INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Harmonic maps of trivalent trees

Stockton, George F., M.A.
Rice University, 1991
RICE UNIVERSITY

HARMONIC MAPS OF TRIVALENT TREES

by

GEORGE F. STOCKTON

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE MASTER OF ARTS

APPROVED, THESIS COMMITTEE

Michael Wolf, Director
Assistant Professor of Mathematics

Robert Hardt
Professor of Mathematics

Steven Cox
Assistant Professor of Mathematical Sciences

Houston, Texas

May, 1991
HARMONIC MAPS OF TRIVALENT TREES
by George Stockton

ABSTRACT

This thesis is a study of harmonic maps of trivalent trees into Euclidean space. The existence of such maps is established, and uniqueness is shown to hold up to a certain isotopy condition. Moreover, within its particular isotopy class, each harmonic map is shown to be a local minimum for the energy functional.

A harmonic map of a trivalent tree is determined by its associated nodes. Collectively, these nodes are a function of the lengths of the parameter spaces of the paths which comprise the map. It is shown that this node function can be continuously extended to certain parts of the boundary of its domain; these parts of the boundary are closely related to the geometry of the trivalent tree which serves as the domain of the given harmonic map.
ACKNOWLEDGEMENTS

I would like to thank the Rice University Mathematics Department for providing me with the opportunity to undertake this project. I am particularly grateful to my thesis advisor, Michael Wolf, for the considerable amount of time and effort which he spent assisting me with my work. His helpful comments and suggestions have undoubtedly improved the quality of this thesis. I would also like to thank Robert Hardt and Steven Cox for serving on my thesis committee. Thanks also to Sharon McDonough for handling the administrative details, and Maxine Turner for typesetting assistance.

This thesis would not have been possible without the wealth of support which I received from friends and family. To these people I am most grateful. This includes many people, but most notably Rick Moser, Jean Buchanan, and my parents, Barbara and Fred. Their kindness and generosity throughout the writing of this thesis far exceeded my expectations.
TABLE OF CONTENTS

1. Introduction. 1
2. Existence, Uniqueness, and Linearity of Harmonic Maps of Trivalent Trees. 5
3. A Continuous Extension of a Related Function to the Boundary of Its Domain. 16
4. Bibliography. 40
1. Introduction

A harmonic map is a map between Riemannian manifolds which is critical for the energy functional. Such maps have arisen in various branches of mathematics since the development of variational calculus, although it was not until the middle of this century that a general theory for harmonic maps was sought. Currently, there is a general theory consisting of numerous special cases. A thorough summary of this theory can be found in [E-L].

This thesis deals with harmonic maps of trivalent trees into Euclidean space. A study of harmonic maps begins with the fundamental questions of existence and uniqueness; the second chapter is devoted to the establishment of existence and uniqueness of harmonic maps of trivalent trees up to a certain isotopy condition. In addition, it is proved that such maps consist of linear paths and an explicit formula for each path is given. Finally, it is shown that these harmonic maps are local minima for the energy functional. The main tools used are the calculus of variations and linear algebra.

During the development of the second chapter, it becomes clear that the harmonic maps can be characterized in terms of their associated nodes (to be defined in the next section); collectively, these nodes are a function of the lengths of the parameter spaces of the linear paths which comprise the harmonic maps. In the third chapter, a continuous extension of this node function to parts of the boundary of its domain is obtained. In addition, a moduli space for the domain of the extended node function is determined.

Before proceeding to the results of this thesis, a few preliminary definitions and remarks are necessary. A graph consists of two finite sets called vertices and edges, along with a rule which assigns a pair of vertices to each edge. A trivalent tree is defined to be a tree (i.e. a connected, acyclic graph) whose vertices have valency 1 or 3. Those
vertices having valency 1 shall be called boundary points; those vertices having valency 3 shall be called nodes. Two results found in [S-T] allow us to characterize trivalent trees according to the number of edges, boundary points, and nodes. The first result states that the sum of the valencies of the vertices of any graph is 2e, where e is the number of edges; the second states that a graph is a tree if and only if the graph is connected and satisfies e = v - 1, where v denotes the number of vertices. The following proposition is a consequence of these results.

**Proposition 1.1.** A trivalent tree with e edges, b boundary points, and n nodes satisfies

\[(1-1) \quad n = b - 2 \quad \text{and} \quad e = 2b - 3.\]

Moreover, any connected graph satisfying (1-1) must be a trivalent tree.

**Proof.** Suppose T is a trivalent tree. Then \(2e = 3n + b\), where the right-hand side is the sum of the valencies of the vertices of T. Since \(e = v - 1 = b + n - 1\), we have

\[3n + b = 2b + 2n - 2 \quad \text{or, equivalently,} \quad n = b - 2; \quad e = 2b - 3 \quad \text{then follows.}

Conversely, suppose T is a connected graph satisfying (1-1). Then

\[4b - 6 = 2e = \text{the sum of the valencies of the vertices of T} \]

\[\geq b + 3(b - 2) \]

\[= 4b - 6.\]

Thus, T has no vertices other than its b boundary points and n nodes, and

\[e = 2b - 3 = (2b - 2) - 1 = v - 1; \quad \text{consequently, T must be a trivalent tree. Q.E.D.}\]
The proposition provides the impetus for the following definition.

**Definition 1.1** Given $P = \{p_1, \ldots, p_b\}$, a collection of distinct points in $\mathbb{R}^N$, along with a vector $a = (a_1, \ldots, a_{2b-3})$ consisting of positive real numbers, a map $T$ of a trivalent tree is defined to be a collection of $C^\infty$ paths $\gamma_i : [0, a_i] \to \mathbb{R}^N$ subject to the following conditions:

1. $\gamma_i(a_i) = p_i$ for $i \in \{1, \ldots, b-1\}$
2. $\gamma_b(0) = p_b$
3. For each $i \in \{b, \ldots, 2b-3\}$, there exist two distinct indices $i_1$ and $i_2$ such that $\gamma_i(a_i) = \gamma_{i_1}(0) = \gamma_{i_2}(0)$; we assume no other identifications of path endpoints.
4. $T$ is connected.

If we let the $C^\infty$ paths constitute edges and the endpoints of these paths be the vertices, then, in light of Proposition 1.1, a map of a trivalent tree may itself be regarded as a trivalent tree. As a result of this fact, we shall henceforth refer to such maps as trivalent trees. Let $T^n_P$ denote the collection of all trees with $a$ and $P$ specified.

Given $T \in T^n_P$, it is possible that two distinct nodes or a node and a boundary point might coincide. We wish to classify trivalent trees according to whether or not this happens.

**Definition 1.1b.** We say that $T$ is **nondegenerate** if, for $i > b$ and $j > b$, we have $\gamma_i(a_i) \notin P$ and $\gamma_i(a_i) \neq \gamma_j(a_j)$ whenever $i \neq j$. Otherwise, we say that $T$ is **degenerate**.

We may subdivide the collection $T^n_P$ into subcollections as follows. Given $T \in T^n_P$...
and \( j \in \{1, \ldots, b-2\} \), let \( I_j = \{i, i_1, i_2\} \) where \( i = j+b-1 \). Note that each \( I_j \) is merely the set of indices of the paths corresponding to the identification set forth in condition (3) of Definition 1.1. We shall call \( I_j \) the \textit{jth node index set} of \( T \).

**Definition 1.2.** Given \( T_1, T_2 \in T_p^a \), let \( I_j^1 \) and \( I_j^2 \) be their respective \( j \)th node index sets. We say that \( T_1 \) and \( T_2 \) are \textit{isotopic} if and only if \( I_j^1 = I_j^2 \) for each \( j \in \{1, \ldots, b-2\} \).

It should be noted that isotopy, as defined here, is not an equivalence relation, since degenerate trees may lie in several isotopy classes. We shall henceforth denote a given isotopy class of \( T_p^a \) by \( C_p^a \).

In the case of a single \( C^\infty \) path \( \gamma : [0, a_0] \to \mathbb{R}^N \) where \( a_0 > 0 \), the energy is defined to be

\[
E(\gamma) = \frac{1}{2} \int_0^{a_0} \left| \frac{d\gamma}{dt} \right|^2 dt.
\]

where \( \left| \cdot \right| \) denotes the Euclidean norm. We may extend this definition to trivalent trees in the following way:

**Definition 1.3.** Given \( T \in T_p^a \), the energy of \( T \) is defined to be

\[
E(T) = \frac{1}{2} \sum_{i=1}^{2b-3} \int_0^a \left| \frac{d\gamma_i}{dt} \right|^2 dt.
\]

It should be noted that \( E \geq 0 \). An obvious question arises: is it possible to minimize \( E \) over \( C_p^a \), and if so, can we characterize the minimizer? A partial answer to this question is provided in the next chapter.

This chapter establishes the existence, uniqueness, and linearity of harmonic maps \( T \in C^\infty_p \). The primary tool used is the calculus of variations. In many ways, this chapter is merely an extension of the argument found in [S], which establishes the result for \( C^\infty \) paths in manifolds; however, some subtleties arise when dealing with maps of trivalent trees as opposed to \( C^\infty \) paths, as we shall soon see.

Before discussing the existence and uniqueness of harmonic maps within each \( C^\infty_p \), we first define the concept of a variation. The definition which follows is a fairly straightforward extension of the definition of a variation of a \( C^\infty \) path.

**Definition 2.1** Given \( T \in C^\infty_p \), a variation \( V_T \) of \( T \) is defined to be a collection of \( C^\infty \) maps \( \alpha_i : (-\epsilon, \epsilon) \times [0, a_i] \rightarrow \mathbb{R}^N, \epsilon > 0 \), subject to the following conditions:

1. \( \alpha_i (0, t) = \gamma_i(t) \) for \( i \in \{1, \ldots, 2b-3\} \)
2. \( \alpha_i(u, a_i) = p_i \) for \( i \in \{1, \ldots, b-1\} \), all \( u \in (-\epsilon, \epsilon) \)
3. \( \alpha_b(u, 0) = p_b \) for all \( u \in (-\epsilon, \epsilon) \)
4. For each \( i \in \{b, \ldots, 2b-3\} \), \( \alpha_i(u, a_i) = \alpha_i(u, 0) = \alpha_i(u, 0) \) for all \( u \in (-\epsilon, \epsilon) \).
5. For each \( u \in (-\epsilon, \epsilon) \), the collection \( T_u = \{\alpha_i(u, t)\} \) is connected.
6. \( \frac{\partial \alpha_i}{\partial u}(0, t) \neq 0 \) for some \( i \in \{1, \ldots, 2b-3\} \), some \( a_i > 0 \), and some \( t \in [0, a_i] \).

