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A probabilistic model for fatigue damage accumulation in composite laminates

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Rice University, 1993
RICE UNIVERSITY

A Probabilistic Model for Fatigue Damage Accumulation in Composite Laminates

by

John David Rowatt

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

Master of Science

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Abstract

A Probabilistic Model for Fatigue Damage Accumulation in Composite Laminates

by

John David Rowatt

A stochastic model for the accumulation of fatigue damage in composite laminates is proposed. The model considers the accumulation of fatigue damage as an evolutionary stochastic process and attempts to predict the distribution of damage in terms of the stiffness loss in the laminate as a function of the number of load cycles. The cumulative damage process is modeled as an embedded discrete time, finite state, nonstationary Markov process utilizing a discrete time, finite state, stationary model in conjunction with a time transformation-condensation method. Furthermore, the mechanisms of the complex damage development process in composite laminates are incorporated into the model. This feature affords a more complete and accurate representation of the fatigue damage accumulation process in which the states of the model can be directly related to the stiffness loss in the laminate.
Acknowledgments

The author wishes to thank Dr. Pol D. Spanos for his guidance, support, and confidence throughout this project. His enthusiasm and knowledge are a great inspiration.

In addition, many thanks are due to Dr. Enrique V. Barrera and Dr. Joel P. Conte for serving as thesis committee members and whose courses, comments, and suggestions proved invaluable in the preparation of this work. Sincere thanks are also extended to Dr. Scott White and Dr. Larry Bergman at the University of Illinois at Urbana-Champaign for the many opportunities which were extended to the author during his undergraduate studies.

The author also wishes to thank Mr. Scott Miller for his guidance in the presentation of this work and for the fine example of hard work and dedication he has represented. Special gratitude is also extended to the author’s family and his fiancée Ms. Cynthia Spencer for their unending support, patience, and understanding throughout this undertaking. A final thanks is extended to the One who has blessed the author’s life so richly and given him the knowledge and wisdom to pursue his goals.
Table of Contents

Abstract ii
Acknowledgments iii
List of Figures vii
List of Tables ix
Nomenclature x

1 Introduction 1

2 Fatigue of Engineering Materials 4
  2.1 Basic Concepts ...................................... 4
  2.2 Fatigue of Isotropic Engineering Materials ............. 7
  2.3 Fatigue of Composite Laminates ......................... 10
    2.3.1 The Laminate Code ................................ 12
    2.3.2 Matrix Cracking .................................. 13
    2.3.3 Delamination ...................................... 15
    2.3.4 Fiber Fracture and Longitudinal Splitting .......... 16
  2.4 Summary ............................................ 19

3 Fatigue Models 20
  3.1 Fatigue Models for Isotropic Engineering Materials .... 20
    3.1.1 Deterministic Models ............................ 20
3.1.2 Stochastic Models ........................................ 23
3.2 Fatigue Models for Composite Laminates .................. 25
  3.2.1 Deterministic Models .................................... 26
  3.2.2 Stochastic Models ....................................... 33
3.3 Summary ................................................... 36

4 Concepts from Discrete Probability Theory ................. 38
  4.1 Discrete Random Variables ............................... 38
  4.2 Discrete Probability Distributions ....................... 41
  4.3 Moments of Discrete Random Variables .................. 42
  4.4 Discrete Random Processes ................................ 46
  4.5 Summary ................................................ 48

5 Markov B-Models ............................................. 49
  5.1 Markov Chains ........................................... 49
  5.2 Markov B-Models ......................................... 53
    5.2.1 Stationary B-Models ................................... 55
    5.2.2 Nonstationary B-Models ............................... 64
  5.3 Summary ................................................ 69

6 B-Models for Composite Laminates .......................... 70
  6.1 Modeling Concepts ...................................... 70
  6.2 Extension of the Markov B-Model ........................ 74
  6.3 Numerical Example ...................................... 78
  6.4 Summary ................................................ 87
7 Concluding Remarks 88

A Derivation of Moment Formulae 90

Bibliography 94
List of Figures

2.1 Constant amplitude sinusoidal fatigue load .................. 5
2.2 General time varying fatigue load .......................... 6
2.3 Two stages of crack growth .............................. 8
2.4 Mechanism of fatigue crack growth ....................... 9
2.5 Chronology of damage development in composite laminates under 
fatigue loading. Regions 1 through 5 correspond to: 1. matrix 
cracking and fiber breakage, 2. crack coupling, interfacial debonding, 
and fiber breakage, 3. delamination and fiber breakage, 4. 
delamination growth and fiber breakage, 5. laminate fracture. .... 11
2.6 Schematic representation of the relationship between matrix cracks 
and delamination ........................................... 15
2.7 Schematic representation of longitudinal splitting ............... 18

3.1 Illustration of the damage vector concept. .................. 27
3.2 Chain of bundles model .................................... 34

4.1 Probability mass function for the dice rolling example ............ 43
4.2 Cumulative distribution function for the dice rolling example .... 43
4.3 Example of a discrete random process. ........................ 47
5.1 Flow diagram representation of a “unit-jump” Markov chain ........ 53

6.1 Secant stiffness loss versus load cycles for [0/90/ ± 45]s
  graphite/epoxy laminate ......................................... 80

6.2 Relationship between the percent secant stiffness loss and the damage
  states of the model................................................. 82

6.3 Comparison of the experimental empirical distribution function
  (solid) and cumulative distribution function predicted by the model
  (dotted) for a 4 percent loss in the secant modulus of the laminate. 1
  duty cycle = 10 load cycles. ...................................... 83

6.4 Comparison of the experimental empirical distribution function
  (solid) and cumulative distribution function predicted by the model
  (dotted) for an 8 percent loss in the secant modulus of the laminate.
  1 duty cycle = 10 load cycles. .................................... 84

6.5 probability mass function of damage after 1400 duty cycles........ 85

6.6 probability mass function of damage after 4600 duty cycles ........ 86
List of Tables

4.1 Possible outcomes of throwing two dice. ....................... 39

6.1 Model Input ............................................. 80
6.2 Model Output ............................................ 81
# Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(n)$</td>
<td>surface area of edge delamination</td>
</tr>
<tr>
<td>$A^*$</td>
<td>total interfacial surface area</td>
</tr>
<tr>
<td>$A_{LD}(n)$</td>
<td>surface area of interior delamination</td>
</tr>
<tr>
<td>$a$</td>
<td>crack length</td>
</tr>
<tr>
<td>$a_i$</td>
<td>coefficients of time transformation</td>
</tr>
<tr>
<td>$C$</td>
<td>set of material parameters</td>
</tr>
<tr>
<td>$D$</td>
<td>deterministic damage variable, magnitude of the damage vector, stochastic damage variable</td>
</tr>
<tr>
<td>$D(x)$</td>
<td>discrete random process</td>
</tr>
<tr>
<td>$E$</td>
<td>random event</td>
</tr>
</tbody>
</table>
\( E[-] \) expectation operator

\( E_1 \) longitudinal Young's modulus of a laminate

\( E_1^{\prime} \) longitudinal Young's modulus under total edge delamination

\( E_{1LD} \) longitudinal Young's modulus under total interior delamination

\( E_2 \) transverse Young's modulus of a laminate

\( E_s \) shear modulus of a laminate

\( \tilde{f} \) limit state function

\( F_D(d) \) cumulative distribution function of a random variable D

\( F(\sigma) \) cumulative distribution function of fiber strength in the chain of bundles model

\( f_c \) nondimensional constraint factor for a matrix crack array

\( G_c \) critical strain energy release rate
$G_h(\sigma)$ cumulative distribution function of bundle strength in the chain of bundles model

g(x) time transformation

$H_{\bar{m},\bar{n}}(\sigma)$ cumulative distribution function of chain strength in the chain of bundles model

$h$ individual ply thickness

$\bar{h}$ average height of matrix cracks

$K$ stress intensity factor

$K_{max}$ maximum stress intensity factor

$K_{min}$ minimum stress intensity factor

$K_1^*$ load concentration factor in the chain of bundles model

$k_n$ $n^{th}$ moment of a random variable with respect to the mean

$l$ length of laminate
\( \bar{l} \) average length of matrix cracks

\( m \) number of plies in a laminate

\( m_n \) \( n^{th} \) moment of a random variable with respect to the origin

\( \hat{n} \) number of sublaminates formed under total delamination

\( \hat{m} \) number of bundles in the chain of bundles model

\( N \) number of cycles in a load history at a given load level

\( n \) number of load cycles

\( \mathbf{n} \) unit normal vector

\( \hat{n} \) number of fibers in a given bundle in the chain of bundles model

\( N_f \) number of cycles to failure at a given load level under constant amplitude fatigue loading

\( \mathbf{P} \) stationary probability transition matrix
\[ P[\cdot] \quad \text{probability measure} \]

\[ P_D(d_i) \quad \text{probability mass function of a random variable D} \]

\[ P_D(z) \quad \text{probability generating function of a random variable D} \]

\[ p_i \quad \text{probability of staying in a given state after one duty cycle} \]

\[ P_n(\sigma) \quad \text{cumulative distribution function of two or more} \]
\[ \text{fibers with strength less than or equal to } \sigma \text{ existing} \]
\[ \text{side by side in a bundle of fibers in the chain of bundles model} \]

\[ p_r \quad \text{fraction of components requiring replacement} \]

\[ P_x \quad \text{probability mass function of damage after } x \text{ duty cycles} \]

\[ P_y \quad \text{probability mass function of damage after } y \text{ duty cycles} \]

\[ P_0 \quad \text{probability mass function of initial damage} \]

\[ Q \quad \text{nonstationary probability transition matrix} \]

\[ Q(n) \quad \text{laminate stiffness tensor} \]
\(Q^{(0)}\) undamaged laminate stiffness tensor

\(Q^{(i)}(n)\) stiffness tensor representing damage due to matrix cracking in the \(i^{th}\) crack array

\(q_i\) probability of jumping to the next state after one duty cycle

\(R\) cyclic stress ratio

\(\bar{R}\) resistance term in the limit state function

\(r_i\) ratio of stay to jump probabilities

\(S\) range of a random variable

\(\bar{S}\) load effect in the limit state function

\(\text{SDD}[\cdot]\) standard deviation of a random variable

\(T\) temperature

\(t\) laminate thickness

\(t_c\) thickness of a crack array
\( t_i \)  \hspace{1cm} \text{thickness of sublaminates formed under total edge delamination}

\( t_{LD} \)  \hspace{1cm} \text{thickness of interior delaminated region}

\( \text{VAR}[\cdot] \)  \hspace{1cm} \text{variance of a random variable}

\( \mathbf{y} \)  \hspace{1cm} \text{damage vector}

\( W_{i,b} \)  \hspace{1cm} \text{time to failure in damage state } b \text{ given that the cumulative damage process began in damage state } i

\( X_i \)  \hspace{1cm} \text{laminate coordinate system}

\( Y(a) \)  \hspace{1cm} \text{specimen geometry factor}

\( \alpha \)  \hspace{1cm} \text{ratio of load cycles at a given level in load history to the number of cycles to failure under constant amplitude fatigue loading at that level, parameter related to the state of damage at failure}

\( \alpha' \)  \hspace{1cm} \text{parameter related to the state of damage at failure}

\( \beta \)  \hspace{1cm} \text{parameter related to the state of damage at failure}
**ΔK**  
stress intensity range  

**Δσ**  
cyclic stress range  

**ε**  
strain vector  

**ε_{1c}**  
nominal longitudinal midplane strain on the laminate at the onset of delamination  

**θ_i**  
orientation angle of *i*th ply of a laminate with respect to the longitudinal axis of the laminate  

**λ**  
matrix crack density  

**μ**  
friction coefficient between a fiber and the reinforcing matrix material  

**ν_{12}**  
longitudinal Poisson’s ratio  

**ν_{21}**  
transverse Poisson’s ratio  

**ρ**  
pmf of the state of damage at failure
\( \rho_0 \) \hspace{1cm} \text{Weibull scale parameter}

\( \mathbf{\sigma} \) \hspace{1cm} \text{stress vector}

\( \sigma_a \) \hspace{1cm} \text{cyclic stress amplitude}

\( \sigma_f \) \hspace{1cm} \text{axial fiber stress}

\( \sigma_m \) \hspace{1cm} \text{mean value of the cyclic stress}

\( \sigma_{\text{max}} \) \hspace{1cm} \text{maximum cyclic stress}

\( \sigma_{\text{min}} \) \hspace{1cm} \text{minimum cyclic stress}

\( \sigma_r \) \hspace{1cm} \text{residual stress}

\( \sigma_z \) \hspace{1cm} \text{interlaminar normal stress}

\( \sigma_0 \) \hspace{1cm} \text{Weibull shape parameter}

\( \mathbf{r} \) \hspace{1cm} \text{vector of inspection qualities}

\( \Omega \) \hspace{1cm} \text{sample space of a random variable}
$\omega$ an element of a sample space
Chapter 1

Introduction

Due to their high strength, high stiffness, and light weight, composite materials are becoming an increasingly attractive alternative to conventional structural materials. Their applications are both extensive and diverse ranging from sport and recreation equipment to automotive and aerospace components. With their use, however, comes a departure from conventional structural analysis due to their general anisotropic behavior. This behavior, although adding some inherent complexity to their analysis, allows for composite materials to be tailored for specific applications. The utilization of this feature in an optimum manner requires a thorough knowledge of the response of the material under both static and dynamic loading.

In an attempt to develop a concise understanding of the mechanical behavior of composite materials, an extensive research effort has been put forth over the last few decades. An appreciable amount of this research has centered around composite laminates which lend themselves naturally to applications involving beams, plates, and shells. From this research laminated plate theory has emerged. This theory in conjunction with an appropriate failure criterion has been widely accepted as a successful predictor of the static behavior of composite laminates. Unfortunately, no one particular method for characterizing the behavior of composite laminates under general dynamic loading has proven as successful.

Nevertheless, progress has been made for the case when the dynamic load is cyclic in nature. That is, fatigue loading. Under fatigue loading the response of a laminate is governed by a complex cumulative damage process. This process is the result of the
interaction of several distinct damage modes including matrix cracking, fiber/matrix debonding, delamination, and fiber fracture. The combined effect of these damage mechanisms is the degradation with fatigue loading of the laminate strength and stiffness on both local and global levels. This degradation in material properties naturally hinders the performance of a laminate over time. Consequently, structural components made from these materials will deteriorate as well. The engineer working with such components must be able to predict their lifetime and to develop inspection and replacement policies for safety and maintenance purposes. To accomplish these tasks a suitable mathematical model is needed. One possible model is proposed herein.

The model is based on a class of stochastic processes known as Markov B-Models. These models consider a cumulative damage process as an evolutionary random phenomenon in which the accumulation of damage is randomly distributed with time. They are based on an embedded discrete time, finite state Markov chain and are both probabilistic and phenomenological in nature. These attributes are a reflection of the general absence of the appropriate physical laws needed to completely describe a cumulative damage process and the fact that variations from mean behavior often occur in such processes. The strength of the Markov B-Models lies in their comprehensive and coherent structure. The models are simple, unified, and accurately predict behavior for conditions not covered by data. In addition, these models address the inherent sources of variability in a cumulative damage process. However, they do not address the issue of how damage develops during the cumulative damage process and how information on the damage development process can be incorporated into the model.
The subject of the present work is to present a probabilistic model for the accumulation of fatigue damage in composite laminates. This model extends the Markov B-Models to a particular material system by incorporating the mechanics of damage development into the model. This extension serves to enhance the model's practical applicability by utilizing the type of data often recovered in field inspections using nondestructive testing techniques. Furthermore, it affords a more complete and accurate representation of the fatigue damage accumulation process.

Since the model is used to describe the cumulative damage process associated with fatigue, a discussion of the basic principles of the fatigue of engineering materials is included. To complement this discussion, some of the more prominent fatigue models for engineering materials are presented. In conjunction with the proposed model, a review of the basic concepts from discrete probability theory is presented along with a discussion of discrete time, finite state Markov chains and Markov B-Models in both a stationary and a nonstationary context. Finally, the proposed modeling procedure is developed and illustrated by a numerical example.
Chapter 2

Fatigue of Engineering Materials

In an effort to establish continuity between the physical aspects of the fatigue of engineering materials and the mathematical models used to describe them, a review of the important concepts concerning fatigue are presented in this chapter. This chapter discusses the physical aspects common to both isotropic materials and composite laminates as well as those specific to each.

2.1 Basic Concepts

Engineering components ultimately fail when they are unable to bear the load to which they are subjected. The inability of a component to carry a load may be the result of any one or a combination of improper design or manufacturing, over stressing, environmental effects, or fatigue.

