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Design and analysis of direct-sequence multiuser receivers

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DESIGN AND ANALYSIS OF DIRECT-SEQUENCE MULTI-USER RECEIVERS

by

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Abstract

Matched filter receivers are commonly used for Direct-Sequence Spread-Spectrum Multiple-Access communication systems. These receivers are easy to implement and analyze, and are optimal for single user Gaussian noise channels. However, for applications in spread-spectrum networks, lower average bit-error probability can be achieved by a receiver which takes into account the effect of the interfering users. Two such receivers which form Maximum-Likelihood decisions given an observation vector consisting of chip correlator outputs are shown to perform better than the matched filter. The performance of these receivers is analyzed via Monte Carlo simulations using Importance Sampling. The level of improvement over the matched filter is dependent upon the relative levels of Gaussian noise and multiple-access interference. These receivers demonstrate less deterioration of performance in near-far situations than the matched filter, and find application in wide-band radio networks.
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CHAPTER 1

Introduction

Receivers for asynchronous Direct-Sequence Spread-Spectrum Multiple-Access (DS/SSMA) communications using binary phase-shift-keyed modulation are studied. Direct-Sequence Spread-Spectrum is a modulation technique used for providing interference rejection, increasing immunity to multipath effects, lowering probability of detection and interception as well as providing a means of multiple access [3, 10]. In DS/SSMA systems multiple users share a common communications channel and each user’s carrier is phase modulated by a distinct binary signature sequence which spreads the data over a wide bandwidth. The goal of the receiver is to demodulate a particular user’s signal in the presence of the other users’ signals which occupy the same bandwidth at the same time. The receiver is also assumed to operate in the presence of additive channel noise.

Much work has been done in analyzing the performance of linear correlation receivers for DS/SSMA systems [6, 11, 12]. While the linear correlation receiver does not achieve the minimum probability of error due to the presence of the interfering users, it is used in a great majority of DS/SSMA systems. This is because the linear correlation receiver is much more easily implemented than the more complex optimal receiver. Other nonlinear receivers have been considered for use in non-Gaussian noise environments and while the nonlinearity is seen to improve perfor-
mance with respect to non-Gaussian impulsive noise, linear receivers perform better against multiple access interference [1, 2]. Recently Verdu studied a receiver for DS/SSMA which assumed synchronization with all users and processed the output of a bank of matched filters to form a Maximum a Posteriori estimate of each users bit sequence [13]. That is, the purpose of his receiver was to simultaneously detect each user's data signal after acquisition of each user's carrier phase and delay. In contrast, the intent of this thesis is to detect the data sequence of one user in a multiple access environment without having knowledge of the interfering user's carrier phases or relative time delays, after having observed the received signal for the period of one bit. Enge and Sarwate studied the performance of a SSMA system in which each user is assigned a set of $M$ orthogonal sequences, as opposed to each user having only one sequence [4, 5]. Hence, each sequence transmitted conveys $\log_2 M$ bits of information instead of one bit as in most systems. They analyzed the performance of linear receivers in Gaussian noise as well as impulsive noise, and a locally optimum nonlinear receiver is found which is seen to be effective against impulsive noise.

Park's work with DS/SSMA systems in $\varepsilon$-mixture noise also involves a locally optimum nonlinear receiver [9]. In addition, he assumes the spreading sequences of the interfering users are random. While random spreading sequences are also considered here, the receivers discussed will make a maximum likelihood decision with the received vector instead of using the locally optimum Bayes detector.

The goal of this thesis is to derive receivers which take into account the effect of the interfering users. This effect appears in the form of samples of the multiple-
access interference. Two receivers will be considered in this thesis. First the samples of the multiple-access interference will be assumed to be independent random variables. The receiver will be based on a likelihood ratio, which requires that we be able to compute the first order density function of each component of our received vector. The second receiver will also be based on a likelihood ratio, but the samples of the multiple-access interference will no longer be assumed to be independent, as was commonly done in the past. While it is known that the samples are dependent, the independence assumption simplifies the analysis. However by exploring the dependency structure of the samples of the multiple-access interference we can arrive at a receiver which utilizes this dependency in making its decision. The emphasis will be on analyzing systems with two users, which is an approximation to the case where there are additional interfering users present but the received power from one of them is much larger than the received power from the others. The performance of these proposed receivers will be analyzed via Monte Carlo simulation. The theory of importance sampling is applied to the analysis in order to reduce the number of trials needed to achieve a close estimate of the error probability of these receivers [8].

In the following chapter the signals which each user transmits will be described and the general form of the receiver given. Chapter 3 deals with the first order statistics of the multiple-access interference as well as the Gaussian noise. The second order statistics of the multiple-access interference plus Gaussian noise are considered in Chapter 4. In Chapter 5 the two proposed receivers are given, which are based on the noise statistics covered in Chapters 3 and 4. Chapter 6 describes the performance
of both receivers and compares the performance with that of the linear receiver.
CHAPTER 2

System Definition

In this chapter we give a description of DS/SSMA signals and the system model to be used. We also describe the general receiver design as a discrete-time detection problem. The $k^{th}$ user's data signal $b_k(t)$ is a sequence of unit amplitude, positive and negative pulses of duration $T$ and can be written as

$$b_k(t) = \sum_{m=-\infty}^{\infty} b_m^{(k)} P_T(t-mT), \quad k = 1,2,...,K,$$

where $b_m^{(k)} \in \{-1,+1\}$ is the $m^{th}$ data symbol of the $k^{th}$ user, and $P_T(t)$ denotes the unit rectangular pulse of duration $T$. Each user's signal is phase-modulated by a distinct signature sequence. The signature sequence assigned to the $k^{th}$ user forms code waveform $a_k(t)$, which can be written as

$$a_k(t) = \sum_{j=-\infty}^{\infty} a_j^{(k)} P_{T_c}(t-jT_c), \quad k = 1,2,...,K,$$

where $a_j^{(k)} \in \{-1,+1\}$ is the value of the $j^{th}$ chip in the $k^{th}$ user's code waveform. While only rectangular chip waveforms are considered here, the extension of these results for more general signals should be straightforward. The signature sequences (also called spreading sequences) are assumed to be periodic with period $N$, so that $a_j^{(k)} = a_j^{(k)}$ for all $j$ and $k$ and for some integer $N$. We also assume that there is one spreading sequence period per data bit, so that $T = NT_c$. The signal transmitted by the
The user is the product of the user's data sequence, code waveform and carrier, and is given as

\[ s_k(t) = \sqrt{2\pi_k} a_k(t) b_k(t) \cos(\omega_c t + \theta_k), \quad k = 1, 2, \ldots, K, \]  

where \( \pi_k \) is the transmitted power of the \( k \)-th user, \( \omega_c \) is the carrier frequency common to all users, and \( \theta_k \) is the phase angle of the \( k \)-th user. The phase angles of the \( K \) users may be distinct since the transmitters used in DS/SSMA systems are not usually synchronized. A delayed version of each user's signal arrives at the receiver plus additive channel noise. The delay associated with the \( k \)-th user's signal, \( \tau_k \), accounts for the lack of synchronism between transmitters and the propagation delay. The noise process \( n(t) \) is assumed to be a white Gaussian random process with spectral height \( N_0/2 \). The received signal is given by

\[ r(t) = n(t) + \sum_{k=1}^{K} \sqrt{2P_k} b_k(t - \tau_k) a_k(t - \tau_k) \cos(\omega_c t + \phi_k), \]  