Remark: Condition(4) guarantees that each trivalent tree \( T_u = \{\alpha_i(u, t)\} \) is an element of \( C^\infty_p \).
Definition 1.3 may be modified in an obvious way to obtain:

\[ E_{V_T}(u) = \frac{1}{2} \sum_{i=1}^{2b^3} \int_0^{a_i} \left| \frac{\partial \alpha_i}{\partial t}(u,t) \right|^2 dt. \]

If \( T \) is a local minimum for \( E \) over \( C_p^a \), then certainly \( 0 = \frac{d}{du} \left[ E_{V_T}(u) \right] \bigg|_{u=0} \) for all variations \( V_T \) of \( T \) (\( E \) is a smooth function of \( u \) since the \( \alpha_i \) are smooth functions of \( u \)). Any \( T \) satisfying the latter condition is said to be critical for \( E \). The second lemma of this chapter asserts that if \( T \) is critical for \( E \), then the \( C^\infty \) paths comprising \( T \) are linear (this is what we mean by the linearity of a harmonic map of a tree). Before presenting the details, however, the following standard lemma is necessary.

**Lemma 2.1** Given \( T, V_T \),

\[
\frac{dE_{V_T}(0)}{du} = \sum_{i=1}^{2b^3} \left( \left\langle \frac{\partial \alpha_i}{\partial t}(0,t), \frac{\partial \alpha_i}{\partial u}(0,t) \right\rangle_0^{a_i} - \int_0^{a_i} \left\langle \frac{\partial^2 \alpha_i}{\partial t^2}(0,t), \frac{\partial^2 \alpha_i}{\partial t \partial u}(0,t) \right\rangle dt \right). \]

**Proof.**

\[
\frac{dE_{V_T}(0)}{du} = \frac{1}{2} \sum_{i=1}^{2b^3} \int_0^{a_i} \left| \frac{\partial}{\partial u} \left[ \frac{\partial \alpha_i}{\partial t}(u,t) \right]^2 \right|_{u=0} dt
\]

\[
= \sum_{i=1}^{2b^3} \int_0^{a_i} \left\langle \frac{\partial \alpha_i}{\partial t}(0,t), \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \right\rangle dt
\]

\[
= \sum_{i=1}^{2b^3} \left( \left\langle \frac{\partial \alpha_i}{\partial t}(0,t), \frac{\partial \alpha_i}{\partial u}(0,t) \right\rangle_0^{a_i} - \int_0^{a_i} \left\langle \frac{\partial \alpha_i}{\partial u}(0,t), \frac{\partial^2 \alpha_i}{\partial t^2}(0,t) \right\rangle dt \right),
\]
a consequence of Stokes' theorem. Q.E.D.

Lemma 2.2 If \( T \in C^0_\mathbb{R} \) is critical for \( F \), then

\[
\gamma_i'(t) = \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} t + \gamma_i(0) \quad \text{for all } i \in \{b, \ldots, 2b-3\}.
\]

Proof. Assume \( T \in C^0_\mathbb{R} \) is critical for \( F \). Given \( \eta \in \{1, \ldots, 2b-3\} \), choose \( V_T \) so that

\[
\frac{\partial \alpha_i}{\partial u}(0, t) \equiv 0, \quad \text{except for } i = \eta, \text{ in which case we require that}
\]

\[
\frac{\partial \alpha_\eta}{\partial u}(0, t) = \varphi(t) \frac{\partial^2 \alpha_\eta}{\partial t^2}(0, t), \quad \text{where } \varphi(t) : [0, a_\eta] \to \mathbb{R}^N, \text{ is continuous, positive on}
\]

\((0, a_\eta)\), and zero at the endpoints of its domain. Apply the previous lemma to obtain

\[
0 = \frac{dE_{V_T}(t)}{du} = \int_0^{a_\eta} \varphi(t) \left| \frac{\partial^2 \alpha_\eta}{\partial t^2}(0, t) \right|^2 dt.
\]

Consequently, \( \varphi(t) \left| \frac{\partial^2 \alpha_\eta}{\partial t^2}(0, t) \right|^2 = 0 \), which forces \( \frac{\partial \gamma_\eta}{\partial t}(t) = \frac{\partial^2 \alpha_\eta}{\partial t^2}(0, t) = 0 \). It follows that

\( \gamma_\eta(t) = c_\eta t + d_\eta \) for some constants \( c_\eta, d_\eta \in \mathbb{R}^N \). But

\[
\gamma_\eta(0) = d_\eta
\]

\[
\gamma_\eta(a_\eta) = c_\eta a_\eta + \gamma_\eta(0);
\]

thus, \( \gamma_\eta(t) = \frac{\gamma_\eta(a_\eta) - \gamma_\eta(0)}{a_\eta} t + \gamma_\eta(0) \). Since \( \eta \in \{1, \ldots, 2b-3\} \) was arbitrary, the lemma follows. Q.E.D.
The previous lemma reduces our search for harmonic maps to linear trivalent trees. We now turn our attention to the matter of existence and uniqueness. To begin, we assume that $T$ is linear and critical for $E$. For each $V_T$,

\begin{align}
0 = \frac{dE_u}{du} T(0) &= \sum_{i=1}^{2b-3} \left( \left\langle \frac{\partial \gamma_i}{\partial t}(t), \frac{\partial \gamma_i}{\partial u}(0, t) \right\rangle \right)^a - \int_0^a \left( \left\langle \frac{\partial^2 \gamma_i}{\partial u^2}(0, t), \frac{\partial^2 \gamma_i}{\partial t^2}(t) \right\rangle \right) dt \tag{2-1}
\end{align}

by Lemma 2.1. As $\gamma_i(t) = \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} t + \gamma_i(0)$ (Lemma 2.2), we see that

\[ \frac{\partial \gamma_i}{\partial t} = \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} \] and \[ \frac{\partial^2 \gamma_i}{\partial t^2} = 0. \] Hence, (2-1) reduces to

\begin{align}
0 &= \sum_{i=1}^{2b-3} \left( \left\langle \gamma_i(0), \frac{\partial \gamma_i}{\partial u}(0, t) \right\rangle \right)^a \\
&= \sum_{i=1}^{2b-3} \left( \left\langle \gamma_i(0), \frac{\partial \gamma_i}{\partial u}(0, 0) \right\rangle - \left\langle \gamma_i(0), \frac{\partial \gamma_i}{\partial u}(0, 0) \right\rangle \right) \\
&= \sum_{i=1}^{2b-3} \left( \left\langle \gamma_i(0), \frac{\partial \gamma_i}{\partial u}(0, 0) \right\rangle \right) \\
&= \sum_{i=1}^{2b-3} \left( \left\langle \gamma_i(0), \frac{\partial \gamma_i}{\partial u}(0, 0) \right\rangle \right). \tag{2-2}
\end{align}

Recall from Definition 2.1 that $\alpha_i(u, a_i) = p_i$ for $i \in \{1, \ldots, b-1\}$, and $\alpha_b(u, 0) = p_b$ for all $u \in (-\epsilon, \epsilon)$. Consequently, $\frac{\partial \alpha_i}{\partial u}(0, a_i) = 0$, and $\frac{\partial \alpha_b}{\partial u}(0, 0) = 0$. Also, since $\alpha_i(u, a_i) = \alpha_i(u, 0) = \alpha_i(u, 0)$ for each $i \in \{b, \ldots, 2b-3\}$, we have

\[ \frac{\partial \alpha_i}{\partial u}(0, a_i) = \frac{\partial \alpha_i}{\partial u}(0, 0) = \frac{\partial \alpha_i}{\partial u}(0, 0). \]

Thus, (2-2) becomes
\[ (2-3) \quad 0 = \sum_{i=b}^{2b-3} \left( \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} \cdot \frac{\partial \alpha_i}{\partial u}(0, a_i) \right) - \sum_{i \neq b} \left( \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} \cdot \frac{\partial \alpha_i}{\partial u}(0, 0) \right) \]

\[ = (\text{upon regrouping}) \sum_{i=b}^{2b-3} \left( \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} + \frac{\gamma_i(0) - \gamma_i(a_i)}{a_i} + \frac{\gamma_i(0) - \gamma_i(a_i)}{a_i} \cdot \frac{\partial \alpha_i}{\partial u}(0, a_i) \right). \]

Given \( \eta \), choose \( V_T \) so that \( \frac{\partial \alpha_i}{\partial u}(0, a_i) = 0 \) for \( i \neq \eta \); the equation (2-3) reduces to

\[ 0 = \left( \frac{\gamma_{\eta}(a_{\eta}) - \gamma_{\eta}(0)}{a_{\eta}} + \frac{\gamma_{\eta}(0) - \gamma_{\eta}(a_{\eta})}{a_{\eta}} + \frac{\gamma_{\eta}(0) - \gamma_{\eta}(a_{\eta})}{a_{\eta}} \cdot \frac{\partial \alpha_{\eta}}{\partial u}(0, a_{\eta}) \right). \]

As \( \frac{\partial \alpha_{\eta}}{\partial u}(0, a_{\eta}) \) is arbitrary, we have

\[ (2-4) \quad 0 = \frac{\gamma_{\eta}(a_{\eta}) - \gamma_{\eta}(0)}{a_{\eta}} + \frac{\gamma_{\eta}(0) - \gamma_{\eta}(a_{\eta})}{a_{\eta}} + \frac{\gamma_{\eta}(0) - \gamma_{\eta}(a_{\eta})}{a_{\eta}} \]

for all \( \eta \in \{b, ..., 2b-3\} \). The equation (2-4) is a system of \( b-2 \) linear equations in the \( b-2 \) unknowns \( \gamma_{\eta}(a_{\eta}), \eta \in \{b, ..., 2b-3\} \). For convenience, we let \( n_j = \gamma_{j+b-1}(a_{j+b-1}) \) for \( j \in \{1, ..., b-2\} \). The equation (2-4) now becomes the \( j \)-th equation in the system given by

\[ (2-5) \quad 0 = \left( \frac{1}{a_{i}} + \frac{1}{a_{i}} + \frac{1}{a_{i}} \right) n_j - \left( \frac{\gamma_{i}(0)}{a_{i}} + \frac{\gamma_{i}(a_{i})}{a_{i}} + \frac{\gamma_{i}(a_{i})}{a_{i}} \right) \quad \text{for} \quad i = j+b-1. \]

If we move all constant terms in (2-5) to the left-hand side, the resulting system of \( b-2 \) equations in \( n_j \), now nonhomogeneous, may be expressed as

\[ (2-6) \quad An^t = c^t \]
where $A$ is a $b-2 \times b-2$ matrix of coefficients, $n = (n_1, ..., n_{b-2})$, and $c \in (\mathbb{R}^{N})^{b-2}$ is the matrix of constant terms.

