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Soil-structure interaction effects of simple structures supported on rectangular foundations

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Soil-Structure Interaction Effects of Simple Structures Supported on Rectangular Foundations

by

Wen-Hwa Wu

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July, 1991
Soil-Structure Interaction Effects of Simple Structures Supported on Rectangular Foundations

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Abstract

This thesis deals with the effects of soil-structure interaction, both kinematic and inertial, on the responses of seismically excited rectangular foundations and simple structures supported on such foundations. The ground motion considered is defined stochastically by a local power spectral density function and a spatial incoherence function. The structures examined are considered to have one lateral and one torsional degree of freedom in their fixed-base condition. The response quantities investigated include the ensemble means of the peak values of the lateral and torsional components of the foundation input motion and of the associated structural deformations. These responses are evaluated and compared with those obtained for no soil-structure interaction and for kinematic interaction only. The information and concepts presented elucidate the nature and relative importance of the two effects and make it possible to assess readily the influences of the more important parameters involved.
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Finally, I wish to dedicate this thesis to my parents in gratitude for their sacrifices, love and encouragement throughout my life.
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Chapter 1

INTRODUCTION

It is generally recognized that the motion experienced by the foundation of a structure during an earthquake may differ substantially from the free-field ground motion. The latter term refers to the motion that the ground would experience at its interface with the foundation if the structure were not present.

Two factors are responsible for this difference: (i) the inability of a rigid foundation to conform to the generally non-uniform, spatially varying free-field ground motion; and (ii) the dynamic interaction or coupling between the vibrating structure, foundation and supporting soils.

The motion actually experienced by the foundation is normally evaluated in two steps. First, the so-called foundation input motion is computed. This is the motion that the foundation would experience if both it and the superimposed structure were massless. Next, the inertial effects of the structure and the foundation are provided for, giving due regard to their interaction or coupling with the supporting soils.

The difference in the responses of the structure computed for the foundation input motion and the free-field ground motion at some reference or control point is known as the kinematic interaction effect, whereas the difference in the responses computed for the actual foundation motion and the foundation input motion is known as the inertial interaction effect. The total soil-structure interaction is clearly given by the sum of the kinematic and inertial interaction effects.

The objective of this dissertation is to elucidate the effects of kinematic and inertial interaction effects for simple structures that are supported at the ground surface through rigid, rectangular foundations and are excited by representative earthquake ground motions. The structures examined are presumed to have one lateral and one torsional degree of freedom in their fixed-base condition, and the spatial variation of the free-field ground motion is specified stochastically in terms of an incoherence function. The response quantities examined include the ensemble means of the peak values of the lateral and torsional components of the foundation input motion and
of the corresponding structural deformations. These deformations are displayed in
the form of pseudo-velocity response spectra and compared, over wide range of the
parameters involved, with those obtained for no soil-structure interaction and for
kinematic interaction only. Simple, physically motivated interpretations are given for
the observed differences.

A fundamental step in the evaluation of the kinematic interaction effects is the
computation of the transfer functions of the foundation. Defined for harmonically
excited massless foundations, these functions relate the amplitudes of the horizontal
and torsional components of the foundation input motion to the amplitudes of the
free-field ground motion. In Chapter 2, these functions are evaluated for rigid rect-
angular foundations subjected to spatially varying horizontal ground motions. The
incoherence of the ground motions is represented by an exponentially decaying func-
tion of the square of the exciting frequency and of the separation between points.
The kinematic interaction effects for the structure-foundation systems are examined
in Chapter 3.

An important step in the evaluation of the inertial interaction effects is the com-
putation of the foundation impedances. Defined again for harmonically excited
massless foundations, these functions represent the complex-valued amplitudes of the
forces or moments necessary to induce unit linear or angular displacement ampi-
tudes in each of the directions of possible motion. In Chapter 4, a re-evaluation is
made of the impedances for rigid rectangular foundations presented recently by Wong
and Luco [19]. It is shown that when plotted as a function of a modified frequency
parameter, the real-valued amplitudes of the foundation impedances are practically
independent of the ratio of side-lengths of the foundation. The inertial interaction
effects for the structure-foundation systems and their relationship to the kinematic
interaction effects are examined in Chapter 5.

The justification, importance and the scope of the various studies are discussed in
further detail in the introductions of the chapters in which the individual studies are
reported, and the conclusions of the component studies are summarized at the ends
of the relevant chapters.
Chapter 2

TRANSFER FUNCTIONS

2.1 Introduction

A fundamental step in the analysis of the dynamic response of a foundation or of a foundation-structure system to spatially varying ground motions is the evaluation of the transfer functions of the foundation. Defined for harmonically excited massless foundations, these functions represent the ratios of the amplitudes of the components of the steady-state motion actually experienced by the foundation to the amplitude of the free-field ground motion at some reference or control point.

For surface-supported, rigid, circular foundations subjected to horizontal ground motions of a stochastically specified spatial variation, these functions have been evaluated recently by Prasad and Veletsos [1] [2]. An approximate method of analysis, based on the averaging technique employed by Iguchi [3] and Scanlan [4] in their studies of wave passage effects for plane waves, was used. The objectives of the present study are to evaluate the corresponding functions for rigid, rectangular foundations, and to elucidate the effects and relative importance of the various parameters involved.

A uni-directional free-field ground motion directed parallel to one of the foundation sides is considered. The motion is characterized by a space-invariant power spectral density function and an incoherence function of a particular form. The response quantities examined include the amplitudes of the lateral and torsional components of the resulting foundation motion; these amplitudes are evaluated over wide ranges of the parameters involved, and the more important trends are displayed graphically.

The problem examined herein was studied previously by Luco and Wong [5], who formulated the more nearly exact integral expressions for the transfer functions of the foundation, and presented numerical solutions for square foundations excited by vertically incident, incoherent shear waves. As far as it can be determined, however, no numerical solutions have been reported for rectangular foundations.
The advantages over the Luco-Wong formulation of the approximate method employed here are that: (a) it leads to relatively simple, closed-form expressions for the desired functions; and (b) it reduces the number of independent parameters that must be considered, thereby simplifying the interpretation of the resulting solutions. In view of the uncertainties involved in the characterization in practice of the free-field ground motion, the solutions presented herein are believed to be sufficiently accurate for practical purposes.

2.2 System and Excitation Consideration

The system investigated is shown in Figure 2-1. It is a rigid, massless, rectangular foundation that is supported at the surface of an elastic half-space and is bonded to the supporting medium so that no sliding or uplifting may occur. The foundation is referred to a Cartesian system of coordinates, with its origin taken at the center of the foundation, and its axes parallel to the foundation sides. The lengths of the foundation sides in the $x$- and $y$-directions are denoted by $2a$ and $2b$, respectively. The half-space is characterized by its shear modulus, $G$, Poisson's ratio, $\nu$, and mass density, $\rho$. The shear wave velocity for the medium is then given by $v_s = \sqrt{G/\rho}$.

The free-field ground motion for all points of the foundation-soil interface is presumed to be directed along the $x$-axis, with the waves impinging on the foundation at an angle $\alpha_y$ with the vertical and propagating along the positive $y$-axis. The motion at any point is specified stochastically by a space-invariant, local power spectral density (psd) function, $S_p(\omega)$, in which $\omega$ is the circular frequency of the harmonic component of the ground motion under consideration, and the spatial correlation of the amplitudes of the component motions at two arbitrary points defined by the position vectors $\vec{r}_1$ and $\vec{r}_2$ is specified by the cross psd function,

$$S(\vec{r}_1, \vec{r}_2, \omega) = \Gamma(\vec{r}_1, \vec{r}_2, \omega) \exp\left[-i\omega \left(\frac{y_1 - y_2}{c_y}\right)\right] S_p(\omega) \quad (2.1)$$

Referred to as the incoherence function, the quantity $\Gamma$ is a dimensionless, generally decreasing function of $\omega$ and of the distance between points; $i = \sqrt{-1}$; $y_1$ and $y_2$ are the components (projections) of $\vec{r}_1$ and $\vec{r}_2$ in the direction of propagation of the waves; and $c_y$ is the apparent horizontal velocity of propagation of the waves. The latter quantity is related to the velocity of shear wave propagation in the medium, $v_s$, by the expression
Fig. 2-1 System Considered
\[ c_y = \frac{v_s}{\sin \alpha_y} \] (2.2)

The product of \( S_g \) and the exponential term in Equation 2.1 represents the component of ground motion variability associated with the wave passage effect, while the product \( \Gamma S_g \) represents the component due to the incoherence of the component waves. The peak value of \( \Gamma \) is unity and occurs at \( \vec{r}_1 = \vec{r}_2 \).

There is no general agreement at present on the form of the incoherence function that may be appropriate for earthquake ground motions. For the solutions presented herein, \( \Gamma \) is taken in the form proposed by Der et al. [11] as

\[ \Gamma(\vec{r}_1, \vec{r}_2, \omega) = \exp \left\{ -\left( \frac{\omega}{v_s} \right)^2 \left[ \gamma_x^2 (x_1 - x_2)^2 + \gamma_y^2 (y_1 - y_2)^2 \right] \right\} \] (2.3)

in which \( x_1 \) and \( x_2 \) are the \( x \)-coordinates of the points defined by the position vectors \( \vec{r}_1 \) and \( \vec{r}_2 \); \( y_1 \) and \( y_2 \) are the corresponding \( y \)-coordinates; and \( \gamma_x \) and \( \gamma_y \) are dimensionless factors with values typically in the range between 0 and 0.5. This form of incoherence will henceforth be referred to as orthotropic. For \( \gamma_x = \gamma_y = \gamma \), Equation 2.3 reduces to the isotropic form considered in References [1], [2], [5] and [6]. Furthermore, for \( \gamma_x = 0 \) and \( \gamma_y = \gamma \), it defines an one-dimensional incoherence in the \( y \)-direction; in this case, points with the same \( y \)-coordinate experience motions that are identical in both amplitude and phase.

### 2.3 Expressions for Transfer Functions

Let \( S_{hl} \) be the psd function for the lateral or horizontal component of the foundation displacement, and \( S_{ss} \) be the corresponding function for the displacement component induced by the torsional component of the foundation motion along the edges parallel to the direction of the ground shaking. Denoted by \( s(t) \), the latter component is defined by

\[ s(t) = b \psi(t) \] (2.4)

in which \( \psi(t) \) is the rotation at any time \( t \) of the foundation about a vertical centroidal axis, and \( b \) is the half-length of the foundation side normal to the direction of the free-field ground motion. With \( s(t) \) defined in this manner, the effects of the lateral and torsional components of foundation motion may be compared readily.
These functions were evaluated from the cross psd function of the free-field ground motion by application of the averaging technique employed by Iguchi [3] and Scanlan [4]. This approach leads to

\[
S_{ll} = \frac{1}{A^2} \int_A \int_A S(\vec{r}_1, \vec{r}_2, \omega) \, dA_1 \, dA_2
\]  

(2.5)

\[
S_{ss} = \frac{b^2}{I_\phi} \int_A \int_A y_1 y_2 S(\vec{r}_1, \vec{r}_2, \omega) \, dA_1 \, dA_2
\]  

(2.6)

The cross psd function for the two components of the foundation motion, \(S_{ls}\), is given similarly by

\[
S_{ls} = b S_{l\phi} = \frac{b}{I_\phi A} \int_A \int_A y_2 S(\vec{r}_1, \vec{r}_2, \omega) \, dA_1 \, dA_2
\]  

(2.7)

in which \(dA_1\) and \(dA_2\) are elemental areas of the foundation; \(A = 4ab\) is the area of the foundation; and \(I_\phi = \frac{1}{2} A(a^2 + b^2)\) is the polar moment of inertia of the foundation about a vertical centroidal axis.