where \( P_k \) is the received power of the \( k \)-th user, and where \( \phi_k \triangleq \theta_k - \omega_c \tau_k, k = 1, 2, \ldots, K. \) The receiver studied here is designed to recover the data sequence of a particular user. Without loss of generality we choose the first user. We assume synchronization with the first user, and consider only relative time delays and phase angles. This allows us to set \( \phi_1 = 0 \) and \( \tau_1 = 0 \) in our analysis. We also assume \( \phi_k \in [0, 2\pi) \) and \( \tau_k \in [0, T), 2 \leq k \leq K, \) since we are only concerned with time delays modulo \( T \) and phase angles modulo \( 2\pi \). In the absence of any other knowledge about \( \phi_k \) and \( \tau_k \) we assume they are random variables which are uniformly distributed over the range of their possible values.
The general form of the receiver is given in Figure 1. The received signal is multiplied with $a_1(t)\cos\omega_c t$, integrated for $T_c$ seconds and then dumped. This is repeated $N$ times over one complete data bit yielding a length-$N$ received observation vector. This received vector is then passed to a decision rule block which decides which data bit was sent by the first user during the observation interval. It is this decision rule block which is to be examined in detail. In a linear correlation receiver, the received process is multiplied with a coherent copy of the first user's signal and integrated over a complete data bit period. The output of the integrator is then sampled and compared to a threshold. This receiver attains the minimum average probability of error in the presence of Gaussian noise alone. This is equivalent to simply adding together the samples obtained using the receiver discussed here. However, the sum of these samples is not a sufficient statistic in the multi-user case, since the samples of multiple-access interference are not Gaussian. By examining the statistics of the multiple-access interference we will determine how to optimally process these samples in order to minimize the average error probability. In other words, we will take the received vector and make a maximum-likelihood decision based on the probability density function of the channel noise plus the multiple-access interference.

The output of the integrator from the $j^{th}$ chip can be viewed as a sum of three components; signal, additive noise, and multiple-access interference. The signal component is the output due to the first user's signal, and is $\pm \sqrt{P_1 T_c}$ depending on whether a "+1" or "-1" was transmitted. The output of the integrator due to the first user's signal will be the same during each chip since we are examining
only one data bit. The additive noise component is a Gaussian random variable \( \eta_j \) which results from integrating and sampling the Gaussian noise process and is given by

\[
\eta_j = a_j^{(1)} \int_{fT_c}^{(j+1)fT_c} n(t) \cos \omega_c t \, dt, \quad j = 0, 1, 2, \ldots, N-1.
\] (5)

The noise samples are independent and identically distributed random variables with zero mean and variance \( N_0 T_c / 4 \). The multiple-access interference component is due to the other \( K-1 \) signals. This component is denoted by \( I_j \) where

\[
I_j = \sum_{k=2}^{K} a_j^{(1)} \sqrt{P_k / 2} \cos \phi_k \int_{fT_c}^{(j+1)fT_c} a_k(t-\tau_k) b_k(t-\tau_k) \, dt \quad j = 0, 1, 2, \ldots, N-1.
\] (6)

The length-\( N \) observation vector will be denoted \( \mathbf{R} \). The decision rule block must choose one of the two following statistical hypotheses:

\[
H_0 : \mathbf{R} = \mathbf{S}_0 + \eta + I
\]

\[
H_1 : \mathbf{R} = \mathbf{S}_1 + \eta + I
\]

where \( \mathbf{S}_0 \) and \( \mathbf{S}_1 \) are vectors with elements \( \pm \sqrt{P_1 / 2} T_c \) and \( -\sqrt{P_1 / 2} T_c \), respectively. The random vectors \( \eta \) and \( I \) denote the length-\( N \) additive noise and multiple-access interference vectors. Since we assume the statistics of \( \eta \) are Gaussian, in order to specify the statistics of \( \mathbf{R} \) under either hypothesis we need to find the joint density function of \( I \). This will enable us to compute the likelihood ratio which our receiver will be based on. It will be shown in Chapter 4 that the joint density function of \( I \) depends only on the first and second order density functions. Therefore the goal of the following two chapters is to find the first and second order density functions of
the components of $I$. 
CHAPTER 3

First Order Statistics

The first order probability density function of the multiple-access interference from the \( j^{th} \) chip interval, denoted by \( p_{j}(x) \), is derived in this chapter. The first order density function of \( Z_j \), the sum of the Gaussian noise sample and the multiple-access interference sample, will also be examined.

The density function of \( I_j \) will be obtained via the characteristic function method. A common method of treating the spreading sequences of the interfering users is to assume that they are independent, randomly generated binary waveforms. In other words, we assume \( a_{j}^{(k)} \) and \( a_{j}^{(k)} \) are independent random variables for \( k=2,3,\ldots,K \) and any \( j \neq i \), and also that \( a_{j}^{(k)} \) and \( a_{j}^{(l)} \) are independent random variables for \( j=0,1,2,\ldots,N-1 \) and any \( k \neq l \). This assumption is well suited in analyzing systems where the spreading sequences of the other users are unknown. The random spreading sequence assumption simplifies the computation of the first order density function of \( I_j \) and is intuitively reasonable in the sense that while one may know the spreading sequences of the other users, the relative delays associated with each user are still random so the amount of information one has about the other users spreading sequences is still small. Detailed analysis is done which verifies the claim that the random spreading sequence assumption has little effect upon the first order density function of \( I_j \). In the following sections the density function of \( I_{j,k} \) is found, where \( I_{j,k} \)
is the sample of multiple-access interference present in the $j^{th}$ chip from the $k^{th}$ user. The density of $I_{j,k}$ is seen to be independent of $k$. If $K$, the number of users, is greater than two then the density function of $I_j$ is found by performing $K-2$ convolutions of the density of $I_{j,k}$ with itself. First we obtain $p_{I_{j,k}}(x)$ assuming random spreading sequences.

3.1. Random Spreading Sequences

In this section the characteristic function of $I_{j,k}$ will be derived for a rectangular chip waveform. We begin with an expression for $I_{j,k}$ in terms of the partial autocorrelation functions for the general chip waveform $\Gamma(t)$. These chip autocorrelation functions are denoted $R_\Gamma(\cdot)$ and $\hat{R}_\Gamma(\cdot)$ and are as in [11]. They are defined as

$$R_\Gamma(s) = \int_0^s \Gamma(t) \Gamma(t+T_c-s) \, dt$$

and

$$\hat{R}_\Gamma(s) = \int_s^{T_c} \Gamma(t) \Gamma(t-s) \, dt.$$  \hfill (7)

We then find the characteristic function of $I_{j,k}$ for the specific case of a rectangular chip waveform, i.e. $\Gamma(t) = P_{T_c}(t)$. We begin by restating the definition of $I_j$ as

$$I_j = \sum_{k=2}^K I_{j,k}$$

where

$$I_{j,k} = a_j^{(1)} \sqrt{\frac{P}{k^2}} \cos \phi_k \int_{-T_c}^{(j+1)T_c} b_k(t-\tau_k) a_{k} t-\tau_k dt$$

for $j=0,1,\ldots,N-1$ and $k=2,3,\ldots,K$. \hfill (9)
Due to the fact that the random variables present in the expression for $I_{j,k}$ are independent for different $k$, we see that $I_j$ is a sum of $K$ independent random variables. Under the random spreading sequence assumption it is easy to see that the integral in (9) will take on only four values with probability $1/4$. This allows us to write $I_{j,k}$ as

$$I_{j,k} = a_j^{(1)} \sqrt{P_k} \cos \phi_k \left[ c_j R_T(\tau_k) + c_{j+1} \hat{R}_T(\tau_k') \right]$$

where $\tau_k' = \tau_k - \left[ \frac{\tau_k}{T_c} \right]$. The two random variables $c_j$ and $c_{j+1}$ are functions of the spreading sequence and data bits of the $k^{th}$ user. Due to the random spreading sequence assumption $c_j$ and $c_{j+1}$ are independent and take on the values +1 and -1 with equal probability. Since $\tau_k$ was assumed to be uniformly distributed over $[0,T]$, it is clear that $\tau_k'$ is uniformly distributed over $[0,T_c]$. We can then write $\Phi_{I_{j,k}}(u)$ as

$$\Phi_{I_{j,k}}(u) = E_{c_j,\tau_k,\tau_k'} \left\{ \exp \left[ i u a_j^{(1)} \sqrt{P_k} \cos \phi_k \left( c_j R_T(\tau_k') + c_{j+1} \hat{R}_T(\tau_k') \right) \right] \right\}$$

where $i = \sqrt{-1}$. After taking the expectation with respect to $c_j$ and $c_{j+1}$, $\Phi_{I_{j,k}}(u)$ is seen to be

$$\Phi_{I_{j,k}}(u) = \frac{1}{2} E_{\tau_k',\phi_k} \left\{ \cos \left[ u \sqrt{P_k} \cos \phi_k (R_T(\tau_k') + \hat{R}_T(\tau_k')) \right] \right\}.$$

