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Hypervelocity impact on strain-rate sensitive shielded plates

Smith, James Pope, M.S.
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Hypervelocity Impact on Strain-Rate Sensitive Shielded Plates

by

James P. Smith

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science

Approved, Thesis Committee:

Y.C. Angel, Director
Associate Professor
Mechanical Engineering and Materials Science

Enrique U. Barrera
Assistant Professor
Mechanical Engineering and Materials Science

G. M. Pharr
Professor
Mechanical Engineering and Materials Science

Houston, Texas
November, 1992
Hypervelocity Impact on Strain-Rate Sensitive Shielded Plates

James P. Smith

Abstract
A ballistic limit equation for hypervelocity impact on thin plates is derived analytically. This equation applies to cases of impulsive impact on a plate that is protected by a multi-shock shield, and is valid in the range of velocity above 6 km/s. Experimental tests were conducted at the NASA Johnson Space Center on square aluminum plates. Comparing the center deflections of these plates with the theoretical deflections of a rigid-plastic plate subjected to a blast load, one determines the dynamic yield strength of the plate material. The analysis is based on a theory for the expansion of the fragmented projectile and on a simple failure criterion. Curves are presented for the critical projectile radius versus the projectile velocity, and for the critical plate thickness versus the velocity. These curves are in good agreement with curves that have been generated empirically.
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Chapter 1

Introduction

Debris and meteoroids of various sizes may impact space vehicles at relative velocities as high as 70 km/s, and the impacts may have catastrophic effects on the integrity of the vehicles. Thus, it is important that proper shielding systems be designed to protect space structures. The basic idea underlying the design of all shielding systems is to intercept the potentially damaging objects and to shatter them into a cloud of solid, molten, or vaporized fragments. This can be achieved by using a dual-plate shield (Kinslow [1], Zukas et al. [2], and Rajendran and Elfer [3]), or one can use a multi-shock shield (Cour-Palais and Crews [4]). In either case, the cloud of fragments impacts the structure over an area much larger than that of the cross-section of the unshattered object.

The evaluation of the damage caused to a plate by a cloud of fragments has been the object of many experimental tests. With these experimental results, empirical ballistic limit equations have been derived, with little analytical work to support the conclusions. If it were possible to theoretically predict the response of a plate to
hypervelocity impact, less time would be spent on experiment, and more time could be spent on the design of new shielding concepts.

In this thesis, we derive a ballistic limit equation for hypervelocity impact on thin plates. This equation applies to cases of impulsive impact on a plate that is protected by a multi-shock shield; it is valid when the projectile velocity is in the range above 6 km/s, and when the area of the impact zone is much smaller than the area of the plate. The analysis that leads to the ballistic limit equation (equation (5.9) below) is discussed in detail in the following sections. For convenience, we reproduce here equation (5.9) in the form

$$\frac{\rho_p U_m^2 r^3}{\sigma_0 D h^2} = C, \quad C = \frac{9\lambda\mu(1 + K) \sqrt{K}}{2(1 - K)}, \quad (1.1)$$

where the parameters $K$, $\lambda$, $\mu$, and $C$ take the values

$$K = 0.04, \quad \lambda = 4.5, \quad \mu = 50, \quad C = 219.4. \quad (1.2)$$

The parameters $\rho_p$, $r$, and $U_m$ denote the mass density, the radius, and the velocity of the projectile; $h$ and $\sigma_0$ are the half-thickness and the static yield strength of the plate; $D$ is the spacing between the outer sheet of the multi-shock shield and the plate; the dimensionless $K$ measures the expansion of the fragmented projectile, $\lambda$ is the ratio of the dynamic yield strength to the static yield strength of the plate material, and $\mu$ is the dimensionless load applied to the plate. In equation (1.1),
the two groups \( \rho_p U_m^2 / \sigma_0 \) and \( r^3 / (Dh^2) \) are dimensionless; the parameter \( C \) is also dimensionless.

Equation (1.1) represents a relation between the critical values of the parameters. If we let any two of the parameters \( \rho_p, U_m, \sigma_0, r, D, \) and \( h \) vary, while keeping all the other parameters fixed, we obtain from (1.1) a plane curve. The curve divides the plane in two regions: one of them is the safe region, and the other is the failure region.

It will be seen below that equation (1.1), together with (1.2), yields numerical results that are in good agreement with those obtained from the empirical equation of Christiansen [7]. Thus, equation (1.1) can be used to guide further experiments and shield design studies.

In a previous work, Angel and Whitney [5] were able to obtain a ballistic limit equation by using a simple analytical solution for the permanent deflection of a rigid-plastic beam. In this work, we replace the beam solution by a plate solution. This yields equation (1.1), which is simpler than the previous ballistic limit equation.

To arrive at equation (1.1), we discuss first, in the next section, the plate solution of Hopkins and Prager [8]. This solution corresponds to a simply supported circular plate uniformly loaded over its entire area by a blast load. The plate is made of a rigid perfectly-plastic material. We give expressions for the center deflection of the
plate. Next, we compare the theoretical deflection with the experimental deflections of six aluminum square plates supplied by the Hypervelocity Impact Test Facility of NASA/JSC. In this process, we invoke the debris-expansion theory of Swift et al. [9], and we determine the dynamic yield strength of the plate material. Then, we choose a failure criterion based on the maximum permissible deflection, and we establish that the critical values are related by equation (1.1). We present curves for the critical projectile radius \( r \) versus the projectile velocity \( U_m \), and for the critical plate half-thickness \( h \) versus \( U_m \). Finally, we comment on the value of the dynamic yield strength of the aluminum plates. Appendix A is devoted to a detailed derivation of the permanent deflection of a rigid perfectly-plastic circular plate subjected to a time-dependent rectangular load.
Chapter 2

Permanent Deflection of a Circular Plate

Consider a simply supported circular plate of radius \( a \) and thickness \( 2h \) subjected to a uniformly distributed circular load of radius \( R \), as shown in Figure 2.1. The plate is made of a rigid perfectly-plastic material, and the load \( P(t) \) has a rectangular time-dependence as in Figure 2.2. Let \( P_m \) be the maximum value of the load, and let \( T \) be the time at which the load returns instantaneously to zero.

Assuming the deformations are small, the axisymmetric equation of dynamic equilibrium for the plate, which is derived in Appendix A (equation (A.6)), can be written in cylindrical coordinates in the form

\[
[r M_r(r, t)]' - M_\theta(r, t) = - \int_0^r [p(\alpha, t) - 2 \rho h \ddot{w}(\alpha, t)] \alpha d\alpha,
\]

where \( w \) is the deflection of the plate in the \( z \)-direction of Figure 2.1; \( M_r \) and \( M_\theta \) are bending moments per unit length caused by the radial and circumferential stresses, respectively; \( p \) is a force per unit area along the \( z \)-direction; and \( \rho \) is the mass density of the plate. In (2.1), the prime superscript denotes differentiation with respect to the
radial coordinate \( r \), and the superimposed dots denote differentiations with respect to the time \( t \).

Using the Tresca yield criterion that is discussed in Appendix A, one finds that the moments \( M_r \) and \( M_\theta \) take values in the \((M_r, M_\theta)\) plane inside, or on the boundary of, a hexagon containing the origin. The size of the hexagon is determined by the plastic collapse moment \( M_0 \) per unit length, which can be expressed in terms of the static yield strength \( \sigma_0 \) of the plate as

\[
M_0 = \sigma_0 h^2. \tag{2.2}
\]

When the circular load of Figure 2.1 is applied over the entire area of the plate \((R = a)\), the permanent deflection \( W_e \) at the center of the plate has been calculated by Hopkins and Prager [8] (Eqn (22) p. 324 and Eqn (44) p. 329). The results of these authors, as shown in Appendix A, can be written in the form

\[
\delta_e = \begin{cases} 
0, & 0 \leq \mu \leq 1, \\
\mu(\mu - 1), & 1 \leq \mu \leq 2, \\
\frac{\mu}{4}(3\mu - 2), & 2 \leq \mu,
\end{cases} \tag{2.3}
\]

where the dimensionless deflection \( \delta_e \) and the dimensionless load \( \mu \) are defined by

\[
\delta_e = \frac{\rho a^2}{3\sigma_0 h T^2} W_e, \quad \text{and} \quad \mu = \frac{P_m}{6\pi \sigma_0 h^2}. \tag{2.4}
\]

For large values of \( \mu \), the equation of \( \delta_e \) in terms of \( \mu \) is the equation of a parabola, as can be seen from (2.3). Next, we recall that the static deflection \( W_e \) at the center
Figure 2.1: Plate subjected to a uniformly distributed circular load.

of the plate of Figure 2.1, when the total distributed load is equal to $P_m$, is given by

(Timoshenko and Woinowsky-Krieger [11], pp. 64-67)

$$W_s = \frac{P_m}{16\pi F} \left[ \frac{3 + \nu}{1 + \nu} \frac{a^2}{R^2} - R^2 \ln \frac{a}{R} - \frac{7 + 3\nu}{4(1 + \nu)} R^2 \right], \quad (2.5)$$

where $\nu$ is Poisson's ratio and $F$ is the flexural rigidity of the plate. The flexural rigidity can be expressed in terms of Young's modulus $E$ as: $F = 2Ef^2/[3(1 - \nu^2)]$.

