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Analysis of a Liquid Droplet Radiator by Galerkin’s method with an improved profile

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Rice University, 1989
RICE UNIVERSITY

ANALYSIS OF A LIQUID DROPLET RADIATOR
BY GALERKIN'S METHOD WITH AN IMPROVED PROFILE

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
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MASTER OF SCIENCE

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ABSTRACT

The Liquid Droplet Radiator is a proposed lightweight radiator for the dissipation of waste heat generated by power plants in space. The hot working fluid is sprayed into space as coherent streams of tiny, discrete droplets, which then cool by transient radiative heat transfer, and are later collected for recirculation. An improved, hybrid trial function is presented for the solution by Galerkin's method of the equation of radiative transfer, which governs the emission, scattering, and absorption within the layer formed by the droplet streams. The trial function compensates for the effects of the particular solution corresponding to the inhomogeneous source term. The method of analysis is demonstrated for a source function having a polynomial profile. The improved-profile Galerkin solution is then applied to a non-dimensional analysis of gray, isotropically, scattering droplet layers which are asymmetrically heated by normal, isotropic, external radiation. The analysis identifies a critical value for the magnitude of external radiation. Droplet layers exposed to external radiation greater than this critical value will heat initially, and then either recover and cool, or for magnitudes sufficiently stronger than the critical value will achieve a state of equilibrium.
ACKNOWLEDGEMENTS

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I reserve a special note of gratitude and indebtedness to my advisor, Professor Yildiz Bayazitoglu. Her ideas and concepts are the foundation upon which this work is built. Now as I look back, I recognize and appreciate her efforts, insight, and analysis which have pulled this work out of the fire on several occasions, and has guided it to a successful conclusion. My family and I will always remember the graciousness that she and her family extended to us. Her presence has made my stay at Rice a thoroughly enjoyable and satisfying experience.

My final and most deepest appreciation goes to my wife Suki and daughter Christina. The successful completion of this study, which culminates the attainment of a long-held personal dream, has come at a price of many lost weekends and missing nights as husband and father. The work presented here is as much a product of their sacrifice and support, and it is to them that I dedicate it.
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NOMENCLATURE

\( a_n \) coefficients of the Legendre polynomials

\( A \) emission term in the boundary condition

\( A_j \) first coefficient of the improved-profile component related to the complementary solution

\( B_j \) second coefficient of the improved-profile component related to the complementary solution

\( B_{il} \) orthogonality function; Eq. (3.11)

\( c \) speed of propagation in a medium

\( C_j \) coefficient of the improved-profile component related to the particular solution

\( d \) orthogonality function comprised of the source term and boundary conditions; Eq. (3.11)

\( d_i \) orthogonality function comprised of the source term and boundary conditions; Eq. (3.11)

\( f \) intensity of external radiation

\( F \) non-dimensional intensity of external radiation

\( F_c \) critical magnitude of the non-dimensional intensity of external radiation

\( E_n \) exponential integral of order \( n \)

\( E_i \) exponential integral of negative arguments \( ( = E_1 ( -x ) ) \)

\( G \) incident radiation

\( H \) emission component of the source term in the equation of radiative transfer

\( I \) intensity of radiation

\( J \) order of the improved-profile approximation of the complementary solution for the equation of radiative transfer

\( K \) kernel of the integral form of the equation of radiative transfer

\( L \) physical depth of the participating medium
M  order of the improved-profile approximation of the particular solution of the equation of radiative transfer
n  real refractive index of a medium
NP number of gaussian quadrature nodes
p  phase function; directional distribution of scattered radiation
P  incident radiation variable; Eq. (3.22a)
q  heat flux
R  residual term
S  source function
s  path of travel within a participating medium
T  temperature
t  time \( \tilde{t} \) non-dimensional time parameter \( (= \frac{\rho c p B}{\sigma T_i^3}) \)
\( \tilde{t}_s \)  Siegel's time parameter \( (= 4 \tilde{t} / \tau) \)
x  space variable
\( X_i \)  incident radiation variable variable; Eq. (4.4)
V  orthogonality function; Eq. (3.11)
\( V_i \)  orthogonality function; Eq. (3.11)
W  net volumetric gain of radiative energy
\( w_j \)  gaussian quadrature weight
Y  free term of the integral form of the equation of radiative transfer
\( Y_i \)  exit heat flux variable; Eq. (3.18a)
\( Z_i \)  exit heat flux variable; Eq. (3.18b)
### Greek Symbols

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<tr>
<td>$\alpha$</td>
<td>absorptivity</td>
</tr>
<tr>
<td>$\beta$</td>
<td>extinction coefficient ($= \kappa + \sigma$)</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>emissivity</td>
</tr>
<tr>
<td>$\phi$</td>
<td>azimuthal angle between the incident and scattered beams of radiation</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>forward-direction heat flux variable; Eq. (3.19a)</td>
</tr>
<tr>
<td>$\Gamma_i$</td>
<td>incident radiation variable; Eq. (3.22b)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>spectral absorption coefficient</td>
</tr>
<tr>
<td>$\lambda_j$</td>
<td>positive, natural eigenvalues derived from Chandrasekhar's polynomials</td>
</tr>
<tr>
<td>$\theta$</td>
<td>polar angle of a beam of radiation with respect to the normal axis</td>
</tr>
<tr>
<td>$\theta_o$</td>
<td>polar angle between the incident and scattered beams of radiation</td>
</tr>
<tr>
<td>$\mu$</td>
<td>directional cosine ($= \cos \theta$)</td>
</tr>
<tr>
<td>$\mu_o$</td>
<td>directional cosine between the incident and scattered beams of radiation ($= \cos \theta_o$)</td>
</tr>
<tr>
<td>$\nu$</td>
<td>spectral wavelength</td>
</tr>
<tr>
<td>$\rho$</td>
<td>reflectivity</td>
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<td>$\sigma$</td>
<td>Stefan-Boltzmann constant</td>
</tr>
<tr>
<td>$\tilde{\sigma}$</td>
<td>spectral scattering coefficient</td>
</tr>
<tr>
<td>$\tau$</td>
<td>optical depth ($= \beta x$)</td>
</tr>
<tr>
<td>$\tau_o$</td>
<td>total optical depth of a medium ($= \beta L$)</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>direction of travel within a participating medium</td>
</tr>
<tr>
<td>$\omega$</td>
<td>albedo of the medium ($= \sigma/\beta$)</td>
</tr>
<tr>
<td>$\psi$</td>
<td>backward-direction heat flux variable; Eq. (3.19b)</td>
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Subscripts

0  inner surface, upper boundary wall
1  outer surface, upper boundary wall
2  outer surface, lower boundary wall
3  inner surface, lower boundary wall
r  radiative heat transfer
b  black-body
v  spectrally dependent

Superscripts

\cdot  dummy variable
\rightarrow  forward direction
\rightarrow^\rightarrow  backwards direction
\star  spectrally dependent incident radiation
\sim  non-dimensionalized with respect to the initial energy generation rate ($= \sigma T_i^4$)
CHAPTER 1

INTRODUCTION

1.1 The Liquid Droplet Radiator

The power requirements for spacecraft have increased dramatically since 1957 when man first ventured into space. Current satellites use 1 to 10 KW of power, with the Space Shuttle requiring about 20 KW. Plans for the manned Space Station propose power demands at 75 KW initially, growing to 300 KW when full development is reached. Operating levels for Strategic Defense Initiative platforms could average ten to fifteen MW, with burst requirements for hundreds of megawatts. [1] One concern with these large power requirements is the rejection of the proportionally large quantity of waste heat generated. The heat pipe radiator, the technology currently in use, is complex and heavy. Although future improvements could reduce the heat pipe's specific mass, it would still comprise a large fraction of the power plant mass. [2]

The Liquid Droplet Radiator (LDR) is an advanced radiator concept proposed to significantly reduce the mass requirements associated with heat rejection in space. The LDR radiates heat by exposing the working fluid directly to space. As illustrated in Figure 1.1, the hot working fluid is sprayed into space as coherent streams of tiny, discrete droplets which are collected for recirculation at some distance from the injector. Since the droplets have a collectively large surface area in comparison to their weight, the LDR is more efficient at radiating heat into space than a solid radiating surface. Early studies have shown that the LDR could be one-fifth the weight of a comparable heat pipe, and be easier to transport and deploy in space. [3]
Figure 1.1 Liquid Droplet Radiator system model.
The early studies of the radiative performance of the droplet sheet by Mattick, et al. [3-7] was based on a lumped parameters analysis. A numerical procedure was used to determine a value for an effective sheet emissivity ($\varepsilon_s$) based on the optical depth of the sheet and droplet emissivity. An effective sheet temperature ($T_s$) was computed as a function of the mean inlet and outlet temperatures. The heat flux emitted by the LDR was then computed as $\varepsilon_s \sigma T_s^4$. Siegel [8] presented a more precise analysis by solving the equation of radiative transfer with homogeneous boundary conditions. Siegel's separation of variables solution is limited to an LDR operating in black space, as expressed by the homogeneous boundary conditions. However, the actual operating environment is not black space, but one subjected to solar radiation and diffuse radiation reflected and emitted from the sun, earth and other objects in orbit. Although the LDR can be "tuned" by orienting the droplet sheet edge towards an external heat source, the radiating area of the sheet would still be subject to incident radiation from the other sources. In addition, the contamination of spacecraft by errant droplets is a growing concern that may necessitate enclosing the LDR with a thin plastic film [2,4]. A precise analysis of LDR performance in this more complex environment requires a solution for the equation of radiative transfer with a set of general boundary conditions.

1.2 Solution of the Equation of Radiative Transfer

The solution of problems involving radiative heat transfer in a participating medium poses a significant challenge, and has been the subject of numerous studies. The difficulty is due primarily to the governing equation, an integro-differential equation in terms of the radiative intensity, for radiative heat transfer in an absorbing, emitting, scattering medium. The radiative intensity in turn is a function of position, direction, frequency, and time. The problem becomes further complicated by the
introduction of boundary conditions, which are expressed as integral equations. The
general form of the problem is therefore extremely formidable. In practice, the problem
is simplified by introducing assumptions, such as a gray medium or steady state.
Further simplification is often achieved through the use of limiting cases, such as a one-
dimensional medium. The method of solution will therefore depend on the constraints
peculiar to the problem being studied.

Even with the use of these assumptions, the problem is still quite general in
nature, and several methods of solution, most involving approximations, have been
studied. An excellent overview of several methods is presented in the text by Özisik
[9]. Some are applicable only at extreme limits of the problem, such as the Rosseland
approximation for optically thick media. Other methods approximate the solution to the
problem by simplifying the integro-differential equation. One popular approximation
solution is the method of spherical harmonics, which reduces the integro-differential
equation to a set of coupled ordinary differential equations. The first order
approximation P-1 yields satisfactory results for an optically thick medium, but is only
moderately accurate for a thin medium. Increasing the order of the approximation
improves the accuracy of the method, but with greater difficulty and computational
time.

The spherical harmonics method is closely related to variational methods, which
work with a function related to, instead of directly with, the governing equation and
boundary conditions [10]. In the case of spherical harmonics, the functions used are
Legendre polynomials. The method of weighted residuals on the other hand works
directly with the governing equation and boundary conditions. One advantage to this
approach is that higher order approximations can be achieved with far less work than is
required for variational methods. One of the best known methods of weighted residuals
was developed by the Russian engineer Galerkin. Özisik and Yener [11] introduced
Galerkin's method for the solution of radiation problems in an absorbing, emitting, isotropically scattering, plane-parallel medium, using a polynomial in the space variable $x$ as a trial function. Cengel and Özisik [12-14]. However, using Galerkin's method with their trial function leads to convergence difficulties for an optically thick medium. Thynell and Özisik [15] improved the convergence of Galerkin's method through the introduction of an exponential trial function based on the natural eigenvalues of the problem, which are obtained from Chandrasekhar's polynomials. [16] Their trial function included a particular solution related to the inhomogeneous source term. Although the use of the particular solution was optional, they demonstrated that its inclusion greatly enhanced the rate of convergence. As they pointed out, the particular solution is often difficult to find, therefore unless the particular solution is known beforehand, the trial function consists only of the exponential function related to the complementary solution.

1.3 Objectives

Several alternative LDR configurations have been considered by NASA [3], with the rectangular droplet array favored due to its simplicity and large radiating area. This study assumes a rectangular geometry, and will model the droplet sheet as a finite, plane-parallel, enclosed medium. The introduction of walls to the model allows incorporation of general boundary conditions. Since the droplets are uniformly distributed and each droplet receives radiation from many directions, scattering within the sheet is assumed isotropic. Previous studies show that the effect of anisotropic scattering is such that this assumption will yield satisfactory results. [17] As envisioned, the LDR sheet width and length are very large compared to the sheet's
thickness. Thus for small optical depths the temperature gradients along the width and length are assumed insignificant compared to the gradient across the sheet thickness. Therefore, a one-dimensional approximation will be used in the analysis of the droplet layer. Although the optical properties and other variables involved in this study are all frequency-dependent, due to the complexities involved in an analysis of the non-gray case, the model's medium and boundaries will be assumed gray. Temperature gradients within a droplet and variations in optical properties with changes in temperature will not be considered.