At this point we will restrict our analysis to rectangular chip waveforms, i.e. $\Gamma(t) = P_{T_c}(t)$. For a rectangular chip waveform, it is easily shown that $R_T(\tau_k') = \tau_k'$
and \( \hat{\rho}(\tau_k') = T_c - \tau_k' \) for \( 0 \leq \tau_k' \leq T_c \). So for the case of a rectangular chip waveform, \( \Phi_{I,\alpha}(u) \) is given by

\[
\Phi_{I,\alpha}(u) = \frac{1}{4\pi T_c} \int_0^{T_c} 2\pi \int_0^{T_c} \left[ \cos \left( u \sqrt{\frac{P_c}{2}} \cos \phi_k(T_c) \right) + \cos \left( u \sqrt{\frac{P_c}{2}} \cos \phi_k(2\tau_k' - T_c) \right) \right] d\phi_k d\tau_k' \\
= \frac{1}{2T_c} \int_0^{T_c} \left[ J_0(u \sqrt{\frac{P_c}{2}} T_c) + J_0(u \sqrt{\frac{P_c}{2}} (T_c - 2\tau_k')) \right] d\tau_k' \\
= \frac{1}{2} J_0(u \varepsilon_k) + \frac{1}{2} u \varepsilon_k \int_0^u J_0(z) dz,
\]

(12)

where \( \varepsilon_k = \sqrt{\frac{P_c}{2}} T_c \) and \( J_0(\cdot) \) is the Bessel function of the first kind, zero order.

Now we can obtain the probability density function of \( I_{j,k} \), namely \( p_{I,\alpha}(x) \), by taking the Fourier transform of \( \Phi_{I,\alpha}(u) \) and dividing by \( 2\pi \). Upon evaluating the integrals we find that

\[
p_{I,\alpha}(x) = \begin{cases} \\
1/2\pi \varepsilon_k \left[ \frac{1}{\sqrt{1 - (x/\varepsilon_k)^2}} + \ln \left( \frac{1 + \sqrt{1 - (x/\varepsilon_k)^2}}{|x/\varepsilon_k|} \right) \right] & 0 \leq |x| \leq \varepsilon_k \\
0 & \varepsilon_k < |x| \end{cases}
\]

(13)

This density function is symmetric about the origin and is nonzero only in the interval \([-\varepsilon_k, \varepsilon_k]\), as can be seen from Figure 2. Figure 2 shows the density function of \( I_{j,k} \) for the case where \( \varepsilon_k = 1 \). The function \( p_{I,\alpha}(x) \) increases without bound as \( x \) approaches \( \pm \varepsilon_k \). This behavior is due to the fact that the integral in the expression for \( I_{j,k} \)

\[
l_{jT_c} \int_{jT_c}^{(j+1)T_c} a_k(t - \tau_k) b_k(t - \tau_k) dt,
\]

takes on the discrete values \( \pm T_c \) with probability
1/4. This concludes the discussion of the first order density under the assumption of random spreading sequences.

3.2. Known Spreading Sequences

In this section, the case of known spreading sequences will be considered. This consideration will also determine the effect of the random spreading sequence assumption upon $p_{I_{jk}}(x)$. The form of $p_{I_{jk}}(x)$ obtained using the random spreading sequence assumption will be compared with the form of $p_{I_{jk}}(x)$ obtained assuming the spreading sequences of the other users are known.

For a rectangular chip waveform the sample of multiple access interference from the $j^{th}$ chip of the $k^{th}$ user can be written in a form equivalent to equation (9)

$$ I_{jk} = a_j^{(1)} \cos \phi_k \sqrt{P_d/2} \left[ B(j, m_k) a_{j-1-m_k}^{(k)} \tau_k + \hat{B}(j, m_k) (T_c - \tau_k') a_{j-m_k}^{(k)} \right] $$  (14)

where

$$ B(j, m_k) \triangleq \begin{cases} b_{-1}^{(k)} & 0 \leq j \leq m_k \\ b_0^{(k)} & m_k + 1 \leq j \leq N-1 \end{cases} $$

$$ \hat{B}(j, m_k) \triangleq \begin{cases} b_{-1}^{(k)} & 0 \leq j \leq m_k - 1 \\ b_0^{(k)} & m_k \leq j \leq N-1 \end{cases} $$

and where $m_k \triangleq \left\lfloor \frac{\tau_k}{T_c} \right\rfloor$. We can evaluate the characteristic function $\Phi_{I_{jk}}(u)$ by taking the expectation in three steps.
\[ \Phi_{I,\mu}(u) = E_{\tau'_k} \left[ E_{\Phi_k} \left[ E_{B(j,m_n),B'(j,m_n),m_n} \left[ e^{iu I_{\mu}} \right] \right] \right] \] (15)

Taking the innermost expectation yields

\[
\Phi_{I,\mu}(u) = E_{\tau'_k} \left[ E_{\Phi_k} \left[ \frac{1}{2N} \left( \cos(u \sqrt{P_k/2} \cos \phi_k (\tau_k' a_{-1}^{(k)} + a_0^{(k)} (T_c - \tau_k')) \right) + \cos(u \sqrt{P_k/2} \cos \phi_k (\tau_k' a_{-1}^{(k)} - a_0^{(k)} (T_c - \tau_k'))) \right] \right. \\
\left. + \frac{1}{N} \sum_{l=0}^{N-1} \cos(u \sqrt{P_k/2} \cos \phi_k (\tau_k' a_{j-1}^{(k)} + a_{j-1}^{(k)} (T_c - \tau_k'))) \right] \right] \\
\]

Now we take the expectation with respect to \( \phi_k \) and obtain

\[
\Phi_{I,\mu}(u) = E_{\tau'_k} \left[ \frac{1}{2N} J_0(u \sqrt{P_k/2} (\tau_k' a_{-1}^{(k)} + a_0^{(k)} (T_c - \tau_k'))) \right. \\
\left. + \frac{1}{2N} J_0(u \sqrt{P_k/2} (\tau_k' a_{j-1}^{(k)} + a_{j-1}^{(k)} (T_c - \tau_k'))) \right] \\
\left. + \frac{1}{N} \sum_{l=0}^{N-1} J_0(u \sqrt{P_k/2} (\tau_k' a_{j-1}^{(k)} + a_{j-1}^{(k)} (T_c - \tau_k'))) \right] \\
\left. = \frac{1}{2N} \int_0^{T_c} J_0(u \sqrt{P_k/2} (\tau_k' a_{-1}^{(k)} + a_0^{(k)} (T_c - \tau_k'))) d\tau_k' \\
\left. + \frac{1}{N} \int_0^{T_c} J_0(u \sqrt{P_k/2} \cos(\tau_k' a_{-1}^{(k)} - a_0^{(k)} (T_c - \tau_k'))) d\tau_k' \right] \\
\]
\[ + \frac{1}{NT_c} \sum_{l=0}^{N-1} \int_{T_c} J_0(u \sqrt{P_k/2} (\tau_k' a_{j-l-1}^{(l)} + a_{j-l}^{(l)} (T_c - \tau_k'))) \, d\tau_k'. \]