On introducing the dimensionless distances \(\xi_1 = x_1/\alpha\), \(\xi_2 = x_2/\alpha\), \(\eta_1 = y_1/\beta\), and \(\eta_2 = y_2/\beta\), and on making use of Equation 2.1, Equations 2.5 through 2.7 may be rewritten as

\[
\frac{S_{ll}}{S_g} = \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \tilde{\Gamma} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2
\]  

(2.8)

\[
\frac{S_{ss}}{S_g} = \left(\frac{b^2 A}{I_\psi}\right)^2 \left\{ \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \eta_1 \eta_2 \tilde{\Gamma} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2 \right\}
\]  

(2.9)

\[
\frac{S_{ls}}{S_g} = \left(\frac{b^2 A}{I_\psi}\right) \left\{ \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \eta_2 \tilde{\Gamma} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2 \right\}
\]  

(2.10)

in which

\[
\tilde{\Gamma} = \exp \left\{ - \left[ d_{x}^2 (\xi_1 - \xi_2)^2 + d_{y}^2 (\eta_1 - \eta_2)^2 \right] - i \epsilon_y (\eta_1 - \eta_2) \right\}
\]  

(2.11)

\[
d_{x} = \gamma_x \frac{\omega a}{v_s}
\]  

(2.12)
\[ d_y = \gamma_y \frac{\omega b}{v_s} \]  
(2.13)

\[ e_y = \frac{\omega b}{c_y} = \sin \alpha_y \frac{\omega b}{v_s} \]  
(2.14)

Equations 2.8 through 2.10 can be integrated exactly in terms of the standard error function of complex argument to yield:

\[ \frac{S_{ll}}{S_g} = f_1 (d_y, e_y) g_1 (d_x) \]  
(2.15)

\[ \frac{S_{ss}}{S_g} = \frac{9}{\left[ 1 + (a/b)^2 \right]^2} f_2 (d_y, e_y) g_1 (d_x) \]  
(2.16)

\[ \frac{S_s}{S_g} = \frac{3}{\left[ 1 + (a/b)^2 \right]} f_3 (d_y, e_y) g_1 (d_x) \]  
(2.17)

in which the functions \( f_1, f_2, f_3, \) and \( g_1 \) are given by

\[ f_1 (d_y, e_y) = B_1 (d_y, e_y) - B_3 (d_y, e_y) - \frac{e_y}{4d_y^2} B_2 (d_y, e_y) \]  
(2.18)

\[ f_2 (d_y, e_y) = \frac{1}{3} \left\{ B_1 (d_y, e_y) - B_3 (d_y, e_y) - \frac{1}{2d_y^2} [1 - B_3 (d_y, e_y)] \right\} \]  
\[ - \frac{e_y \left( 24d_y^4 - 6d_y^2 + e_y^2 \right)}{32d_y^6} B_2 (d_y, e_y) \]  
\[ - \frac{e_y^2}{8d_y^4} [B_3 (d_y, e_y) - 2B_4 (d_y, e_y)] \]  
(2.19)

\[ f_3 (d_y, e_y) = i \left\{ \frac{e_y}{4d_y^2} [B_1 (d_y, e_y) - B_3 (d_y, e_y)] \right\} \]  
\[ + \frac{2d_y^2 - e_y^2}{16d_y^4} B_2 (d_y, e_y) \]  
(2.20)

\[ g_1 (d_x) = f_1 (d_x, 0) = \frac{\sqrt{\pi}}{2d_x} \Phi (2d_x) - \frac{1 - \exp \left( -4d_x^2 \right)}{4d_x^2} \]  
(2.21)

and
\begin{align}
B_1 (d_y, e_y) &= \frac{\sqrt{\pi}}{2d_y} \exp \left( -\frac{e_y^2}{4d_y^2} \right) \Re \left[ \phi \left( 2d_y + i \frac{e_y}{2d_y} \right) \right] \\
B_2 (d_y, e_y) &= \frac{\sqrt{\pi}}{2d_y} \exp \left( -\frac{e_y^2}{4d_y^2} \right) \Im \left[ \phi \left( 2d_y + i \frac{e_y}{2d_y} \right) - \phi \left( i \frac{e_y}{2d_y} \right) \right] \\
B_3 (d_y, e_y) &= \frac{1 - \exp \left( -4d_y^2 \right) \cos (2e_y)}{4d_y^2} \\
B_4 (d_y, e_y) &= \frac{\exp \left( -4d_y^2 \right) \sin (2e_y)}{2e_y}
\end{align}

The function \( \phi(z) \) in Equations 2.22 and 2.23 is the error function with complex argument \( z \), defined by
\begin{equation}
\phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp (-v^2) \, dv
\end{equation}

and the symbols \( \Re[.] \) and \( \Im[.] \) represent the real and imaginary parts of the bracketed quantities.

The quantities \( \sqrt{S_{ll}}/S_g \) and \( \sqrt{S_{ss}}/S_g \) represent the transfer functions for the lateral and torsional components of the foundation motion, whereas the modulus or amplitude of \( S_{ll}/\sqrt{S_{ll}S_{ss}} \) is a measure of the degree of correlation between the two components of the motion. A numerical value of unity for the latter quantity indicates that the component motions are fully correlated, i.e., with one component known, the other can be predicted; whereas a zero value indicates that the component motions are uncorrelated, i.e., their interrelationship is random.

Equation 2.15 reveals that the lateral transfer function, \( \sqrt{S_{ll}}/S_g \), depends on the parameters \( d_x, d_y \) and \( e_y \). When normalized with respect to the factor
\begin{equation}
C_o = \frac{3}{1 + (a/b)^2}
\end{equation}

the same is also true of the transfer function for the torsional component of foundation motion, defined by Equation 2.16. By contrast, the amplitude of \( S_{lt}/\sqrt{S_{ll}S_{ss}} \), which from equations 2.15 through 2.17 can be shown to be given by
\[
\left| \frac{S_{ls}}{\sqrt{S_{ll}S_{ss}}} \right| = \frac{|f_3(d_y, e_y)|}{\sqrt{f_1(d_y, e_y) f_2(d_y, e_y)}}
\]

is independent of \(d_x\) and a function of \(d_y\) and \(e_y\) only. The symbol \(|\cdot|\) represents the amplitude of the enclosed quantity. Considering that the free-field ground motion is presumed to be directed along the \(x\)-axis, the motions in that direction for points with the same \(x\)-coordinate but different \(y\)-coordinates are clearly independent of \(\gamma_x\). The same is also true of the lateral and torsional components of the resulting foundation motions.

### 2.3.1 Reduction to Special Cases

For vertically incident incoherent waves, for which \(\alpha_y = 0\) and hence \(e_y = 0\), Equations 2.15 through 2.17 reduce to

\[
\frac{S_{ll}}{S_g} = g_1(d_y) g_1(d_x)
\]

\[
\frac{S_{ss}}{S_g} = \frac{C_o^2}{3} \left\{ g_1(d_y) - \frac{1}{2d_y^2} [1 - B(d_y)] \right\} g_1(d_x)
\]

\[
S_{ls} = 0
\]

in which

\[
B(d_y) = B_3(d_y, 0) = \frac{1 - \exp\left(-4d_y^2\right)}{4d_y^2}
\]

The value of \(\left|S_{ls}/\sqrt{S_{ll}S_{ss}}\right|\) in this case is zero, and \(\sqrt{S_{ll}/S_g}\) and the quantity \(\sqrt{S_{ss}/S_g/C_o}\) depend on \(d_x\) and \(d_y\) only.

For obliquely incident coherent waves, for which \(\gamma_x = \gamma_y = d_x = d_y = 0\), Equations 2.15 through 2.17 can be expressed solely in terms of \(e_y\) as

\[
\frac{S_{ll}}{S_g} = \left(\frac{\sin e_y}{e_y}\right)^2
\]

\[
\frac{S_{ss}}{S_g} = C_o^2 \left[ \frac{1}{e_y} \left( \frac{\sin e_y}{e_y} - \cos e_y \right) \right]^2
\]
\[
\frac{S_{ts}}{S_g} = iC_o \left( \frac{\sin e_y}{e_y} \right) \left[ \frac{1}{e_y} \left( \frac{\sin e_y}{e_y} - \cos e_y \right) \right]
\]  
(2.35)

Note that the value of \( S_{ts}/\sqrt{S_{ss}S_{ss}} \) is unity in this case. Equations 2.33 through 2.35 have been presented previously by Luco and Sotiropoulos [12].

### 2.4 Results for Vertically Incident Incoherent Waves

The transfer function \( \sqrt{S_{tt}/S_g} \) and the quantity \( \sqrt{S_{ss}/S_g}/C_o \) are plotted in Figure 2-2 as a function of \( d_y \) for fixed values of

\[
\varepsilon = \frac{d_x}{d_y} = \frac{\gamma_x}{\gamma_y} \frac{a}{b}
\]  
(2.36)

Referred to as the effective ratio of foundation side-lengths, the factor \( \varepsilon \) may also be viewed as a measure of the relative importance of the ground incoherence in the \( x- \) and \( y- \) directions. Note that the solution for orthotropic incoherence (i.e., values of \( \gamma_x \) and \( \gamma_y \) that are different) is governed by the same number of dimensionless parameters as that for isotropic incoherence (i.e., values of \( \gamma_x = \gamma_y = \gamma \)). As a matter of fact, the results for the former case may be obtained from those for the latter by merely changing the ratio of the side lengths of the foundation, \( a/b \), so that the values of \( \varepsilon \) in the two cases are the same.

For ease of comparison with the transfer functions for the lateral component of foundation motion displayed in the upper part of Figure 2-2, the corresponding functions for the torsional component are replotted in Figure 2-3 without the normalizing factor \( C_o \). These particular data are for isotropic incoherence only.

The following trends are worth noting in Figures 2-2 and 2-3:

1. For a fixed value of \( \varepsilon \), the transfer functions for the lateral component of foundation motion decreases monotonically with increasing \( d_y \), whereas the torsional component increases from zero to a peak value and then decreases monotonically. These trends are similar to those of the corresponding functions for circular foundations presented in References [1] and [2].

2. Increasing the effective ratio of foundation sides, \( \varepsilon \), decreases the values of both \( \sqrt{S_{tt}/S_g} \) and \( \sqrt{S_{ss}/S_g} \). These trends may be explained by reference to the
Fig. 2-2 Normalized Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Rectangular Foundations Subjected to Vertically Incident Incoherent Waves: Plotted Against Frequency Parameter $d_z$.
Fig. 2-3 Transfer Functions for Torsional Component of Foundation Input Motion for Rectangular Foundations Subjected to Vertically Incident Incoherent Waves: Isotropic Incoherence, $\gamma_x = \gamma_y = \gamma$
results for isotropic incoherence, keeping in mind that the length $b$ is effectively constant in these plots. Increasing $a/b$ increases the length $a$ over which the incoherence of the ground motion must be averaged, and this increase decreases the effective or weighted values of both the lateral and torsional components of the resulting foundation motion.

3. For isotropic incoherence ($\gamma_x = \gamma_y = \gamma$), the reductions with increasing $a/b$ of the torsional component of foundation motion shown in Figure 2-3 are consistently greater than the corresponding reductions for the lateral component shown in the upper part of Figure 2-2. Two factors are responsible for this trend: (a) the torsional component of the foundation motion is more closely related to the moment of inertia of the foundation about its vertical centroidal axis than to its area, and (b) an increase in $a$ increases the moment of inertia of the foundation more rapidly than the area. The increase in moment of inertia is effectively represented by the factor $C_o$, defined by Equation 2.27.

4. The curves for $\varepsilon = 0$ in Figures 2-2 and 2-3 are applicable to foundations of arbitrary ratios of sides, provided the ground motion incoherence is one-dimensional, i.e., $\gamma_x = 0$ and $\gamma_y \neq 0$. Because of the reduced interference between the components of the free-field ground motion in this case, the reduction with increasing $d_y$ in the lateral component of foundation motion is smaller than for the two-dimensional incoherence, and the corresponding increase in the torsional component of motion is greater.

For the vertically incident incoherent wave fields examined in this section, the lateral and torsional components of the foundation motion are uncorrelated, and $S_{t_l} = 0$.

2.4.1 Alternative Presentation of Results

For the non-vertically incident waves considered in Equations 2.15 through 2.17, the transfer functions were expressed in terms of $d_x$, $d_y$ and $e_y$, whereas for the vertically incident waves examined in Figures 2-2 and 2-3, for which $e_y = 0$, they were expressed in terms of $d_y$ and the effective side-lengths ratio, $\varepsilon = d_x/d_y$. In general, the results can most effectively be displayed in terms of $\varepsilon$, the generalized frequency parameter,

$$\tilde{a}_e = \sqrt{d_x d_y} + e_y^2 = \frac{\omega b}{v_s} \sqrt{\gamma_x \gamma_y \left( \frac{a}{b} \right) + \sin^2 \alpha_y}$$

(2.37)
and the generalized incoherence parameter,

\[
\tilde{\gamma} = \frac{\sqrt{d_x d_y}}{e_y} = \frac{\sqrt{\gamma_x \gamma_y}}{\sin \alpha_y} \sqrt{\frac{a}{b}}
\] (2.38)

In addition to the frequency of the motion and the characteristics of the foundation and the supporting medium, the parameter \( \tilde{a}_o \) is a measure of the overall spatial variation of the ground motion, whereas \( \tilde{\gamma} \) is a measure of the relative importance of the components of that variation due to random incoherence and wave passage. These parameters are analogous to those used in the study of circular foundations reported in Reference [1] and [2]. A value of \( \tilde{\gamma} = 0 \) refers to plane, coherent waves with arbitrary angle of incidence, \( \alpha_y \), whereas \( \tilde{\gamma} = \infty \) refers to vertically incident, incoherent wave fields.

In comparing the behavior of rectangular foundations of different proportions, it is also desirable to relate them to that of an equivalent square foundation having the same area. On noting that the half-length of the side for the equivalent square foundation, \( b_c \), is given by

\[
b_c = \sqrt{ab} = b \sqrt{\frac{a}{b}}
\] (2.39)

Equations 2.37 and 2.38 may be rewritten in the form

\[
\tilde{a}_o = \frac{\omega b_c}{v_s} \sqrt{\gamma_x \gamma_y + (\sin^2 \alpha_y)(b/b_c)^2}
\] (2.40)

and

\[
\tilde{\gamma} = \frac{\sqrt{\gamma_x \gamma_y} b_c}{\sin \alpha_y} \frac{b_c}{b}
\] (2.41)

For the vertically incident incoherent waves examined in this section,

\[
\tilde{a}_o = \frac{\sqrt{\gamma_x \gamma_y}}{v_s} \frac{\omega b_c}{v_s}
\] (2.42)

and \( \tilde{\gamma} = \infty \). The parameters \( d_x \) and \( d_y \) are then related to \( \tilde{a}_o \) and \( \varepsilon \) by

\[
d_x = \tilde{a}_o \sqrt{\varepsilon}
\] (2.43)
\[ d_y = \frac{\bar{a}_o}{\sqrt{\bar{\varepsilon}}} \]  

(2.44)

The results presented in Figure 2-2 are replotted in Figure 2-4 as a function of \( \bar{a}_o \) for fixed values of \( \varepsilon \). Note that when displayed in this format, the curves for the lateral transfer function almost merge into a single curve. Accordingly, this function may, to a reasonable degree of approximation, be considered to be independent of \( \varepsilon \) and expressible only in terms of the frequency parameter \( \bar{a}_o \), which, being a function of \( b_c \), depends on the total area of the foundation. The high-frequency limit of the normalized version of the torsional transfer function is also nearly independent of \( \varepsilon \), indicating that this limit too is a function of the total area of the foundation.