Equation (2.5) can be used to evaluate the static deflection of a plate loaded over its entire area ($R = a$). The ratio $W$ of the deflection (2.5) to the deflection that corresponds to $R = a$ is given by

$$W = \frac{1}{5 + \nu} \left[ 4(3 + \nu) \frac{a^2}{R^2} - 4(1 + \nu) \ln \left( \frac{a}{R} \right) - (7 + 3\nu) \right] \frac{R^2}{a^2}. \quad (2.6)$$
Next, let $W$ be the permanent deflection at the center of the plate of Figure 2.1 when the load $P(t)$ has the time-dependence of Figure 2.2. In order to evaluate $W$, we assume that the ratio of $W$ to the deflection $W_e$ corresponding to (2.3) - (2.4) is equal to the ratio $\bar{W}$ of (2.6). Thus, one has

$$W = \bar{W}W_e.$$  

(2.7)

We introduce now a dimensionless measure $\delta$ of the permanent deflection $W$, and a parameter $c$ that is equal to the ratio of the plate radius to the load radius. These parameters are defined by

$$\delta = \frac{\rho R^2}{3\sigma_0 h T^2}W, \quad \text{and} \quad c = \frac{a}{R}.$$  

(2.8)
Then, we infer from equations (2.3)\textsubscript{3}, (2.4), (2.6), (2.7), and (2.8) that the expression of $\delta$ in terms of the parameter $\mu$ of (2.4) is

$$\delta = \frac{\mu}{4}(3\mu - 2)B(c, \nu),$$  \hspace{1cm} (2.9)

where

$$B(c, \nu) = \frac{4(3 + \nu)c^2 - 4(1 + \nu)\ln c - (7 + 3\nu)}{(5 + \nu)c^4},$$  \hspace{1cm} (2.10)

and $\mu \geq 2$, as indicated in (2.3)\textsubscript{3}. Observe that the dimensionless $\delta$ in (2.8) is defined in terms of the load radius $R$, whereas the dimensionless $\delta_e$ of (2.4) is defined in terms of the plate radius $a$. 
Chapter 3

Experimental Results

Six multi-shock shields and aluminum target plates were supplied by the Hypervelocity Impact Research Laboratory of NASA/JSC. The experimental tests are labeled A1229, A1230, A1233, A1235, A1237, and A1253. For each of the six experiments, the target consists of a 15.24 cm (6 in.) square aluminum backwall protected by a series of four evenly-spaced Nextel sheets, also 15.24 cm square. Nextel is a lightweight ceramic fiber that is woven into a cloth fabric. The distance from the outer sheet to the target plate is called $D$, as illustrated in Figure 3.1.

In the experiments, aluminum spheres with a diameter of 3.175 mm (1/8 in.) and a mass density $\rho_p = 2796$ kg/m$^3$ are fired from a light-gas gun at velocities near 6.5 km/s. The projectile impacts the outer Nextel sheet normally, leaving a hole slightly larger than the projectile diameter, and shatters into fragments. Because the fragments move inside a cloud of expanding radius, the following sheets have increasingly larger holes. Finally, the expanding cloud impacts the backwall, causing a permanent plastic deformation.
Figure 3.1: Impact of a projectile on a multi-shock shield.

Several parameters were varied in the experiments in order to see how each parameter affects the backwall deflection. The parameters (projectile velocity and mass, backwall thickness, shield spacing, permanent deflection, and load radius) of the six experiments are shown in Table 3.1. Four experiments were conducted with a shield spacing of 10.16 cm, and two experiments have a reduced shield spacing. The yield strength of the backwall material is equal to 344.74 MPa (Al 2024-T3) and the mass density is $\rho = 2768$ kg/m$^3$ for all experiments, except for A1230, where the yield strength is $\sigma_0 = 275.79$ MPa (Al 6061-T6) and the mass density is $\rho = 2713$ kg/m$^3$. Also, the plate thickness takes four different values.
<table>
<thead>
<tr>
<th>Shot</th>
<th>$U_m$(km/s)</th>
<th>$m_p$(mg)</th>
<th>$2h$(mm)</th>
<th>$D$(cm)</th>
<th>$W$(mm)</th>
<th>$R$(cm)</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1229</td>
<td>6.49</td>
<td>47.04</td>
<td>0.508</td>
<td>10.16</td>
<td>4.018</td>
<td>2.08</td>
<td>0.0402</td>
</tr>
<tr>
<td>A1230</td>
<td>6.32</td>
<td>46.92</td>
<td>0.635</td>
<td>10.16</td>
<td>2.515</td>
<td>2.39</td>
<td>0.0522</td>
</tr>
<tr>
<td>A1233</td>
<td>6.26</td>
<td>46.84</td>
<td>0.813</td>
<td>10.16</td>
<td>1.102</td>
<td>2.12</td>
<td>0.0417</td>
</tr>
<tr>
<td>A1235</td>
<td>6.24</td>
<td>46.86</td>
<td>0.635</td>
<td>10.16</td>
<td>1.834</td>
<td>2.19</td>
<td>0.0442</td>
</tr>
<tr>
<td>A1237</td>
<td>6.20</td>
<td>46.84</td>
<td>0.813</td>
<td>7.620</td>
<td>2.118</td>
<td>2.16</td>
<td>0.0744</td>
</tr>
<tr>
<td>A1253</td>
<td>6.51</td>
<td>46.90</td>
<td>1.000</td>
<td>5.080</td>
<td>1.283</td>
<td>1.92</td>
<td>0.1250</td>
</tr>
</tbody>
</table>

Table 3.1: Experimental data.

The permanent deflections of the backwalls, which are recorded in Table 3.1 as $W$, were measured using a precision lathe to a precision of three hundredths of a millimeter. The deflections $W$ represent the centerline deflections of the backwalls.

The backwall in each experiment is tarnished and pitted by the impact. The tarnished area is nearly identical to the area of the backwall that is permanently deformed. Measurements of the load diameter were taken by carefully tracing the region of impact onto guideline tracing paper, where actual diameters could be more accurately determined and measured. Eight different diameters were measured and averaged for each of the six experiments to obtain the experimental values of the radius $R$ listed in Table 3.1.
Chapter 4

Debris Cloud Dynamics

In the range of projectile velocities above 6 km/s, which contains the six velocities of Table 3.1, the impact pressure applied by the Nextel sheets on the projectile is sufficiently high to melt the projectile completely before it impacts the backwall (Christiansen [6,7]). Thus, the backwall is subjected to the impulse of a cloud of molten fragments. In the following, we consider only the case of impulsive loading caused by molten fragments.

The impulse on the backwall depends on two opposite effects. First, the rebounding effect discussed by Gehring [12] tends to increase the impulse, and second the impacts on the successive sheets tend to reduce it. Thus, we write as an approximation that the total impulse $I_{bw}$ on the backwall is equal to the momentum of the incoming projectile. It follows that

\[
I_{bw} = m_p U_m = \frac{4}{3} \pi \rho_p r^3 U_m, \tag{4.1}
\]

where $m_p$, $U_m$, $\rho_p$, and $r$ denote, respectively, the mass, the velocity, the mass density, and the radius of the projectile.
Next, we assume that the impulse $I_{bw}$ is equivalent to that of a rectangular load $P(t)$ as in Figure 2.2. The load is applied during a time interval $T$; it jumps instantaneously from zero to $P_m$ at $t = 0$, remains constant with magnitude $P_m$, and jumps back to zero at $t = T$. It follows that the equivalent maximum load $P_m$ is given by

$$P_m = \frac{I_{bw}}{T}. \quad (4.2)$$

We now recall that Swift et al. [9] have proposed a theory for the debris expansion. This theory can be applied to our problem, provided that the four Nextel sheets are replaced by an equivalent shield. The equivalent shield is made of a single plate; it is located at the place of the outer Nextel sheet, and its material properties are such that the conical debris expansion is identical to that produced by the four sheets.