In general, fatigue failure is the progressive failure of a component due to the irreversible accumulation of damage under a time varying load. These loads are generally repetitive or cyclic in nature and can arise from several sources including ocean waves, wind, mechanical vibrations, acoustical vibrations, and thermal cycling to name a few. Under such loads a component may fail at stresses well below the ultimate static strength of its material constituent(s). Thus, the understanding of the fatigue behavior of a material is critical in properly designing a component against fatigue failure.

Recognition of this fact has led researchers to study the fatigue problem in a wide variety of materials. Much of this work has centered around common metals
and other isotropic materials due to their extensive use in engineering applications. However, as the interest in composite materials has grown, so has the research effort into understanding their fatigue behavior.

Central to all materials are some basic concepts concerning fatigue. Consider first the description of the fatigue load. Under a constant amplitude load, such as that depicted in figure 2.1, a component is subjected to a maximum and minimum cyclic stress, $\sigma_{max}$ and $\sigma_{min}$ respectively. The algebraic average of $\sigma_{max}$ and $\sigma_{min}$ is known as the mean stress, $\sigma_m$. The absolute value of the difference between $\sigma_m$ and $\sigma_{max}$ or $\sigma_{min}$ is the stress amplitude, $\sigma_a$. In addition, the stress range, $\Delta \sigma$, is defined as the absolute value of the difference between $\sigma_{max}$ and $\sigma_{min}$ while the stress ratio, $R$, is the ratio of $\sigma_{min}$ to $\sigma_{max}$. For a general time varying load, such as that in figure 2.2, the above quantities become functions of time and vary from cycle to cycle. The question then arises as to how to define a stress cycle for such a general load. One possible solution is to employ the use of cycle counting methods.

![Figure 2.1: Constant amplitude sinusoidal fatigue load](image)

**Figure 2.1: Constant amplitude sinusoidal fatigue load**
Figure 2.2: General time varying fatigue load

Cycle counting methods generally attempt to reduce a time varying load to a sequence of events which are compatible with constant amplitude fatigue data. Three of the more prominent cycle counting methods are peak counting, range counting, and rain-flow counting. Rain-flow methods are widely accepted as the method of choice in the analysis of general time varying fatigue loads. Their primary advantage is that they determine both the number of equivalent load cycles and the stress ranges associated with each cycle. In addition, several algorithms for rain-flow cycle counting are available thus enhancing the applicability of the method. A more detailed discussion of all three methods can be found in Madsen et al. (1986) and Sobczyk and Spencer (1992).

The application of a fatigue load to a component generally results in the formation of a damage zone which propagates and eventually causes failure due to a decrease in the load carrying capability of the component. Thus, three issues must be addressed regarding the relationship between the fatigue load and the fatigue damage accumu-
lation process. These topics are the nucleation and propagation of damage and the final fracture event.

2.2 Fatigue of Isotropic Engineering Materials

The literature on the fatigue behavior of isotropic engineering materials, most notably metals, is vast and readily accessible. Discussions of fatigue in such materials can be found in a variety of sources including texts on properties of engineering materials, fracture mechanics, and specific discourses on the subject itself. A few references pertinent to this section which may be particularly useful are Madayag (1969), Ashby and Jones (1980), and Broek (1991).

The fatigue of isotropic materials can be categorized into three areas including the low cycle fatigue of undamaged components, high cycle fatigue of undamaged components, and fatigue of damaged components. Common to all three areas and the necessary condition for fatigue damage accumulation is the simultaneous action of the fatigue load, tensile stresses, and plastic strain.

The first two fatigue categories are nucleation controlled processes, and are dominated by the initiation of fatigue cracks. The components are generally small and manufactured under strict standards to ensure the absence of flaws. For low cycle fatigue, the service loads are usually above the yield strength of the constituent material, while for high cycle fatigue the loads are normally below the material yield strength.

For low cycle fatigue the plastic behavior of the material governs the formation of fatigue cracks. Since plastic deformation is not completely reversible, materials subjected to cyclic loads above the yield strength are prone to the formation of slipband extrusions and intrusions. Under continued cyclic loading the intrusions may grow
into a crack by further plastic flow along the slipband. Such a crack first grows along a slipband, stage I, then changes direction and follows a path of high tensile stress, stage II, perpendicular to the applied load. This process is shown in figure 2.3.

For high cycle fatigue the initiation of fatigue cracks is somewhat different. Since the service loads on the component are generally below the yield strength of the constituent material most of the fatigue life is spent in the initiation of cracks. Although plastic behavior does not occur on a global level, it does appear locally in the presence of stress raisers. Stress raisers occur whenever a scratch, subsurface defect such as a void or inclusion, or a sharp change in the cross section of the component is present. In these local plastic regions fatigue cracks form in a similar manner as those due to low cycle fatigue. The primary difference between the low and high cycle fatigue

![Figure 2.3: Two stages of crack growth](image)
mechanisms is the time involved in the crack initiation due to the general magnitude of the applied load.

Once fatigue cracks are present in a component, whether they were initially present or formed later, the mechanism by which the cracks propagate is the same. Once a stage II crack is formed, it grows under the action of the tensile and compressive load excursions of the fatigue load. Under tensile stress a plastic zone is formed at the crack tip which causes the crack tip to open by an amount $\delta$ thus creating new surface area at the crack tip. When the tensile load is reversed, the compressive action forces the crack to close and the newly formed surface area is pushed forward thus extending the crack by approximately $\delta$. This process is depicted in figure 2.4 and continues until a critical crack size is reached. This critical crack size represents the minimum size of a crack in a component such that the stresses at the crack tip

![Diagram](image)

**Figure 2.4:** Mechanism of fatigue crack growth
exceed the fracture strength of the material and the total potential energy of the system decreases with advancement of the crack. Crack growth past this critical size is unstable and culminates in the fracture of the component.

The fracture surface of a component undergoing fatigue failure exhibits several distinct regions; they are a fatigue zone and a fracture zone. The fatigue zone contains the origin of the fatigue crack and an area of fatigue crack growth. The area of crack growth is marked by the presence of striations due to the intermittent propagation of the fatigue crack. Between these striations the material surface is quite smooth due to the contact of the two crack surfaces during closure. In contrast, the fracture zone is quite rough due to the instantaneous growth of the crack across the cross section of the component. These features provide insight into the direction and the rate of crack growth and are important in understanding the fatigue behavior of a material.

2.3 Fatigue of Composite Laminates

The fatigue damage process in isotropic materials is generally the result of the growth of a single dominant crack. This process in composite materials is much more complex due to the interaction of the damage mechanisms mentioned earlier.

The accumulation of fatigue damage in composite laminates is characterized by the degradation of laminate strength and stiffness with the number of load cycles from their respective static values predicted by laminated plate theory. The loss of strength and stiffness is the direct result of the action of the damage mechanisms. Although the effect of the individual mechanisms varies from laminate to laminate a general pattern concerning their chronology is evident. The typical chronology of these damage modes, as noted by Reifsnider and Talug (1980), Reifsnider et al. (1983a, 1983b), Jamison et al. (1984), Jamison (1986), and Stinchcomb and Reifsnider (1988) is illus-
trated in figure 2.5. From this figure the underlying pattern of damage development can be seen as a sequence of matrix related effects, each in conjunction with fiber breakage, leading ultimately to laminate fracture.

This section will focus on the development and duration of these damage modes in continuous fiber, polymer matrix composite laminates under cyclic loading with peak cyclic stresses, $\sigma_{\text{max}}$, of approximately 60 to 70 percent of their ultimate tensile strengths. The choice of this material system is driven by three issues. First, the development of fatigue damage in these materials has been well documented. Second, they are experiencing widespread growth in both the diversity and criticality of their applications, particularly in the aerospace industry. Finally, the magnitude of loading is typical of those seen in actual practice.

![Graph showing damage development over percent of fatigue life](image)

**Figure 2.5:** Chronology of damage development in composite laminates under fatigue loading. Regions 1 through 5 correspond to: 1. matrix cracking and fiber breakage, 2. crack coupling, interfacial debonding, and fiber breakage, 3. delamination and fiber breakage, 4. delamination growth and fiber breakage, 5. laminate fracture.
2.3.1 The Laminate Code

As a prelude to discussing the fatigue of composite laminates it is prudent to briefly address the structure of the laminates themselves. A continuous fiber, polymer matrix composite laminate is composed of reinforcing fibers and a viscous resin. The fibers are arranged in a unidirectional array and are impregnated with the resin to form a prepreg tape. This tape is cut into sheets which are then stacked upon one another with the fibers oriented in various directions. The stack is then processed in a vacuum under heat and pressure to produce the final laminate.

The laminate is designated by three quantities including the fiber material, the matrix material, and a code describing the stacking sequence of the individual plies. The code for a general laminate is given as \([\theta_1, \theta_2, \ldots, \theta_m]\) where \(\theta_i (i = 1, 2, \ldots, m)\) denotes the orientation of the \(i^{th}\) ply with respect to the longitudinal axis of the laminate. Positive values of \(\theta\) represent counterclockwise rotations. Here, \(i = 1\) denotes the bottom ply and \(i = m\) the top ply.

When the laminate has symmetry with respect to its midplane two other notations are commonly used to designate the stacking sequence. If the number of plies in the laminate is even then the laminate is given by \([\theta_1, \theta_2, \ldots, \theta_{\frac{m}{2}}]\) where the subscript \(S\) denotes midplane symmetry. Similarly, if the number of plies in the laminate is odd then the laminate is given by \([\theta_1, \theta_2, \ldots, \bar{\theta}_{\frac{m+1}{2}}]\) where the bar above the last value of \(\theta\) represents midplane symmetry with respect to that ply. That is, the midplane of the laminate is located in the center of the ply whose orientation is \(\bar{\theta}\).

The static properties of the laminate, including its strength and stiffness, are a function of the constituent materials and stacking sequence and can be found via
laminated plate theory. A thorough discussion of laminated plate theory can be found in Christensen (1979) and Tsai and Hahn (1980).

2.3.2 Matrix Cracking

Composite laminates are usually constructed in such a manner that the fiber orientation is parallel to the applied load. However, if plies with off-axis orientations exist within the laminate they will be subjected to a load component perpendicular to their respective fiber directions. If the magnitude of this load component exceeds the transverse strength of the plies, transverse matrix cracks may form parallel to the fibers in the plies. These cracks extend through the thickness of the ply and span the entire width of the ply on formation.

The introduction of a transverse crack into the matrix changes the local geometry of the ply. This in turn brings about changes in the local compliance. The later of these changes is reflected by the loss of transverse and shear stiffness in the ply in the vicinity of the crack. The location of these transverse cracks, that is in which plies the cracks will appear, is governed by the maximum cyclic stress, $\sigma_{\text{max}}$, and can be predicted with laminated plate theory by determining the individual ply stress at the peak cyclic load.

The number of transverse cracks in a given ply is dependent upon the number of load cycles. As the number of load cycles increases so does the number of transverse matrix cracks. The density of these cracks increases rapidly at first but eventually rolls off and attains a stable value. This saturation level is known as the characteristic damage state. The characteristic damage state is dependent upon the material constituents and stacking sequence of the laminate. The number of load cycles required to reach the characteristic damage state is dependent upon the fatigue load.
The spacing between successive transverse matrix cracks at the characteristic damage state can be accurately predicted using a shear lag model developed by Reifsnider (1977) and Masters and Reifsnider (1982).

In addition to transverse matrix cracks in the off-axis plies, longitudinal matrix cracks may form in the load bearing plies. The nature of the longitudinal cracks is somewhat different than that of the transverse cracks. The key to understanding their development lies in the state of stress in the load bearing plies. Consider a laminate subjected to a uniaxial tensile load in the longitudinal direction. Under such a load the transverse stress which develops in the load bearing plies is highly tensile due to the large Poisson mismatch between the load bearing plies and the adjacent off-axis plies. The magnitude of the transverse stress at the aforementioned load levels is generally on the same order of magnitude as the transverse strength of the load bearing plies. Consequently, longitudinal cracks may form perpendicular to this stress. Unlike transverse matrix cracks, the longitudinal matrix cracks grow intermittently with cyclic loading due to a high degree of interlaminar constraint.

With matrix cracks present in the laminate, crack coupling and interfacial debonding begin to occur. At free edges additional cracks form in the resin rich interface between the off-axis plies and connect the transverse matrix cracks in adjacent plies. This coupling of matrix cracks creates regions of high interlaminar shear and normal, out-of-plane, stresses. If the magnitude of these interlaminar stresses is sufficient, interfacial debonding, that is edge delamination, may occur.

A similar effect is seen in the interior of the laminate. As longitudinal cracks form and begin to grow they may at some point intersect a transverse matrix crack. At points where transverse and longitudinal matrix cracks intersect out-of-plane stresses
develop. These stresses are usually large enough to facilitate the development of interior delaminations.

### 2.3.3 Delamination

Matrix cracking and delamination are intimately related. This relationship is shown schematically in figure 2.6 for a [0/90]₅ cross-ply laminate. Matrix cracks provide the driving force in initiating both edge and interior delamination. A laminate's susceptibility to delamination is related to the material constituents of the laminate and its stacking sequence. This fact allows the location of delaminations to be predicted by laminated plate theory.

Aside from how they form, the two types of delamination are quite different. Edge delaminations form under constant strain and develop in such a way that they often

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**Figure 2.6:** Schematic representation of the relationship between matrix cracks and delamination
span the entire length of the laminate. Under cyclic loading a delaminated strip begins to grow from the edge of the laminate towards its center. Since delamination prevents the transfer of loads between plies the load carried by the delaminated strip is reduced. This in turn causes an increase in the load carried by the undamaged portion of the laminate. If delamination growth continues until complete delamination occurs, then the load on the entire laminate is reduced. Since edge delaminations form at constant strain this implies that the modulus of the laminate must decrease with increasing delamination size. O'Brien (1982) has shown that the relationship between the modulus of the laminate and the size of the edge delamination is linear. Furthermore, it has been shown that edge delamination is a stable fracture process.

In contrast, interior delaminations form at constant stress and serve to isolate cracked and uncracked plies from one another. This action increases the strain locally in the uncracked plies. This strain concentration often causes the uncracked plies to fracture soon after the formation of interior delaminations. If the interior delaminations grow to span the entire width of the laminate, the local strain increase will lead to an increase in the nominal strain on the laminate. Since interior delaminations form under constant stress, the modulus of the laminate must decrease with increasing delamination size. O'Brien (1985) determined the nature of this relationship and has shown that interior delamination is an unstable fracture process.

2.3.4 Fiber Fracture and Longitudinal Splitting

As previously noted, the fatigue damage process in composite laminates is characterized by a sequence of matrix related damage mechanisms accompanied by fiber breakage. Matrix damage mechanisms are progressive in nature in the sense that as the laminate is subjected to a fatigue load, the ability of the matrix to carry its share
of the load is diminished. In contrast, reinforcing fibers are uneffected by the fatigue load itself. Due to their brittle nature, the fibers fail only when the load to which they are subjected exceeds their fracture strength. In this sense, fibers are impervious to fatigue damage. In the presence of matrix cracks, however, fiber fracture often occurs due to the stress enhancement near the tips of matrix cracks.

In the section on matrix cracking it was noted that the transverse matrix cracks which form in off-axis plies extend through the entire thickness of the ply. This places the crack tips in the interface between adjacent plies. It is here that matrix cracks encounter reinforcing fibers. The stresses near the crack tip are greater than the global stress on the laminate and if they exceed the fracture strength of the fibers breakage will occur. The relationship between matrix cracks and fiber breakage is illustrated by the banded nature of fiber failures in a laminate subjected to fatigue loading. Under such loading, the distance between the bands of failed fibers is approximately equal to the saturation spacing of the matrix cracks predicted by Reifsnider’s shear lag model.

Fiber fracture is experimentally observed to occur throughout the fatigue life of a laminate, but the rate at which the fibers fail is not constant. Fiber fractures accumulate quite rapidly in the early stages of fatigue loading due to the formation of matrix cracks. As loading continues and delaminations begin to form the rate of fiber breakage decreases and tends to approach a saturation level. The reason for this behavior lies in the effect of the delaminations on the laminate. When delaminations form and grow they isolate adjacent plies from one another. In general, the interlaminar stresses which drive delamination are highest between plies with the large differences in their orientations. Thus, the plies which separate are usually the plies with transverse matrix cracks and the load bearing plies. This action segregates the
fibers from the high stresses at the tips of the matrix cracks and hence the rate at which the fibers fail decreases. The effect of fiber breakage on the laminate strength and stiffness is minimal until the final fracture event. This is due to the relatively low number of fiber fractures on a global scale and the ability of a fractured fiber to be reloaded over a critical distance by load transfer between the fiber and matrix.