Also shown in Figure 2-4 in dashed lines is the lateral transfer function for rigid circular foundations and isotropic incoherence presented previously by Veletsos and Prasad [2]. The half-length of the side of the equivalent square foundation in this case is

\[ b_c = \frac{\sqrt{\pi}}{2} R \]  

(2.45)

in which \( R \) is the radius of the circular foundation. The close agreement of the results for the circular and square foundations is a further confirmation of the dependence of the lateral transfer functions on the total area of the foundation.

An effort was also made to replot the results for the torsional transfer function in terms of parameters which might merge them into a single curve. However, it did not prove possible to achieve this objective. In particular, foundations with the same moment of inertia about a vertical centroidal axis, \( I_{\psi} \), did not generally yield comparable results. In retrospect, this result is not surprising, since the torsional component of the foundation motion depends not only on \( I_{\psi} \), but also on the moment of inertia of the foundation about a horizontal centroidal axis parallel to the direction of free-field ground motion (see Equation 2.9).

### 2.4.2 Properties of Lateral Transfer Functions

The following two properties may be established for the non-dimensionalized spectral density function, \( S_{uu}/S_g \), of the lateral component of the foundation motion:
Fig. 2-4 Normalized Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Rectangular Foundations Subjected to Vertically Incident Incoherent Waves: Plotted Against Frequency Parameter $\tilde{a}_o$. 

\[ \varepsilon = \frac{\gamma_x a}{\gamma_y b} = 1 \]

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\[ \frac{1}{C_o} \sqrt{\frac{S_{ss}}{S_g}} \]

\[ \frac{1}{4}, 4 \]

\[ \frac{1}{2}, 2 \]

\[ \frac{1}{4}, 4 \]

\[ \varepsilon = \frac{\gamma_x a}{\gamma_y b} = 4 \]
1. On noting that Equations 2.43 and 2.44 for \( d_x \) and \( d_y \) yield the same result if the value of \( \varepsilon \) in one is taken as its reciprocal in the other, it is concluded from Equation 2.29 that

\[
\left( \frac{S_{ll}}{S_g} \right)_\varepsilon = \left( \frac{S_{ll}}{S_g} \right)^{1/\varepsilon}
\]

(2.46)

in which the subscripts identify the effective side-length ratios considered. It follows that, even for the orthotropic ground-motion incoherence, the transfer function for the lateral component of the foundation motion, \( \sqrt{S_{ll}/S_g} \), is not altered by a 90 degree change in the orientation of the foundation.

2. On first squaring and then taking the square root of its right hand members, Equation 2.29 may, after regrouping of terms, be rewritten as

\[
\frac{S_{ll}}{S_g} = \sqrt{\frac{S_{ll}(d_x, d_x)}{S_g}(d_y, d_y)}
\]

(2.47)

The first term on the right hand side of this equation represents the value of \( S_{ll}/S_g \) for a square foundation of side length \( 2a \) and isotropic incoherence characterized by a value of \( \gamma_x \), whereas the second term represents the corresponding value for a square foundation of side length \( 2b \) and isotropic incoherence with \( \gamma_y \). On identifying the two sets of conditions with the subscripts \( (a, \gamma_x) \) and \( (b, \gamma_y) \), Equation 2.47 can be rewritten as

\[
\frac{S_{ll}}{S_g} = \sqrt{\left( \frac{S_{ll}}{S_g} \right)_{a, \gamma_x} \left( \frac{S_{ll}}{S_g} \right)_{b, \gamma_y}}
\]

(2.48)

In other words, \( S_{ll}/S_g \), and hence the lateral transfer function \( \sqrt{S_{ll}/S_g} \), for a rectangular foundation and orthotropic incoherence is equal to the geometric mean of the corresponding functions of square foundations with sides equal to each of the sides of the rectangular foundation and the indicated isotropic incoherences.
Note should finally be taken of the fact that as either $a$ or $\gamma_e$ tends to zero, $(S_{ll}/S_g)_{a, \gamma_e}$ tends to unity, and hence

$$\frac{S_{ll}}{S_g} = \sqrt{\left(\frac{S_{ll}}{S_g}\right)_{b, \gamma_y}} \quad (2.49)$$

It follows that, both for very long, narrow foundations subjected to ground motions characterized by orthotropic incoherence, and for foundations of arbitrary proportions with one-dimensional incoherence defined by the parameter $\gamma_y$, the transfer function for lateral motion is equal to the square root of the corresponding function for a square foundation of side length $2b$ and isotropic incoherence with $\gamma = \gamma_y$.

2.5 Results for Obliquely Incident Incoherent Waves

Representative transfer functions for foundations subjected to obliquely incident incoherent waves are shown in Figure 2-5. Plotted for fixed values of $\tilde{\gamma}$ as a function of the generalized frequency parameter defined by Equation 2.40, these curves are for systems with an effective side-length ratio $\varepsilon = 1$. As before, the transfer functions for the torsional component of the foundation motion are normalized by the factor $C_0$.

Whereas the curves for the larger values of $\tilde{\gamma}$ (i.e., when ground motion incoherence dominates wave passage) vary smoothly in Figure 2-5, those for values of $\tilde{\gamma}$ equal to or close to zero (i.e., when wave passage dominates) are undulatory. These trends are similar to those presented in Reference [2] for circular foundations and isotropic incoherence. Incidentally, the curves for the limiting value of $\tilde{\gamma} = 0$ are independent of $\varepsilon$, and, therefore, also apply to a wider range of conditions than those considered for these plots.

The effect of the factor $\varepsilon$ of the transfer functions of foundations to obliquely incident waves is shown in Figure 2-6. These plots refer to systems with $\tilde{\gamma} = 1$. It is noteworthy that, unlike the corresponding plots for vertically incident incoherent waves ($\tilde{\gamma} = \infty$) presented in Figure 2-4, the curves for the lateral transfer functions in this figure do not merge into a single curve, neither do the high-frequency limits of the curves for the torsional transfer functions. The reason for these differences may be appreciated from Equations 2.33 and 2.34, which define the wave passage effect for coherent, plane waves. Being functions of $e_y$, these expressions depend on the length $b$ rather than the length $b_c$ involved in the definition of $\tilde{a}_c$. 
Fig. 2-5 Normalized Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Rectangular Foundations with $\varepsilon = 1$
Fig. 2-6 Normalized Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Rectangular Foundations Subjected to Obliquely Incident Incoherent Waves with $\bar{\gamma} = 1$
It may be recalled that the correlation amplitude for the lateral and torsional components of foundation motion, \( |S_{ls}/\sqrt{S_{ll}S_{sl}}| \), depends on only two parameters, \( d_y \) and \( e_y \). This amplitude is plotted in Figure 2-7 as a function of the modified frequency parameter

\[
\tilde{a}_y = \sqrt{d_y^2 + e_y^2} = \frac{\omega b}{v_s} \sqrt{\gamma^2_y + \sin^2\alpha_y}
\]  

(2.50)

and the modified incoherence parameter

\[
\tilde{\gamma} = \frac{d_y}{e_y} = \frac{\gamma_y}{\sin \alpha_y}
\]  

(2.51)

It is noteworthy that these results are independent of \( \varepsilon \), and apply, therefore, to foundations of arbitrary ratio of sides and arbitrary values of the ground motion incoherence parameter \( \gamma_x \). The results in Figure 2-7 are similar to those presented in Reference [2] for circular foundations and isotropic incoherence.

### 2.6 Effect of Two-component Ground Motion

The free-field ground motion in the discussion so far was presumed to be directed parallel to one of the foundation sides. For a two-dimensional excitation for which the components along the \( x \)- and \( y \)-axes may be considered to be independent of each other, the horizontal components of the foundation motion will also be independent, and the transfer functions for both the lateral and torsional responses can be obtained from the expressions already presented by superposing the effects of each component excitation. However, great care must be exercised in the application of these expressions properly to interpret the parameters involved. In this connection, it should be recalled that, for the one-dimensional excitation considered, the ground motion is directed parallel to the \( x \)-axis, the propagation of the waves is parallel to the \( y \)-axis, and the distances \( a \) and \( b \) are, respectively, parallel to and normal to the direction of the ground motion.

### 2.7 Other Meanings for Results

Although defined specifically for the displacement histories of the foundation input motion, the spectral density ratios \( S_{ll}/S_g \), \( S_{ss}/S_g \) and \( S_{ls}/S_g \) also define the ratios \( S_{ll}/S_{\tilde{y}} \), \( S_{ss}/S_{\tilde{y}} \), \( S_{ls}/S_{\tilde{y}} \) and \( S_{ll}/S_{\tilde{\gamma}} \), \( S_{ss}/S_{\tilde{\gamma}} \), \( S_{ls}/S_{\tilde{\gamma}} \) of the corresponding velocity and
\[ \gamma = \frac{\gamma_y}{\sin \alpha_y} = 0 \]

\[ \bar{a}_o = \frac{\omega b}{v_s} \sqrt{\gamma_y^2 + \sin^2 \alpha_y} \]

Fig. 2-7 Normalized Cross PSD Functions for Lateral and Torsional Components of Foundation Input Motion for Rectangular Foundations Subjected to Obliquely Incident Incoherent Waves
acceleration traces. It may be recalled that the psd function for the first derivative of a stationary random process is given by the product of \((2\pi f)^2\) and the corresponding function of the original process.

2.8 Conclusion

Information and concepts has been presented which elucidate the effects and relative importance of the various factors that control the transfer functions of surface-supported, rigid rectangular foundations excited by horizontally polarized, incoherent shear waves with motions parallel to one of the foundation sides. For vertically incident incoherent wave fields, the transfer function for the lateral component of foundation motion for a rectangular foundation and orthotropic incoherence has been shown to be equal to the geometric mean of the corresponding functions for square foundations with sides equal to each of the sides of the rectangular foundation and appropriate isotropic incoherences. The lateral transfer function of a square foundation may, in turn, be approximated accurately by that of an equivalent circular foundation. The transfer function for the torsional component of foundation motion is sensitive to the foundation side-length ratio, decreasing in value with increasing relative length of the foundation in the direction of the ground motion.
Chapter 3

KINEMATIC INTERACTION EFFECTS

3.1 Introduction

This section deals with the kinematic interaction effects for simple structures subjected to spatially varying horizontal free-field ground motions representative of those induced by earthquakes. The structures are presumed to be supported on the ground surface through rigid rectangular foundations, and the free-field ground motion is specified stochastically by a space-invariant local power spectral density function and the incoherence function considered in the proceeding chapter.

The mean peak values of foundation input motions and those of structural responses are presented. Some important insights can be extracted from these results and compared with those for transfer functions in Chapter 2.

3.2 System and Ground Motion Considered

The system investigated is shown in Figure 3-1. It is a linear structure of mass \( m \) and height \( h \), which is supported through a rigid, rectangular foundation of mass \( m_f \) at the surface of a homogeneous elastic halfspace. The columns are presumed to be massless and axially inextensible. The plan areas of the structure and foundation are considered to be identical, and both \( m \) and \( m_f \) are assumed to be uniformly distributed over their respective areas. The side lengths of the structure and foundation along the \( x \) - and \( y \) - coordinate axes are denoted, as in Chapter 2, by \( 2a \) and \( 2b \), respectively. Complete bonding is assumed between the foundation and supporting medium, which, as in Chapter 2, is characterized by its mass density, \( \rho \), shear velocity, \( v_s \), and Poisson's ratio, \( \nu \). The circular natural frequencies of the lateral and torsional modes of vibration for the structure when fixed at its base are denoted by \( \omega_i = 2\pi f_i \) and \( \omega_\theta = 2\pi f_\theta \), respectively, in which \( f_i \) and \( f_\theta \) are the associated frequencies in cycles per unit of time; and the corresponding percentages of critical damping are denoted by \( \zeta_i \) and \( \zeta_\theta \), respectively. This structure may be viewed either
Fig. 3-1 System Considered
as the direct model of a single-story building or, more generally, as the model of a multistory, multimode structure that responds as a system with one lateral and one torsional degree of freedom in its fixed-base condition.

The free-field ground motion for all points of the foundation-soil interface is considered to be a uni-directional excitation directed parallel to the horizontal \( x \)-axis, as shown in Figure 3-1, with the detailed histories of the motions varying from point to point. Such motions may be induced by horizontally polarized incoherent shear waves propagating either vertically or at an arbitrary angle with the vertical, \( \alpha_y \). The intense portion of the motion at any point is represented by an ergodic, Gaussian random process with zero mean and limited duration, \( t_o \), and the spatial variation of the motion is specified by the incoherence function considered in Chapter 2. The local psd function for the acceleration traces of the motions is taken, in the form employed by Pais and Kausel in a related study [8],

\[
S_g = \begin{cases} 
\frac{f^4}{0.5 + f^4} \left( 1 - \frac{f^2}{f_o^2} \right) S_o & \text{for } f \leq f_o \\
0 & \text{for } f \geq f_o
\end{cases}
\]  

(3.1)

in which \( S_o \) is a constant, \( f = \omega/2\pi \) is the exciting frequency in cps, and \( f_o \) is the cut-off frequency which is taken as 15 cps.

Let \( \bar{X}_g \) be the mean of the absolute maximum peaks of the acceleration traces of the free-field motion, and \( \bar{X}_g \) and \( X_g \) be the corresponding means of the velocity and displacement traces. These quantities were computed from Der Kiurghian’s empirical expressions [14] taking \( t_o = 20 \text{sec} \). The resulting values are \( \bar{X}_g = 26.173\sqrt{S_o} \), \( \bar{X}_g = 1.4174\sqrt{S_o} \), and \( X_g = 0.24676\sqrt{S_o} \) [2].