This study will begin with a derivation of the governing equation for the one-dimensional case of a gray, isotropically scattering, finite plane-parallel medium. For transient problems, the use of the particular solution in Thynell's and Özisik's profile is numerically cumbersome and time intensive to implement. However, the greatly enhanced rate of convergence associated with its inclusion is still desired. Therefore, an improved profile will be introduced, using the same natural eigenfunctions, together with a trial function to compensate for the effects of the particular solution. The solution of the governing equation by Galerkin's method with the improved profile will then be presented. Analytical expressions for several radiation parameters of interest follow. Next, the rate of convergence and accuracy of the improved solution will be demonstrated by the computation of incident radiation and exit heat fluxes for several steady-state problems, and the results compared to previous studies. The limitations of Galerkin's method with various trial functions will then be discussed. Finally the transient problem of an LDR with radiation incident on one side will be analyzed. Results will be presented for a variety of LDRs and intensities of external radiation.
CHAPTER 2

THE EQUATION OF RADIATIVE TRANSFER

2.1 Introduction

Several methods for the derivation of the equation of radiative heat transfer exist. In this chapter, the Eulerian approach will be presented. Once derived, the general equation will be simplified using azimuthal symmetry, based on the assumption of a plane-parallel medium. The general boundary conditions will then be introduced by modeling the medium as a slab enclosed by two surfaces. Due to the difficulty in solving an integro-differential equation, the governing equation will be reduced to a simpler form through a formal solution. The final result will be a single integral equation for radiative heat transfer which incorporates the general boundary conditions. This equation will be the basis for the analysis which follows in the subsequent chapters of this study.

2.2 Formulation of the Equation of Radiative Transfer

In general, a participating medium absorbs, emits, and scatters radiative energy. These properties are characterized by the spectral coefficients for absorption, \( \kappa_\nu \), and scattering, \( \sigma_\nu \). In the Eulerian approach an elemental control volume is used, as illustrated in Fig. 2.1.[9] Consider a beam of monochromatic radiation \( I_\nu(s,\Omega,t) \) propagating in the medium along a path \( s \) in direction \( \Omega \). Next, define a right angle cylindrical control volume of cross-sectional area \( dA \) and length \( ds \), with its longitudinal axis aligned along \( s \) and in direction \( \Omega \). Let \( I_\nu \) be the intensity of radiation incident on the surface \( dA \) at \( s \), and \( I_\nu + dI_\nu \) be the intensity exiting the control volume at \( s + ds \) and direction \( \Omega \). Then \( dI_\nu \) is
the net change in spectral intensity, and the difference in radiative energy between the surfaces at \( s \) and \( s + ds \), in the time interval \( dt \), frequency interval \( dv \), and within the elemental solid angle \( d\Omega \) is

\[
\frac{dI_v(s, \Omega, t) \ dA \ d\Omega \ dv \ dt}{ds}
\] (2.1)

Let \( W_v \) be the net gain of volumetric radiative energy by the beam in this control volume. Then the net gain of radiative energy by the beam within the solid angle \( d\Omega \), in time interval \( dt \), frequency interval \( dv \), and volume \( dA \ ds \) is

\[
W_v \ dA \ ds \ d\Omega \ dv \ dt
\] (2.2)

Equating Eqs. (2.1) and (2.2) results in the relation

\[
W_v = \frac{dI_v(s, \Omega, t)}{ds}
\] (2.3)

If \( c \) is the speed of propagation of radiation in the medium, then the distance \( ds \) is

\[
ds = c \ dt
\]

Substitution for \( ds \) into Eq. (2.3) yields

\[
W_v = \frac{1}{c} \ \frac{DI_v(s, \Omega, t)}{dt}
\]

\[
= \frac{1}{c} \left[ \frac{\partial I_v}{\partial t} + c \ \Omega \cdot \nabla I_v \right]
\]

\[
= \frac{1}{c} \ \frac{\partial I_v}{\partial t} + \frac{\partial I_v}{\partial s}
\] (2.4)
Figure 2.1 Control volume for the derivation of the equation of radiative transfer
For an arbitrary medium, \( W_\nu \) is composed of gains and losses of radiant energy due to absorption, emission, and scattering. This relation may be expressed as

\[
W_\nu = W_{\text{emission}} - W_{\text{absorption}} + W_{\text{in-scattering}} - W_{\text{out-scattering}} \tag{2.5a}
\]

The gain in radiant energy by the beam due to radiation emitted by the medium is

\[
W_{\text{emission}} = \kappa_\nu(s) \frac{n^2 \sigma T^4(s)}{\pi} \tag{2.5b}
\]

The loss in the beam’s radiant energy due to absorption by the medium is

\[
W_{\text{absorption}} = \kappa_\nu(s) I_\nu(s, \Omega, t) \tag{2.5c}
\]

The third term in Eq. (2.5a) is the gain in radiant energy due to radiation incident on the control volume that is scattered from all directions by the medium, in the same direction \( \Omega \) as the beam. For purely coherent scattering in an isotropic medium, this gain due to in-scattering is

\[
W_{\text{in-scattering}} = \frac{1}{4\pi} \bar{\sigma}_\nu(s) \int_0^{4\pi} p(\Omega', \Omega) I_\nu(s, \Omega, t) \, d\Omega' \tag{2.5d}
\]

The final term in Eq. (2.5a) is the radiant energy lost by the beam due to scattering by the medium in all directions other than \( \Omega \), and is given by

\[
W_{\text{out-scattering}} = \bar{\sigma}_\nu(s) I_\nu(s, \Omega, t) \tag{2.5e}
\]
The equation of radiative transfer is obtained by substituting Eq. (2.5) into Eq. (2.4), yielding

\[
\frac{1}{c} \frac{\partial I_v(s, \Omega, t)}{\partial t} + \frac{\partial I_v(s, \Omega, t)}{\partial s} + \left[ \kappa_v(s) + \sigma_v(s) \right] I_v(s, \Omega, t)
\]

\[
= \kappa_v(s) I_{vb}(T) + \frac{1}{4\pi} \sigma_v(s) \int_0^{4\pi} p(\Omega' \cdot \Omega) I_v(s, \Omega', t) \, d\Omega'
\]  

(2.6)

For almost all problems of practical interest, the speed of propagation \(c\) within the medium is much larger than the transient change in the radiosity, and so the first term of Eq. (2.6) may be neglected yielding the simplified expression

\[
\frac{\partial I_v(s, \Omega, t)}{\partial s} + \left[ \kappa_v(s) + \sigma_v(s) \right] I_v(s, \Omega, t)
\]

\[
= \kappa_v(s) I_{vb}(T) + \frac{1}{4\pi} \sigma_v(s) \int_0^{4\pi} p(\Omega' \cdot \Omega) I_v(s, \Omega', t) \, d\Omega'
\]  

(2.7)

If transient effects are neglected for the moment, and our attention is restricted to a gray medium (i.e. the radiative properties are independent of frequency), then Eq. (2.7) may be expressed in the form

\[
\frac{1}{\beta(s)} \frac{\partial I(s, \Omega)}{\partial s} + I(s, \Omega) = (1 - \omega) \frac{n^2 \sigma T^4(s)}{\pi} + \frac{\omega}{4\pi} \int_0^{4\pi} p(\Omega' \cdot \Omega) I(s, \Omega', t)
\]  

(2.8)
The $\beta(s)$ is the extinction coefficient and $\omega$ is the albedo -- the ratio of the scattering coefficient to the extinction coefficient. The right hand side of Eq. (2.8) is called the source function, $S(s)$. Recalling that $\theta$ is the angle between the beam's direction $\Omega$ and the normal axis, then

$$\Omega' \cdot \Omega = \cos \theta_0 = \mu_o$$

where $\theta_0$ is the angle between the directions of the incident and scattered beams of radiation. Substituting $\mu_o$ for $(\Omega' \cdot \Omega)$, Eq. (2.8) becomes

$$\frac{1}{\beta(s)} \frac{\partial I(s,\Omega)}{\partial s} + I(s,\Omega) = (1 - \omega) \frac{n^2 \sigma T^4(s)}{\pi} + \frac{\omega}{4\pi} \int_0^{4\pi} p(\mu_o) I(s,\mu',\phi') \, d\phi' \quad (2.9)$$

where $\phi$ is the azimuthal angle of the beam. A complete expression for the phase function as determined by Mie scattering theory would require the calculations be performed for a large number of scattering angles. Chu and Churchill [19] developed an expression for the phase function of unpolarized radiation in terms of Legendre polynomials of the form

$$p(\mu_0) = \sum_{n=0}^{N} a_n P_n(\mu_0)$$

This relation may be expressed as

$$p(\mu_0) = p(\mu, \mu') = \sum_{n=0}^{N} a_n P_n(\mu) P_n(\mu')$$

$$a_0 = 1$$
For the special case of isotropic scattering, the order N is zero, and therefore the phase function is exactly 1. Thus the resulting equation for radiative transfer with azimuthal symmetry and isotropic scattering is

\[
\frac{1}{\beta(s)} \frac{\partial I(s, \mu)}{\partial s} + I(s, \mu) = (1 - \omega) \frac{n^2 \sigma T^4(s)}{\pi} + \frac{\omega}{2} \int_{-1}^{1} I(s, \mu') d\mu' \quad (2.10)
\]

For a plane-parallel medium, Eq. (2.10) may be written in terms of the space coordinate as

\[
\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = (1 - \omega) \frac{n^2 \sigma T^4(\tau)}{\pi} + \frac{\omega}{2} \int_{-1}^{1} I(\tau, \mu') d\mu' \quad (2.11a)
\]

2.3 **Boundary Conditions**

A participating medium, however large, itself exists within some environment. In general, this environment surrounding the participating medium also absorbs, emits, and scatters radiation. Since the environment encloses the medium, the plane-parallel medium can be modelled as a slab (Fig. 2.2), with its boundaries represented by walls having the same radiative properties as the environment. For the one-dimensional slab illustrated in Fig. 2.2, the intensity of radiation has two components, one in the forward direction, \(I^+\), and in the backward direction, \(I^-\). The general conditions at either boundary may be expressed as

\[
I^+(0, \mu) = (1 - \rho_0) f_1(\mu) + A_1 + 2\rho_1 K_1 \quad (2.11b)
\]

\[
I^-(\tau_{o}, -\mu) = (1 - \rho_3) f_2(\mu) + A_2 + 2\rho_2 K_2 \quad (2.11c)
\]
Figure 2.2  Geometry of the plane-parallel medium bounded by diffuse, gray surfaces
with

\[ K_1 = \int_0^1 I^-(0, -\mu') \mu' \, d\mu' \]  

(2.12a)

\[ K_2 = \int_0^1 I^+ (\tau_0, \mu') \mu' \, d\mu' \]  

(2.12b)

and

\[ A_1 = \varepsilon_1 \frac{n^2 \sigma T_1^4}{\pi}; \quad A_2 = \varepsilon_2 \frac{n^2 \sigma T_2^4}{\pi} \]

The first term in Eqs. (2.11b) and (2.11c) represents the portion of external, azimuthally-symmetric radiation not absorbed (i.e. transmitted) by the environment, and thus is incident on the medium's outer surfaces. Note that the external radiation is directional, and may be visualized as cones of radiation coming from all directions. The second term represents the diffuse emission of radiation by the environment at some temperature T. The third term is the portion of radiation leaving the medium that is scattered, or diffusely reflected back into the medium by it's surroundings.