Aside from transverse matrix cracks, fiber fracture may be associated with longitudinal matrix cracks. The growth of longitudinal cracks in the load bearing plies and the coalescence of delaminations often isolates a volume of fibers in the load bearing plies from the rest of the laminate. The fibers in this volume support the load on the laminate in parallel independent of the rest of the laminate. If fiber fracture occurs in this volume, the net section of the volume is reduced and the volume itself may fail. If such failure does occur a global net section decrease follows. Accumulations of such volume failures may then lead to global failure of the laminate. This phenomenon occurs first on the edges of the laminate and then in its interior. It is known as longitudinal splitting and is shown schematically in figure 2.7.

![Figure 2.7: Schematic representation of longitudinal splitting](image-url)
2.4 Summary

The above discussion has highlighted the key aspects of the fatigue process in both isotropic engineering materials and composite laminates. As evident from this discussion, the mechanics problems involved in the fatigue of composite laminates, and composite materials in general, are much more complicated than those associated with isotropic materials. Understanding these problems and the ability to address their role in the damage accumulation process is of great importance in characterizing the fatigue behavior of composite laminates.
Chapter 3

Fatigue Models

The prime motivation for the study of the fatigue of engineering materials is the desire to understand and predict the behavior of a material or a component under complex loading conditions. The ability to predict such behavior lies in the development of fatigue models. This chapter draws attention to some of the more prominent methods of modeling the fatigue process. In keeping with the structure of chapter 2, this chapter considers the fatigue models for general isotropic engineering materials and composite laminates separately. Furthermore, both deterministic and stochastic models are discussed within this framework.

3.1 Fatigue Models for Isotropic Engineering Materials

3.1.1 Deterministic Models

As mentioned in chapter 2, fatigue is a cumulative damage process and for isotropic engineering materials is often characterized by the growth of a single dominant crack. Consequently, two methods of modeling fatigue in such materials have come about, namely cumulative damage models and fatigue crack growth models.

The foundation for the vast majority of cumulative damage models is the empirical Miner’s rule as proposed by Miner (1945). The rule assumes that a component is subjected to a series of sinusoidal fatigue loads of maximum cyclic stress $\sigma_{\text{max}}$. Under such loading the damage accumulation in the component is equal to the sum of the ratios of the number of stress cycles, $N_i$, in the load history at a particular load level,
to the number of cycles to failure, $N_{fi}$, of the component subjected to that particular load level under constant amplitude loading. Stated mathematically, Miner’s rule is

$$D = \sum_{i=1}^{j} \frac{N_i}{N_{fi}},$$

where $D$ represents the damage in the component and $j$ the number of distinct sinusoidal load levels. Failure is assumed to occur when $D = 1$. The lifetime, $N_f$, of the component can be found from

$$N_f = \sum_{i=1}^{j} \frac{\alpha_i}{N_{fi}},$$

where $\alpha_i$ is the ratio of load cycles at a particular load level to the total number of load cycles in the load history.

It should be noted that Miner’s rule is a linear damage accumulation rule and is load independent. This implies that the damage accruing from a particular group of sinusoidal loads is not dependent on the location of the group in the load history. Miner’s rule takes a cycle equivalent approach to the fatigue process that considers a load history which produces failure and establishes the contribution of each load level to failure based on the number of load cycles in the load history at a given load level. Since the rule is linear, it can also be formulated as a stress equivalent rule which considers the existence of a particular stress level resulting from some applied load which will cause failure in the same number of load cycles as the more complex load history.

The weaknesses of Miner’s rule are obvious. First and foremost, it neglects the effects of load interaction, crack closure, and crack growth retardation and acceleration which are known to contribute to the accumulation of damage. Furthermore, the only material property the model considers is the fatigue life under a given load
level. In addition, the model often gives values of \( D > 1 \) at failure when compared to experimental data.

To circumvent these shortcomings numerous modifications of Miner’s rule have been proposed. These modifications include the introduction of a stress dependent, nonlinear structure into the model (Maroco-Starkey model), the consideration of the fatigue process in nucleation and propagation stages (Grover model), and the introduction of a load interaction rule (Corten-Dolan model) to name a few. A review of these models can be found in Madayag (1969).

Unfortunately, none of these modifications can be relied upon to produce accurate predictions of lifetime under complex load histories. However, Miner’s rule as well as its numerous variations do provide a useful analytical criterion for comparing different designs of a component.

As alluded to earlier, an alternative modeling scheme is to model the fatigue process in terms of the rate of fatigue crack propagation. The physical aspects of fatigue crack growth were presented in section 2.2 and are governed by the principles of fracture mechanics within the limits of the theories of elasticity and plasticity. These principles relate the applied loads on a component to the stress distribution near a crack tip.

In general, fatigue crack growth models have the basic form

\[
\frac{da}{dN} = f(a, \Delta \sigma, C, T, \zeta), \tag{3.3}
\]

where \( a \) is the crack length, \( \Delta \sigma \) is the applied stress range, \( C \) is a material constant, \( T \) is temperature, and \( \zeta \) is the set of all other parameters including specimen geometry which effect the growth of the crack. The most widely accepted crack growth model is the Paris-Erdogan model as proposed by Paris and Erdogan (1963) which has the
form
\[ \frac{da}{dN} = C(\Delta K)^m, \]  
(3.4)

where \( m \) is a material constant generally ranging between \( m = 2 \) and \( m = 4 \) for most metals, and \( \Delta K = K_{max} - K_{min} \) is the stress intensity range. The symbol \( K \) is the stress intensity factor. The stress intensity factor is the coefficient of the singularity term defining the elastic stress field at the tip of a crack. The factor \( K \) can be expressed as
\[ K = Y_0 \sigma \sqrt{\pi a}, \]  
(3.5)

where \( Y = Y(a) \) is a function of both the geometry of the component and the crack itself.

This model, while applicable for moderate fatigue crack growth rates, does not take the sequence of loading into account. Further, crack growth retardation and acceleration caused by load excursions is ignored. Several modifications to this model have been presented and are summarized by Sobczyk and Spencer (1992).

### 3.1.2 Stochastic Models

Due to the shortcomings of the deterministic models, the often random nature of damage accumulation, and the present lack of understanding of the underlying physical mechanisms associated with fatigue, stochastic concepts have also been considered for addressing the fatigue phenomenon. There are two possible means for introducing a probabilistic perspective into fatigue models. The first is to modify existing deterministic models by replacing some or all of the deterministic parameters with stochastic parameters to characterize random effects. Alternatively, the fatigue process can be characterized as an evolutionary stochastic process. The first method is discussed
here in conjunction with the fatigue crack growth model of Paris and Erdogan. The second is propounded in chapter 5 in the context of Markov B-Models.

Since there exists a considerable amount of scatter in fatigue data for most engineering materials, it seems quite appropriate to model the fatigue phenomenon as a stochastic process. Replacing $\Delta K$ in equation 3.4 with its equivalent expression of equation 3.5, allows the Paris-Erdogan model to be rewritten as

$$\frac{da}{dN} = CY(a)^m(\Delta \sigma)^m(\sqrt{\pi}a)^m.$$  \hspace{1cm} (3.6)

Equation 3.6 has the general solution

$$f(a) = C(\Delta \sigma)^mn,$$  \hspace{1cm} (3.7)

where

$$f(a) = \int_{a_0}^{a} \frac{dz}{Y(z)^m(\sqrt{\pi}z)^m}$$  \hspace{1cm} (3.8)

and $a_0$ is the initial crack length. The crack length as a function of the number of load cycles can be found by solving equation 3.7 for $a(n)$.

Madsen et al. (1986) noted that this model can be randomized by allowing $C$ to be a random quantity written as

$$C = \frac{C_1}{C_2(a)},$$  \hspace{1cm} (3.9)

where $C_1$ is a random variable describing variations in the material parameters of different components. The symbol $C_2(a)$ denotes a stationary random process describing variations in the said parameters along the crack path in each component. This yields a differential equation of the form

$$\frac{da}{dN} = \frac{C_1}{C_2(a)}Y(a)^m(\Delta \sigma)^m(\sqrt{\pi}a)^m,$$  \hspace{1cm} (3.10)
whose solution is
\[ f_1(a) = C_1(\Delta \sigma)^m n, \]  
(3.11)

where \( f_1(a) \) has the form,
\[ f_1(a) = \int_{a_0}^a C_2(z) \frac{dz}{Y(z)^m \sqrt{xz}^m}. \]  
(3.12)

The use of such a model has proven quite successful and has been adopted by a number of researchers. As noted by Sobczyk and Spencer (1992), however, the solution of the resulting stochastic differential equations is generally not attainable except for a few simple cases. Further, the probability distributions needed for reliability studies can only be found under certain restrictive assumptions. In addition, such a model is well suited primarily to situations involving a single dominant crack. Such is not the case with composite laminates.

### 3.2 Fatigue Models for Composite Laminates

Due to the inherently complex nature of the damage development process, the aforementioned models fail to provide an acceptable description of damage accumulation in composite laminates. Consequently, more sophisticated models are needed. As of date, the more pertinent modeling procedures available in the literature focus on specific aspects of the damage accumulation process such as matrix cracking, delamination, and fiber fracture. The approaches to modeling these individual damage mechanisms represent a wide variety of disciplines including continuum mechanics, laminated plate theory, and reliability methods. The first two are represented by deterministic models for predicting the residual stiffness of a damaged laminate due to matrix cracking and delamination. The later is represented by a probabilistic method for determining the upper bound on the strength of unidirectional laminates.
The first two methods are used in conjunction with the proposed cumulative damage model. The later is included as a perspective of traditional probabilistic models for cumulative damage in composite laminates.

3.2.1 Deterministic Models

Stiffness Loss Due to Matrix Cracking

Talreja (1987) has presented a continuum mechanics approach to predicting the changes in the stiffness of a composite laminate due to transverse matrix cracking. This model serves to describe the events in the early portions of the damage accumulation process as described in section 2.3 and shown in figure 2.5. The model uses as input the matrix crack density per unit surface area versus the number of load cycles. This is a quantity readily determined from nondestructive testing techniques and knowledge of the loading conditions. As output the model determines the four independent elastic moduli, which define the stiffness of the laminate, as a function of the number of load cycles.

To establish this model consider a two dimensional, multidirectional laminate containing \( j \) sets of oriented crack arrays. Here "array" is taken to mean a ply or group of adjacent plies with the same orientation containing matrix cracks. For each crack array a damage vector \( \mathbf{v}^{(i)} \) for \( i = 1, 2, \ldots, j \) is assigned whose orientation is perpendicular to the cracks in the array themselves. The damage vector has the form

\[
\mathbf{v}^{(i)} = D^{(i)} \hat{n}^{(i)} \quad \text{for} \quad i = 1, 2, \ldots, j, \tag{3.13}
\]

where \( \hat{n}^{(i)} \) is the unit normal to the \( i^{th} \) crack array and \( D^{(i)} \) is the magnitude of \( \mathbf{v}^{(i)} \).

This concept is illustrated in figure 3.1 for a cross ply laminate. For a given array,
Figure 3.1: Illustration of the damage vector concept.

the magnitude of $\mathbf{v}^{(i)}$ is written as

$$D^2 = \mathbf{v}\mathbf{v}^T = \lambda \overline{h} f_c,$$

(3.14)

where $\lambda$ is the crack density per unit surface area of the array, $\overline{l}$ is the average length of the matrix cracks in the array, and $\overline{h}$ is the average height of the matrix cracks in the array. The symbol $f_c$ represents a nondimensional constraint factor due to the finite thickness of the cracked ply or plies which comprise the array and the stiffness of the adjacent constraining plies. Taking only thickness effects into account, $f_c$ is
expressed as

$$f_c = \frac{t_c}{t},$$

(3.15)

where $t$ is the total thickness of the laminate and $t_c$ is the thickness of the crack array.

Due to the nature of matrix cracks as noted in section 2.3.2, approximate values for $\bar{I}$ and $\bar{h}$ can be found. The matrix cracks which form in off-axis plies run parallel to the fibers in the plies and generally span the entire width of the laminate. Therefore, $\bar{I}$, for a given crack array, can be taken as the length of the fibers in the array. Likewise, the height of the cracks is approximately equal to the thickness of the crack array which implies $\bar{h} = t_c$. For example, if one considers a cross ply laminate with a layup of $[0/90_2]_S$ then the height $\bar{h}$ of the transverse matrix cracks in the 90 degree plies is approximately $4h$ due to the four adjacent 90 degree plies of individual thickness $h$.

The damage vector can now be incorporated into the in-plane stress-strain relationship for a two dimensional laminate as

$$\sigma = Q(n)\varepsilon,$$  

(3.16)

where $\sigma$ is the in-plane stress vector, $\varepsilon$ is the in-plane strain vector, and $Q(n)$ is the stiffness tensor of the laminate as a function of the number of load cycles.

The tensor $Q(n)$ can be expanded as the sum of an undamaged stiffness tensor, $Q^{(0)}$, and the damaged stiffness tensors, $Q^{(i)}(n)$, of each crack array representing the stiffness loss due to matrix cracking. Then, $Q(n)$ becomes

$$Q(n) = Q^{(0)} + \sum_{i=1}^{j} Q^{(i)}(n).$$  

(3.17)

For an orthotropic laminate, such as a cross ply or bidirectional angle ply laminate, $Q^{(0)}$ is found from laminated plate theory given the material constituents and stacking
sequence. Specifically,

\[ Q^{(0)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \begin{bmatrix} 2k_1 & k_2 & 0 \\ k_2 & 2k_6 & 0 \\ 0 & 0 & 2k_{10} \end{bmatrix}, \tag{3.18} \]

where \( Q_{ij} \) are the stiffness components of the laminate which can be related to the elastic moduli and the Poisson's ratios of the laminate by the equations

\[
\begin{align*}
E_1 &= \frac{Q_{11}Q_{22} - Q_{12}^2}{Q_{22}} \\
E_2 &= \frac{Q_{11}Q_{22} - Q_{12}^2}{Q_{11}} \\
E_6 &= Q_{66} \\
\nu_{12} &= \frac{Q_{12}}{Q_{22}} \\
\nu_{21} &= \frac{Q_{12}}{Q_{11}}. \tag{3.19} 
\end{align*}
\]

Further, \( Q^{(i)}(n) \) has the form

\[ Q^{(i)}(n) = \begin{bmatrix} (2k_3v_1^2 + 2k_4v_2^2) & (k_{13}v_1^2 + k_{14}v_2^2) & k_5v_1v_2 \\ (k_{13}v_1^2 + k_{14}v_2^2) & (2k_7v_1^2 + 2k_8v_2^2) & k_9v_1v_2 \\ k_5v_1v_2 & k_9v_1v_2 & (2k_{11}v_1^2 + 2k_{12}v_2^2) \end{bmatrix}, \tag{3.20} \]

where \( v_1 \) and \( v_2 \) are the components of \( \mathbf{v} \), and \( k_i \) are material constants which are determined from the initial moduli, the residual moduli at the characteristic damage state, and the matrix crack density at the characteristic damage state. The residual moduli at the characteristic damage state can be found from laminated plate theory using the ply-discount method or experimentally. The saturation crack density is found using Reifsnider's shear lag model or experimentally. From equations 3.17 through 3.20 it can be seen that an initially orthotropic laminate will become generally anisotropic as a result of matrix cracking.
Stiffness Loss Due to Delamination

O'Brien (1982, 1985) has established two models for determining the amount of stiffness loss in graphite/epoxy composite laminates cyclically loaded in tension due to both edge and local delamination. Both models approach the delamination process from an energy related standpoint taking the critical strain energy release rate, $G_c$, as an input. As output the models provide the residual longitudinal modulus for a damaged laminate.