### 3.3 Foundation Input Motion

Before examining the response of the structure, it is desirable to compute the mean peak values of the acceleration, velocity and displacement traces of both the lateral and torsional components of foundation input motion. The relevant values for the lateral or horizontal component of foundation input motion are denoted by \( \bar{X}_l \), \( \bar{X}_l \), and \( X_l \), whereas those for the torsional or circumferential component along the foundation side parallel to the direction of the free-field ground motion are denoted by \( \bar{X}_t \), \( \bar{X}_t \), and \( X_t \). More specifically, \( X_s = b\psi \), where \( \psi \) represents the mean peak value of the rotation of the massless foundation about a vertical axis. Computed from the
appropriate psd functions by use of Der Kiurghian's approximation, the values of \( \tilde{X}_1 \), \( \tilde{X}_l \) and \( X_1 \), normalized with respect to the mean peak values of the corresponding histories of the free-field ground motion are displayed in the upper part of Figure 3-2, and the values of \( \tilde{X}_s \), \( \tilde{X}_s \) and \( X_s \), normalized with respect to the product of the constant \( C_0 \) and the mean peak values of the corresponding histories of the free-field ground motion are shown in the lower part of the figure. The particular plots are for rectangular foundations with several different values of the effective side-lengths ratio, \( \varepsilon \), subjected to vertically propagating incoherent wave fields (\( \tilde{\gamma} \geq \infty \)). The results for \( \varepsilon = 1 \) (square foundations with isotropic incoherence) are compared in Figure 3-3 with those obtained for \( \tilde{\gamma} = 0 \) and \( \tilde{\gamma} = 1 \). As might be expected, the latter plots are similar to those presented in Reference [2] for structures supported on circular foundations.

Instead of the modified frequency parameter, \( \alpha_{\sigma} \), which is applicable to incoherent harmonic wave fields, the quantities displayed in Figures 3-2 and 3-3 are functions of the generalized transit time

\[
\tilde{\tau} = \sqrt{\frac{\gamma_x \gamma_y}{b^2} + \sin^2 \alpha_y} \frac{b}{v_s} = \sqrt{\frac{\gamma_x \gamma_y}{b^2} + \sin^2 \alpha_y} \tau \tag{3.2}
\]

in which \( \tau = b/v_s \), usually referred to as transit time, is the time required for a shear wave in the supporting medium to traverse a distance \( b \) (the half-length of the side of foundation normal to the direction of the free-field ground motion).

The following trends are worth observing in these plots:

- The reduction in the lateral component of the foundation input motion and the corresponding increase in the torsional component are greatest for acceleration, much smaller for velocity, and almost negligible for displacement. In effect, the foundation acts as a low-pass filter, filtering the high-frequency wave components. Therefore, the acceleration traces of the ground motion, which are richer in high-frequency content than the velocity and displacement traces, are influenced much more than the latter traces. This trend is similar to that found for the circular foundations examined in Reference [2].

- On comparing Figures 3-2 with Figure 2-4 and Figures 3-3 with Figure 2-5, it is observed that the effects of \( \tilde{\gamma} \), \( \varepsilon \) and \( \tilde{\tau} \) on the mean peak values of the foundation input motion are similar to those for the corresponding transfer functions.
Fig. 3-2 Effects of $\varepsilon$ and $\bar{\tau}$ on Mean Peak Values of Lateral and Torsional Components of Foundation Input Displacements, Velocities and Accelerations for Vertically Incident Incoherent Waves
Fig. 3-3 Effects of $\tilde{\gamma}$ and $\tilde{\tau}$ on Mean Peak Values of Lateral and Torsional Components of Foundation Input Displacements, Velocities and Accelerations for Systems with $\epsilon = 1$
• Considering that the response of high-frequency systems is acceleration-sensitive and that of low-frequency systems is displacement-sensitive, it should be clear that the effects of kinematic interaction would be important for high-frequency systems and negligible for low-frequency systems. Furthermore, the medium-frequency systems which are velocity-sensitive would be expected to be affected moderately. That this is indeed the case is confirmed by the data presented in the following section.

### 3.4 Structural Response

With the psd functions of the foundation input motion established, the corresponding functions of the structural response can be determined by well-established procedures (e.g., [13]). Let \( S_{u_{ui}} \) be the psd function of the structural deformation, \( u_i \), induced by the lateral component of the foundation input motion; and let \( S_{u_{us}} \), be the corresponding function of the deformation, \( u_s = \dot{b}\theta \), induced along the side of the structure parallel to the direction of the free-field ground motion by the torsional component of the foundation input motion. The quantity \( \theta \) represents the angular deformation of the structure. The quantities \( S_{u_{ui}} \) and \( S_{u_{us}} \) are related to the psd functions of the foundation input accelerations, \( S_{\dddot{u}} \) and \( S_{\dddot{s}} \), by

\[
S_{u_{ui}} = |H_{u_i}|^2 S_{\dddot{u}}
\]

(3.3)

and

\[
S_{u_{us}} = |H_{u_s}|^2 S_{\dddot{s}}
\]

(3.4)

in which \( H_{u_i} \) is the structure transfer function for lateral response, given by

\[
H_{u_i} = -\frac{1}{\nu_i^2} \frac{1}{1 - (\omega/\nu_i)^2 + i2\zeta_i(\omega/\nu_i)}
\]

(3.5)

and \( H_{u_s} \) is the corresponding function for torsional response, given by

\[
H_{u_s} = -\frac{1}{\nu_s^2} \frac{1}{1 - (\omega/\nu_s)^2 + i2\zeta_s(\omega/\nu_s)}
\]

(3.6)

and the quantities enclosed by vertical bars indicate their real-valued moduli. Furthermore, the psd function \( S_{uu} \) for the total deformation at the mid-point of the
side of the structure parallel to the direction of free-field ground motion, \( u = u_l + u_s \), is given by [13]

\[
S_{uu} = S_{u_l u_l} + S_{u_s u_s} + 2 \Re \left( H_{u_l} H_{u_s}^* S_{f\tilde{f}} \right)
\]  

(3.7)

where \( S_{f\tilde{f}} \) is the cross psd function of the foundation input acceleration; \( \Re \) denotes the real part of the indicated quantity; and the superscript \(*\) denotes the complex conjugate of the quantity to which it is attached.

Let \( U_l \) be the mean of the maximum values of structural deformations induced by the ensemble of the lateral components of the foundation input motions, and \( U_s \) be the corresponding mean of the deformations induced by the torsional components along the side of the structure parallel to the direction of free-field ground motion. Further, let \( V_l = p_l U_l \) and \( V_s = p_s U_s \) represent the corresponding pseudo-velocities. Pseudo-velocity response spectra for structures subjected to vertically incident incoherent shear waves (\( \gamma = \infty \)) are presented in Figure 3-4. The results are normalized with respect to the mean peak value of the free-field ground velocity. Several values of \( \varepsilon \) and three values of the effective transit time, \( \tilde{\tau} \), are considered, including the limiting case of \( \tilde{\tau} = 0 \) for which there is no kinematic interaction. The damping factors for both the lateral and torsional modes of vibration are taken as \( \zeta_l = \zeta_s = 0.02 \).

The following trends can be observed from Figure 3-4:

- As anticipated from examination of the peak values of the foundation input motion, the lateral component of the response of high-frequency systems is reduced significantly by ground motion incoherence, and this reduction is particularly large within the practically important region of the response spectrum for which the pseudo-acceleration attains its maximum value. The reduction is less pronounced for medium-frequency systems and practically negligible for low-frequency systems.

- The general trends of the response spectra for the torsional deformation in Figure 3-4 are consistent with those of the corresponding curves for the foundation input motion displayed in Figure 3-2. In the low-frequency, displacement-sensitive region, the response increases with increasing values of the effect transit time, \( \tilde{\tau} \), whereas in the high-frequency, acceleration-sensitive region, the response decreases with the increasing values of \( \tilde{\tau} \).
Fig. 3-4 Effects of $\epsilon$ and $\bar{\tau}$ on Normalized Pseudo-Velocity Response Spectra for Lateral and Torsional Modes of Vibration of Simple Structures Subjected to Vertically Propagating Incoherent Waves; $\zeta_l = \zeta_\theta = 0.02$
• The effect of the effective ratio of side lengths, $\varepsilon$, is similar to that for the transfer function displayed in Figure 2-2 and that for the foundation input motion shown in Figure 3-2.

• The component of the response contributed by the torsional motion of the foundation is relatively small, and the combined effect of lateral and torsional response is usually only slightly greater than that due solely to the lateral response. This is shown in Figure 3-5 for structures with square foundations considering that $p_0/p_l = 1.5$. These results are computed from Equation 3.7 making use of Der Kiurghian’s approximation. Decreasing the value of $\varepsilon$ tends to increase the contribution of the torsional component of the response. This is consistent with the results shown in Figure 3-2, which show that the torsional deformation increases with decreasing $\varepsilon$.

In Figure 3-6, the response spectra representing the effects of the ground motion incoherence are compared with those considering only wave passage effects, and those for a combination of wave passage and ground motion incoherence effects represented by a value $\tilde{\gamma} = 1$. These results are for systems with $\varepsilon = 1$. It is observed that the responses are not sensitive to the value of the parameter $\tilde{\gamma}$, and that this insensitivity is fully consistent with that observed in Figure 3-3 for the foundation input motions. Therefore, to a reasonable degree of approximation, the effect of wave passage can be replaced by the effect of ground motion incoherence and vice versa. This approximation was first suggested by Luco and Wong [5] from examination of the relevant foundation input motions and further confirmed by the studies reported in Reference [2]. In implementing this replacement, it is important that the value of $\tilde{\tau}$ be the same in both cases.

### 3.5 Conclusions

• The kinematic interaction effects for systems subjected to incoherent transient ground motions are characterized by the effective transit time parameter, $\tilde{\tau}$, the incoherence parameter, $\tilde{\gamma}$, and the effective ratio of side lengths, $\varepsilon$.

• The kinematic interaction effects on the mean peak values of structural responses can be related with reasonable accuracy to the corresponding values of the acceleration, velocity, and displacement traces of the lateral and torsional components of the the foundation input motion.
Fig. 3-5 Comparison of Maximum Lateral and Maximum Total Deformations of Simple Structures Subjected to Vertically Propagating Incoherent Waves; $a/b = 1$, $\gamma_x = \gamma_y = \gamma$, $\zeta_l = \zeta_g = 0.02$, $f_0/f_l = 1.5$
Fig. 3-6 Effects of $\tilde{\gamma}$ and $\tilde{\tau}$ on Normalized Pseudo-Velocity Response Spectra for Lateral and Torsional Modes of Vibration of Simple Structures with $\varepsilon = 1$ and $\zeta_l = \zeta_\theta = 0.02$
• Kinematic interaction may reduce significantly the peak responses of high-frequency systems, but has little influence on the corresponding responses of low-frequency systems.

• The kinematic interaction effects due to ground motion incoherence are similar to those due to wave passage and the two effects may be interrelated.

These conclusions are similar to those drawn from a previous investigation of the structures supported on circular foundations. The effect of the particular parameter for the structures supported on rectangular foundations, ε, on the kinematic interaction effects is found to be similar to that for the corresponding transfer functions, which was shown in Chapter 2.
Chapter 4

DYNAMIC IMPEDANCE FUNCTIONS

4.1 Introduction

This section deals with the dynamic impedances of rigid, rectangular foundations. As was the case with the transfer functions examined in Chapter 2, the dynamic impedances considered in this chapter also refer to harmonically excited, massless foundations. However, whereas the transfer functions are for based-excited systems, the impedances are for force-excited systems. More specifically, they represent the complex-valued amplitudes of the exciting forces or moments necessary to induce unit linear or angular displacement amplitudes.

The real part of an impedance function represents the component of the generalized force that is in phase with the displacement component under consideration, whereas the imaginary part represents the force component that is 90° out of phase. The in-phase components define the elastic restraining actions of the medium and may be modeled by a set of linear springs, whereas the out-of-phase components represent the effects of the radiational energy dissipation and may be modeled by a set of viscous dampers.

For the rigid foundation considered herein, the dynamic impedances are represented by a $6 \times 6$ matrix. The evaluation of these functions has been the subject of numerous investigations over the years [16] [17] [18]. For a review of this effort, reference may be made to Gazetas [15]. Solutions of different degrees of accuracy, based on different assumptions and calculation schemes, have been presented. The most comprehensive and accurate numerical solutions for rectangular foundations appear to be those reported recently by Wong and Luco [19]. In the presentation of these data, a special effort was made to so express the results that the coefficients for the real and imaginary parts of the impedances are reasonably independent of the ratio of side-lengths of the foundation. However, when expressed in this manner, the moduli or real-valued amplitudes of the impedances, which are the quantities of greatest
interest in practice, are found to depend importantly on the foundation side-length ratio.

The objectives of this section are: (i) to identify the source of this dependence; and (ii) to present the modifications which for all practical purposes eliminate this dependence. Consideration is also given to how the modified presentations affect the coefficients for the real and imaginary parts of the impedances.

4.2 Statement of Problem

The system considered is shown in Figure 4-1. It is a rigid, massless, rectangular foundation of infinitesimal thickness supported at the surface of a homogeneous, isotropic, elastic half-space. The foundation is presumed to be bonded to the supporting medium so that no sliding or uplifting may occur, and it is referred to a Cartesian coordinate system, \( x, y, z \), with its origin taken at the foundation center. The lengths of sides of the foundation in the \( x \)- and \( y \)-directions are denoted by \( 2a \) and \( 2b \), respectively, where \( 2b \) is considered to be the shorter side length; accordingly, the solutions are limited to values of result \( a/b \geq 1 \). The half-space is characterized by its shear modulus, \( G \), Poisson’s ratio, \( \nu \), and its mass density, \( \rho \). The shear wave velocity for the medium is then given by \( v_s = \sqrt{G/\rho} \).