**2.4 Reduced Form of the Equation of Radiative Transfer**

Equation (2.11a) is both an integral and a differential equation, an integro-differential equation. As the solution methods differ for either family of equations, the solution process would be simplified if the governing equation could be reduced to either an integral or a differential equation. Compatibility with the boundary conditions suggests
the reduced form should be an integral equation of some sort. The reduction in form starts with the multiplication of both sides of Eq. (2.11a) by an integrating factor \( e^{\tau/\mu} \)

\[
e^{\tau/\mu} \left[ \mu \frac{\partial I}{\partial \tau} + I \right] = e^{\tau/\mu} \left( 1 - \omega \right) \frac{n^2 \sigma T^4(\tau)}{\pi} + \frac{\omega}{4\pi} e^{\psi_\mu} G
\]

where the incident radiation \( G \) is defined as

\[
G(\tau) = 2\pi \int_{-1}^{1} I(\tau, \mu') d\mu'
\]

Consolidation of the terms on the left-hand side leads to the expression

\[
\frac{\partial}{\partial \tau} \left( I e^{\tau/\mu} \right) = \frac{1}{\mu} e^{\tau/\mu} \left( 1 - \omega \right) \frac{n^2 \sigma T^4}{\pi} + \frac{\omega}{4\pi} \frac{1}{\mu} e^{\tau/\mu} G
\]

Integrating both sides of the above equation and dividing by \( e^{\tau/\mu} \) yields

\[
I(\tau, \mu) = \int_{0}^{\tau} \left( 1 - \omega \right) \frac{1}{\mu} \frac{n^2 \sigma T^4(\tau')} e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \frac{\omega}{4\pi} \int_{0}^{\tau} \frac{1}{\mu} G(\tau') e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + C e^{-\frac{\tau}{\mu}}
\]

The constant of integration is determined by the boundary conditions Eqs. (2.11b) and (2.11c), respectively

\[
I^+(\tau, \mu) = \int_{0}^{\tau} \frac{1}{\mu} \left( 1 - \omega \right) \frac{n^2 \sigma T^4(\tau')} e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \frac{\omega}{4\pi} \int_{0}^{\tau} \frac{1}{\mu} G(\tau') e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + I^+(0, \mu) e^{-\frac{\tau}{\mu}}
\]
and

\[
I^- (\tau, \mu) = \int_{\tau_0}^{\frac{\tau}{\mu}} \frac{1}{1 - \omega} \frac{n^2 \sigma T^4(\tau')}{\pi} e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \frac{\omega}{4\pi} \int_{\tau_0}^{\tau} \frac{1}{\mu} G(\tau') e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + I^-(\tau_0, \mu) e^{-\frac{(\tau - \tau_0)}{\mu}}
\]

Reversing the limits of integration in the equation for \( I^- (\tau, \mu) \) (due to the presence of 
- \( 1/\mu \) in the first two terms) results in the relation

\[
I^- (\tau, \mu) = \int_{\tau}^{\tau_0} \frac{1}{1 - \omega} \frac{n^2 \sigma T^4(\tau')}{\pi} e^{-\frac{(\tau' - \tau)}{\mu}} d\tau' + \frac{\omega}{4\pi} \int_{0}^{\tau_0} \frac{1}{\mu} G(\tau') e^{-\frac{(\tau' - \tau)}{\mu}} d\tau' + I^-(\tau_0, -\mu) e^{-\frac{(\tau_0 - \tau)}{\mu}}
\]

(2.13b)

To complete the expressions for \( I^+ \) and \( I^- \), explicit relations must be obtained for \( K_1 \) and \( K_2 \). The expression for \( K_1 \) is developed by setting \( \tau = 0 \) in Eq. (2.13b) and substituting
the resulting relation into Eq. (2.12a), yielding,

\[
K_1 = \int_{0}^{1} I^- (0, -\mu') \mu' d\mu'
\]

\[
= \int_{0}^{\tau_0} (1 - \omega) \frac{n^2 \sigma T^4(\tau')}{\pi} \int_{0}^{\frac{\tau}{\mu}} e^{-\frac{\tau'}{\mu}} d\mu' d\tau' + \frac{\omega}{4\pi} \int_{0}^{\tau_0} G(\tau') \int_{0}^{\frac{\tau}{\mu}} e^{-\frac{\tau'}{\mu}} d\mu' d\tau' + \frac{\tau_0}{\mu} \int_{0}^{1} I^- (\tau_0, -\mu) e^{-\frac{\tau_0}{\mu}} d\mu
\]
Substituting the boundary condition relation for \( I^-(\tau_0, -\mu) \) from Eq. (2.11c), and introducing the exponential integral notation \( E_n \) results in the expression

\[
K_1 = \int_0^{\tau_0} H(\tau) \int_0^1 e^{-\frac{\tau'}{\mu}} d\mu d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_0} G(\tau') \int_0^1 e^{-\frac{\tau'}{\mu}} d\mu d\tau' + (1 - \rho_3) \int_0^1 f_2(\mu) e^{-\frac{\tau_0}{\mu}} d\mu
\]

\[
\quad + A_2 \int_0^{\tau_0} e^{-\frac{\tau_0}{\mu}} d\mu + 2\rho_2 K_2 \int_0^{\tau_0} e^{-\frac{\tau_0}{\mu}} d\mu
\]

\[
= \int_0^{\tau_0} H(\tau') E_2(\tau') d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_0} G(\tau') E_2(\tau') d\tau' + (1 - \rho_3) \int_0^1 f_2(\mu) e^{-\frac{\tau_0}{\mu}} d\mu
\]

\[
\quad + A_2 E_3(\tau_0) + 2\rho_2 K_2 E_3(\tau_0)
\]

(2.14a)

where

\[
E_n(\tau) = \int_0^1 e^{-\frac{\tau}{\mu}} \mu^{n-2} d\mu; \quad H(\tau) = (1 - \omega) \frac{n^2 \sigma T^4(\tau)}{\pi}
\]
In similar fashion, $K_2$ is computed by first setting $\tau = \tau_o$ in Eq. (2.13a), and then substituting back into Eq. (2.12b), with the result,

$$K_2 = \int_0^1 I^+ (\tau_o, \mu) \, d\mu$$

$$= \int_0^{\tau_o} H (\tau') \int_0^1 \frac{1}{\mu} e^{-\frac{(\tau_o - \tau)}{\mu}} \, d\mu \, d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_o} G (\tau') \int_0^1 \frac{1}{\mu} e^{-\frac{(\tau_o - \tau)}{\mu}} \, d\mu$$

$$+ \int_0^1 I^+ (0, \mu) e^{-\frac{\tau_o}{\mu}} \, d\mu$$

Substituting the boundary condition relation for $I^+ (0, \mu)$ from Eq. (2.11b), yields

$$K_2 = \int_0^{\tau_o} H (\tau') \int_0^1 e^{-\frac{\tau_o}{\mu}} \, d\mu \, d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_o} G (\tau') \int_0^1 e^{-\frac{\tau_o}{\mu}} \, d\mu \, d\tau'$$

$$+ (1 - \rho_o) \int_0^1 f_1(\mu) e^{-\frac{\tau_o}{\mu}} \, d\mu + A_1 \int_0^1 e^{-\frac{\tau_o}{\mu}} \, d\mu + 2\rho_1 K_1 \int_0^1 e^{-\frac{\tau_o}{\mu}} \, d\mu$$

$$= \int_0^{\tau_o} H (\tau') E_2 (\tau_o - \tau') \, d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_o} G (\tau') E_2 (\tau_o - \tau') \, d\tau'$$

$$+ (1 - \rho_o) \int_0^1 f_1(\mu) e^{-\frac{\tau_o}{\mu}} \, d\mu + A_1 E_3 (\tau_o) + 2\rho_1 K_1 E_3 (\tau_o)$$

(2.14b)
Equations (2.14a) and (2.14b) are solved simultaneously to obtain the following explicit expressions for $K_1$ and $K_2$,

\[
K_1 = B \left\{ a_2 \left( 1 - \rho_0 \right) \int_0^1 f_1(\mu) e^{-\frac{\tau_0}{\mu}} \mu \, d\mu + \left( 1 - \rho_3 \right) \int_0^1 f_2(\mu) e^{-\frac{\tau_0}{\mu}} \mu \, d\mu + a_2 A_1 E_3(\tau_0) \right. \\
+ A_2 E_3(\tau_0) + \int_0^{\tau_0} [E_2(\tau') + a_2 E_2(\tau_0 - \tau')] H(\tau') \, d\tau' \\
+ \frac{\omega}{4\pi} \int_0^{\tau_0} [E_2(\tau') + a_2 E_2(\tau_0 - \tau')] G(\tau') \, d\tau' \left\} \right.
\]

(2.15a)

\[
K_2 = B \left\{ (1 - \rho_0) \int_0^1 f_1(\mu) e^{-\frac{\tau_0}{\mu}} \mu \, d\mu + (1 - \rho_3) \int_0^1 f_2(\mu) e^{-\frac{\tau_0}{\mu}} \mu \, d\mu + A_1 E_3(\tau_0) \right. \\
+ a_1 A_2 E_3(\tau_0) + \int_0^{\tau_0} [E_2(\tau_0 - \tau') + a_1 E_2(\tau')] H(\tau') \, d\tau' \\
+ \frac{\omega}{4\pi} \int_0^{\tau_0} [E_2(\tau_0 - \tau') + a_1 E_2(\tau')] G(\tau') \, d\tau' \left. \right\} \]

(2.15b)

where

\[
a_1 = 2 \rho_1 E_3(\tau_0); \quad a_2 = 2 \rho_2 E_3(\tau_0); \quad B = \frac{1}{1 - a_1 a_2}
\]
These explicit expressions for $K_1$ and $K_2$ are substituted into Eqs. (2.13a) and (2.13b) to fully specify the forward and backward intensities of radiation. The incident radiation $G(\tau)$ at a point $\tau$ in the slab may now be determined from

\[
G(\tau) = 2\pi \int_{-1}^{1} I(\tau, \mu) \, d\mu = 2\pi \left[ \int_{0}^{1} I^{+}(\tau, \mu) \, d\mu + \int_{-1}^{0} I^{-}(\tau, \mu) \, d\mu \right]
\]

\[
= 2\pi \left[ \int_{0}^{1} I^{+}(\tau, \mu) \, d\mu + \int_{0}^{1} I^{-}(\tau, -\mu) \, d\mu \right]
\]

(2.16)

Substituting for $I^{+}(\tau, \mu)$ and $I^{-}(\tau, \mu)$ from Eqs. (2.13a) and (2.13b) yields

\[
G(\tau) = 2\pi \left[ \int_{0}^{\tau} H(\tau') \int_{0}^{1} \frac{1}{\mu} e^{-\frac{(\tau - \tau')}{\mu}} \, d\mu \, d\tau' + \frac{\omega}{4\pi} \int_{0}^{\tau} G(\tau') \int_{0}^{1} \frac{1}{\mu} e^{-\frac{(\tau - \tau')}{\mu}} \, d\mu \, d\tau' \right]
\]

\[
+ \int_{0}^{1} I^{+}(\tau, \mu) \, d\mu + \int_{0}^{\tau} H(\tau') \int_{0}^{1} \frac{1}{\mu} e^{-\frac{(\tau - \tau')}{\mu}} \, d\mu \, d\tau'
\]

\[
+ \frac{\omega}{4\pi} \int_{0}^{\tau} G(\tau') \int_{0}^{1} \frac{1}{\mu} e^{-\frac{(\tau - \tau')}{\mu}} \, d\mu \, d\tau' + \int_{0}^{1} I^{+}(\tau', -\mu) \, d\mu + \int_{0}^{\tau} I^{-}(\tau', -\mu) \, d\mu
\]
Using the boundary condition expressions for $I^+(0, \mu)$ and $I^-(\tau_0, - \mu)$, Eqs. (2.11b) and (2.11c), the above equation for the incident radiation may be rewritten as

$$
G(\tau) = 2\pi \left[ \int_0^\tau H(\tau') \int_0^{\frac{\tau}{\mu}} e^{-\frac{(\tau - \tau')}{\mu}} d\mu \, d\tau' + \frac{\omega}{4\pi} \int_0^\tau G(\tau') \int_0^{\frac{\tau}{\mu}} e^{-\frac{(\tau - \tau')}{\mu}} d\mu \, d\tau' 
+ (1 - \rho_0) f_1(\mu) e^{\frac{\tau}{\mu}} d\mu + A_1 \int_0^{\frac{\tau}{\mu}} e^{-\frac{\tau}{\mu}} d\mu + 2\rho_1 K_1 \int_0^{\frac{\tau}{\mu}} e^{-\frac{\tau}{\mu}} d\mu 
+ \int_0^{\tau_0} H(\tau') \int_0^{\frac{\tau_0 - \tau}{\mu}} e^{-\frac{(\tau - \tau')}{\mu}} d\mu \, d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_0} G(\tau') \int_0^{\frac{\tau_0 - \tau}{\mu}} e^{-\frac{(\tau - \tau')}{\mu}} d\mu \, d\tau' 
+ (1 - \rho_3) f_2(\mu) e^{-\frac{(\tau_0 - \tau)}{\mu}} d\mu + A_2 \int_0^{\frac{\tau_0 - \tau}{\mu}} e^{-\frac{(\tau_0 - \tau)}{\mu}} d\mu + 2\rho_2 K_2 \int_0^{\frac{\tau_0 - \tau}{\mu}} e^{-\frac{(\tau_0 - \tau)}{\mu}} d\mu \right]
$$

By consolidating like terms, and using the exponential integral notation $E_n$ the above expression may be simplified as

$$
G(\tau) = 2\pi \left[ \int_0^\tau H(\tau') E_1((\tau - \tau') d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_0} G(\tau') E_1((\tau - \tau') d\tau' + (1 - \rho_0) f_1(\mu) e^{\frac{\tau}{\mu}} d\mu 
+ A_1 E_2(\tau) + 2\rho_1 K_1 E_2(\tau) + (1 - \rho_3) f_2(\mu) e^{-\frac{(\tau_0 - \tau)}{\mu}} d\mu + A_2 E_2(\tau_0 - \tau) 
+ 2\rho_2 K_2 E_2(\tau_0 - \tau) \right]
$$