A qualitative description of edge delamination was presented in section 2.3.3. The locations where edge delaminations are likely to occur can be predicted by laminated plate theory by examining the interlaminar normal stresses which develop in the laminate. From laminated plate theory these stresses can be expressed as

$$\sigma_z(z) = \frac{90}{7t^2} \int_z^{t/2} \frac{t}{2} \sigma_2(\zeta)(z - \zeta) d\zeta,$$

(3.21)

where $\sigma_2$ is the applied stress in the longitudinal direction, and $t$ is the total thickness of the laminate. This expression yields the normal interlaminar stresses in the laminate as a function of the distance from the midplane of the laminate. At ply interfaces where $\sigma_z$ has larger values delamination is most probable. Once these locations are known, the total delaminated longitudinal modulus, $E_1^*$, can be found by assuming that the laminate completely separates into $\hat{n}$ sublaminates at the points of highest interlaminar normal stress. Then, $E_1^*$ is determined from a simple rule-of-mixtures analysis as

$$E_1^* = \frac{\sum_{i=1}^{\hat{n}} E_{1i} t_i}{t},$$

(3.22)

where $E_{1i}$ and $t_i$ are the corresponding longitudinal modulus and thickness for each sublamine.
O'Brien has established experimentally that the stiffness loss in composite laminates which exhibit edge delamination is a linear function with respect to the size of the delaminated area. The relationship between the stiffness and the delaminated area is

\[ E_1(n) = (E_1^* - E_1) \frac{A(n)}{A^*} + E_1 \]  

(3.23)

where \( E_1(n) \) is the residual longitudinal modulus, \( A^* \) is the total interfacial area, and \( A(n) \) is the delaminated area. Differentiating \( E_1(n) \) with respect to the number of load cycles, \( n \), yields an expression relating the rate of stiffness loss with the rate of delamination growth. Specifically,

\[ \frac{dE_1(n)}{dn} = \frac{(E_1^* - E_1) dA(n)}{A^*} \]  

(3.24)

O'Brien further established that the onset of delamination can be predicted using the critical strain energy release rate, \( G_c \). The strain energy release rate is the difference between the rate at which work is done and the rate at which elastic strain energy is stored in an elastic body as a flaw propagates. O'Brien found that \( G_c \) is related to nominal strain at the onset of delamination, \( \varepsilon_1^* \), by the equation

\[ G_c = \frac{\varepsilon_1^* t}{2(\tilde{m} - 1)} (E_1 - E_1^*) \]  

(3.25)

The critical strain energy release rate is a laminate property related to the material constituents of the laminate and the stacking sequence of the plies. However, O'Brien's analysis concluded that \( G_c \) may be independent of the ply orientations of the delaminating surface(s). It is clear, that once \( G_c \) is determined for some reference laminate of a given material makeup, it can be utilized to determine the nominal strain at the onset of delamination in other laminates of the same material makeup by equation 3.25. This is feasible since the dependence of the stacking sequence is
taken into account in the term \((E_1 - E_1^*)\). This allows data for the reference laminate to be used to predict the delamination behavior of other laminates.

O'Brien performed a similar analysis on interior delaminations in graphite/epoxy composites. This analysis yields a relationship between the longitudinal modulus of the laminate and the size of the interior delamination, and the ability to predict the onset of interior delaminations.

To begin the analysis the potential sites for interior delaminations must be determined. These locations are generally the interfaces between the off-axis plies where matrix cracks occur and the load-bearing on-axis plies. Once these locations are known, the longitudinal modulus of the locally delaminated cross section, \(E_{1,LD}\), and the corresponding thickness of the cross section, \(t_{LD}\), can be found by laminated plate theory.

Performing an elastic fracture mechanics analysis allows \(E_{1,LD}\) and \(t_{LD}\) to be related to the residual stiffness of the damaged laminate as

\[
E_1(n) = \frac{l}{A^*} \left[ \frac{A_{LD}(n)E_{1,LD}}{a} - \frac{A^*E_1}{(l-a)} \right],
\]

(3.26)

where \(A_{LD}\) is the area of the locally delaminated region and is equal to the product of the width of the laminate and \(t_{LD}\). The symbol \(a\) is the length of the local delamination, and \(l\) is the length of the laminate. Differentiating \(E_1(n)\) with respect to the number of load cycles, \(n\), yields a relationship between the stiffness loss and the rate of growth of the interior delaminations. Specifically,

\[
\frac{dE_1(n)}{dn} = \frac{l}{a} \left( \frac{E_{1,LD}}{A^*} \right) \frac{dA_{LD}(n)}{dn}.
\]

(3.27)
Furthermore, these parameters can be related to the strain energy release rate and the nominal strain on the laminate at the onset of interior delamination as,

\[ G_c = \frac{E_l^2}{2(\hat{m} - 1)} \left( \frac{1}{t_{LD}E_{1,LD}} - \frac{1}{t E_1} \right). \]  \hspace{1cm} (3.28)

These results can be used to predict the onset of interior delaminations in other laminates of the same material constituents.

It should be noted that in the development of these relationships for both edge and interior delaminations $E_1$ was taken as the longitudinal modulus for an undamaged laminate as predicted by laminated plate theory. However, matrix cracking in the off-axis plies often precedes delamination in the development of fatigue damage. To account for this prior damage, $E_1$ can simply be replaced by the reduced longitudinal modulus at the characteristic damage state as predicted by either Talreja’s model or by the ply discount scheme as saturation of matrix cracks generally occurs before delaminations begin to develop.

### 3.2.2 Stochastic Models

Relatively few stochastic models have been presented for general composite laminates for either static or dynamic loading. The majority of stochastic models have been concerned either with the static strength alone of unidirectional laminates or with transformation of fatigue data into equivalent static strength data. In each case the static strength data are analyzed by assuming that they follow some given probability distribution; in general it is a Weibull distribution. The first of these two approaches is exemplified by the models of Harlow (1979), Harlow and Phoenix (1979), Phoenix (1979), and Tenn (1981), while the later is illustrated by the model of Sendeckyj (1981). Attention in this section will be focused on the models of Harlow and Phoenix.
The model presented by Harlow and Phoenix is a reliability based model for the strength of unidirectional laminates. It utilizes a Weibull weakest link rule for brittle materials in conjunction with the classic chain of bundles model. A unidirectional composite can be represented as a chain of bundles as shown in figure 3.2, where \( n \) parallel fibers compose a chain of \( \tilde{n} \) bundles each with \( \tilde{n} \) fibers and a length \( \delta \). This gives the total length of the composite as \( l = \tilde{n}\delta \) where \( \delta \) is the ineffective length. As noted by Courtney (1990), \( \delta \) represents the smallest length over which the matrix reloads a discontinuous fiber to the value of the load carried by a continuous fiber. This reloading takes place by shear transfer in the fiber/matrix interface. For resin type materials \( \delta \) is found to be

\[
\delta = \frac{\sigma_f(e)}{4\mu\sigma_r},
\]

(3.29)

where \( \sigma_f(e) \) is the axial fiber stress at some given composite strain, \( \mu \) is the friction coefficient between the fiber and the matrix, and \( \sigma_r \) is the residual stress in the laminate due in part to thermal mismatch and chemical shrinkage.

![Figure 3.2: Chain of bundles model](image)
The strength of the fibers which comprise the bundles is assumed to follow a two
parameter Weibull distribution whose cumulative distribution function is given by
the equation
\[ F(\sigma) = 1 - e^{-\left(\frac{\sigma}{\sigma_0}\right)^{\rho_0}} \quad \text{for} \quad \sigma \geq 0, \quad (3.30) \]
where \( \sigma_0 \) and \( \rho_0 \) are respectively the scale and shape parameters of the distribution.
When fibers fail within a given bundle, the remaining fibers share the applied load
on the bundle according to some specified load rule. The cumulative distribution
function of the bundle strength, \( G_{\tilde{n}}(\sigma) \), is then determined from \( F(\sigma) \) and the load
sharing rule. Once \( G_{\tilde{n}}(\sigma) \) is known, the cumulative distribution function for the
strength of the chain can be expressed as
\[ H_{\tilde{n},\tilde{n}}(\sigma) = 1 - [1 - G_{\tilde{n}}(\sigma)]^{\tilde{n}} \quad \text{for} \quad \sigma \geq 0, \quad (3.31) \]
by relying on the weakest link rule. While \( H_{\tilde{n},\tilde{n}}(\sigma) \) and \( G_{\tilde{n}}(\sigma) \) are readily related,
the computation of \( G_{\tilde{n}}(\sigma) \) proves quite difficult.

Harlow and Phoenix have determined an upper bound on the probability of failure
of a unidirectional laminate based on the occurrence of at least two adjacent failed
fibers which represents a necessary but not sufficient requirement for laminate failure.
Their results are based on a load sharing rule within a bundle that assumes that
only fibers adjacent to failed fibers are subjected to increased loads. Given these
assumptions, the upper bound on \( G_{\tilde{n}}(\sigma) \) denoted by \( G^*_{\tilde{n}}(\sigma) \) is
\[ G^*_{\tilde{n}}(\sigma) = \min \left\{ 1, P_{\tilde{n}}(\sigma) + 1 - \left[ 1 - F(\sigma) + F(\sigma) \left[ \frac{1 - F(K^*_1 \sigma)}{1 - F(\sigma)} \right]^{27\tilde{n}} \right] \right\}, \quad (3.32) \]
where \( P_{\tilde{n}}(\sigma) \) is the cumulative distribution function of the probability that two or more
fibers with strength \( \leq \sigma \) exist side by side in a planar array of \( \tilde{n} \) fibers. The symbol
\( K^*_1 \) represents a load concentration factor determined from the load sharing rule.
The cumulative distribution function $P_n(\sigma)$ is related to the cumulative distribution function $F(\sigma)$ as

$$P_n(\sigma) = 1 - \frac{[1 - F(\sigma)]^{\hat{a}}}{2^{\hat{a}+2} \alpha(\sigma)} \left\{ \left[ 1 + \alpha(\sigma) \right]^{\hat{a}+2} - \left[ 1 - \alpha(\sigma) \right]^{\hat{a}+2} \right\},$$

(3.33)

where $\alpha(\sigma) = \sqrt{1 + 4F(\sigma)/[1 - F(\sigma)]}$. Substituting $G_n(\sigma)$ into equation 3.31 then yields an upper bound $H_{m,n}^*(\sigma)$ on the cumulative distribution distribution of the strength of the unidirectional laminate $H_{m,n}^*(\sigma)$ as

$$H_{m,n}^*(\sigma) = 1 - \{1 - G_n^*(\sigma)\}^{\hat{a}} \quad \text{for} \quad \sigma \geq 0.$$  

(3.34)

Analysis of the asymptotic behavior of these bounds suggests that as the size of the laminate grows, the bound on the strength of the laminate approaches a Weibull distribution. The shape parameter of the distribution is approximately double that of the individual fibers, and the scale parameter is somewhat less than those of the fibers. This approach could possibly be modified to address the fatigue of a unidirectional laminate by considering the effect on the local load sharing rule by the growth of a local failure zone caused by fiber failure and other damage mechanisms. Furthermore, the approach could be possibly extended to multidirectional laminates by considering the constraint effect of the off-axis plies on the damage development in the load-bearing on-axis plies.

### 3.3 Summary

This chapter has presented an overview of some of the more common models for studying the fatigue of engineering materials. Models have been presented for both isotropic materials and composite laminates. Those models for composite laminates pertinent to the present study have been discussed in some detail. The models relating
the stiffness loss due to matrix cracking and delamination will be used at a later stage to transform data obtained from nondestructive testing techniques into data on stiffness loss. These data will serve as input into the proposed cumulative damage model.
Chapter 4

Concepts from Discrete Probability Theory

As noted earlier, the model presented herein is based on an embedded discrete time, finite state Markov chain. To facilitate the understanding of this model some fundamental concepts from discrete probability theory are reviewed. This chapter serves as a background of those concepts pertinent to the present study.

As a starting point the notion of a random event and the probability associated with it is discussed. This concept is then naturally applied to a random variable and finally to a random process. In addition, the important idea of a probability generating function and how it relates to the probabilistic description of a random variable is discussed. A thorough review of these topics can be found in a number of references including Benjamin and Cornell (1970), Ang and Tang (1975), and Soong and Grigoriu (1993). A discussion of how these topics are related to Markov chains can be found in Isaacson and Madsen (1976) and Bogdanoff and Kozin (1985).

4.1 Discrete Random Variables

To discuss the concept of random event, consider an experiment consisting of a number of trials, for example rolling a die a given number of times. Associated with each trial, or roll, is an outcome, or event, which in this case is an integer representing the number of “dots” appearing on the top surface of the die. Assuming that the die is not loaded, the outcome of each roll is random and constitutes an elementary random event.
The set of all possible outcomes is known as the sample space and is denoted by \( \Omega \), with an arbitrary element of \( \Omega \) denoted by \( \omega \). If the number of elements in \( \Omega \) is finite or countably infinite, then \( \Omega \) is called a discrete sample space. If the number of elements is infinite, then \( \Omega \) is known as a continuous sample space. Attention in this chapter is restricted to \( \Omega \) being discrete.

To elucidate these ideas, consider an experiment which consists of rolling two standard six-sided dice. Let the outcomes of this experiment be the sum of the number of “dots” on the top faces of the two dice. The sample space for this experiment is

\[
\Omega = \{ \omega : \omega = d_1 + d_2 \text{ where } d_i \in \{1,2,\ldots,6\} \text{ for } i = 1,2 \},
\]

where \( d_i \) is the realization of each die. Clearly for this experiment there are 11 possible outcomes corresponding to the integers 2, 3, \ldots, 12. Thus, \( \Omega \) is a discrete sample space whose elements \( \omega_i \) are given in table 4.1.

![Table 4.1: Possible outcomes of throwing two dice.](image)
probability measure $p_i$ given by the equation

$$p_i = P[\omega_i] \quad \text{and} \quad \sum_{\omega_i \in \Omega} P[\omega_i] = 1. \quad (4.1)$$

The pair $(\Omega, P)$ consisting of the sample space $\Omega$ and the probability measure $P$ on $\Omega$ is known as a probability space.

For the dice rolling example, the probability of each $\omega_i \in \Omega$ can be found using the concept of relative frequency. The relative frequency of an outcome is simply the ratio of the number of times that a particular outcome occurs to the total number of possible outcomes. For example, there are 36 possible ordered pairs $(d_1, d_2)$. The probability that $d_1 + d_2 = \omega_i$ for $\omega_i = 2, 3, \ldots, 12$ is simply the ratio of pairs $(d_1, d_2)$ such that $d_1 + d_2 = \omega_i$ to the total number of pairs. Thus, if one were interested in the probability of rolling a 7, that is $d_1 + d_2 = 7$, then from table 4.1 one would find

$$P[\omega_i = 7] = \frac{6}{36} = \frac{1}{6}.$$ 

If $(\Omega, P)$ is a probability space then define a random event to be any subset of $\Omega$ which is denoted by $E = \{\omega_1, \omega_2, \ldots, \omega_j\}$ with $j = 1$ denoting the elementary random event. The probability of a random event is thus,

$$P[E] = P[\{\omega_1, \omega_2, \ldots, \omega_j\}]. \quad (4.2)$$

Furthermore, if the elementary events which comprise the random event are independent, equation 4.2 simplifies to

$$P[E] = \prod_{i=1}^{j} P[\omega_i]. \quad (4.3)$$

Returning to the dice rolling example, it is clear that the outcome of one roll of the dice is independent of the outcome of any other roll. Thus, if one were interested
in the probability of rolling a 3 on the first roll of the dice and a 7 on the second roll, the probability of the random event \( E = (\omega_1 = 3, \omega_2 = 7) \) is obtained using the multiplication rule of equation 4.3 as

\[
P[E] = P[\omega_1 = 3, \omega_2 = 7] = P[\omega_1 = 3]P[\omega_2 = 7] = \frac{2}{36} \cdot \frac{6}{36} = \frac{1}{18} \cdot \frac{1}{6} = \frac{1}{108}.
\]

The elements of the sample space may or may not be numbers. To associate numbers with the elements of the given sample space a random variable \( D(\omega) \) is defined. A random variable simply maps the elements of the sample space onto the real line and is mathematically stated as,

\[
D : \Omega \mapsto \mathbb{R}.
\]  \hspace{1cm} (4.4)

\( D \) is a discrete random variable if it maps the elements of \( \Omega \) into a finite or countably infinite set of values on \( \mathbb{R} \). If \( D \) maps the elements of \( \Omega \) into an infinite set of values on \( \mathbb{R} \), \( D \) is said to be a continuous random variable. Attention is restricted here to discrete random variables. The rolling of a pair of dice is an example of a discrete random variable in which the sum of the number of "dots" on the top sides of the two dice is mapped to an integer on \( \mathbb{R} \) corresponding to the value of sum.

### 4.2 Discrete Probability Distributions

If the range of \( D \) is defined as \( S \), equation 4.4 can be written as \( D : \Omega \mapsto S \). If the probability measure \( P \) can be mapped into a new probability measure \( P^* \) a new probability space \( (S, P^*) \) can be constructed from \((\Omega, P)\). The mapping between \( P \) and \( P^* \) is the probability mass function of the random variable \( D \). The probability mass function is defined as

\[
P^*[s_i] = P[\omega : D(\omega) = s_i], \hspace{1cm} (4.5)
\]
where \( s_i \in S \). Closely related to the probability mass function is the cumulative distribution function defined as

\[
F^*[s] = P[\omega : D(\omega) \leq s] = \sum_{s_i \leq s} P^*[s_i].
\]  

Equations 4.5 and 4.6 are generally given in the following respective forms in the literature

\[
P_D(d_i) = P[D = d_i] \quad F_D(d) = P[D \leq d] = \sum_{d_i \leq d} P_D(d_i),
\]

where \( d_i = s_i \in S \) and \( D(\omega) = D \) for convenience. Thus the random variable \( D \) is completely defined given either \( P_D(d_i) \) or \( F_D(d) \).