Using the geometry of $T$, we shall characterize $A$ in such a way that we may establish the invertibility of $A$ without having to write out $|A|$ explicitly. The invertibility of $A$ ensures a unique solution for (2-6), and thus, uniqueness for harmonic maps of trivalent trees within a given isotopy class.

Let $A = (\alpha_{jk})$ be the matrix representation of $A$. We wish to write each $\alpha_{jk}$ in terms of the $a_i$, $i = 1, ..., 2b-3$. Clearly,

$$\alpha_{jj} = \frac{1}{a_i} + \frac{1}{a_{i_1}} + \frac{1}{a_{i_2}} \quad \text{where } i = j+b-1.$$

Suppose $j \neq k$ and $i = j+b-1$. As a consequence of condition (3) of Definition 1.1, we see that $n_j$ and $n_k$ are adjacent (that is, they are endpoints of a common path) if and only if $n_k \in \{ \gamma_1(0), \gamma_1(n_{i_1}), \gamma_2(n_{i_2}) \}$. Thus, if $n_j$ and $n_k$ are endpoints of some $\gamma_m$, then $m \in I_j = \{ i, i_1, i_2 \}$, and (2-5) implies $\alpha_{jk} = -\frac{1}{a_m}$. If $n_j$ and $n_k$ are not adjacent, then (2-5) implies $\alpha_{jk} = 0$. The entries of $A$ are, therefore, given by:

$$\alpha_{jk} = \begin{cases} 
\frac{1}{a_i} + \frac{1}{a_{i_1}} + \frac{1}{a_{i_2}} & \text{if } j=k \\
-\frac{1}{a_m} & \text{if } j \neq k \text{ and } n_j \text{ and } n_k \text{ are endpoints of } \gamma_m \\
0 & \text{if } j \neq k \text{ and } n_j \text{ and } n_k \text{ are not adjacent}
\end{cases}$$

(2-7)

Note that $A$ is symmetric.

Since any given node is adjacent to at most three other nodes, it follows that

$$\alpha_{jj} \geq \sum_{k \neq j} |\alpha_{jk}| > 0$$

(2-8)
for all \( j \), and since some nodes must be adjacent to boundary points, equality does not hold for all \( j \). We are now ready to apply the following standard result.

**Lemma 2.3.** If a real, symmetric \( n \times n \) matrix \( A = (\alpha_{jk}) \) satisfies (2-8) for all \( j \in \{1, ..., n\} \), with inequality for some such \( j \), then \( A \) is invertible.

**Proof.** We first show that \( A \) is positive definite. Let \( v = (v_1, ..., v_n) \in \mathbb{R}^n \sim \{0\} \) be arbitrary. Then

\[
\langle Av, v \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} v_j v_k = \sum_{j=1}^{n} \alpha_{jj} v_j^2 + \sum_{j=1}^{n} \sum_{k \neq j} \alpha_{jk} v_j v_k.
\]

It suffices to show that \( \left| \sum_{j=1}^{n} \sum_{k \neq j} \alpha_{jk} v_j v_k \right| < \sum_{j=1}^{n} \alpha_{jj} v_j^2 \). We have

\[
\left| \sum_{j=1}^{n} \sum_{k \neq j} \alpha_{jk} v_j v_k \right| \leq \sum_{j=1}^{n} \sum_{k \neq j} |\alpha_{jk}| |v_j v_k| \\
\leq \sum_{j=1}^{n} \sum_{k \neq j} |\alpha_{jk}| (v_j^2 + v_k^2) \\
= \text{(by symmetry)} \frac{1}{2} \left( \sum_{j=1}^{n} \sum_{k \neq j} |\alpha_{jk}| v_j^2 + \sum_{j=1}^{n} \sum_{k \neq j} |\alpha_{jk}| v_k^2 \right) \\
= \frac{1}{2} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_{jk}| v_j^2 - \sum_{j=1}^{n} |\alpha_{jj}| v_j^2 \right) + \frac{1}{2} \left( \sum_{k=1}^{n} \sum_{j=1}^{n} |\alpha_{kk}| v_k^2 - \sum_{k=1}^{n} |\alpha_{kk}| v_k^2 \right) \\
= \frac{1}{2} \left( 2 \sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_{jk}| v_j^2 - 2 \sum_{j=1}^{n} |\alpha_{jj}| v_j^2 \right).
\]
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_{jk}| v_j^2 = \sum_{j=1}^{n} |\alpha_{jj}| v_j^2
\]

\[
= \sum_{j=1}^{n} \sum_{k \neq j} |\alpha_{jk}| v_j^2
\]

\[
= \sum_{j=1}^{n} v_j^2 \sum_{k \neq j} |\alpha_{jk}|
\]

\[
< \sum_{j=1}^{n} |\alpha_{jj}| v_j^2
\]

Thus, A is positive definite.

Let \( \lambda \) be an eigenvalue of A, with corresponding eigenvector \( v \in \mathbb{R}^n \). Then

\[
0 < \langle \lambda v, v \rangle = \langle v, v \rangle = \lambda |v|^2
\]

and so \( \lambda > 0 \). Hence, \( |\lambda| > 0 \), since \( |\lambda| \) is the product of the eigenvalues of A, and consequently, A is invertible. Q.E.D.

The invertibility of A insures a unique solution to the system of equations given by (2-6). Thus, we have found a unique \( T_0 \in C^n_0 \) whose nodes are given by the solution to (2-6). As the nodes of \( T_0 \) solve (2-6), they must also satisfy (2-1), thereby rendering \( T_0 \) critical for energy.

**Theorem 2.1.** There exists a unique \( T_0 \in C^n_0 \) which is harmonic.

We now wish to show that \( T_0 \) in fact locally minimizes energy. To achieve this, it suffices to show that \( \frac{d^2 E_{T_0}}{d u^2} (0) > 0 \). This is provided in the
Theorem 2.2. \( T_0 \in C_p^a \) is a local minimum for the energy functional.

Proof. For each variation \( V_{T_0} \) of \( T_0 \).

\[
\frac{d^2 E_{V_{T_0}}}{du^2}(0) = \frac{1}{2} \sum_{i=1}^{2b-3} \int_0^{a_i} \left( \frac{\partial^2}{\partial u \partial t} \left( \frac{\partial \alpha_i}{\partial t}(u,t) \right) \right)^2 \, dt
\]

\[
= \sum_{i=1}^{2b-3} \int_0^{a_i} \left( \left| \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \right|^2 + \left\langle \frac{\partial \alpha_i}{\partial t}(0,t), \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \right\rangle \right) \, dt.
\]

We see that \( \sum_{i=1}^{2b-3} \int_0^{a_i} \left| \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \right|^2 \, dt \geq 0 \). Suppose equality occurs. Then \( \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \equiv 0 \).

Thus, there exists a constant \( c_i \) with \( \frac{\partial \alpha_i}{\partial u}(0,t) \equiv c_i \) for each \( i \). Since trivalent trees are connected, we observe that \( c_1 = c_2 = \ldots = c_{2b-3} \). Condition (3) of Definition 2.1 assures us that \( \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \equiv 0 \); hence, \( c_b = 0 \) and, moreover, \( c_i = 0 \) for each \( i \), by the above observation; this is contrary to condition (6) of Definition 2.1. Consequently,

\[
(2-9) \quad \sum_{i=1}^{2b-3} \int_0^{a_i} \left| \frac{\partial^2 \alpha_i}{\partial u \partial t}(0,t) \right|^2 \, dt > 0.
\]

As for the rest of the summand, since \( \frac{\partial \alpha_i}{\partial t}(0,t) = \frac{d\gamma_i}{dt}(t) = \frac{\gamma_i(a_i) - \gamma_i(0)}{a_i} \), we have
\begin{equation}
\sum_{i=1}^{2b-3} \int_0^{a_i} \left\langle \frac{\partial \gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0)}{\partial u^2 \partial t} \right\rangle \, dt = \sum_{i=1}^{2b-3} \int_0^{a_i} \left\langle \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0)}{\partial u^2 \partial t} \right\rangle \, dt
\end{equation}

by Lemma 2.2. Moreover,

\begin{equation}
\sum_{i=1}^{2b-3} \int_0^{a_i} \left\langle \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0)}{\partial u^2 \partial t} \right\rangle \, dt.
\end{equation}

\begin{align*}
&= \sum_{i=1}^{2b-3} \left\langle \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0,0)}{\partial u^2 \partial t} \right\rangle - \sum_{i=1}^{2b-3} \int_0^{a_i} \left\langle \frac{\partial}{\partial t} \left( \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0,0)}{\partial u^2 \partial t} \right) \right\rangle \, dt \\
&= \sum_{i=1}^{2b-3} \left\langle \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0,0)}{\partial u^2 \partial t} \right\rangle.
\end{align*}

The first equality is a consequence of Stokes' theorem; the second follows from the fact that \( \frac{\partial}{\partial t} \frac{\gamma_i(a_i)}{a_i} = 0 \). Using the conditions of Definition 2.1, we obtain

\[ \frac{\partial^2 \alpha_i(0,0)}{\partial u^2} = 0 \quad \text{for } i \in \{1, \ldots, b-1\}, \quad \frac{\partial^2 \alpha_b(0,0)}{\partial u^2} = 0, \]

and

\[ \frac{\partial^2 \alpha_i(0,a_i)}{\partial u^2} = \frac{\partial^2 \alpha_i(0,0)}{\partial u^2} \]

for \( i \in \{b, \ldots, 2b-3\} \). Equation (2-11) now becomes

\begin{equation}
\sum_{i=1}^{2b-3} \int_0^{a_i} \left\langle \frac{\partial \gamma_i(0,0)}{a_i}, \frac{\partial^3 \alpha_i(0,0)}{\partial u^2 \partial t} \right\rangle \, dt
\end{equation}

\begin{align*}
&= \sum_{i=b}^{2b-3} \left\langle \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0,0)}{\partial u^2} \right\rangle - \sum_{i \neq b} \left\langle \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0,0)}{\partial u^2} \right\rangle \\
&= \sum_{i=b}^{2b-3} \left\langle \frac{\gamma_i(0,0)}{a_i} + \frac{\gamma_i(0,0)}{a_i}, \frac{\partial^2 \alpha_i(0,0)}{\partial u^2} \right\rangle.
\end{align*}
But \( \frac{\gamma_i(a_i) - \gamma_i(a)}{a_i} + \frac{\gamma_i(a) - \gamma_i(a_i)}{a_i} + \frac{\gamma_i(a) - \gamma_i(a_i)}{a_i} = 0 \) for each \( i \in \{b, \ldots, 2b-3\} \), as in (2-4), whence

\[
(2-13) \sum_{i=1}^{2b-3} \int_0^a \left\langle \frac{\partial \alpha_i}{\partial t} - t, \frac{\partial^3 \alpha_i}{\partial u^2 \partial t} - (0, t) \right\rangle \, dt = 0.
\]

Upon juxtaposing (2-9) and (2-13), we see that \( \frac{d^2 E_{V_{T_0}}}{du^2} (0) > 0 \) for all variations \( V_{T_0} \) of \( T_0 \), the assertion of the theorem. \textbf{Q.E.D.}

We may write \( E_{T_0} = E_{V_{T_0}} \) in terms of the vertices of \( T_0 \) as follows

\[
E_{T_0} = E_{V_{T_0}} = \frac{1}{2} \sum_{i=1}^{2b-3} \int_0^a \left| \frac{\partial \alpha_i}{\partial t} \right|^2 dt
\]

\[
= \frac{1}{2} \sum_{i=1}^{2b-3} \int_0^a \left| \frac{d\gamma_i(t)}{dt} \right|^2 dt
\]

\[
= \frac{1}{2} \sum_{i=1}^{2b-3} \int_0^a \left| \gamma_i(a) - \gamma_i(0) \right|^2 dt
\]

\[
= \frac{1}{2} \sum_{i=1}^{2b-3} \left| \gamma_i(a) - \gamma_i(0) \right|^2 \int_0^a dt
\]

\[
= \frac{1}{2} \sum_{i=1}^{2b-3} \left| \gamma_i(a) - \gamma_i(0) \right|^2.
\]

This gives a rather nice formula for computing the energy of \( T_0 \).