(2.17)
Substituting for $K_1$ and $K_2$ from Eqs. (2.15a) and (2.15b) into the above expression for $G(\tau)$ yields

$$G(\tau) = 2\pi \left[ \int_0^{\tau_o} H(\tau') E_1(\tau - \tau') d\tau' + \frac{\omega}{4\pi} \int_0^{\tau_o} G(\tau') E_1(\tau - \tau') d\tau' \right]$$

$$+ (1 - \rho_o) \frac{1}{0} f_1(\mu) e^{-\frac{\tau_o}{\mu}} d\mu + A_1 E_2(\tau) + (1 - \rho_3) \frac{1}{0} f_2(\mu) e^{-\frac{\tau_o}{\mu}} d\mu$$

$$+ A_2 E_2(\tau_o - \tau) + 2\rho_1 E_2(\tau) B \left\{ a_2 (1 - \rho_o) \frac{1}{0} f_1(\mu) e^{-\frac{\tau_o}{\mu}} \mu d\mu \right\}$$

$$+ (1 - \rho_3) \frac{1}{0} f_2(\mu) e^{-\frac{\tau_o}{\mu}} \mu d\mu + \int_0^{\tau_o} \left[ E_2(\tau') + a_2 E_2(\tau_o - \tau') \right] H(\tau') d\tau'$$

$$+ a_2 A_1 E_3(\tau_o) + A_2 E_2(\tau_o) + \frac{\omega}{4\pi} \int_0^{\tau_o} \left[ E_2(\tau') + a_2 E_2(\tau_o - \tau') \right] G(\tau') d\tau'$$

$$+ 2\rho_2 E_2(\tau_o - \tau) B \left\{ (1 - \rho_o) \frac{1}{0} f_1(\mu) e^{-\frac{\tau_o}{\mu}} \mu d\mu + a_1 (1 - \rho_3) \frac{1}{0} f_2(\mu) e^{-\frac{\tau_o}{\mu}} \mu d\mu \right\}$$

$$+ A_1 E_3(\tau_o) + a_1 A_2 E_3(\tau_o) + \int_0^{\tau_o} \left[ E_2(\tau_o - \tau') + a_1 E_2(\tau') \right] H(\tau') d\tau'$$

$$+ \frac{\omega}{4\pi} \int_0^{\tau_o} \left[ E_2(\tau_o - \tau') + a_1 E_2(\tau') \right] G(\tau') d\tau' \}$$
Collecting like terms in the above expression, the incident radiation $G(\tau)$ may be rewritten as the integral equation

$$G(\tau) = Y(\tau) + \omega \int_0^{\tau_o} K(\tau, \tau^') G(\tau^') \, d\tau'$$  \hspace{1cm} (2.18a)$$

Equation (2.18a) is a Fredholm integral equation of the second kind. The kernel of this integral equation is

$$K(\tau, \tau^') = \frac{1}{2} E_1(|\tau - \tau^'|) + \rho_1 B E_2(\tau) \left[ E_2(\tau^') + a_2 E_2(\tau_o - \tau^') \right]$$

$$+ \rho_2 B E_2(\tau_o - \tau) \left[ E_2(\tau_o - \tau^') + a_1 E_2(\tau^') \right]$$  \hspace{1cm} (2.18b)$$

The free term $Y(\tau)$ is comprised of the boundary conditions and the source function $S(\tau)$ and is expressed as

$$Y(\tau) = 2\pi \left\{ 2 \int_0^{\tau_o} K(\tau, \tau^') H(\tau^') \, d\tau^' + (1 - \rho_o) \int_0^{\tau_o} f_1(\mu) e^{-\frac{\tau}{\mu}} d\mu + (1 - \rho_3) \int_0^{\tau_o} f_2(\mu) e^{-\frac{(\tau_o - \tau)}{\mu}} d\mu ight. \right.$$ \hspace{1cm} (2.18c)$$

$$+ \left[ 2\rho_2 B \left( a_1 E_2(\tau) + E_2(\tau_o - \tau) \right) \right] (1 - \rho_o) \int_0^{\tau_o} f_1(\mu) e^{-\frac{\tau_o}{\mu}} \mu d\mu$$

$$+ \left[ 2\rho_1 B \left( a_2 E_2(\tau_o - \tau) + E_2(\tau) \right) \right] (1 - \rho_3) \int_0^{\tau_o} f_2(\mu) e^{-\frac{\tau_o}{\mu}} \mu d\mu$$

$$+ \left[ (1 + a_1 a_2 B) E_2(\tau) + a_2 B E_2(\tau_o - \tau) \right] A_1$$

$$+ \left[ (1 + a_1 a_2 B) E_2(\tau_o - \tau) + a_1 B E_2(\tau) \right] A_2$$
Thus, the integro-differential equation has been reduced to the pure integral relation of Eq. (2.18). Solving for $G(\tau)$ in the above Fredholm integral equation rather than $I(\tau,\mu)$ in Eq. (2.11a) offers several advantages. The mixed form of Eq. (2.11a) restricts the choice of solution methods, while a wide variety of approaches have been established for the solution of equations of the form of Eq. (2.18a). By reducing the form of the equation of radiative transfer the number of independent variables has been decreased from two to one, clearly enhancing the possibility of obtaining an accurate solution. Finally, incorporating the boundary conditions and source function into a single expression serves to streamline the governing equation, making the solution process more efficient. In the next chapter, it will be demonstrated that once $G(\tau)$ has been computed, all other radiative properties of interest, such as radiative intensity or the heat flux, can be determined anywhere within the medium.
NOTES

1 The final expression may be derived from the definition of the total derivative, and recalling that \( c = ds/dt \),

\[
\frac{DL_v}{Dt} = \frac{\partial L_v}{\partial t} + \frac{\partial L_v}{\partial s} \frac{ds}{dt} = \frac{\partial L_v}{\partial t} + c \frac{\partial L_v}{\partial s}
\]

Note that when the direction \( \Omega \) is along the path \( s \), we have

\[
\nabla \cdot (\Omega L_v) = \frac{\partial L_v}{\partial s}
\]

2 For problems of radiative transfer in a participating medium, local thermodynamic equilibrium is assumed. The assumption is that a small volumetric element in the medium is in local thermodynamic equilibrium, and so each point can be characterized by a local temperature \( T(x) \), allowing Kirchoff’s law to be applied. Kirchoff’s law states that the emission of radiation \( J_v^c \) in the medium is related to the spectral absorption coefficient \( \kappa_v(x) \) and the black-body radiation intensity \( I_{vb}(T) \) and \( T(x) \) by

\[
J_v^c(x) = \kappa_v(x) I_{vb}[T(x)]
\]

The expression for the total black-body radiation intensity \( I_b \) may be derived by integrating Planck’s black-body radiation intensity function:

\[
I_b(T) = \int_0^\infty I_{vb}(T) \, dv = \frac{2h}{c_0^2} \int_0^\infty \frac{n^2 v^3}{e^{hv/kT} - 1} \, dv
\]

\[
= \frac{2h}{c_0^2} \frac{n^2}{h} \left( \frac{kT}{h} \right)^3 \int_0^\infty \frac{(vh/kT)^3}{e^{vh/kT} - 1} \, dv
\]

\[
= \frac{2k^4}{c_0^2 h^3} (n^2 T^4) \int_0^\infty \frac{n^3}{e^n - 1} \, dn
\]

\[
= \frac{2k^4}{c_0^2 h^3} (n^2 T^4) \left( \frac{\pi^4}{15} \right)
\]

\[
= n^2 \sigma T^4 \frac{4}{\pi}
\]

Where \( \sigma \) is the Stefan-Boltzmann constant.
3 The quantity $\sigma_\nu(s) I_\nu(s, \Omega') d\Omega'$ is the portion of the incident radiation scattered by the medium in all directions. In order to determine the direction distribution of this scattered radiation, the phase function $p_\nu(\Omega' \rightarrow \Omega)$ is introduced [18], and normalized such that

$$\frac{1}{4\pi} \int_0^{4\pi} p_\nu(\Omega' \rightarrow \Omega) d\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} p_\nu(\mu', \phi' \rightarrow \mu, \phi) d\mu d\phi = 1$$

where $\mu = \cos \theta$, with $\theta$ is the angle between the incident and scattering directions, as illustrated in Fig. (2.3). The quantity $(1/4\pi) p_\nu$ is the probability that the incident radiation will be scattered into an elemental solid angle $d\Omega$ about the direction $\Omega$. Then

$$\left[ \sigma_\nu(s) I_\nu(s, \Omega') d\Omega' \right] \frac{1}{4\pi} p_\nu(\Omega' \rightarrow \Omega) d\Omega$$

is the portion of the incident radiation scattered by the medium into the elemental solid angle $d\Omega$. Since the radiation incident on the control volume is from all directions in spherical space, the above equation is integrated over all solid angles of incidence, yielding

$$\frac{1}{4\pi} \sigma_\nu(s) d\Omega \int_0^{4\pi} I_\nu(s, \Omega') p_\nu(\Omega' \rightarrow \Omega) d\Omega'$$

When the medium consists of a homogenous, isotropic material with spherical symmetry, and has no preferential scattering direction (noncoherent), then the phase function depends only on the angle between the directions $\Omega'$ and $\Omega$. This finally yields

$$\frac{1}{4\pi} \sigma_\nu(s) \int_0^{4\pi} p(\Omega' \cdot \Omega') I_\nu(s, \Omega', t) d\Omega'$$

with the normalized phase function now defined as

$$\frac{1}{4\pi} \int_0^{4\pi} p(\Omega' \cdot \Omega) d\Omega' = 1$$
Figure 2.3  Geometry for the volumetric scattering of radiation
Applying solid geometry relations to the geometry in Fig (2.3), results in the following expression for the angle \( \theta_0 \) between the incident and scattered rays

\[
\cos \theta_0 = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')
\]

or

\[
\mu_0 = \mu \mu' + \sqrt{1 - \mu^2} \sqrt{1 - \mu'^2} \cos (\phi - \phi')
\]

With this expression for \( \mu_0 \) the following relation existing among Legendre polynomials is introduced [20, pp. 326-328]

\[
P_n(\mu_0) = P_n(\mu)P_n(\mu') + 2 \sum_{m=1}^{\infty} \frac{(n - m)!}{(n + m)!} P_n^m(\mu)P_n^m(\mu') \cos m(\phi - \phi')
\]

The \( P_n^m(\mu) \) are the associated Legendre functions. This relation is substituted into the equation for \( p(\mu_0) \) to yield

\[
p(\mu_0) = \sum_{n=0}^{N} a_n P_n(\mu)P_n(\mu') + 2 \sum_{n=1}^{N} \sum_{m=1}^{n} a_n^m P_n^m(\mu)P_n^m(\mu') \cos m(\phi - \phi') \quad (A)
\]

with

\[
a_n^m = \frac{a_n (n - m)!}{(n + m)!}; \quad n = m, ..., N; \quad a_0 = 1
\]

For azimuthal symmetry, Eq. (A) is integrated over \( \phi' \) from 0 to \( 2\pi \), with the result

\[
\int_{0}^{2\pi} p(\mu_0) \, d\phi' = 2\pi \sum_{n=0}^{N} a_n P_n(\mu) P_n(\mu'); \quad a_0 = 1
\]

Note that the integration of \( \cos m(\phi - \phi') \) over \( 2\pi \) is zero for integer values of \( m \).

A plane-parallel system is a medium which can be stratified into planes perpendicular to the normal axis (vector \( \mathbf{x} \) in Fig. 2.2), such that the radiative properties are uniform over each layer. \( S \) is the length measured along an arbitrary direction \( \Omega \), with \( \theta \) the polar angle.
between the direction $\Omega$ and the positive normal axis. The directional derivative $d/ds$ can be expressed in terms of the derivative of the space coordinate $x$ as

$$\frac{d}{ds} = \frac{\partial}{\partial x} \frac{dx}{ds} = \mu \frac{\partial}{\partial x}$$

where $\mu$ is the cosine of $\theta$. In radiative heat transfer, a common variable used is the optical depth, which is defined as

$$d\tau \equiv \beta \, dx \quad \text{or} \quad \tau \equiv \int_{0}^{x} \beta \, dx'$$

The optical variable is a function of frequency, but here a gray medium is assumed.

6 For general scattering, the boundaries are taken as diffusely reflecting surfaces, that reflect all incident radiation backwards in all directions. Thus the reflection components of the backward and forward intensities at the boundaries are expressed as [9, p. 274]

$$I^+(0, \mu) = \rho_1 \frac{\int_{0}^{2\pi} \int_{-1}^{0} I^-(0, \mu') \mu' \, d\mu' \, d\phi'}{\int_{0}^{2\pi} \int_{0}^{-1} \mu' \, d\mu' \, d\phi'} = 2\rho_1 \int_{0}^{1} I^-(0, -\mu') \mu' \, d\mu'$$

$$I^-(L, -\mu) = \rho_2 \frac{\int_{0}^{2\pi} \int_{0}^{1} I^+(L, \mu') \mu' \, d\mu' \, d\phi'}{\int_{0}^{2\pi} \int_{0}^{1} \mu' \, d\mu' \, d\phi'} = 2\rho_2 \int_{0}^{1} I^+(L, \mu') \mu' \, d\mu'$$

7 The following relation is used to simplify the integrals involving $H(\tau)$ and $G(\tau)$,

$$\int_{0}^{1} e^{-\frac{(\tau - \tau_1)}{\mu}} \, d\mu + \int_{0}^{1} e^{-\frac{(\tau - \tau)}{\mu}} \, d\mu = E_1(\tau - \tau_1) + E_1(\tau' - \tau) = E_1(\tau' - \tau')$$
CHAPTER 3

THE SOLUTION OF THE
EQUATION OF RADIATIVE TRANSFER
BY GALERKIN'S METHOD

3.1 Introduction

Integral equations, such as the Fredholm equation of Eq. (2.18a), are usually solved with an approximation method. These approximation techniques may fall into one of two categories; variational methods or the method of weighted residuals (MWR). The methods of weighted residuals are usually easier to apply as they work directly with the governing equation and boundary conditions. Variational methods on the other hand require more mathematical manipulation since they operate on a function which is related to the governing equation and boundary conditions [21]. In MWR, an approximate solution (that satisfies the boundary conditions) consisting of a trial function multiplied by unknown constants is selected and substituted into the governing equation. Since this solution is only an approximation, there is a residual value remaining after its substitution into the governing equation. If the approximate solution were the exact solution, the residual would be zero. In MWR, the constants are chosen in such a way that the residual is forced to be zero (in an average sense) by setting the weighted integrals of the residual to zero. It is the selection of a weighting function that differentiates the various methods of weighted residuals. Probably the best known of the MWR family is Galerkin's method which chooses as its weighting function the trial function. The next section will provide a more general explanation of Galerkin's method, although a rigorous proof of Galerkin's method will not be attempted in this work. Such a proof however, can be found in the
text by Kantorovich [22]. The third section of this chapter will introduce an improved trial function for the solution of Eq. (2.18). Galerkin's method will then be applied, using this trial function, to arrive at an explicit relation for \( G(\tau) \). With this result, expressions will be derived for the intensity of incident radiation and the heat fluxes.