At this point we define the set \( A^{(k)} \) to be the set of all nonnegative integers \( j \) less than \( N-1 \) such that \( a_j^{(k)} a_{j+1}^{(k)} = 1 \) and the set \( B^{(k)} \) to be the set of nonnegative integers less than \( N-1 \) such that \( a_j^{(k)} a_{j+1}^{(k)} = -1 \), as in [7]. We note that

\[
\frac{1}{T_c} \int_{0}^{T_c} J_0 \left[ u \sqrt{P_k/2} (\tau_k' a_{j-l-1}^{(k)} + a_{j-l}^{(k)} (T_c - \tau_k')) \right] d\tau_k' = \begin{cases} J_0(u \varepsilon_k) & l \in A^{(k)} \\ \frac{u \varepsilon_k}{1/u \varepsilon_k} \int_{0}^{J_0(z)} dz & l \in B^{(k)} \end{cases}
\]

and

\[
\frac{1}{T_c} \int_{0}^{T_c} J_0(u \sqrt{P_k/2} (\tau_k' a_{j-l-1}^{(k)} + J_0(u \sqrt{P_k/2} (\tau_k' a_{j-l}^{(k)}) - a_{j-l}^{(k)} (T_c - \tau_k')))) d\tau_k' = \]

\[ J_0(u \varepsilon_k) + \frac{u \varepsilon_k}{1/u \varepsilon_k} \int_{0}^{J_0(z)} dz \]

regardless of the values of \( a_{j-l-1}^{(k)} \) and \( a_{j-l}^{(k)} \). We can now write \( \Phi_{I_{\mu}}(u) \) in terms of \( A^{(k)} \) and \( B^{(k)} \)

\[
\Phi_{I_{\mu}}(u) = \left[ \frac{2||A^{(k)}|| + 1}{2N} \right] J_0(u \varepsilon_k) + \left[ \frac{2||B^{(k)}|| + 1}{2N} \right] \frac{u \varepsilon_k}{1/u \varepsilon_k} \int_{0}^{J_0(z)} dz \quad (16)
\]

This implies that

\[
P_{I_{\mu}}(x) = \frac{1}{2 \pi \varepsilon_k} \left[ \frac{2||A^{(k)}|| + 1}{N \sqrt{1 - (x/\varepsilon_k)^2}} + \frac{2||B^{(k)}|| + 1}{N} \ln \left( \frac{1 + \sqrt{1 - (x/\varepsilon_k)^2}}{|x/\varepsilon_k|} \right) \right] \quad (17)
\]
for \( 0 \leq |x| \leq \varepsilon_k \) and is zero elsewhere. In the above equation, \( \|A^{(k)}\| \) denotes the number of elements contained in \( A^{(k)} \). This expression is identical to equation (13) except for the factors \( (2\|A^{(k)}\| + 1)/N \) and \( (2\|B^{(k)}\| + 1)/N \) which appear in equation (17). The form of \( p_{I_{jk}}(x) \) obtained assuming known spreading sequences is the same as the form of \( p_{I_{jk}}(x) \) obtained assuming random spreading sequences, the only difference being the relative weights of the two terms. However, in the case of maximal length sequences, the factors \( (2\|A^{(k)}\| + 1)/N \) and \( (2\|B^{(k)}\| + 1)/N \) are easily shown to be equal to \((N-1)/2N\) or \((N+1)/2N\), and the two factors add up to one. This result is proven by using properties of maximal length sequences given in [3]. Clearly for large values of \( N \) these factors are very nearly equal to \( 1/2 \) and equations (13) and (19) are very nearly identical. This demonstrates the small effect upon \( p_{I_{jk}}(x) \) due to assuming known spreading sequences. Because of the relatively small effect upon \( p_{I_{jk}}(x) \) which results from knowing each user’s spreading sequence, throughout the rest of this thesis the spreading sequences of the interfering users will be assumed to be random as described in Section 3.1.

3.3. Cumulative Distribution Function

In this section the distribution function of \( I_j \) is given. Upon integrating \( p_{I_{jk}}(x) \) we obtain

\[
F_{I_{jk}}(b) = Pr(I_{jk} \leq b) = \int_{-\varepsilon_k}^{b} p_{I_{jk}}(x) \, dx
\]
\[
F_{I_{jk}}(b) = \begin{cases} 
0 & b \leq -\varepsilon_k \\
\frac{1}{2} + \frac{1}{\pi} \sin^{-1} b/\varepsilon_k + b/2\pi\varepsilon_k \ln \left| \cot \left( \frac{\sin^{-1} b/\varepsilon_k}{2} \right) \right| & -\varepsilon_k < b < \varepsilon_k \\
1 & \varepsilon_k \leq b 
\end{cases}
\]

This distribution function is plotted in Figure 3 for \( \varepsilon_k = 1 \). The shape of the distribution function appears very similar to that of a uniformly distributed random variable, which is misleading since the density function is unbounded at three points.

3.4. Density Function of Noise Plus Multiple-Access Interference

In this section the density of \( Z_j \) is discussed, where \( Z_j = I_j + \eta_j \). The statistics of the multiple-access interference sample, \( I_j \), are discussed in the preceding section.

The Gaussian noise sample, \( \eta_j \), is zero mean with variance equal to \( N_0 T_c / 4 \) and is assumed to be independent of \( I_j \). Therefore the density of \( Z_j \) is equal to the convolution of the densities of \( I_j \) and \( \eta_j \). Recognizing \( I_j \) as a sum of the independent random variables \( I_{j1}, I_{j2}, I_{j3}, ..., I_{jk} \) yields

\[
p_{Z_j}(x) = p_{\eta_j}(x) * p_{I_{j1}}(x) * ... * p_{I_{jk}}(x) \quad (19)
\]

where \(*\) denotes convolution. An alternative way of obtaining \( p_{Z_j}(x) \) is by taking the Fourier transform of the characteristic function of \( Z_j \), denoted \( \Phi_{Z_j}(u) \), where the
characteristic function is the product of the characteristic functions of $I_j$ and $\eta_j$. This is written as

$$p_{Z_j}(x) = 1/2\pi \int_{-\infty}^{\infty} e^{-iux} \Phi_{\eta_j}(u) \Phi_{\eta_j}(u) \cdots \Phi_{\eta_j}(u) \, du$$  \hspace{1cm} (20)$$

where $i = \sqrt{-1}$. While an analytical solution for $p_{Z_j}(x)$ is not known, there are several numerical methods for evaluating $p_{Z_j}(x)$. One method is to compute the convolution integral numerically. Another is to compute a large set of samples of $\Phi_{Z_j}(u)$ and take the discrete Fourier transform to obtain a set of samples of $p_{Z_j}(x)$. This method has the advantage of being easily adapted to the situation where more than two users are present, since instead of performing multiple convolutions we need only multiply the characteristic functions and then compute the DFT. However, this method has two types of error. First there is aliasing error due to the fact that the Gaussian density is nonzero over the entire real line. There is also error introduced by the fact that we must use a finite number of samples to represent a nonperiodic function. With attention paid to these two issues, this method will yield samples of the density with arbitrarily small error.

The emphasis of this thesis will be on the case of two users. While the extension of the results obtained is conceptually straightforward, it is computationally cumbersome. Graphs of $p_{Z_j}(x)$ for the case of two users are given in Figures 4-8. These figures reflect the dependence on the shape of $p_{Z_j}(x)$ on the quantity $SNR_2$, where $SNR_2 = E_{b_2}/N_0$. The bit energy of the second user, denoted $E_{b_2}$, is equal to $TP_2 = NT_cP_2$. For relatively small levels of $SNR_2$, $p_{Z_j}(x)$ looks very similar to a
Gaussian density. However, for large values of $SNR_2$ the shape of $p_{Z_i}(x)$ looks distinctly non-Gaussian. The density becomes tri-modal as $SNR_2$ is increased and eventually approaches the density of $I_j$ as $N_0$ approaches zero.
CHAPTER 4

Second Order Statistics

In this chapter the second order statistics of the multiple-access interference are considered. The joint density function of $I_j$ and $I_{j+1}$, denoted $p_{I_j, I_{j+1}}(x,y)$, is derived using the joint characteristic function. The joint density of $Z_j$ and $Z_{j+1}$ is discussed, where $Z_j = \eta_j + I_j$ is the sum of the multiple-access interference and the Gaussian noise. It will also be shown that the sequence of random variables $I_0, I_1, \cdots, I_{N-1}$ is first order Markov and hence the joint density of $I$ can be written as a product of first and second order densities. Since the elements of the vector $\eta$, the Gaussian noise samples, are independent, the vector $Z$ will also be first order Markov. Hence, once the first and second order statistics of the combined noise are known, we may make a maximum likelihood decision with the received samples.