Given $C^a_p$, we have shown the existence of a unique map $T_0 \in C^a_p$ which is a local minimum for energy. We note that $T_0$ is determined by its nodes which are given by the solution to (2-6). Equation (2-6) may be solved as follows. Let $A_i$ denote the matrix obtained by replacing the $i$th column of $A$ by $c^i$, where $c$ is as given in (2-6). Cramer's rule gives us:

$$n = \left| \begin{array}{c} A_1 \\ \vdots \\ A_{b-1} \\ A_{b-2} \end{array} \right|.$$\[\left| A \right|\]

For each $i$, $\left| \begin{array}{c} A_i \\ \vdots \\ A_{b-1} \end{array} \right| \left| A \right|$ is an expression in terms of $a_j$ for $j = 1, ..., 2b-3$. Thus, we may regard $n$ as a function on $(R^+)^{2b-3}$ where $R^+$ denotes the set of positive real numbers. That is, $n: (R^+)^{2b-3} \to (R^N)^{b-2}$. Moreover, it is clear that $\left| A_i \right|$ and $\left| A \right|$ are polynomial functions in the variables $\frac{1}{a_1}, \ldots, \frac{1}{a_{2b-3}}$ for $i = 1, ..., b-2$. Since $\left| A \right| > 0$, $n$ is continuous on $(R^+)^{2b-3}$.

Before proceeding any further, the following definition is necessary.

**Definition 3.1.** A subgraph $\Gamma$ of $T_0 \in C^a_p$ is *unrestrained* if $\Gamma$ does not connect two boundary points and contains no isolated vertices.

In this chapter, we exhibit a continuous extension of $n$ to certain parts of the boundary of $(R^+)^{2b-3}$. The key result which facilitates this extension is Theorem 3.2 which states that if $G$ is an unrestrained subgraph of $T_0$ with edges $\{\gamma_{11}, ..., \gamma_{m}\}$, then $\lim_{a_{11}, ..., a_{m} \to 0} n(a)$ exists.
The theorem is established via several lesser results, including Theorem 3.1 which gives a precise formula for \(|A|\). In addition to an extension of \(n\), we show that \(n\) is constant along all rays emanating from, but not including, the origin which lie in the extended domain of \(n\). By identifying such rays in the extended domain, we obtain a moduli space, \(M\), along with a function \(\tilde{\tau}\) defined on \(M\) which completely characterizes the extended function \(n\).

Let \(I = I_1 \times \ldots \times I_{b-2}\) and let \(I^* = \{ \lambda = (\lambda_j)_{j=1}^{b-2} \in I : \lambda_j \neq \lambda_k \text{ for } j \neq k \}\) where \(I_j\) is the \(j\)th node index set of \(T_0\), as defined in Chapter 1. Note that \(I^*\) is merely the collection of all vectors in \(I\) which have distinct entries. By definition,

\[
|A| = \sum_{\sigma \in S_{b-2}} (-1)^{\sigma} \prod_{j=1}^{b-2} a_{j\sigma(j)}
\]

where \(S_{b-2}\) is the alternating group on \(\{1, \ldots, b-2\}\) and where

\[
(-1)^{\sigma} = \begin{cases} 
-1 & \text{if } \sigma \text{ is odd} \\
1 & \text{if } \sigma \text{ is even} 
\end{cases}
\]

If \(\sigma\) is the identity permutation, then

\[
(-1)^{\sigma} \prod_{j=1}^{b-2} a_{j\sigma(j)} = \prod_{i=b}^{2b-3} \left( \frac{1}{a_i} + \frac{1}{a_{i+1}} + \frac{1}{a_{i+2}} \right) = \sum_{\lambda \in \Lambda^*} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda_j}}
\]

\[
= \sum_{\lambda \in \Lambda^*} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda_j}} + \sum_{\lambda \in \Lambda^*} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda_j}}
\]

For \(\lambda \in \Lambda \setminus I^*\), there exist indices \(j_1, j_2\) such that \(j_1 \neq j_2\) but \(\lambda_{j_1} = \lambda_{j_2}\). Let \(v_{\lambda} = \lambda_{j_1}\) in this case. Then
\[
\sum_{\lambda \in \mathcal{N}} \prod_{j=1}^{b-2} \frac{1}{a_j} = \sum_{\lambda \in \mathcal{N}} \left( \frac{1}{a_{\lambda}} \right)^2 q_\lambda
\]

where \( q_\lambda \) is some polynomial in \( \frac{1}{a_1}, \ldots, \frac{1}{a_{2b-3}} \). For \( \sigma \neq \text{id} \), we use the following:

**Lemma 3.1.** If \( \sigma \) is not the identity permutation, then

\[
(-1)^\sigma \prod_{j=1}^{b-2} a_j = \left( \frac{1}{a_{\nu}} \right)^2 q_\sigma
\]

for some index \( \nu_\sigma \) and some polynomial \( q_\sigma \) in \( \frac{1}{a_1}, \ldots, \frac{1}{a_{2b-3}} \).

**Proof.** If \( (-1)^\sigma \prod_{j=1}^{b-2} a_j = 0 \), let \( \nu_\sigma \) be arbitrary and let \( q_\sigma = 0 \). If \( (-1)^\sigma \prod_{j=1}^{b-2} a_j \neq 0 \), we first show that \( \sigma \) can be written as a product of disjoint 2-cycles.

Suppose \( \sigma \), when decomposed as a product of disjoint cycles, has a cycle of length \( m \geq 3 \). Then we may write \( \sigma = \tau(j_1, \ldots, j_m) \) where \( \tau \) is a permutation leaving \( j_1, \ldots, j_m \) fixed. Clearly, we must have

\[
0 \neq \alpha_{\tau_{j_1}} = \begin{cases} 
\alpha_{j_i \sigma(j_{i+1})} & \text{for } 1 \leq m - 1 \\
\alpha_{j_m} & \text{for } i = m .
\end{cases}
\]

From (2-7), we see that \( n_{j_i} \) and \( n_{j_{i+1}} \) must be adjacent for \( 1 \leq i \leq m \), and \( n_{j_m} \) and \( n_{j_1} \) must also be adjacent. But then \( T_0 \) has a circuit, contrary to its acyclicity. Thus, \( \sigma \) cannot have a cycle of length \( m \geq 3 \), and consequently, \( \sigma \) can be written as a product of disjoint 2-cycles.
Since $\alpha_{j_0(j)} \neq 0$ for each $j$, either $\sigma(j) = j$ or $n_j$ and $n_{\sigma(j)}$ are adjacent. As $\sigma$ is not the identity, there exists some $j_0$ such that $j_0 \neq \sigma(j_0)$. Denote $\sigma(j_0)$ by $k_0$. Let $\gamma_{k_0}$ be that path whose endpoints are $n_{j_0}$ and $n_{k_0}$. Then $\alpha_{j_0k_0} = \frac{1}{a_{\gamma_{k_0}}}$. Moreover, we must have

$\sigma(k_0) = j_0$; otherwise, $\sigma$ would have a cycle of length 3 or greater. Thus,

$\alpha_{k_0\sigma(k_0)} = \alpha_{k_0l_0} = \frac{1}{a_{\gamma_{l_0}}}$ by symmetry of $A$, and we may write

$$(-1)^{\sigma} \prod_{j=1}^{b_2} \alpha_{j,k_0} = \left(\frac{1}{a_{\gamma_{l_0}}}\right)^2 q_{\sigma},$$

where $q_{\sigma}$ is some polynomial in $\frac{1}{a_1}, \ldots, \frac{1}{a_{2b-3}}$. Q.E.D.

The lemma allows us to write

$$|A| = \sum_{\lambda \in I} \prod_{j=1}^{b_2} \frac{1}{a_{\lambda_j}} + \sum_{\lambda \in I \setminus I^*} \left(\frac{1}{a_{\lambda_j}}\right)^2 q_{\lambda} + \sum_{\sigma \in S_{b_2} \setminus I_d} \left(\frac{1}{a_{\gamma_{\sigma}}}\right)^2 q_{\sigma}$$

or

$$(3-1) \quad |A| - \sum_{\lambda \in I^*} \prod_{j=1}^{b_2} \frac{1}{a_{\lambda_j}} = \sum_{\lambda \in I \setminus I^*} \left(\frac{1}{a_{\lambda_j}}\right)^2 q_{\lambda} + \sum_{\sigma \in S_{b_2} \setminus I_d} \left(\frac{1}{a_{\gamma_{\sigma}}}\right)^2 q_{\sigma}$$

Given a polynomial $q$ in the variables $x_1, \ldots, x_n$ we define $d(x_i, q)$ to be the degree of $q$ as a polynomial in the single variable $x_i$. Using this definition and equation (3-1) we will show that

$$|A| \equiv \sum_{\lambda \in I} \prod_{j=1}^{b_2} \frac{1}{a_{\lambda_j}}.$$

Our first step in that direction will be to show that, for each $m \in \{1, \ldots, 2b-3\}$, the following equation holds:
\[ d \left( \frac{1}{a_m}, \sum_{\lambda \in I^*} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda_j}} \right) = 1. \]