3.2 Galerkin's Method

Galerkin's method is demonstrated by its application to a general equation represented formally as

\[ L(F(x)) = 0 \]  

(3.1)

Where \( L \) is an operator which characterizes all the operations involved in the equation for \( F(x) \). In selecting a trial function to approximate \( F(x) \), the only restriction is that it be a member of a complete set of functions. A set of functions \( \phi(x) \) is complete if any function can be expanded in terms of the set

\[ f(x) = \sum_{i=1}^{N} C_i \phi_i(x) \]  

(3.2)

Because \( f(x) \) is an approximation, its insertion in Eq. (3.1) will not satisfy the relation exactly; thus there will be a residual of the form

\[ R(x,\phi) = L(f(x)) = L\left( \sum_{i=1}^{N} C_i \phi_i(x) \right) \]  

(3.3)

As the number \( N \) is increased in successive approximations, the residuals will hopefully become smaller. The exact solution is reached when the residual is identically zero. This
is where the requirement for trial functions that are members of a complete set of functions is important, because a continuous function \( \phi(x) \) is zero if it is orthogonal to every member of a complete set. Thus Galerkin's method forces the residual to zero by making it orthogonal to each member of a complete set of functions (as \( N \to \infty \)). This is called the orthogonality condition, and is expressed as

\[
< w_j, R(x,\phi) > = 0
\]  

(3.4)

where

\[
< u, v > \equiv \int_V u v \, dV
\]

represents an inner product or spatial average, and \( w_j \) is a weighting function. In Galerkin's method, the weighting function is chosen to be the trial function \( \phi(x) \). Equation (3.4) becomes

\[
< w_j, R(x,\phi) > = < \phi, L(f(x)) >
\]

\[
= < \phi, L(\sum_{i=1}^{N} C_i \phi_i(x)) >
\]

\[
= \int_{0}^{L} L(C_i \phi_i(x)) \phi(x) \, dx = 0; \quad 0 < x < L
\]

(3.5)

Equation (3.5) yields a system of \( N \) equations from which the \( N \) unknown constant coefficients \( C_i \) are determined. Once the \( C_i \)'s are known, Eq. (3.2) can be inserted into Eq. (3.1) to obtain the approximate solution to the problem. The orthogonality condition ensures that the minimum residual results for the trial function selected. Successive
approximations are obtained by increasing N, and solving Eq. (3.5) again. The convergence of the successive approximations is an indicator, but not a definitive one, of the performance of the trial function. The main influence of choosing a better trial function is the rate of convergence rather than the eventual solution, as even a less than optimal profile will reach the exact solution for a sufficiently large enough N. Selection of the trial function is therefore the critical step in Galerkin's method, as it provides the power of the method. As will be demonstrated later in this work, the selection of a trial function based on some physical insights into the problem being solved provides the best results.

3.3 Application of Galerkin's Method to the Equation of Radiative Transfer

Applying Galerkin's method to the solution of the reduced form of the equation of radiative transfer Eq. (2.18a), the Fredholm integral equation becomes the linear operation of Eq. (3.1); i.e. the function F(x) now becomes the incident radiation G(τ). The next step is the selection of the trial function φ(τ). In general, for functions involving a space variable such as G(τ), a simple power series in terms of the space variable can be used for φ(x). This choice requires no real insight into the problem, which simplifies the solution process for complex expressions such as Eq. (2.18). In their work introducing the use of Galerkin's method for the solution of the equation of radiative heat transfer, Özisik and Yener [11] used such a trial function of the form,

\[ G(\tau) = \sum_{i=0}^{N} C_i \tau^i \]  

(3.6)
The order of the approximation $N$ is increased until convergence to the desired accuracy is obtained. As the optical depth of the medium $\tau_0$ increases, so does the number of terms required to achieve the same degree of convergence. However, for reasons that will be discussed in Chapter 4, the solution starts to diverge when $N$ exceeds about 20. Thus the polynomial profile is limited to optical thicknesses less than 15 due to its inability to converge within 20 terms. Often an improved trial function for the Galerkin method is suggested by other solution methods applied to similar problems. Özisik and Thynell [15] obtained an improved profile from the application of the method of spherical harmonics to the solution of the equation of radiative transfer. Their trial function took the form

$$G(\tau) = \sum_{j=1}^{I} \left[ A_j e^{-\tau \lambda_j} + B_j e^{-(\tau_0 - \tau)\lambda_j} \right] + G_0(\tau) \quad (3.7)$$

Where $G_0(\tau)$ is the particular solution corresponding to the inhomogenous source term $H(\tau)$ in Eq. (2.18), and the $\lambda_j$ are the natural eigenvalues of the problem. The procedure to compute the eigenvalues from Chandrasekhar's polynomials [16] is discussed in Appendix B. Özisik and Thynell found that the exponential series, which they related to the complementary solution of the homogeneous problem, could be used by itself with results superior to that obtained with the polynomial profile. However, they found that the inclusion of the particular solution improved the rate of convergence. This of course requires the solution of a separate problem be performed first in order to determine the particular solution. Unfortunately, the particular solution is often difficult to find, particularly in transient radiation problems or problems involving combined modes of heat transfer. Thus the inclusion of the particular solution in Eq. (3.7) appears to be limited to well-known problems where the particular solution is available, and is not of a form such that it cancels the source term in the integral equation. [15]
The difficulty in determining the exact particular solution, and that its presence in the trial function greatly enhances convergence, suggests that Eq. (3.7) could be improved by eliminating $G_o$ and compensating for the effects of the particular solution. Thus, the following trial function is proposed

$$G(\tau) = \sum_{j=1}^{I} \left[ A_j e^{-\tau/\lambda_j} + B_j e^{-(\tau_0 - \tau)/\lambda_j} \right] + \sum_{i=0}^{M} \Phi_i(\tau)$$

The series $\phi(\tau)$ compensates for the effects of the particular solution, and its profile depends on the form of the source term $H(\tau)$. For example, when $H(\tau)$ is a polynomial in $\tau$, the particular solution is also a polynomial [p.386,9]. Since most any $H(\tau)$ could be approximated by a polynomial, it is proposed it is proposed that $\phi(\tau)$ is also a polynomial, which yields the trial function.

$$G(\tau) = \sum_{j=1}^{I} \left[ A_j e^{-\tau/\lambda_j} + B_j e^{-(\tau_0 - \tau)/\lambda_j} \right] + \sum_{i=0}^{M} C_i \tau^i \quad (3.8)$$

It will be demonstrated later that the order of the polynomial $M$ is the same as the order of $H(\tau)$, if it is known. If the order of $H(\tau)$ is not known, $M$ is increased until no further improvement in convergence is obtained. The form of Eq. (3.8) offers two advantages over Eq. (3.7). First, no prior knowledge of the form of $H(\tau)$ is required, and so this profile is applicable to a wider range of problems. The second is that the coefficients $C_i$ can be computed simultaneously with the $A_j$ and $B_j$ by the orthogonality condition. Thus the additional step required for Eq. (3.7) to determine the particular solution is eliminated,
speeding up the solution process. The procedure starts by expressing Eq. (2.18a) as a residual, and replacing \( G(\tau) \) by its trial solution Eq. (3.8) to yield

\[
R = Y(\tau) - \sum_{j=1}^{J} \left [ A_j e^{-\frac{\tau}{\lambda_j}} + B_j e^{\frac{(\tau_0 - \tau)}{\lambda_j}} \right ] + \sum_{i=0}^{M} C_i \tau^i \\
+ \omega \int_{0}^{\tau_0} K(\tau, \tau') \left \{ \sum_{j=1}^{J} \left [ A_j e^{-\frac{\tau'}{\lambda_j}} + B_j e^{\frac{(\tau_0 - \tau)}{\lambda_j}} \right ] + \sum_{i=0}^{M} C_i \tau'^i \right \} d\tau'
\] (3.9)

As before, if Eq. (3.8) was the exact solution of \( G(\tau) \), then \( R = 0 \). The orthogonality condition of Galerkin's method is then applied by operating on the residual with first

\[
\int_{0}^{\tau_0} R(\tau, \tau') e^{-\frac{\tau}{\lambda_k}} d\tau = 0; \quad k = 1,2,..., J
\] (3.10a)

then with

\[
\int_{0}^{\tau_0} R(\tau, \tau') e^{\frac{\tau}{\lambda_k}} d\tau = 0; \quad k = 1,2,..., J
\] (3.10b)

and finally with

\[
\int_{0}^{\tau_0} R(\tau, \tau') \tau^l d\tau = 0; \quad l = 0,1,..., M
\] (3.10c)
The result is a series of \((2\times J) + M + 1\) linear algebraic equations for the \((2\times J) + M + 1\) unknown expansion coefficients, and may be expressed as

\[
\sum_{j=1}^{J} [A_j V(\lambda_j, \lambda_k) + B_j e^{-\frac{\tau_0}{\lambda_j}} V(-\lambda_j, \lambda_k)] + \sum_{i=0}^{M} C_i V_i(\lambda_k) = d(\lambda_k)
\]

\[
\sum_{j=1}^{J} [A_j V(\lambda_j, \lambda_k) + B_j e^{-\frac{\tau_0}{\lambda_j}} V(-\lambda_j, -\lambda_k)] + \sum_{i=0}^{M} C_i V_i(-\lambda_k) = d(-\lambda_k); \quad k = 1, 2, \ldots, J
\]

\[
\sum_{j=1}^{J} [A_j V_1(\lambda_j) + B_j e^{-\frac{\tau_0}{\lambda_j}} V_1(-\lambda_j)] + \sum_{i=0}^{M} C_i B_{il} = d_l; \quad l = 0, 1, \ldots, M \quad (3.11)
\]

Where

\[
d(\eta) = \int_{0}^{\tau_0} Y(\tau) e^{-\frac{\tau}{\eta}} d\tau
\]

\[
d_i = \int_{0}^{\tau_0} Y^i(\tau) d\tau
\]

\[
V(\eta, \nu) = \int_{0}^{\tau_0} e^{-\frac{\tau}{\eta} - \frac{\nu}{\nu}} d\tau - \frac{\omega}{2} \int_{0}^{\tau_0} \int_{0}^{\tau_0} e^{-\frac{\tau}{\eta}} E_1(|\tau - \tau'|) e^{-\frac{\nu}{\nu}} d\tau d\tau'
\]

\[
V_n(\eta) = \int_{0}^{\tau_0} \tau^n e^{-\frac{\tau}{\eta}} d\tau - \frac{\omega}{2} \int_{0}^{\tau_0} \int_{0}^{\tau_0} e^{-\frac{\tau}{\eta}} E_1(|\tau - \tau'|) \tau^n d\tau d\tau'
\]

\[
B_{mn} = \int_{0}^{\tau_0} \tau^m \tau^n d\tau - \frac{\omega}{2} \int_{0}^{\tau_0} \int_{0}^{\tau_0} \tau^m E_1(|\tau - \tau'|) \tau^n d\tau d\tau'
\]
Explicit expressions for $V$, $V_n$, and $B_{mn}$ are provided in Appendix A. The $d$ and $d_i$ for specific source functions and boundary conditions are presented in later chapters. The series of algebraic equations may be expressed in matrix form as

$$[A][C] = [D]$$

(3.12)

Where $A$ is the square-symmetric matrix

$$[A] = \begin{bmatrix}
V(\lambda_j, \lambda_k) & \frac{\tau_0}{\lambda_j} V(\lambda_j, \lambda_k) & V_i(\lambda_k) \\
\frac{\tau_0}{\lambda_j} V(\lambda_j, \lambda_k) & e^{\frac{\lambda_j}{V_1(\lambda_j)}} & V_i(\lambda_k) \\
V_1(\lambda_j) & e^{\frac{\lambda_j}{\tau_0V_1(\lambda_j)}} & B_i
\end{bmatrix}$$

And the vectors $C$ and $D$ are given by

$$[C] = \begin{bmatrix} A_j \\ B_j \\ C_i \end{bmatrix}; \quad [D] = \begin{bmatrix} d(\lambda_j) \\ d(-\lambda_j) \\ d_i \end{bmatrix}$$
The expansion coefficients in Eq. (3.8) can now be determined from the solution of the matrix relation, Eq. (3.12). Once the coefficients are computed, \( G(\tau) \) is specified by Eq. (3.8), and the remaining radiative properties of the problem may be computed.

