state is reached. Within the time loop

   (a) Solve for $\bar{u}$ - the function PSOR is called to solve equation 2.52 for $\bar{u}$. All the necessary information is passed to the PSOR function by the main program.

   (b) Solve for $\bar{v}$ - the function PSOR is called to solve equation 2.53 for $\bar{v}$. All the necessary information is passed to the PSOR function by the main program.

   (c) Solve for $\bar{w}$ - the function PSOR is called to solve equation 2.54 for $\bar{w}$. All the necessary information is passed to the PSOR function by the main program.

   (d) Solve for $ppp$ - the function PSOR is called to solve equation 2.55 for $ppp$. All the necessary information is passed to the PSOR function by the main program.

   (e) Distribute $ppp$ to get $p$ - the newly calculated values of $ppp$ at the mesh cell centers is distributed to the mesh cell nodes in order to obtain $p$. 
(f) Correct \( \bar{u}, \bar{v} \) and \( \bar{w} \) to obtain \( u, v \) and \( w \) - using equations 2.27, 2.31 and 2.35, the main program corrects for the fluid velocities at the mesh nodes.

(g) Calculate the kinetic energy of the fluid - the kinetic energy of the fluid is calculated at this time using the magnitude of the fluid velocities calculated in the previous step. It is calculated by determining the energy content of every grid cell of volume \( \Delta x \Delta y \Delta z \), and then adding the energies of all the cells in the grid.

(h) Solve for \( \theta \) - the function PSOR is called to solve equation 2.56 for \( \theta \). All the necessary information is passed to the PSOR function by the main program.

(i) Check for steady state - the main program uses the number of iterations the PSOR went through to calculate the unknowns as a measure of how close the system of equations is to steady state. When the number of iterations required to calculate the value of the unknowns at the next time step is close to one (1), the current solution is not changing significantly and therefore is either very close to or the steady state solution itself.

(j) Recalculate \( \Delta t \) - the value of \( \Delta t \) is recalculated such that it satisfies the experimental stability criteria for the solution algorithm.

(k) Dump output to screen and to file - the current maximum velocities of the flow, the maximum pressure and temperature, together with the kinetic energy of the system, and the number of iterations required for their solution are all stored in one of the output files specified by the user at startup, and also displayed to the standard output, usually, the screen.
5. Close the time loop and repeat - the time loop will continue until steady state is reached. At that time, the loop condition is no longer satisfied and the loop ends.

6. Dump steady state solution to file - the value of all unknowns at all the mesh nodes for the steady state is then stored in one of the output files specified by the user at startup.

2.2.4 The Post-Processor

Post-processing of the data generated and stored by the main program includes

1. Contour plots of $u$, $v$, $w$, $p$ and $\theta$ at different cross sections of the system’s volume.

2. Vector plots of the fluid motion in two (2) and three (3) dimensions.

3. Streamline plots of the fluid motion at different cross sections of the system’s volume.

4. Kinetic energy vs. non-dimensional time from $t = 0$ up to the steady state.

5. Maximum $u$, $v$ and $w$ vs. non-dimensional time from $t = 0$ up to the steady state.

6. Number of PSOR iterations vs. non-dimensional time from $t = 0$ up to the steady state, for $u$, $v$, $w$, $p$ and $\theta$.

7. Nusselt number ($Nu$) values for hot and cold walls in two (2) and three (3) dimensions.

The main post-processing tool is a set of M files created to be used with MATLAB. MATLAB generates all the two (2) and three (3) dimensional plots presented in the
study. In addition to MATLAB, a set of external C functions has been created. One of them has been written such that it can call the generalized PSOR function and solve for the value of the stream function at the grid nodes. The stream function equation in three (3) dimensions can be written as

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z}
\]

(2.58)

where

\[
u = \frac{\partial \Psi}{\partial y} + \frac{\partial \Psi}{\partial z}
\]

(2.59)

\[
v = -\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial z}
\]

(2.60)

\[
w = -\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}
\]

(2.61)

This equation is elliptic, and therefore can be solved by using the same generalized PSOR scheme used to solve for the primitive variables of the system.

In addition to the stream function C routine, another one has been written in order to calculate the local Nu in two (2) and three (3) dimensions. For that purpose, a one-sided third-order finite difference approximation of the local Nu definition is used [8]. In our study, the local Nu is defined as:

\[
Nu = - \left( \frac{\partial \theta}{\partial x} \right)_{wall}
\]

(2.62)

For the left wall, the derivative of \( \theta \) can be approximated by

\[
\left( \frac{\partial \theta}{\partial x} \right)_{i,j} = \frac{\theta_{i+2,j} + 6\theta_{i+1,j} - 3\theta_{i,j} - 2\theta_{i-1,j}}{6\Delta x}
\]

(2.63)

and for the right wall, it can be approximated by

\[
\left( \frac{\partial \theta}{\partial x} \right)_{i,j} = \frac{\theta_{i-2,j} - 6\theta_{i-1,j} + 3\theta_{i,j} + 2\theta_{i+1,j}}{6\Delta x}
\]

(2.64)
As we can see, there is one phantom node in each of these equations. Because we can reduce equation 2.13 to

$$\frac{\partial^2 \theta}{\partial x^2} = 0$$  \hspace{1cm} (2.65)

at the heated or cooled walls, we can use a second-order central difference for

$$\theta_{i-1,j} = 2\theta_{i,j} - \theta_{i+1,j}$$  \hspace{1cm} (2.66)

$$\theta_{i+1,j} = 2\theta_{i,j} - \theta_{i-1,j}$$  \hspace{1cm} (2.67)

and insert equation 2.67 into equation 2.63, and equation 2.67 into equation 2.64.

We end up with two expressions of the local $Nu$. For the left wall we get

$$Nu = \frac{7\theta_{i,j} - 8\theta_{i+1,j} + \theta_{i+2,j}}{6\Delta x}$$  \hspace{1cm} (2.68)

and for the right wall we end up with

$$Nu = \frac{-\theta_{i-2,j} + 8\theta_{i-1,j} - 7\theta_{i,j}}{6\Delta x}$$  \hspace{1cm} (2.69)

We have studied three different problems of increasing complexity to thoroughly validate the code and to extend available two (2) dimensional studies into the three-dimensional realm. The next few sections intend to mathematically and graphically describe these problems.

2.3 **Bench Mark Problem: Thermally Driven Cavity**

One classic problem represented by the governing equations is the buoyancy-driven flow of a viscous fluid in an unit square enclosure. This two-dimensional problem is used as a bench mark and code validation tool due to the existence of very accurate and complete studies such as those by de Vahl Davis [1] and Saitoh and Hirose [2]. Figure 2.2 is a three-dimensional representation of the geometry of the problem.
Figure 2.2: Geometry for the benchmark problem where an unit cube cavity is heated from the left, cooled from the right, and insulated on the top, bottom, front and back walls.

The boundary conditions of the two-dimensional benchmark are

\begin{align}
  u(0, y, t) = 0, & \quad v(0, y, t) = 0, \quad \theta(0, y, t) = 0.5, \quad \frac{\partial p(0, y, t)}{\partial x} = 0 \quad (2.70) \\
  u(1, y, t) = 0, & \quad v(1, y, t) = 0, \quad \theta(1, y, t) = -0.5, \quad \frac{\partial p(1, y, t)}{\partial x} = 0 \quad (2.71) \\
  u(x, 0, t) = 0, & \quad v(x, 0, t) = 0, \quad \frac{\partial \theta(x, 0, t)}{\partial y} = 0, \quad \frac{\partial p(x, 0, t)}{\partial y} = 0 \quad (2.72) \\
  u(x, 1, t) = 0, & \quad v(x, 1, t) = 0, \quad \frac{\partial \theta(x, 1, t)}{\partial y} = 0, \quad \frac{\partial p(x, 1, t)}{\partial y} = 0 \quad (2.73)
\end{align}

In our study we solve the three-dimensional version of the thermally driven cavity problem whose boundary conditions can be written as

\begin{equation}
  u(0, y, z, t) = 0, \quad v(0, y, z, t) = 0, \quad w(0, y, z, t) = 0, \quad (2.74)
\end{equation}
\[ \theta(0, y, z, t) = 0.5, \quad \frac{\partial p(0, y, z, t)}{\partial x} = 0 \]

\[ u(1, y, z, t) = 0, \quad v(1, y, z, t) = 0, \quad w(1, y, z, t) = 0, \quad (2.75) \]

\[ \theta(1, y, z, t) = -0.5, \quad \frac{\partial p(1, y, z, t)}{\partial x} = 0 \]

\[ u(x, 0, z, t) = 0, \quad v(x, 0, z, t) = 0, \quad w(x, 0, z, t) = 0, \quad (2.76) \]

\[ \frac{\partial \theta(x, 0, z, t)}{\partial y} = 0, \quad \frac{\partial p(x, 0, z, t)}{\partial y} = 0 \]

\[ u(x, 1, z, t) = 0, \quad v(x, 1, z, t) = 0, \quad w(x, 1, z, t) = 0, \quad (2.77) \]

\[ \frac{\partial \theta(x, 1, z, t)}{\partial y} = 0, \quad \frac{\partial p(x, 1, z, t)}{\partial y} = 0 \]

\[ u(x, y, 0, t) = 0, \quad v(x, y, 0, t) = 0, \quad w(x, y, 0, t) = 0, \quad (2.78) \]

\[ \frac{\partial \theta(x, y, 0, t)}{\partial z} = 0, \quad \frac{\partial p(x, y, 0, t)}{\partial z} = 0 \]

\[ u(x, y, 1, t) = 0, \quad v(x, y, 1, t) = 0, \quad w(x, y, 1, t) = 0, \quad (2.79) \]

\[ \frac{\partial \theta(x, y, 1, t)}{\partial z} = 0, \quad \frac{\partial p(x, y, 1, t)}{\partial z} = 0 \]

The three-dimensional scheme can solve the two-dimensional bench mark when the number of nodes in the \( z \) direction is equal to three. With this many nodes in the \( z \) direction, the three-dimensional problem collapses into its two-dimensional counterpart, so it can be validated. The bench mark problem gives solutions for a \( Pr = 0.71 \) and a range of \( Ra \) from \( 10^3 \) to \( 10^6 \). We have solved the 2-D bench mark for a \( Pr = 1.0 \) and the same \( Ra \) range, plus an extra solution for \( Ra = 10^7 \) for meshes with \( 11 \times 11, 21 \times 21 \) and \( 41 \times 41 \) grid points. In addition, we have solved the fully
three-dimensional problem for $Pr = 1.0$, $Ra$ ranging from $10^3$ to $10^5$ and meshes such as those with $11 \times 11 \times 11$ grid points. The difference in $Pr$ is not significant in regards to the overall behavior of the fluid system, although the numerical values are slightly different. The results of this problem are discussed in chapter 3.

2.4 Problem Two: Cold Cavity with Two Internal Heated Bodies

A more complicated problem is the one in which a set of two bodies heated to a constant temperature is placed inside an unit cube cavity that is kept cooled to a constant temperature on its left, right, top and bottom walls, and insulated on its front and back walls. Figure 2.3 is a three-dimensional view of the problem geometry. The two internal bodies are identical rectangular boxes of 0.1 units of length in the $x$-direction, 0.5 units of length in the $y$-direction, and unit length in the $z$-direction. Adlam [7] has worked on the two-dimensional version of this problem for a $Pr = 5.39$, which corresponds to water at some realistic experimental temperature, and $Ra$ between $10^6$ and $2 \times 10^7$. We have solved the two-dimensional projection of the three-dimensional version, as well as the fully three-dimensional problem for $Pr = 1.0$, which corresponds to air at room temperature, and $Ra$ between $10^3$ and $10^5$. We provide steady-state information, together with some time dependent information concerning the fluid. The full time evolution of the problem has been calculated and can be obtained, but is not presented due to the enormous amount of information we would have to deal with.

The boundary conditions for the three-dimensional version of this problem are very involved so we describe them in a simpler way as follows:

*enclosing cavity:*
Figure 2.3: Geometry for the problem where two internal blocks heated to a constant temperature are surrounded by a containing cavity cooled to a constant temperature on its left, right, top and bottom walls, and insulated on its front and back walls.

left wall \((x = 0)\):

\[ u = v = w = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \theta = 0, \quad (2.80) \]

right wall \((x = 1)\):

\[ u = v = w = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \theta = 0, \quad (2.81) \]

bottom wall \((y = 0)\):

\[ u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \theta = 0, \quad (2.82) \]

top wall \((y = 1)\):

\[ u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \theta = 0, \quad (2.83) \]
front wall \((z = 0)\):
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial z} &= 0, & \frac{\partial \theta}{\partial z} &= 0, \\
\end{align*}
\] (2.84)

back wall \((z = 1)\):
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial z} &= 0, & \frac{\partial \theta}{\partial z} &= 0,
\end{align*}
\] (2.85)

internal blocks:

left wall:
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial x} &= 0, & \theta &= 10,
\end{align*}
\] (2.86)

right wall:
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial x} &= 0, & \theta &= 10,
\end{align*}
\] (2.87)

bottom wall:
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial y} &= 0, & \theta &= 10,
\end{align*}
\] (2.88)

top wall:
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial y} &= 0, & \theta &= 10,
\end{align*}
\] (2.89)

front wall:
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial z} &= 0, & \frac{\partial \theta}{\partial z} &= 0,
\end{align*}
\] (2.90)

back wall:
\[
\begin{align*}
u &= v = w = 0, & \frac{\partial p}{\partial z} &= 0, & \frac{\partial \theta}{\partial z} &= 0,
\end{align*}
\] (2.91)

The results of this problem are discussed in chapter 4.
2.5 Problem Three: Two Heated and Interconnected Cavities of Different Sizes

The final problem we have analyzed is the one in which two cavities of different sizes but which are interconnected by a shared surface are submitted to differential heating from the sides and are insulated on the bottom, top, front and back. Figure 2.4 is a three-dimensional view of the geometry of the problem. We solve the fully three-dimensional problem for various mesh sizes, as well as a two-dimensional version of it. Evren-Selamet, Arpacı and Borgnakke [8] have worked on a two-dimensional version of the same problem, and report solutions for a $Pr = 0.71$ and $Ra$ ranging from $10^3$ and $3\times10^6$. Their results are used for comparison with the solution to our two-dimensional projection. We present solutions for a $Pr = 1.0$, and a range of $Ra$ between $10^5$ and $10^6$.

The boundary conditions for the three-dimensional version of this problem are as follows:

*Bottom cavity:

Left wall ($x = 0$):

$$u = v = w = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \theta = 0.5,$$

(2.92)

Right wall ($x = 1$):

$$u = v = w = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \theta = -0.5,$$

(2.93)

Bottom wall ($y = 0$):

$$u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial \theta}{\partial y} = 0,$$

(2.94)

Top wall ($y = 1$):

$$u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \theta = 0,$$

(2.95)
Figure 2.4: Geometry for the problem where two cavities of different sizes are interconnected, and the system is heated from the left on both cavities, cooled from the right on both cavities, and insulated on the front, back, top and bottom. The top portion of the bottom cavity is also heated, cooled and insulated in the same manner.

\[ u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \theta = 0, \quad (2.96) \]

\[ u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \theta = 0, \quad (2.97) \]

\[ u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \theta = 0, \quad (2.98) \]

**front wall** \( z = 0 \):

\[ u = v = w = 0, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0, \quad (2.99) \]

**back wall** \( z = 1 \):

\[ u = v = w = 0, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0, \quad (2.100) \]

**top block** :
left wall:

\[ u = v = w = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \theta = 0.5, \quad (2.101) \]

right wall:

\[ u = v = w = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \theta = -0.5, \quad (2.102) \]

top wall:

\[ u = v = w = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial \theta}{\partial y} = 0, \quad (2.103) \]

front wall:

\[ u = v = w = 0, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0, \quad (2.104) \]

back wall:

\[ u = v = w = 0, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0, \quad (2.105) \]

The results of this problem are discussed in chapter 5.
Chapter 3

Bench Mark Problem: Thermally Driven Cavity

3.1 Two-dimensional Problem

In the two-dimensional projection of the unit cube thermally driven cavity problem, we can examine the effect of the $Ra$ on the flow pattern and the heat flux profile. Due to the nature of the problem we expect certain general behavior for the different $Ra$, as well as for the fluid itself no matter what the $Ra$ is.

In our study, an unit value of the $Pr$ is used. This differs from the bench mark solutions presented by de Vahl Davis [1], and Saitoh and Hirose [2], where they use $Pr = 0.71$. Nevertheless the differences, although present, are minimal and the overall behavior of the fluid in terms of the flow pattern and the heat transfer phenomena is almost identical.

Both the $x$ momentum and $y$ momentum equations have a common nonlinear convective term. The $y$ momentum equation, however, has an extra convective term related to the buoyant force. We call this term the convective buoyant term. The term $s = RaPr$ in the momentum equation 2.11 is a measure of how intense the convective buoyant effect is in the fluid system. In addition, the differential heating of the cavity produces a temperature difference between the fluid and the heated right wall, and the fluid and the cooled left wall, and therefore, a heat flux as well. As the $RaPr$ product grows, the same temperature differential applied to the cavity has more influence on the vertical $v$ velocity of the fluid, than on the $u$ velocity. As time progresses and heat is transferred from the walls to the fluid, the vertical and horizontal velocities initiate
flow circulation. As the fluid gets hotter near the right wall, the value of $\theta$ of the fluid close to the wall becomes positive. This produces a positive convective buoyant term, thus causing a pronounced movement of the fluid against the gravity gradient, in our case, upwards. In a similar manner, the fluid gets colder near the left wall, the value of $\theta$ of the fluid close to the wall becomes negative. This produces a negative buoyant term, causing the fluid to move in the direction of the gravity gradient, in our case, downwards. This fluid movement extends throughout the cavity, and therefore we expect a clockwise rotation of the fluid as the system evolves in time. This effect is expected for any $Ra$, but is of course more pronounced for higher $Ra$.