The probability mass function and cumulative distribution function can be visualized graphically by plotting the ordinate \( P_D(d) \) and \( F_D(d) \) respectively versus the abscissa \( d \). This is shown in figures 4.1 and 4.2 for the dice rolling example.

### 4.3 Moments of Discrete Random Variables

As noted above, the probability mass function or cumulative distribution function of a random variable completely characterizes its behavior. An alternative approach to characterizing a random variable is through its moments. The moments of a random variable are indicators of the dominant features of the behavior of the random variable. These indicators are weighted averages of certain functions of a random variable where the weight is the probability mass function of the random variable. These weighted averages are known as expectations of the random variable.
**Figure 4.1**: Probability mass function for the dice rolling example

**Figure 4.2**: Cumulative distribution function for the dice rolling example
For a random variable $D$ with a probability mass function $P_D(d_i)$ the expectation operation is defined as

$$E[g(D)] = \sum_{i=-\infty}^{\infty} g(d_i) P_D(d_i),$$

(4.8)

where $g(D)$ is an arbitrary discrete function for which the sum exists. For the case when $g(D)$ has the form $g(D) = D^n$ equation 4.8 becomes

$$m_n = E[D^n] = \sum_{i=-\infty}^{\infty} d_i^n P_D(d_i),$$

(4.9)

and is known as the $n^{th}$ moment of $D$. For $n = 1$ equation 4.9 is the mean value of $D$ often denoted as $m_1 = E[D]$. Another important case arises when $g(D) = (D - m_1)^n$. When $g(D)$ has this form equation 4.8 becomes

$$k_n = E[(D - m_1)^n] = \sum_{i=-\infty}^{\infty} (d_i - m_1)^n P_D(d_i),$$

(4.10)

and is known as the $n^{th}$ central moment of $D$. Note for $n = 1$, $k_1 = 0$ and for $n = 2$, $k_2 = \text{VAR}[D] = \text{SDD}^2[D]$ where $\text{VAR}[D]$ and $\text{SDD}[D]$ are the variance and standard deviation of $D$ respectively. Clearly, the moments describe the distribution of $D$ about the origin, while the central moments characterize the distribution with respect to the mean. For the case of zero mean the moment and central moment expressions are equivalent.

An alternative method for generating the moments of a random variable is to consider its probability generating function. To illustrate this idea, let $D$ be a random variable whose range space $S$ is the set of nonnegative integers, and whose probability mass function is given by the equation

$$P_D(d_i) = P[D = i] = p_i,$$

(4.11)
where \( p_i \geq 0 \) and \( \sum_i p_i = 1 \) for \( i = 0, 1, \ldots \) With \( D \) described in this manner, the probability generating function of \( D \) is

\[
P_D(z) = E \left[ z^D \right] = \sum_{i=0}^{\infty} p_i z^i. \tag{4.12}
\]

Comparing this expression with equation 4.8 implies that the probability generating function of \( D \) is the expected value of \( g(D) = z^D \). Further examination of equation 4.12 reveals that the probability generating function of \( D \) is nothing more than the unilateral \( z \)-transform of the probability mass function of \( D \).

Invoking some of the properties of the \( z \)-transform, allows some fundamental results to be derived. Namely if the moments \( m_n \) of \( D \) exist, differentiating the probability generating function of \( D \) with respect to \( z \) yields

\[
\frac{d^n P_D(z)}{dz^n} = P_D^{(n)}(z) = E \left[ (D(D-1) \cdots (D-(n-1)) z^{D-n} \right]. \tag{4.13}
\]

Now evaluating \( P_D^{(n)}(z) \) at \( z = 1 \), gives

\[
P_D^{(n)}(z) \bigg|_{z=1} = E \left[ D(D-1) \cdots (D-(n-1)) \right]
= E \left[ D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_1 D \right]. \tag{4.14}
\]

Noting that the expectation operator is a linear operator, equation 4.14 allows for the \( n^{th} \) moment of \( D \) to be evaluated using the probability generating function of \( D \). For the case when \( n = 1 \), equation 4.14 reduces to

\[
P_D^{(1)}(z) \bigg|_{z=1} = E[D] = m_1. \tag{4.15}
\]

Similarly for \( n = 2 \)

\[
P_D^{(2)}(z) \bigg|_{z=1} = E[D(D-1)] = E \left[ D^2 - D \right] = E[D^2] - E[D] = m_2 - m_1, \tag{4.16}
\]
and thus

\[ m_2 = P_D^{(2)}(z)|_{z=1} + P_D^{(1)}(z)|_{z=1}. \]  

(4.17)

Alternatively, one can find the probability mass function of \( D \) by expanding the probability generating function of \( D \) in a Taylor series about the origin. Specifically,

\[ P_D(z) = P_D(0) + P_D^{(1)}(0)z + \frac{1}{2!}P_D^{(2)}(0)z^2 + \cdots = \sum_{i=0}^{\infty} P_D^{(i)}(0)\frac{z^i}{i!}, \]

(4.18)

which when compared to equation 4.12 implies that

\[ P_D(d_i) = P[D = d_i] = p_i = \frac{1}{i!} P_D^{(i)}(0). \]

(4.19)

From equations 4.12, 4.14, and 4.19 the relationship between the probability generating function of a random variable and the moments and probability mass function of the random variable is established. Specifically, the moments of a random variable, as well as the probability mass function of a random variable, can be found by differentiating the probability generating function.

### 4.4 Discrete Random Processes

In the previous sections the concept of utilizing a random variable to model the outcome or set of outcomes from an experiment was introduced. Since the random variable maps the outcome to the real line each outcome could then be characterized by a simple real number. The time at which the outcome occurred was neglected. Often, however, it is desirable to know the outcome of an experiment with respect to time. Thus, a second parameter, time, must be introduced into the stochastic model of the experiment. This leads to the concept of a random process \( D(\omega,t) \).

Consider an experiment in which the outcome is modeled by a random variable with a finite range space \( S \) denoted by the set of nonnegative integers. Now suppose
the experiment is conducted \( X \) times and that the outcome of the experiment is observed at discrete points in time indexed by the set of nonnegative integers. For example, consider that the elastic modulus of a component in a constant amplitude fatigue test is measured at intervals of a given number of load cycles until failure occurs. As one would expect, the realization of each experiment may vary but if the experiments are tightly controlled the realizations may be similar. The collection of the \( X \) realizations from the experiment constitutes a discrete random process \( D(x) \). This concept is illustrated in figure 4.3. Each realization \( d_j(x) \) of \( D(x) \) for \( j = 1, 2, \ldots, X \) is known as a sample function. For a fixed \( x \) the outcomes of the \( X \) experiments form a random variable. Thus, a random process can be viewed as either an ensemble of random variables or an ensemble of sample functions. An important aspect of random processes which plays a key role in several models is the idea of stationarity. Loosely speaking, a random process is said to be stationary if for each

\[ d_j(x) \]

\[ d_2(x) \]

\[ d_3(x) \]

\[ d_1(x) \]

\[ x_i \]

\[ x \]

**Figure 4.3:** Example of a discrete random process.
fixed $x$ the statistics, such as mean and variance, of the random variable denoting the outcome of the $X$ experiments are invariant. It is nonstationary otherwise. The importance of this concept will become evident in chapter 5.

4.5 Summary

In this chapter concepts from discrete probability theory as applied to random variables and random processes have been reviewed. These concepts are required for the development of a new cumulative damage model for composite laminates which is discussed in the ensuing chapters.
Chapter 5

Markov B-Models

The notion of a discrete random process was introduced in the previous chapter. In this chapter attention will focus on a specific class of discrete random processes known as Markov chains. The basic properties of Markov chains will be discussed along with a cumulative damage model, known as a Markov B-model, based on the theory of Markov chains. The discussion of this model will center on understanding and estimating its fundamental parameters in both a stationary and a nonstationary context.

5.1 Markov Chains

The theory of Markov chains forms the basis of the cumulative damage model presented herein. The theory is well developed and has been applied to many random phenomena. The literature on Markov chain theory is both rich in content and number and it provides a sound foundation on which to build the proposed cumulative damage model. This section focuses on aspects of Markov chains which pertain to the present study. A thorough introduction to Markov chain theory can be found in references such as Isaacson and Madsen (1976) and Ross (1983).

Markov chains are a class of random processes characterized by certain restrictions on the random process itself. The first of these restrictions is that the process is considered to be a discrete time process indexed on the set on nonnegative integers. The second restriction is that the process has a finite or countably infinite state
space. Finally, the process must satisfy the Markov property which will be explained momentarily.

To illustrate these ideas consider an experiment consisting of a sequence of \( n \) trials whose outcomes, or states, are the random events \( E_1, E_2, \ldots, E_b \) which are mutually exclusive and collectively exhaustive. Let the outcome of each trial be modeled by a random variable \( D \). Clearly, the events \( E_i \) for \( i = 1, 2, \ldots, b \) constitute a partition of the sample space \( \Omega \) of \( D \), and thus only one event occurs for any given trial. If multiple trials are performed, then \( D \) is a function of the number of trials, \( n \), and thus constitutes a random process \( D(n) \).

To initiate the experiment some knowledge of the probability of being in each state \( E_i \) initially is required. Let these probabilities be given by \( P[E_i] = \pi_i \). They can be written in vector form by defining the \((1 \times b)\) row vector \( \mathbf{p}_0 \) as follows

\[
\mathbf{p}_0 = [P[E_{i_0}], P[E_{i_2}], \ldots, P[E_{i_b}] = [\pi_1, \pi_2, \ldots, \pi_b]. \quad (5.1)
\]

Now suppose interest is centered on finding the probability of being in some specific state after the \( n^{th} \) trial. If the outcome of the sequence of \( n \) trials is denoted by \( E_{i_0}, E_{i_1}, \ldots, E_{i_n} \), the probability of \( E_{i_n} \) can be found from

\[
P[E_{i_n}] = \Sigma_{i_0} \Sigma_{i_1} \cdots \Sigma_{i_{n-1}} P[E_{i_0}E_{i_1} \ldots E_{i_n}]. \quad (5.2)
\]

The right hand side of equation 5.2 can be readily evaluated if the events \( E_i \) are statistically independent which implies that the individual outcomes are not related. Alternatively, the right hand side of equation 5.2 can be evaluated if the \( E_i \) satisfy the Markov property. That is, if the sequence of trials is Markovian, the probability of each event in the sequence is dependent only on the probability of the event itself and the probability of the last known event in the sequence. Mathematically stated
the Markov property implies that

\[ P[E_{i_0} E_{i_1} \ldots E_{i_n}] = P[E_{i_0}] P[E_{i_1} \mid E_{i_0}] P[E_{i_2} \mid E_{i_1}] \ldots P[E_{i_n} \mid E_{i_{n-1}}]. \]  

(5.3)

If the conditional probabilities \( P[E_{ij} \mid E_{ij-1}] \), also known as transition probabilities, are denoted as

\[ P[E_{ij} \mid E_{ij-1}] = p_{ij-1ij}, \]

(5.4)

and are assumed to be constant, the probability of a specific sequence of \( n \) trials is given by the equation

\[ P[E_{i_0} E_{i_1} \ldots E_{i_n}] = p_{i_0} p_{i_0i_1} \ldots p_{i_{n-1}i_n}. \]

(5.5)

Substituting this equation into equation 5.2 gives the probability for a specific event after \( n \) trials as

\[ P[E_{i_n}] = \sum_{i_0} \sum_{i_1} \ldots \sum_{i_{n-1}} p_{i_0} p_{i_0i_1} \ldots p_{i_{n-1}i_n}. \]

(5.6)

Clearly, if there are \( b \) possible outcomes in the experiment, there are \( b^2 \) possible transition probabilities. These transition probabilities can be conveniently written in matrix form as

\[ P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1b} \\ p_{21} & p_{22} & \cdots & p_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ p_{b1} & p_{b2} & \cdots & p_{bb} \end{bmatrix}, \]

(5.7)

where \( P \) is a \((b \times b)\) matrix known as the probability transition matrix.

In a similar manner as in equation 5.5 let the probability of an event \( E_i \) after the \( n^{th} \) trial be written as a \((1 \times b)\) row vector of the form

\[ \mathbf{p}_n = [p_n(1), p_n(2), \ldots, p_n(b)]. \]

(5.8)
With this notation, equation 5.6 reduces to matrix multiplication and can be written in the general form

$$P_n = P_0 P^n,$$  \hspace{1cm} (5.9)

where $P[E_{i_n}]$ is the $i^{th}$ component of the vector $P_n$.

The structure of the probability transition matrix is of direct interest in the formulation of the proposed cumulative damage model. The elements $p_{ij}$ of the probability transition matrix represent the probability of the random process $D(n)$ having a realization $E_j$ after a trial, given that it had a realization $E_i$ before the trial. Thus, a fully populated probability transition matrix describes a truly random Markov chain in the sense that any state may be obtained from any other state after a given trial.

Consider now the case where the probability transition matrix is nonzero only on the main diagonal and the superdiagonal. Let the nonzero entries be denoted by $p_i$ and $q_i$, respectively. The probability transition matrix then has the form

$$P = \begin{bmatrix} p_1 & q_1 & 0 & \ldots & \ldots & 0 \\
0 & p_2 & q_2 & 0 & \ldots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & \ldots & p_{b-1} & q_{b-1} \\
0 & \ldots & \ldots & \ldots & 0 & 1 \end{bmatrix}. \quad (5.10)$$

This matrix implies that after a given trial, the model can only occupy the same state or the next higher state. A model with such a probability transition matrix can be visualized by the flow diagram as shown in figure 5.1. In this figure, the states 1 through $(b-1)$ are referred to as transient states since the model can both enter and exit these states. In contrast, state $b$ is known as an absorbing state since once it is
entered in can not be vacated. Such a probability transition matrix is referred to as a "unit-jump" probability transition matrix.

5.2 Markov B-Models

The Markov B-Model discussed in this section is an extension of Markov chain theory to the physical process of damage accumulation. This section reviews some of the more fundamental concepts associated with this model. A much more thorough description of these aspects, as well as a discussion of the model's finer points can be found in references by Bogdanoff (1978a, 1978b), Bogdanoff and Krieger (1978), Bogdanoff and Kozin (1980), and in particular Bogdanoff and Kozin (1985).

Cumulative damage refers to the irreversible accumulation of damage in a component under time varying loads. This damage may be manifested by features such as loss of material (wear), propagation of a flaw, or loss of strength or stiffness. In each case some physical property deteriorates with the loading until some critical level of damage accumulation occurs at which point the component is retired or fails. Monitoring this physical property with time, or the number of load cycles for the case of fatigue loading, results in a realization of the damage accumulation process. If the cumulative damage process exhibits random behavior, each realization can be considered as a sample function of the cumulative damage process. From a collection
of sample functions empirical statistics for the cumulative damage process can be found.

It is from this sample function point of view that the Markov B-Model of cumulative damage has been formulated. This model attempts to describe the probabilistic distribution of damage in a component as a function of time, or load cycles, by assuming that the cumulative damage process is an evolutionary stochastic process. This model is phenomenological in nature requiring the parameters of the model to be determined from numerical data. The primary advantages of the model are summarized by Madsen (1986) as follows:

1. The model is flexible and can be used for almost all models of continuous damage accumulation problems.

2. It is possible to separately assess the major sources of uncertainty in manufacturing standards, severity and order of loading, specification of failure, and the effect of inspection and replacement strategies.

3. The model is simple to use and statistics for random variables such as lifetime and state of damage after a given time can be readily determined.

4. The model is well suited to suggest further test programs for its verification.

The physical situation which the model attempts to describe is as follows. A component is subjected to some sort of repetitive or cyclic loading which is possibly random in nature. The time over which the load acts is discretized and is measured in units of duty cycles. A duty cycle represents some repetitive period of operation in which damage can accumulate and could be a specific number of load cycles, revolutions, or hours of operation. Damage is assumed to monotonically increase
with the number of duty cycles and takes on discrete values (states). The initial state of damage and the state of damage at failure may be random as well. The level of damage is considered only at the end of each duty cycle and is assumed to depend only on the severity of the duty cycle itself and the level of damage prior to the duty cycle. These assumptions are the Markov assumptions and such a cumulative damage process can be modeled as an embedded discrete time, finite state Markov chain.

5.2.1 Stationary B-Models

Basic Structure

To examine the basic structure of the model, consider the case when the damage in a component is discretized into a finite number of damage states indexed on the non-negative set of integers $i = 1, 2, \ldots, b$, with the magnitude of the damage increasing as the index of the damage state. Furthermore, assume that during any given duty cycle the level of damage in the component can remain unchanged or increase by one unit to the next higher damage level culminating in failure when damage state $b$ is attained. Under such conditions a unit-jump Markov B-Model is obtained. For such a model the probability mass function of the damage distribution is completely determined by the number and severity of the duty cycles and the initial distribution of damage in the component before the cumulative damage process began.