Let \( m \in \{1, \ldots, 2b-3\} \) be given. Suppose \( j_m \) is such that \( m \in I_{j_m} \) and let \( \lambda \in I^* \) satisfy \( \lambda_{j_m} = m \). Then

\[ \prod_{j=1}^{b-2} \frac{1}{a_{\lambda_j}} = \frac{1}{a_m} \left( \prod_{j \neq j_m} \frac{1}{a_{\lambda_j}} \right). \]

Since \( \lambda \in I^* \) implies \( \lambda_i \neq \lambda_j \) for \( i \neq j \), it follows that \( \frac{1}{a_{\lambda_j}} \neq \frac{1}{a_m} \) for \( j \neq j_m \), whence

\[ d \left( \frac{1}{a_m}, \prod_{j \neq j_m} \frac{1}{a_{\lambda_j}} \right) = 0. \]

We have, for our particular \( \lambda \),

\[ d \left( \frac{1}{a_m}, \prod_{j=1}^{b-2} \frac{1}{a_{\lambda_j}} \right) = d \left( \frac{1}{a_m}, \frac{1}{a_m} \prod_{j \neq j_m} \frac{1}{a_{\lambda_j}} \right) \]
\[ = d \left( \frac{1}{a_m}, \frac{1}{a_m} \right) + d \left( \frac{1}{a_m}, \prod_{j \neq j_m} \frac{1}{a_{\lambda_j}} \right) \]
\[ = 0 + 1 = 1. \]

For each \( \lambda \) such that \( \lambda_i \neq m \) for all \( i \), the expression \( \prod_{j \neq j_m} \frac{1}{a_{\lambda_j}} \) must be independent of \( a_m \); in this case we conclude that
\[
d \left( \frac{1}{a_m}, \prod_{j=1}^{b-2} \frac{1}{a_{b_j}} \right) = 0.
\]

Thus,

\[
(3-2) \quad d \left( \frac{1}{a_m}, \sum_{\lambda \in I^*} \prod_{j=1}^{b-2} \frac{1}{a_{b_j}} \right) = \max_{\lambda \in I^*} d \left( \frac{1}{a_m}, \prod_{j=1}^{b-2} \frac{1}{a_{b_j}} \right) = 1.
\]

Assume \(|A| - \sum_{\lambda \in I^*} \prod_{j=1}^{b-2} \frac{1}{a_{b_j}}\) is not identically zero. Then

\[
\sum_{\lambda \in I \setminus I^*} \left( \frac{1}{a_{b_\lambda}} \right)^2 q_{b_\lambda} + \sum_{\sigma \in S_{b-2 \setminus I^*}} \left( \frac{1}{a_{b_\sigma}} \right)^2 q_{b_\sigma}
\]

is not identically zero, by (3.1). Also, for each \(m\),

\[
(3-3) \quad d \left( \frac{1}{a_m}, |A| - \sum_{\lambda \in I^*} \prod_{j=1}^{b-2} \frac{1}{a_{b_j}} \right) \in \{0, 1\},
\]

a consequence of equation (3-2) and the following

**Lemma 3.2.** For each \(m = 1, \ldots, 2b-3\), there exist polynomials \(r_m\) and \(s_m\) in the variables \(\frac{1}{a_1}, \ldots, \frac{1}{a_{2b-3}}\), which are independent of \(a_m\), and are such that

\[
|A| = \frac{1}{a_m} r_m + s_m.
\]

**Proof.** Suppose \(1 \leq m \leq b\) is given. Then one endpoint of \(\gamma_m\) is a boundary point and the other is a node. Assume the node is \(n_{j_0}\); thus, \(m \in I_{j_0}\) and \(m \notin I_j\) for \(j \neq j_0\). Recall that \(I_{j_0} = \{i, i_1, i_2\}\) where \(i = j_0 + b - 1\). Without a loss of generality, we may assume
m = i_1. By equation (2-7), \( \alpha_{j_0} = \frac{1}{a_i} + \frac{1}{a_m} + \frac{1}{a_{j_2}} \), and \( \alpha_{jk} \) is independent of \( a_m \) for \( j \neq j_0 \) or \( k \neq j_0 \). We may write

\[
|A| = \sum_{k=1}^{b-2} \alpha_{j_0 k} M_{j_0 k}
\]

where \( M_{j_0 k} \) denotes the cofactor of \( \alpha_{j_0 k} \). Rewriting, we obtain

\[
|A| = \frac{1}{a_m} M_{j_0 l_0} + \left( \frac{1}{a_i} + \frac{1}{a_{j_2}} \right) M_{j_0 l_0} + \sum_{k \neq j_0} \alpha_{j_0 k} M_{j_0 k}
\]

Since \( \alpha_{jk} \) is independent of \( a_m \) for \( j \neq j_0 \) or \( k \neq j_0 \), it follows that \( M_{j_0 l_0}, M_{j_0 k} \), and \( \alpha_{jk} \) are polynomials independent of \( a_m \) for \( k \neq j_0 \). Let

\[
r_m = M_{j_0 l_0}
\]

\[
s_m = \left( \frac{1}{a_i} + \frac{1}{a_{j_2}} \right) M_{j_0 l_0} + \sum_{k \neq j_0} \alpha_{j_0 k} M_{j_0 k}.
\]

Then \( r_m \) and \( s_m \) are polynomials independent of \( a_m \) and \( |A| = \frac{1}{a_m} r_m + s_m \), as desired.

Now suppose \( b+1 \leq m \leq 2b-3 \). Then each endpoint of \( \gamma_m \) must be a node. By condition (3) of Definition 1.1, one node must be \( n_{j_1} \) where \( j_1 = m-b+1 \). Suppose the other node is \( n_{j_2} \) where \( j_1 \neq j_2 \). We have \( m \in I_{j_1} \) and \( m \in I_{j_2} \), as well as \( m \notin I_j \) for \( j \notin \{ j_1, j_2 \} \); also, \( I_{j_1} = \{ m, m_1, m_1 \} \) and \( I_{j_2} = \{ i, i_1, i_2 \} \) where \( i = j_2+b-1 \). Without a loss of generality, we may assume that \( i_1 = m \). Let \( B \) be the matrix obtained by adding the \( j_1 \)th row of \( A \) to the \( j_2 \)th row of \( A \); that is \( B = (\beta_{jk}) \) where

\[
\beta_{jk} = \begin{cases} 
\alpha_{jk} & \text{for } j \neq j_2 \\
\alpha_{j_1 k} + \alpha_{j_2 k} & \text{for } j = j_2
\end{cases}
\]
Since \( m \neq j \) for \( j \notin \{ j_1, j_2 \} \), we see from (2-7) that \( \alpha_{jk} \) is independent of \( a_m \) for \( j \notin \{ j_1, j_2 \} \). As \( A \) is symmetric, \( \alpha_{jk} \) must be independent of \( a_m \) for \( k \notin \{ j_1, j_2 \} \). In short, \( \beta_{jk} \) may depend on \( a_m \) only if both \( j \) and \( k \) are elements of \( \{ j_1, j_2 \} \). However,

\[
\beta_{j_2 j_1} = \alpha_{j_1 j_1} + \alpha_{j_2 j_1} = \left( \frac{1}{a_m} + \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} \right) + \frac{1}{a_m} = \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} \\
\beta_{j_2 j_2} = \alpha_{j_1 j_2} + \alpha_{j_2 j_2} = -\frac{1}{a_m} + \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) = \frac{1}{a_1} + \frac{1}{a_2}
\]

Thus, \( \beta_{j_2 j_1} \) and \( \beta_{j_2 j_2} \) are independent of \( a_m \); by a previous remark, we must have \( \beta_{jk} \) dependent upon \( a_m \) only if \( j = j_1 \) and \( k \notin \{ j_1, j_2 \} \).

As before, we may write

\[
|B| = \sum_{k=1}^{b-2} \beta_{j_1 k} M_{j_1 k},
\]

where \( M_{j_1 k} \) is the cofactor of \( \beta_{j_1 k} \). Rewriting, we obtain

\[
|B| = \frac{1}{a_m} \left( \alpha_{j_1 j_1} - \alpha_{j_1 j_2} \right) + \left( \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} \right) M_{j_1 j_1} + \sum_{k \notin \{ j_1, j_2 \}}^{b-2} \beta_{j_1 k} M_{j_1 k}.
\]

Since \( \beta_{jk} \) is independent of \( \frac{1}{a_m} \) for \( j \neq j_1 \) or \( k \notin \{ j_1, j_2 \} \), it follows that \( \beta_{j_1 k}, M_{j_1 k}, M_{j_2 k}, \) and \( M_{j_2 j_2} \) are polynomials independent of \( a_m \) for \( k \notin \{ j_1, j_2 \} \). Let

\[
r_m = M_{j_1 j_1} - M_{j_2 j_2} \\
s_m = \left( \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} \right) M_{j_1 j_1} + \sum_{k \notin \{ j_1, j_2 \}}^{b-2} \beta_{j_1 k} M_{j_1 k}.
\]
Then \( r_m \) and \( s_m \) are polynomials independent of \( \frac{1}{a_m} \) and \(|A| = |B| = \frac{1}{a_m} r_m + s_m\). \textbf{Q.E.D.}

After cancellations and reorganization of terms, the right-hand side of (3.1) may be expressed as \( \sum_{\nu \in J} \left( \frac{1}{a_{\nu}} \right)^2 q_{\nu} \), where \( J \) is some index set and \( q_{\nu} \) some nonzero polynomial.

Thus, for \( \mu \in J \)

\[
d \left( \frac{1}{a_{\mu}} \sum_{\nu \in J} \left( \frac{1}{a_{\nu}} \right)^2 q_{\nu} \right) \geq 2,
\]

contrary to the fact that we must have

\[
d \left( \frac{1}{a_{\mu}} \sum_{\nu \in J} \left( \frac{1}{a_{\nu}} \right)^2 q_{\nu} \right) \in (0, 1)
\]

by (3.1) and (3.3). Consequently,

\[
|A| - \sum_{\lambda \in \Lambda} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda j}} = 0 \text{ or } |A| = \sum_{\lambda \in \Lambda} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda j}}
\]

We state this result as

\textbf{Theorem 3.1.} \( |A| = \sum_{\lambda \in \Lambda} \prod_{j=1}^{b-2} \frac{1}{a_{\lambda j}} \)

Since \( a_i > 0 \) for \( i = 1, \ldots, 2b-3 \), it follows that \(|A| > 0\). Note that this theorem was derived independently of Lemma 2.3. More importantly, we shall use the theorem to prove Theorem 3.2, which asserts that \( \lim n(a) \) exists as we approach certain parts of the boundary of \( (R^+)^{2b-3} \). The "certain parts" are related to unrestrained subgraphs of \( T_0 \). We now turn our attention to these subgraphs. The following lemma establishes a
correspondence between the indices of the paths of the unrestrained subgraphs of \( T_0 \) and elements of \( \Gamma^* \).