We now use the theory of Swift et al. [9] to calculate the load duration $T_L$ and the load radius $R$ on the backwall. To obtain these results, we consider first a particle of mass $m_p$ and velocity $U_m$ that strikes the equivalent shield. Let $m_b$ be the mass that is removed from the bumper during the perforation of the shield by the projectile. Then conservation of momentum gives

$$m_p U_m = (m_p + m_b) U_c, \quad (4.3)$$

where $U_c$ is the velocity of the combined mass at the mass center. Letting $K$ be the ratio of the mass per unit area of the bumper to the mass per unit area of the incoming
particle and $G$ the ratio of the bumper hole diameter to the particle diameter, one finds from (4.3) that the velocity $U_e$ is given by

$$U_e = \frac{U_m}{1 + KG^2},$$

(4.4)

where

$$K = \frac{3 \rho_p h_b}{4 \rho_p r}, \quad G = \frac{r_b}{r}.$$  

(4.5)

In (4.5), $\rho_b$ is the mass density of the single equivalent shield, $h_b$ is the shield thickness, and $r_b$ is the radius of the hole in the equivalent shield. Next, using conservation of energy, one finds that the extra energy $\Delta E$ that can be dissipated as heat or used to expand the gas cloud is

$$\Delta E = m_p U_m^2/2 - (m_p + m_b) U_e^2/2.$$  

(4.6)

Then, using equations (4.4), (4.5), and (4.6), we have

$$\Delta E = \frac{1}{2} m_p U_m^2 \left( \frac{KG^2}{1 + KG^2} \right).$$  

(4.7)

If $U_e$ is the velocity of expansion of the molten fragments away from the mass center, then the kinetic energy of expansion $E_e$ is given by

$$E_e = (m_p + m_b) U_e^2/2.$$  

(4.8)
Then we write that \( E_c = Q \Delta E \), where \( Q \) is close to unity, and we use (4.7) and (4.8) to obtain the velocity \( U_e \) in the form

\[
U_e = \frac{U_m G \sqrt{QK}}{1 + KG^2}. \tag{4.9}
\]

Figure 4.1 shows the cloud as it impacts the backwall. The radius \( r_d \) at this instant and the velocity \( U_{\text{max}} \) at the cloud front are given by

\[
r_d = \frac{U_e D}{U_{\text{max}}}, \quad U_{\text{max}} = U_e + U_c = \frac{U_m(1 + G \sqrt{QK})}{1 + KG^2}. \tag{4.10}
\]

From the geometry of Figure 4.1, the load radius \( R \), which corresponds to the intersection of the expansion cone with the backwall, is given by

\[
R = \frac{Dr_d}{(D^2 - 2Dr_d)^{1/2}} = \frac{DG \sqrt{QK}}{(1 - G^2 QK)^{1/2}}, \tag{4.11}
\]
where (4.9) and (4.10) have been used. The velocity \( U_{\text{min}} \) at the back of the cloud is equal to \( U_{\text{min}} = U_c - U_e \). Thus, the load duration \( T_L \) on the backwall is given by

\[
T_L = \frac{D}{U_{\text{min}}} - \frac{D}{U_{\text{max}}} = \frac{2DG\sqrt{QK}(1 + KG^2)}{U_m(1 + G\sqrt{QK})(1 - G\sqrt{QK})}.
\]  

(4.12)

For the special case where the hole radius in the shield is equal to the projectile radius \( (G = 1) \), and where the kinetic energy expended in the fragments expansion is equal to the total available kinetic energy \( (Q = 1) \), equations (4.11) and (4.12) yield

\[
R = D \left( \frac{K}{1 - K} \right)^{1/2}, \quad T_L = \frac{2D(1 + K)\sqrt{K}}{U_m(1 - K)}.
\]

(4.13)

Equation (4.13) shows that the values of \( K \) must satisfy the condition \( 0 < K < 1 \).

By using (4.13), together with the \( R \) and \( D \) values of Table 3.1, we have calculated the corresponding value of \( K \) for each of the six experiments. These values are recorded in the last column of Table 3.1. We observe here that the value of \( K \) increases when the shield spacing \( D \) decreases, as indicated by Shots A1229, A1237, and A1253. We now select from the list of values of \( K \) a lower bound \( K^* \) such that

\[
K^* = 0.04.
\]

(4.14)

The time \( T_L \) of (4.13) is the time necessary for the particle at the back of the spherical debris cloud to travel the length of the sphere diameter, starting at the instant when the first particle touches the backwall. The time \( T \) of (4.2) is the
duration of the rectangular load in the theoretical approach of Chapter 2. In order to determine a relation between $T_L$ and $T$, we write that the impulse of an isosceles triangular load of duration $T_L$ and height $P_m$ is equal to the impulse of a rectangular load of duration $T$ and height $P_m$. Then, one finds that

$$T = \frac{1}{2} T_L.$$  \hspace{1cm} (4.15)

The argument that leads to (4.15) is consistent with the numerical results of Perzyna [13]; these results show that impact loads of equal impulses on rigid perfectly-plastic plates cause approximately the same permanent deflections.
Chapter 5

Ballistic Limit Equation

Let \( \sigma \) be the dynamic yield strength of the backwall material. Then, using \( \sigma \) instead of the static yield strength \( \sigma_0 \), we can rewrite the dimensionless measure \( \delta \) of the permanent deflection and the dimensionless measure \( \mu \) of the maximum load in the form (see (2.8) and (2.4))

\[
\delta = \frac{\rho R^2}{3\lambda \sigma_0 h T^2} W, \quad \mu = \frac{P_m}{6\pi \lambda \sigma_0 h^2}, \tag{5.1}
\]

where \( \sigma = \lambda \sigma_0 \), and \( \lambda \) is a multiplicative factor. The parameters \( \delta \) and \( \mu \) can be expressed in terms of \( \rho_p, r, U_m, D, K, \rho, h, W, \sigma_0, \) and \( \lambda \). To see this, it suffices to substitute (4.1), (4.2), (4.13), and (4.15) into equations (5.1). The result is

\[
\delta = \frac{1 - K}{3\lambda(1 + K)^2} \frac{\rho U_m^2}{\sigma_0} \frac{W}{h}, \quad \mu = \frac{2(1 - K)}{9\lambda(1 + K)\sqrt{K}} \frac{\rho_p U_m^2}{\sigma_0} \frac{r^3}{Dh^2}. \tag{5.2}
\]

We have determined the factor \( \lambda \) by substituting the experimental values of Table 3.1 into (5.2), and by plotting the corresponding points in a \((\mu, \delta)\) system of axes. For each experiment in Table 3.1, and for varying values of \( \lambda \), the points \((\mu, \delta)\) describe a straight line through the origin. If, on the other hand, a fixed value of \( \lambda \) is chosen, then the six points corresponding to the six experiments of Table 3.1
are located on a parabola-like curve. Three such curves (for \( \lambda = 1.7, 4.5, \) and 7.0) are shown in dotted lines in Figure 5.1. Also shown in Figure 5.1 is a solid line. The solid line is obtained from equation (2.9) for \( \mu \geq 2, \) with a value of Poisson’s ratio \( \nu = 0.3 \) and a choice of the parameter \( c \) of (2.8) such that

\[ c = c^* = 3.18. \tag{5.3} \]

The value \( c^* \) of (5.3) is a lower bound for the six experiments of Table 3.1; it is obtained by taking the values \( a = 7.62 \text{ cm} \) and \( R = 2.39 \text{ cm} \) of Shot A1230. For the other five experiments, the values of \( c \) are greater than \( c^* \). Now, returning to equation (2.9), one can see that \( B(c, \nu) \) takes the value 1.0 at \( c = 1.0 \) and decreases monotonically to zero as \( c \) approaches infinity (for fixed values of \( \nu \)). Thus, the deflection \( \delta \) of (2.9), for fixed values of \( \nu \) and \( \mu \), decreases as \( c \) increases. It follows that the lower bound \( c^* \) yields a conservative upper bound for the deflection \( \delta \).

Figure 5.1 shows that the solid line, which represents the theoretical permanent deflection at the center of the plate, is very close to the dotted line representing the experimental deflection when \( \lambda = \lambda^* = 4.5 \). It is also shown in Figure 5.1 that the experimental values for \( \lambda = 1.7 \) and \( \lambda = 7.0 \) are not near the theoretical curve.

Consequently, in all our subsequent calculations, we choose

\[ \sigma = 4.5\sigma_0 = \lambda^*\sigma_0. \tag{5.4} \]
Figure 5.1: Theoretical (solid line) and experimental backwall deflections for dynamic yield strengths of $1.7\sigma_0(\circ)$, $4.5\sigma_0(\bullet)$, and $7\sigma_0(\circ)$.

The ballistic limit equation can now be obtained from equation (5.2). We begin by rewriting $\delta$ of (5.2) in the form

$$\delta = \frac{1 - K}{3\lambda(1 + K)^2} \frac{\rho U_m^2}{\sigma_0} \Omega,$$  \hspace{1cm} (5.5)

where $\Omega = W/h$ is a measure of the deformation of the backwall. This measure is independent of the width $2a$ of the backwall. The values of $\Omega$ for the six experiments of Table 3.1 are: $\Omega = 15.82, 7.921, 2.711, 5.776, 5.210$, and $1.604$. Based on these results, we choose an upper limit

$$\Omega^* = 15.82.$$  \hspace{1cm} (5.6)
Equation (5.6) can be interpreted as a failure criterion. For values of $\Omega$ less than $\Omega^*$, the backwall has sustained the impact; for values greater than $\Omega^*$, the backwall has failed.