The foundation is excited at its center by harmonic forces and moments, the positive directions of which and of the resulting displacements are identified in the figure. The temporal variation of these forces and displacements is proportional to \( e^{i\omega t} \), where \( i = \sqrt{-1} \) and \( \omega \) is the circular frequency of the excitation and response. For brevity, the factor \( e^{i\omega t} \) is omitted in the following discussion.

The generalized forces \( (F_x, M_y, F_y, M_x, F_z, M_z) \) and the associated displacements \( (u_x, \theta_y, u_y, \theta_x, u_z, \theta_z) \) are interrelated by the equations,

\[
\begin{bmatrix}
F_x \\
M_y \\
F_y \\
M_x \\
F_z \\
M_z
\end{bmatrix}
= \begin{bmatrix}
K_1 & K_{12} & 0 & 0 & 0 & 0 \\
K_{21} & K_2 & 0 & 0 & 0 & 0 \\
0 & 0 & K_3 & K_{34} & 0 & 0 \\
0 & 0 & K_{43} & K_4 & 0 & 0 \\
0 & 0 & 0 & 0 & K_5 & 0 \\
0 & 0 & 0 & 0 & 0 & K_6
\end{bmatrix}
\begin{bmatrix}
u_x \\
\theta_y \\
u_y \\
\theta_x \\
u_z \\
\theta_z
\end{bmatrix}
\]

(4.1)

in which \( K_1, K_2 \) and \( K_{12} = K_{21} \) are, respectively, the lateral, rocking and coupling impedances of the foundation for motion in the \( xz \) plane; \( K_3, K_4 \) and \( K_{34} = K_{43} \).
Fig. 4-1 System Considered
are the corresponding quantities for motion in the \( yz \) plane; and \( K_z \) and \( K_\phi \) are the vertical and torsional impedances. These quantities are complex-valued functions of the ratio of side-lengths of the foundation, \( \varepsilon = a/b \), Poisson’s ratio, \( \nu \), and the dimensionless frequency parameter

\[
a_o = \frac{\omega b}{v_s}
\]

(4.2)

In the following sections, the coupling terms are not considered, and Poisson’s ratio is taken as \( \nu = 1/3 \).

### 4.3 Dynamic Impedances

In the comprehensive study by Wong and Luco [19], the foundation impedance for the \( j \)-th mode of vibration, \( K_j \), was expressed in the form

\[
K_j = Gb^n \left[ g_j(\varepsilon)\tilde{k}_j(\varepsilon, a_o) + i\alpha_\theta h_j(\varepsilon)\tilde{c}_j(\varepsilon, a_o) \right], \quad j = 1 \sim 6
\]

(4.3)

where \( \tilde{k}_j \) and \( \tilde{c}_j \) are dimensionless dynamic stiffness and damping coefficients; \( g_j \) and \( h_j \) are dimensionless functions of \( \varepsilon \) which are introduced so as to reduce the dependence of the two terms in the equation on the aspect ratio of the foundation; and \( n \) is an integer which assumes the value of \( n = 1 \) for the translational impedances and the value of \( n = 3 \) for the rocking and torsional impedances. The static stiffness of the foundation is then given by

\[
K_j^s = Gb^n g_j(a/b)\tilde{k}_j(a/b, 0)
\]

(4.4)

At this stage, it is desirable to relate the static stiffnesses of the rectangular foundation to those of an equivalent square foundation. For the translational modes (two lateral components and one vertical component), the side length of the equivalent square foundation is obtained by equating the areas of the two foundations, whereas for each of the rotational modes (two rocking components and one torsional component), it is obtained by equating the appropriate moments of inertia. The half-lengths of the sides, \( b_o \), of the equivalent square foundations for the different modes of vibration are then given by
\[ b_o = \begin{cases} 
\sqrt{\varepsilon} b, & j = 1, 3, 5 \\
\varepsilon^{3/4} b, & j = 2 \\
\varepsilon^{1/4} b, & j = 4 \\
\left[ \frac{a(\varepsilon^2+1)}{2} \right]^{1/4} b, & j = 6 
\end{cases} \] (4.5)

where it may be recalled that \( b \) refers to the shorter side of the original, rectangular foundation. In the notation of Reference [19], the interrelationship of \( b_o \) and \( b \) may be expressed as

\[ b_o^{n+1} = b^{n+1} h_j(\varepsilon) \] (4.6)

The static stiffness of the equivalent square foundation, \( K_j^o \), is determined from Equation 4.4 to be

\[ K_j^o = G b_o^n \bar{r}_j(1, 0) \] (4.7)

where use has been made of the fact that the value of \( g_j(\varepsilon) \) for \( \varepsilon = 1 \) is unity [19]. The static stiffnesses of the rectangular foundations may then be related to those of the equivalent square foundations by

\[ K_j^s = C_j(\varepsilon) K_j^o \] (4.8)

in which \( C_j(\varepsilon) \) is a dimensionless factor that depends on \( \varepsilon \) and the mode of vibration under consideration. A listing of the values of \( C_j(\varepsilon) \), recalculated from previous solutions [15] [19] [22], is presented in Table 4-1. Note that these values are close to, but not identically equal to, unity.

On making use of Equations 4.4, 4.6 and 4.7, Equation 4.3 can be rewritten as

\[ K_j = K_j^s \alpha_j(\varepsilon, a_o) + i K_j^o a_o^* \beta_j(\varepsilon, a_o) \] (4.9)

where \( \alpha_j(\varepsilon, a_o) \) and \( \beta_j(\varepsilon, a_o) \) are the renormalized stiffness and damping coefficients given by

\[ \alpha_j(\varepsilon, a_o) = \frac{\bar{k}_j(\varepsilon, a_o)}{\bar{k}_j(\varepsilon, 0)} \] (4.10)
Table 4-1 Factors $C_j$ Interrelating Static Stiffnesses of Rectangular and Equivalent Square Foundations

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$C_1$ for Lateral Stiffness Along Long Sides</th>
<th>$C_2$ for Rocking Stiffness Along Long Sides</th>
<th>$C_3$ for Lateral Stiffness Along Short Sides</th>
<th>$C_4$ for Rocking Stiffness Along Short Sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>0.990</td>
<td>0.997</td>
<td>1.009</td>
<td>0.997</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>1.023</td>
<td>-</td>
<td>1.018</td>
</tr>
<tr>
<td>4</td>
<td>1.007</td>
<td>1.053</td>
<td>1.048</td>
<td>1.041</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$C_5$ for Vertical Stiffness</th>
<th>$C_6$ for Torsional Stiffness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>1.002</td>
<td>1.019</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
\[ \beta_j(\varepsilon, a_\omega) = \frac{\xi_j(\varepsilon, a_\omega)}{k_j(1, 0)} \quad (4.11) \]

and \( a_\omega^* \) is a modified frequency parameter given by

\[ a_\omega^* = \frac{\omega b_\omega}{v_s} = a_\omega \frac{b_\omega}{b} \quad (4.12) \]

### 4.3.1 Moduli of Dynamic Impedances

The modulus or real-valued amplitude of a complex-valued impedance is given by the square root of the sum of the squares of its component terms. In the left-hand parts of Figures 4–2 through 4–7, the moduli \(|K_j^*|\) of the impedances for the various modes of vibration are plotted as a function of \( a_\omega \) in two different formats. The plots at the top of the figures are normalized with respect to \( K_j^* \), the actual static stiffnesses of the rectangular foundation; hence, they all have unit values at \( a_\omega = 0 \). The curves at the bottom are normalized with respect to \( K_j^0 \), the corresponding stiffnesses of the equivalent square foundation; therefore, for \( a_\omega = 0 \), they are close to but not identical to unity. Note that when displayed in either of these forms, the curves for the different side-lengths are widely separated; i.e., the results depend importantly on the ratio of side-lengths, \( a/b \).

Motivated by the observation that Equation 4.9 is a function of both \( a_\omega \) and \( a_\omega^* \), the normalized moduli of the impedances are replotted in the right-hand parts of Figures 4–2 through 4–7 against the modified frequency parameter \( a_\omega^* \). Note that when plotted in this manner, the curves for the different side-length ratios are significantly closer to each other than before. Expressed differently, this form of presentation practically eliminate the dependence of the moduli of the foundation impedances on the side-length ratio of the foundation. The significance of this result may further be appreciated by reference to Equation 4.12. On noting that \( a_\omega^* \) is proportional to \( b_\omega \), and recalling that, for the translational modes of vibration, \( b_\omega \) is a measure of the area of the foundation, whereas for the rocking and torsional modes, it is a measure of the relevant moment of inertia of the foundation, it is concluded that the moduli of the foundation impedances for the translational modes of vibration are controlled by the foundation area, whereas for the rocking and torsional modes, they are controlled by the appropriate moment of inertia of the foundation. This conclusion, which is approximately true over the entire range of frequency, is a generalization of the widely
Fig. 4-2 Normalized Amplitudes of Foundation Impedance for Lateral Motion in Direction of Long Sides
Fig. 4-3 Normalized Amplitudes of Foundation Impedance for Rocking Motion in Direction of Long Sides
Fig. 4-4 Normalized Amplitudes of Foundation Impedance for Lateral Motion in Direction of Short Sides
Fig. 4-5  Normalized Amplitudes of Foundation Impedance for Rocking Motion in Direction of Short Sides
Fig. 4-6 Normalized Amplitudes of Foundation Impedance for Vertical Motion
Fig. 4-7 Normalized Amplitudes of Foundation Impedance for Torsional Motion
recognized interrelationship between the static stiffnesses of rectangular and square or circular foundations.

Closer examination of the plots on the right-hand parts of Figures 4-2 through 4-7 reveals that, at lower values of $a_{\infty}^s$, the individual curves are closer to each other when the ordinates are normalized with respect to $K_j^s$, while for the higher values of $a_{\infty}^s$, they are closer to each other when the ordinates are normalized with respect to $K_j^o$. This trend may be appreciated from the following considerations. At low frequencies, the moduli of the complex-valued impedances are controlled by their real parts, which are close to the true static stiffnesses of the foundation. Thus, when normalized with respect to the actual static stiffnesses, the amplitudes would have very low dependence on the aspect ratio. At high frequencies, on the other hand, they are controlled by the imaginary parts, which are measures of the radiational energy dissipating capacity of the supporting medium. It has been shown by Gazetas [20] that, at high frequencies, the equivalent viscous damping for foundations in translational modes of vibration are proportional to the actual area of the foundation and those for the rotational modes of vibration are proportional to the appropriate moments of inertia of the foundation. Accordingly, when normalized with respect to the static stiffness of the equivalent square foundation, the moduli of the foundation should be independent of the side-length ratio of the foundation.

### 4.3.2 Dimensionless Coefficients of Stiffness and Damping

Emphasis in the presentation so far has been placed on the moduli of the foundation impedances. It is also of interest to examine the dependence of the dimensionless stiffness and damping coefficients, $\alpha_j$ and $\beta_j$, on the ratio of side lengths of the foundation. The variations of these coefficients with $a_o$ for several different side-length ratios, $a/b$, are shown in Figures 4-8 through 4-13, and the corresponding variations with $a_{\infty}^s$ are shown by the top two sets of curves in the right-hand parts of these figures. The first form of display corresponds to that employed by Wong and Luco [19].

It can be seen that, excepting the plots for the vertical mode of vibration, the results plotted against the modified frequency parameter, $a_{\infty}^s$, are less sensitive to variations in $a/b$ than are those plotted against the original frequency parameter, $a_o$. The reduction in this sensitivity is substantive, however, only for a few modes of
Fig. 4-8  Stiffness and Damping Coefficients for Foundations Vibrating Laterally in Direction of Long Sides
Fig. 4.9 Stiffness and Damping Coefficients for Foundations Vibrating in Rocking in Direction of Long Sides
Fig. 4.10 stiffness and damping coefficients for foundations vibrating laterally in direction of short sides.
Fig. 4-11 Stiffness and Damping Coefficients for Foundations Vibrating in Rocking in Direction of Short Sides
Fig. 4-12 Stiffness and Damping Coefficients for Foundations Vibrating Vertically
Fig. 4-13 Stiffness and Damping Coefficients for Foundations Vibrating Torsionally
vibration, and even for these modes, it is not nearly as important as for the moduli of the foundation impedances.

If the foundation impedance $K_j$ is expressed in terms of its corresponding static stiffness, $K_j^s$, Equation 4.9 can be rewritten in the form

$$K_j = K_j^s \left[\alpha_j + i\alpha_j^*\beta_j'\right]$$

(4.13)

in which

$$\beta_j' = \frac{K_j^e}{K_j^s} \beta_j$$

(4.14)

The variations of $\beta_j'$ with $a_o^*$ for several different values of $a/b$ are shown by the bottom curves in the right-hand parts of Figures 4-8 through 4-13. As would be expected, within the lower range of $a_o^*$ values, these plots are less sensitive to variations in $a/b$ than are the corresponding plots for $\beta_j$.

### 4.4 Conclusion

It has been shown that when plotted against the modified frequency parameter, $a_o^*$, the moduli of the impedances of rigid rectangular foundations are practically independent of the ratio of sides of the foundation, $a/b$. Excepting foundations in vertical motion, the dependence on $a/b$ of the dynamic stiffness and damping coefficients of the foundation is also reduced by plotting the results against $a_o^*$. The reduction in that dependence, however, is not as important for these coefficients as for the moduli of the foundation impedances.
Chapter 5

INERTIAL INTERACTION EFFECTS

5.1 Introduction

This section deals with the inertial interaction effects for structures supported on rectangular foundations. These effects are evaluated through the procedure employed by Prasad [2] to study the corresponding effects for structures resting on circular footings. The numerical values of the foundation impedances for the various modes of vibration are taken from the recent contribution by Wong and Luco [19], and the cross coupling terms between lateral and rocking motions are presumed to be negligible.