In order to visualize the conductive and convective effects, temperature contour plots are ideal. In an insulated wall, the heat transfer is purely conductive. We can use the isotherm pattern of such a wall as a reference with which to compare the temperature contours at different cross-sections of the system, as to assess the changes in the way heat is transferred in the fluid. Figure 3.1a presents such a pattern. The isotherms in a purely conductive cross-section have a particular curvature. Deviations from this isotherm pattern indicate changes in the energy transfer mechanisms.

For $Ra = 10^3$, figure 3.1b shows that the isotherm pattern is very close to that of the insulated wall. That indicates that heat is being mainly transferred by conduction. This can also be inferred from the flow pattern in figure 3.2a. The flow is perfectly symmetric at this $Ra$, therefore there are no areas in the flow where the velocities are significantly different from the rest. The slight change in the curvature of the isotherms indicates though that convection is beginning to have an effect. Figure 3.3 represents the velocity vector field for this $Ra$. Notice that there is not much difference in the magnitudes of the $u$ velocities near the top and bottom walls and the $v$ velocities near the left and right walls.
Figure 3.1: Temperature contours or isotherms for a) pure conduction, b) $Ra = 10^3$, c) $Ra = 10^4$, d) $Ra = 10^5$, e) $Ra = 10^6$, f) $Ra = 10^7$. 
Figure 3.2: Stream line pattern for a) $Ra = 10^3$, b) $Ra = 10^4$, c) $Ra = 10^5$, d) $Ra = 10^6$, e) $Ra = 10^7$. 
Figure 3.3: Velocity vector plot for $Ra = 10^3$.

For $Ra = 10^4$, the isotherms in figure 3.1c show a definite change from pure conduction. The hot and cold isotherms begin to elongate clockwise in the top and the bottom of the cavity, respectively. This indicates that hotter fluid can now be found close to the left and top walls, and cold fluid can be found close to the right and bottom walls, as expected from the fluid circulation. In addition, notice that the isotherms are now closer to each other near the left and right walls. This indicates that the temperature differences at the walls are larger, therefore, the heat flux is now larger than before, also as expected. We therefore expect an increase in the local $Nu$ in the regions where the isotherms are closely packed. The flow pattern has also changed as we can see in figure 3.2b. The flow is still symmetric but the streamlines are closer together near the left and right walls, which means that the fluid velocity in the vertical direction is increasing faster than the horizontal velocity due to the added effect of the $s\theta$ term. We can appreciate this more clearly in figure 3.5 where
Figure 3.4: Pressure contours or isobars for a) $Ra = 10^3$, b) $Ra = 10^4$, c) $Ra = 10^5$, d) $Ra = 10^6$, e) $Ra = 10^7$. 
the $v$ velocity vectors at the heated walls are becoming larger than the $u$ velocity vectors near the insulated ones.

![Velocity vector plot for $Ra = 10^4$.](image)

For $Ra = 10^5$, figure 3.1d shows that the isotherm pattern is an extension of what is observed at $10^4$. The isotherms are even closer together at the hot and cold walls, so the heat flux is still increasing, and their curvature indicates that the flow is much more convective. Nevertheless, there is a basic difference between the flow pattern at $10^5$ and that at $10^4$. As we can observe in figure 3.2c, the streamline pattern has decentralized and in fact created two (2) identical circulation cells. The streamline pattern near the heated walls is more closely packed, indicating an increase in $v$ velocity due to the even stronger effect of the $Ra$. Figure 3.6 supports that conclusion. Notice the lengthening of the $v$ vector near the heated walls. As a final observation, one can see that the isotherm pattern follows the streamline pattern closely, that is, near the new circulation cells, the isotherms are bent to follow the stream lines.
For $Ra = 10^6$, the isotherm pattern in figure 3.1e is extremely bent near the heated walls. The heat flux is very large near the walls as one can observe from the closeness of the isothermal lines. In terms of the fluid motion, figure 3.2d shows that the two cells from the $10^5$ case still exist but have elongated in the $y$ direction, and have moved farther apart in the $x$ direction. The fluid's $v$ velocity at the heated walls has increased again as we can see in figure 3.7, and therefore the streamlines are closer to each other in those areas. Notice that the flow circulation at the cells is still not strong enough to create local mixing of fluid, that is, we are not observing folding of the isotherms or formation of more than one rotation cell.

Finally, for $Ra = 10^7$, figure 3.1f shows that the isotherm pattern is still more compact near the heated walls. The heat flux has again increased, as well as the reach of the hot isotherm near the top wall and the cold one near the bottom wall. The flow field has again changed. The circulation cells in figure 3.2e have elongated
again in the $y$ direction, and have separated in the $z$ direction. The stream lines are very closely packed near the heated walls, indicating very high vertical velocities.

The increase in the heat flux due to the increase in the $Ra$, evidenced in the way the isotherms are packed together near the heated walls, can be better observed in the change of the local $Nu$ at the hot and cold walls as the $Ra$ increases. Figures 3.9 and 3.10, show the value of the local $Nu$ at the hot right wall and the cold left wall respectively.

As we can see, the local $Nu$ increases as the $Ra$, as expected. We can also observe that the local $Nu$ pattern of the cold left wall is the inverted mirror image of the $Nu$ pattern of the hot right wall. This is due to our definition of non dimensional temperature and the choice of its values at the walls. It might be important to also mention that, as can be observed in the plots, the maximum local $Nu$ occurs always
Figure 3.8: Velocity vector plot for $Ra = 10^7$.

Figure 3.9: Hot wall $(0, y)$ $Nu$ for a) $Ra = 10^3$, b) $Ra = 10^4$, c) $Ra = 10^5$, d) $Ra = 10^6$, e) $Ra = 10^7$. 
close to the bottom left hand corner on the hot right wall, and close to the top left hand corner on the cold left wall.

![Graph showing variations in Nu with different Ra values](image)

**Figure 3.10: Cold wall $(1,y)$ Nu for**
a) $Ra = 10^3$, b) $Ra = 10^4$, c) $Ra = 10^5$, d) $Ra = 10^6$, e) $Ra = 10^7$.

Quantitatively, tables 3.1 and 3.2 present and compare our numerical results to those of de Vahl Davis (1983). We want to point out that the discrepancies are due in some extent to the difference in $Pr$, but mostly to the fact that the bench mark results have been extrapolated by Richardson’s extrapolation techniques to meshes with $81 \times 81$ grid points, when our most precise results have been calculated for meshes with $41 \times 41$ grid points.

Qualitatively, figure 3.11 is a excerpt from de Vahl Davis’ paper (1983). It is clear that our results are in agreement.
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Table 3.1: Comparison of the present numerical results of the simulation (P) and the bench mark solutions (BM).

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Table 3.2: Comparison of the present numerical results of the simulation (P) and the bench mark solutions (BM).
Figure 3.11: Streamlines and isotherm patterns by de Vahl Davis (1983) for the benchmark problem. From top left to bottom right: 1) streamlines for $Ra = 10^3$, 2) streamlines for $Ra = 10^4$, 3) isotherms for $Ra = 10^3$, 4) isotherms for $Ra = 10^4$, 5) streamlines for $Ra = 10^5$, 6) stream lines for $Ra = 10^6$, 7) isotherms for $Ra = 10^5$, 8) isotherms for $Ra = 10^6$.

Figure 3.12: Color isotherm pattern for the benchmark problem when the $Ra = 10^3, Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
Figure 3.13: Color isotherm pattern for the benchmark problem when the $Ra = 10^4$, $Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.

Figure 3.14: Color isotherm pattern for the benchmark problem when the $Ra = 10^5$, $Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
Figure 3.15: Color isotherm pattern for the benchmark problem when the $Ra = 10^7, Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.

Figure 3.16: Color isotherm pattern for the benchmark problem when the $Ra = 10^6, Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
3.2 Three-dimensional Problem

In the full three-dimensional version of the thermally driven cavity, the flow develops in the $x$, $y$ and $z$ directions, giving rise to complex fluid flow. We present results for $Ra = 10^3$ in a $11 \times 11 \times 11$ mesh, together with results for $Ra = 10^4$ and $10^5$ in meshes with $17 \times 17 \times 9$ grid points. Due to the nature of the problem geometry and its governing equations, we can infer certain aspects of the fluid behavior in three dimensions.

First of all, the flow is symmetric with respect to the central cross-section at $z = 0.5$, i.e. cross-sections at $z = 0.5 \pm \Delta z$ are identical. This reduces the number of cross-sections we need to present. Secondly, for every $Ra$, the $xy$ cross-section at $z = 0.5$ is almost identical to the two-dimensional solution for the same $Ra$. This behavior can be explained by the fact that the two-dimensional driven cavity problem can be interpreted as the solution to the three-dimensional problem at the central $xy$ cross-section when the $z$ dimension is infinite. We also expect that as we get closer to the insulated cavity walls such as the front and back ones, the fluid velocities will decrease in magnitude due to the no-slip boundary condition. This translates into a reduction of the intensity of convective heat transfer near those cavity walls. In our presentation of the data, isotherm patterns at the cavity walls are not presented due to their non-convective behavior.

The heat transfer behavior of the three dimensional fluid system as the $Ra$ increases is almost identical to that of the two dimensional one. As the $Ra$ increases, the fluid’s vertical velocity $v$ increases in magnitude faster than its $u$ and $w$ velocities in the regions where the temperature gradient is non-zero, and specially in the regions were the gradient is actually large. This will induce an increase in the local $Nu$ at the heated walls as the $Ra$ increases. This behavior is identical to that of the fluid in the two dimensional simulation.
In terms of the fluid motion, a three-dimensional flow field view would be best but, although available, is not appropriate for enclosed flows, due to their multi-directional nature. Instead, $xy$, $zy$ and $xz$ cross-sections are presented for every $Ra$ between $10^3$ and $10^5$.

Table 3.3 presents a series of values obtained from the numerical simulations of the three (3) dimensional bench mark problem. We will refer to the values in this table as we discuss the behavior of the flow.

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<td>$z$</td>
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Table 3.3: Important numerical results for the three (3) dimensional bench mark problem simulation.

We begin by examining the numerical results for $Ra = 10^3$. Figure 3.17 is a collection of temperature contours or isotherm plots of $x$ vs. $y$ cross-sections of the flow, arranged along the $z$ axis of the system’s volume.

As expected, the curvature of the isotherms increases as we move away from the front or back wall. The isotherms at $x = 0.5$ are almost identical to those in figure 3.1b, also as expected. Now, if we look closely at the isotherm pattern, the differences in it as we move along the $z$ axis are minimal. Moreover, this minimal difference can be attributed to the fact that we are approaching or receding from an insulated wall.
Figure 3.17: Temperature contour plots at three different cross-sections ($x = 0.1$, $x = 0.3$ and $x = 0.5$) for the fully three-dimensional flow inside a cubic cavity for $Ra = 10^3$.

Therefore, it follows that if there is any effect due to the newly introduced $w$ velocity in the $z$ direction, it is almost negligible. This could happen if the $w$ velocity is really small in comparison to the $u$ and $v$ velocities. It turns out that this is exactly the case.

Figures 3.18 to 3.20 present velocity vector field plots at different cross-sections in the three (3) major axes, $z$, $x$ and $y$ respectively. As we can observe in table 3.3, the $u$ and $v$ velocities are much larger than the $w$ velocity, as we inferred from the isotherm patterns. The $w$ velocity is both seven (7) times smaller than the $u$ velocity, and about nine (9) times smaller than the $v$ velocity. That is why the temperature contour plots do not change in shape much as we move along the $z$ axis. We also observe that the effect of $w$ is minimal when we examine the available vector field plots. Figure 3.19 is a collection of $y$ vs. $z$ cross-sections of the flow along the $x$ axis. Therefore the arrows represent the resultants of the $v$ velocity vector in the $y$
direction and the $w$ velocity vector in the $z$ direction. Nevertheless, the resultants are mostly parallel to the $z$ axis. In figure 3.20 we have $z$ vs. $x$ cross-sections of the flow along the $y$ axis, such that the arrows represent the resultants of the vector sum of the $w$ vector in the $z$ direction and the $u$ velocity in the $z$ direction. Notice that the resultants are mostly parallel to the $x$ axis. Therefore, we conclude without a doubt that the effect of $w$ in the flow is negligible for $Ra = 10^3$. Moreover, figures 3.19c and 3.20c show that when the $v$ and $u$ velocities are small, the resultant vectors are almost zero, in other words, $w$ is negligible. So, the flow rotates clockwise almost uniformly around the $z$ axis without any significant change as we move in the $z$ direction.

Figures 3.21 and 3.22 show the local $Nu$ surface over the hot left wall and the cold right wall of the cavity, respectively. We can see that the distribution of the local $Nu$ is very similar to that of the two (2) dimensional benchmark. The $Nu$ at the hot wall is a maximum close to its bottom, and is a minimum at its top. Inversely, the local $Nu$ at the cold wall is maximum near its top portion, and a minimum at the bottom. In fact, as one can see in table 3.3 the maximum value of the local $Nu$ is almost identical to that of the two (2) dimensional solution. It also occurs at the $z = 0.5$ cross-section, as predicted from the geometric configuration.

For $Ra = 10^4$, the flow's behavior is very similar to that at $10^3$. Figure 3.23 is also a collection of temperature contours of $x$ vs. $y$ cross-sections along the $z$ axis.

Again we observe an increase in isotherm curvature as we move away from the insulated front wall along the $z$ axis. Nevertheless, the curvature increase is too low to be a result of $w$ velocity influence. So, we conclude as before that the $w$ velocity must be much smaller in magnitude than either the $u$ or $v$ velocities. The vector plots in figures 3.24, 3.25 and 3.26 support that conclusion.

Resultant vectors parallel to the $y$ axis in plots of $y$ vs. $z$ cross-sections indicate $v$ velocities much larger than $w$. In the same fashion, resultant vectors parallel to the $x$
Figure 3.18: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.1$, b) $z = 0.2$, c) $z = 0.3$, d) $z = 0.4$, e) $z = 0.5$ for $Ra = 10^3$. 
Figure 3.19: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.1$, b) $x = 0.3$, c) $x = 0.5$, d) $x = 0.7$, e) $x = 0.9$ for $Ra = 10^3$. 
Figure 3.20: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.1$, b) $y = 0.3$, c) $y = 0.5$, d) $y = 0.7$, e) $y = 0.9$ for $Ra = 10^3$. 
Figure 3.21: Hot wall \((0, y, z)\) \(Nu\) for \(Ra = 10^3\).

Figure 3.22: Cold wall \((0, y, z)\) \(Nu\) for \(Ra = 10^3\).
axis in $z$ vs. $x$ cross-section plots indicate $u$ velocities much larger than $w$ velocities. Vector plots in which the resultants are almost zero support the conclusion that the $w$ velocity has a minimal effect on the fluid flow. As inferred from the graphic information, table 3.3 shows that the $w$ velocity is in fact both eleven (11) times smaller than the $u$ velocity, and about thirteen (13) times smaller than the $v$ velocity. Therefore, due to the negligible effect of $w$ on the flow, the fluid rolls clockwise about the $z$ axis, very much like it does when the $Ra = 10^3$. At $Ra = 10^4$ though, the circulation cell elongates in the $x$ direction such that the central region of stationary and slowly moving fluid becomes almost twice as wide as that observed at a $Ra = 10^3$.

The local $Nu$ surfaces for both the hot and cold walls can be seen in figures 3.27 and 3.28, respectively. The average value of the local $Nu$ and its maximum value, together with its location can be found in table 3.3. Notice that these values are very close to those of the two (2) dimensional problem at the same $Ra$. 

Figure 3.23: Temperature contour plots at four different cross-sections ($x = 0.125$, $x = 0.25$, $x = 0.375$ and $x = 0.5$) for the fully three-dimensional flow inside a cubic cavity for $Ra = 10^4$. 


Figure 3.24: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.125$, b) $z = 0.25$, c) $z = 0.375$, d) $z = 0.5$ for $Ra = 10^4$. 
Figure 3.25: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.1875$, b) $x = 0.3125$, c) $x = 0.5$, d) $x = 0.625$, e) $x = 0.75$, f) $x = 0.875$ for $Ra = 10^4$. 
Figure 3.26: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.1875$, b) $y = 0.3125$, c) $y = 0.5$, d) $y = 0.625$, e) $y = 0.75$, f) $y = 0.875$ for $Ra = 10^4$. 
Figure 3.27: Hot wall $(0, y, z) \, Nu$ for $Ra = 10^4$.

Figure 3.28: Cold wall $(0, y, z) \, Nu$ for $Ra = 10^4$. 
For $Ra = 10^5$, the flow exhibits certain significant changes when compared with previous cases with different $Ra$. Figure 3.29 is again a collection of temperature contours of $x$ vs. $y$ cross-sections along the $z$ axis.