Substituting $\Omega^*$ into (5.5), using (5.4) and the values of Table 3.1, we find that the values of $\delta$ for the six experiments are: $\delta = 351.55, 394.18, 325.62, 321.16, 290.02,$ and $275.69$. We now select an upper bound $\delta^*$ for $\delta$ such that

$$\delta^* = 395.$$  \hfill (5.7)

Then, we use equation (2.9), together with $c = c^*$ as in (5.3), $\nu = 0.3$, and $\delta = \delta^*$ as in (5.7), to deduce an upper bound $\mu^*$ for $\mu$. Assuming that $\mu^*$ is much greater than 2, one infers from (2.9), that

$$\mu^* = \left[ \frac{4 \delta^*}{3 B(c^*, 0.3)} \right]^{1/2} = 50.$$  \hfill (5.8)

The ballistic limit equation follows now from (5.2), (5.4), (5.8) and (4.5). One has

$$\frac{\rho p U_m^2}{\sigma_0} \frac{r^3}{D h^2} = C, \quad C = \frac{9 \lambda \mu (1 + K) \sqrt{K}}{2(1 - K)},$$  \hfill (5.9)

where the parameters $K, \lambda, \mu$, and $C$ take the values

$$K = 0.04, \quad \lambda = 4.5, \quad \mu = 50, \quad C = 219.4.$$  \hfill (5.10)

Equation (5.9) represents a relation between the critical values of the parameters. If we let any two of the parameters $\rho_p, U_m, \sigma_0, r, D$, and $h$ vary, while keeping all the
other parameters fixed, we obtain from (5.9) a plane curve. The curve divides the plane in two regions: one of them is the safe region, and the other one is the failure region.
Chapter 6

Ballistic Limit Curves

We now compare the ballistic limit curves corresponding to (5.9) to those developed by Christiansen [7]. The ballistic limit equation of [7] can be written in the form

\[ r = 0.177 \left( \frac{2h \rho D^2}{\rho_p U_m} \right)^{1/3} \left( \frac{\sigma_0}{275.79} \right)^{1/6} , \]  

(6.1)

where \( r \), \( D \) and \( h \) are in centimeters, \( U_m \) is in km/sec, and \( \sigma_0 \) is in MPa. We first select the parameters \( r \) and \( U_m \) as variables. Using the experimental values for Shots A1229, A1237, and A1253, we obtain the ballistic limit curves shown in Figures 6.1 to 6.3. The dotted curves correspond to (6.1), while the solid curves correspond to (5.9). The actual experimental point is also shown in each of the three figures. The region below each curve is the safe region; the region above each curve is the failure region.

In Figure 6.1, the ballistic limit curves are plotted for a backwall thickness \( 2h = 0.508 \text{ mm (0.02 in.)} \) and a shield spacing \( D = 10.16 \text{ cm} \). Notice that the two curves are very close and the solid line is more conservative than the dotted one. In Figure 6.2, the backwall thickness is \( 0.813 \text{ mm (0.032 in.)} \) and the shield spacing
is 7.62 cm. The curves cross near $U_m = 12$ km/s. Thus, (6.1) is less conservative than (5.9) at the higher velocities, and it is more conservative than (5.9) at the lower velocities. In Figure 6.3, the solid line is above the dotted one and predicts that Shot A1253 is not near failure, which is consistent with the experimental observation. For this case, the backwall thickness is 1.6 mm (0.063 in.) and the shield spacing is 5.08 cm.

Finally, we plot in Figure 6.4 a ballistic limit curve for the backwall half-thickness $h$ versus the projectile velocity $U_m$. The projectile radius is 1.588 mm (1/16 in.) and the shield spacing is 10.16 cm. The regions above the lines are the safe regions. The four experimental data points corresponding to the first four experiments in Table 3.1 are shown in Figure 6.4. All four points lie above both curves, indicating that the plate has not failed. The solid line predicts that A1229 is very near failure, which is consistent with the experimental observation. Figures 6.1 to 6.4 show that the solid lines corresponding to (5.9) are consistent with the six experiments of Table 3.1.
Figure 6.1: Critical projectile radius for backwall thickness $2h = 0.508$ mm and shield spacing $D = 10.16$ cm (Christiansen: dotted line).

Figure 6.2: Critical projectile radius for backwall thickness $2h = 0.813$ mm and shield spacing $D = 7.62$ cm (Christiansen: dotted line).
Figure 6.3: Critical projectile radius for backwall thickness $2h = 1.6$ mm and shield spacing $D = 5.08$ cm (Christiansen: dotted line).

Figure 6.4: Critical backwall half-thickness for projectile radius $r = 1.588$ mm and shield spacing $D = 10.16$ cm: $\diamondsuit$A1230, $+$A1233, $\times$A1235 (Christiansen: dotted line).
Chapter 7

Dynamic Yield Strength

It is shown in Chapter 5 that if the multiplier $\lambda$ takes the value $\lambda^* = 4.5$ (or the dynamic yield strength of the backwall is four and a half times the static yield strength), then the theoretical and experimental deflections of the backwall are very close. This multiplier allows us to take into account the high strain-rates that develop in the backwall.

In this chapter, we discuss another method of estimating $\lambda$. Uniaxial tension tests performed by Cowper and Symonds (Jones [10]) show that the dynamic yield multiplier $\lambda$ has the form

$$\lambda = 1 + \left( \frac{\dot{\varepsilon}}{S} \right)^{1/q}, \quad (7.1)$$

where $\dot{\varepsilon}$ is the strain-rate, and $q$ and $S$ are experimentally determined constants. Values of $S$ and $q$ for various metals are listed in Table 7.1. Notice that the strain-rate of aluminum must be more than 100 times that of steel to give the same $\lambda$ from (7.1). Also, according to the values given in Table 7.1, a strain-rate of nearly $10^6 \text{ s}^{-1}$ is necessary to achieve a yield multiplier of $\lambda = 4.5$ in aluminum.
<table>
<thead>
<tr>
<th>Material</th>
<th>$S(s^{-1})$</th>
<th>$\dot{q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mild Steel</td>
<td>40.4</td>
<td>5</td>
</tr>
<tr>
<td>Al Alloy</td>
<td>6500</td>
<td>4</td>
</tr>
<tr>
<td>Ti 50A</td>
<td>120</td>
<td>9</td>
</tr>
<tr>
<td>304 Steel</td>
<td>100</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 7.1: Strain-rate properties of four metals.

\[ \sqrt{W^2 + R^2} \]

\[ 2R \]

\[ \frac{W}{\sqrt{W^2 + R^2}} \]

Figure 7.1: Deformed shape of the backwall.

We now evaluate the strain rate applied by the hypervelocity impact to the backwall. First, consider the middle plane of the backwall and assume that the deformation is confined to the area inside the load radius. Furthermore, let the deformed shape be approximated by the conical shape of Figure 7.1. Using the deformed and undeformed lengths of the backwall radius, one finds that the radial strain $\epsilon$ and the strain rate $\dot{\epsilon}$ are given by

\[ \epsilon = \left(1 + \frac{W^2}{R^2}\right)^{1/2} - 1, \quad \dot{\epsilon} = \frac{W}{R^2} \left(1 + \frac{W^2}{R^2}\right)^{-1/2} \dot{W}, \tag{7.2} \]
where $\dot{W}$ is the backwall velocity at the center. Next, we write that the impulse $I_{bw}$ of (4.1) is absorbed entirely by the circular portion of radius $R$ of the backwall. Assuming that the velocity imparted to the plate varies linearly from $\dot{W}$ at $r = 0$ to zero at $r = R$, one finds that

$$I_{bw} = \int_{-h}^{h} \int_{0}^{2\pi} \int_{0}^{R} \rho \dot{W} \left(1 - \frac{r}{R}\right) r dr d\theta dz = \frac{2}{3} \pi \rho R^2 \dot{W}. \quad (7.3)$$

Thus, it follows from (7.3) and (4.1) that the velocity $\dot{W}$ at the center of the backwall is

$$\dot{W} = \frac{3I_{bw}}{2\pi \rho h R^2} = \frac{3m_p U_m}{2\pi \rho h R^2}. \quad (7.4)$$

Since the term $W^2 / R^2$ in equation (7.2), based on the experimental data of Table 3.1, is not greater than 0.04, one infers from using (7.2), (7.4), and (4.1) that $\dot{e}$ is given approximately by

$$\dot{e} = \frac{2W \rho \pi^3 U_m}{\rho h R^4}. \quad (7.5)$$

Values for $\epsilon$, $\dot{e}$, and $\lambda$ obtained from (7.2), (7.5), (7.1), and Table 3.1 are given in Table 7.2 for the six experiments. Notice that the values of $\lambda$ are less than the value $\lambda^* = 4.5$ of Chapter 5.

Now, we perform the procedure of Chapter 5 using the new value $\lambda = 1.7$, which is close to the average of the values listed in Table 7.2. The same $\Omega^*$ as in (5.6) is used ($\Omega^* = 15.82$), and it is found that the new maximum allowable dimensionless
Table 7.2: Strain-rate effects based on the Cowper-Symonds equation.