**Lemma 3.3.** Suppose \( \Gamma \) is an unrestrained subgraph of \( T_0 \). Then there exists some \( \lambda = (\lambda_i)_{i=1}^{b-2} \in \Gamma^* \) such that the edges of \( \Gamma \) are given by \( \{\gamma_{\lambda i_1}, \ldots, \gamma_{\lambda i_m}\} \) for some subset \( \{\lambda_i\}_{j=1}^m \) of \( (\lambda_i)_{i=1}^{b-2} \).

**Proof.** The idea of the proof is as follows. We begin by showing that \( m \leq b-2 \), where \( m \) is the number of edges of \( \Gamma \). For \( \Gamma \) such that \( m < b-2 \), we show that \( \Gamma \) may be extended to an unrestrained subgraph \( \Gamma' \) of \( T_0 \) consisting of exactly \( b-2 \) edges. We then construct a bijective function \( \Psi \) from the collection of nodes of \( T_0 \) to the collection of edges of \( \Gamma' \). Using this bijection, we find a \( \lambda \in \Gamma^* \) such that the indices of the edges of \( \Gamma' \) are given by the components of \( \lambda \). We then use the fact that \( \Gamma \subseteq \Gamma' \) to obtain the lemma.

Assume \( \Gamma \) consists of \( m \) edges. Let \( \Gamma_1, \ldots, \Gamma_r \) be the components of \( \Gamma \). For each \( i \in \{1, \ldots, r\} \), let \( e_i, v_i, \) and \( b_i \) denote the number of edges, nodes and boundary points, respectively, associated with \( \Gamma_i \). Recall that any tree must satisfy \( e = v-1 \), where \( e \) is the number of edges and \( v \) the number of vertices. As \( \Gamma_i \) is a subtree of \( T_0 \), it follows that \( e_i = v_i + b_i - 1 \) or, equivalently, \( e_i - v_i + 1 = b_i \). Clearly, \( \Gamma_i \) cannot connect two boundary points, and thus, \( b_i \leq 1 \). This yields \( e_i \leq v_i \), and moreover,

\[
m = \sum_{i=1}^{r} e_i \leq \sum_{i=1}^{r} v_i \leq b-2,
\]

since \( T_0 (a) \) has \( b-2 \) nodes.

We shall now show that, if \( m < b-2 \), then it is possible to extend \( \Gamma \) to an unrestrained subgraph \( \Gamma' \) consisting of exactly \( b-2 \) edges.
case i: \( \sum_{i=1}^{r} v_i = b-2 \). Suppose each \( \Gamma_i \) contains a boundary point. Then \( e_i = v_i \), and
\[
\sum_{i=1}^{r} v_i = \sum_{i=1}^{r} e_i = m < b-2,
\]
a contradiction. Thus, there exists some \( \Gamma_{i_0} \) which does not contain a boundary point. Choose any edge \( \gamma \in T_0(a) \) which is adjacent to some edge of \( \Gamma_{i_0} \). Note that \( \gamma \notin \Gamma \): if \( \gamma \in \Gamma \) then \( \gamma \in \Gamma_i \) for some \( i \neq i_0 \); thus \( \Gamma_i \) and \( \Gamma_{i_0} \) must share a node, contrary to the fact that different components cannot share vertices. The subgraph \( \Gamma^{m+1} \) obtained by adjoining \( \gamma \) and its endpoints to \( \Gamma \) has \( m+1 \) edges and does not connect two boundary points, since \( \Gamma_{i_0} \) lacks a boundary point.

case ii: \( \sum_{i=1}^{r} v_i < b-2 \). In this case, there exists some node \( n \) such that \( n \notin \Gamma \). Let \( \gamma \in T_0(a) \setminus \Gamma \) be any edge which has \( n \) as an endpoint; let \( v \) denote the other endpoint of \( \gamma \). Then \( v \) is an element of at most one \( \Gamma_i \), and thus, the subgraph \( \Gamma^{m+1} \) obtained by adjoining \( \gamma \) and its endpoints to \( \Gamma \) does not connect two boundary points. Clearly, \( \Gamma^{m+1} \) has \( m+1 \) edges.

If \( m+1 < b-2 \), we repeat the above process until we obtain a subgraph \( \Gamma' \) of \( T_0(a) \) which is unrestrained and contains \( b-2 \) edges.

Let \( \Gamma'_1, \ldots, \Gamma'_s \) be the components of \( \Gamma' \). As before, \( e_i = v_i + b_i - 1 \), \( e_i \leq v_i \), and
\[
b-2 \leq \sum_{i=1}^{s} e_i \leq \sum_{i=1}^{s} v_i \leq b-2,
\]
whence \( \sum_{i=1}^{s} v_i = b-2 \). It then follows that \( \Lambda \subseteq \Gamma' \) where \( \Lambda = \{ n_1, \ldots, n_{b-2} \} \) is the collection of nodes of \( T_0 \). Suppose there exists some \( i_0 \in \{ 1, \ldots, s \} \) such that \( \Gamma_{i_0}' \) does not contain a boundary point. Then \( e_{i_0} = v_{i_0} - 1 < v_{i_0} \). Since \( e_i \leq v_i \) for each \( i \), this implies that \( \sum_{i=1}^{s} e_i < \sum_{i=1}^{s} v_i = b-2 \), a contradiction. Thus, for each \( i \), \( \Gamma_i' \) contains a boundary point, \( p_{r_i} \), where \( r_i \in \{ 1, \ldots, b \} \).

Let \( W' \) denote the collection of edges of \( \Gamma' \). We construct a function \( \Psi: \Lambda \rightarrow W' \) as follows. Given \( n_i \in \Lambda \), there exists exactly one \( j \in \{ 1, \ldots, s \} \) such that \( n_i \in \Gamma_j' \). Theorem 2.1 in [ST] confirms the existence of a unique path between any two vertices of a tree. Since \( \Gamma_j' \) is a tree, there exists a unique path \( p_i \subseteq \Gamma_j' \) between \( p_{r_j} \) and \( n_i \). Let \( \gamma_{r_i} \) denote the edge of \( \Gamma_j' \) which is incident on \( n_i \) and is an element of \( p_{r_j} \). Set \( \Psi(n_i) = \gamma_{r_i} \). We wish to show that \( \Psi \) is a one-to-one correspondence between \( \Lambda \) and the edges of \( \Gamma' \). Both sets are finite, so it suffices to show injectivity.

Suppose \( \Psi(n_i) = \Psi(n_j) \); that is, \( \gamma_{r_i} = \gamma_{r_j} \). Then \( n_i \) and \( n_j \) clearly lie in the same component of \( \Gamma' \), since they are endpoints of the same path. Suppose \( n_i \neq n_j \). Consider the path \( \rho \) obtained by removing \( n_j \) and, thus, \( \gamma_{r_j} \) from \( \rho_{r_i} \). Note that \( n_i \in \rho \). Thus, \( \rho \) is a path between \( p_{r_j} \) and \( n_i \) and clearly \( \rho \neq \rho_{r_i} \), contrary to the uniqueness of \( \rho_{r_i} \). It follows that \( n_i = n_j \) whence \( \Psi \) is injective.

Let \( \lambda = (\lambda_j)_{j=1}^{b-2} \). Since \( \gamma_{r_i} \) has \( n_i \) as an endpoint, we must have \( \lambda_i \in I_i \) for each \( i \) and so \( \lambda \in I^* \). Let \( W \) denote the collection of edges of \( \Gamma \). As \( W \subseteq W' \), we see that \( W = \{ \gamma_{r_i}, \ldots, \gamma_{r_{b-2}} \} \) for certain indices \( \{ \lambda_i \}_{j=1}^{m} \) of \( \{ \lambda_i \}_{i=1}^{b-2} \), as desired. Q.E.D.

Theorem 3.2 asserts that if \( \Gamma \) is an unrestrained subgraph of \( T_0 \), with edges given by
\{y_1, \ldots, y_m\}, then \(\lim_{a_1, \ldots, a_m \to 0} n(a)\) exists. To prove the theorem, it suffices to show that
\[
\lim_{a_1, \ldots, a_m \to 0} \frac{|A_v(a)|}{|A|}
\]
exists for each \(v \in \{1, \ldots, b-2\}\), where \(A_v\) is the matrix obtained by replacing the \(v\)th column of \(A\) by \(c^t\), and \(c\) is as given in (2-6). Thus, it is necessary to gather information about each \(|A_v|\) which will allow us to take the above limit. This is the purpose of Lemma 3.4.

**Lemma 3.4.** For each \(v = 1, \ldots, b-2\), and each \(m = 1, \ldots, 2b-3\), there exist functions \(r_m\) and \(s_m\), independent of \(m\), such that \(|A_v| = \frac{1}{a_m} r_m + s_m\).

**Proof.** By definition,

\[
(3-3) \quad A_v = (\beta_{jk}) = \begin{cases} 
\alpha_{jk} & \text{if } k \neq v \\
c_j & \text{if } k = v
\end{cases}
\]

where \(c_j\) is the \(j\)th component of \(c\) and \(A = (\alpha_{jk})\). Note that for each \(j \in \{1, \ldots, b-2\}\), if \(n_j\) is not adjacent to a boundary point, then \(c_j = 0\); otherwise \(c_j = \sum_{\mu} p_{\mu} a_{\mu} \), where the sum ranges over all \(\mu\) such that \(p_{\mu}\) is adjacent to \(n_j\). Equivalently,

\[
(3-4) \quad c_j = \begin{cases} 
0 & \text{if } I_j^* = \emptyset \\
\sum_{\mu \in I_j^*} \frac{p_{\mu}}{a_{\mu}} & \text{if } I_j^* \neq \emptyset
\end{cases}
\]

where \(I_j^* = I_j \cap \{1, \ldots, b\}\). Note that \(I_j^*\) has at most two elements, since a node may be adjacent to at most two boundary points.
Suppose $1 \leq m \leq b$. Then one endpoint of $\gamma_m$ is a boundary point, $p_m$, and the other is a node. Assume this node is $n_{j_0}$. Recall that $I_{j_0} = \{i_0, i_1, i_2\}$ where $i_0 = j_0 + b - 1$. Since the other endpoint of $\gamma_m$ is a boundary point, it follows that $m \in I_{j_0}$ and $m \notin I_j$ for $j \neq j_0$.