![Temperature contour plots](image)

**Figure 3.29:** Temperature contour plots at four different cross-sections ($x = 0.125$, $x = 0.25$, $x = 0.375$ and $x = 0.5$) for the fully three-dimensional flow inside a cubic cavity for $Ra = 10^5$.

The already observed increase in isotherm curvature as we move away from the insulated front wall along the $z$ axis is now more pronounced. In contrast with cases with lower $Ra$, the curvature increase is too high to be solely explained by the closeness of the insulated walls, so we must conclude that it must be influenced by the $w$ velocity. We also conclude that the $w$ velocity must have increased with the $Ra$, although, as before, it must be much smaller in magnitude than either the $u$ or $v$ velocities. This is because although the curvature has changed prominently, the overall isotherm pattern has not changed qualitatively. The vector plots in figures 3.30, 3.31 and 3.32 support that conclusion. Resultant vectors parallel to the $y$ axis in plots of $y$ vs. $z$ cross-sections indicate $v$ velocities much larger than $w$. Nevertheless,
when compared to similar plots for lower $Ra$ the effect of $w$ on the fluid motion is more visible now, specially in the bottom near the insulated front wall, and at the top near the insulated back wall. Similarly, resultant vectors parallel to the $x$ axis in $z$ vs. $x$ cross-section plots indicate $u$ velocities much larger than $w$ velocities. Table 3.3 indicates that the $w$ velocity is five (5) times smaller than the $u$ velocity, and at the same time ten (10) times smaller than the $v$ velocity. This is a considerable increase in the relative magnitude of $w$, if we compare it with that at lower $Ra$. The local $Nu$ pattern reflects the effect of the $w$ velocity on the fluid's behavior. For lower $Ra$ the change in the local $Nu$ in the $z$ direction is very smooth and gradual. However, for $Ra = 10^5$, the local $Nu$ changes rapidly in the $z$ direction to a maximum, specially in the regions where the $w$ velocity seems to have a comparable effect with to that of the $v$ and $u$ velocities, as can be seen in figures 3.33 and 3.34. Table 3.3 provides values and locations for the maximum and minimum local $Nu$.

Although the $w$ velocity starts to play an important role on the fluid's behavior, the overall flow movement is very much like that at $Ra = 10^3$ and $Ra = 10^4$. The fluid rolls clockwise around the $z$ axis. However, the circulation cell present in previous cases finally changes shape into two (2) interconnected cells which are inverted mirror images of each other.

Our results can be qualitatively compared to those of Pepper [6]. In his work, the thermally driven cavity problem with its primitive variable equations is solved by using a time-split finite element scheme similar to the Petrov-Galerkin formulation. There are no quantitative results in his paper, nevertheless, his qualitative results and ours are in agreement.
Figure 3.30: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.125$, b) $z = 0.25$, c) $z = 0.375$, d) $z = 0.5$ for $Ra = 10^5$. 
Figure 3.31: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.1875$, b) $x = 0.3125$, c) $x = 0.5$, d) $x = 0.625$, e) $x = 0.75$, f) $x = 0.875$ for $Ra = 10^5$. 
Figure 3.32: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.1875$, b) $y = 0.3125$, c) $y = 0.5$, d) $y = 0.625$, e) $y = 0.75$, f) $y = 0.875$ for $Ra = 10^5$. 
Figure 3.33: Hot wall \((0, y, z)\) \(Nu\) for \(Ra = 10^5\).

Figure 3.34: Cold wall \((0, y, z)\) \(Nu\) for \(Ra = 10^5\).
Chapter 4

Problem Two: Cold Cavity with Two Internal Heated Bodies

Now that we have validated the three-dimensional solver, we can increase the problem's complexity by introducing additional bodies to the problem's geometry, together with larger temperature gradients.

A cubic cavity with two static blocks placed inside as described in figure 2.3 is filled with a fluid with a $Pr = 1.0$, static at $t = 0$ when suddenly, the surface temperature of both blocks changes to $\theta = 10$, the surface temperature of the bottom, top, left and right walls of the surrounding cavity change to $\theta = 0$, and the front and back walls become insulated.

We solve the two-dimensional version of this problem for $Ra = 10^3$ and $10^4$ in meshes with $21 \times 21$ grid points, and for $Ra = 10^5$ in a mesh with $41 \times 41$ grid points. We then solve the same problem in three (3) dimensions for both $Ra = 10^3$ and $Ra = 10^4$ in meshes with $21 \times 21 \times 11$ grid points.

4.1 Two-dimensional Version

Due to the nature of the problem at hand, we can point out a series of properties of the solution a priori. To begin with, the problem's geometry as well as the boundary conditions are symmetric with respect to the $x = 0.5$ line, so we expect the solution to be symmetric with respect to this line as well. Secondly, the temperature gradient between the hot inner blocks and the cold surrounding cavity is large, specially when compared to that present in the bench mark problem, where the temperature
difference between hot and cold walls was of unit magnitude. As a result we expect large $v$ velocities even for low $Ra$. Thirdly, we can expect certain fluid motions. Both the region between the left block and the cold left wall of the enclosing cavity, and the region between the right block and the right wall of the enclosing cavity are very similar to the benchmark problem itself. We expect the formation of a circulation cell in both regions. Finally, the region between the two heated blocks is subject to large positive non-dimensional $\theta$ temperatures. This will create larger $v$ velocities directed against the gravity gradient, so we expect flow directed upwards in this region.

Figures 4.1 and 4.3 are the temperature contour and vector plots for $Ra = 10^3$, respectively.

![Temperature contours and isotherms for $Ra = 10^3$.](image)

**Figure 4.1: Temperature contours or isotherms for $Ra = 10^3$.**

As we can observe from the isotherm pattern the heat transfer is very much conductive at this low $Ra$. The isotherms are almost parallel to each other and to the nearest wall, mainly in the region between the cooled left wall and the hot left block,
Figure 4.2: Color isotherm pattern for the problem where two hot blocks are placed inside a cooled cavity ($Ra = 10^4, Pr = 1.0$). Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
Figure 4.3: Velocity vector plot for $Ra = 10^3$. Maximum resultant velocity = 20.8847 at (0.5,0.65).

in the region between the cooled right wall and the hot right block, and the region between the top of both hot blocks and the cooled top wall. The region between the two hot blocks is reminiscent of the pure conduction isotherm pattern encountered in a flat plate whose boundaries are subjected to constant temperatures. We also observe that the isotherms are closely packed together at the bottom of each of the heated blocks. This indicates a large temperature gradient or local $Nu$.

The flow pattern is quite interesting. The flow is symmetric with respect to the $x = 0.5$ line, as expected. The flow is composed of two strong circulation cells, each of them surrounding one heated block. The flow rises along the $x = 0.5$ line and splits left and right near the top of the blocks. To the right of the centerline, the fluid rotates clockwise around the right block. To the left of the centerline, the fluid rotates counterclockwise around the left block. This describes the flow in general. If
one looks more closely, we can observe several other subtle aspects of it. Near the left wall of the left block, and the right wall of the right block, a very weak circulation cell seems to be forming, which extends along the height of each of the blocks. We will observe an increase in the strength of these circulations as the \( Ra \) increases.

Figures 4.4 and 4.5 are plots which represent the distribution of the local \( Nu \) at the walls of the enclosing cavity as well as that at the walls of the heated internal blocks.

![Graphs showing Nu distribution](image)

**Figure 4.4:** \( Nu \) of the enclosing cavity for \( Ra = 10^3 \). From top left to bottom right: left wall, right wall, top wall and bottom wall.

As one can see, the local \( Nu \) seems to be a maximum near the center of the top wall of the enclosing cavity. The \( Nu \) at this location is about 60 units in magnitude. On the blocks, the \( Nu \) is minimum near the center of the vertical walls (\( Nu = 15 \)).

Figures 4.6 and 4.8 are the temperature contour and vector plots for \( Ra = 10^4 \), respectively. The isotherm pattern has changed considerably from that observed at \( Ra = 10^3 \). The region in between the left cold wall and the left hot block, and the
Figure 4.5: $Nu$ of the heated blocks for $Ra = 10^3$. From top left to bottom right: left wall, right wall, top wall and bottom wall.

Figure 4.6: Temperature contours or isotherms for $Ra = 10^4$. 
Figure 4.7: Color isotherm pattern for the problem where two hot blocks are placed inside a cooled cavity ($Ra = 10^4, Pr = 1.0$). Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
region between the right cold wall and the right hot block have isotherms that are not longer parallel to each other. This increase in their curvature is an indication that convection is beginning to play a larger role in the overall heat transfer process. The isotherm pattern in the region between the two hot blocks is still very similar to that of pure conduction in a heated plate, as it was for \( Ra = 10^3 \). In the top region of the cavity, the isotherms are still parallel to each other, so the heat transfer is mainly conductive, but the isotherms are also more closely packed, therefore, there has been an increase in the temperature gradient, so we expect an increase in the local \( Nu \) of the top wall. In the same manner, the isotherms close to the bottom of each of the hot blocks are now closer to each other when compared to those at \( Ra = 10^3 \). This indicates that the local \( Nu \) in these regions has increased.

![Figure 4.8: Velocity vector plot for \( Ra = 10^4 \). Maximum resultant velocity = 57.2712 at both (0.45,0.7) and (0.55,0.7).](image-url)
The flow pattern has also changed. The flow is still symmetric with respect to the $x = 0.5$ line. The flow moves against the gravity gradient along the symmetry line and splits right and left close to the top of the hot blocks ($y = 0.75$). The flow to the right of the symmetry line moves clockwise around the right block. The flow to the left of the symmetry line rotates counterclockwise around the left block. Notice however that to the left of the left block and to the right of the right block, the rotation cell present in the $Ra = 10^3$ case has grown and has become stronger. The center of rotation of the left side cell used to be near the point $(0.19, 0.50)$ for $Ra = 10^3$, but has now moved closer to $(0.13, 0.60)$. Because the flow is symmetric, the same behavior is seen on the right hand cell. Notice also, that the flow velocities have increased near the top left and top right corners of the cavity.

Figures 4.9 and 4.10 represent the $Nu$ distribution at the wall of the enclosing cavity as well as that at the walls of the heated internal blocks for $Ra = 10^4$.

Figure 4.9: $Nu$ of the enclosing cavity for $Ra = 10^4$. From top left to bottom right: left wall, right wall, top wall and bottom wall.
The local $Nu$ at the surface of the enclosing cavity is maximum near the areas on top of the two internal blocks. The local $Nu$ magnitude at these locations is about 115 units, which represents a 90% increase in the magnitude of the $Nu$ when compared to that at $Ra = 10^3$. On the blocks, the maximum local $Nu$ again occurs at the bottom walls, as expected from the isotherm distribution. The $Nu$ has a value of 145 units at these locations, which represents a 70% increase in magnitude with respect to the value obtained for $Ra = 10^3$. The minimum $Nu$ at the blocks occurs at the top walls ($Nu = 10$).

![Graphs](image)

**Figure 4.10:** $Nu$ of the heated blocks for $Ra = 10^4$. From top left to bottom right: left wall, right wall, top wall and bottom wall.

For $Ra = 10^5$, we observe significant qualitative and quantitative changes in both the manner by which heat is transferred around the cavity and in the overall fluid flow pattern. Figures 4.11 and 4.13 are the temperature contour and vector plots for this $Ra$, respectively. As one can see, the isotherm pattern is still symmetric with
Figure 4.11: Temperature contours or isotherms for $Ra = 10^5$.

Figure 4.12: Color isotherm pattern for the problem where two hot blocks are placed inside a cooled cavity ($Ra = 10^5, Pr = 1.0$). Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
Figure 4.13: Velocity vector plot for $Ra = 10^5$. Maximum resultant velocity = 284.8239 at both (0.375,0.775) and (0.625,0.775).

respect to the $x = 0.5$ line. The curvature and complexity of the isotherm pattern is evidence of the convective nature of the flow at this $Ra$. In the regions between the left block and the left wall of the enclosing cavity, and between the right block and the right wall of the enclosing cavity, the isotherms are bent in a manner that indicates the presence of a strong rotational cell in those areas. The change in curvature is very dramatic, as one can appreciate by comparing these results to those for $Ra = 10^4$. The region between the top wall of the enclosing cavity and the top of the internal blocks is the region which undergoes the most changes. The parallel isotherms of $Ra = 10^4$ give way to bent isotherms that penetrate the region in between the two heated blocks, and also describe two small rotational cells centered at (0.4125,0.875) and (0.5875,0.875). This penetration of fluid indicates a fluid flow in the direction of
the gravity gradient, the opposite behavior to that observed at lower $Ra$. Therefore, the principal mechanism of heat transfer at this $Ra$ is natural convection.

The fluid motion has also changed considerably. The maximum resultant velocity for $Ra = 10^5$ is about 284.8239, which is about five (5) times larger than the maximum resultant velocity for $Ra = 10^4$ and about thirteen (13) times larger than the maximum resultant velocity for $Ra = 10^3$. The fluid flow is still symmetric with respect to the $x = 0.5$ line. The fluid, however, moves very differently from the fluid at lower $Ra$. First of all, the fluid moves in the direction of the gravity gradient along the $x = 0.5$ line, which is a complete reversal of flow direction. In addition, the two small, but strong rotation cells formed in the region between the top wall of the enclosing cavity and the top walls of the blocks circulate the fluid. The cell to the left of the symmetry line rotates clockwise, and the cell to the left of the symmetry line rotates counterclockwise. Now, in between these circulation cells and the blocks, the fluid moves as that at lower $Ra$. The fluid to the left of the symmetry line moves up close to the right wall of the left block, passes under the small circulation cell on the left, reaches the top left hand corner of the cavity, moves downwards along the left wall of the enclosing cavity, later moves under the left block, and moves back up near the right wall of the left block. The flow on the other side of the $x = 0.5$ line is similar. Notice that the circulation cells described for $Ra = 10^4$, located in both the region between the left block and the left wall of the cavity and the right block and the right wall of the cavity are still centered at the same location, although their strength and reach have increased.

Figures 4.14 and 4.15 represent the distribution of the local $Nu$ over the surface of the enclosing cavity and the surface of the heated internal blocks for $Ra = 10^5$.

The local $Nu$ of the surface of the enclosing cavity is a maximum in the areas directly above the two hot blocks. The value of the $Nu$ in these regions is approxi-
Figure 4.14: $Nu$ of the enclosing cavity for $Ra = 10^5$. From top left to bottom right: left wall, right wall, top wall and bottom wall.

Figure 4.15: $Nu$ of the heated blocks for $Ra = 10^5$. From top left to bottom right: left wall, right wall, top wall and bottom wall.
mately 248. On the blocks, the $Nu$ has a maximum value of 295 at the bottom walls, and a minimum value of 14 at the top walls.

Tables 4.1 and 4.2 contain the values of the median $Nu (N_{u0})$, the maximum $Nu (N_{u_{max}})$, the minimum $Nu (N_{u_{min}})$, and their respective locations $(x, y)$, at all the cavity walls and block walls, respectively. They also include the maximum $u$ and $v$ velocities $(u_{max}, v_{max})$ at their corresponding $Ra$.

Figure 4.16 is an excerpt from Adlam's paper [7] in which he has worked on the same two-dimensional problem for a $Pr = 5.39$, which corresponds to water at some realistic experimental temperature. We include the plot for qualitative comparison with our own results. The plot in question is that of the isotherm pattern of the flow in a $51 \times 51$ mesh, when the normalized time in his formulation reaches the value $t_{normalized} = 0.039$, and the $Ra = 10^9$. He presents no other significant qualitative nor quantitative information we can use for meaningful comparison. Nevertheless, it is obvious from the pattern that the behavior predicted by our approach and his is very similar.