In order to show that $I$ is Markov, consider the sequence of $I_j$'s conditioned on the values of $\phi_2$ and $\tau_{2}'$. Since we are considering the case of two users, $I_j$ can be written as

$$I_j = a_j^{(1)} \sqrt{P_2} \cos \phi_2 \left[ c_j \tau_{2}' + c_{j+1}(T_c - \tau_{2}') \right]$$

If we condition on $\tau_{2}'$ and $\phi_2$, then only $c_j$ and $c_{j+1}$ are random. Notice that $I_{j+2}$ conditioned on $\tau_{2}'$ and $\phi_2$ depends only on $c_{j+2}$ and $c_{j+3}$, both of which are independent of $c_j$ and $c_{j+1}$. This implies that $I_j$ and $I_{j+2}$ are independent, conditioned on $\tau_{2}'$ and $\phi_2$. 

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Moreover, this is true for any $I_j$ and $I_{j+1}$ for $l>1$. This means that the density of $I_j$ given $\tau_2, \tau_2'$ and the rest of the sequence $I$ is equal to the density of $I_j$ given $\tau_2, \phi_2$ and $I_{j+1}$. In other words,

$$P_{I_j|\phi_2, \tau_2', I_{j+1}, I_{j+2}, \ldots, I_N}(i_j|\phi_2, \tau_2', i_{j+1}, i_{j+2}, \ldots, i_N) = P_{I_j|I_{j+1}, \phi_2, \tau_2'}(i_j|i_{j+1}, \phi_2, \tau_2').$$

Furthermore, we can multiply both sides of this equality by $p_{\phi_2, \tau_2'}(\phi_2, \tau_2')$ and integrate both sides with respect to $\phi_2$ and $\tau_2'$ which yields

$$P_{I_j|I_{j+1}, I_{j+2}, \ldots, I_N}(i_j|i_{j+1}, i_{j+2}, \ldots, i_N) = P_{I_j|I_{j+1}}(i_j|i_{j+1}).$$

This establishes that $I$ is first order Markov. If we add the vector $\eta$ to $I$ to obtain $Z$ we note that $Z$ is also first order Markov, because the elements of $\eta$ are independent.

As in the previous chapter, both random spreading sequences and known spreading sequences will be considered. The second order density function will be shown to have the same form in either case, with the only differences being in the relative weights of the different terms. This is similar to the case of the first order density function as seen in the previous chapter.

4.1. Random Spreading Sequences

In this section the joint density function of $I_{j,k}$ and $I'_{j+1,k}$ is derived assuming the spreading sequences of the interfering users are random sequences. As in the previous chapter, the chip waveform will be assumed to be rectangular. First the joint characteristic and density functions of $I'_{j,k}$ and $I'_{j+1,k}$ will be derived, where $I'_{j,k} = I_{j,k}/a_j^{(1)}$ and $I'_{j+1,k} = I_{j+1,k}/a_{j+1}^{(1)}$. That is, $I'_{j,k}$ and $I'_{j+1,k}$ are normalized versions
of $I_{j,k}$ and $I_{j+1,k}$ which do not depend on the first user's spreading sequence. $p_{I_{j},I_{j+1}}(x,y)$ will then be given in terms of $p_{I_{j},I_{j+1}}(x,y)$ for the two cases $a_j^{(1)} = a_{j+1}^{(1)}$ and $a_j^{(1)} \neq a_{j+1}^{(1)}$. Assuming random spreading sequences, $I'_{j,k}$ can be written as

$$I'_{j,k} = \sqrt{P_k / 2} \cos \phi_k \left[ c_j \tau_k' + c_{j+1} (T_c - \tau_k') \right]$$

where $c_j$ and $c_{j+1}$ are independent random variables which take on the values +1 and -1 with equal probability. The joint characteristic function of $I'_{j,k}$ and $I'_{j+1,k}$ denoted $\Phi_{I'_{j,k} I'_{j+1,k}}(u,v)$, is given as

$$\Phi_{I'_{j,k} I'_{j+1,k}}(u,v) = E_{\omega_{\phi_k c}} \left[ \exp i \left\{ u a_j^{(1)} \sqrt{P_k / 2} \cos \phi_k (c_j \tau_k' + c_{j+1} (T_c - \tau_k')) 
+ v a_{j+1}^{(1)} \sqrt{P_k / 2} \cos \phi_k (c_{j+1} \tau_k' + c_{j+2} (T_c - \tau_k')) \right\} \right]$$

$$= E_{\omega_{\phi_k c}} \left[ \cos \left\{ u \sqrt{P_k / 2} \cos \phi_k \tau_k' \right\} \cos \left\{ v \sqrt{P_k / 2} \cos \phi_k (T_c - \tau_k') \right\} \right] \cdot \cos \left\{ v \sqrt{P_k / 2} \cos \phi_k (T_c - \tau_k') \right\}$$

(21)

where $E$ in the expectation operator denotes the vector $[c_j, c_{j+1}, c_{j+2}]$. This can be written as

$$\Phi_{I'_{j,k} I'_{j+1,k}}(u,v) = 1/4 E_{\omega_{\phi_k c}} \left[ \cos \left\{ \sqrt{P_k / 2} \cos \phi_k (uT_c - vT_c) \right\} + \cos \left\{ \sqrt{P_k / 2} \cos \phi_k (uT_c + v(2\tau_k' - T_c)) \right\} \right]$$
\[ + \cos \left\{ \sqrt{P_k/2} \cos \phi_k (u(2\tau_k' - T_c) + v(T_c - 2\tau_k')) \right\} + \cos \left\{ \sqrt{P_k/2} \cos \phi_k (u(2\tau_k' - T_c) - vT_c) \right\} \]

After performing the integrals associated with taking the expectations, \( \Phi_{l_{i,j},l'_{j+1,k}}(u,v) \) can be written

\[
\Phi_{l_{i,j},l'_{j+1,k}}(u,v) = 1/4 J_0(\epsilon_k(u+v)) + 1/(4\epsilon_k(u-v)) \int_0^{\epsilon_k(u-v)} J_0(z) \, dz \tag{22}
\]

\[
+ 1/8 \mu \epsilon_k \int_{-\epsilon_k(u+v)}^{\epsilon_k(u-v)} J_0(z) \, dz + 1/8 v \epsilon_k \int_{-\epsilon_k(u-v)}^{\epsilon_k(u+v)} J_0(z) \, dz
\]

where \( \epsilon_k \triangleq \sqrt{P_k/2T_c} \). The joint density function of \( l'_{j,k} \) and \( l'_{j+1,k} \) assuming random spreading sequences is obtained by taking the Fourier transform of the joint characteristic function and is given as

\[
P_{l_{i,j},l'_{j+1,k}}(x,y) = 1/4 \left[ \frac{\delta(x-y)}{\pi \epsilon_k \sqrt{1-(x/\epsilon_k)^2}} + \frac{\delta(x+y)}{\pi \epsilon_k} \ln \frac{1+\sqrt{1-(x/\epsilon_k)^2}}{x/\epsilon_k} \right]
\]

\[
+ \frac{\text{sgn}(x-y) + \text{sgn}(x+y)}{4\pi \epsilon_k x \sqrt{1-(x/\epsilon_k)^2}} + \frac{\text{sgn}(x+y) - \text{sgn}(x-y)}{4\pi \epsilon_k y \sqrt{1-(y/\epsilon_k)^2}} \right]\tag{23}
\]

for \( 0 \leq |x| \leq \epsilon_k, \ 0 \leq |y| \leq \epsilon_k \) and zero elsewhere. If \( a_j^{(1)} = a_{j+1}^{(1)} \) then \( p_{l_{i,j},l'_{j+1,k}}(x,y) = p_{l_{i,j},l'_{j+1,k}}(x,y) \). If \( a_j^{(1)} \neq a_{j+1}^{(1)} \) then \( p_{l_{i,j},l'_{j+1,k}}(x,y) = p_{l_{i,j},l'_{j+1,k}}(-y,x) \). Note the impulses along the lines \( x=y \) and \( x=-y \). These reflect the fact that two consecutive samples of multiple-access interference have a non-zero probability of having equal magnitude.
4.2. Known Spreading Sequences