At this stage assume that the duty cycles are repetitive and of constant severity thus defining a stationary cumulative damage process. For the unit-jump model this implies that the $(b \times b)$ probability transition matrix which defines the duty cycles
has the form

\[
P = \begin{bmatrix}
p_1 & q_1 & 0 & \ldots & \ldots & 0 \\
0 & p_2 & q_2 & 0 & \ldots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & p_{b-1} & q_{b-1} \\
0 & \ldots & \ldots & \ldots & 0 & 1
\end{bmatrix},
\]

(5.11)

with \( p_i \geq 0, q_i \geq 0, \) and \( p_i + q_i = 1 \). The banded and upper triangular form of the probability transition matrix reflects the assumption that the damage level in a component may "stay" at the current level or "jump" to the next higher level during a duty cycle. Further, the damage level is monotonically increasing. The "stay" and "jump" probabilities are given by \( p_i \) and \( q_i \), respectively. The magnitude of these probabilities is a measure of the severity of the duty cycle with larger "jump" probabilities indicating a greater probability for the damage level to increase during a duty cycle.

Let the initial distribution of damage be represented by a \((1 \times b)\) row vector

\[
P_0 = [\pi_1, \pi_2, \ldots, \pi_b],
\]

(5.12)

with \( \pi_i \geq 0 \) and \( \sum_{i=1}^{b} \pi_i = 1 \) where \( \pi_i \) denotes the probability that damage was in state \( i \) before the process began. Furthermore, express the distribution of damage after \( x \) duty cycles in a similar form

\[
P_x = [p_x(1), p_x(2), \ldots, p_x(b)],
\]

(5.13)

with \( p_x(i) \geq 0 \) and \( \sum_{i=1}^{b} p_x(i) = 1 \) where \( p_x(i) \) is the probability that damage is in state \( i \). Clearly \( P_0 \) and \( P_x \) constitute probability mass functions. It follows then from
the results of the preceding section on Markov chains that

$$P_x = P_0 P^x. \quad (5.14)$$

Given $P_x$, statistical information on the state of damage, the lifetime of the component, and other quantities of interest may be found. If the probability of damage, denoted by the random variable $D_x$, of being in state $i$ after $x$ duty cycles is given by the equation

$$P[D_x = i] = p_x(i) \quad \text{for} \quad i = 1, 2, \ldots, b, \quad (5.15)$$

then the results of chapter 4 allow the cumulative distribution function of damage as well as its second order moment data, to be expressed as

$$F_D(j; x) = P[D_x \leq j] = \sum_{i=1}^{j} p_x(i)$$

$$E[D_x] = \sum_{i=1}^{b} i p_x(i) \quad (5.16)$$

$$\text{VAR}[D_x] = \sum_{i=1}^{b} i^2 p_x(i) - E^2[D_x].$$

Information on the time to failure at state $b$, denoted by the random variable $W_b$, can be found in a similar fashion. The cumulative distribution function of the time to failure is

$$F_W(x; b) = P[W_b \leq x] = p_x(b) \quad \text{for} \quad x = 1, 2, \ldots \quad (5.17)$$

If the reliability function, that is the probability that failure does not occur after $x$ duty cycles, is defined as

$$\bar{F}_W(x; b) = 1 - F_W(x; b), \quad (5.18)$$

then the mean and variance of the time to failure are given by the equation

$$E[W_b] = \sum_{x=1}^{\infty} \bar{F}_W(x; b)$$

$$\text{VAR}[W_b] = 2 \sum_{x=1}^{\infty} x \bar{F}_W(x; b) + E[W_b] - E^2[W_b]. \quad (5.19)$$
Similar results can be found analytically for the time to failure from any other state using the notion of probability generating functions as discussed in chapter 4. These results are derived in Appendix A and are used to determine the parameters of the Markov B-Model.

**Inspection and Replacement**

Inspection and replacement procedures are designed to insure the reliable performance of a component. However, the effectiveness of such procedures is dependent upon the inspection times, the quality of the inspections, the replacement policy, and the safety requirements for the component. This section examines how inspection and replacement policies can be incorporated into the Markov B-Model.

For illustration, consider a stationary cumulative damage process as defined by equation 5.14, where inspection is scheduled to take place after $x_1$ duty cycles. Let the quality of this inspection be described by the $(1 \times b)$ row vector $\tau$

$$\tau = [\tau_1, \tau_2, \ldots, \tau_b], \quad (5.20)$$

where $\tau_i$ denotes the probability of detecting damage given that damage is in state $i$.

If state $j$ is the smallest level of damage which can be detected

$$\tau_i = \begin{cases} 0 & \text{for } i = 1, 2, \ldots, j - 1 \\ > 0 & \text{for } i = j, j + 1, \ldots, b. \end{cases} \quad (5.21)$$

Note that $\tau$ does not constitute a probability mass function since its components do not necessarily sum to unity.

If inspection occurs after $x_1$ duty cycles, the state of damage at this time is given by equation 5.14 upon replacing $x$ with $x_1$. Element by element multiplication of $\tau$
and $p_{x_1}$ then yields a new vector whose components $\tau_i p_{x_1}(i)$ are the probabilities that damage is in state $i$ and that it is detected.

Consider the case where a number of identical components are placed into service at the same time and are subjected to the same loading environment. Now assume that state $j$ represents the highest acceptable level of damage in a component at inspection time $x_1$. Further, assume that a component with a damage level greater than the acceptable level will be replaced by a new component. Thus, the fraction of components requiring replacement is then

$$p_r^{(1)} = \sum_{i=j}^b \tau_i p_{x_1}(i). \quad (5.22)$$

However, due to the variability in inspection quality the actual fraction of components which is replaced is given by the equation

$$p_r^{(1)} = \sum_{i=j}^b \tau_i p_{x_1}(i). \quad (5.23)$$

The fraction of components which replace the unsatisfactory components themselves have an initial distribution of damage with elements $\pi_i^{(1)}$. Given this distribution an "new" initial distribution of damage after the inspection can be found. Specifically,

$$p_0^{(1)} = \begin{cases} p_{x_1}(i) + \pi_i^{(1)} p_r^{(1)} & \text{for } i = 1, 2, \ldots, j - 1 \\ (1 - \tau_j) p_{x_1}(i) + \pi_j^{(1)} p_r^{(1)} & \text{for } i = j, j+1, \ldots, b \end{cases} \quad (5.24)$$

where $p_0^{(1)}$ constitutes a probability mass function. The cumulative damage process can now be restarted by modifying equation 5.14 to account for inspection after $x_1$ duty cycles as

$$p_x = p_0^{(1)} p^{x-x_1} \quad \forall \ x \geq x_1. \quad (5.25)$$

Equation 5.25 is then used until inspection at time $x_2$ at which point the above process is repeated.
Variability of Failure

As seen in equation 5.14, the initial distribution of damage in a component at the time it enters service is directly related to the damage distribution after a given number of duty cycles. However, in the formulation of equation 5.14 it was assumed that failure occurred only when the absorbing state b was reached. That is, the state of damage in which failure occurred was deterministic. It may be such that failure can occur in other damage states as well. Equation 5.14 can be easily modified to account for this variability.

Let $\mathbf{p}$ be a $(1 \times b)$ row vector,

$$\mathbf{p} = [p_1, p_2, \ldots, p_b]$$

(5.26)

with $p_i \geq 0$ and $\sum_{i=1}^{b} p_i = 1$ which describes the distribution of the damage state at failure. The $i^{th}$ component of $\mathbf{p}$ represents the probability that failure occurs in damage state $i$. Clearly $\mathbf{p}$ is a probability mass function.

From a physical standpoint it is noted that $\mathbf{p}_0$ and $\mathbf{p}$ should not have overlapping elements. That is, for a given $i$ both $\pi_i$ and $p_i$ should not be greater than zero as this implies that a failed component may be placed into service. For practical situations only the the first few elements of $\mathbf{p}_0$ and the last few elements of $\mathbf{p}$ are usually nonzero.

With this mild restriction on the initial and final damage distributions, let $k = b - \beta, b - \beta + 1, \ldots, b$ be the last $\beta$ components of $\mathbf{p}$ where $\beta$ is an integer greater that the largest integer for which $\pi_i > 0$. To incorporate $\mathbf{p}$ into the unit-jump probability transition matrix given by equation 5.11, the last $(b - \beta - 1)$ rows and columns of the
probability transition matrix must be replaced with the submatrix

\[
\begin{bmatrix}
  p_{b-\beta-1} & \alpha'_{b-\beta-1}q_{b-\beta-1} & 0 & \cdots & \cdots & \alpha_{b-\beta-1}q_{b-\beta-1} \\
  0 & p_{b-\beta} & \alpha'_{b-\beta}q_{b-\beta} & 0 & \cdots & \alpha_{b-\beta}q_{b-\beta} \\
  0 & 0 & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & \cdots & \cdots & 0 & 1 \\
\end{bmatrix}, \quad (5.27)
\]

where

\[
\alpha_i + \alpha'_i = 1
\]

\[
p_{b-\beta} = \alpha_{b-\beta-1}
\]

\[
p_{b-\beta+1} = \alpha'_{b-\beta-1}\alpha_{b-\beta}
\]

\[
\vdots
\]

\[
p_b = \alpha'_{b-\beta-1}\alpha'_{b-\beta} \cdots \alpha'_{b-2}\alpha_{b-1}.
\]

Thus, by specifying \( p \), or equivalently the \( \alpha \)'s, it is possible to consider a distribution on the state of damage at failure.

Estimating Model Parameters

Since the Markov B-Model is phenomenological in nature, its parameters must be found from numerical data. For the stationary cumulative damage process these parameters are the "stay" and "jump" probabilities, \( p_i \) and \( q_i \), and the size \( b \) which define the probability transition matrix. The other aspects of the model such as the initial distribution of damage, the distribution of the state of damage at failure, and the quality of inspections can be found in a variety of ways. Some of the more common of these methods are fitting a histogram to experimental results, adopting some
assumptions on the underlying process, or utilizing subjective information such as expert opinion. The question then arises as to how numerical data on the cumulative damage process can be related to the model parameters. One possible approach is to utilize the method of moments.

With a simple probability transition matrix as given by equation 5.11, it is possible to obtain concise analytical results relating the model parameters to the moments of the times to transfer from one state to another. The derivation of these results is given in Appendix A. Fortunately, such expressions are in accordance with the kinds of data commonly associated with cumulative damage processes such as the time to failure or to reach a specific damage state.

For example, consider a constant amplitude fatigue test on a number of identical specimens. Assume that all of the specimens are damage free prior to the testing and that they are cyclicly loaded until failure occurs. Further, assume that the number of cycles to failure is recorded for each specimen along with the mean and variance of the time to failure for the group. If the damage free state is taken as state 1 and the failure state as state $b$, then from equation A.14 of Appendix A one finds

$$
E[W_{1,b}] = \sum_{i=1}^{b-1} (1 + r_i)
$$

$$
\text{VAR}[W_{1,b}] = \sum_{i=1}^{b-1} r_i (1 + r_i),
$$

(5.29)

where $W_{1,b}$ is the random variable representing the time to failure in state $b$, given that the process began in state 1 and $r_i = \frac{\mu_i}{\sigma_i}$ is as noted earlier. This yields two equations in $b$ unknowns. One possibility in solving this system of equations is to choose $r_i = r$ as a constant leaving only the parameters $r$ and $b$ to be determined. This leads to a rather simple model. Simple models such as this may not be sufficient to model a complex cumulative damage process, but they serve to illustrate some important points about
Markov B-Models. The first point, as it is known from probability theory, is that the availability of the first few moments is not sufficient to model a cumulative damage process uniquely; as several different processes may have the same or very similar distributions on the time to failure. This feature does not pose a serious problem if interest is directed on ultimate failure alone. However, one of the primary goals of any model is to predict behavior not covered by experimental data, and therefore a unique model is essential. Additional information on the cumulative damage process is therefore required. This information must be obtained from nondestructive testing techniques and it must be decided how measured inspection quantities can be related to model damage states.

Heretofore, the damage in a component has been indexed solely by "state" without regard as to what the "state" physically represented, with the exception of state \( b \) for failure. If additional data, that is second order moment data, are known on the time to reach subcritical damage levels, such as a particular crack length, then the results of Appendix A can be modified to associate a physical property with the individual damage states. Furthermore, the additional information serves to uniquely define the cumulative damage process.

The modification of the results of Appendix A can be carried out as such. Consider the case where the mean and variance of the time to reach \( j \) distinct damage levels are known. The unit-jump probability transition matrix can then be modeled as having \( j \) "blocks", where the parameters to be determined are the ratio of the "stay" to "jump" probabilities, \( r_i \), and the size, \( b_i - b_{i-1} (i = 1, 2, \ldots, b) \) of each "block" where the \( r_i \) of each "block" are assumed to be constant. The expressions relating these parameters to the experimental data then become
\[ E[W_{1,b_1}] = (b_1 - 1)(1 + r_1) \]
\[ \text{VAR}[W_{1,b_1}] = (b_1 - 1)r_1(1 + r_1) = r_1E[W_{1,b_1}] \]

(5.30)

\[ E[W_{1,b_i}] = E[W_{1,b_{i-1}}] + (b_i - b_{i-1})(1 + r_i) \quad \text{for} \quad i > 1 \]
\[ \text{VAR}[W_{1,b_i}] = \text{VAR}[W_{1,b_{i-1}}] + r_i\{E[W_{1,b_i}] - E[W_{1,b_{i-1}}]\} \quad \text{for} \quad i > 1. \]

Thus, for each "block" the parameter estimation process reduces to solving two equations in two unknowns for each "block" of the probability transition matrix. For example, the structure of such a probability transition matrix with three "blocks" of size \( b_i - b_{i-1} \) for \( i = 1, 2, 3 \) with \( b_0 = 0, b_1 = 1, b_2 = 3, b_3 = 6 \) respectively, is

\[
P = \begin{bmatrix}
p_1 & q_1 & 0 & 0 & 0 \\
0 & p_2 & q_2 & 0 & 0 \\
0 & 0 & p_2 & q_2 & 0 \\
0 & 0 & 0 & p_3 & q_3 \\
0 & 0 & 0 & 0 & p_3 & q_3 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

(5.31)

5.2.2 Nonstationary B-Models

The prior sections developed the basic properties of the Markov B-Models with respect to a stationary cumulative damage process. In this section the extension of the model to nonstationary cumulative damage processes is examined. Nonstationary models must be used when duty cycle severity and/or environment and/or material properties change with time. This discussion plays an important role in the forthcoming chapter where the model is used in connection with composite laminates whose strength
and stiffness both degrade with cyclic loading. There are several methods by which to introduce a nonstationary structure into the B-Models. However, one method has proven particularly useful and is discussed herein; it is the time transformation-condensation method.

**Time Transformations**

The premise of the time transformation method is to transform a known or postulated nonstationary cumulative damage process into a stationary cumulative damage process using a nonlinear time transformation. Thus, the simple concepts of the stationary Markov B-Models may be used to describe the nonstationary process. This transformation is generally expressed in the form of a polynomial.

Disregarding momentarily the fact that time is measured in discrete units, let x-time refer to time in the “stationary” domain and y-time to time in the “nonstationary” domain. The transformation between x-time and y-time is

$$ y = g(x) \geq 0 \quad \text{with} \quad g(0) = 0, \quad (5.32) $$

where \( g(x) \) has a continuous first derivative and \( \frac{dg(x)}{dx} > 0 \) for \( x > 0 \). Thus, \( g^{-1}(y) \) exists and is unique for all \( x > 0 \).

To apply the time transformation method it is first assumed that an empirical distribution function \( F_n \), which is nothing more than an experimental cumulative distribution function, for the time to failure in y-time is known. Further, the corresponding second order moment data are assumed known as well. Given these data, a stationary Markov B-Model is constructed. At this point experience with the model is particularly useful as a reasonable amount of knowledge concerning the behavior of the stationary models with respect to their various parameters facilitates choosing
the parameters of the stationary model at hand. Once the parameters are chosen, a cumulative distribution function of the time to failure in x-time is computed using equation 5.14.

A general polynomial of the form

\[ y = g(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{5.33} \]

is chosen for the transformation. The objective is to transform the x-time cumulative distribution function of the time to failure, \( F_W \), into a y-time cumulative distribution function of the time to failure, \( F_Y \), such that \( F_Y \) is an accurate description of the empirical distribution function \( F_n \). This is accomplished by requiring that \( F_Y \) and \( F_n \) have the same ordinate values at a given number of points. That is, \( F_Y = F_n = 0.2, 0.4, \ldots \) The number of points corresponds to the number of unknowns, \( n \), in the transformation of equation 5.33. These unknown coefficients are found by finding first the abscissa values in x-time and y-time respectively for which \( F_W \) and \( F_n \) have the chosen ordinate values. This yields an ordered set of \( n \) x and y values \( \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \). These ordered pairs are then placed into equation 5.32 to derive a set of \( n \) linear equations in the unknown coefficients \( a_1, a_2, \ldots, a_n \). This system is then solved to determine the unknown coefficients, and the time transformation itself.