The consideration is first given to the inertial interaction effects for systems subjected to uniform harmonic free-field ground motions. Next, the total soil-structure interaction effects, both kinematic and inertial, are evaluated for systems excited by transient ground motions.

The objectives of this chapter are: (i) to identify the parameters which best characterize the inertial interaction effects; (ii) to define the conditions under which these effects are of sufficient importance to warrant consideration in design; and (iii) to present a simple model of a viscously damped oscillator to account for the inertial interaction effects in design. Comprehensive response spectra are presented for a wide range of the important parameters involved, and the results are used to assess the accuracy of the replacement oscillator.

5.2 System Considered and Modeling of Halfspace

5.2.1 System Considered

The system considered is essentially the same as that considered in Chapter 3 and shown in Figure 3-1. It is supported by a rectangular foundation resting on the surface of a half-space made of a homogeneous, purely elastic or viscoelastic material. The supporting medium is characterized by its shear modulus, $G$, Poisson's ratio,
\( \nu, \) mass density, \( \rho, \) and the specific loss factor, \( \Delta W/W. \) For a soil specimen in harmonic motion, \( \Delta W \) is the area of the elliptical hysteresis loop in the stress-strain diagram [25], and \( W \) is the strain energy stored in a purely elastic, linear material which is subjected to the same maximum stress and strain as the viscoelastic material. For a linearly elastic material, \( \Delta W = 0, \) whereas, depending on the property of the viscoelastic medium, \( \Delta W \) may be a function of, or independent of, the frequency of excitation. In the following discussions, \( \Delta W/W \) will be considered to be independent of frequency and be expressed in the form:

\[
\frac{\Delta W}{W} = 2\pi \tan \delta \quad (5.1)
\]

in which \( \delta \) is a constant representing the phase angle between the stress and the associate strain of the harmonically oscillating soil specimen.

The base excitation is specified by the foundation input motion, which, as indicated in Chapter 2, has a lateral and a torsional component, denoted by \( x_i(t) \) and \( \psi(t) \), respectively. The lateral and rocking components of the actual foundation motion are denoted by \( y_i(t) \) and \( \phi(t) \), respectively, and the torsional component of the actual foundation motion is denoted by \( \psi_o(t) \). The lateral configuration of the coupled system can then be specified by \( y_i(t), \phi(t), \) and the resulting interfloor deformation, \( u(t), \) whereas the torsional configuration can be specified by \( \psi_o(t) \) and the resulting torsional deformation of the structure, \( \theta(t). \) The former is shown in Figure 5-1.

### 5.2.2 Modeling of Halfspace

Neglecting the small coupling [22] between lateral and rocking motions, each mode of the restraining action of the halfspace harmonically excited may be represented by a combination of a linear or rotational spring and a viscous damper, as shown in Figure 5-2. The foundation impedances can then be given by:

\[
K_j = k_j + i\omega c_j \quad (5.2)
\]

where \( k_j \) is the stiffness of the spring and \( c_j \) is the damping coefficient of the damper. In this model, the spring accounts for the flexibility of the soil and the damper accounts for the effect of energy dissipation.

It is important to recall that the properties of these elements depend not only on the characteristics of the halfspace, but also on the exciting frequency. As shown
Fig. 6-1 System Considered
Fig. 5-2 Replacement Oscillator
in Figure 4-8 through 4-13, the normalized stiffness and damping coefficients for lateral components of motion are practically independent of the frequency parameter, whereas those for rocking components of motion are quite sensitive to variations of the frequency parameter. Furthermore, the normalized damping coefficients for rocking components of motion are significantly smaller than those for lateral components of motion, especially when the values of the frequency parameter is small. Then, it follows that, for a close-to-resonance condition, the rocking components of the response will be amplified by a much greater factor than the lateral components.

Because of the important contribution of the rocking motions, which is controlled by the moment of inertia of the foundation and of the structure with respect to the rocking axis, it is more convenient to choose $a$, the half side-length of the rectangular foundation parallel to the direction of the free-field ground motion, than $b$, which was used in the previous chapters as a reference length to present the numerical results. In the following, all the parameters previously related to $b$ will be expressed in terms of the length $a$ and marked by the subscript $a$.

5.3 Response to Harmonic Motion

In evaluating the inertial interaction effect on harmonically excited foundation-structure systems, the base excitation is specified by a horizontal free-field ground motion. For the sake of simplicity, the subscript $l$ attached to a quantity related to the lateral mode of vibration will be omitted in this section. Evaluation of the effect induced by a corresponding torsional motion is similar and simpler, and is reported in the appendices.

The harmonic response of a laterally excited system corresponds to the superposition of the effects of lateral and rocking components of the foundation motion. It depends on the properties of the supporting soil and the foundation, the properties of the superstructure, and on the characteristics of the excitation. The dimensionless parameters which can be used conveniently to characterize the system with a circular foundation have been identified in previous studies [23] [24]. For the system with a rectangular foundation, the only difference is the use of the half side-lengths, $a$ and $b$, to characterize the rectangular foundations instead of the radius, $r$, in the case of the circulation foundations. These dimensionless parameters are reproduced or modified hereafter in order of more or less decreasing importance:

• The wave parameter
\[ \sigma = \frac{hf}{v_s} \] (5.3)

which is a measure of the relative flexibility of the supporting medium and the superstructure.

- The ratio \( h/a \) of the height of the structure to the half side-length along the direction of foundation input motion, \( a \).

- The ratio \( \omega/p \) of the exciting frequency to the fixed-base natural frequency of the system.

- The material damping factor of the supporting medium, \( \tan \delta \), denoted by Equation 5.1.

- The damping factor for the structure in its fixed-based condition, \( \zeta \).

- The aspect ratio of the rectangular foundation, \( \varepsilon_a = b/a \).

- The relative mass density of the structure and the supporting medium,

\[ \mu = \frac{m}{4\rho ab} \] (5.4)

- The ratio \( m_f/m \) of the foundation mass to the mass of the superstructure.

- Poisson's ratio for the supporting medium, \( \nu \).

In the following, \( \mu \) is taken as 0.15, a representative value for buildings; the foundation mass is neglected compared to the mass of the superstructure; \( \nu \) is taken as 1/3; and \( \zeta \) is taken as 0.05. In practical cases, the structural response is generally insensitive to variations of these parameters.

In Figure 5-3, response spectra for the deformation amplitude, \( u_o \), of families of foundation-structure systems excited by a harmonic free-field ground motion, \( x_g(t) = x_o e^{i\omega t} \), are presented. Three different values of \( h/a \) are considered, one corresponding to short structures (\( h/a = 1 \)), another corresponding to moderately high structures (\( h/a = 3 \)), and the other corresponding to tall structures (\( h/a = 5 \)). For each value of \( h/a \), a family of curves with different values of the aspect ratio of foundation
Fig. 5-3 Pseudo-acceleration Response Spectra for Harmonically Excited Systems with $\sigma=0.2$ and Different Values of $b/a$ and $h/a$
sides are plotted using different types of dash lines. The solid lines, labeled by $\sigma = 0$, correspond to structures resting on an absolutely rigid medium, whereas the remaining curves are plotted for $\sigma = 0.2$, which corresponds to the systems with a flexible soil relative to the structure.

The spectra in these figures are displayed on logarithmic paper with all scales nondimensionalized. The abscissas represent the frequency ratio $p/\omega$; and the ordinates represent the pseudo-acceleration of the structure, $p^2 u_o$, normalized with respect to the amplitude of ground acceleration amplitude, $\ddot{x}_o = \omega^2 x_o$. A small value of $p/\omega$ corresponds either to a low frequency, flexible structure or to a high-frequency excitation, whereas a large value of $p/\omega$ corresponds to a high-frequency, stiff structure or for a low-frequency excitation.

Some important trends of the inertial interaction effect can be observed from Figure 5-3:

- It decreases the resonant frequency of the system relatively to the fixed-base case, moving the locations of the peak responses to the right. This increased flexibility is due to the added flexibility of the supporting soil.

- It changes the magnitude of the peak response, decreasing the value for short structures, whereas increasing the value for tall structures. This is the result of two opposing mechanisms. The dissipation of the energy radiating into the supporting soil caused by the lateral foundation motion increases the effective damping of the system and then decreases the response of the interacting system. On the other hand, the rocking component of the actual foundation motion increases the acceleration of the superstructure and leads to an increase in response. For short structures (e.g., $h/a = 1$), the contribution of the rocking of the foundation and the associated increase in response is apparently small. Instead, the radiation damping effect of the lateral component of foundation motion predominates and the associated reduction in response is more significant. As a result, the peak response is reduced. On the other hand, for tall structures (e.g., $h/a = 5$), the rocking component effect of the actual foundation motion is more important than the lateral component effect and, therefore, increases the peak response.

These two trends are the same as the ones identified for the system resting on circular foundations [23]. In addition, it should be noticed that the effect of material damping
in the halfspace is not included in this solution. Consideration of this effect will increase the effective damping of the interacting system and decrease its response. This will be shown in the following discussions.

Based on the conclusions in Chapter 4, a further study of the effect of the ratio of foundation side-lengths on the inertial interaction can be made. It is recalled that rectangular foundations of same area, regardless of their foundation side-lengths ratios, have very close moduli of the foundation impedance in lateral modes of vibration, whereas foundations with the same moment of inertia have similar moduli of the foundation impedance in rocking modes. Thus, for a fixed value of $h/a$, a larger value of $b/a$ implies a larger area as well as a larger moment of inertia and leads to a more flexible system. This argument is confirmed by the curves presented in Figure 5-3, whose peaks are displaced to the right for increasing values of $b/a$. On the other hand, because of the two mechanisms changing the magnitude of the peak response mentioned above, the larger values of the foundation area have a stronger effect on reduction of peak response, whereas larger values of the foundation moment of inertia induce larger increase in the peak response. For short structures, which are controlled primarily by the lateral impedance functions, the curves for larger values of $b/a$ should reach the higher peaks. But when the value of $h/a$ increases, the rocking impedance functions would become more dominant and then induce more increase in peak responses for the curves with larger values of $b/a$. The compensation of these two mechanisms can be observed in Figure 5-3, which agree with the above reasoning.

Furthermore, by the concept of equivalent area or moment of inertia mentioned before, when the half side-length of the square foundation having the same area as that of a rectangular foundation, $a_l$, is chosen as the reference length, all the curves for different aspect ratios are expected to become very close for short structures. Similarly, when the half side-length of the square foundation having the same moment of inertia, $a_r$, is used as the reference length, all the curves for different aspect ratios are also expected to come together for tall structures. This statement can be examined in Figure 5-4. For tall structures as $h/a_r = 5$, it works perfectly. But for short structures as $h/a_l = 0.5$, it is not as good as expected. This is because the dampings of the rocking components are always less than those of the lateral components. So, even for a short structure as $h/a_l = 0.5$, the effects by the dampings of rocking components are still nonnegligible. If a much smaller value of $h/a_l$ is taken, all the curves would really come together, but that is an impractical case. For the cases in between (moderately high structures), both the lateral and rocking impedance functions (in
turn, the area and the moment of inertia of the foundation) have nonnegligible effects on the inertial interaction.

5.3.1 Approximation via Replacement Oscillator

To simplify the procedure to account for the inertial interaction effects in design, a single-degree-of-freedom (SDF) replacement oscillator can be used to correspond to the physical model [23] [24]. It is shown in Figure 5-2. The spring connected to the mass represents the elastic resistance of the structure, and its stiffness, $k$, is equal to that of the fixed-base structure. The spring connected to the base, with stiffness $k_f$, accounts for the lateral and rocking flexibilities of the foundation, whereas the dashpot accounts for the overall damping of the system, including structural damping, radiation damping, and material soil damping. The mass of the oscillator, $m$, is taken equal to that of the superstructure, and the base motion is taken equal to the lateral free-field ground motion. The total system can be characterized by the natural period of the oscillator, $\tilde{T}$ (or the associated frequency, $\tilde{f} = \tilde{p}/2\pi = 1/\tilde{T}$), and by the fraction of critical damping, $\tilde{\zeta}$. The relationship of these quantities to those of the fixed-base structure, $T$ (or the associated frequency, $f = p/2\pi = 1/T$) and $\zeta$, is shown in the following discussion. Let $\tilde{u}(t)$ represent the total deformation of the replacement oscillator. It is composed of the deformations developed in the two springs, which are in inverse proportion to their respective stiffnesses. Since the stiffness of the structure spring is proportional to $f^2$, whereas the total stiffness of the oscillator is proportional to $\tilde{f}^2$, it follows that the structural deformation, $u(t)$, is related to $\tilde{u}(t)$ by the equation

$$u(t) = \left(\frac{\tilde{f}}{f}\right)^2 \tilde{u}(t) = \frac{\tilde{u}(t)}{\left(\frac{\tilde{T}}{T}\right)^2}$$

(5.5)

The parameters $\tilde{T}$ and $\tilde{\zeta}$ of the replacement oscillator are determined in such a way that the absolute maximum or resonant amplitude of the structural deformation and the period at which this maximum occurs are the same as the corresponding quantities for the actual system when subjected to the same foundation input motion. The relationship of $\tilde{T}$ and $\tilde{\zeta}$ to $T$ and $\zeta$ can be expressed in terms of the basic dimensionless parameters by the following two equations:
\[ \tilde{T} = T \left( 1 + \frac{k}{K_I} + \frac{kh^2}{K_\phi} \right)^{1/2} \]

\[ = T \left( 1 + \frac{Ga}{K_I} \frac{16\pi^2 \mu}{a_l} \frac{b/a}{\sigma^2 h/a} \left[ 1 + \frac{\alpha_l}{\alpha_\phi} \left( \frac{h}{a} \right)^2 \left( \frac{K_I}{K_\phi} \right) \right] \right) \]  

\[ \zeta = \left( \frac{T}{\tilde{T}} \right)^3 \left\{ \zeta + \frac{16\pi^2 \mu}{\sigma^3} \left[ \frac{Ga}{K_I} \frac{\beta'_l}{a_l (\alpha_l + ia_\phi \beta'_l) (h/a)^2} \right. \right. \]

\[ + \left. \frac{Ga^3}{K_\phi} \frac{\beta'_\phi}{\alpha_\phi (\alpha_\phi + ia_\phi \beta'_\phi) a} \right\} \]  

(5.6) \]

where \( K_I \), \( K_\phi \), \( \alpha_l \), \( \alpha_\phi \), \( \beta'_l \), and \( \beta'_\phi \) are the same notations as those used in Equation 4.13 but the subscripts \( l \) and \( \phi \) denote the lateral mode and rocking mode, respectively, and the frequency parameter \( a_o \) is redefined as \( a_o = \omega a / v_s \). The derivation of Equations 5.6 and 5.7 are given in Appendix B.