In this section the joint density function of \( I_{j,k} \) and \( I_{j+1,k} \) is given assuming the spreading sequences of all users are known. The density function is seen to have a form similar to the density function derived assuming random spreading sequences, with the only difference being the relative weights of the four terms which appear in (23). The weights of the four terms depend upon parameters of the spreading sequences of the interfering users. These parameters are examined for maximal-length sequences and are seen to be very nearly equal to 1/4, which reduces the expression for the density function to (23).

For the case of \( a_j(1) = a_{j+1}(1) \) the joint characteristic function of \( I_{j,k} \) and \( I_{j+1,k} \) is given by

\[
\Phi_{I_{j,k}I_{j+1,k}}(u,v) = W_1 J_0(\varepsilon_k(u+v)) + W_2/2\varepsilon_k\nu \int_{\varepsilon_k(u-v)}^{\varepsilon_k(u+v)} J_0(z) \, dz \\
+ W_3/2\varepsilon_k\mu \int_{\varepsilon_k(-u+v)}^{\varepsilon_k(u+v)} J_0(z) \, dz + W_4/\varepsilon_k(u-v) \int_{0}^{\varepsilon_k(u-v)} J_0(z) \, dz
\]

where \( W_1, W_2, W_3 \) and \( W_4 \) are weights which are determined by the interfering user’s spreading sequence. This leads to a joint density of \( I_{j,k} \) and \( I_{j+1,k} \) identical to (23) but with different weighting of the four terms. This also holds for the case where \( a_{j+1}(1) \neq a_{j}(1) \). These weights can be written in terms of parameters of the interfering user’s spreading sequence as given below.
\[
W_1 = \frac{1}{2N} \left[ 2\|C\| + \delta_1 + \delta_2 \right] \\
W_2 = \frac{1}{2N} \left[ 2\|D\| + \delta_1 + 1 - \delta_2 \right] \\
W_3 = \frac{1}{2N} \left[ 2\|E\| + \delta_2 + 1 - \delta_1 \right] \\
W_4 = \frac{1}{2N} \left[ 2\|F\| + 2 - \delta_1 - \delta_2 \right]
\]

where \( C^{(k)} \) is defined as the set of all positive integers \( l \) less than \( N-2 \) such that \( a^{(k)}_l = a^{(k)}_{l+1} = a^{(k)}_{l+2} \). Similarly, \( D^{(k)} \) is defined as the set of all positive integers \( l \) less than \( N-2 \) such that \( a^{(k)}_l = -a^{(k)}_{l+1} \), \( E^{(k)} \) is the set of all positive integers \( l \) less than \( N-2 \) such that \( -a^{(k)}_l = a^{(k)}_{l+1} = a^{(k)}_{l+2} \) and \( F^{(k)} \) is the set of all positive integers \( l \) less than \( N-2 \) such that \( -a^{(k)}_l = -a^{(k)}_{l+1} = a^{(k)}_{l+2} \). The two variables \( \delta_1 \) and \( \delta_2 \) are given as

\[
\delta_1 \triangleq \begin{cases} 
1 & a^{(k)}_2 = a^{(k)}_1 \\
0 & a^{(k)}_2 \neq a^{(k)}_1 
\end{cases}
\]

and

\[
\delta_2 \triangleq \begin{cases} 
1 & a^{(k)}_0 = a^{(k)}_1 \\
0 & a^{(k)}_0 \neq a^{(k)}_1 
\end{cases}
\]

It is easily shown, using properties of maximal-length sequences given in [3], that \( \|C\| = 2^{N-2} - 1 \) and \( \|D\| = \|E\| = \|F\| = 2^{N-2} \). Therefore for large \( N \) the factors \( W_1, W_2, W_3 \) and \( W_4 \) are seen to be very close to 1/4, yielding a characteristic function which is similar in form to that for random spreading sequences. Deriving the expression for the second order characteristic function assuming known spreading sequences involves the same manipulations used in deriving the first order characteristic
function. However the parameters which determine the four weights \( W_1, W_2, W_3, \) and \( W_4 \) are found by examining groups of three consecutive chips of the interfering user's spreading sequence instead of only looking at groups of two consecutive chips as was the case with the first order characteristic function. As with the first order case, the effect of assuming knowledge of each user's spreading sequence is very small for systems using maximal-length sequences of reasonable length. For this reason the spreading sequences will be treated as random during the rest of this thesis.

4.3. Joint Density of Noise Plus Multiple-Access Interference

In this section the joint density of \( Z_j \) and \( Z_{j+1} \) is discussed. This density function can be found through a two-dimensional convolution of \( p_{\eta_1,\eta_2}(x,y) \) and \( p_{f_{j,j+1}}(x,y) \) or by finding the two-dimensional Fourier transform of the product of the second order characteristic functions. Neither method yields an analytical solution for \( p_{Z_jZ_{j+1}}(x,y) \), making it necessary to evaluate this density numerically. Another form for \( p_{Z_jZ_{j+1}}(x,y) \) more suited to numerical evaluation is

\[
p_{Z_jZ_{j+1}}(x,y) = \int_{0}^{2\pi T_c} \int_{0}^{2\pi T_c} p_{Z_jZ_{j+1}|\tau_2',\phi_2}(x,y|\tau_2',\phi_2)p_{\tau_2',\phi_2}(\tau_2',\phi_2) \, d\tau_2' \, d\phi_2 \quad (26)
\]

The joint density of \( \tau_2' \) and \( \phi_2 \) is simply \( 1/2\pi T_c \) over the range of \( \tau_2' \) and \( \phi_2 \) and zero elsewhere. The conditional density of \( Z_j \) and \( Z_{j+1} \) given \( \tau_2' \) and \( \phi_2 \) is a sum of eight two-dimensional independent Gaussian densities with means equal to the eight possible realizations of \( Z_j \) and \( Z_{j+1} \) with \( \tau_2' \) and \( \phi_2 \) treated as constants. Using a numerical
integration routine to perform the double integral yields different tables of values
used for the joint density in the simulations and plotted in Figures 8-12. These
figures all depict the density for the case where \( a_j^{(1)} = a_{j+1}^{(1)} \). Note that in Figures 9
and 10, the density looks very similar to a two-dimensional Gaussian density with
independent variables. In these two figures, the level of multiple-access interference
is small compared to the Gaussian noise. The presence of the larger levels of the
multiple-access interference in Figures 11-13 is evident along the lines \( x = y \) and
\( x = -y \). This is as expected due to the impulses along these lines in the second order
density of \( I_j \) and \( I_{j+1} \). This effect is most noticeable in Figure 13 where \( SNR_2 \) is larg-
est.