<table>
<thead>
<tr>
<th>Shot</th>
<th>$\varepsilon$</th>
<th>$\dot{\varepsilon}$ (s(^{-1}))</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1229</td>
<td>.0185</td>
<td>4450.9</td>
<td>1.91</td>
</tr>
<tr>
<td>A1230</td>
<td>.0055</td>
<td>1267.0</td>
<td>1.66</td>
</tr>
<tr>
<td>A1233</td>
<td>.00135</td>
<td>678.8</td>
<td>1.57</td>
</tr>
<tr>
<td>A1235</td>
<td>.0035</td>
<td>1266.6</td>
<td>1.66</td>
</tr>
<tr>
<td>A1237</td>
<td>.00480</td>
<td>1199.1</td>
<td>1.66</td>
</tr>
<tr>
<td>A1253</td>
<td>.00223</td>
<td>621.6</td>
<td>1.57</td>
</tr>
</tbody>
</table>

The deflection based on the experimental results is

$$\delta = 1043.43,$$

which, from (5.8), yields a dimensionless load of

$$\mu^* = 80.$$  \hfill (7.7)

Next, we choose a value of $K$ such that the term $C$ of (5.9) takes the same value as in (5.10). We find that

$$K = 0.09.$$  \hfill (7.8)

This value of $K$ is in the range of values listed in Table 3.1. We conclude that the ballistic limit equation of (5.9) remains unchanged when the parameters $K$, $\lambda$, and $\mu$ take the values

$$K = 0.09, \quad \lambda = 1.7, \quad \mu = 80.$$  \hfill (7.9)
For any given $\lambda$, there are infinitely many $(K, \mu)$ pairs that yield the value of $C$ listed in (5.9). These pairs satisfy the equation

$$
\mu = \frac{2(1 - K)}{9(1 + K)\sqrt{K}} \frac{C}{\lambda}, \quad (C = 219.4). \tag{7.10}
$$

For fixed $\lambda$, the values of $K$ and $\mu$ of Figure 7.2 yield ballistic limit curves that are identical to the solid lines of Figures 6.1 to 6.4.

The value of $\lambda = 1.7$ in this chapter is lower than the value $\lambda = 4.5$ that is determined in Chapter 5. Further analysis is needed to evaluate the coefficient $\lambda$ more accurately.
Chapter 8

Conclusions and Future Work

We have derived a new ballistic limit equation for a thin plate subjected to normal hypervelocity impact. This equation applies to cases of impulsive impact on a plate that is protected by a multi-shock shield; it is valid when the projectile velocity is in the range above 6 km/s, and when the area of the impact zone is much smaller than the area of the plate.

The ballistic equation contains two dimensionless groups \( (\rho_p U_m^2 / \sigma_0 \text{ and } r^3/(Dh^2)) \), and the product of these groups when the critical conditions of impact are reached must be equal to a constant \( C \). We have found that the value \( C = 219.4 \) gives good agreement with the results of Christiansen [7].

To arrive at this expression of the ballistic equation, it was necessary to determine the dynamic yield strength of the plate. This was achieved by comparing the experimental deflections of six aluminum square plates supplied by the Hypervelocity Impact Test Facility of NASA/JSC with the theoretical deflections of a rigid perfectly-plastic plate subjected to a blast load.
In the course of this work, we have replaced the multi-shock shield by a single equivalent shield, and we have applied the debris-expansion theory of Swift et al. [9] to the equivalent configuration. We have selected a failure criterion that places an upper bound on the ratio of the plate deflection to the plate thickness.

Taking into account the rebounding effect and the loss of energy as the projectile moves across the multi-shock shield, we have assumed that the impulse on the plate is equal to the momentum of the projectile. For simplicity, it was also assumed that the load transferred to the plate by the impacting cloud is uniformly distributed.

In future attempts to derive ballistic limit equations, particular attention should be given to the dynamics of the debris cloud expansion. This is a complicated issue, especially when the cloud moves across a multi-shock shield, but it must be examined carefully because recent experimental evidence (Piekutowski [15]) has shown that debris-cloud fragments are not uniformly distributed on the surface of an expanding sphere. Further, the dynamics of debris-cloud expansion for oblique incidence should also be investigated, because impacts of space debris and meteoroids are expected to occur under all angles of incidence, not only under normal incidence.
Bibliography


Appendix A

Plastic Analysis of Plate Deformations

In this Appendix, we derive an equation of equilibrium for circular plates subjected to axisymmetric loads. Then, assuming that the plate deforms plastically, we discuss a yield condition based on the Tresca criterion. Finally, we derive expressions for the permanent deflection of a rigid perfectly-elastic plate subjected to a uniform load during a finite time interval.

A.1 Equation of Equilibrium

The element $abcd$ shown in Figure A.1 is a cut of a circular plate of thickness $2h$ that is subjected to an axisymmetric load $p(r, t)$ per unit area. The surface load $p(r, t)$ acts normally to the plate in the $z$-direction. The moments and the shear force are functions of $r$ only (and independent of $\theta$) since the plate is symmetrically loaded. The moments $M_r$ and $M_\theta$ are moments per unit length, and the shear force $Q$ is a circumferential edge force per unit length. The axisymmetry of the load and of the geometry implies that there are no shearing forces acting in the plane of the figure on sides $ab$ and $cd$ and the moments $M_\theta$ are identical on these sides. Recall that the
shear force $Q$ and the moments $M_r$ and $M_\theta$ are defined in terms of the stresses $\sigma_{rz}$, $\sigma_{rr}$, and $\sigma_{\theta\theta}$ by
\[ Q = \int_{-h}^{h} \sigma_{rz} dz, \quad M_r = \int_{-h}^{h} \sigma_{rr} dz, \quad M_\theta = \int_{-h}^{h} \sigma_{\theta\theta} dz. \] (A.1)

To establish the equation of equilibrium, we write that the forces and moments acting on the element $abcd$ are balanced. Summation of forces in the $z$-direction and of moments about an axis perpendicular to the $rz$ plane at $O$ yield, respectively,
\[ -Qr d\theta + (Q + \frac{dQ}{dr})(r + dr)d\theta + pr dr d\theta = 0, \] (A.2)

and
\[ -Qr dr d\theta - M_r rd\theta + \left(M_r + \frac{dM_r}{dr} dr\right) (r + dr)d\theta - M_\theta dr d\theta = 0. \] (A.3)

In (A.3), the contribution of the load $p$ to the moments is neglected since it is of higher order than the moment and edge shear terms. After simplification, by neglecting small quantities of higher order, equations (A.2) and (A.3) yield
\[ \frac{d(Qr)}{dr} = -pr, \quad \frac{d(r M_r)}{dr} - M_\theta = Qr. \] (A.4)

Combining the last two equations and assuming that the shear force $Q$ is bounded at $r = 0$, one finds that the equation of equilibrium for the plate is
\[ [r M_r(r)]' - M_\theta(r) = -\int_0^r p(\alpha) d\alpha. \] (A.5)
Figure A.1: Plate element.
For the dynamic case, the inertia term is included. Thus, (A.5) becomes

\[ [r M_r(r, t)]' - M_\theta(r, t) = - \int_0^r [p(\alpha, t) - 2 \rho h \ddot{w}(\alpha, t)] d\alpha, \]  

(A.6)

where \( w \) is the deflection of the plate in the \( z \)-direction of Figure A.1, \( \rho \) is the mass density, \( h \) is the plate half-thickness, and \( p \) is a force per unit area in the \( z \)-direction.

In equations (A.5) and (A.6), the prime superscripts denote differentiations with respect to \( r \), and the superposed dots denote time differentiations.

### A.2 Fully Plastic Moment

Consider now a plate made of a material with static yield strength \( \sigma_0 \), and let \( \sigma \) be any of the stress components. We define the yield moment \( M_0 \) corresponding to \( \sigma \) as the moment of a stress distribution equal to \( \sigma_0 \) in the upper half of the plate, and

![Figure A.2: Plate cross-section.](image-url)
\( -\sigma_0 \) in the lower half, as in Figure A.2. Thus, one has

\[
M_0 = \int_{-h}^{h} \sigma z \, dz = \int_{-h}^{0} (-\sigma_0) z \, dz + \int_{0}^{h} \sigma_0 z \, dz = \sigma_0 h^2. \tag{A.7}
\]

Notice that \( M_0 \) has the dimensions of a moment per unit length, which is consistent with the plate theory developed earlier.

### A.3 Yield Criterion

Next, we consider a yield criterion for a circular plate subjected to an axisymmetric loading. This criterion is derived from the Tresca yield criterion (Lubliner [14]) and is represented in the \((M_r, M_\theta)\) plane by the hexagon \(ABCDEF\) of Figure A.3. In the following, we use this criterion for a rigid perfectly-plastic plate. If the moments \(M_r\) and \(M_\theta\) at a given point of the plate are inside the hexagon \(ABCDEF\), the plate is rigid at that point; when the moments \(M_r\) and \(M_\theta\) lie on the hexagon, the plate has yielded. States of stress giving moments outside the Tresca hexagon are not permissible.