### 4.2 Three-dimensional Version

In the three-dimensional scenario, the geometric arrangement and the boundary conditions are symmetric with respect to both the $yz$ plane at $x = 0.5$, and the $xy$ plane at $z = 0.5$. We expect the solution to be symmetric with respect to these planes. As explained in chapter 3, for every $Ra$, the $xy$ cross-section at $z = 0.5$ is almost identical to the solution of the analog two-dimensional problem. One can recognize that the two-dimensional problem is the same as the three-dimensional problem when the $z$ dimension is infinite, i.e. the front and back walls are infinitely apart. We can also expect that the closer the fluid is to the insulated back and front walls, the smaller the magnitude of the fluid velocities will be due to the imposed no-slip boundary
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**Cavity's Left wall**

| \(N_{u_0}\) | -27.1018 | -38.6544 | -63.9388 |
| \(N_{u_{\text{max}}}\) | -36.0416 | -71.7945 | -142.6654 |
| \(y\) | 0.5000  | 0.8500  | 0.8750  |
| \(N_{u_{\text{min}}}\) | 0.0000  | 0.0000  | 0.0000  |
| \(y\) | 0.0000  | 0.0000  | 0.0000  |

**Cavity's Right wall**

| \(N_{u_0}\) | 27.1023 | 38.5705 | 63.9393 |
| \(N_{u_{\text{max}}}\) | 36.0583 | 70.1761 | 142.6665 |
| \(y\) | 0.5000  | 0.8500  | 0.8750  |
| \(N_{u_{\text{min}}}\) | 0.0000  | 0.0000  | 0.0000  |
| \(y\) | 0.0000  | 0.0000  | 0.0000  |

**Cavity's Top wall**

| \(N_{u_0}\) | 40.9772 | 107.2169 | 193.7386 |
| \(N_{u_{\text{max}}}\) | 59.5684 | 115.5602 | 247.9880 |
| \(x\) | 0.5000  | 0.2000  | 0.8000  |
| \(N_{u_{\text{min}}}\) | 0.0000  | 0.0000  | 0.0000  |
| \(x\) | 0.0000  | 1.0000  | 0.0000  |

**Cavity's Bottom wall**

| \(N_{u_0}\) | -17.5672 | -8.2928  | -5.0390  |
| \(N_{u_{\text{max}}}\) | -21.1617 | -10.3218 | -6.7731  |
| \(x\) | 0.3000  | 0.7000  | 0.2500  |
| \(N_{u_{\text{min}}}\) | 0.0000  | 0.0000  | 0.0000  |
| \(x\) | 0.0000  | 1.0000  | 0.0000  |

Table 4.1: Important numerical results of the simulation of the two (2) dimensional problem of the cavity with internal heated bodies.
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*Bock’s Left wall*

| $Nu_0$       | 20.6089  | 50.2810  | 81.1948  |
| $Nu_{max}$   | 67.2966  | 105.1971 | 210.2952 |
| $y$          | 0.2500    | 0.2500   | 0.2500   |
| $Nu_{min}$   | 14.7814  | 34.2702  | 68.4978  |
| $y$          | 0.6500    | 0.7500   | 0.6750   |

*Bock’s Right wall*

| $Nu_0$       | 36.4911  | 15.2904  | 21.9310  |
| $Nu_{max}$   | 51.0695  | 18.7599  | 43.7806  |
| $x$          | 0.2500    | 0.3500   | 0.3500   |
| $Nu_{min}$   | 31.6920  | 10.7688  | 13.9864  |
| $x$          | 0.3500    | 0.3000   | 0.3000   |

*Bock’s Top wall*

| $Nu_0$       | -84.6488 | -131.3487 | -251.0902 |
| $Nu_{max}$   | -84.7026 | -145.4591 | -294.7911 |
| $x$          | 0.3500    | 0.3500    | 0.2500    |
| $Nu_{min}$   | -77.7713 | -131.1712 | -237.7013 |
| $x$          | 0.3000    | 0.2500    | 0.3000    |

Table 4.2: Important numerical results of the simulation of the two (2) dimensional problem of the cavity with internal heated bodies.
condition, thus reducing the intensity of convective heat transfer near the insulated walls.

Figure 4.17 is a collection of temperature contour plots at different $xy$ cross-sections for $Ra = 10^3$. Due to the symmetry of the problem, only isotherm patterns up to the $xy$ plane at $z = 0.5$ are necessary. The isotherm distribution at this $Ra$ is very similar to the one observed in the two-dimensional case at the same $Ra$, as expected. As one can observe, there are four main areas were the isotherms are parallel to each other: the region between the top wall of the enclosing cavity and the two hot blocks, the region between the left wall of the enclosing cavity and the left hot block, the region between the bottom wall of the enclosing cavity and the two heated blocks, and the region between the right wall of the enclosing cavity and the right heated block. Parallel isotherms are associated with regions where the transfer
Figure 4.17: Temperature contours or isotherms for $Ra = 10^3$ at a) $z = 0.0$, b) $z = 0.1$, c) $z = 0.2$, d) $z = 0.3$, e) $z = 0.4$, f) $z = 0.5$. 
of heat is mainly due to conduction. In the region between the two heated blocks, the isotherms are curved. Nevertheless, the pattern in this region is similar to that observed in the conduction of heat in a plate subjected to specified temperatures at its boundaries. We therefore conclude that in this region, the heat transfer is also conductive in nature. The variation in the isotherm pattern is minimal as we move along the z axis. This is an indication that the influence of the fluid velocity component w in the z direction is minimal. We must conclude that the magnitude of the w velocity is small when compared to that of the u and v velocities. This conclusion can be corroborated by studying the velocity vector plots for this Ra.

Figures 4.18 to 4.20 present velocity vector field plots at different cross-sections in the three (3) major axes: z, x and y, respectively. Figure 4.18 is a collection of vector plots at different y vs. x cross-sections along the z axis. The arrows in these plots represent the resultants of the vector sum of the v velocity in the y direction, and the u velocity in the x direction at every node in the mesh. The flow pattern at Ra = 10^3 is similar to that observed for the same Ra in the two-dimensional simulation, as expected. Notice that related plots have the same scale. Observe how the size of the resultant vectors at the z = 0.1 cross-section is slightly smaller than that at the remaining ones. This is due to the no-slip boundary condition at the z = 0.0 cross-section, as previously explained. The flow is symmetric with respect to the x = 0.5 plane, also as expected. The fluid rises against the gravity gradient along the symmetry plane and splits right and left near the top of the heated blocks (y = 0.75). The flow to the right of the symmetry plane moves to the right at the top of the right hot block, moves downward to the left of the right block, then turns left at the bottom of the right block and finally rises along the symmetry plane, thus creating a clockwise rotating circulation cell around the right heated block. The flow to the left of the symmetry plane behaves in a similar fashion, creating a counter-
Figure 4.18: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.1$, b) $z = 0.2$, c) $z = 0.3$, d) $z = 0.4$, e) $z = 0.5$ for $Ra = 10^3$. 


clockwise rotating cell around the left heated block. It is important to notice that in both the region closest to the right wall of the right heated block and the one closest to the left wall of the left heated block, the resultants are small. In fact, there is a weak circulation cell forming in both these regions, which will gain strength as the \( Ra \) increases. The negligible effect of the \( w \) velocity in the nature of the flow at this \( Ra \) can be best assessed in the accompanying figures.

Figure 4.19 is a collection of vector plots at different \( y \) vs. \( z \) cross-sections along the \( z \) axis such that the arrows represent the resultants of the vector sum of \( v \) and \( w \) velocities. Nevertheless, the resultants are mostly either extremely small or mostly parallel to the \( y \) axis, indicating \( v \) velocities many times larger than their corresponding \( w \) velocities. Figure 4.20 is a collection of vector plots at different \( z \) vs. \( x \) cross-sections along the \( y \) axis such that the arrows represent the resultants of the vector sum of \( w \) and \( u \) velocities. In fact, table 4.3 compares the maximum values of \( u \), \( v \) and \( w \). As expected, the maximum value of \( w \) is both four (4) times smaller than the maximum value of \( u \) and seven (7) times smaller than the maximum value of \( v \). Therefore, we conclude without doubt that the effect of the \( w \) velocity on the flow for \( Ra = 10^3 \) is negligible.

Figure 4.21 is a collection of Nusselt number distribution plots over the surface of the enclosing cavity. In it, the local \( Nu \) appears to be a maximum near the center of the top wall. The distribution of the local \( Nu \) can be inferred from the isotherm patterns describe in figure 4.17. In the regions where the isotherms are more closely packed we can expect large temperature gradients and therefore a larger local \( Nu \). In the regions where the isotherms are more loosely packed, in this case the areas near the corners of the enclosing cavity, we have lower temperature gradients, and therefore lower local \( Nu \). Table 4.3 reports a value of 59 at this location. Notice the smoothness of the \( Nu \) surfaces corresponding to the enclosing cavity. This will change when the
Figure 4.19: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.15$, b) $x = 0.35$, c) $x = 0.5$, d) $x = 0.65$, e) $x = 0.8$, f) $x = 0.95$ for $Ra = 10^3$. 
Figure 4.20: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.15$, b) $y = 0.35$, c) $y = 0.5$, d) $y = 0.65$, e) $y = 0.8$, f) $y = 0.95$ for $Ra = 10^3$. 
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**Cavity's Left wall**

| $Nu_0$ | -28.9099 | -38.1771 |
| $Nu_{max}$ | -38.4019 | -56.2517 |
| $y$ | 0.4500 | 0.8000 |
| $z$ | 0.5000 | 0.4000, 0.6000 |
| $Nu_{min}$ | 0.0000 | 0.0000 |
| $y$ | 0.0000, 1.0000 | 0.0000, 1.0000 |
| $z$ | all | all |

**Cavity's Right wall**

| $Nu_0$ | 28.9099 | 38.1771 |
| $Nu_{max}$ | 38.4019 | 56.2512 |
| $y$ | 0.4500 | 0.8000 |
| $z$ | 0.5000 | 0.4000, 0.6000 |
| $Nu_{min}$ | 0.0000 | 0.0000 |
| $y$ | 0.0000, 1.0000 | 0.0000, 1.0000 |
| $z$ | all | all |

**Cavity's Top wall**

| $Nu_0$ | 35.9753 | 57.2146 |
| $Nu_{max}$ | 59.1149 | 124.9805 |
| $\bar{x}$ | 0.5000 | 0.4500, 0.5500 |
| $z$ | 0.5000 | 0.2000, 0.8000 |
| $Nu_{min}$ | 0.0000 | 0.0000 |
| $x$ | 0.0000, 1.0000 | 0.0000, 1.0000 |
| $z$ | all | all |

**Cavity's Bottom wall**

| $Nu_0$ | -21.5739 | -11.6334 |
| $Nu_{max}$ | -25.1210 | -16.8746 |
| $x$ | 0.3000, 0.7000 | 0.3000, 0.7000 |
| $z$ | 0.5000 | 0.0000, 1.0000 |
| $Nu_{min}$ | 0.0000 | 0.0000 |
| $x$ | 0.0000, 1.0000 | 0.0000, 1.0000 |
| $z$ | all | all |

Table 4.3: Important numerical results of the simulation of the three (3) dimensional problem of the cavity with internal heated bodies.
Figure 4.21: \( Nu \) of the enclosing cavity for \( Ra = 10^3 \). From top left to bottom right: left wall, right wall, top wall and bottom wall.

\( w \) velocity starts to play a major role in the fluid motion. Figure 4.22 is a collection of Nusselt number distribution plots over the surface of the heated blocks. It can be seen that the local \( Nu \) is a maximum at the bottom of the blocks, specially in the corners where the bottom wall meets the front and back walls (\( Nu = 82 \)). Nevertheless, the \( Nu \) does not vary significantly on the wall in question. On the other hand, the local \( Nu \) seems to be a minimum near the center line of both the right wall of the left block and the left wall of the right block (\( Nu = 1.4, y = 0.4, z = 0.0, 1.0 \)). Table 4.4 includes both the exact location and the corresponding value of the \( Nu \) for these surfaces.

For \( Ra = 10^4 \), figure 4.23 is a collection of temperature contour plots at different \( xy \) cross-sections. The isotherm distribution at this \( Ra \) is very similar to the one observed in the two-dimensional case at the same \( Ra \), as expected. However, when compared
<table>
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<th></th>
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<tr>
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<td>$10^4$</td>
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<tr>
<td></td>
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<td></td>
</tr>
<tr>
<td><em>Block's Left wall</em></td>
<td></td>
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<td>$N_u_0$</td>
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<td>-43.3403</td>
</tr>
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<td>-95.9861</td>
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<tr>
<td>$z$</td>
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<td>0.1000, 0.9000</td>
</tr>
<tr>
<td>$N_{u_{min}}$</td>
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<td>-19.3242</td>
</tr>
<tr>
<td>$y$</td>
<td>0.2500</td>
<td>0.7500</td>
</tr>
<tr>
<td>$z$</td>
<td>0.0000, 1.0000</td>
<td>0.0000, 1.0000</td>
</tr>
</tbody>
</table>

| *Block's Right wall* |
| $N_u_0$  | 14.2900  | 44.3693  |
| $N_{u_{max}}$ | 59.8298  | 96.3870  |
| $y$      | 0.2500  | 0.2500 |
| $z$      | 0.1000, 0.9000 | 0.1000, 0.9000 |
| $N_{u_{min}}$ | -1.4143  | 2.0692  |
| $y$      | 0.4000  | 0.7500 |
| $z$      | 0.0000, 1.0000 | 0.0000, 1.0000 |

| *Block's Top wall* |
| $N_u_0$  | 38.3017  | 19.6896  |
| $N_{u_{max}}$ | 66.8484  | 56.6772  |
| $x$      | 0.2500  | 0.2500 |
| $z$      | 0.1000, 0.9000 | 0.1000, 0.9000 |
| $N_{u_{min}}$ | 21.4135  | 15.6943  |
| $x$      | 0.2500  | 0.3000 |
| $z$      | 0.0000, 1.0000 | 0.4000, 0.6000 |

| *Block's Bottom wall* |
| $N_u_0$  | -74.3842 | -110.2744 |
| $N_{u_{max}}$ | -81.6641 | -126.9324 |
| $x$      | 0.2500  | 0.3500 |
| $z$      | 0.1000, 0.9000 | 0.1000, 0.9000 |
| $N_{u_{min}}$ | -24.2493 | -27.8494 |
| $x$      | 0.2500  | 0.2500 |
| $z$      | 0.0000, 1.0000 | 0.0000, 1.0000 |

Table 4.4: Important numerical results of the simulation of the three (3) dimensional problem of the cavity with internal heated bodies.
Figure 4.22: $Nu$ of the heated blocks for $Ra = 10^3$. From top left to bottom right: left wall, right wall, top wall and bottom wall.

to the pattern observed at $Ra = 10^3$, it has considerably changed. The isotherms show a definite change in curvature in four main areas: the region between the top wall of the enclosing cavity and the two hot blocks, the region between the bottom wall of the enclosing cavity and the left hot block, the region between the right wall of the enclosing cavity and the right heated block. The change from mainly parallel isotherms in these regions to bent isotherms is a clear indication of the convective nature of the heat transfer at this $Ra$. As a matter of fact, the isotherm pattern in the later two regions is similar to that of the solution to the benchmark problem described in chapter 3 and found in figure 3.1b. As one can see, the convective effect can also be observed in the region between the heated blocks. In addition, the temperature variation in both the $x$ and $y$ directions cannot be attributed to conduction anymore, as the temperature contours deviate from those expected of pure conduction. As we
Figure 4.23: Temperature contours or isotherms for $Ra = 10^4$ at a) $z = 0.0$, b) $z = 0.1$, c) $z = 0.2$, d) $z = 0.3$, e) $z = 0.4$, f) $z = 0.5$. 
move along the $z$ axis, changes in the isotherm pattern are noticeable. This indicates that the $w$ velocity in the $z$ direction is influencing the flow. We must conclude that the magnitude of the $w$ velocity is comparable to that of the $u$ or $v$ velocities in some region of the flow. This conclusion can be corroborated by studying the velocity vector plots for this $Ra$.

Figures 4.24 to 4.26 present velocity vector field plots at different cross-sections in the three (3) major axes: $z$, $x$ and $y$, respectively. Figure 4.24 is a collection of vector plots at different $y$ vs. $x$ cross-sections along the $z$ axis. The arrows in these plots represent the resultants of the vector sum of the $v$ and $u$ velocities. The flow pattern at $Ra = 10^4$ is similar to that observed for the same $Ra$ in the two-dimensional simulation, as expected. Observe how the size of the resultant vectors at the $z = 0.1$ cross-section is slightly smaller than that at the remaining ones. This is due to the no-slip boundary condition at the front and backs walls of the enclosing cavity, as previously explained. The flow is symmetric with respect to the $x = 0.5$ plane as expected. The fluid behaves in the same manner as it did for $Ra = 10^3$. There is clockwise rotating cell around the right heated block, as well as a counter-clockwise rotating cell around the left heated block. Nevertheless, the additional two weak cells observed at $Ra = 10^3$ forming near the walls of the heated blocks are now well developed. Contrary to the lower $Ra$ case, for $Ra = 10^4$, the fluid next to both the left wall of the left block and the right wall, which is used to be almost static, is now moving against the gravity gradient at velocities similar to those experienced by the fluid near the $x = 0.5$ symmetry plane. In addition, the centers of rotation of these cells used to be close to the $x = 0.5, 0.375$ plane. The significant effect of the $w$ velocity in the nature of the flow at this $Ra$ can be best assessed in accompanying figures.
Figure 4.24: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.1$, b) $z = 0.2$, c) $z = 0.3$, d) $z = 0.4$, e) $z = 0.5$ for $Ra = 10^4$. 
Figure 4.25 is a collection of vector plots at different $y$ vs. $z$ cross-sections along the $x$ axis such that the arrows represent the resultants of the vector sum of $v$ and $w$ velocities. Notice that the resultants are mostly either extremely small or parallel to the $y$ axis, indicating $v$ velocities many times larger than their corresponding $w$ velocities. However, in the region between the top wall of the enclosing cavity and the top of the heated blocks, the resultants are no longer parallel to the $y$ axis, indicating significant $w$ velocities in this region. On the other hand, Figure 4.26 is a collection of vector plots at different $z$ vs. $x$ cross-sections along the $y$ axis such that the arrows represent the resultants of the vector sum of $w$ and $u$ velocities. Observe how the resultants are either extremely small or mostly parallel to the $x$ axis in cross-sections below $y = 0.8$. Nevertheless, above and including the $xz$ plane at $y = 0.8$, which corresponds to the region between the top of the heated blocks and the top wall of the enclosing cavity, the resultant vectors are no longer negligible nor parallel to the $x$ axis. This indicates comparable and significant $u$ and $w$ velocities. In fact, table 4.3 compares the maximum values of $u$, $v$ and $w$ velocities. As expected, the maximum value of $w$ is both two (2) times smaller than the maximum value of $u$ and four (4) times smaller than the maximum value of $v$. This effectively shows $w$ to be twice as strong with respect to $u$ and $v$ as it was for $Ra = 10^3$. We are forced to conclude that the effect of the $w$ velocity on the flow is now longer negligible at $Ra = 10^4$.