Once the transformation is determined, the cumulative distribution function of the time to failure in x-time, \( F_W \), is then transformed point by point into the cumulative distribution function of the time to failure in y-time, \( F_Y \), utilizing equation 5.33. If \( F_Y \) sufficiently describes \( F_n \), the procedure is complete. Otherwise, a new transformation based on a new stationary Markov B-Model is chosen. This method has somewhat of a trial and error nature but if a suitable transformation is found,
much can be deduced about the nature of the nonstationary cumulative damage process as both the properties of the stationary Markov B-Model and of the polynomial time transformation are well understood. This information may address such issues as whether the duty cycles in y-time are more or less severe than those in x-time, the nature of the asymptotic behavior of the cumulative damage process, whether or not the cumulative damage process was stopped at some point, or if there was variability in the initial state of damage to name a few. Bogdanoff and Kozin (1985) provide a more thorough discussion of the time transformation method and the reader is referred to this source for further information.

Condensation Method

In the section on the time transformation method, a stationary Markov B-Model was chosen with constant duty cycle severity in x-time described by the probability transition matrix $P$. This probability transition matrix was used in conjunction with $P_0$ to find the distribution of damage after a given integer number of duty cycles, $x$. The condensation method provides a means for finding the probability transition matrices of the corresponding nonstationary process in discrete y-time from the stationary probability transition matrix $P$ and the time transformation $y = g(x)$.

Recall that for the cumulative damage process to be modeled as a Markov chain, that time, in terms of duty cycles, must be measured on the set of nonnegative integers. However, placing integer values of $x$ into equation 5.33 does not necessarily produce integer values for $y$ as needed. This problem can be circumvented by the following discussion which summarizes the condensation method.

Let $x_1, x_2, \ldots$ be determined from $i = g(x_i)$ for $i = 1, 2, \ldots$. Given the $x_i$'s the following intervals are formed $(0, x_1], (x_1, x_2], \ldots$ Given these intervals the probability
transition matrices in \( y \)-time, \( Q_i \), are then defined by the equation

\[
Q_i = I \quad \text{if} \quad (x_{i-1}, x_i) \text{ contains no integer} \\
= P \quad \text{if} \quad (x_{i-1}, x_i) \text{ contains one integer} \\
= P^2 \quad \text{if} \quad (x_{i-1}, x_i) \text{ contains two integers} \tag{5.34} \\
= P^3 \quad \text{if} \quad (x_{i-1}, x_i) \text{ contains three integers}, \\
= : 
\]

where \( I \) is the identity matrix. The nonstationary cumulative damage process can then be modeled as

\[
P_y = p_0 \prod_{i=1}^{y} Q_i. \tag{5.35}
\]

This equation is analogous to equation 5.14 for stationary cumulative damage processes; and \( p_0 \) is the initial distribution of damage as before.

Note for any positive integer \( y \), that \( \prod_{i=1}^{y} Q_i \) is the integer power of \( P \) which is the largest integer equal to or just less than \( g^{-1}(y) \). This implies that the condensation method, with its restriction to produce integer values of \( y \), does not produce the exact powers of \( P \) as required by \( g^{-1}(y) \), however, this difference never exceeds \( P \).

Furthermore, it is shown by Bogdanoff and Kozin (1985) that the error between \( p_y \) and the exact distribution of damage is related to the “jump” probabilities of the probability transition matrix \( P \) which are generally quite small. Consequently, the error is often negligible. In essence, the condensation method takes integer jumps of \( F_W \) and condenses them into unit jumps of \( F_V \), thus preserving the assumption of discrete time in both the stationary and nonstationary domains.
5.3 Summary

In this chapter the basic concepts associated with Markov chains and Markov B-Models that are utilized to describe a cumulative damage process have been discussed. These concepts included the determination of model parameters, incorporating sources of variability into the B-Models, and the construction of nonstationary B-Models from stationary B-Models. These concepts will be utilized and extended to develop a cumulative damage model of the fatigue damage accumulation in composite laminates.
Chapter 6

B-Models for Composite Laminates

The previous chapters have served to present a useful background for fatigue as a damage accumulation process, models of the fatigue process, a general class of cumulative damage models known as Markov B-Models, and the concepts from discrete probability theory needed to understand the models. In this chapter these concepts are combined to establish a model for the accumulation of fatigue damage in composite laminates. Further, a simple, yet illustrative, example of the application of the model will be presented.

6.1 Modeling Concepts

The vast majority of models for the accumulation of fatigue damage in composite laminates currently available in the literature are solely phenomenological in nature. While these models provide insight into the nature of the fatigue phenomenon in composite laminates they fail to address the mechanics of damage development. To characterize the damage accumulation process in composite laminates most accurately, models based on the mechanics of the damage development process are needed. Furthermore, since the damage accumulation process in composite laminates is random in nature, these models should be based on stochastic processes.

Given these considerations, several possibilities for a model which combined all of these aspects were explored. However, due to the complexity of the fatigue damage accumulation process in composite laminates a framework for selecting such a model was needed. This framework was provided by the critical element model as
proposed by Reifsnider (1986) and Reifsnider and Stinchcomb (1986). The critical element model serves as a modeling philosophy whose premise is to develop a unified mechanistic approach to modeling the fatigue of composite laminates through precise problem identification, physical insight, and the mechanics of the damage development process.

The model approaches the fatigue damage accumulation process as such. First, an accurate description of the failure mode(s) which controls the ultimate failure of the laminate must be obtained. With the failure mode(s) defined, a representative volume element is chosen such that the response of the laminate as a whole is characterized by the response of the representative volume element. This representative volume element is then divided into critical and subcritical elements. The subcritical elements are those damage mechanisms whose realization causes a redistribution of stress and changes in the local geometry of the laminate. The magnitude of these effects are determined from some parameter such as the laminate stiffness which can be directly correlated to the subcritical elements and can be measured directly or modeled using micro-mechanical analysis. The critical elements are those damage mechanisms whose realization causes global failure or fracture of the laminate. The critical elements describe the state of the material itself and are characterized by phenomenological information on their response under fatigue loading. While the number of damage mechanisms is generally quite large, the number of failure modes, and consequently critical elements, is few. Thus, the critical element model provides a means for reducing the otherwise complex problem of characterizing the fatigue behavior of composite laminates into a well defined and greatly simplified problem. For the composite laminates and fatigue loadings considered here, the subcritical el-
ements are matrix cracking and delamination. The critical element is the fracture of the load bearing plies.

The framework provided by the critical element model led to the choice of the Markov B-Model for the basis of the cumulative damage model presented herein. Of and by itself, the Markov B-Model is phenomenological and probabilistic in nature. However, as discussed in chapter 5, to develop a unique Markov B-Model of a cumulative damage process, information on damage prior to failure must be known. This information is generally the second order moment data on the time to reach some given level of damage as indicated by some physically measurable quantity. Bogdanoff and Kozin (1985) have addressed this issue for stationary cumulative damage processes and have shown how to relate the model damage states to the measured quantity. To the best of the author’s knowledge, however, this issue has not been addressed for nonstationary cumulative damage processes. A method for addressing this issue in terms of the fatigue of composite laminates and in conjunction with the ideas put forth by the critical element model is proposed.

Specifically, since the effects of the failure of the subcritical elements are characterized by some quantity, typically changes in the laminate stiffness, it is proposed that information on stiffness changes in a laminate be used to provide the additional required information for determining a unique Markov B-Model. Since changes in the stiffness of a laminate are determined from the mechanics of the damage development process, utilizing stiffness changes to determine a unique Markov B-Model allows the mechanics of the damage development process to be incorporated into the cumulative damage model. This allows for a more complete and accurate description of the damage accumulation process.
Information regarding the stiffness loss in a laminate subjected to fatigue loading may be obtained directly in the laboratory by simply monitoring the stiffness of the laminate continuously by the slope of the dynamic stress-strain curve. That is, measuring the secant modulus. Alternatively, the stiffness may be monitored by periodically suspending the loading and performing a static measurement of the laminate modulus. In the field, direct measurement of the stiffness of a laminate is generally not possible. As a result, measurable quantities as obtained through nondestructive testing techniques must be used to determine the stiffness loss in a laminate.

From the discussion of section 2.3 and figure 2.5 it is clear that for the laminates considered here matrix cracking and delamination dominate the fatigue damage accumulation process and constitute the subcritical elements in the laminate. Stiffness loss due to the realization of matrix cracking and delamination can be obtained through the models of Talreja and O'Brien as discussed in section 3.2.1. These models allow for quantities obtainable through nondestructive testing, namely matrix crack density and delamination size, to be used to determine laminate stiffness changes.

Given the stiffness loss in a laminate as a damage indicator, the additional information necessary for determining a unique B-Model of the damage accumulation process can be found. This information is taken here as the second order moment data and the corresponding empirical distribution functions on the time to reach a sufficient number of levels of stiffness loss prior to failure or some critical value. From this information the parameters of a stationary B-Model and an ensemble of time transformations for use with the time transformation-condensation method can be found.
6.2 Extension of the Markov B-Model

In chapter 5 it was shown how to construct a nonstationary B-Model given an appropriate stationary B-Model and utilizing the time transformation-condensation method. In that discussion the choice of the stationary B-Model was somewhat arbitrary with one's experience with the model driving the choice of the stationary model. Here, the arbitrary nature of the stationary model is removed. Specifically, the second order moment data for the stationary process are taken to be proportional to the second order moment data for the nonstationary process. This allows nonstationary data on the time to reach a given damage levels below failure to be used in the construction of the stationary Markov B-Model. The significance of this procedure is that it allows the "states" of both the stationary and nonstationary Markov B-Models to be directly related to a physical property, such as the stiffness loss in the laminate. The model can then be used to determine the evolutionary distribution of damage in the laminate in terms of the stiffness loss.

To elucidate the procedure assume that the mean and standard deviation on the time to reach $j$ distinct damage levels are known. The damage levels are some given percentage of stiffness loss. Let these quantities be denoted by $E[D_{y_i}]$ and $SDD[D_{y_i}]$. It is now desired to construct the mean and standard deviation for the $j$ damage levels for a stationary process. Let these quantities be denoted by $E[D_{x_i}]$ and $SDD[D_{x_i}]$. One then arbitrarily assumes a value of $E[D_{x_1}]$ and determines the ratio between $E[D_{x_1}]$ and $E[D_{y_1}]$ denoting this ratio by $r^*$. Given this ratio, the means for the $j - 1$ remaining damage levels $E[D_{x_i}]$ for $i = 2, 3, \ldots, j$ are found as

$$E[D_{x_i}] = r^*E[D_{y_i}] \quad \text{for} \quad i = 2, 3, \ldots, j.$$ (6.1)
The standard deviations \( SDD[D_{x_i}] \) are then determined by requiring the coefficient of variation for each damage level in x-time to be equal to the corresponding coefficient of variation in y-time. Stated mathematically,

\[
SDD[D_{x_i}] = \gamma_{yi} E[D_{x_i}] \quad \text{for} \quad i = 1, 2, \ldots, j,
\]

(6.2)

where \( \gamma_{yi} = SDD[D_{y_i}]/E[D_{y_i}] \) is the coefficient of variation in y-time. Requiring this equality establishes a one-to-one relationship between the damage states in the stationary model to those in the nonstationary model.

Given \( E[D_{x_i}] \) and \( \text{VAR}[D_{x_i}] \) one can now construct the stationary Markov B-Model using the results of Appendix A and section 5.2.1 to determine the required parameters. Namely, if a unit-jump stationary B-Model is chosen with a probability transition matrix, \( P \), containing \( j \) "blocks", where \( b_i - b_{i-1} \) is the size of each individual "block" and \( r_i \) is the ratio of the "stay" to "jump" probabilities in each "block", the parameters are related to the data as follows

\[
E[W_{1,b_i}] = (b_i - 1)(1 + r_i) \\
\text{VAR}[W_{1,b_i}] = (b_i - 1)r_i(1 + r_i) = r_i E[W_{1,b_i}]
\]

(6.3)

\[
E[W_{1,b_i}] = E[W_{1,b_{i-1}}] + (b_i - b_{i-1})(1 + r_i) \quad \text{for} \quad i > 1 \\
\text{VAR}[W_{1,b_i}] = \text{VAR}[W_{1,b_{i-1}}] + r_i\{E[W_{1,b_i}] - E[W_{1,b_{i-1}}]\} \quad \text{for} \quad i > 1.
\]

The symbol \( W_{1,b_i} \) stands for the random variable representing the number of duty cycles required for the damage to reach state \( b_i \) given that the cumulative damage process began in state 1. The total number of states in the stationary B-Model is \( b = b_j \) with each of the \( b_i \) for \( i = 1, 2, \ldots, j \) corresponding to a specific damage level. The other states in the model can be associated with a given level of damage by simply
interpolating between the \( b_i \). The structure of such a probability transition matrix with three "blocks" of size \( b_i - b_{i-1} \) for \( i = 1, 2, 3 \) with \( b_0 = 0, b_1 = 1, b_2 = 3, b_3 = 6 \) respectively is shown in equation 5.31 as an example.

With the stationary model, that is the probability transition matrix \( P \), determined the time transformation-condensation method is employed to determine the nonstationary model, that is the probability transition matrix \( Q \). For the proposed model this procedure differs from that of section 5.2.2. The procedure is as follows. If \( y \) is the number of duty cycles for which the distribution of damage is desired, one first determines between which experimental mean values, \( E[D_{y_i}] \) and \( E[D_{y_{i+1}}] \), \( y \) lies. Once this interval is identified, one determines an appropriate two parameter unit-jump stationary B-Model based solely on \( E[D_{x_{i+1}}] \) and \( SDD[D_{x_{i+1}}] \) as determined from equations 6.1 and 6.2. The parameters of this B-Model, denoted as \( \hat{b} \) and \( \hat{r} \), are found by solving the equations

\[
E[W_{1,h_{i+1}}] = (\hat{b}_{i+1} - 1)(1 + \hat{r})
\]

\[
VAR[W_{1,h_{i+1}}] = (\hat{b}_{i+1} - 1)^2(1 + \hat{r}),
\]

(6.4)

for \( \hat{b} \) and \( \hat{r} \). They are used to determine the corresponding probability transition matrix \( \hat{P} \). In this case, \( \hat{P} \) is a \((\hat{b} \times \hat{b})\) matrix and has the form

\[
\hat{P} = \begin{bmatrix}
\hat{p} & \hat{q} & 0 & \ldots & \ldots & 0 \\
0 & \hat{p} & \hat{q} & 0 & \ldots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & \hat{p}
\end{bmatrix},
\]

(6.5)

where \( \hat{r} = \hat{p}/\hat{q} \) and \( \hat{p} + \hat{q} = 1 \).
Once $\hat{P}$ is found, the corresponding stationary cumulative distribution function $\hat{F}_W$ must be determined. Given this cumulative distribution function a time transformation, $y = \hat{g}(x)$, is constructed as described in section 5.2.2 based on the experimental empirical distribution function $F_{n+1}$. The transformation is then used in conjunction with the probability transition matrix $P$ to determine the $Q_i$ for the nonstationary model as such. Let $x_1, x_2, \ldots$ be determined from $i = \hat{g}(x_i)$ for $i = 1, 2, \ldots$

Given the $x_i$'s the following intervals are then formed $(0, x_1], (x_1, x_2], \ldots$ Given these intervals the probability transition matrices in $y$-time, $Q_i$, are then defined by

$$Q_i = I \quad \text{if} \quad (x_{i-1}, x_i] \text{ contains no integer}$$

$$= P \quad \text{if} \quad (x_{i-1}, x_i] \text{ contains one integer}$$

$$= P^2 \quad \text{if} \quad (x_{i-1}, x_i] \text{ contains two integers} \quad (6.6)$$

$$= P^3 \quad \text{if} \quad (x_{i-1}, x_i] \text{ contains three integers,}$$

$$= \vdots$$

where $I$ is the identity matrix. The nonstationary cumulative damage process can then be modeled as

$$P_y = P_0 \prod_{i=1}^{y} Q_i, \quad (6.7)$$

where $P_0$ is the initial distribution of damage. This procedure allows for the most accurate description of the nonstationary process to be obtained by utilizing a time transformation based on an empirical distribution function of experimental data nearest the number of duty cycles for which the distribution of damage is desired. Further, it can be extended to include inspection and replacement, as well as variability in the state of damage at failure as discussed in section 5.2.1 by modifying the probability transition matrix $P$ of the stationary model accordingly.
As a further extension to the Markov B-Model, the results of equation 6.7 can be used in conjunction with the probability mass function of the state of damage at failure to calculate the probability of failure of the laminate. Here, "failure" corresponds to the damage level associated with model damage state $b_j$. To develop this idea consider a limit state function defined as

$$
\tilde{f}(\mathbf{p}_0, \mathbf{p}, \mathbf{P}, y) = \tilde{R}(\mathbf{p}) - \tilde{S}(\mathbf{p}_0, \mathbf{P}, y),
$$

(6.8)

where $\tilde{R}$ is a resistance and $\tilde{S}$ is a load effect such that $\tilde{f} < 0$ implies failure will occur. The probability mass functions of $\tilde{R}$ and $\tilde{S}$ are respectively

$$
P_{\tilde{R}}(\tilde{r}_i) = p
$$

$$
P_{\tilde{S}}(\tilde{s}_i) = p_y.
$$

(6.9)

Given these distributions, the probability of failure $P_F$ can be found by summing over all $i$ the probabilities that $\tilde{f} < 0$. That is,

$$
P_F = \sum_{i=1}^{b-1} P[\tilde{R} = \tilde{s}_i]P[\tilde{S} > \tilde{s}_i].
$$

(6.10)

Noting that $P[\tilde{R} = \tilde{s}_i] = P_{\tilde{R}}(\tilde{s}_i)$ and $P[\tilde{S} > \tilde{s}_i] = \sum_{j=i+1}^{b} P_{\tilde{S}}(\tilde{s}_i)$, equation 6.10 becomes

$$
P_F = \sum_{i=1}^{b-1} P_{\tilde{R}}(\tilde{s}_i) \left[ \sum_{j=i+1}^{b} P_{\tilde{S}}(\tilde{s}_i) \right].
$$

(6.11)

The quantity $P_F$ then gives a convenient measure of the reliability of the laminate and can be used to determine when inspection and replacement should occur.