Without a loss of generality, we may assume $m = i_1$. We may write

$$|A_v| = \sum_{k=1}^{b-2} \beta_{j_0k} M_{j_0k}$$

where $M_{j_0k}$ denotes the cofactor of $\beta_{j_0k}$. We must consider two cases.

case i: $j_0 = v$. In this case, we have $\beta_{vv} = \frac{p_m}{a_m} + \xi$ where

$$\xi = \begin{cases} 
0 & \text{if } i_2 > b \\
\frac{p_{i_2}}{a_{i_2}} & \text{if } i_2 \leq b.
\end{cases}$$

From (2-7), we see that $\alpha_{jk}$ is independent of $a_m$ if $k \neq v$. Since $\alpha_{jk} = \beta_{jk}$ for $k \neq v$, it follows that $\beta_{jk}$ is independent of $a_m$ if $k \neq v$. If $j \neq v$, then $\beta_{jv} = c_j$, and since $m \notin I_j^*$ for $j \neq n$, $\beta_{jv}$ is independent of $a_m$. In short, $\beta_{jk}$ is independent of $a_m$ for $(j,k) \neq (v,v)$.

Rewriting the cofactor expansion of $|A_v|$, we obtain

$$|A_v| = \frac{p_m}{a_m} M_{vv} + \xi M_{vv} + \sum_{k \neq v} \beta_{vk} M_{vk}$$

As $\beta_{jk}$ is independent of $a_m$ for $(j,k) \neq (v,v)$, it follows that $M_{vk}$ is independent of $a_m$ for all $k$. Let
\[ r_m = p_m M_{vv} \quad \text{and} \quad s_m = \xi M_{vv} + \sum_{k \neq v} \beta_{vk} M_{vk} \]

Then \( r_m \) and \( s_m \) are independent of \( a_m \) and 
\[ |A_v| = \frac{1}{a_m} r_m + s_m. \]

Case ii: \( j_0 \neq v \). In this case, we have \( \alpha_{j_0v} = \frac{1}{a_i} + \frac{1}{a_m} + \frac{1}{a_2} \), and \( \beta_{j_0v} = \frac{p_m}{a_m} + \xi \), where \( \xi \) is as defined in case i. We see that \( \alpha_{jk} \) is independent of \( a_m \) for \( (j,k) \neq (j_0, j_0) \). As before, \( \beta_{j_0v} \) is independent of \( a_m \) if \( j \neq v \). From (3-3), we see that \( \beta_{jk} \) is independent of \( a_m \) for \( (j,k) \notin \{ (j_0, j_0), (j_0, v) \} \). Rewriting the cofactor expansion of \( |A_v| \), we obtain

\[ |A_v| = \frac{1}{a_m} \left( M_{j_0j_0} + p_m M_{j_0v} \right) + \left( \frac{1}{a_i} + \frac{1}{a_2} \right) M_{j_0j_0} + \xi M_{j_0v} + \sum_{k \notin \{ j_0, v \}} \beta_{j_0k} M_{j_0k}. \]

As \( \beta_{jk} \) is independent of \( a_m \) for \( (j,k) \notin \{ (j_0, j_0), (j_0, v) \} \), it follows that \( M_{j_0k} \) is independent of \( a_m \) for all \( k \). Let

\[ r_m = M_{j_0j_0} + p_m M_{j_0v} \quad \text{and} \quad s_m = \left( \frac{1}{a_i} + \frac{1}{a_2} \right) M_{j_0j_0} + \xi M_{j_0v} + \sum_{k \notin \{ j_0, v \}} \beta_{j_0k} M_{j_0k} \]

Then \( r_m \) and \( s_m \) are independent of \( a_m \) and 
\[ |A_v| = \frac{1}{a_m} r_m + s_m. \]

Now suppose \( b < m \leq 2b-3 \). Then each endpoint of \( \gamma_m \) must be a node. By condition (3) of Definition 1.1, one node must be \( n_{j_1} \) where \( j_1 = m-b+1 \). Suppose the other node is \( n_{j_2} \) where \( j_1 \neq j_2 \). We have \( m \in I_j \) if and only if \( j \in \{ j_1, j_2 \} \), since the endpoints of \( \gamma_m \) are \( n_{j_1} \) and \( n_{j_2} \). Also, \( I_{j_1} = \{ m, m_1, m_2 \} \) and \( I_{j_2} = \{ i_1, i_2, i_3 \} \) where \( i = j_2+b-1 \). Without a loss of generality, we may assume that \( i_1 = m \). We must consider two cases.
case i: $v \in \{j_1, j_2\}$. Without a loss of generality, we may assume $j_1 = v$. Since $m > b$, $\beta_{jv}$ is independent of $a_m$ for each $j \in \{1, \ldots, b-2\}$. Recall that for $k \neq v$, $\beta_{jk} = \alpha_{jk}$. Since $m \notin I_k$ for $k \notin \{j_1, j_2\}$, we see from (2-7) that $\alpha_{jk}$ is independent of $a_m$ for $k \notin \{j_1, j_2\}$. Thus, $\beta_{jk}$ is independent of $a_m$ if $k \neq j_2$. Moreover, if $j \notin \{j_1, j_2\}$, then $\beta_{jj_2} = \alpha_{jj_2}$ is independent of $a_m$. In short, $\beta_{jk}$ may be dependent upon $a_m$ only if $(j,k) \in \{(j_1,j_2), (j_2,j_2)\}$. Note that $\beta_{jj_2} = \frac{1}{a_m}$ and $\beta_{j_2j_2} = \frac{1}{a_i} + \frac{1}{a_m} + \frac{1}{a_{j_2}}$. We may write

$$|A_v| = \sum_{j=1}^{b-2} \beta_{jj_2} M_{jj_2}$$

where $M_{jj_2}$ denotes the cofactor of $\beta_{jj_2}$. Rewriting, we obtain

$$|A_v| = \frac{1}{a_m} (M_{jj_2} - M_{j_1j_2}) + \left(\frac{1}{a_i} + \frac{1}{a_m}\right) M_{j_2j_2} + \sum_{j \notin \{j_1,j_2\}} \beta_{jj_2} M_{jj_2}.$$  

Since $\beta_{jk}$ depends upon $a_m$ if and only if $(j,k) \in \{(j_1,j_2), (j_2,j_2)\}$, it follows that $M_{jj_2}$ is independent of $a_m$ for all $j$. Let

$$r_m = M_{jj_2} - M_{j_1j_2}$$

$$s_m = \left(\frac{1}{a_i} + \frac{1}{a_m}\right) M_{jj_2} + \sum_{j \notin \{j_1,j_2\}} \beta_{jj_2} M_{jj_2}$$

Then $r_m$ and $s_m$ are independent of $a_m$ and $|A_v| = \frac{1}{a_m} r_m + s_m$.

case ii: $v \notin \{j_1, j_2\}$. Since $m > b$, $\beta_{jv}$ is independent of $a_m$ for each $j \in \{1, \ldots, b-2\}$. As before, since $m \notin I_k$ for $k \notin \{j_1, j_2\}$, we see from (2-7) that $\alpha_{jk}$ is independent of $a_m$ for
\( k \not\in \{j_1, j_2\} \). As \( \beta_{jk} = \alpha_{jk} \) for \( k \neq v \), it follows that \( \beta_{jk} \) is independent of \( a_m \) if \( k \not\in \{j_1, j_2\} \). Moreover, if \( j \not\in \{j_1, j_2\} \), then \( \beta_{ji} = \alpha_{ji} \) and \( \beta_{jj} = \alpha_{jj} \) are independent of \( a_m \).

In short, \( \beta_{jk} \) may be dependent on \( a_m \) only if \( (j,k) \in \{(j_1, j_1), (j_1, j_2), (j_2, j_1), (j_2, j_2)\} \).

Let \( B_v \) be the matrix obtained by adding the \( j_1 \)th column of \( A_v \) to the \( j_2 \)th column of \( A_v \); that is, \( B_v = (\delta_{jk}) \) where

\[
\delta_{jk} = \begin{cases} 
\beta_{jk} & \text{for } k \neq j_2 \\
\beta_{jj} + \beta_{jj_2} & \text{for } k = j_2
\end{cases}
\]

Clearly then, \(|B_v| = |A_v|\), and \( \delta_{jk} \) may be dependent upon \( a_m \) only if \( (j,k) \in \{(j_1, j_1), (j_1, j_2), (j_2, j_1), (j_2, j_2)\} \). However,

\[
\delta_{j_1j_2} = \beta_{j_1j_1} + \beta_{j_1j_2} = \left( \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} \right) + \frac{1}{a_m} = \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}}
\]
\[
\delta_{j_2j_2} = \beta_{j_2j_1} + \beta_{j_2j_2} = -\frac{1}{a_{m_1}} + \left( \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} + \frac{1}{a_{m_2}} \right) = \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}}
\]

As before, we may write

\[
|A_v| = |B_v| = \sum_{j=1}^{b-2} \delta_{jj_1} M_{jj_1}
\]

where \( M_{jj_1} \) is the cofactor of \( \delta_{jj_1} \). Rewriting, we obtain

\[
|A_v| = \frac{1}{a_m} (M_{j_1j_1} - M_{j_2j_1}) + \left( \frac{1}{a_{m_1}} + \frac{1}{a_{m_2}} \right) M_{jj_1} + \sum_{j \notin \{j_1, j_2\}} \delta_{jj_1} M_{jj_1}
\]
Since $\delta_{jk}$ depends on $a_m$ if and only if $(j,k) \in \{(j_1,j_1), (j_2,j_1)\}$, it follows that $M_{jj}$ is independent of $a_m$ for all $j$. Let

\[
    r_m = M_{jj} - M_{jj_m}
\]
\[
    s_m = \left(\frac{1}{a_m} + \frac{1}{a_m'}\right) M_{jj} + \sum_{j \neq j_1, j_2} \delta_{jj} M_{jj'}
\]

Then $r_m$ and $s_m$ are independent of $a_m$ and $|A_v| = \frac{1}{a_m} r_m + s_m$. Q.E.D.

We are now ready to state and prove Theorem 3.2 which will provide a continuous extension of $n$ to parts of $\partial[(R^+)^{2b-3}]$.

**Theorem 3.2.** Suppose $\Gamma$ is an unrestrained subgraph of $T_0$ with edges $\{\gamma_{1}, ..., \gamma_{m}\}$.

Then $\lim_{a_{1},...,a_{m} \to 0} n(a)$ exists.