The $w$ velocity effect is clearly observable when we analyze the local $Nu$ distribution across the cavity. In figure 4.27 we have a collection of four (4) Nusselt number distribution plots over the surface of the enclosing cavity. In contrast with the distribution observed at $Ra = 10^3$, the local $Nu$ near both the $x = 0.0, y = 1.0$ line, and the $x = 1.0, y = 1.0$ line has increased in magnitude. With the increase in the $Ra$ we can also detect a decrease in the magnitude of the local $Nu$ over the surface of the bottom wall of the enclosing cavity. All of these observations are consistent
Figure 4.25: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.15$, b) $x = 0.35$, c) $x = 0.5$, d) $x = 0.65$, e) $x = 0.8$, f) $x = 0.95$ for $Ra = 10^4$. 
Figure 4.26: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.15$, b) $y = 0.35$, c) $y = 0.5$, d) $y = 0.65$, e) $y = 0.8$, f) $y = 0.95$ for $Ra = 10^4$. 
with the isotherm patterns seen in figure 4.23. However, the most drastic change in
$Nu$ distribution is lost. This is an indication that there has been a drastic change in
the way heat is being transferred near the top of the cavity. As we have seen before,
for $Ra = 10^4$, the $w$ velocity plays an important role in moving the fluid, specially
near the top wall. This effect is responsible for the irregular heat flux across the top
wall surface. As the $Ra$ increases beyond $Ra = 10^4$ we expect this effect to gradually
spread mostly throughout the top half of the cavity, enhancing the heat flux across
the top wall of the enclosing cavity as well as the top half of both the left and right
walls of the cavity. We also expect reduced flux across the bottom wall.

Figure 4.28 is a collection of four (4) Nusselt number distribution plots at the
non-insulated surfaces of the heated blocks. It is clear from the plots that the $Nu$ is
a maximum at bottom walls of the blocks. The $Nu$ on the surface of the blocks has
Figure 4.28: $Nu$ of the heated blocks for $Ra = 10^4$. From top left to bottom right: left wall, right wall, top wall and bottom wall.

decreased in all the walls above the $y = 0.5$ plane. The maximum value of the $Nu$ is about 127, while the minimum is about 2 and occurs near the top of both the right wall of the left block and the left wall of the right block. Exact values and locations of the local $Nu$, its mean, minimum and maximum values can be found in table 4.4.
Chapter 5

Problem Three: Two Heated and Interconnected Cavities of Different Sizes

The motion of fluids enclosed in heated cavities is an important problem, because the parallelepiped cavity in question can be used as a building block for far more complex fluid systems. For example, the flow of air in a structure can be simulated if we can describe the structure in question as a collection of interconnected heated blocks of different sizes. Or we can simulate the pasteurization process of bottled fluids by describing the container as a collection of interconnected blocks of different cross-sectional areas.

A simple example of the use of the heated parallelepiped as a building block is the problem described by figure 2.4. Two cubic cavities of different dimensions are interconnected as shown. The temperature is then specified at different locations, while the heat flux is specified at others, as described in the same figure.

The objective is to solve for the primitive variables of the flow in the region described by these cavities. In this chapter we present the simulation results obtained for both the two-dimensional and three-dimensional versions of such a problem. These calculations have been performed assuming a Prandtl number of 1.0 for various values of the Rayleigh number between $10^3$ and $10^6$.

5.1 Two-dimensional Version

In the two-dimensional version of this problem, two rectangular cavities of different sizes are placed one on top of the other. In detail, a square cavity of $0.4 \times 0.4$ square
units is centered on top of a rectangular cavity of $1.0 \times 0.6$ square units, such that the
distance from the top left hand corner of the bottom block to the bottom left hand
corner of the top block is exactly 0.3 units.

When the block-to-block interface is removed, we are left with an irregular cavity
which can be easily discretized with a regularly spaced grid. The cavity is differentially
heated, that is, the left walls of both the bottom and top blocks are heated to a
constant temperature ($\theta = 0.5$), while the right walls of both blocks are cooled to a
constant temperature ($\theta = -0.5$). At the same time, the top wall of the top block
and the bottom wall of the bottom block are insulated. The remaining segment to
the left of the top cavity is heated ($\theta = 0.5$), while the remaining segment to the right
of the top cavity is cooled ($\theta = -0.5$).

In contrast with the problem described in the previous chapter, although there
is geometric symmetry with respect to the $x = 0.5$ line, the solution will not be
symmetric due to the differences in boundary conditions across the cavity. However,
notice how both cavities are subjected to the same conditions we observed in the
bench mark problem. For this reason, we expect the overall solution to be some kind
of superposition of the bench mark solutions for each of the blocks.

Figure 5.1 is a temperature contour plot for $Ra = 10^3$. We have used a fine mesh
with $41 \times 41$ grid points for the solution at this $Ra$. This explains the smoothness
of the solution hereby presented. The isotherms in the top cavity are mainly parallel
to each other at this $Ra$. This indicates that the main heat transfer mechanism in
this region is conductive at $Ra = 10^3$. We will observe a change from conductive
to convective heat transfer in the region in question as the $Ra$ increases, due to
the increase in the magnitude of the buoyancy term in the $y$-momentum equation.
Another interesting fact is that the isotherms are nearly perpendicular to the interface
between the two heated blocks. This indicates that the heat flux at the interface is
Figure 5.1: Temperature contours or isotherms for $Ra = 10^3$.

Figure 5.2: Color isotherm pattern for the irregular cavity problem when the $Ra = 10^3, Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
almost zero. We will notice a change in this behavior as the $Ra$ increases. With respect to the bottom cavity, we can observe that the isotherms are curved, especially near the top left and top right corners. This is an indication of convective heat transfer in action. Nevertheless, the curvature of the isotherms is not pronounced. As mentioned in previous chapters, the isotherm curvature can be used as a measure of the intensity of convective heat transfer. Again, as with the other system behaviors, we expect an increase in the isotherm curvature as the $Ra$ increases. Finally, the temperature contour pattern in the bottom cavity is very similar to that observed in the two-dimensional benchmark problem for the same $Ra$.

Figure 5.3 is a velocity vector plot for $Ra = 10^3$. As in the problems discussed in previous chapter, we observe a dominant $v$ velocity in the $y$ direction near the heated walls in both the top and bottom cavities. We can observe that a circulation cell has formed and is centered close to the center point of the bottom cavity. We can also
observe that the effect of the circulation cell extends into the top cavity. In it, the flow enters the interface near the bottom left corner of the top cavity, moves close to the left wall, turns right near the top left corner until it reaches the top right corner, turns right again and follows the left wall until it leaves through the interface near the bottom right hand corner of the top cavity. Due to the no slip condition, all the fluid velocities diminish as the fluid approaches the system boundaries. But also notice that the fluid is almost static near the center of the circulation cell. This is characteristic behavior of circulation cells throughout the study.

From the nature of the problem, one might expect the formation of a circulation cell in the top cavity as well. The boundary conditions are very similar to those for the bottom cavity, so why has not a circulation cell formed in the region? In our study we have only worked with a specific aspect ratio between the top and bottom cavities for different values of the Rayleigh number. Nevertheless, Selarnet [11] has worked with different aspect ratios, Rayleigh numbers and Prandtl numbers as well, and when we examine his data it appears that the formation of the cell in the smallest cavity is dependent on both the aspect ratio, the $Ra$ and the $Pr$. Our choice of $Pr$ and aspect ratio seems to preclude the formation of a circulation cell in the region.

Figures 5.4 to 5.6 represent the local Nusselt number distribution over the heated two-dimensional surfaces of the problem at hand. The sign of the $Nu$ values indicate whether the heat flux is in the positive or negative direction of the corresponding coordinate axis. Figure 5.4 corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the bottom cavity, in that order. As expected, the flux at the hot wall is in the positive $x$ direction, from the hot wall to the less hot fluid, and as expected from the separation of the isotherms, the $Nu$ is maximum near $y = 0.0$, with a value of about 1.1, and is minimum close to $y = 0.6$, with a value of 0. At the cold wall, the flux is also in the positive $x$ direction, from the fluid to the colder wall.
Figure 5.4: Nusselt number distributions for $Ra = 10^3$. From top to bottom: a) hot left wall of bottom cavity, b) cold right wall of bottom cavity.

In a similar fashion, the flux is maximum near $y = 0.0$, with a value close to 0.85, and is minimum close to $y = 0.6$, with a magnitude of 0.

Figure 5.5 corresponds to the local $Nu$ at both the hot left top wall and the cold right top wall of the bottom cavity, in that order. The flux at the hot wall is in the

Figure 5.5: Nusselt number distributions for $Ra = 10^3$. From top to bottom: a) hot left top wall of bottom cavity, b) cold right top wall of bottom cavity.

negative $y$ direction, from the hot wall to the less hot fluid. As expected from the isotherm distribution, the $Nu$ is maximum near $x = 0.3$, the starting point of the interface between the two interconnected blocks. The $Nu$ at this location has a value
of about 3.2. The $Nu$ is a minimum close to $x = 0.0$, where it has a value of 0. At the cold wall, the flux is in the positive $y$ direction, from the fluid to the colder wall. The $Nu$ is maximum near the interface, specifically at $x = 0.7$, with a value close to 3.5, and is minimum at $x = 1.0$, where its magnitude is 0.

Figure 5.6 corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the top cavity, in that order. The flux at the hot wall is in the positive $x$

![Graph](image)

Figure 5.6: Nusselt number distributions for $Ra = 10^3$. From top to bottom: a) hot left wall of top cavity, b) cold right wall of top cavity.

direction, into the cavity. The $Nu$ is maximum near $y = 0.6$, the starting point of the interface between the two interconnected blocks. The $Nu$ at this location has a value of about 3.6. The $Nu$ is a minimum close to the middle of the wall, at $y = 0.8$, where it has a value of about 2.35. At the cold wall, the flux is also in the positive $x$ direction, out of the cavity. The $Nu$ is again maximum near the interface, specifically at $y = 0.6$, with a value of about 4.3, and is minimum at $y = 1.0$, where its magnitude is close to 2.6.

Figures 5.7 and 5.9 are the temperature contour and velocity vector plots for $Ra = 10^4$, respectively. As in the $10^3$ case, heat is being transferred mainly by conduction in the top cavity. Nevertheless, the heat flux across the interface is now non-zero, especially near the center of the interface. As one can see, the isotherms are
Figure 5.7: Temperature contours or isotherms for $Ra = 10^4$.

Figure 5.8: Color isotherm pattern for the irregular cavity problem when the $Ra = 10^4$, $Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
no longer perpendicular to the interface in that region. In addition, we see a change in the isotherm distribution. The contours are now more closely packed near the cold right wall of the top cavity. This indicates an increase in the heat flux, and therefore we expect an increase in the local $Nu$ over that wall. As a matter of fact, the contours are now less closely packed near the hot left wall of the top cavity, indicating a decrease in the heat flux and therefore in the local $Nu$ over the hot wall. In the bottom cavity, the curvature of the isotherms has increased, indicating an increase in the convective heat transfer in the region. We begin to notice an increase in the proximity of the isotherms near the bottom of the hot left wall of the cavity, indicating an increase in the heat flux and the local $Nu$ in that region. This is very similar to the behavior observed in the benchmark problem.

![Figure 5.9: Vector plots for $Ra = 10^4$.](image)

The flow pattern at this $Ra$ is not very different from that at $10^3$. The only significant changes are related to the magnitude of the velocities, which have increased, and
related to the extension of the circulation cell in the bottom cavity, which has elongated in the $x$ direction, becoming somewhat elliptical in shape, rather than circular as before. Notice that the center of rotation has not moved even with the elongation. Another subtle aspect to notice now in the top cavity is that the region next to the hot left wall in which the fluid moves upwards is now somewhat larger than the region next to the cold left wall where the fluid moves downwards. For the $10^3$ case, these regions were about the same size. We will notice a progressive change of this form as the $Ra$ increases.

Figures 5.10 to 5.12 represent the local Nusselt number distribution over the heated two-dimensional surfaces of the two interconnected cavities in the case when the $Ra = 10^4$. The first figure corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the bottom cavity, in that order. The flux at the hot wall

![Figure 5.10: Nusselt number distributions for $Ra = 10^4$. From top to bottom: a) hot left wall of bottom cavity, b) cold right wall of bottom cavity.](image)

is in the positive $x$ direction, in other words, into the cavity. As expected from the isotherm pattern, the $Nu$ is maximum near $y = 0.15$, with a value of about 2.5, and is minimum close to $y = 0.6$, with a value of 0 at that location. These values indicate an increase of 127% in the maximum $Nu$ at the wall, due to the change in the $Ra$. 
At the cold wall, the flux is also in the positive $x$ direction, in this case indicating heat leaving the cavity. The $Nu$ at this wall is maximum near its center ($y = 0.3$), with a value close to 0.77, and is minimum at $y = 0.6$, with a magnitude of 0. These values represent a 12% decrease in the maximum $Nu$ at this wall when compared to the values obtained for $Ra = 10^3$.

Figure 5.11 corresponds to the local $Nu$ at both the hot left top wall and the cold right top wall of the bottom cavity, in that order. The flux at the hot wall is in the negative $y$ direction, into the cavity. As expected from the isotherm distribution, the $Nu$ is again maximum near $x = 0.3$, the starting point of the interface between the two interconnected blocks. The $Nu$ at this location now has a value of about 2.9. This represents a 10% decrease in the maximum $Nu$. The $Nu$ is a minimum at $x = 0.0$, where it has a value of 0. At the cold wall, the flux is in the positive $y$ direction, towards the exterior. The $Nu$ is maximum near the interface, at $x = 0.7$, with a value close to 3.6, and is minimum at $x = 1.0$, where its magnitude is 0. There is no significant change in the maximum magnitude of the $Nu$ at this wall when the $Ra$ changes from $10^3$ to $10^4$.

Figure 5.11: Nusselt number distributions for $Ra = 10^4$. From top to bottom: a) hot left top wall of bottom cavity, b) cold right top wall of bottom cavity.
Figure 5.12 corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the top cavity, in that order. The flux at the hot wall is in the positive $x$ direction, into the cavity. The $Nu$ is maximum near $y = 0.6$, the starting point of the interface between the two interconnected blocks. The $Nu$ at this location has a value of about 2.6, which is 27\% less than the value observed at $Ra = 10^9$. The $Nu$ is a minimum close to the middle of the wall, at $y = 0.85$, where it has a value of about 1.66, a decrease of about 30\%. At the cold wall, the flux is also in the positive $x$ direction, towards the outside of the cavity. The $Nu$ is again maximum near the interface at $y = 0.6$, with a value of about 5.27, and is minimum at $y = 1.0$, where its magnitude is close to 3.4. These values indicate a 22\% increase in the magnitude of the maximum $Nu$ and a 28\% increase in the magnitude of the minimum $Nu$, both at this wall. All these numerical measures are consistent with the conclusions drawn from the isotherm distribution.

For $Ra = 10^5$, we begin to observe significant qualitative and quantitative changes in both the manner by which heat is transferred around the cavity and in the overall fluid flow pattern. Figures 5.13 and 5.15 are the temperature contour and velocity vector plots for this $Ra$, respectively. In the top cavity there is a definite change in the
temperature contour distribution. The curvature of the isotherms has increased, indicating the transition from conductive to convective heat transfer. This is a definite change from the mainly conductive behavior at previous $Ra$. Again we observe that the contour lines are very closely packed near the cold right wall indicating a further increase in heat flux and local $Nu$. The opposite can be seen occurring at the opposite hot left wall where the isotherms have separated from each other even further, especially in the top $\frac{1}{3}$ of the wall, indicating a further decrease in heat flux and local $Nu$. The heat flux through the interface has again increased as the isotherms cross it at steeper angles at this $Ra$. In the bottom cavity, the convective heat transfer is the dominating process. Cold fluid can be found closer to the hot wall now than in any of the previous cases. In the same manner, hot fluid is now closer to the cold right wall. The heat flux and therefore the local $Nu$ have substantially increased near the bottom half of the hot left wall, as indicated by the isotherm distribution. In addition, we can see that the isotherms are getting closer together near the center of the cold right wall, indicating a significant increase in heat flux and local $Nu$ not seen before in this region.

The flow pattern has also changed substantially. The sole, well-defined circulation cell found in previous cases at lower Rayleigh numbers has become irregular. The center of rotation has shifted up and to the left from where it used to be in previous cases. In fact, as we will see later, this is the beginning of the formation of a second very week circulation cell in the same region. As can be seen, there is a region in the bottom cavity where the flow is dominated by the $u$ velocity. This region has progressively grown in size as the $Ra$ has increased. At this $Ra$ the flow velocities have increased in magnitude again, while the static region around which the flow rotates has also increased in size. With respect to the top cavity, the flow penetrates it as before, but the flow velocities are larger at this time. Finally, we observe that the
Figure 5.13: Temperature contours or isotherms for $Ra = 10^5$.

Figure 5.14: Color isotherm pattern for the irregular cavity problem when the $Ra = 10^5, Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.
region near the left wall where the fluid is moving upwards is about three (3) times as large as the region next to the right wall where the fluid is moving downwards.