We consider now a simply supported circular plate of radius \(a\) that is subjected to a static axisymmetric load. Thus, the boundary conditions at \(r = a\) are

\[
w(a) = 0, \quad M_r(a) = 0. \tag{A.8}
\]

Now, given (A.8) and the yield criterion of Figure A.3, we investigate the variations of \(M_r\) and \(M_\theta\). Observe first that if \(M_r\) and \(M_\theta\) are both less than \(M_0\) in absolute
value but their difference is equal to $M_0$, then plastic yielding can occur (segments $DC$ and $FA$). On the other hand, plastic yielding on the segments $CB$, $AB$, $EF$, and $FA$ implies that one of the moments is equal to $M_0$ (or $-M_0$), and the other moment is less than $M_0$ in absolute value. Next, we infer from (A.5) that, if $M'_r$ and $p$ are bounded at $r = 0$, then the moments $M_r$ and $M_0$ are equal at the plate center. Thus, one has

$$M_r(0) - M_0(0) = 0. \quad (A.9)$$

Now we assume that the load $p$ is uniform (positive in the $z$-direction of Figure A.1) and is applied over the entire area of the plate, so that

$$p(r) = p, \quad 0 \leq r \leq a. \quad (A.10)$$
Further, we assume that $p$ is sufficiently large to cause yielding of the plate, and that yielding first occurs at the plate center $r = 0$. Thus, we infer from (A.9) and from the yield hexagon of Figure A.3 that

$$M_r(0) = M_\theta(0) = M_0, \quad (A.11)$$

and the point $r = 0$ corresponds to point $B$ on the hexagon. Then, if plastic yielding occurs in the neighborhood of $r = 0$, as well as at $r = 0$, we have by continuity one of two cases

$$M_r = M_0, \quad M_\theta < M_0, \quad \text{if yielding is along } BA, \quad (A.12)$$

$$M_r < M_0, \quad M_\theta = M_0, \quad \text{if yielding is along } BC. \quad (A.13)$$

If (A.12) holds in some neighborhood of $r = 0$, we infer from the equation of equilibrium (A.5) and from (A.10) that

$$M_\theta(r) = M_0 + \frac{pr^2}{2}. \quad (A.14)$$

If (A.13) holds in some neighborhood of $r = 0$, one finds that (A.5), (A.10), and (A.11) yield

$$M_r(r) = M_0 - \frac{pr^2}{6}. \quad (A.15)$$

Since (A.14) implies that $M_\theta$ is greater than $M_0$ in a neighborhood of $r = 0$, we conclude that for values of $r$ close to $r = 0$, the yield state is represented by a point on
BC near B. Next, we assume that the plate yields for all values of \( r \) in the interval \([0, a]\). Then, by continuity, we infer that the yield state is represented by a point on the segment BC. Further, it can be seen from the boundary condition (A.8) that the point of the boundary \( r = a \) is represented by the point C. We conclude from this analysis that, when the plate yields for all values of \( r \), the moment \( M_\theta \) is such that

\[
M_\theta(r) = M_0, \quad 0 \leq r \leq a. \tag{A.16}
\]

### A.4 Static Collapse Pressure

Consider a simply supported plate of radius \( a \) that is subjected to a uniform pressure \( p \) over its entire area as in (A.10). Then, the boundary conditions at \( r = a \) are given by (A.8), and the equation of equilibrium (A.5) yields

\[
[r M_r(r)]' - M_\theta(r) = -\frac{pr^2}{2}, \quad 0 \leq r \leq a. \tag{A.17}
\]

Assume now that the pressure \( p \) is sufficiently high to cause yielding of the plate for all values of \( r \). Then (A.16) holds, and (A.17) can be integrated in the interval \([0, r]\).

Using (A.11), one has

\[
M_r(r) = -\frac{pr^2}{6} + M_0. \tag{A.18}
\]

Since \( M_r(a) = 0 \) by (A.8), one infers from (A.18) and (A.7) that the pressure \( p = p_0 \) for which yielding occurs for all values of \( r \) and for which the boundary condition on
$M_r$ is satisfied is such that

$$p_0 = \frac{6M_0}{a^2} = \frac{6\sigma_0 h^2}{a^2}. \quad (A.19)$$

The pressure $p_0$ of (A.19) is the static collapse pressure of the plate. If the pressure $p$ applied to the plate is less than $p_0$, then the plate remains rigid.

### A.5 Dynamic Analysis

We now determine the permanent deformation of a simply supported circular plate of radius $a$ subjected to a uniform load $p(t)$ with a rectangular time-dependence, as shown in Figure 2.2. In the interval $[0, T]$, the load takes the constant value $P_m/(\pi a^2)$, and vanishes elsewhere. The plate is rigid perfectly-plastic and satisfies both the yield criterion of Figure A.3 and the dynamic equation of equilibrium (A.6). The analysis that follows is taken in parts from Hopkins and Prager [8] and Perzyna [13].

If the magnitude of the uniform load is always lower than the static collapse pressure $p_0$, then the plate remains rigid and the permanent deflection is zero. If the magnitude of the uniform load is above $p_0$, then the plate yields and has a non-zero permanent deflection. It will be shown that the response of the plate depends on the magnitude of the load relative to the value $2p_0$. 
A.5.1 Medium Pressures, $p_0 \leq P_m/(\pi a^2) \leq 2p_0$

The boundary and initial conditions of the problem are

\begin{align*}
M_\theta(r, t) &= M_0, \quad 0 \leq r \leq a, \quad (A.20) \\
M_r(r, t) &= M_0, \quad r = 0, \quad (A.21) \\
M_r(r, t) &= 0, \quad r = a, \quad (A.22) \\
w(r, t) &= 0, \quad r = a, \quad (A.23) \\
w(r, 0) &= 0, \quad 0 \leq r \leq a, \quad (A.24) \\
w'(r, 0) &= 0, \quad 0 \leq r \leq a. \quad (A.25)
\end{align*}

First, we choose a kinematically admissible displacement field $w(r, t)$ that satisfies (A.23) - (A.25) such that

\begin{equation}
 w(r, t) = f(t) \left(1 - \frac{r}{a}\right), \quad (A.26)
\end{equation}

together with

\begin{equation}
 f(0) = 0, \quad \dot{f}(0) = 0. \quad (A.27)
\end{equation}

The displacement field (A.26) corresponds to a deformation with a plastic hinge at the center. Substituting (A.26) into (A.6) and using (A.20), one finds for a uniform load $p(t)$ that

\begin{equation}
 [rM_r(r, t)]' - M_0 = -p(t) \frac{r^2}{2} + 2\rho h \ddot{f}(t) \left(\frac{r^2}{2} - \frac{r^3}{3a}\right). \quad (A.28)
\end{equation}
Next, we integrate (A.28) in the interval \([0, r]\) and we use (A.21) to obtain

\[
M_r(r, t) = M_0 - p(t) \frac{r^2}{6} + 2\rho h \tilde{f}(t) \left( \frac{r^2}{6} - \frac{r^3}{12a} \right). \tag{A.29}
\]

Then, evaluating (A.29) at \(r = a\) and using (A.22) and (A.19), we have

\[
\rho h \tilde{f}(t) = p(t) - \frac{6M_0}{a^2} = p(t) - p_0. \tag{A.30}
\]

To find the time function \(f(t)\), we integrate (A.30) twice with respect to time and apply the initial conditions (A.27). The time integration of (A.30) in the interval \([0, t]\) gives

\[
\rho h \dot{f}(t) = \int_0^t p(\alpha) d\alpha - p_0 t = I(t) - p_0 t, \tag{A.31}
\]

where the impulse \(I(t)\) of the load \(p(t)\) is defined by

\[
I(t) = \int_0^t p(\alpha) d\alpha. \tag{A.32}
\]

The time integration of (A.31) yields

\[
\rho h f(t) = \int_0^t I(\alpha) d\alpha - p_0 \frac{t^2}{2} = \int_0^t (t - \alpha) p(\alpha) d\alpha - p_0 \frac{t^2}{2}. \tag{A.33}
\]

The time \(t_s\) when the plate comes to rest is given by \(\dot{f}(t_s) = 0\). Using (A.31), one has

\[
t_s = \frac{I(t_s)}{p_0}. \tag{A.34}
\]

The permanent deflection \(W_s = w(0, t_s)\) at the plate center is obtained from (A.26), (A.33), and (A.34) in the form

\[
\rho h W_s = \int_0^{t_s} I(\alpha) d\alpha - p_0 \frac{t_s^2}{2}. \tag{A.35}
\]
For the special case where the uniform load \( p(t) \) is defined by

\[
p(t) = \begin{cases} 
  \frac{P_m}{(\pi a^2)}, & 0 \leq t \leq T, \\
  0, & t > T,
\end{cases}
\]  
(A.36)

then the impulse \( I(t) \) of (A.32) is such

\[
I(t) = \begin{cases} 
  \frac{P_m t}{(\pi a^2)}, & 0 \leq t \leq T, \\
  \frac{P_m T}{(\pi a^2)}, & t > T.
\end{cases}
\]  
(A.37)