It should be noticed that the coefficients \( \alpha_l \) and \( \alpha_\phi \) in Equation 5.6 must be evaluated at a frequency equal to the desired natural frequency, \( \tilde{f} \). Since this frequency is unknown, Equation 5.6 has to be used by iteration to determine \( \tilde{T} \). This computation, however, may be practically simplified to a single step by use of the static stiffness, \( K_I \) and \( K_\phi \), instead of the actual dynamic stiffness, \( K_I \) and \( K_\phi \), in the first expression of Equation 5.6. This simplification corresponds to taking \( \alpha_l = \alpha_\phi = 1 \) in the second expression of Equation 5.6.

In Figure 5-5, Comparison between the exact solution and the approximation by SDF replacement oscillator is made for the normalized pseudo-acceleration response spectra of systems supported on square foundations and with \( \sigma = 0.2 \) for different values of \( h/a \). The solid lines represent the actual responses, whereas the dashed lines are for the SDF oscillators. It is clear that, excepting the left-hand portions of the spectra, the response values for the replacement SDF oscillators are in excellent agreement with those for the actual coupled systems. The agreement deteriorates, however, in the left-hand regions. As \( p/\omega \) tends to zero, the ratio \( u_o / x_o \) for the actual coupled systems approaches unity, whereas for the replacement SDF oscillators, \( u_o / x_o \) tends to \( (\tilde{p} / p)^2 = (\tilde{f} / f)^2 \). Depending on the relative values of \( \tilde{f} \) and \( f \), the difference between the exact and the approximate response asymptotes may be quite significant.

In Figure 5-6, the ratio \( \tilde{T} / T \) is plotted as a function of the relative flexibility parameter of the halfspace, \( \sigma \), for systems having different values of \( h/a \) and different values of \( b/a \). The solid lines are determined by the exact procedure, using the peak
Fig. 5-5 Comparison of Exact and Approximate Pseudo-acceleration Response Spectra for Systems with b/a=1, σ=0.2 and Different Values of h/a
Fig. 5-8 Comparison of Exact and Approximate Values of Effective Natural Period for Interacting Systems with Different Values of $h/a$
responses of the harmonically excited systems, whereas the dashed lines are calculated from Equation 5.6 by taking $\alpha_i = \alpha_\phi = 1$. When the relevant dynamic values of $\alpha_i$ and $\alpha_\phi$ are used, the results computed from Equation 5.6 would be closer to the exact solutions and are shown in Figure 5-7. The agreement achieved even by use of the static stiffness values may be more than adequate for many practical applications. The fraction of critical damping for the replacement oscillator, $\tilde{\zeta}$, is shown in Figure 5-8. The solid lines also represent the values determined by the exact procedure, whereas the dashed lines are computed from Equation 5.7 using the values of $\tilde{T}/T$ obtained by iteration.

From Figures 5-5 through 5-8, the following trends for the replacement oscillator are noteworthy:

- The accuracy of the approximation by using Equations 5.6 and 5.7 improves with increasing values of $h/a$, which correspond to the taller structures.

- The period ratio $\tilde{T}/T \geq 1$, and it increases with increasing values of $\sigma$ and increasing values of $h/a$.

- The damping value $\tilde{\zeta}$ may be greater or smaller than the values applicable to a fixed-base structure depending primarily on $h/a$. Inertial interaction increases the apparent damping of short structures but decreases the apparent damping of tall structures. These changes are particularly significant for systems with large values of $\sigma$, which correspond to more serious inertial interaction.

The second and third trends stated above are similar to the two trends obtained from harmonic response spectra mentioned in before. But, with the model of replacement oscillator and Equations 5.6 as well as 5.7, they can be explained more precisely and quantitatively.

From Equation 5.6, it is obvious that $\tilde{T}/T$ must be greater than 1, i.e., the inertial interaction increases the flexibility of the system. In addition, since $\alpha_\phi$ is always less than $\alpha_i$ and the practical value of $K_1^*a^2/K_{\phi}^*$ is smaller than 1.2 [19], it follows that the rotational flexibility of the foundation contributes more to increase the system flexibility when $h/a$ is approximately greater than 1. Thus, rotation of the foundation is the more important factor in reducing the resonant frequency of tall structures.

In Equation 5.7, the first term represents the contribution of the structural damping, whereas the second and third terms represent the contributions of the foundation
Fig. 5-7 Comparison of Exact and Approximate Values of Effective Natural Period for Interacting Systems with Different Values of $h/a$
Fig. 5-8 Effective Damping Factor, $\zeta$, for Structures Supported on Elastic Halfspace
damping, including both radiation and material damping, associated with the horizontal and rocking components of foundation motion, respectively. Since $\tilde{T}/T$ is greater than 1, it leads to a decrease in the structural damping and this reduction may be especially important for the tall structures, which have larger values of $\tilde{T}/T$. As explained previously, this reduction is due to the effect of foundation rocking, which trends to increase the inertia force on the structure and the resulting deformation. On the other hand, when the value of $h/a$ is small, the second term in Equation 5.7 would result in a tremendous increase in foundation damping and it could compensate the reduction of structural damping to make the total damping larger than that of the fixed-base system. This increase comes from the dissipation of energy by the lateral component of foundation motion. The third term, which comes from the dissipation of energy by the rocking component of foundation motion and also results in an increase in foundation damping, is usually of secondary effect if the structures are extremely tall or short.

Equation 5.7 can be rewritten as

$$\tilde{\zeta} = \frac{\zeta}{(\tilde{T}/T)^2} + \zeta_o$$

(5.8)

whose second term represents the total contribution of the foundation damping. In Figure 5-9, $\zeta_o$ is shown as a function of $\tilde{T}/T$ for different values of $h/a$ and $b/a$. The dashed lines, which correspond to systems resting on an elastic medium, represent the contribution of radiation damping only, whereas the solid lines, which refer to an viscoelastic medium with $\tan \delta = 0.1$, represent the combined effect of radiation and material dampings. From this figure, it is obvious that the foundation damping may contribute significantly for a short structure. This can also be reasoned from Equation 5.7 as discussed previously. Moreover, the contribution by hysteretic action in foundation damping increases with the increasing values of $h/a$. Therefore, for tall structures, the material damping is always more important than the radiation damping.

### 5.4 Response to Transient Excitation

In this section, the transient excitation based on the psd function of free-field ground motion specified in Chapter 3 is considered. Therefore, the psd functions of the foundation input motion obtained in Chapter 3 can be used so that the relative
Fig. 5-9 Foundation Damping Factor, $\ddot{\zeta}$, for Structures Supported on Elastic or Viscelastic Halfspace
importance of the kinematic interaction effects and the inertial interaction effects under certain free-field ground motion can be assessed to complete the framework of the whole soil-structure interaction problem.

The psd functions of the lateral and torsional components of the structural response are related to the psd functions of the corresponding foundation input motion by Equations A.13 and A.20, which are described in details in Appendix A. The desired mean peak values of the response can then be computed from Der Kiurghian's approximation.

The principal parameters that influence the mean peak response of a system with soil-structure interaction are the characteristics of the free-field ground motion; the fixed-base natural frequencies of the structure, \( f_l \) and \( f_\theta \), and the associated damping factors, \( \zeta_l \) and \( \zeta_\theta \); the ratio of \( h/a \); the aspect ratio, \( b/a \); the relative mass density ratio, \( \mu \); the modified incoherence parameter, \( \bar{\gamma} \); the transit time,

\[
\tau_a = \frac{a}{v_s}
\]  

(5.9)

and the effective transit time,

\[
\bar{\tau}_a = \sqrt{\gamma_l \gamma_y + (b/a)^2 \sin^2 \alpha_y} \left( \frac{a}{v_s} \right) = \sqrt{\gamma_l \gamma_y + (b/a)^2 \sin^2 \alpha_y} \tau
\]  

(5.10)

Other secondary parameters are the same as those stated in section 5.3.

**5.4.1 Results for Vertically Propagating Incoherent Waves**

Figures 5-10 through 5-12 show the representative pseudo-velocity response spectra for lateral and torsional response for a family of interacting systems subjected to vertically propagating incoherent waves (\( \bar{\gamma} = \infty \)), taking \( \gamma_x = \gamma_y = \gamma = 0.4 \), \( \varepsilon = b/a \), and \( \tau_a = a/v_s = 0.05 \) second. The same results also correspond to the more generally orthotropic cases with \( \sqrt{\gamma_x \gamma_y} = 0.4 \) and \( \varepsilon = \gamma_y b/\gamma_x a \). Three sets of solutions considering different degrees of interaction effects are displayed:

1. no soil-structure interaction effects, i.e., considering the foundation motion to be equal to the free-field ground motion at some reference or control point.

2. only the kinematic interaction (KI) effects, i.e., using the foundation input motion of a rigid footing as the foundation motion.
Fig. 5-10 Effects of Kinematic and Inertial Interaction on Maximum Deformations of Structures with \( b/a = \frac{1}{4} \) and Different \( h/a \); \( \gamma = 0.4 \) and \( \tau_a = 0.05 \) sec
Fig. 5-11 Effects of Kinematic and Inertial Interaction on Maximum Deformations of Structures with b/a=1 and Different h/a; γ=0.4 and τa=0.05 sec
Fig. 5-12 Effects of Kinematic and Inertial Interaction on Maximum Deformations of Structures with b/a=4 and Different h/a; \( \gamma = 0.4 \) and \( \tau_c = 0.05 \) sec
3. total soil-structure interaction (SSI) effects including both the kinematic and inertial interaction (II), i.e., considering the actual foundation motion, which is also influenced by its own inertia and the inertia of the structure, and by the coupling between them and the supporting soil.

In each figure, two different values of $h/a$ are considered, one corresponding to short structures ($h/a = 1$), and the other corresponding to tall structures ($h/a = 3$). Solutions 1 and 2 are independent of $h/a$, whereas solution 3 are valid for all combination of $\gamma$ (or $\sqrt{\gamma_\nu \gamma_\tau}$ for orthotropic cases) and $\tau_\nu$ for which $\tilde{\tau}_\nu = \gamma \tau_\nu = 0.02$ second. Three different values of aspect ratio are considered.

For solution 3, the corresponding response spectra for the replacement oscillator are also presented. Just as for the harmonically excited systems investigated in the proceeding section, these spectra are computed by application of Equations 5.6, 5.7, B.5, and B.6.

To assess the effect of aspect ratio, the results shown in Figures 5-10 through 5-12 are rearranged in Figures 5-13 and 5-14. Figure 5-13 shows two plots for the lateral response spectra with the values of $h/a$ equal to 1 and 3, respectively, whereas Figure 5-14 presents the corresponding spectra for the torsional response. In each plot, three values of aspect ratio ($b/a = 0.25, 1, 4$) are considered.

At last, Figure 5-15 is presented to evaluate the contribution by the torsional rotation of the foundation to the total response. The mean maximum values of the total deformation at the mid-point of the side of the structure parallel to the direction of free-field ground motion, which are induced by the combined effect of the lateral and torsional responses, are shown by the solid lines. These results are computed by taking $p_\phi/p_\tau = 1.5$ and using Equation 3.7 to get the psd function of the combined motion. The lateral response spectra, which have been shown in the previous figures, are plotted in dashed lines for comparison.

From Figures 5-10 through 5-15, the following trends are observed:

- The II effects for low-frequency, highly compliant structures are negligible because the halfspace of such systems can be viewed as composed of a very stiff, effectively rigid medium.

- Similarly, the KI effects are also negligible in the low-frequency spectral regions. Therefore, the total SSI effects can be neglected for highly compliant structures. However, in the medium- and high-frequency spectral regions, both the KI and II effects are considerable to the system responses.
Fig. 5-13 Effects of Ratio of Foundation Side-lengths on Maximum Value of Lateral Component of Deformation of Rigidly and Elastically Supported Systems with Different Values of $h/a$; $\gamma=0.4$ and $\tau_e=0.05$ sec
Fig. 5-14 Effects of Ratio of Foundation Side-lengths on Maximum Value of Torsional Component of Deformation of Elastically Supported Systems with Different Values of h/a; γ=0.4 and τₜ=0.05 sec
Fig. 5-15 Contributions of Lateral and Torsional Components of Response to Maximum Deformation of Systems with Different Values of b/a and h/a; γ=0.4, τ=0.05 and $f_o/f_s=1.5$
• Unlike KI effects, which generally reduce the lateral response, II effects may increase the lateral response in the high-frequency spectral region. This increase is especially significant for tall structures. Nevertheless, such structures usually fall in the medium-frequency region of the spectrum, for which the II effects are relatively small.