Figures 5.16 to 5.18 represent the local Nusselt number distribution over the heated two-dimensional surfaces of the two interconnected cavities in the case when the $Ra = 10^5$. The first figure corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the bottom cavity, in that order. The flux at the hot wall is in the positive $x$ direction, in other words, into the cavity. As expected from the isotherm pattern, the $Nu$ is maximum near $y = 0.1$, with a value of about 7.3, and is minimum close to $y = 0.6$, with a value of 0 at that location. These values indicate an increase of 190% in the maximum $Nu$ at the wall, due to the change in the $Ra$. At the cold wall, the flux is also in the positive $x$ direction, in this case indicating heat leaving the cavity. The $Nu$ at this wall is maximum at $y = 0.325$, with a value close to 2, and is minimum at $y = 0.6$, with a magnitude of 0. These values represent a
Figure 5.16: Nusselt number distributions for $Ra = 10^5$. From top to bottom: a) hot left wall of bottom cavity, b) cold right wall of bottom cavity.

167% increase in the maximum $Nu$ at this wall when compared to the values obtained for $Ra = 10^4$.

Figure 5.17 corresponds to the local $Nu$ at both the hot left top wall and the cold right top wall of the bottom cavity, in that order. The flux at the hot wall is in

Figure 5.17: Nusselt number distributions for $Ra = 10^5$. From top to bottom: a) hot left top wall of bottom cavity, b) cold right top wall of bottom cavity.

the negative $y$ direction, into the cavity. As expected from the isotherm distribution, the $Nu$ is again maximum near $x = 0.3$, the starting point of the interface between
the two interconnected blocks. The $Nu$ at this location now has a value of about 5.4. This represents a 90% increase in the maximum $Nu$. The $Nu$ is a minimum at $x = 0.0$, where it has a value of 0. At the cold wall, the flux is in the positive $y$ direction, indicating that heat is leaving the cavity. The $Nu$ is maximum near the interface, at $x = 0.7$, with a value close to 6, and is minimum at $x = 1.0$, where its magnitude is 0. These represent a 71% increase in the maximum magnitude of the $Nu$ at this wall when the $Ra$ changes from $10^4$ to $10^5$.

Figure 5.18 corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the top cavity, in that order. The flux at the hot wall is in the positive $x$

![Figure 5.18: Nusselt number distributions for $Ra = 10^5$. From top to bottom: a) hot left wall of top cavity, b) cold right wall of top cavity.](image)

direction, into the cavity. The $Nu$ is maximum near $y = 0.6$, the starting point of the interface between the two interconnected blocks. The $Nu$ at this location has a value of about 4, which is 54% more than the value observed at $Ra = 10^4$. The $Nu$ is now a minimum close to the top of the wall, at $y = 1.0$, where it has a value of about 0.88, a decrease of about 47%. At the cold wall, the flux is also in the positive $x$ direction, which in this case indicates heat flowing out of the cavity. However, the $Nu$ is no longer maximum near the interface, but maximum near $y = 0.9$ with a magnitude of 9.1, and is no longer minimum at $y = 1.0$, but minimum near $y = 0.675$, where its
magnitude is close to 6.75. These values indicate a 73% increase in the magnitude of the maximum $Nu$ and a 98% increase in the magnitude of the minimum $Nu$, both at this wall. All these numerical measures are consistent with the conclusions drawn from examination of the isotherm distribution.

When the $Ra = 10^6$, what we observe is an extension of the results for $Ra = 10^5$. Figures 5.19 and 5.21 are the temperature contour and velocity vector plots for this $Ra$, respectively. The curvature of the isotherms in the top cavity has increased indicating an increase in the transfer of heat by convection. The isotherms near the cold right wall of the top cavity are almost indistinguishable from each other due to their closeness. This indicates a very large heat flux and local $Nu$ on that surface. On the other hand, we can see that on the bottom $\frac{3}{4}$'s of the hot left wall the isotherms are now closer to each other if compared to those when the $Ra = 10^5$. This indicates that although the heat flux had been decreasing in that area, it is now increasing.

Figure 5.19: Temperature contours or isotherms for $Ra = 10^6$. 
Figure 5.20: Color isotherm pattern for the irregular cavity problem when the $Ra = 10^5$, $Pr = 1.0$. Dark blue corresponds to coldest temperature, while dark red corresponds to hottest temperature.

Figure 5.21: Vector plots for $Ra = 10^5$. 
However, in the top $\frac{1}{4}$ of the same wall, the heat flux and local $Nu$ is still decreasing. The isotherm pattern in the top cavity now resembles that obtained as the solution of the benchmark problem. In the bottom cavity we again see an increase in the heat flux and local $Nu$ at both the left and right walls. Finally, the curvature of the contours indicates that convective heat transfer is again the dominant heat transfer process.

The flow pattern is very similar to that at a $Ra = 10^5$. Nevertheless, there are a few differences. The magnitude of the velocities has again increased with the Rayleigh number. In addition, there are two circulation cells in action instead of two. The second cell is hard to see in the figure because it is very weak, i.e. the velocities associated with it are small in comparison with the fluid velocities in other regions of the cavity. The center of rotation of the stronger cell is close to the point $(0.80, 0.25)$, while the center of rotation of the weaker cell is close to the point $(0.45, 0.45)$. The fluid velocities near the boundaries are extremely large, but the maximum velocities can be found near the interface of the cavities, especially near the two corners. Finally, it can be seen that in the top cavity, the region of fluid moving upwards and the region of fluid moving downwards are now almost the same size. This is a change in the trend previously observed. We think this is due to the formation of the new weak rotation cell near the interface. In fact, it could well be that for higher $Ra$, this cell could move past the interface and into the top cavity.

Figures 5.22 to 5.24 represent the local Nusselt number distribution over the heated two-dimensional surfaces of the two interconnected cavities in the case when the $Ra = 10^5$. The first figure corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the bottom cavity, in that order. The flux at the hot wall is in the positive $x$ direction, in other words, into the cavity. As expected from the isotherm pattern, the $Nu$ is maximum near $y = 0.08$, with a value of about 15.2, and
Figure 5.22: Nusselt number distributions for $Ra = 10^6$. From top to bottom: a) hot left wall of bottom cavity, b) cold right wall of bottom cavity.

is minimum close to $y = 0.6$, with a value of 0 at that location. These values indicate an increase of 108% in the maximum $Nu$ at the wall, due to the change in the $Ra$. At the cold wall, the flux is also in the positive $x$ direction, in this case indicating heat leaving the cavity. The $Nu$ at this wall is maximum near its center ($y = 0.325$), with a value close to 4.8, and is minimum at $y = 0.6$, with a magnitude of 0. These values represent a 128% increase in the maximum $Nu$ at this wall when compared to the values obtained for $Ra = 10^5$.

Figure 5.23 corresponds to the local $Nu$ at both the hot left top wall and the cold right top wall of the bottom cavity, in that order. The flux at the hot wall is in the negative $y$ direction, into the cavity. As expected from the isotherm distribution, the $Nu$ is again maximum near $x = 0.3$. The $Nu$ at this location now has a value of about 8.8. This represents a 63% decrease in the maximum $Nu$. The $Nu$ is a minimum at $x = 0.0$, where it has a value of 0. At the cold wall, the flux is in the positive $y$ direction, towards the exterior. The $Nu$ is maximum near the interface, at $x = 0.7$, with a value close to 6.7, and is minimum at $x = 1.0$, where its magnitude
Figure 5.23: Nusselt number distributions for $Ra = 10^6$. From top to bottom: a) hot left top wall of bottom cavity, b) cold right top wall of bottom cavity.

is 0. These values represent a 12% increase in the maximum $Nu$ at this wall when compared to the values obtained for $Ra = 10^5$.

Figure 5.24 corresponds to the local $Nu$ at both the hot left wall and the cold right wall of the top cavity, in that order. The flux at the hot wall is in the positive $x$

Figure 5.24: Nusselt number distributions for $Ra = 10^6$. From top to bottom: a) hot left wall of top cavity, b) cold right wall of top cavity.

direction, into the cavity. The $Nu$ is maximum near $y = 0.6$. The $Nu$ at this location has a value of about 5.5, which is 34% larger than the one observed at $Ra = 10^5$. The $Nu$ is now a minimum at $y = 1.0$, where it has a value of about 1.2, an increase
of about 36%. At the cold wall, the flux is also in the positive \( z \) direction, towards the outside of the cavity. The \( Nu \) is again no longer maximum near the interface, but at \( y = 0.925 \) with a value of about 18.2, and is minimum at \( y = 0.625 \), where its magnitude is close to 12.1. These values indicate a 100% increase in the magnitude of the maximum \( Nu \) and a 79% increase in the magnitude of the minimum \( Nu \), both at this wall. Again, all these numerical measures are consistent with the conclusions drawn from the isotherm distribution. A collection of all the Nusselt number values and their locations can be found in tables 5.1 and 5.2.

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Table 5.1: Important numerical results of the two-dimensional version of the problem where two parallelepiped cavities of different sizes are differentially heated while interconnected.
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**Top Cavity's Cold Right wall**

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**Bottom Cavity's Cold Right Top wall**

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**Bottom Cavity's Cold Right wall**

Table 5.2: Important numerical results of the two-dimensional version of the problem where two parallelepiped cavities of different sizes are differentially heated while interconnected.
5.2 Three-dimensional Version

It is important to recognize the relation of both the two-dimensional and three-dimensional problems in order to understand both the similarities and the differences between both versions of the same problem. As in the problems dealt with in previous chapters, the two-dimensional problem can be viewed as the corresponding three-dimensional problem where the front and back walls of both interconnected blocks are located at infinity. Due to that fact, we expect the solutions of the two-dimensional problem for a specific $Ra$ to be very similar to that of the three-dimensional problem at the $xy$ cross-section at $z = 0.5$ for the same $Ra$. On the other hand, the three-dimensional solution at $xy$ cross-sections will differ from the two-dimensional solution the closer it is to the front and back walls of the irregular cavity.

As with the benchmark problem, as the $Ra$ increases, we expect an increase in the fluid velocities, and a progressively more important role played by the $w$ velocity in the $z$ direction. This increase in velocity has been observed together with an increase in heat flux near the heated walls, hence we expect larger local $Nu$ as the $Ra$ increases.

Figure 5.25 is a collection of temperature contour plots at different $xy$ cross-sections for $Ra = 10^3$. Due to the problem’s symmetry, only $xy$ cross-sections between $z = 0.0$ and $z = 0.5$ need to be presented. At the first few cross-sections, the boundary conditions are very similar to the ones observed in the study of the benchmark problem in chapter 3. It is then reasonable to expect a very similar isotherm pattern, especially at $z = 0.1$. This is exactly what we observe at this cross-section. Nevertheless, due to the change in the boundary conditions on the top wall of the bottom cavity from Von Neumann to Dirichlet as we approach the open interface between the two blocks starting at the $z = 0.3$ cross-section and ending at the $z = 0.7$ cross-section, the isotherms near the top of the cold right wall tend to bend to the
Figure 5.25: Temperature contours or isotherms for $Ra = 10^3$ at a) $z = 0.0$, b) $z = 0.1$, c) $z = 0.2$, d) $z = 0.3$, e) $z = 0.4$, f) $z = 0.5$. 
left, towards the interface, and diverge, instead of bending to the right, towards the wall, and converge. This effect is clear in plots a through c. As we observe the region directly under the interface, we notice how the isotherms from the bottom cavity straighten as they penetrate the top cavity. In that region, the isotherms become mostly parallel. In addition, the curvature of the isotherms in the bottom cavity can be attributed mostly to the presence of the block interface. All of these imply that conduction is the predominant heat transfer process in both cavities at this $Ra$. Also, we note there is no significant change in the isotherm pattern of both the bottom and top cavities as one moves in the $z$ direction. This uniformity along the $z$ axis is an indication of the weak effect of the $u$ velocity on the fluid flow. This will be corroborated when we examine the velocity vector plots for this specific $Ra$.

Figures 5.26 to 5.28 present velocity vector plots at different cross-sections along the three (3) major axes: $z$, $x$ and $y$, respectively, for the case where $Ra = 10^3$. Figure 5.26 is a collection of vector plots at different $y$ vs. $x$ cross-sections along the $z$ axis. The arrows in this plot represent the vector sum of the $v$ velocity in the $y$ direction and the $u$ velocity in the $x$ direction at every mesh point in the grid. In both the $z < 0.3$ and $z > 0.7$ regions of the system, the boundary conditions are very similar to those observed in the bench mark problem. In fact, observe how similar the flow patterns of these two problems are in these regions. There is a single circulation cell very close to the center of the cavity, which rotates clockwise. The fluid velocities seem to decrease as we look at cross-sections closer to the system walls, which enforce a no-slip boundary condition. All of these are common to both problems. On the other hand, let us concentrate in the region $0.3 \leq z \leq 0.7$ immediately under the two block interface. As we can see in plots d and e, the rotation cell in the bottom cavity is still present at the same central location. Nevertheless, the fluid now penetrates the top cavity. The fluid enters the cavity near its bottom left hand corner, rises
Figure 5.26: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.1$, b) $z = 0.2$, c) $z = 0.3$, d) $z = 0.4$, e) $z = 0.5$ for $Ra = 10^3$. 
along its left wall, turns right near its top wall, follows the wall and turns right at the right wall and moves downwards until leaving the top cavity through the interface, at the bottom right hand corner. The fluid then joins the circulation cell in the bottom cavity to repeat the process. Notice that there is no independent circulation cell in the top cavity. This only occurs for certain values of the Rayleigh number, the Prandtl number and certain aspect ratios between the top and bottom cavities. Due to the orientation of the resultant vectors, we can establish a relationship between the magnitudes of its vector components. In this figure, it is clear that the $u$ and $v$ velocities have mostly the same magnitude. We will see in the following figures that this is not the case with the $w$ velocity in the $z$ direction.

Figure 5.27 is a collection of velocity vector plots at different $y$ vs. $z$ cross-sections along the $x$ axis such that the arrows represent the resultants of the vector sum of $v$ and $w$ velocities. Observe, however, that most of the resultant vectors in the figure are either negligible (like in plot $c$) or mostly parallel to the $y$ axis, indicating $v$ velocities much larger in magnitude than their corresponding $w$ velocities. It is only near the insulated front and back walls, where the buoyant effects are less pronounced, that the $w$ velocity seems to be comparable to the $v$ velocity at all. In fact, on the top cavity, the $w$ velocity is always many times smaller than the $v$ velocity. Figure 5.28 is a collection of velocity vector plots at different $z$ vs. $x$ cross-sections along the $y$ axis such that the arrows represent the resultants of the vector sum of $w$ and $u$ velocities. Notice how the resultants are either negligible (as in plot $b$) or mostly parallel to the $x$ axis, indicating now $u$ velocities much larger than their corresponding $w$ velocities. In the top cavity this is always the case. It is again only near the insulated walls of the bottom cavity that the $w$ velocity is at all comparable in magnitude with the non-zero $u$ velocity. In fact, tables 5.3 and 5.4 compares the maximum values of the $u$, $v$ and $w$ velocities. As expected, from the vector plots, the maximum value of $w$ is
Figure 5.27: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.2$, b) $x = 0.4$, c) $x = 0.5$, d) $x = 0.6$, e) $x = 0.8$ for $Ra = 10^3$. 
Figure 5.28: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.1$, b) $y = 0.3$, c) $y = 0.5$, d) $y = 0.7$, e) $y = 0.8$, f) $y = 0.9$ for $Ra = 10^3$. 
both 3.2 times smaller than the maximum value of \( u \) and 3.5 times smaller than the maximum value of \( v \). Thus, we conclude that the effect of the \( w \) velocity on the fluid flow for \( Ra = 10^3 \) is negligible in most regions of the system.