At this point both the first and second order density functions of the noise plus
multiple-access interference have been analyzed, and it has been shown that the joint
density can be completely specified in terms of the first and second order densities.
Using the joint density function of the noise plus multiple-access interference we can
now derive the optimal receiver structure.
CHAPTER 5

Receiver Design

This chapter will discuss two different receivers for DS/SSMA systems based on the statistics of the multiple-access interference plus Gaussian noise. First a receiver which optimally processes the received vector at the output of the chip integrator under the assumption of independent multiple-access interference samples is derived. A second receiver which takes into account the second order statistics of the samples of multiple-access interference will be examined also. The performance of both receivers will be determined by computing estimates of the average error probability through computer simulations. These receivers take the form of a likelihood ratio test which is shown below:

\[
\frac{p_R | H_0(r | H_0)}{p_R | H_1(r | H_1)} \begin{cases} H_0 \\ H_1 \end{cases} > 1
\]

where \( p_R | H_i(r | H_i) \) denotes the joint probability density function of the received vector \( R \) under the hypothesis \( H_i \). The two hypotheses \( H_0 \) and \( H_1 \) are as follows:

\( H_0: \ b_0^{(1)} = +1 \)
\( H_1: \ b_0^{(1)} = -1 \)

where \( b_0^{(1)} \) is the data bit from the first user sent during the interval \([0, T]\) and it is assumed that both hypotheses have equal a priori probabilities. This form of receiver
minimizes the average probability of error and is also the form which arises from using the Bayes criterion with zero cost assigned to correct decisions, equal cost assigned to incorrect decisions, and equal a priori probabilities of either data bit. The two receivers discussed in this chapter differ in the way $p_{R|H_i}(r | H_i)$ is computed. In the first receiver, the joint density function is written as a product of first order density functions. The second receiver models the multiple access samples as a first order Markov sequence and the joint density function is given as a product of conditional density functions.

5.1. First Order Receiver

In this section the receiver based on independent samples of multiple-access interference is discussed. While it is clear that the samples are dependent, the receiver for this case has a very simple form and if the dependency between samples is small will have an average probability of error approaching that of the optimal receiver. The joint probability density function $p_{R|H_0}(r | H_0)$ can be written as a product of first order densities as given below:

$$p_{R|H_0}(r | H_0) = \prod_{j=1}^{N} p_{R_j|H_0}(r_j | H_0)$$

where $p_{R_j|H_0}(r_j | H_0) = p_{Z_j}(r_j - \varepsilon_1)$. The received signal component assuming the first user transmitted a +1 is denoted $\varepsilon_1$ and is equal to $\sqrt{P_1/2T_c}$. $Z_j$ is the composite noise sample from the $j^{th}$ chip and is given as $Z_j = N_j + \sum_{k=2}^{K} I_{j,k}$ where $N_j$ is Gaussian noise and $I_{j,k}$ is the sample of multiple-access interference from the $k^{th}$ user during the $j^{th}$
chip. After taking the logarithm of the likelihood ratio the receiver for the case of independent noise samples is seen to have the form of Figure 14. As seen in Figure 14, the receiver subtracts \( +\epsilon_1 \) and \( -\epsilon_1 \) from the received signal and then passes these differences through the memoryless transformation \( \ln[p_{Z'}(\cdot)] \). After the transformation the samples are summed and compared to a threshold. The function \( p_{Z'}(\cdot) \) which appears was discussed in Chapter 3. This receiver has a simple form and has performance which approaches the optimal receiver when the noise samples are relatively uncorrelated. Next a more complex receiver which provides optimal performance for correlated noise samples is studied.

5.2. Second Order Receiver

This section discusses a receiver design based on a second order model of the noise statistics. The joint probability density function \( p_{R,H_0}(r | H_0) \) is now given as

\[
p_{R,H_0}(r | H_0) = p_{R_1,R_2,H_0}(r_1 | r_2,H_0) \cdot p_{R_3,R_4,H_0}(r_2 | r_3,H_0) \cdots \\
p_{R_{n-1},R_n,H_0}(r_{n-1} | r_n,H_0) \cdot p_{R_n,H_0}(r_n | H_0).
\]

(27)

As was shown earlier, this is the actual joint density of \( R \). Consequently, the receiver based on this model of the noise statistics will be optimal, as opposed to the one discussed in the previous section which is based on a model of independent noise samples. With this expression for \( p_{R,H_0}(r | H_0) \) substituted into the likelihood ratio test the form of the receiver is given in Figure 15. Note that in this receiver two consecutive chip samples are processed together as opposed to the first order receiver where only one sample at a time is needed. As with the first order receiver, the receiver
subtracts $+\varepsilon_1$ and $-\varepsilon_1$ from the received samples, but now these are passed to the nonlinearity $\ln[p_{z_j} | z_{j_n} (\cdot | \cdot)]$. This nonlinearity is a function of the densities discussed in sections 3.4 and 4.3.

Next the performance of both the first order and second order receivers are considered, and the simulation techniques used to estimate the performance are discussed.
CHAPTER 6

Results and Conclusions

In this chapter the performance of the optimal receiver is compared with that of the linear receiver as well as that of the receiver based on the assumption of independent multiple-access interference samples. The measure of performance is chosen to be the average probability of error. The average error probability of each receiver is estimated by using Monte Carlo simulations. Importance Sampling is used to reduce the number of trials necessary to achieve a reasonable level of variance in the estimate of the error probability. Orsak recently showed that by generating the noise samples from a shifted version of the original noise density and then weighting the observed errors accordingly, an estimate of the error probability with a given variance may be obtained with a drastic reduction in the number of trials needed [8]. In the simulations run for this thesis, the noise density was shifted halfway toward the signal vector, corresponding to a value of $\alpha = 1/2$ in the notation of [8]. In the simulations a look-up table was used to evaluate both the first and second order density functions. This was necessary to reduce the computer time required, since evaluation of these functions numerically is quite cumbersome. For the first order density, 1025 points of the density were computed a priorily and stored in the table. When a particular value of the density was needed, linear interpolation of the values in the look-up table was used to calculate the desired value. A similar approach was used in
evaluating the second order density function, where a 101 by 101 point table was used. Careful choice of the spacing of the samples used in the look-up table ensured that the values called for in the simulation would not fall outside of the table.

Tables 1-4 contain results obtained through the simulations. Each table contains Monte Carlo estimates of error probability for a fixed value of the first user’s signal to noise ratio, denoted by $SNR_1$. This quantity is defined as $SNR_1 \triangleq E_{b_1}/N_0$, where $E_{b_1}$ is the bit energy of the first user and is given by $E_{b_1} = TP_1 = NT_cP_1$. The signal to noise ratio of user two, denoted $SNR_2$, is defined in a similar manner.

First the performance of the linear receiver is computed as $P_e = Q(\sqrt{E_{b_1}T_c}/2\sigma^2)$ where $\sigma^2$ is the variance of $Z_j$, and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$. This is equivalent to assuming that the total variance present in each sample is from a Gaussian random variable, as opposed to the real situation where the total variance is due to both Gaussian and multiple-access random variables. Treating the multiple-access interference as Gaussian noise is one simple way of estimating the performance of the linear receiver. The simulation results are then given for the linear receiver, the receiver based on a first order model of the noise density, and the optimal receiver. From now on, the latter two receivers will be referred to as the first order receiver and the second order receiver, respectively.

The top row of each table lists the different values of $SNR_2$ for which the error probability is computed. As is visible in each table, the improvement of the second order receiver over the linear receiver is very small for $SNR_2 = 12.77$dB. The
improvement is more noticeable for higher values of $SNR_2$. This is due to the fact that larger values of $SNR_2$ correspond to larger levels of multiple-access interference with respect to the Gaussian noise. When the total noise is predominantly Gaussian, the linear receiver, which is optimal for Gaussian noise, has performance very close to that of the optimal receiver. The variance present in the estimates of error probability obtained through simulations makes it impossible for us to distinguish between the performance of the linear receiver and that of the second order receiver when $SNR_2 = 12.77$dB. However for larger values of $SNR_2$, the noise becomes more non-Gaussian in nature and the second order receiver shows a definite improvement over the linear receiver, particularly for the cases where $SNR_2 \geq 21.90$dB. Note that the situations in which the second order receiver shows noticeable improvement over the linear receiver are those in which the near-far problem is most severe. That is, the received power of the interfering user is much larger than that of the intended user.