### 3.4 Computation of the Intensities of Radiation

Using the trial function for \( G(\tau) \) Eq. (3.8), the backward and forward intensities of radiation, \( I^- (\tau, \mu) \) and \( I^+ (\tau, \mu) \) respectively, can be computed from the expressions derived in Chapter 2. Equation (3.8) is substituted for \( G(\tau) \) in Eqs. (2.13a) and (2.13b), along with the boundary conditions, Eqs. (2.11b) and (2.11c), yielding

\[
I^+ (\tau, \mu) = \int_0^\tau \frac{1}{\mu} H(\tau') e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \frac{\omega}{4\pi} \int_0^\tau \frac{1}{\mu} \left[ \sum_{j=1}^J \left\{ A_j e^{\frac{\tau'}{\lambda_j}} + B_j e^{-\frac{\tau'}{\lambda_j}} \right\} \right]
\]

\[+ \sum_{i=0}^M C_i \tau_i^i \right\} e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \left[(1 - \rho_1) f_1(\mu) + A_1 + 2\rho_1 K_1 \right] e^{-\frac{\tau}{\mu}} \]  

(3.13a)

\[
I^- (\tau, -\mu) = \int_\tau^0 \frac{1}{\mu} H(\tau') e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \left[(1 - \rho_2) f_2(\mu) + A_2 + 2\rho_2 K_2 \right] e^\frac{(\tau - \tau')}{\mu}
\]

\[+ \frac{\omega}{4\pi} \int_\tau^0 \frac{1}{\mu} \left[ \sum_{j=1}^J \left\{ A_j e^{\frac{\tau'}{\lambda_j}} + B_j e^{-\frac{\tau'}{\lambda_j}} \right\} \right] e^\frac{(\tau - \tau')}{\mu} d\tau' \]  

(3.13b)
Performing the integrations of the terms involving the trial functions results in the expressions

\[
I^+ (\tau, \mu) = \int_0^\tau \frac{1}{\mu} H(\tau') e^{-\frac{(\tau - \tau')}{\mu}} d\tau' + \frac{\omega}{4\pi} \sum_{j=1}^I \lambda_j \left[ A_j \left( e^{-\frac{\tau}{\mu}} - e^{-\frac{\tau}{\lambda_j}} \right) + B_j \left( 1 - e^{-\frac{\tau}{\mu + \lambda_j}} \right) \right] \\
+ \frac{\omega}{4\pi} \sum_{i=0}^M C_i i! \left\{ (-1)^{i+1} e^{-\frac{\tau}{\mu}} \mu^i + \sum_{l=0}^i (-1)^l \frac{\tau^i - 1}{(i-l)!} \mu^l \right\} 
\]

and

\[
I^- (\tau, -\mu) = \int_\tau^{\tau_o} \frac{1}{\mu} H(\tau') e^{-\frac{(\tau' - \tau)}{\mu}} d\tau' \\
+ \frac{\omega}{4\pi} \sum_{j=1}^I \lambda_j \left[ A_j \left( e^{-\frac{\tau}{\lambda_j}} - e^{-\frac{\tau}{\mu + \lambda_j}} \right) + B_j \left( e^{-\frac{\tau}{\mu}} - e^{-\frac{\tau}{\mu - \lambda_j}} \right) \right] \\
+ \frac{\omega}{4\pi} \sum_{i=0}^M C_i i! \left[ \sum_{l=0}^i \frac{\mu^l}{(i-l)!} \left\{ (\tau_o - \tau)^{i-l} - e^{-\frac{(\tau - \tau_o)}{\mu}} \tau_o^{i-l} \right\} \right]
\]

(3.14a)

(3.14b)

3.4 Computation of the Radiative Heat Fluxes

The forward and backward radiation heat fluxes within the medium are defined, respectively by

\[
q^+ (\tau) = 2\pi \int_0^1 I^+ (\tau, \mu) \mu d\mu 
\]

(3.15a)

\[
q^- (\tau) = 2\pi \int_0^1 I^- (\tau, -\mu) \mu d\mu 
\]

(3.15b)
Equations (3.13a) and (3.13b) are substituted for $I^+\tau$ and $I^-\tau$ in Eqs. (3.15a) and (3.15b) respectively, as this allows introduction of the exponential integrals, thus simplifying the notation. The resulting expressions are

$$q^+ (\tau) = 2\pi \int_0^\infty H(\tau') \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' + 2\pi \int_0^\infty \left\{ (1 - \rho_1 \text{e}_1(\mu) + A_1 + 2\rho_1 K_1 e^{-\frac{\tau}{\mu}} d\mu \right\}$$

$$+ \frac{\omega}{2} \sum_{j=1}^J \left\{ A_j \int_0^\infty e^{-\frac{\tau}{\lambda_j}} \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' + B_j \int_0^\infty e^{-\frac{\tau}{\lambda_j}} \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' \right\}$$

$$+ \sum_{i=0}^M C_i \int_0^\tau e^{-\frac{\tau}{\lambda_i}} \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' \right\} \quad (3.16a)$$

$$q^- (\tau) = 2\pi \int_0^\infty H(\tau') \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' + 2\pi \int_0^\infty \left\{ (1 - \rho_2 \text{e}_2(\mu) + A_2 + 2\rho_2 K_2 e^{-\frac{\tau}{\mu}} d\mu \right\}$$

$$+ \frac{\omega}{2} \sum_{j=1}^J \left\{ A_j \int_0^\infty e^{-\frac{\tau}{\lambda_j}} \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' + B_j \int_0^\infty e^{-\frac{\tau}{\lambda_j}} \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' \right\}$$

$$+ \frac{\omega}{2} \sum_{i=0}^M C_i \int_0^\tau e^{-\frac{\tau}{\lambda_i}} \int_0^\infty e^{-\frac{(\tau - \tau')}{\mu}} d\mu d\tau' \right\} \quad (3.16b)$$
The introduction of the exponential integral notation yields

\[ q^+ (\tau) = 2\pi \int_0^\tau H(\tau') E_2(\tau - \tau') \, d\tau' + 2\pi \int_0^1 \left[ (1 - \rho_o) f_1(\mu) + A_1 + 2\rho_1 K_1 \right] e^{-\frac{\tau}{\mu}} \mu \, d\mu \]

\[ + \frac{\omega}{2} \sum_{j=1}^I \left\{ A_j \int_0^{\tau} e^{-\frac{\tau'}{\lambda_j}} E_2(\tau - \tau') \, d\tau' + B_j e^{-\frac{\tau_0}{\lambda_j}} \int_0^{\tau} e^{-\frac{\tau'}{\lambda_j}} E_2(\tau - \tau') \, d\tau' \right\} \]

\[ + \frac{\omega}{2} \sum_{i=0}^M C_i \int_0^\tau \tau^i E_2(\tau - \tau') \, d\tau' \]  \hspace{1cm} (3.17a)

\[ q^- (\tau) = 2\pi \int_0^{\tau_0} H(\tau') E_2(\tau' - \tau) \, d\tau' + 2\pi \int_0^1 \left[ (1 - \rho_3) f_2(\mu) + A_2 + 2\rho_2 K_2 \right] e^{-\frac{(\tau_0 - \tau)}{\mu}} \mu \, d\mu \]

\[ + \frac{\omega}{2} \sum_{j=1}^J \left\{ A_j \int_{\tau}^{\tau_0} e^{-\frac{\tau'}{\lambda_j}} E_2(\tau' - \tau) \, d\tau' + B_j e^{-\frac{\tau_0}{\lambda_j}} \int_{\tau}^{\tau_0} e^{-\frac{\tau'}{\lambda_j}} E_2(\tau' - \tau) \, d\tau' \right\} \]

\[ + \frac{\omega}{2} \sum_{i=0}^M C_i \int_{\tau}^{\tau_0} \tau^i E_2(\tau' - \tau) \, d\tau' \]  \hspace{1cm} (3.17b)
The exit heat fluxes, \( q^+ (\tau_o) \) and \( q^- (0) \), may be expressed as

\[
q^+ (\tau_o) = 2\pi \int_0^{\tau_o} \left[(1 - \rho_2) f_1(\mu) + A_1 + 2\rho_1 K_1 \right] e^{-\frac{\tau_o}{\mu}} \mu \, d\mu
+ \frac{\omega}{2} \left[ \sum_{j=1}^{J} \left\{ A_j \Phi(\lambda_j, \tau_o) + B_j e^{-\frac{\tau_o}{\lambda_j}} \Phi(-\lambda_j, \tau_o) \right\}
+ \sum_{i=0}^{M} C_i Y_i \right] \tag{3.18a}
\]

\[
q^- (0) = 2\pi \int_0^{\tau_o} \left[(1 - \rho_2) f_2(\mu) + A_2 + 2\rho_2 K_2 \right] e^{-\frac{\tau_o}{\mu}} \mu \, d\mu
+ \frac{\omega}{2} \left[ \sum_{j=1}^{J} \{ A_j \psi(\lambda_j, 0) + B_j e^{-\frac{\tau_o}{\lambda_j}} \psi(-\lambda_j, 0) \}
+ \frac{\omega}{2} \sum_{i=0}^{M} C_i Z_i \right] \tag{3.18b}
\]

Where

\[
\Phi(\eta, \tau) = \int_0^{\tau} e^{-\frac{\tau'}{\eta}} E_2(\tau' - \tau') \, d\tau' \tag{3.19a}
\]

and

\[
\psi(\eta, \tau) = \int_\tau^{\tau_o} e^{-\frac{\tau'}{\eta}} E_2(\tau' - \tau) \, d\tau' \tag{3.19b}
\]

Explicit analytical expressions for \( \Phi, \psi, Y_i, \) and \( Z_i \) are presented in Appendix A.
3.5 Computation of the Incident Radiation

Up to this point, the incident radiation $G(\tau)$ has been represented by the approximation function of Eq. (3.8). However, when used to compute the incident radiation, the approximation converges to a number that is one or two percent lower than the exact value.\(^1\) This occurs because the contribution by the boundary conditions is included only in an approximate manner in the computations for the expansion coefficients, $A_j$, $B_j$, and $C_i$. Therefore, a more accurate relation for the incident radiation should include an explicit expression of the boundary conditions. This is done by substituting the approximation profile of Eq. (3.8) into the right-hand side of the Fredholm equation for $G(\tau')$, Eq. (2.18 a), with the result

$$G(\tau) = Y(\tau) + \omega \int_0^{\tau_o} K(\tau,\tau') \left\{ \sum_{j=1}^J \left[ A_j e^{\frac{\tau'}{\lambda_j}} + B_j e^{-\frac{(\tau_o - \tau')}{\lambda_j}} \right] + \sum_{i=0}^M C_i \tau' i \right\} d\tau' \quad (3.20)$$

Equation (2.18b) is substituted for $K(\tau,\tau')$ in Eq. (3.20), yielding the expression

$$G(\tau) = Y(\tau) + \omega \sum_{j=1}^J \left\{ A_j \nu(\lambda_j, \tau) + B_j e^{\frac{\tau_o}{\lambda_j}} \nu(-\lambda_j, \tau) \right\}$$

$$+ \omega \sum_{i=0}^M C_i \Gamma_i(\tau) \quad (3.21)$$

Where

$$P(\eta, \tau) = \frac{1}{2} T(\eta, \tau) + \rho_1 B E_2(\tau) \left[ \nu(\eta, 0) + a_2 \Phi(\eta, \tau_o) \right]$$

$$+ \rho_2 B E_2(\tau_o - \tau) \left[ \Phi(\eta, \tau_o) + a_1 \nu(\eta, 0) \right] \quad (3.22a)$$
\[ \Gamma_i (\tau) = \frac{1}{2} \int_0^{\tau_0} \tau' E_1 (\tau - \tau') \, d\tau' + \rho_1 B E_2 (\tau) \int_0^{\tau_0} \tau' E_2 (\tau') \, d\tau' + a_2 \int_0^{\tau_0} \tau' E_2 (\tau) \, d\tau' + \rho_2 B E_2 (\tau_0 - \tau) \int_0^{\tau_0} \tau' E_2 (\tau_0 - \tau') \, d\tau' + a_1 \int_0^{\tau_0} \tau' E_2 (\tau_0) \, d\tau' \]  \hspace{1cm} (3.22b) \\

Explicit expressions for \( T \) and \( \Gamma_i \) are provided in Appendix A, and \( \Phi \) and \( \psi \) are defined by Eqs. (3.19 a) and (3.19 b) respectively. Thus Eq. (3.21) provides a relation by which the incident radiation may be computed more accurately due to the explicit expression of the boundary conditions' contribution in the free term \( Y(\tau) \), with the approximation involved only in the expansion terms.