Figure 5.29 is a collection of Nusselt number distribution plots over the six (6) different heated surfaces of the irregular cavity problem. Tables 5.3 and 5.4 include the locations and values of the maximum, median and minimum value of the \( Nu \) at each of

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**Figure 5.29:** Nusselt number distribution for \( Ra = 10^3 \) over different heated surfaces of the irregular cavity formed by two interconnected three-dimensional cavities. From top left to bottom right: distributions for a) left hot wall of bottom cavity, b) right cold wall of bottom cavity, c) hot left top wall of bottom cavity, d) cold right top wall of bottom cavity, e) hot left wall of top cavity, and f) cold right wall of top cavity.
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**Bottom Cavity's Hot Left wall**

| $Nu_0$ | 0.9374 | 1.6409 | 5.0564 |
| $Nu_{\text{max}}$ | 1.0767 | 2.3462 | 7.0479 |
| $y$ | 0.0000 | 0.1000 | 0.1000 |
| $z$ | 0.5000 | 0.5000 | 0.4000 |
| $Nu_{\text{min}}$ | 0.0000 | 0.0000 | 0.0000 |
| $y$ | 0.6000 | 0.6000 | 0.6000 |
| $z$ | 0.3, 0.4, 0.5, 0.6, 0.7 | 0.3, 0.4, 0.5, 0.6, 0.7 | 0.3, 0.4, 0.5, 0.6, 0.7 |

**Bottom Cavity's Cold Right wall**

| $Nu_0$ | 0.8190 | 0.7518 | 2.1417 |
| $Nu_{\text{max}}$ | 0.9028 | 1.0388 | 3.7062 |
| $y$ | 0.0000 | 0.4000 | 0.4500 |
| $z$ | 1.0000 | 0.9000 | 0.2000 |
| $Nu_{\text{min}}$ | 0.0000 | 0.0000 | 0.0000 |
| $y$ | 0.6000 | 0.6000 | 0.6000 |
| $z$ | 0.3, 0.4, 0.5, 0.6, 0.7 | 0.3, 0.4, 0.5, 0.6, 0.7 | 0.3, 0.4, 0.5, 0.6, 0.7 |

**Bottom Cavity's Hot Left Top wall**

| $Nu_0$ | -0.2716 | -0.3763 | -0.4059 |
| $Nu_{\text{max}}$ | -1.9770 | -2.0712 | -3.1454 |
| $z$ | 0.7000 | 0.7000 | 0.7000 |
| $x$ | 0.3000 | 0.3000 | 0.3000 |
| $Nu_{\text{min}}$ | 0.0000 | 0.0000 | 0.0242 |
| $z$ | all | all | 0.9000 |
| $x$ | 0.0000 | 0.0000 | 0.3000 |

Table 5.3: Important numerical results of the three-dimensional version of the problem where two parallelepiped cavities of different sizes are differentially heated while interconnected.
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**Bottom Cavity’s Cold Right Top wall**

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<td>0.9000</td>
</tr>
<tr>
<td>(z)</td>
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<tr>
<td>(z)</td>
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<td>0.3000</td>
</tr>
</tbody>
</table>

Table 5.4: Important numerical results of the three-dimensional version of the problem where two parallelepiped cavities of different sizes are differentially heated while interconnected.
the surfaces. For $Ra = 10^3$, it is clear that the maximum local $Nu$ can be found near the intersection of the cold right wall of the top cavity and the two block interface, as expected from the isotherm patterns. At this location $(x, y, z) = (0.7, 0.6, 0.3)$ the $Nu$ has a magnitude of 3.1. In addition, the median $Nu$ is also largest at the cold right wall of the top cavity. At this wall, it has an approximate value of 2.7. In both these regions heat flows out of the irregular cavity. On the other hand, the minimum local $Nu$ can be found at the insulated walls (the bottom wall of the bottom cavity, and the top wall of the top cavity), where the heat flux, and therefore the Nusselt number are both equal to zero (0). Other walls where the heat flux and the local $Nu$ are usually low are the cold right wall of the bottom cavity, and both the hot left top wall and the cold right top wall of the bottom cavity.

For $Ra = 10^4$, figure 5.30 is a collection of temperature contour plots at different $xy$ cross-sections along the $z$ axis. Although the mesh used to obtain solutions for this $Ra$ is somewhat coarse, the solution is still valid if not as smooth as desired. In the regions away from the two block interface, the isotherm pattern is again very similar to that observed in the bench mark problem for the same $Ra$. The main difference, however is again that the isotherms near the top of the bottom cavity now tend to bend towards the interface, a behavior not present in the bench mark problem. This is most pronounced near the top right corner of the bottom cavity, where the isotherms used to approach the cold right wall and each other in the region as the $Ra$ increased, and now tend to separate and approach the interface instead. The curvature of the isotherms in the bottom cavity has increased from that observed at $Ra = 10^3$. This is an indication of an increase in convective heat transfer. The same is true in the top cavity. The isotherms are no longer parallel close to the bottom left hand corner of the top cavity. In this region the isotherms seem to converge, and are closely packed, indicating an increase in the local heat flux, and therefore in the local $Nu$. This
Figure 5.30: Temperature contours or isotherms for $Ra = 10^4$ at a) $z = 0.0$, b) $z = 0.1$, c) $z = 0.2$, d) $z = 0.3$, e) $z = 0.4$, f) $z = 0.5$. 
transition from conduction dominated heat transfer to convection dominated heat transfer will continue as the $Ra$ increases. This transition is reflected by the fluid flow pattern in the role played by the $u$ velocity in the $z$ direction, as we will see in subsequent figures.

Figures 5.31 to 5.33 present velocity vector plots at different cross-sections in the three (3) major axes: $z$, $x$ and $y$, respectively, for the case when the $Ra = 10^4$. Figure 5.31 is a collection of velocity vector plots at different $y$ vs. $x$ cross-sections along the $z$ axis. Again, the arrows represent resultants of the corresponding $v$ and $u$ velocities. As in the $Ra = 10^3$ case, the flow pattern inside the bottom cavity and away from the two block interface is dominated by a rotation cell located close to the center of the cavity. The only significant change at this $Ra$ is that the cell has elongated in the $x$ direction, as seen in plots $a$ through $c$. In the region under and above the two block interface, the flow is very similar to that at $Ra = 10^3$. Again, the only main difference is the elongation of the circulation cell in the bottom cavity. The flow in the top cavity has not changed significantly with the increase in Rayleigh number.

Figure 5.32 is a collection of velocity vector plots at different $y$ vs. $z$ along the $x$ axis such that the arrows represent the resultant of the vector sum of $u$ and $w$ velocities. The flow pattern has changed in the sense that the $w$ velocity effect has diminished in certain areas in which it was strong at $Ra = 10^3$ and has increased in other regions where it was weak before. Notice how the resultants are mostly parallel to the $y$ axis near the hot left wall (plot $a$) suggesting a reduction in the $v$ to $u$ ratio in the region. The same can be observed near the cold right wall (plot $c$). On the other hand, notice how the resultants near the interface in plots $b$ through $d$ have changed from those at $Ra = 10^3$, this indicates that the $w$ velocity is becoming more significant in the region. Figure 5.33 is a collection of velocity vector plots at different $z$ vs. $x$ cross-sections along the $y$ axis, such that the arrows represent the resultant
Figure 5.31: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.1$, b) $z = 0.2$, c) $z = 0.3$, d) $z = 0.4$, e) $z = 0.5$ for $Ra = 10^4$. 
Figure 5.32: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.2$, b) $x = 0.4$, c) $x = 0.5$, d) $x = 0.6$, e) $x = 0.8$ for $Ra = 10^4$. 
vector sum of the $w$ and $u$ velocities at every node in the grid. It is clear from plots $c$ through $f$ that the effect of the $w$ velocity can now be felt even in the top cavity. Notice how the resultants are no longer parallel to the $x$ axis. On the other hand, it seems now that the $u$ velocity to $w$ velocity ratio has increased in the bottom cavity because the $w$ is now less pronounced than at $Ra = 10^3$. However, the overall effect is more widespread now than before. Tables 5.3 and 5.4 compares the maximum values of the $u$, $v$ and $w$ velocities. As expected form the vector plots, the maximum value of $w$ is both 4.6 times smaller than the maximum value of $u$ and 3.6 times smaller than the maximum value of $v$. We conclude that the effect of the $w$ velocity on the fluid flow for $Ra = 10^4$ is more pronounced and extensive than at previous Rayleigh numbers.

Figure 5.34 is a collection of Nusselt number distribution plots over the six (6) different heated surfaces of the irregular cavity problem. Tables 5.3 and 5.4 include the locations and values of the maximum, median and minimum value of the $Nu$ at each of the surfaces. For $Ra = 10^4$, the maximum local $Nu$ can be found near the intersection of the cold right wall of the top cavity and the two block interface, as expected from the isotherm patterns. The exact location has changed to $(x, y, z) = (0.7, 0.6, 0.5)$. At this location the $Nu$ has a magnitude of 5.0, which represents a 61% increase in the magnitude of the maximum $Nu$ of the system. In addition, the median $Nu$ is again largest at the cold right wall of the top cavity. At this wall, it has an approximate value of 3.3, which represents a 22% increase in the magnitude of the largest median $Nu$ of the system. In both these regions heat flows out of the irregular cavity. Again the minimum local $Nu$ can be found at the insulated walls (the bottom wall of the bottom cavity, and the top wall of the top cavity), but other walls where the heat flux and the median $Nu$ are usually low are again the cold right wall of the bottom cavity, and both the hot left top wall and the cold right top wall of the bottom cavity.
Figure 5.33: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.1$, b) $y = 0.3$, c) $y = 0.5$, d) $y = 0.7$, e) $y = 0.8$, f) $y = 0.9$ for $Ra = 10^4$. 
Figure 5.34: Nusselt number distribution for $Ra = 10^4$ over different heated surfaces of the irregular cavity formed by two interconnected three-dimensional cavities. From top left to bottom right: distributions for a) left hot wall of bottom cavity, b) right cold wall of bottom cavity, c) hot left top wall of bottom cavity, d) cold right top wall of bottom cavity, e) hot left wall of top cavity, and f) cold right wall of top cavity.
For $Ra = 10^5$, figure 5.35 is a collection of temperature contour plots at different $xy$ cross-sections along the $z$ axis. The curvature of the isotherms is now much more pronounced than at previous Rayleigh numbers. This implies that the dominant heat transfer process at this $Ra$ is convection. The same is true in the top cavity. The isotherms are now even more deformed than at $Ra = 10^4$. In the region next to the cold right wall of the top cavity, the isotherms are closely packed, indicating an increase in the local heat flux, and therefore in the local $Nu$. This is much more pronounced than at lower Rayleigh numbers, at which the heat flux had increased mainly near the bottom right hand corner. In the bottom cavity we observe an increase in the heat flux near both the hot and cold walls. In addition, we observe colder fluid now closer to the hot left wall as well as hotter fluid closer to the cold right wall. This is due to the enhanced convection at this $Ra$. We still observe the bending of isotherms towards the interface due to the change in the boundary conditions on the top wall of the bottom cavity from Von Neumann to Dirichlet. The now dominating convective heat transfer process has a strong effect on the flow pattern as well, as described in the following discussion.

Figures 5.36 to 5.38 present velocity vector plots at different cross-sections in the three (3) major axes: $z$, $x$ and $y$, respectively, for the case when the $Ra = 10^5$. Figure 5.36 is a collection of velocity vector plots at different $y$ vs. $x$ cross-sections along the $z$ axis. Again, the arrows represent resultants of the corresponding $v$ and $u$ velocities. At this $Ra$, the rotation cell that used to be located at the center of the cavity has moved up to about $y = 0.035$, and has again elongated in the $x$ direction. This can be clearly seen in plots a through c. In the region under and above the interface the flow pattern has changed quite a lot. The rotational cell in the bottom cavity has actually split into two weaker cells. Their centers are close to each other near the point $(0.65, 0.35)$. The overall effect of both these cells is that of a very elongated
Figure 5.35: Temperature contours or isotherms for $Ra = 10^5$ at a) $z = 0.0$, b) $z = 0.1$, c) $z = 0.2$, d) $z = 0.3$, e) $z = 0.4$, f) $z = 0.5$. 


Figure 5.36: Vector plots for $y$ vs. $x$ cross-sections at a) $z = 0.1$, b) $z = 0.2$, c) $z = 0.3$, d) $z = 0.4$, e) $z = 0.5$ for $Ra = 10^5$. 
rotational cell towards the right of the cavity together with a region of rotation that seems to deviate from the horizontal and bend diagonally towards the bottom right hand corner. These effects can be clearly observed in plots d and e. The fluid still moves clockwise around the cavity and penetrates the top cavity through the interface as in previous cases. Notice that the region of fluid that moves the least in the top cavity is now located near the cold right well, when it used to be located near the center of the cavity. However, there is no rotation cell located in this region yet. Next we will examine the intensity of the w velocity in the fluid flow.

Figure 5.37 is a collection of velocity vector plots at different y vs. z along the x axis such that the arrows represent the resultant of the vector sum of v and w velocities. The flow pattern has changed significantly. Most of the resultants for this Ra are either very small, or mostly parallel to the y axis. In fact, it is clear that the v velocities in the top cavity and through the interface are much larger in magnitude than the w velocities in the rest of the cavity. Figure 5.38 is a collection of velocity vector plots at different z vs. x cross-sections along the y axis, such that the arrows represent the resultant vector sum of the w and u velocities at every node in the grid. It is clear from all the plots that the effect of the w velocity has now become negligible in most regions of the cavity. Notice how the resultants are either very small or mostly parallel to the x axis. Again, this is an indication of much larger u velocities than w velocities. Nevertheless, there is a region in which the w velocity has a significant effect: at the two block interface. The w velocity effect is necessary to help the flow move through the sudden contraction between the bottom cavity and the top cavity. Tables 5.3 and 5.4 compares the maximum values of the u, v and w velocities. As expected form the vector plots, the maximum or mean value of w is both 4.9 times smaller than the maximum value of u and 4.7 times smaller than the maximum value of v. We conclude that the effect of the w velocity on the fluid flow
Figure 5.37: Vector plots for $y$ vs. $z$ cross-sections at a) $x = 0.2$, b) $x = 0.4$, c) $x = 0.5$, d) $x = 0.6$, e) $x = 0.8$ for $Ra = 10^5$. 
Figure 5.38: Vector plots for $z$ vs. $x$ cross-sections at a) $y = 0.1$, b) $y = 0.3$, c) $y = 0.5$, d) $y = 0.7$, e) $y = 0.8$, f) $y = 0.9$ for $Ra = 10^5$. 
for $Ra = 10^5$ is significant near the interface in order to ensure continuity and force the flow through the sudden contraction the interface represents.

Figure 5.39 is a collection of Nusselt number distribution plots over the six (6) different heated surfaces of the irregular cavity problem. Tables 5.3 and 5.4 include

Figure 5.39: Nusselt number distribution for $Ra = 10^5$ over different heated surfaces of the irregular cavity formed by two interconnected three-dimensional cavities. From top left to bottom right: distributions for a) left hot wall of bottom cavity, b) right cold wall of bottom cavity, c) hot left top wall of bottom cavity, d) cold right top wall of bottom cavity, e) hot left wall of top cavity, and f) cold right wall of top cavity.

the locations and values of the maximum, median and minimum value of the $Nu$ at each of the surfaces. For $Ra = 10^5$, the maximum local $Nu$ can be found near the
intersection of the cold right wall of the top cavity and the insulated top wall of the
top block, as expected from the isotherm patterns. The exact location has changed
to \((x, y, z) = (0.7, 0.9, 0.5)\). At this location the \(Nu\) has a magnitude of 8.8, which
represents a 76% increase in the magnitude of the maximum \(Nu\) from that observed
at \(Ra = 10^4\). In addition, the median \(Nu\) is again largest at the cold right wall of
the top cavity. At this wall, it has an approximate value of 7.0, which represents a
112% increase in the magnitude of the largest median \(Nu\) of the system from that
observed at \(Ra = 10^4\). In both these regions heat flows out of the irregular cavity.
Again the minimum local \(Nu\) can be found at the insulated walls (the bottom wall of
the bottom cavity, and the top wall of the top cavity), but other walls where the heat
flux and the median \(Nu\) are usually low are now the hot left wall of the top cavity,
and both the hot left top wall and cold right top wall of the bottom cavity except at
the corners next to the interface, where the local \(Nu\) is rather high.

In Evren-Selamet, Arpaci and Borgnakke's paper [8] and in Selamet's doctoral dis-
cavity is thoroughly studied for different aspect ratios and different values of both
the Rayleigh number and the Prandtl number. Our two-dimensional results are for
an aspect ratio not treated in their study. Nevertheless, there is no qualitative dis-
crepancy between our results and theirs in term of the fluid behavior in the cavity. In
fact, we did some test simulations on exactly the same aspect ratio they described,
but with a much coarser mesh and a slightly larger Prandtl number \((Pr = 1.0 in-
stead of \(Pr = 0.71\)). The only difference between our approach and theirs is that
their formulation seems to introduce some artificial diffusion, resulting in an overall
smoothing of the steady state solution. Finally, we have found that the extension
of the problem to three-dimensions is straight-forward, and computationally feasible
with the current computer resources available.
Chapter 6

Conclusions

At this point, certain details related to the solution method and its performance, as well as related to possible future improvements and areas of study should be addressed.

Performance, stability and convergence are certainly some of the most important aspects of any solution scheme. We want to deal with each of them individually, and later suggest some possible improvements for future studies to come.

6.1 PSOR Performance

The PSOR scheme used in our study has proven to be very reliable and adequately accurate. In addition, the scheme was validated in chapter 3, when the results of the benchmark solutions by de Vahl Davis [1] were compared with the ones obtained in our study. It is important however, to understand the scheme performance in terms of how the scheme behaves as computations progress towards the solution of the problem in question. One way of doing this is to plot the number of iterations versus the nondimensional problem time, in order to see how the former vary as the scheme approaches the steady state solution, if it exists. These plots can be generated for each of the primitive and intermediate variables of the problem.

Recall from chapter 2 that the number of iterations the PSOR scheme performs is dependent on how much the solution changes from a particular time step to the other. If there is a steady state solution, then there will be a time at which the current solution is almost the steady state solution, and therefore the difference between
solutions at two consecutive time steps will be almost null. This will be reflected by
the scheme as a very low number of PSOR iterations. In the limit, the numerical
solution has become the steady state solution, when the number of PSOR iterations
for all the explicitly time dependent primitive variables (velocities and temperatures)
is equal to one (1). The pressure does not behave in the same manner because the
pressure equation is an elliptic equation at every time step. In fact, the solution of
the pressure will require a particular number of PSOR iterations which will remain
almost constant after some variation during the first few time steps.

Figures 6.1 to 6.6 are collections of plots that describe the variation in value of
the variables solved for by the PSOR scheme versus the non-dimensional problem
time, as well as the number of PSOR iterations required versus the non-dimensional
problem time for the same variables. The variables in question are five (5): \( \bar{u}, \bar{v},
\bar{w}, ppp \) and \( \theta \). In addition we have included the variation of the kinetic energy of
the system versus the non-dimensional problem time when available. For comparison
purposes, the plots have been chosen for a specific Rayleigh number \( Ra = 10^4 \) and
span the three major three-dimensional problems we have discussed.