### 6.3 Numerical Example

To illustrate the application of the proposed cumulative damage model for the accumulation of fatigue damage in composite laminates the model was applied to data
from the literature. Unfortunately, a complete set of data in the form of second order moment data and empirical distribution functions on the time to reach damage states below failure could not be obtained. Therefore, this example makes use of the data that are available to provide a hypothetical, yet representative, example of the proposed modeling procedure.

Jamison and coworkers (1984) have studied the stiffness loss-damage development relationship in T300/5208 and T300/914C graphite/epoxy [0/90/ ± 45]_S angle ply laminates under constant amplitude, sinusoidal, tension-tension fatigue loading. The specimen dimensions were 25.4 mm wide and 203 mm long with individual ply thicknesses of 0.14 mm. The specimens were cyclically loaded under a stress ratio of \( R = 0.1 \) with a maximum cyclic stress of \( \sigma_{\text{max}} = 0.62\sigma_{\text{ult}} \) at a loading frequency of 10 Hz. The symbol \( \sigma_{\text{ult}} \) is the ultimate static strength of the laminate. Damage was measured continuously by the reduction in the dynamic secant modulus computed from the dynamic stress-strain curve. It should be noted that the secant modulus is not the same quantity as the longitudinal modulus of the laminate. It is, however, a convenient quantity to measure and yields the same results for the stiffness loss with continued cyclic loading.

Jamison and coworkers presented a representative stiffness loss versus load cycle curve for the laminate configuration. The primary features of this curve are reproduced in figure 6.1. The partition of the stiffness reduction curve into regions 1, 2, and 3 served to partition the dominant fatigue damage mechanisms for the given laminate configuration. Region 1 corresponds to matrix cracking in the off-axis plies, region 2 to stable delamination growth, and region 3 to accelerated, unstable delamination growth. For the present example, this curve is taken to represent the mean of the stiffness loss versus load cycle curves for the lot of specimens. The data for the time to
reach four stiffness loss levels were used to construct the model. Respectively, these losses were 4, 8, 12, and 16 percent. Due to the lack of data on the variance and the empirical distribution functions of damage prior to and including failure, values for the coefficient of variation were assigned assuming that the scatter in the data increased with the number of load cycles. Furthermore, for a given damage level the data were assumed to follow a two parameter Weibull distribution with shape parameter $\sigma_0$ and scale parameter $\rho_0$. The input into the model is summarized in table 6.1.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
D_{yi} & \frac{E_i}{E_1} & E[D_{yi}] & \frac{1}{\delta_{yi}} & \sigma_{i0} & \rho_{i0} \\
\hline
1 & 0.96 & 1400 & 4.00 & 4.54 & 1533.34 \\
2 & 0.92 & 4600 & 3.75 & 4.23 & 5058.78 \\
3 & 0.88 & 6550 & 3.50 & 3.92 & 7234.65 \\
4 & 0.84 & 7650 & 3.25 & 3.61 & 8488.30 \\
\hline
\end{array}
$$

Table 6.1: Model Input
A computer program was written to find the parameters of the stationary B-Model and the corresponding time transformations. A time transformation of the form

\[ y = \hat{g}(x) = x + a_2x^2 + a_3x^3 + a_4x^4 \]  

(6.12)

was assumed. These results are summarized in table 6.2. Further, these results were used to determine the relationship between the various stiffness loss levels and the states of the stationary and nonstationary B-models by linear interpolation as shown in figure 6.2. The stationary model was then used in conjunction with the time transformation-condensation method to determine the cumulative distribution functions for the first two damage levels. As shown in figures 6.3 and 6.4 these cumulative distribution functions yield an accurate description of the empirical distribution functions as modeled by Weibull distributions particularly in the central portions of the distributions where interest is primarily centered. The time transformation-condensation method was utilized again to find the distribution of damage after 1400 and 4600 duty cycles respectively with 1 duty cycle corresponding to 10 load cycles. This was done to determine if the results predicted from the nonstationary B-Model agreed with the experimental data. These results are shown in figures 6.5 and 6.6.

From these figures it is apparent that the model performs quite well in the sense that the greatest values of the respective probability mass functions occur at the

<table>
<thead>
<tr>
<th>$D_{m}$</th>
<th>$b_i$</th>
<th>$r_i$</th>
<th>$a_{1i}$</th>
<th>$a_{2i}$</th>
<th>$a_{3i}$</th>
<th>$a_{4i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13</td>
<td>3.17</td>
<td>1.0000</td>
<td>1.4052</td>
<td>-0.0225</td>
<td>0.0001</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>15.33</td>
<td>1.0000</td>
<td>0.4176</td>
<td>-0.0020</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>33.82</td>
<td>1.0000</td>
<td>0.2935</td>
<td>-0.0010</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
<td>38.29</td>
<td>1.0000</td>
<td>0.2635</td>
<td>-0.0008</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 6.2: Model Output
Figure 6.2: Relationship between the percent secant stiffness loss and the damage states of the model.

appropriate damage states. For example, after 1400 duty cycles the largest value in probability mass function for the distribution of damage occurs at damage state 13 which corresponds to a 4 percent stiffness loss as was reported in the literature.
Figure 6.3: Comparison of the experimental empirical distribution function (solid) and cumulative distribution function predicted by the model (dotted) for a 4 percent loss in the secant modulus of the laminate. 1 duty cycle = 10 load cycles.
Figure 6.4: Comparison of the experimental empirical distribution function (solid) and cumulative distribution function predicted by the model (dotted) for an 8 percent loss in the secant modulus of the laminate. 1 duty cycle = 10 load cycles.
Figure 6.5: probability mass function of damage after 1400 duty cycles.
Figure 6.6: probability mass function of damage after 4600 duty cycles
6.4 Summary

The foregoing discussion has addressed both the physical aspects and the implementation procedure of a model for the accumulation of fatigue damage in composite laminates. It was shown how to construct the model based on an extension of the Markov B-Model, given information on the cumulative damage process in terms of stiffness loss in a laminate. Furthermore, a simple hypothetical, but representative, example was presented based on limited information available in the open literature. The results of this example are encouraging to the extent that further development of the model may be warranted.
Chapter 7

Concluding Remarks

This work has attempted to develop a probabilistic model for the accumulation of fatigue damage in composite laminates based on an extension of the Markov B-Models. This extension was utilized to incorporate information on stiffness loss prior to and including failure for developing an appropriate nonstationary Markov B-Model whose damage states can be directly associated with some measurable physical quantity, in this case stiffness loss.

Chapters 1 through 5 discussed the background necessary for understanding the concepts associated with the model. Chapters 2 and 3 included the basics of the fatigue process for both common engineering materials and composite laminates and discussed some of the more prominent models used to describe the fatigue phenomenon. Particular attention was focused on the models proposed by Talreja and O’Brien for relating matrix cracking and delamination to stiffness loss in a laminate as these two damage mechanisms are the most prominent in the fatigue of composite laminates. Furthermore, these models provide a link between data obtained in the field to input for the proposed model. Thus, they greatly increase the applicability of the model to “real life” situations.

Chapters 4 and 5 are concerned with the aspects of the Markov B-Models and the concepts from discrete probability theory that were needed to understand the models. The Markov B-Models provide a versatile class of models for a wide variety of cumulative damage processes and incorporate the major sources of variability in such processes into their structure. Furthermore, the Markov B-Models have a strong
analytical basis and results relating their parameters to experimental data can be found in Appendix A.

Chapter 6 contains the crux of the present work. It brings together the contents of the previous chapters into a unified structure by presenting a modeling philosophy for the problem at hand in addition to describing the proposed model itself. To the best of the author’s knowledge this is the first attempt to approach the fatigue process in composite laminates in such a manner.

Several remarks should be made, however, concerning the proposed model. First, the model uses as input data on stiffness loss which is either measured directly in the laboratory or is output from a collection of other models. Hence, the error associated with this data will carry over into the proposed cumulative damage model. This issue must be addressed by analyzing the other models in more detail. Furthermore, there is a clear absence in the literature of the kind of data that are needed to verify the given model. In light of this predicament, a need for further testing is clear. Only from such testing can the proposed model be verified.

In conclusion, the proposed model provides for a simple, yet effective, means of modeling the complex fatigue damage accumulation process in composite laminates by utilizing as input data on stiffness loss which can be correlated to the individual damage mechanisms. From this model a wealth of information on the cumulative damage process can be obtained. Furthermore, as composite laminates continue to increase in both the number and criticality of their applications such a model is needed. It is hoped that this work is a useful step in this direction. Note that both the framework upon which the proposed model is built as well as the modeling philosophy adopted in its development can be extended to other cumulative damage processes and material systems.
Bibliography


Appendix A

Derivation of Moment Formulae

It is feasible to obtain analytical results for stationary Markov chains relating the parameters of the Markov chain to the moments for the times to transfer from one state to another in the Markov chain. These results are obtained through probability generating functions. This appendix focuses on the derivation of these results as related to the subject of this work. A rigorous treatment of this topic can be found in Bogdanoff and Kozin (1985).

To begin the discussion, consider the stationary Markov chain with \( b \) possible states given by the equation

\[ \mathbf{p}_x = \mathbf{p}_0 \mathbf{P}^x. \]  \hspace{1cm} (A.1)

The symbol \( \mathbf{p}_x \) is the probability mass function given as a \((1 \times b)\) row vector whose components \( p_x(i) \) for \( i = 1, 2, \ldots, b \) represent the probability of being in a particular state after \( x \) trials. Similarly, \( \mathbf{p}_0 \) is a probability mass function given as a \((1 \times b)\) row vector whose components \( \pi_i \) for \( i = 1, 2, \ldots, b \) represent the probability of being in a particular state before any trials are performed. The probability transition matrix, \( \mathbf{P} \), is a \((b \times b)\) matrix whose elements \( p_{ij} \) represent the probabilities of the process having an outcome in state \( j \) after a trial given, that the process had an outcome in state \( i \) before the trial.

Taking the unilateral z-transform of equation A.1 yields

\[ \chi(z) = \mathbf{p}_0 \Psi(z), \]  \hspace{1cm} (A.2)
where \( \chi(z) \) and \( \Psi(z) \) are defined as

\[
\chi(z) = \sum_{i=0}^{\infty} p_x(i)z^i \quad \text{and} \quad \Psi(z) = \sum_{i=0}^{\infty} P^i z^i.
\]  
(A.3)

A component \( p(y; z) \) of \( \chi(z) \) is the probability generating function of being in state \( y \).

A component \( \psi_{i,j}(z) \) of \( \Psi(z) \) is the probability generating function of being in state \( j \) given that state \( i \) was occupied at the beginning of the process. The relationship between these two probability generating functions is

\[
p(y; z) = \sum_{i=1}^{b} \pi_i \psi_{i,y}(z) \quad \text{for} \quad y = 1, 2, \ldots, b.
\]  
(A.4)

Utilizing a Neumann series, \( \Psi(z) \) in equation A.3 can be written as

\[
\Psi(z) = \sum_{i=0}^{\infty} P^i z^i = (I - Pz)^{-1},
\]  
(A.5)

where \( I \) is the \((b \times b)\) identity matrix. Thus, \( \psi_{i,j}(z) \) can be written as,

\[
\psi_{i,j}(z) = (-1)^{i+j} \frac{|m_{ij}|}{|I - Pz|},
\]  
(A.6)

where \(|I - Pz|\) is the determinant of the matrix \((I - Pz)\). The symbol \(|m_{ij}|\) represents the determinant of the \((ij)\)th minor of \((I - Pz)\) and is the \(((b-1) \times (b-1))\) determinant found by removing the \(i\)th row and \(j\)th column of \(|I - Pz|\).

If \( P \) is the unit-jump probability transition matrix with only nonzero elements on its main diagonal, \( p_i \), and its super diagonal, \( q_i \), then \((I - Pz)\) has the form

\[
I - Pz = \begin{bmatrix}
1 - p_1 z & -q_1 z & 0 & \ldots & \ldots & 0 \\
0 & 1 - p_2 z & -q_2 z & 0 & \ldots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
& \vdots & \ddots & \ddots & \ddots & \vdots \\
& & & 1 - p_{b-1} z & -q_{b-1} z \\
0 & \ldots & \ldots & \ldots & 0 & 1 - z
\end{bmatrix}.
\]  
(A.7)
It is readily seen that
\[ \psi_{i,j}(z) = 0 \quad \text{for} \quad i > j, \quad (A.8) \]
and the determinant \( |I - Pz| \) is simply the product of the elements of the main diagonal
\[ |I - Pz| = (1 - p_1 z)(1 - p_2 z) \cdots (1 - p_{b-1} z)(1 - z). \quad (A.9) \]
Therefore, evaluating equation A.6 for \( i = y \) and \( j = b \) yields
\[ \psi_{y,b}(z) = \frac{q_y \cdots q_{b-1} z^{b-y}}{(1 - p_1 z)(1 - p_2 z) \cdots (1 - p_{b-1} z)(1 - z)} \quad \text{for} \quad y = 1, 2, \ldots, b - 1. \quad (A.10) \]
Consider a new probability generating function \( \phi_{y,b}(z) \) of a random variable \( W_{y,b} \) defined by the equation
\[ \phi_{y,b}(z) = (1 - z)\psi_{y,b}(z). \quad (A.11) \]
Then, from equation A.10, \( \psi_{y,b}(z) \) can be rewritten as
\[ \phi_{y,b}(z) = \frac{q_y \cdots q_{b-1} z^{b-y}}{(1 - p_1 z)(1 - p_2 z) \cdots (1 - p_{b-1} z)} \quad \text{for} \quad y = 1, 2, \ldots, b - 1. \quad (A.12) \]
Recall from probability theory that, if the moments of a random variable exist, the moments can be found simply by differentiating the probability generating function with respect to \( z \) then evaluating the result at \( z = 1 \). Specifically, given a random variable \( D \), the mean and variance of \( D \) can be expressed as
\[ E[D] = P_D^{(1)}(z)|_{z=1} \]
\[ \text{VAR}[D] = P_D^{(2)}(z)|_{z=1} + P_D^{(1)}|_{z=1}^2 - \{P_D^{(1)}|_{z=1}\}^2, \quad (A.13) \]
where \( P_D^{(n)}(z) \) is the \( n \)th derivative of the probability generating function \( P_D(z) \) of \( D \).
Applying this procedure to \( \phi_{y,b}(z) \), as in equation A.12, yields the following results
for the mean and variance of $W_{y,b}$

\[
E[W_{y,b}] = \sum_{i=y}^{b-1} (1 + r_i)
\]

\[
\text{VAR}[W_{y,b}] = \sum_{i=y}^{b-1} r_i(1 + r_i),
\] (A.14)

where $r_i = \frac{p_i}{q_i}$ and again where $W_{y,b}$ is the time to reach the absorbing state given that the process began in state $y$. 