While the performance of the first order receiver generally falls somewhere between that of the linear receiver and the second order receiver, in some cases the linear receiver actually outperforms the first order receiver. This stems from the fact that the assumption of independent noise samples used in the first order receiver is not very accurate in these cases.

The estimates of the error probability computed assuming that the total noise variance was Gaussian are close to the error rate estimates obtained through simulation for $SNR_1 = 8$dB and $SNR_1 = 12$dB. However for larger $SNR_1$ the error rate estimates assuming all Gaussian noise are lower than the simulated error rates. For these
situations the multiple-access interference deteriorates the performance of the linear receiver more than an equivalent Gaussian noise term with equal variance.

In conclusion, the second order receiver is found to provide a lower average bit-error rate than the linear receiver, particularly in situations where the total noise is dominated by multiple-access interference. In situations where the noise is predominately Gaussian, the linear receiver provides performance very close to that of the second order receiver. For situations where all users have large signal-to-noise ratios the second order receiver provides improved performance, at a cost of a more complex receiver structure.
TABLE 1. AVERAGE ERROR PROBABILITY FOR LINEAR, FIRST ORDER AND SECOND ORDER RECEIVERS, \( N=31, K=2, SNR_1 = 8\text{dB} \).

<table>
<thead>
<tr>
<th>SNR_2 (dB)</th>
<th>12.77</th>
<th>18.00</th>
<th>21.90</th>
<th>24.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINEAR (GAUSSIAN)</td>
<td>1.34×10^{-3}</td>
<td>1.03×10^{-2}</td>
<td>4.40×10^{-2}</td>
<td>9.98×10^{-2}</td>
</tr>
</tbody>
</table>

**SIMULATION RESULTS**

<table>
<thead>
<tr>
<th></th>
<th>1.37×10^{-3}</th>
<th>1.38×10^{-2}</th>
<th>4.08×10^{-2}</th>
<th>1.19×10^{-1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINEAR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FIRST ORDER</td>
<td>1.53×10^{-3}</td>
<td>1.12×10^{-2}</td>
<td>3.89×10^{-2}</td>
<td>1.03×10^{-1}</td>
</tr>
<tr>
<td>SECOND ORDER</td>
<td>1.34×10^{-3}</td>
<td>7.85×10^{-3}</td>
<td>2.25×10^{-2}</td>
<td>3.53×10^{-2}</td>
</tr>
</tbody>
</table>

TABLE 2. AVERAGE ERROR PROBABILITY FOR LINEAR, FIRST ORDER AND SECOND ORDER RECEIVERS, \( N=31, K=2, SNR_1 = 12\text{dB} \).

<table>
<thead>
<tr>
<th>SNR_2 (dB)</th>
<th>12.77</th>
<th>18.00</th>
<th>21.90</th>
<th>24.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINEAR (GAUSSIAN)</td>
<td>1.04×10^{-6}</td>
<td>1.23×10^{-4}</td>
<td>3.42×10^{-3}</td>
<td>2.10×10^{-2}</td>
</tr>
</tbody>
</table>

**SIMULATION RESULTS**

<table>
<thead>
<tr>
<th></th>
<th>1.16×10^{-6}</th>
<th>1.48×10^{-4}</th>
<th>5.61×10^{-3}</th>
<th>3.45×10^{-2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINEAR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FIRST ORDER</td>
<td>1.14×10^{-6}</td>
<td>1.24×10^{-4}</td>
<td>4.23×10^{-3}</td>
<td>4.40×10^{-2}</td>
</tr>
<tr>
<td>SECOND ORDER</td>
<td>9.26×10^{-7}</td>
<td>8.64×10^{-5}</td>
<td>1.45×10^{-3}</td>
<td>1.10×10^{-2}</td>
</tr>
</tbody>
</table>
TABLE 3. AVERAGE ERROR PROBABILITY FOR LINEAR, FIRST ORDER AND SECOND ORDER RECEIVERS, N=31, K=2, $SNR_1 = 16\text{dB}$.

<table>
<thead>
<tr>
<th>$SNR_2$(dB)</th>
<th>18.00</th>
<th>21.90</th>
<th>24.91</th>
<th>27.92</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINEAR (GAUSSIAN)</td>
<td>$3.08\times10^{-9}$</td>
<td>$9.07\times10^{-6}$</td>
<td>$6.35\times10^{-4}$</td>
<td>$9.21\times10^{-3}$</td>
</tr>
<tr>
<td>SIMULATION RESULTS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LINEAR</td>
<td>$7.14\times10^{-9}$</td>
<td>$2.34\times10^{-5}$</td>
<td>$1.65\times10^{-3}$</td>
<td>$1.63\times10^{-2}$</td>
</tr>
<tr>
<td>FIRST ORDER</td>
<td>$5.04\times10^{-9}$</td>
<td>$1.42\times10^{-5}$</td>
<td>$2.14\times10^{-3}$</td>
<td>$1.53\times10^{-2}$</td>
</tr>
<tr>
<td>SECOND ORDER</td>
<td>$2.81\times10^{-9}$</td>
<td>$9.14\times10^{-6}$</td>
<td>$6.54\times10^{-4}$</td>
<td>$1.29\times10^{-2}$</td>
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</table>

TABLE 4. AVERAGE ERROR PROBABILITY FOR LINEAR, FIRST ORDER AND SECOND ORDER RECEIVERS, N=31, K=2, $SNR_1 = 20\text{dB}$.

<table>
<thead>
<tr>
<th>$SNR_2$(dB)</th>
<th>18.00</th>
<th>21.90</th>
<th>24.91</th>
<th>27.92</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINEAR (GAUSSIAN)</td>
<td>$1.60\times10^{-20}$</td>
<td>$5.47\times10^{-12}$</td>
<td>$1.63\times10^{-7}$</td>
<td>$9.37\times10^{-5}$</td>
</tr>
<tr>
<td>SIMULATION RESULTS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LINEAR</td>
<td>$4.12\times10^{-19}$</td>
<td>$1.14\times10^{-10}$</td>
<td>$1.36\times10^{-6}$</td>
<td>$1.11\times10^{-3}$</td>
</tr>
<tr>
<td>FIRST ORDER</td>
<td>$4.05\times10^{-19}$</td>
<td>$6.70\times10^{-11}$</td>
<td>$1.01\times10^{-6}$</td>
<td>$1.49\times10^{-3}$</td>
</tr>
<tr>
<td>SECOND ORDER</td>
<td>$2.40\times10^{-19}$</td>
<td>$3.48\times10^{-11}$</td>
<td>$2.35\times10^{-7}$</td>
<td>$5.65\times10^{-4}$</td>
</tr>
</tbody>
</table>
Figure 1. Direct-Sequence Receiver Structure
Figure 2. Density function of $l_{j,k}, \varepsilon_k = 1$. 
Figure 3. Distribution function of $I_{j,k}, \varepsilon_k = 1$. 
Figure 4. Density function of $Z_p$, $SNR_2 = 12.77$dB.
Figure 5. Density function of $Z_j$, $SNR_2 = 18.00$dB.
Figure 6. Density function of $Z_p, SNR_2 = 21.90\text{dB}$. 
Figure 7. Density function of $Z_j$, $SNR_2 = 24.90$dB.
Figure 8. Density function of $Z_j, SNR_2 = 27.90$dB.
Figure 9. Joint density of $Z_j$ and $Z_{j+1}$, $SNR_2 = 12.77\text{dB}$.
Figure 10. Joint density of $Z_j$ and $Z_{j+1}$, $SNR_2 = 18.00$ dB.
Figure 11. Joint density of $Z_j$ and $Z_{j+1}$, $SNR_2 = 21.90$dB.
Figure 12. Joint density of $Z_j$ and $Z_{j+1}$, $SNR_2 = 24.90$dB.
Figure 13. Joint density of $Z_j$ and $Z_{j+1}, SNR_2 = 27.90\text{dB}$. 
Figure 14. First Order Receiver Structure
Figure 15. Second Order Receiver Structure
Bibliography