• As for the harmonically excited systems, the simple procedures to evaluate the maximum transient structural response by use of the replacement oscillator and the corresponding approximate equations provide reliable solutions with good accuracy. Furthermore, from Figure 5-13, it is obvious that this approximation underestimate the lateral responses for short structures (h/a = 1) but overestimate those for tall structures (h/a = 3). This is consistent with the results presented in Figure 5-7, where the natural frequencies of the tall structures are overestimated by the approximate formula and could move the transient response spectra to the right, whereas the natural frequencies of the short structures are underestimated and could move the transient response spectra to the left.

• As shown in Figure 5-15, the component contributed by the torsional motion of the foundation with total SSI effects is relatively small, and the combined effect of lateral and torsional response is usually only slightly greater than that due solely to the lateral response. And similar to the corresponding results considering only the KI effects in Chapter 3, this torsional contribution increases with the increasing values of the aspect ratio, b/a. Moreover, when the value of h/a increases, the contribution also becomes more significant.

5.5 Conclusions

• The principal effect of inertial interaction is to reduce the resonant frequency of the structure and to modify its effective damping depending primarily on the ratio of h/a involved. The net effect of these changes may be a reduction or increase in the maximum deformation of the structure.

• The other important parameter controlling the II effects is the relationship of the fixed-base natural frequency of the structure to the frequency regions of the response spectrum. This relationship classifies the structure as a low-frequency,
medium-frequency, or high-frequency system and then decides the significance of the II effects.

- The aspect ratio of the rectangular foundations influences the rocking contribution to the lateral response and the torsional response, especially the latter. Therefore, it is also an important factor to be considered in design.

- An excellent approximation to the II effects can be obtained by a SDF replacement oscillator. It accounts for the II effects by changing the natural period and the associated damping of the structure for the mode of vibration and offers simple and time-saving procedures for calculation.
Appendix A

Harmonic Response of Systems with Inertial Interaction Effects

A.1 Laterally Excited System

As shown in Figure 5-1, let \( x_l(t) \) be the lateral component of the foundation input motion, whereas the lateral and angular rocking components of the actual foundation motion are denoted by \( y_l(t) \) and \( \phi(t) \), respectively. Further, let \( u(t) \) be the resulting lateral deformation of structure. Three equations of motion describing the system can be expressed as

\[
\ddot{u} + 2\zeta \dot{p}_l \ddot{u} + p_l^2 u = -(\ddot{y}_l + h \ddot{\phi}) \tag{A.1}
\]

\[
m_f \ddot{y}_l + m(\ddot{y}_l + \ddot{u} + h \ddot{\phi}) + Q_l(t) = 0 \tag{A.2}
\]

\[
I_T \ddot{\phi} + m h (\ddot{y}_l + \ddot{u} + h \ddot{\phi}) + Q_\phi(t) = 0 \tag{A.3}
\]

where \( Q_l \) and \( Q_\phi \) are the horizontal shear and overturning moment at the foundation-soil interface; and \( I_T \) is the total mass moment of inertia of the structure and its foundation with respect to \( y \) axis. Equation A.1 describes the dynamic equilibrium of the lateral forces acting on the structural mass, whereas Equations A.2 and A.3 express the equilibrium of the lateral forces and moments about \( y \) axis for the whole system, respectively.

For the harmonic response considered,

\[
x_l(t) = X_l e^{i\omega t}
\]

\[
y_l(t) = Y_l e^{i\omega t}
\]
\[ \phi(t) = \Phi e^{i\omega t} \]

\[ u(t) = U e^{i\omega t} \]

By the dynamic impedance functions stated in Chapter 4 and neglecting the coupled terms,

\[
\begin{align*}
\begin{cases}
Q_l(t) \\
Q_\phi(t)
\end{cases} &= \begin{bmatrix}
K_l & 0 \\
0 & K_\phi
\end{bmatrix}
\begin{cases}
Y_l - X_l \\
\Phi
\end{cases}
\]
\[= \begin{bmatrix}
K_l^*(\alpha_l + ia_\phi \beta'_l) & 0 \\
0 & K_\phi^*(\alpha_\phi + ia_\phi \beta'_\phi)
\end{bmatrix}
\begin{cases}
Y_l - X_l \\
\Phi
\end{cases}
\]
\[= e^{i\omega t} \quad (A.4)\]

To solve this problem, first, Equation A.1 is taken to express \( U \) in terms of \( \ddot{Y}_l + h\ddot{\Phi} \) in which \( \ddot{Y}_l = -\omega^2 Y_l \) and \( \ddot{\Phi} = -\omega^2 \Phi \). It leads to

\[ U = H_{ui}(\ddot{Y}_l + h\ddot{\Phi}) = -\omega^2 H_{ui}(Y_l + h\Phi) \quad (A.5) \]

in which \( H_{ui} \) is defined in Equation 3.5. Second, the quantity \( U \) is eliminated from Equations A.2 and A.3 by Equation A.5, and the resulting equations are solved for \( Y_l \) and \( \Phi \) by making use of Equation A.4. This procedure leads to

\[
m\omega^2 \left[ \frac{m}{m} + \frac{(TR)_l - \frac{K_l}{m}}{(TR)_h} \frac{h}{h^2 - \frac{K_\phi}{m\omega^2}} \right]
\begin{cases}
Y_l \\
\Phi
\end{cases}
= \begin{cases}
-K_l X_l \\
0
\end{cases} \quad (A.6)
\]

in which \( (TR)_l \) is the transmissibility factor for lateral response, defined by

\[ (TR)_l = -\left(p_l^2 + i2\zeta_l(p_l\omega) \right) H_{ui} = \frac{1 + i2\zeta_l(\omega/p_l)}{1 - (\omega/p_l)^2 + i2\zeta_l(\omega/p_l)} \quad (A.7) \]

At last, Equation A.6 is substituted back into Equation A.5 to obtained the desired expression for \( U \).

On letting \( \epsilon_m = m_f/m, \epsilon_l = I_T/m h^2, a_\phi = \omega a_\phi/v_s, \kappa_l = K_l a^2/(m v^2_s) = K_l a^2/(m\omega^2) \) and \( \kappa_\phi = K_\phi a^2/(m h^2 v^2_s) = K_\phi a^2/(m\omega^2 h^2) \), the solution for \( U \) can be expressed as

\[ U = H_{ui} R_l \ddot{X}_l \quad (A.8) \]
in which $\ddot{X}_l = -\omega^2 X_l$ and $R_l$ is the dimensionless factor that provides for the inertial interaction effect. The latter factor is given by

$$R_l = \frac{B_3}{B_1 a_o^4 - B_2 a_o^2 + B_3} \quad (A.9)$$

where

$$B_1 = (\epsilon_i + \epsilon_m)(TR)_l + \epsilon_i \epsilon_m \quad (A.10)$$

$$B_2 = (\kappa_i + \kappa_\phi)(TR)_l + \epsilon_m \kappa_\phi \quad (A.11)$$

$$B_3 = \kappa_i (\kappa_\phi - \epsilon_i a_o^2) \quad (A.12)$$

The above derivation is for the harmonic response due to sinusoidal excitation. $H_{u_l} R_l$ can be viewed as a frequency response function for the input $x_l(t)$ and output $u(t)$. Therefore, by the theory of random vibration, when the random process for the lateral deformation of the structure reaches stationarity, its psd function, $S_{u_lu_l}$, is given by

$$S_{u_lu_l} = |H_{u_l}|^2 |R_l|^2 S_{\Pi} \quad (A.13)$$

where $S_{\Pi}$ is the psd function of the foundation input motion.

### A.2 Torsionally Excited System

Let $\psi(t)$ and $\psi_o(t)$ be the torsional components of the foundation input motion and the actual foundation motion, respectively, and $\theta(t)$ be the resulting angular torsional deformation of the structure. Two equations of motion describing the system can be written as

$$\ddot{\theta} + 2\zeta_\phi \dot{\theta} + \ddot{\psi}_o = -\ddot{\psi}_o \quad (A.14)$$

$$J(\ddot{\theta} + \ddot{\psi}_o) + J_f \ddot{\psi}_o + Q_s(t) = 0 \quad (A.15)$$
in which $J$ and $J_f$ are the polar mass moments of inertia of the structure and the foundation, respectively; and $Q_\psi$ is the instantaneous torque at the foundation-soil interface. Equation A.14 describes the dynamic equilibrium of the torsional moments acting on the structural mass, whereas Equation A.15 expresses the corresponding equilibrium of the whole system.

For the harmonic response considered,

$$
\psi(t) = \Psi e^{i\omega t}
$$

$$
\psi_\sigma(t) = \Psi_\sigma e^{i\omega t}
$$

$$
\theta(t) = \Theta e^{i\omega t}
$$

By the torsional dynamic impedance function,

$$
Q_\psi(t) = K_\psi (\Psi_\sigma - \Psi) e^{i\omega t} = K_\psi \cdot \left( \alpha_\psi + i \alpha_\psi' \right) (\Psi_\sigma - \Psi) e^{i\omega t}
$$

(A.16)

The three steps to solve this problem are similar to those described in section C.1 and finally leads to

$$
\Theta = H_u R_\psi \bar{\Psi}
$$

(A.17)

In the above expression, $H_u$, is defined in Equation 3.6, $\bar{\Psi} = -\omega^2 \Psi$, and $R_\psi$ is the dimensionless factor that provides for the inertial interaction effect for torsional mode. The latter factor is given by

$$
R_\psi = \frac{\kappa_\psi}{\kappa_\psi - [(TR)_\psi + (J_f/J)]a^2}
$$

(A.18)

where $\kappa_\psi = K_\psi \cdot a^2 / (Jv^2) = K_\psi a^2 / (J\omega^2)$ and

$$
(TR)_\psi = - \left( p_\theta^2 + i2\zeta_\theta p_\theta \omega \right) H_u = \frac{1 + i2\zeta_\theta(\omega/p_\theta)}{1 - (\omega/p_\theta)^2 + i2\zeta_\theta(\omega/p_\theta)}
$$

(A.19)

Furthermore, $H_u, R_\psi$ can be viewed as a frequency response function for the input $\psi(t)$ and output $\theta(t)$. Therefore, the psd function of torsional deformation of the
structure along its periphery, $S_{u,u}$, is given by

$$S_{u,u} = |H_u|^2 |R_\psi|^2 S_{\tilde{S}\tilde{S}}$$

(A.20)

where $S_{\tilde{S}\tilde{S}}$ is the psd function of the foundation input motion.
Appenendix B

Approximate Formulas for Natural Period and Damping of Replacement Oscillator

B.1 Approximation for the Lateral Component of Motion

In the derivation of Appendix A, when the foundation mass and the rotational inertia of the structure mass are assumed to be negligible \((m_f = 0, J_T = 0)\), the combination of Equations A.8 and A.9 can be simplified to

\[
\frac{p_f^2 U}{X_l} = \frac{\kappa_1 \kappa_\phi}{(\kappa_l + \kappa_\phi) [1 + i2 \zeta_l \omega / p_l] a_o^2 - \kappa_1 \kappa_\phi [1 - (\omega / p_l)^2 + i2 \zeta_l \omega / p_l]} \tag{B.1}
\]

Since the amplification factor for an undamped system under resonance is infinite, the undamped natural frequency of the structure-foundation system is obtained by setting the denominator of Equation B.1 to zero and neglecting all terms associated with damping. It leads to

\[
\frac{T_i^\wedge}{T_l} = \frac{\dot{f}}{f} = \sqrt{1 + \frac{k_l}{K_l} + \frac{k_l h^2}{K_\phi}} \tag{B.2}
\]

And by substituting Equations 5.2 through 5.5 into Equation B.2, Equation 5.6 is obtained.

When the amplitude of the pseudo-acceleration defined by Equation B.1 is evaluated at the natural frequency of the system defined by Equation B.2, it reaches the resonant value. Therefore, the fraction of critical damping, \(\zeta\), of the replacement oscillator can be approximated by the following relation

\[
\frac{1}{2 \zeta} = \left. \left\frac{p_f^2 U}{X_l} \right\right| \quad \text{at } \omega = \dot{f} \tag{B.3}
\]

and leads to
\[
2\zeta = \left| 1 - \left( \frac{\omega}{\omega_{pl}} \right)^2 + i2\zeta_l \left( \frac{\omega}{\omega_{pl}} \right) \right| - \left| 1 + i2\zeta_l \left( \frac{\omega}{\omega_{pl}} \right) \left( \frac{1}{\kappa_l} + \frac{1}{\kappa_\phi} \right) a_0^2 \right| \tag{B.4}
\]

Equation B.4 can be simplified by neglecting terms of secondary importance to yield Equation 5.7.

### B.2 Approximation for the Torsional Component of Motion

Following the same approximate procedures as those in section B.1, the similar expressions of the natural period, \( \bar{T}_\theta \), and fraction of critical damping, \( \bar{\zeta}_\theta \), for the torsional motion can be given as:

\[
\bar{T}_\theta = T_{\theta_0} \sqrt{1 + \frac{k_{\psi}}{K_{\psi}}} = T_{\theta_0} \sqrt{1 + 16\pi^2 \mu \frac{(b/a)[1 + (b/a)^2]}{K_{\psi}^s 3\alpha_\psi \sigma_\phi^2 h/a}} \tag{B.5}
\]

\[
\bar{\zeta}_\theta = \left( \frac{T_{\theta_0}}{T_\theta} \right)^3 \left\{ \zeta_\theta + \frac{16\pi^2 \mu \frac{Ga^3(b/a)[1 + (b/a)^2]}{K_{\psi}^s (h/a)^2}}{\sigma_\phi^3 \alpha_\psi \left( \alpha_\psi + \imath a_\phi \beta_\phi \right)} \right\} \tag{B.6}
\]

where

\[
\sigma_\phi = \frac{h f_\theta}{v_s} \tag{B.7}
\]
Bibliography