Since the ratio \( \frac{P_m}{(\pi a^2 p_0)} \) is greater than unity, we conclude from (A.34) and (A.37) that the time \( t_0 \) is greater than \( T \) and is given by

\[
t_0 = \frac{I(t_0)}{p_0} = \frac{P_m}{\pi a^2 p_0} T.
\]  
(A.38)

Using (A.35), (A.37), and (A.38), we find that

\[
W_e = \frac{P_m T^2}{2\pi \rho a^2} \left( \frac{P_m}{\pi a^2 p_0} - 1 \right).
\]  
(A.39)

Using (A.19), one can recast (A.39) in the form

\[
\delta_e = \mu (\mu - 1),
\]  
(A.40)

where the dimensionless deflection \( \delta_e \) and the dimensionless load \( \mu \) are defined by

\[
\delta_e = \frac{\rho a^2}{3\pi \sigma_h T^2} W_e \quad \text{and} \quad \mu = \frac{P_m}{6\pi \sigma_0 h^2}.
\]  
(A.41)

Using (A.29), (A.30), and (A.36), one can study the variations of \( M_r \) as a function of \( r \). Such an investigation shows that the moment \( M_r \) is not greater than \( M_0 \) provided that

\[
\frac{P_m}{(\pi a^2)} \leq 2p_0.
\]  
(A.42)
The significance of equation (A.42) is that for loads of magnitude greater than twice the value of the static collapse pressure, the Tresca yield criterion of Figure A.3 is violated, and a separate analysis is required.

**A.5.2 High Pressures, \( P_m/(\pi a^2) \geq 2p_0 \)**

We now examine the case where the magnitude of the uniform load is greater than or equal to \( 2p_0 \). In this case, we assume that the plate deforms in the shape of a truncated cone. Along the flat portion of the truncated cone, which is a fully plastic zone, the radial moment is \( M_0 \) everywhere. The end of the flat portion corresponds to a plastic hinge, and the radial moment varies from \( M_0 \) at the hinge to zero at the plate edge. The location of the hinge depends on the magnitude of the load. For smaller values of the load, the hinge is closer to the plate center. After the load is completely removed, the hinge settles at the plate center, and later the plate stops moving, deformed by a permanent plastic deformation.

For the first phase of motion, corresponding to the truncated cone, the boundary conditions are

\[
M_\theta(r, t) = M_0, \quad 0 \leq r \leq a, \quad (A.43)
\]

\[
M_r(r, t) = M_0, \quad 0 \leq r \leq a_0, \quad (A.44)
\]

\[
M_r(r, t) = 0, \quad r = a, \quad (A.45)
\]
\[ w(r,t) = 0, \quad r = a, \quad (A.46) \]
\[ w(r,0) = 0, \quad 0 \leq r \leq a, \quad (A.47) \]
\[ \dot{w}(r,0) = 0, \quad 0 \leq r \leq a, \quad (A.48) \]

where \( a_0 \) is the hinge location. We assume as in Perzyna [13] (p. 638) a deformation of the form

\[ w(r,t) = \begin{cases} f_1(t), & 0 \leq r \leq a_0, \\ f_1(t) \frac{a - r}{a_0 - a_0}, & a_0 \leq r \leq a, \end{cases} \quad (A.49) \]

together with

\[ f_1(0) = 0, \quad \dot{f}_1(0) = 0, \quad (A.50) \]

where \( f_1(t) \) is a function to be determined from the boundary conditions. Substituting (A.49) and (A.43) into (A.6), one finds for a uniform load \( p(t) \) that

\[ [r M_r(r,t)]' - M_0 = -p(t) \frac{r^2}{2} + \rho h \ddot{f}_1(t) r^2, \quad 0 \leq r \leq a_0. \quad (A.51) \]

Integrating (A.51) in the interval \([0,r]\), and using (A.44), one has

\[ M_r(r,t) = M_0 - p(t) \frac{r^2}{6} + \rho h \ddot{f}_1(t) \frac{r^2}{3}, \quad 0 \leq r \leq a_0. \quad (A.52) \]

Evaluating (A.52) at \( r = a_0 \) and applying the boundary condition (A.44), we find that

\[ 2\rho h \ddot{f}_1(t) = p(t). \quad (A.53) \]
To determine the function $f_1(t)$, we integrate (A.53) twice and we apply the initial conditions (A.50). Thus, one has

$$2phf_1(t) = \int_0^t I(\alpha)d\alpha = \int_0^t (t - \alpha)p(\alpha)d\alpha,$$  \hspace{1cm} (A.54)

where $I(t)$ is defined in (A.32). Likewise, using (A.49) and (A.53) for the region $a_0 \leq r \leq a$, one finds that (A.6) yields

$$[rM_r(r,t)]' - M_0 = -\frac{p(t)}{a - a_0} \left(\frac{r^3}{3} - \frac{a_0r^2}{2} + \frac{a_0^3}{6}\right), \hspace{1cm} a_0 \leq r \leq a.  \hspace{1cm} (A.55)$$

Integrating (A.55) in the interval $[a_0, r]$, and using $M_r = M_0$ at $r = a_0$, one has

$$M_r(r,t) = M_0 - \frac{p(t)}{a - a_0} \left(\frac{r^3}{12} - \frac{a_0r^2}{6} + \frac{a_0^3}{6} - \frac{a_0^4}{12r}\right), \hspace{1cm} a_0 \leq r \leq a.  \hspace{1cm} (A.56)$$

Evaluating (A.56) at $r = a$ and using (A.45) and (A.19), one finds that

$$1 - \frac{a_0}{a} - \frac{a_0^2}{a^2} + \frac{a_0^3}{a^3} = \frac{2p_0}{p(t)}.  \hspace{1cm} (A.57)$$

For any given fixed value of $p(t)$ greater than $2p_0$, a particular value of $a_0/a$ can be found that satisfies (A.57). A dimensionless graph of $a_0/a$ versus $p(t)/p_0$ is presented in Figure A.4.

At time $T$, when the load $p(t)$ vanishes, the hinge moves towards the plate center. Thus, a separate analysis is needed to describe this phenomenon. We assume that $a_0$ is a function of time and we determine an equation for the position of the hinge.
$r = a_0(t)$. For this purpose, we introduce a velocity field $\dot{w}(r,t)$ in the form

$$
\dot{w}(r,t) = \begin{cases} 
\dot{f}_2(t), & 0 \leq r \leq a_0(t), \quad t \geq T, \\
\dot{f}_2(t) \frac{a-r}{a-a_0(t)}, & a_0(t) \leq r \leq a, \quad t \geq T.
\end{cases}
$$

(A.58)

The boundary conditions are

$$
M_\theta(r,t) = M_0, \quad 0 \leq r \leq a,
$$

(A.59)

$$
M_r(r,t) = M_0, \quad 0 \leq r \leq a_0(t),
$$

(A.60)

$$
M_r(r,t) = 0, \quad r = a,
$$

(A.61)

$$
w(r,t) = 0, \quad r = a.
$$

(A.62)
In addition, the displacement and velocity of the plate must be continuous for all \( r \) at time \( T \). To find the function \( \dot{f}_2(t) \), we use equation (A.58) and (A.59) and the equation of equilibrium (A.6). In the region \([0,a_0(t)]\), one finds for a uniform load \( p(t) \) that

\[
[rM_r(r,t)]' - M_0 = -p(t)\frac{r^2}{2} + \rho h \ddot{f}_2(t)r^2, \quad 0 \leq r \leq a_0(t). \quad (A.63)
\]

Integrating (A.63) in the interval \([0,r]\) and using (A.60), one has

\[
M_r(r,t) = M_0 - p(t)\frac{r^2}{6} + \rho h \ddot{f}_2(t)\frac{r^2}{3}, \quad 0 \leq r \leq a_0(t). \quad (A.64)
\]

Applying (A.60) for \( r = a_0(t) \) to (A.64), we find that

\[
2\rho h \ddot{f}_2(t) = p(t), \quad t \geq T. \quad (A.65)
\]

Integrating (A.65) between \( T \) and \( t \) and using the continuity of \( \dot{w}(r,t) \) at \( t = T \), together with (A.54), one has

\[
2\rho h \ddot{f}_2(t) = \int_T^t p(\alpha)d\alpha + 2\rho h \dot{f}_2(T) = \int_T^t p(\alpha)d\alpha + 2\rho h \dot{f}_1(T) = I(t), \quad t \geq T. \quad (A.66)
\]

Then, one time integration of (A.66) between \( T \) and \( t \) yields

\[
2\rho h f_2(t) = \int_T^t I(\alpha)d\alpha + 2\rho h f_2(T), \quad t \geq T. \quad (A.67)
\]