NOTE

1 Based on computations involving problems for which exact values of the incident radiation have been computed by other methods of solution.
CHAPTER 4

ANALYSIS OF THE
IMPROVED GALERKIN PROFILE

4-1. Introduction

A critical prerequisite to this study's analysis is to determine the performance and limitations of the improved-profile Eq. (3.8). The performance of numerical Galerkin profiles is measured by rate of convergence and accuracy of the computed values. The rate of convergence for the improved-profile is examined by first increasing $J$, the order of the component of the profile approximating the complementary solution. Then the order $M$ is increased to determine the influence on the rate of convergence by the profile component approximating the particular solution. The results of these computations are then compared with previous studies to evaluate accuracy, and also the adequacy of the approximation of the particular solution by the power series. The effect of the particular solution's approximation is demonstrated by presenting numerical results using only the complementary solution component of the improved-profile. In their study using a polynomial trial function, Cengel and Özisik [12] reported a loss in accuracy when Galerkin's method was applied to an optically thick medium. This phenomenon, manifested by a sudden loss in convergence stability, suggests that even with the improved-profile, the range of radiative heat transfer problems that can be solved by Galerkin's method is limited. The last section of this chapter therefore examines the range of optical parameters for which a solution by Galerkin's method with the improved-profile is valid.
4.2. **Numerical Performance of the Improved-Profile**

In this section, the performance of the improved-profile is examined by its application to two cases of steady-state problems; the steady-state restriction allows comparison with previous studies. Since the improved-profile is not time-dependent, its performance will be the same for both the transient and steady-state cases. The performance analysis is based on the values computed for the incident radiation $G(\tau)$ and the exit heat fluxes $q^+(\tau_o)$ and $q^-(0)$, as they are critical parameters for the analysis of the LDR performed in the next chapter. In order to evaluate the effect of the boundary conditions on the performance of the improved profile, each case will be studied for homogeneous boundary conditions and for the condition of external radiation incident only on the boundary $\tau = 0$. These boundary conditions were also selected in deference to the ensuing LDR analysis. Computations will be made for optical depths of 1.0, 5.0, and 10.0; and albedos of 0.1, 0.7, and 0.9.

The case of a gray, plane-parallel medium with a constant internal source ($H = 1/\tau_o$) is initially considered. This case has been studied frequently with many solution methods, and is therefore an excellent problem for verifying the accuracy of the numerical computations. The governing equation from Eq. (2.11a) and boundary conditions from Eqs. (2.11a) and (2.11b) are

$$
\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = \frac{1}{\tau_o} + \frac{\omega}{2} \int_{-1}^{1} I(\tau, \mu') \, d\mu';
$$

$$
0 < \tau < \tau_o, \quad -1 < \mu < 1
$$

(4.1a)

$$
I^+(0, \mu) = f_1(\mu)
$$

(4.1b)

$$
I^-(\tau_o, -\mu) = 0
$$

(4.1c)
From Eq. (2.18a), the reduced form of the governing equation is

\[ G(\tau) = 2\pi \left\{ \frac{1}{\tau_0} \int_0^{\tau_0} E_1(|\tau - \tau'|) \, d\tau' + \int_0^1 f_1(\mu) e^{-\frac{\tau}{\mu}} \, d\mu \right\} \]

\[ + \frac{\omega}{2} \int_0^{\tau_0} G(\tau') E_1(|\tau - \tau'|) \, d\tau' \]  \hspace{1cm} (4.2)

Substitution of the improved-profile for \( G(\tau) \) in Eq. (4.2) \(^1\) leads to the following expressions for the exit heat fluxes from Eqs. (3.18a) and (3.18b) for constant external radiation, \(^2\)

\[ q^+(\tau_o) = 2\pi \left\{ \frac{1}{\tau_0} \left( \frac{1}{2} - E_3(\tau_o) \right) + f_1 E_3(\tau_o) \right\} \]

\[ + \frac{\omega}{2} \left[ \sum_{j=1}^I \left\{ A_j \Phi(\lambda_j, \tau_o) + B_j e^{-\frac{\tau_o}{\lambda_j}} \Phi(-\lambda_j, \tau_o) \right\} + \sum_{i=0}^M C_i Y_i \right] \]  \hspace{1cm} (4.3a)

\[ q^-(0) = 2\pi \left\{ \frac{1}{\tau_0} \left( \frac{1}{2} - E_3(\tau_o) \right) + \frac{\omega}{2} \sum_{j=1}^I \left\{ A_j \psi(\lambda_j, 0) + B_j e^{-\frac{\tau_o}{\lambda_j}} \psi(-\lambda_j, 0) \right\} \]

\[ + \frac{\omega}{2} \sum_{i=0}^M C_i Z_i \]  \hspace{1cm} (4.3b)

The incident radiation from Eq. (3.21) is \(^3\)

\[ G(\tau) = 2\pi \left\{ \frac{1}{\tau_0} \left( 2 - E_2(\tau) - E_2(\tau_o - \tau) \right) + f_1 E_2(\tau) \right\} \]

\[ + \frac{\omega}{2} \sum_{j=1}^I \left\{ A_j T(\lambda_j, \tau) + B_j e^{-\frac{\tau_o}{\lambda_j}} T(-\lambda_j, \tau) \right\} \]

\[ + \frac{\omega}{2} \sum_{i=0}^M C_i X_i(\tau) \]  \hspace{1cm} (4.4)
where

\[ X_i(\tau) = \int_0^{\tau_o} \tau' \ i \ E_1(|\tau - \tau'|) \, d\tau' \]

An explicit expression for \( X_i(\tau) \) is provided in Appendix A.

The computations for homogeneous boundary conditions were performed by setting \( f_1(\mu) \) to zero. The numerical results are presented in Tables (4-1) - (4-3). Due to the symmetry of the problem, the exit fluxes at \( \tau = 0 \) and \( \tau = \tau_0 \) are equal, as are the values of the incident radiation. The third column in the tables presents results using only the complementary solution approximation of the improved-profile. The fourth and fifth columns show the numerical results using the complete improved-profile with increasing order of approximation. The solution using only the complementary solution approximation converges smoothly for an optical depth of 1. However, as the optical depth increases, use of the complementary solution approximation alone produces oscillations which damp with increasing order of the approximation. Physically, the increase in optical depth indicates a greater extinction of radiation, and thus a more non-uniform distribution of incident radiation and heat. The root cause of the oscillation however, lies specifically in the use of only the component related to the complementary solution to approximate the general solution to a non-homogeneous equation. As the optical depth increase, the inadequacy of the complementary solution approximation to represent both homogeneous and particular solutions becomes more pronounced.

The oscillations also increase with larger albedo, which characterizes a highly-scattering medium. Since the scattering is assumed isotropic in these cases, the radiation within the layer is uniformly distributed. As the albedo is increased, less scattering occurs and more of the radiation is absorbed and then re-emitted by the medium. This results in a
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Table 4.1 Incident radiation and exit heat fluxes at τ = 0 and τ = τ₀ for a medium with a constant internal source H(τ) = \( \frac{1}{\tau_₀} \), \( τ_₀ = 1 \).
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Table 4.2 Incident radiation and exit heat fluxes at τ = 0 and τ = τ_0 for a medium with a constant internal source H(τ) = \( \frac{1}{\tau_0} \), τ_0 = 5.
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Table 4.3 Incident radiation and exit heat fluxes at \( \tau = 0 \) and \( \tau = \tau_o \) for a medium with a constant internal source \( H(\tau) = \frac{1}{\tau_o} \), \( \tau_o = 10 \).
greater extinction of incident radiation, and thus a far less uniform distribution. Analytically, the increase in the amplitude of the oscillations with larger optical depth and albedo is related to the difficulties reported by Cengel and Özisik [11]. This will be covered in more detail in the next section. However, one conclusion of a physical nature can be made at this point; the greater the non-uniformity in the distribution of radiation within the medium, the more difficult it is to approximate the distribution analytically using a profile consisting only of the complementary component.

The numerical results obtained using only the complementary solution approximation of the improved-profile confirms the importance of compensating for the effects of the particular solution. Otherwise, a high-order approximation is required to damp the oscillations and ensure an accurate result. The use of the complete improved-profile demonstrates the significance that compensating for the effects of the particular solution has when approximating the general solution, in that the oscillations discussed earlier are no longer present. The improved-profile exhibits smooth convergence with increasing order of approximation \( J \). In addition, the results obtained using a low-order approximation \( J \) are closer to the converged value than those obtained using only the complementary solution approximation. Note that in the fourth column, increasing the order \( M \) produces no improvement in the approximation. In Chapter 3 it was stated that \( M \) is related to the emission term \( H \). In this case, \( H \) is a constant \((1/\tau_0)\) or zero order polynomial in the space variable \( \tau \). The results confirm that the optimal value for \( M \) is the order of \( H \), expressed as the order of a polynomial expansion of \( H \) in the variable \( \tau \). A second conclusion drawn from the numerical results is when the order of \( H \) is unknown, overestimating its order in assigning a value to \( M \) does not degrade the performance of the approximation; it merely results in superfluous computations. Comparison of Table (4-3) with the same case studied by Thynell and Özisik [Table 5, 15] using an analytically derived, exact particular solution shows a near exact match. This confirms the
accuracy of the improved-profile and its ability to attain results comparable to a general solution requiring an analytically-derived, exact particular solution.

The first case of a constant emission term implies that the medium is maintained at a prescribed temperature. Most physical systems, including the LDR, present the more complex case of a temperature profile which varies with position. Therefore, the second case examines a medium with a volumetric source distribution defined by $H = 1 + \tau$. This is a well-documented case often used in linearized studies of planetary atmospheres. The governing equation from Eq. (2.11a) is

$$
\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = (1 + \tau) + \frac{\omega}{2} \int_0^1 I(\tau, \mu') \, d\mu'
$$

$$
0 < \tau < \tau_o, \quad -1 < \mu < 1
$$

and again with boundary conditions from Eqs. (2.11a) and (2.11b)

$$
I^+(0, \mu) = f_1(\mu)
$$

$$
I^-(\tau_o, \mu) = 0
$$

From Eq. (2.18a) the reduced form of the governing equation is

$$
G(\tau) = 2\pi \left\{ \int_0^{\tau_o} \int (1 + \tau') E_1(1\tau - \tau') \, d\tau' + \int_0^{\tau_o} f_1(\mu) \, e^{\frac{\tau}{\mu}} \, d\mu \right\}
$$

$$
+ \frac{\omega}{2} \int_0^{\tau_o} G(\tau') E_1(1\tau - \tau') \, d\tau'
$$

(4.6)
Substitution of the improved-profile for $G(\tau)$ into Eq. (4-6)\textsuperscript{4} leads to the following expressions for the heat fluxes from Eqs. (3.18a) and (3.18b) for constant external radiation.\textsuperscript{5}

\[
q^+(\tau_o) = 2\pi \left\{ \frac{1}{2} (1 + \tau_o) - \frac{1}{3} - E_3(\tau_o) + E_4(\tau_o) + \frac{f}{E_3(\tau_o)} \right\}
+ \frac{\omega}{2} \left\{ \sum_{j=1}^{J} \left[ A_j \Phi(\lambda_j, \tau_o) + B_j e^{-\frac{\tau_o}{\lambda_j}} \Phi(-\lambda_j, \tau_o) \right] + \sum_{i=0}^{M} C_i Y_i \right\} \tag{4.7a}
\]

\[
q^-(\tau_o) = 2\pi \left\{ \frac{1}{2} - (1 + \tau_o) E_3(\tau_o) + \frac{1}{3} - E_4(\tau_o) \right\}
+ \frac{\omega}{2} \left\{ \sum_{j=1}^{J} \left[ A_j \psi(\lambda_j, 0) + B_j e^{-\frac{\tau_o}{\lambda_j}} \psi(-\lambda_j, 0) \right] + \sum_{i=0}^{M} C_i Z_i \right\} \tag{4.7b}
\]

The incident radiation from Eq. (3.21) is\textsuperscript{6}

\[
G(\tau) = 2\pi \left\{ 2 (1 + \tau) - (1 + \tau_o) E_2(\tau_o - \tau) - E_2(\tau) + E_3(\tau) - E_3(\tau_o - \tau) + \frac{f}{E_2(\tau)} \right\}
+ \frac{\omega}{2} \sum_{j=1}^{J} \left[ A_j T(\lambda_j, \tau) + B_j e^{-\frac{\tau_o}{\lambda_j}} T(-\lambda_j, \tau) \right] + \frac{\omega}{2} \sum_{i=0}^{M} C_i X_i(\tau) \tag{4.8}
\]