Figure 6.1 and 6.2 correspond to the bench mark problem described in chapter 3.
In the first one, we can see the time history of the maximum velocities \( u_{max}, v_{max}\)
and \( w_{max} \) together with the number of iterations required for the calculation of the
intermediate variables \( \bar{u}, \bar{v} \) and \( \bar{w} \). In the second one, we can see the number of
iterations over time required for the calculation of the values of the unknowns \( ppp \)
and \( \theta \). Observe how the maximum velocities approach the steady state value as time
progresses. It is clear from the behavior of \( w_{max} \) that at early times the transient
response is not necessarily smooth. Nevertheless, as time progresses, we approach
the steady state very smoothly. The number of iterations is however not smooth
because the number of iterations is always an integer. We see that as we approach
Figure 6.1: Left column: maximum $u$, $v$ and $w$ velocity time histories for the three-dimensional benchmark problem when $Ra = 10^4$. Right column: time histories of the number of iterations required by the PSOR solver at every time step to determine the values of $\bar{u}$, $\bar{v}$ and $\bar{w}$. 
Figure 6.2: Plot 1 and 2: time histories of the number of iterations required by the PSOR solver at every time step to determine the values of $ppp$, $\theta$ for the three-dimensional benchmark problem. Plot 3: time history of the kinetic energy for the same problem.
the steady state solution, the number of iterations required for the calculation of the same unknown decreases. These plots are important because they allow us to relate the number of required iterations with how far away from steady state we are. We know that the minimum number of required iterations is one. It is then correct to suppose that the system of equations has reached its steady state value when all the unknowns require only one iteration of the PSOR scheme and the pressure has reached its steady number of iterations. However, if we examine the iteration plots carefully and compare them with the time histories of the unknowns, the steady state of the system has been reached even before the number of PSOR iterations becomes a minimum. We have concluded that using the number of iterations required by the PSOR as a measure of how close we are to the steady state might be misleading because, although we know that when the number of iterations is close to one we are definitely close to steady state, there is no way of knowing a priori where to draw the line between too many iterations to be at steady state and too little.

How is all this important? The fact is that knowing when you have reached steady state is extremely important from the point of view of the cpu time required for the simulations. For example, notice how in figure 6.1 the maximum velocities have already reached their steady state at \( t_{nd} = 0.05 \). Nevertheless, notice how the simulation continues up to \( t_{nd} = 0.15 \). If the steady state was reached at \( t_{nd} = 0.05 \), why did we proceed further? The reason is simple. In our implementation of the PSOR scheme we used the number of iterations as the criteria for determining if we had reached steady state. It is only at \( t_{nd} = 0.15 \) when all the iterations for the unknowns have reached their minimum value of one (1), when in fact by the time all the iterations where close to five (5), the steady state had already been reached. If we had checked the variations in the unknowns directly, instead of using the number of iterations as a guide, we could have reduced the required cpu time by about 66%.
We intend to use this approach in the future in order to speed the calculation of the steady state solutions.

Figures 6.3 and 6.4 are similar plots for the problem discussed in chapter 4 where two heated blocks are placed inside a cooled cavity. Notice that our previous observations are still valid, indicating we could have saved up to 80% of the cpu time by relaxing the constraint of only stopping the simulation when the PSOR iterations reached their minimum value. In addition to the previously mentioned plots, we have also included the time history of the kinetic energy of the system. This parameter
Figure 6.4: Plot 1 and 2: time histories of the number of iterations required by the PSOR solver at every time step to determine the values of \( ppp \), \( \theta \) for the three-dimensional two-heated-blocks problem. Plot 3: time history of the kinetic energy for the same problem.
takes into account the values of the velocities at all the grid points, not only the maximum value. It is therefore more suitable for detecting whether the system in question has reached steady state, because, instead of keeping track of all primitive variables independently, we can lump at least three (3) of them into one (1) parameter. In fact, this is the value we intend to keep track off in order establish how close to the steady state the solution is.

Figures 6.5 and 6.6 are the same plots corresponding to the irregular cavity problem discussed in chapter 5. The same observations and conclusions previously de-

![Figure 6.5: Left column: maximum u, v and w velocity time histories for the three-dimensional irregular cavity problem when \( Ra = 10^4 \). Right column: time histories of the number of iterations required by the PSOR solver at every time step to determine the values of \( \bar{u}, \bar{v} \) and \( \bar{w} \).]
Figure 6.6: Plot 1 and 2: time histories of the number of iterations required by the PSOR solver at every time step to determine the values of $ppp$, $\theta$ for the three-dimensional irregular cavity problem. Plot 3: time history of the kinetic energy for the same problem.
scribed are valid for this case as well. For this case, we can reduce the cpu time by at least 70% by relaxing the number of PSOR iterations required for program termination.

Finally, we want to mention a peculiar aspect of the PSOR solution method that we observed during the study. There is a wide difference in the number of iterations required for the solution of the cell-centered pressure \( ppp \), depending on the problem. We do not yet completely understand how the steady value of iterations required depends on the other problem parameters, the problem's geometry or its boundary conditions. Even when the values reported are based on solutions using the best possible experimentally calculated over-relaxation parameter, we still observe large required number of iterations. It is then obvious that for all problems, solving for the pressure becomes the most expensive operation as the non-dimensional time progresses. Future work should attack this problem.

6.2 Stability and Convergence Criteria

Due to the nature of the finite difference discretization used, the scheme at hand is conditionally stable, that is, the stability of the scheme is only guaranteed for certain values of the mesh spacing and the time step which, in general, are related by an expression dependent on the value of the unknowns and the problem parameters. For linear problems, one can use methods such as discrete perturbation analysis or the Von Neumann approach in order to obtain the stability constraints. Nevertheless, this is not the case for non linear problems. Being a non linear problem, we had to determine our stability criteria experimentally, with some insight from some similar solutions schemes in the literature.

Hoffman [5] uses the Von Neumann approach to calculate the stability criteria for simple partial differential equations with linear convective and diffusive terms when
one uses second-order central differences in space and first-order forward differences in time. This is similar to our treatment of the momentum equations. He then extends his analysis to multidimensional hyperbolic equations with linear diffusion terms and multidimensional hyperbolic equations with linear conductive terms. By following his derivation we arrive at a set of conditions that must be satisfied to guarantee stability of the scheme.

These conditions can be written as:

\[ c_x + c_y + c_z \leq 1 \quad (6.1) \]

\[ c_x = \frac{\alpha_1 \Delta t}{(\Delta x)^2}, \quad c_y = \frac{\alpha_2 \Delta t}{(\Delta y)^2}, \quad c_z = \frac{\alpha_3 \Delta t}{(\Delta z)^2} \]

\[ d_x + d_y + d_z \leq \frac{1}{2} \quad (6.2) \]

\[ d_x = \frac{\beta_1 \Delta t}{\Delta x}, \quad d_y = \frac{\beta_2 \Delta t}{\Delta y}, \quad d_z = \frac{\beta_3 \Delta t}{\Delta z} \]

where the \( \alpha \)'s are the constant convection coefficients and the \( \beta \)'s are the constant diffusion coefficients.

Given these, at every time step, we replace the \( \alpha_1 \) by the maximum \( u \) velocity, \( \alpha_2 \) by the maximum \( v \) velocity and \( \alpha_3 \) by the maximum \( w \) velocity. On the other hand all the \( \beta \)'s are set equal to the \( Pr \) which in our study is always equal to one (1). With these substitutions, we have at least a set of approximate stability criteria for something similar to the energy and momentum equations at the heart of our study. Furthermore, if we assume that \( \Delta S = \Delta x = \Delta y = \Delta z \) and that \( V = U_{\text{max}} = V_{\text{max}} = W_{\text{max}} \) for simplicity, then we end with the stability criteria:

\[ \Delta t \leq \frac{\Delta S}{3V} \quad (6.3) \]
which we tighten by instead requiring:

\[ \Delta t \leq \frac{\Delta S}{5V} \]  \hspace{1cm} (6.4)

This is, of course, not a rigorous derivation of the stability criteria, but it is adequate for the problems we have worked with.

All the problems we have discussed up to this point are subject to time independent boundary conditions. Due to the nature of both the problem and the boundary conditions, we expect to approach a steady state solution at some time \( t_{steady} \). Recall from the previous section that the number of PSOR iterations is one (1) when the unknown in question has reached the mentioned steady state. It is therefore reasonable to determine whether or not the solution has reach its steady state by examining the number of iterations the PSOR subroutine has to perform to calculate the unknowns at the newest time step. This is exactly what has been implemented in the code. The program is terminated as soon as the number of iterations the PSOR has to perform for all the primitive variables except pressure is less or equal to a threshold value which is usually chosen to be one (1).

### 6.3 Experimental Convergence Rates

In our study, the finite difference approximations used are theoretically first-order in time and second-order in space. To determine the experimental convergence rates for the primitive variables, we have used the inequality:

\[ \| M_{i,j}^{\Delta x, \Delta y} - M_{i,j}^{\Delta x, \Delta y} \|_{2,d} \leq C h_{\Delta x, \Delta y}^2 \]  \hspace{1cm} (6.5)

\[ \| M_{i,j}^{\Delta x, \Delta y} - M_{i,j}^{\Delta x, \Delta y} \|_{2,d} = \left[ \sum_i \sum_j \Delta x \Delta y (M_{i,j}^{\Delta x, \Delta y} - M_{i,j}^{\Delta x, \Delta y})^2 \right]^{\frac{1}{2}} \]  \hspace{1cm} (6.6)
where equation 6.6 describes the form of the discrete $L_2$ norm of the difference between the unknown $M_{i,j}$ in a mesh with $\Delta x, \Delta y$ spacing and the same unknown $M_{i,j}$ in a mesh with $\frac{\Delta x}{2}, \frac{\Delta y}{2}$ spacing. For $\Delta x = \Delta y$ then $h_{\Delta x, \Delta y} = \Delta x$ and $h_{\frac{\Delta x}{2}, \frac{\Delta y}{2}} = \frac{\Delta x}{2}$.

Given this inequality, one can obtain both the experimental convergence rate $s$ and the proportionality constant $C$ if values for the general unknown $M$ are obtained for meshes with $11 \times 11, 21 \times 21$ and $41 \times 41$ grid points. In this case $h_{\Delta x, \Delta y} = 0.1$ and $h_{\frac{\Delta x}{2}, \frac{\Delta y}{2}} = 0.05$. Using results obtained for the bench mark problem described in chapter bench mark chapter, and selecting the results for $Ra = 10^3$, we obtain the convergence rates presented in table 6.1. By using second-order accurate finite difference formulations in space for the discretization of the governing equations, the most we can hope for is quadratic convergence. Due to the nonlinearities of the equations, it is reasonable to expect less than quadratic convergence. This is exactly what we observe. The best convergence rate is obtained for the calculation of the temperature $\theta$, for which the convergence rate is close to 1.8, and the worst convergence rate is obtained for the calculation of the pressure $p$, for which the convergence rate is close to 0.5.

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</tr>
<tr>
<td>$\bar{u}$</td>
<td>0.8919</td>
</tr>
<tr>
<td>$v$</td>
<td>1.1532</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>0.6300</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.7880</td>
</tr>
<tr>
<td>$p$</td>
<td>0.5320</td>
</tr>
</tbody>
</table>

Table 6.1: Experimental convergence rates for the primitive variables $u$, $v$, $\theta$ and $p$, and the intermediate values $\bar{u}$ and $\bar{v}$. These convergence rates were calculated based on the discrete $L_2$ norm.
6.4 Possible Improvements

6.4.1 Stability

A change in the solution scheme could render the system unconditionally stable. For example, the use of an implicit scheme such as the Crank-Nicolson scheme would generate solutions that are theoretically second-order accurate. Nevertheless, the use of this and other implicit schemes requires the use of very efficient solvers and storage algorithms in order to compete with iterative schemes, especially in three-dimensions. Another important factor is that not all schemes are suitable for handling nonlinearities.

Taking all of these factors into account, an appealing scheme would be that one able to generate fast, accurate solutions and still handle nonlinearities efficiently.

We have been working with an explicit MacCormack scheme for the solution of the time dependent equations. This scheme is still conditionally stable but it seems to be more suitable for nonlinear problems, than many of the other schemes available for the solution of hyperbolic partial differential equations. The MacCormack scheme is a multi-step method with a predictor step and a corrector step. The finite difference discretization leads to a set of equations with a 13 point seed, i.e. the MacCormack formulation calculates the value of the general unknown $M_{i,j,k}$ using the values of $M$ two grid points ahead and behind $i, j, k$ in all the three axis $x, y$ and $z$. This requirement creates a problem due to the fact that at the nodes next to the boundary, we do not have enough information to calculate the new value of the unknowns. A change in the formulation is required. We are exploring the possibility of combining both the PSOR scheme with the MacCormack scheme such that the former is used for nodes next to the system boundaries, and the later is used for all remaining nodes in the system. We are debating the effect of using a least accurate scheme for certain
points and a more accurate one for the rest. It is not clear at this point how the error introduced by the PSOR would affect the calculations of the MacCormack solver.

6.4.2 Accuracy

Accuracy seems to be always accompanied by complexity, especially with time dependent problems. To be accurate in both space and time we need to explore second, third and fourth-order finite difference formulations, or multi-step methods. Introducing a second-order scheme such as the MacCormack scheme by itself into the solution algorithm would increase the accuracy, but would also require an increase in code complexity, in order to provide the information required at the nodes next to the boundaries. The intermediate solution is again to combine it with the existing PSOR scheme. Nevertheless, we still do not have a clear idea of how much more accurate the solutions would be in comparison with the ones obtained in this study.

6.4.3 Speed

We can summarize the different possible approaches for increasing the speed of the solution procedure as follows:

1. As discussed before, we can use the value of the kinetic energy of the system as a measure of how close the system variables are to their steady state values, instead of using the number of PSOR iterations for the same purpose. We have shown that the number of PSOR iterations can be misleading if there is no reference value other than the minimum possible value of one (1).

2. In order to reduce the number of PSOR iterations needed to calculate the pressure distribution, which turns out to be the most expensive routine of the procedure, we suggest reducing the internal PSOR tolerance $\epsilon$. In general, we use
a tolerance of 0.01. Increasing this tolerance for the pressure will reduce the number of PSOR iterations. We suggest a new tolerance of about 1.0, which is two orders of magnitude larger.

3. Notice that if the PSOR tolerance remains constant for any mesh size, the scheme becomes much more restrictive for finer meshes than for coarser meshes, because we require that for a fine mesh the sum of much more nodes add to the same value as the sum of less nodes in a coarser mesh. If we want to constraint all meshes in the same way we suggest selecting a required tolerance $\epsilon_0$ and then calculating the PSOR tolerance $\epsilon$ by multiplying $\epsilon_0$ by the number of nodes in the mesh in question.

4. If we are only looking for the steady state solution and are not very concerned with the time history of the problem we can accelerate the solution by using the results of the same problem with a lower Rayleigh number as the initial conditions for a higher Rayleigh number. For example, we can use the solution for $Ra = 10^3$ to get the solutions at $Ra = 10^4$ for the same mesh size. In addition we can use data for a Rayleigh number in a coarse mesh, map it to a finer mesh using a bi-linear interpolation function, and the use this new interpolated fine mesh as an initial condition for the calculation of results at a higher Rayleigh number. For example, a solution for $Ra = 10^3$ is obtained for a $11 \times 11$ mesh. We can map this results to a $21 \times 21$ mesh, nad use these new values as initial conditions to generate results for $Ra = 10^4$ in a $21 \times 21$ mesh.

6.5 Future Work

There are many exiting problems in the area of fluid dynamics and heat transfer which have not yet been solved. However, new techniques and approaches bring their solution closer to us every day. In the near future we would like to and plan to work
with certain topics, which can be best described if we enumerate them. Included are the following:

1. In this study, we have dealt with problems in which the boundary conditions are time invariant. We would like to explore problems where the boundary conditions vary with time. One such problem is to determine the time history and main characteristics of the flow of air inside a three-dimensional structure which convects and conducts heat while it is itself heated by radiation from the sun as it rises and sets on a typical day.

2. In our study we have only dealt with problems where the velocity at the boundaries is always zero. We would like to explore problems where the fluid velocity is specified at the boundaries. These kinds of problems can range in difficulty and complexity from the simulation of a three-dimensional sudden expansion problem, to the determination of the optimal location of air conditioning ducts in a structure in order to minimize power consumption.

3. The scheme described in this study can be extended to include not only the governing equations of chapter 2, but diffusion equations, radiative heat transfer, state equations and electromagnetic equations, in order to solve problems in areas such as magnetohydrodynamics and the like.

4. When the differential heating described in chapter 3 is applied not on the side walls of the cavity but on the top and bottom walls, the problem becomes a problem of Rayleigh-Bénard convection, which has been identified as a very non-linear and inherently unstable physical situation. In this particular problem stable and chaotic time dependent flow behaviors can be observed for ordinary Rayleigh and Prandtl numbers and boundary conditions. Our code is well suited
for the study of this specific type of problem. Preliminary results agree with those reported in the literature [12].

To end, we have been investigating the possibility of using feedforward and recurrent artificial neural networks for the solution of the Navier-Stokes equations and partial differential equations in general. We have had success in developing feedforward neural networks that generate solutions for the problems we have discussed in previous chapters at particular time steps. We are currently working on the development of a recurrent artificial neural network capable of solving the time dependent governing equations for all time. Preliminary work has generated very promising results.
Bibliography