Next, we use the second equation of (A.58) for the region \([a_0(t),a]\). For this region, the acceleration of the plate follows from equations (A.65) and (A.66) in the form

\[
\ddot{w} = \frac{\partial}{\partial t} \left[ \frac{\dot{f}_2(t) a - r}{a - a_0(t)} \right] \quad (A.68)
\]
\[
\begin{align*}
\dot{f}_2(t) \frac{a-r}{a-a_0(t)} + \dot{f}_2(t) \frac{(a-r)}{[a-a_0(t)]^2} \dot{a}_0(t) \\
= \frac{a-r}{2\rho h[a-a_0(t)]} \left[ p(t) + \frac{I(t)\dot{a}_0(t)}{a-a_0(t)} \right], \quad a_0(t) \leq r \leq a.
\end{align*}
\]

Using (A.68), (A.58), (A.65) and the boundary condition (A.59), the equation of equilibrium (A.6) for a uniform load \( p(t) \) gives

\[
[rM_r(r,t)]' - M_0 = p(t)\frac{a_0^2(t) - r^2}{2} + \frac{3a(r^2 - a_0^2(t)) - 2(r^3 - a_0^3(t))}{6[a-a_0(t)]} \left[ p(t) + \frac{I(t)\dot{a}_0(t)}{a-a_0(t)} \right], \quad a_0(t) \leq r \leq a. \tag{A.69}
\]

Equation (A.69) can be rewritten in the form

\[
[rM_r(r,t)]' - M_0 = A_1(r,t)p(t) + A_2(r,t)I(t)\dot{a}_0(t), \quad a_0(t) \leq r \leq a, \tag{A.70}
\]

where

\[
A_1(r,t) = \frac{3r^2a_0(t) - 2r^3 - a_0^3(t)}{6[a-a_0(t)]}, \tag{A.71}
\]

and

\[
A_2(r,t) = \frac{3ar^2 - 2r^3 - 3a_0^2(t) + 2a_0^3(t)}{6[a-a_0(t)]^2}. \tag{A.72}
\]

Integrating (A.70) between \( a_0 \) and \( r \), using (A.60) at \( r = a_0(t) \), and applying the boundary condition (A.61) yields

\[
\frac{I(t)}{p_0} A_3(t) \frac{\dot{a}_0(t)}{a} = \frac{p(t)}{p_0} A_4(t) - 2, \quad t \geq T, \tag{A.73}
\]

where (A.19) has been used, and

\[
A_3(t) = 1 + 2\frac{a_0(t)}{a} - 3\frac{a_0^2(t)}{a^2}, \tag{A.74}
\]
and

\[ A_4(t) = 1 - \frac{a_0(t)}{a} - \frac{a_0^2(t)}{a^2} + \frac{a_0^3(t)}{a^3}. \quad (A.75) \]

Equation (A.73) is a nonlinear differential equation of the first order for the hinge location \( r = a_0(t) \) of the traveling hinge circle.

The value of \( a_0 \), which is constant for all times in the interval \((0, T)\), is given by equation (A.57) where \( p(t) \) takes the value \( P_m/(\pi a^2) \). In the interval \( t > T \), the value of \( a_0 \) depends on time, and we obtain \( a_0(t) \) by solving (A.73). Since \( p(t) \) goes to zero at \( t = T \), then the impulse \( I(t) \) for all \( t \geq T \) is constant. Therefore, one has

\[ p(t) = 0, \quad I(t) = \frac{P_m}{\pi a^2} T, \quad t \geq T. \quad (A.76) \]

Substituting (A.76) into (A.73), one finds that

\[ -\frac{2\pi p_0 a^5}{P_m T} = \dot{a}_0[a^2 + 2a_0(t)a - 3a_0^2(t)], \quad t \geq T, \quad (A.77) \]

which is a separable ordinary differential equation. Equation (A.77) is integrated between \( t \) and \( T \) subject to the initial condition \( a_0(T) = a_0 \). Then, using (A.57), one has

\[ -\frac{2\pi p_0 a^5 t}{P_m T} = a^2 a_0(t) + aa_0^2(t) - a_0^3(t) - a^3. \quad (A.78) \]

Next, the time \( t_c \) when the hinge circle reaches the plate center is determined by solving (A.78) for \( a_0(t) = 0 \), giving

\[ t_c = \frac{P_m T}{2\pi a^2 p_0}. \quad (A.79) \]
The displacement \( w(r, t) \) in the region \([0, a_0(t)]\) can be obtained from (A.58), (A.67), the condition of continuity at \( t = T \), and (A.54). The result is

\[
2\rho hw(r, t) = \int_0^t I(\alpha) d\alpha, \quad 0 \leq r \leq a_0(t), \quad t \geq T. \tag{A.80}
\]

Observe that \( t_c \) is greater than \( T \) since \( P_m/(\pi a^2) \) is greater than \( 2p_0 \). After the hinge reaches the plate center, the deformation has the shape of a cone with the apex at \( r = 0 \). Thus, we write

\[
w(r, t) = \hat{f}_3(t) \left( 1 - \frac{r}{a} \right), \quad 0 \leq r \leq a, \quad t > t_c. \tag{A.81}
\]

The boundary conditions are

\[
M_\theta(r, t) = M_0, \quad 0 \leq r \leq a, \tag{A.82}
\]

\[
M_r(r, t) = M_0, \quad r = 0, \tag{A.83}
\]

\[
M_r(r, t) = 0, \quad r = a, \tag{A.84}
\]

\[
w(r, t) = 0, \quad r = a. \tag{A.85}
\]

In addition, the displacement and velocity of the plate must be continuous for all \( r \) at time \( t_c \). Thus, from the condition \( a_0(t_c) = 0 \), (A.58), and (A.81), we infer that

\[
\hat{f}_3(t_c) = \hat{f}_2(t_c). \tag{A.86}
\]

Substituting (A.81) into (A.6), using (A.82) and the condition \( p(t) = 0 \), one has

\[
[rM_r(r, t)]' - M_0 = 2\rho h \tilde{f}_3(t) \left( \frac{r^2}{2} - \frac{r^3}{3a} \right). \tag{A.87}
\]
We integrate (A.87) in the interval \([0, r]\) and we use (A.83) to obtain

\[
M_r(r, t) = M_0 + 2\rho \dot{r} \ddot{f}_3(t) \left( \frac{r^2}{6} - \frac{r^3}{12a} \right).
\]

(Equation A.88)

Evaluating (A.88) at \(r = a\), using (A.84) and (A.19), one has

\[
\rho \dot{f}_3(t) = -p_0, \quad t \geq t_c.
\]

(Equation A.89)

Integrating (A.89) between \(t_c\) and \(t\), one finds that

\[
\rho \dot{f}_3(t) - \rho \dot{f}_3(t_c) = -p_0(t - t_c).
\]

(Equation A.90)

It follows now from (A.90), (A.86), (A.79), (A.76), and (A.66) for a load with the time dependence of (A.36), that

\[
\rho \dot{f}_3(t) = \frac{1}{2} I(t_c) - p_0 \left( t - \frac{P_m T}{2\pi a^2 p_0} \right) = -p_0 t + \frac{P_m T}{\pi a^2}.
\]

(Equation A.91)

The time \(t_s\), when the plate center stops moving is found by setting the expression \(\dot{f}_3(t)\) in (A.91) equal to zero, giving

\[
t_s = \frac{P_m T}{\pi a^2 p_0}.
\]

(Equation A.92)

One more time integration yields the final plate deformation. Integrating (A.81) between \(t_c\) and \(t\), and using the continuity of the displacement \(w(r, t)\) at \(t = t_c\), (A.80) and (A.91), one has

\[
\rho \omega(0, t) = \frac{1}{2} \int_0^{t_c} I(\alpha)d\alpha - \frac{P_0}{2}(t^2 - t_c^2) + \frac{P_m T}{\pi a^2} (t - t_c).
\]

(Equation A.93)
The final permanent deformation $W_e$ of the plate center is obtained by evaluating (A.93) at $t = t_s$. Using (A.92), (A.79), and the impulse (A.37), one finds that

$$W_e = w(0, t_s) = \frac{P_m T^2}{8\pi a^2 \rho h} \left( 3 \frac{P_m}{\pi a^2 \rho_o} - 2 \right).$$  \hspace{1cm} (A.94)

Equation (A.94) can be recast in the form

$$\delta_e = \frac{\mu}{4} (3\mu - 2),$$  \hspace{1cm} (A.95)

where the dimensionless deflection $\delta_e$ and the dimensionless load $\mu$ are defined by

$$\delta_e = \frac{\rho a^2}{3\sigma_0 h T^2} W_e \quad \text{and} \quad \mu = \frac{P_m}{6\pi \sigma_0 h^2}.$$  \hspace{1cm} (A.96)

In this Appendix, we have obtained the permanent deflection at the center of a rigid perfectly-plastic circular plate subjected to a time-dependent uniform load of the type (A.36). For medium pressures, the permanent deflection $W_e$ is given by (A.39); for higher pressures, the corresponding expression is (A.94).