**Proof.** Given that $n(a) = \left(\frac{|A_1|}{|A|}, ..., \frac{|A_{b-2}|}{|A|}\right)$, it suffices to show that $\lim_{a_{1},...,a_{m} \to 0} \frac{|A_v|}{|A|}$ exists.

\[
    \lim_{a_{1},...,a_{m} \to 0} \prod_{j=1}^{m} a_{ij} |A_v|
\]

or, equivalently, that $\lim_{a_{1},...,a_{m} \to 0} \frac{\prod_{j=1}^{m} a_{ij} |A_v|}{\prod_{j=1}^{m} a_{ij} |A|}$ exists for each $v \in \{1, ..., b-2\}$. We accomplish this in two steps. First, we use Lemmas 3.3 and 3.4 to show that $\lim_{a_{1},...,a_{m} \to 0} \prod_{j=1}^{m} a_{ij} |A_v|$ exists for each $v$. Then, we show that $\lim_{a_{1},...,a_{m} \to 0} \prod_{j=1}^{m} a_{ij} |A|$ exists and is nonzero using Lemma 3.3 and Theorem 3.2.

It is clear from (3-3) and (3-4) that for each $v \in \{1, ..., b-2\}$ we may write
\[ |A_v| = \sum_{i=1}^{b} X_{vi} p_i \]

where \( X_{vi} \) is a polynomial in \( \frac{1}{a_1}, \ldots, \frac{1}{a_{2b-3}} \). As a consequence of Lemma 3.4, we see that for each \( \mu \in \{1, \ldots, 2b-3\} \), there exist polynomials \( r_{\mu vi} \) and \( s_{\mu vi} \), independent of \( a_{\mu} \), such that

\[ (3.5) \quad X_{vi} = \frac{1}{a_{\mu}} r_{\mu vi} + s_{\mu vi}. \]

Also, since \( X_{vi} \) is a polynomial in the specified variables, we may write

\[ (3.6) \quad X_{vi} = \sum b_{vi} \delta_1 \ldots \delta_{2b-3} \prod_{k=1}^{2b-3} \left( \frac{1}{a_k} \right)^{\delta_k} \]

where \( \delta_k \) is a nonnegative integer for each \( k \) and \( b_{vi} \delta_1 \ldots \delta_{2b-3} \) is a real constant. For each \( \mu \),

\[ d(X_{vi}, \frac{1}{a_{\mu}}) = 1, \]

where \( d \) is as defined previously; this last statement is a consequence of (3.5). Hence we must have \( \delta_k \in \{0, 1\} \) for each \( k \).

By Lemma 3.3, there exists some \( \lambda = (\lambda_i)_{i=1}^{b-2} \in 1^* \) such that \( \{t_j\}_{j=1}^{m} \subseteq (\lambda_i)_{i=1}^{b-2} \). We have

\[ \prod_{j=1}^{m} a_{t_j} |A_v| = \sum b_{vi} \delta_1 \ldots \delta_{2b-3} \prod_{j=1}^{m} a_{t_j} \prod_{k=1}^{2b-3} \left( \frac{1}{a_k} \right)^{\delta_k}. \]

Since \( \delta_k \in \{0, 1\} \) for each \( k \), we see that

\[ \lim_{a_1, \ldots, a_m \to 0} \prod_{j=1}^{m} a_{t_j} |A_v| \]
exists. Similarly,

$$\prod_{j=1}^{m} a_{ij} |A| = \sum_{\lambda \in I^*} \prod_{j=1}^{m} a_{ij} \prod_{k=1}^{2b-3} \frac{1}{a_{\lambda_k}}$$

by Theorem 3.1; for each $\lambda \in I^*$ we have $\lambda_j \neq \lambda_k$ for $j \neq k$ and so it follows that

$$\lim_{a_{1}, \ldots, a_{m} \to 0} \prod_{j=1}^{m} a_{ij} |A|$$

exists. Moreover, we shall show that the latter limit is nonzero. By Theorem 3.1, $|A|$ is expressible as a sum of positive terms; thus, the limit in question is necessarily nonnegative. Since

$$\lim_{a_{1}, \ldots, a_{m} \to 0} \prod_{j=1}^{m} a_{ij} |A| = \sum_{\lambda \in I^*} \lim_{a_{1}, \ldots, a_{m} \to 0} \prod_{j=1}^{m} a_{ij} \prod_{k=1}^{2b-3} \frac{1}{a_{\lambda_k}}$$

we need only show that, for one particular $\lambda \in I^*$,

$$\lim_{a_{1}, \ldots, a_{m} \to 0} \prod_{j=1}^{m} a_{ij} \prod_{k=1}^{2b-3} \frac{1}{a_{\lambda_k}} > 0.$$ 

We choose the $\lambda$ guaranteed by Lemma 3.3. In this case,

$$(3.7) \prod_{j=1}^{m} a_{ij} \prod_{k=1}^{2b-3} \frac{1}{a_{\lambda_k}} = \prod_{\lambda \neq \lambda_k \in I^*} \frac{1}{a_{\lambda_k}}$$
since \((t_j)_{j=1}^m \subseteq (\lambda_k)_{k=1}^{b-2}\). Note that if \((t_j)_{j=1}^m = (\lambda_k)_{k=1}^{b-2}\) then the right-hand side of (3-7) is simply 1. Moreover,

\[
\lim_{a_{11}, \ldots, a_m \to 0} \prod_{j=1}^m a_{t_j} \prod_{k=1}^{2b-3} \frac{1}{a_{\lambda_k}} = \prod_{\lambda_k \not\in \{t_j\}_{j=1}^m} \frac{1}{a_{\lambda_k}} > 0.
\]

Juxtaposing these results, we see that

\[
\lim_{a_{11}, \ldots, a_m \to 0} \frac{\prod_{j=1}^m a_{t_j} |A_{\lambda_j}|}{\prod_{j=1}^m a_{t_j} |A|}
\]

exists, and the theorem follows. \textbf{Q.E.D.}

As a consequence of Theorem 3.2, we may extend \(n\) to pieces of the boundary of its domain, \((\mathbb{R}^+)^{2b-3}\), as follows: given an unrestrained subgraph \(\Gamma\) of \(T_0\), let \(x = (x_i)_{i=1}^{2b-3}\) be such that

\[
x_i = \begin{cases} 
0 & \text{for } i \in \{t_j\}_{j=1}^m \\
 a_i > 0 & \text{for } i \not\in \{t_j\}_{j=1}^m 
\end{cases}
\]

where \((t_j)_{j=1}^m\) is the set of indices of the paths of \(\Gamma\). Define

\[
n(x) := \lim_{a_{11}, \ldots, a_m \to 0} n (a).
\]

Then
\[
\lim_{a \to x} n(a) = \lim_{a_1, \ldots, a_n \to 0} n(a) = n(x).
\]

and so \( n \) is continuous at \( x \in \partial[(R^+)^{2b-3}] \).

Let \( K \) denote the collection of all unrestrained subgraphs of \( T_0 \). To each \( \Gamma \in K \) there corresponds an \( x = x(\Gamma) \in \partial[(R^+)^{2b-3}] \) as determined above. Let

\[
D = \left(R^+\right)^{2b-3} \cup \bigcup_{\Gamma \in K} x(\Gamma).
\]

Then clearly \( D \subseteq \partial[(R^+)^{2b-3}] \). As a result of the definition made in the previous paragraph, we have extended the domain of definition of \( n \) from \( (R^+)^{2b-3} \) to \( D \), and this extension is continuous in the sense that

\[
\lim_{a \to x} n(a) = n(x) \quad \text{for all} \quad x \in D \setminus (R^+)^{2b-3}.
\]

Note that for each \( x \in D \) and each real \( \mu > 0 \) we have \( \mu x \in D \). Geometrically, this says that \( D \) is a union of rays emanating from, but not including, the origin. Consider the relation \( \sim \) on \( D \) defined as follows:

\[
x \sim y \quad \text{if and only if} \quad y = \mu x \quad \text{for some} \quad \mu > 0.
\]

It is easily shown that \( \sim \) is an equivalence relation, and thus, partitions \( D \) into equivalence classes. The equivalence classes are rays in \( D \) emanating from, but not including, the origin. Let \( M = D/\sim \); that is, the set of equivalence classes of \( \sim \). We call \( M \) a \textit{moduli space} for \( D \). The following proposition states that \( n \) is constant along rays emanating from the origin of \( R^{2b-3} \) which lie in \( D \); as a consequence of the proposition, we obtain a function \( \tilde{n} \) defined on \( M \) which completely determines \( n \).

\textbf{Proposition 3.1.} For each \( x \in D \), and each \( \mu > 0 \), \( n(\mu x) = n(x) \).
Proof. It suffices to show that \( n(\mu a) = n(a) \) for all \( a \in (\mathbb{R}^+)^{2b-3} \) since, if this is the case, we have for \( x \in Dn(\mathbb{R}^+)^{2b-3} \)

\[
\lim_{a \to x} n(\mu a) = \lim_{a \to x} n(a) = n(x).
\]

Suppose \( a \in (\mathbb{R}^+)^{2b-3} \) and \( i \in \{1, \ldots, \beta-2\} \). Let \( A_i(a) = (\beta_{jk}(a)) \). By (2-7), (3-3), and (3-4) we see that

\[
A_i(\mu a) = (\beta_{jk}(\mu a)) = \left( \frac{1}{\mu} \beta_{jk}(a) \right) = \frac{1}{\mu} A_i(a).
\]

Thus,

\[
|A_i(\mu a)| = \frac{1}{\mu^{b-2}} |A_i(a)|.
\]

Similarly,

\[
A(\mu a) = (\alpha_{jk}(\mu a)) = \left( \frac{1}{\mu} \alpha_{jk}(\mu a) \right) = \frac{1}{\mu} A(a)
\]

whence

\[
|A(\mu a)| = \frac{1}{\mu^{b-2}} |A(a)|.
\]

We thus have

\[
n_i(\mu a) = \frac{|A_i(\mu a)|}{|A(\mu a)|} = \frac{\frac{1}{\mu^{b-2}} |A_i(a)|}{\frac{1}{\mu^{b-2}} |A(a)|} = \frac{|A_i(a)|}{|A(a)|} = n_i(a)
\]

from which \( n(\mu a) = n(a) \) follows. Q.E.D.
The proposition says that $n$ is constant along rays in $D$ which emanate from the origin of $\mathbb{R}^{2b-3}$. We may, therefore, define a function $\tilde{n}$ on $M$ which completely determines $n$, but whose domain is a condensed version of $D$. More specifically, we define $\tilde{n} : M \rightarrow (\mathbb{R}^N)^{2b-2}$ by $\tilde{n}(\bar{x}) = n(x)$ where $\bar{x}$ denotes the equivalence class of $x$ under $\sim$.

We note that $\tilde{n}$ is well-defined since if $\bar{x} = \bar{y}$ then $x = \mu y$ for some $\mu > 0$ and so $\tilde{n}(\bar{x}) = n(x) = n(\mu y) = n(y) = \tilde{n}(\bar{y})$ by the proposition.

Further study of the material presented in this thesis might include a generalization of the results presented here to maps of trees without valency restrictions.
BIBLIOGRAPHY