Tables (4-4) - (4-6) present numerical results for the homogeneous boundary conditions. Performance of the profile consisting only of the complementary solution approximation is similar to that for the first case. Convergence is smooth for $\tau_o = 1$, and the oscillations again appear at larger optical depths, although the scaled amplitude is slightly smaller. The appearance of the convergence oscillations for two cases of
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Table 4.4 Incident radiation and exit heat flux at $\tau = \tau_0$ for a medium with a constant internal source $H(\tau) = 1 + \tau$, $\tau_0 = 1$. 
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Table 4.5 Incident radiation and exit heat flux at \( \tau = \tau_0 \) for a medium with a constant internal source \( H(\tau) = 1 + \tau, \ \tau_0 = 5 \).
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Table 4.6 Incident radiation and exit heat flux at \( \tau = \tau_o \) for a medium with a constant internal source \( H(\tau) = 1 + \tau \), \( \tau_o = 10 \).
physically different systems confirms the earlier observations that the source of the oscillations lies with the analytical method and not to some physical characteristic of the medium. However, the two cases studied here do share a mathematical trait in that the source distribution function for both is a polynomial in the space variable. If only the complementary solution approximation in the improved-profile is used, it must represent both the homogeneous and the particular solutions to the problem. Since the source function for these two cases is a polynomial, the particular solution is also a polynomial of the same order [9, p. 386]. Unfortunately, an exponential function, such as the complementary solution approximation, is not a very efficient approximation for a polynomial. The oscillations are therefore a by-product of the exponential profile approximating the polynomial particular solution. This is shown by using the improved-profile with the order set to zero. In effect this supplements the exponential profile with a free-constant (i.e. a polynomial of order zero). The improvement is quite dramatic as demonstrated by the numerical results in the fourth column of Tables (4-4) - (4-6). The peak amplitude of the oscillations is decreased by at least 80%, and results for the first order approximation \( (J = 1) \) are within 0.5% - 5% of the converged value, compared to a range of 2% - 15% for the exponential profile alone. Increasing \( M \) to one in the improved profile, the order of the exact particular solution, completely eliminates the oscillations and provides a first order approximation that is within 1% of the converged value. That the incremental improvement produced by increasing \( M \) from zero to one is not as great as the introduction of \( M \) equal to zero is a characteristic of Galerkin's method; low-order Galerkin approximations usually produce quite acceptable results. As the order of the approximation increases, the rate of convergence slows, producing less numerical resolution compared to the added computational effort. Applying this concept to the improved-profile, for source functions represented as a high order polynomial, acceptable results may be achieved with a low order \( M \). Even if the order of the source function is
known \textit{a priori}, if the order is large, setting $M$ to this value may not produce results that are worth the large computational effort compared to the expenditure and result for $M$ equal one or two. It may be noted from the numerical results presented in the tables that the convergence oscillations could be eliminated if the order $J$ is restricted to odd values. The reason for this may lie in the origin of the complementary approximation in the improved-profile. Thynell and Özisik derived the exponential series from a solution by the spherical harmonics method [15] which typically has an order restricted to odd number values when applied to the solution of the equation of radiative transfer (this is due to singularities which develop at the boundaries for the intensity of radiation when even orders are used). Thus in an analogous sense, the order $J$ of the improved profile works best when it too is restricted to odd values.

The effect of the boundary conditions on the improved-profile's performance is examined by setting $f_1$ to 1.0. This represents the case of plane-parallel slab with transparent boundaries, subjected to isotropic irradiation of unit intensity at the boundary surface $\tau = 0$. The convergence of values for incident radiation and exit heat fluxes at the far surface $\tau = \tau_o$ are presented in Tables (4-7) - (4-9) for $H(\tau) = 1/\tau_o$, and Tables (4-10) - (4-12) for $H(\tau) = 1 + \tau$. The numerical results tend to indicate little or no effect on rate of convergence, accuracy, nor the amplitude of the oscillations when using only the complementary component of the profile. Additional computations for other boundary conditions, such as gray boundaries and larger values of $f_1$, also demonstrate little change in performance compared to the homogeneous condition. These results show that the oscillation phenomenon is attributable to the numerical process of the method itself. Any effect from the implicit inclusion of the boundary conditions in the application of the orthogonality condition Eq. (3.11), is offset by their explicit inclusion in the expressions for the various radiative properties of the medium. Physically, the boundary conditions affect the strength and distribution of radiative energy within the
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|---|---|---|---|---|---|---|---|

Table 4.7 Incident radiation and exit heat flux at \( τ = τ₀ \) for a medium with a constant internal source \( H(τ) = \frac{1}{τ₀} \), \( τ₀ = 1 \), and external radiation \( F_1(μ) = 1 \) at \( τ = 0 \).
| J | \( \omega \) | G(\( \tau_o \)) | \( q^+(\tau_o) \) |
|---|---|---|---|---|---|---|
|   |   | Complementary Soln Only | M = 0 | M = 1 | Complementary Soln Only | M = 0 | M = 1 |
| 1 |   | 1.3887 | 1.3663 | 1.3659 | 0.686563 | 0.688042 | 0.688025 |
| 2 |   | 1.3563 | 1.3657 | 1.3655 | 0.687735 | 0.687996 | 0.687987 |
| 3 | 0.1 | 1.3675 | 1.3655 | 1.3654 | 0.688054 | 0.687987 | 0.687983 |
| 4 |   | 1.3652 | 1.3655 | 1.3654 | 0.687977 | 0.687984 | 0.687982 |
| 5 |   | 1.3654 | 1.3654 | 1.3654 | 0.687983 | 0.687983 | 0.687982 |
| 6 |   | 1.3654 | 1.3654 | 1.3654 | 0.687982 | 0.687982 | 0.687982 |
| 7 |   | 1.3654 | 1.3654 | 1.3654 | 0.687982 | 0.687982 | 0.687982 |
| 8 |   | 1.3654 | 1.3654 | 1.3654 | 0.687982 | 0.687982 | 0.687982 |
|   | 1   | 3.3417 | 3.0040 | 2.9965 | 1.576726 | 1.571383 | 1.570844 |
| 2 |   | 2.8890 | 2.9886 | 2.9873 | 1.567149 | 1.570243 | 1.570193 |
| 3 | 0.7 | 2.9860 | 2.9856 | 2.9852 | 1.570576 | 1.570145 | 1.570134 |
| 4 |   | 2.9838 | 2.9853 | 2.9851 | 1.570097 | 1.570125 | 1.570122 |
| 5 |   | 2.9851 | 2.9850 | 2.9849 | 1.570121 | 1.570120 | 1.570118 |
| 6 |   | 2.9848 | 2.9849 | 2.9848 | 1.570117 | 1.570118 | 1.570118 |
| 7 |   | 2.9848 | 2.9848 | 2.9848 | 1.570117 | 1.570117 | 1.570117 |
| 8 |   | 2.9848 | 2.9848 | 2.9848 | 1.570117 | 1.570117 | 1.570117 |
|   | 1   | 6.5163 | 5.8183 | 5.8089 | 3.153365 | 3.124327 | 3.123740 |
| 2 |   | 5.7845 | 5.7834 | 5.7827 | 3.119266 | 3.122579 | 3.122549 |
| 3 | 0.9 | 5.8041 | 5.7795 | 5.7791 | 3.122827 | 3.122464 | 3.122459 |
| 4 |   | 5.7763 | 5.7780 | 5.7779 | 3.122425 | 3.122443 | 3.122442 |
| 5 |   | 5.7777 | 5.7774 | 5.7773 | 3.122440 | 3.122437 | 3.122437 |
| 6 |   | 5.7773 | 5.7774 | 5.7771 | 3.122436 | 3.122437 | 3.122435 |
| 7 |   | 5.7772 | 5.7771 | 5.7771 | 3.122435 | 3.122435 | 3.122435 |
| 8 |   | 5.7771 | 5.7770 | 5.7770 | 3.122435 | 3.122434 | 3.122435 |

Table 4.8 Incident radiation and exit heat flux at \( \tau = \tau_o \) for a medium with a constant internal source \( H(\tau) = \frac{1}{\tau_o} \), \( \tau_o = 5 \), and external radiation, \( F_1(\mu) = 1 \), at \( \tau = 0 \).
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Table 4.9 Incident radiation and exit heat flux at \( \tau = \tau_0 \) for a medium with a constant internal source \( H(\tau) = \frac{1}{\tau_0} \), \( \tau_0 = 10 \), and external radiation \( F_1(\mu) = 1 \) at \( \tau = 0 \).
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Table 4.10  Incident radiation and exit heat flux at $\tau = \tau_o$ for a medium with a constant internal source $H(t) = 1 + \tau$, $\tau_o = 1$, and external radiation $F_1(\mu) = 1$ at $\tau = 0$. 
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Table 4.11 Incident radiation and exit heat flux at $\tau = \tau_0$ for a medium with a constant internal source $H(\tau) = 1 + \tau$, $\tau_0 = 5$, and external radiation at $\tau = 0$, $F_1(\mu) = 1$. 
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Table 4.12 Incident radiation and exit heat flux at \( \tau = \tau_0 \) for a medium with a constant internal source \( H(\tau) = 1 + \tau, \tau_0 = 10 \), and external radiation \( F_1(\mu) = 1 \) at \( \tau = 0 \).
medium. Although the existence of boundary conditions influences the magnitude of the solution obtained using Galerkin's method, the computational impact of their existence on the method's performance is nil.

4-3. Limits of the Improved-Profile

When applied to the solution of the equation of radiative transfer, the performance of Galerkin's method is very consistent up to a limit with the profiles currently available. Beyond this limit, the normal convergence pattern suddenly breaks down, and the solution method becomes unstable. Results diverge with increasing order of approximation, and for profiles based purely on exponential functions, the amplitude of the characteristic oscillations increase rather than damp out. The source of the instability lies with the condition of the matrix Eq. (3.11) generated by the orthogonality condition to determine the profile's coefficients. A polynomial profile generates matrix elements that are a function of \( \tau_o^i \), and as \( \tau_o \) and/or \( i \) increases, elements in the lower rows of the matrix can become very large relative to those of the upper rows. When this occurs, the matrix is termed ill-conditioned because its solution involves significant floating-point errors. Eventually, \( \tau_o \) or \( i \) become large enough that the magnitude of some matrix elements exceeds the precision capacity of the CPU. The resulting truncation of these matrix elements is what causes the sudden instability. Using double-precision math, a polynomial profile exhibits instability for order 20 and an optical depth of 15. At optical depth 17, even the first order approximation is unreliable.

Exponential profiles will also eventually generate an ill-conditioned matrix, although the limit is higher than is the case for polynomial profiles. The exponential profile generates matrix elements that are a function of \( \exp ( \pm \tau_o / \lambda_j ) \). The maximum magnitude of the eigenvalues is approximately 1, and their values decrease with
increasing \( \tau_0 \) or \( j \), some matrix elements become relatively large or small, although not as quickly as in the case of a polynomial profile. Truncation begins to occur for order \( J = 20 \) at an optical depth of 21, and the method is completely unstable at optical depth 24. Compensating for the particular solution with a power series does not improve this limit, as it does not affect the values of the matrix elements that are associated with the exponential component of the profile. In fact, if the order \( M \) of the improved-profile is sufficiently large, the limit decreases and approaches that of the polynomial profile. However, compensating for the effects of the particular solution does enhance convergence, allowing accurate results to be obtained before truncation occurs.

In the previous section of this chapter, it was demonstrated that compensating for the effects of the particular solution damped out the oscillations characteristic of a purely exponential profile. The source of the oscillations again is the matrix generated by the orthogonality condition. The exponential profile generates both large and small matrix elements, producing larger floating-point errors during the solution of the matrix than for the case of a polynomial profile. From Table 4-13, the profile consisting only of the complementary solution approximation generates large coefficients, making any floating-point errors significant enough to be manifested as the oscillating pattern of convergence. However, the coefficients of the improved profile are smaller by at least an order of magnitude. The difference is enough to make the floating-point errors sufficiently insignificant that the oscillations appear to be damped out.
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Table 4.13 Coefficients of the improved-profile for $\tau_o = 2$, $\omega = 0.1$, and order $J = 10$. 
NOTES

1 In order to compute the coefficients for the improved profile, analytical expressions for \( d(\eta) \) and \( d_1 \) must be derived. For \( H = 1/\tau_0 \) and the boundary conditions in Eqs. (4.1b) and (4.1c), \( d(\eta) \) may be expressed as

\[
d(\eta) = \int_0^\tau_0 Y(\tau) e^{-\frac{\tau}{\tau_0}} d\tau
\]

\[
= 2\pi \left\{ \int_0^{\tau_0} \int_0^{\tau_0} \frac{1}{\tau_0} E_1(\tau - \tau') e^{-\frac{\tau}{\tau_0}} d\tau d\tau' + \int_0^{\tau_0} \int_0^1 f_1(\mu) e^{-\frac{\mu}{\tau_0}} e^{-\frac{\tau}{\tau_0}} d\mu d\tau \right\}
\]

Using Eqs. (4.41), (4.42), and (4.45) in Ref. [22], the above expression is reduced to

\[
d(\eta) = \frac{2\pi \eta}{\tau_0} \int_0^{\tau_0} \left\{ D(\eta, \tau) - C(\eta, \tau) + E_1(\tau) e^{-\frac{\tau}{\tau_0}} E_1(\tau_0 - \tau) \right\} d\tau
\]

\[
+ 2\pi f_1 \int_0^{\tau_0} e^{-\frac{\tau}{\tau_0}} E_2(\tau) d\tau
\]

Expressions for the variables \( C \) and \( D \) are given in Appendix A. The first integral in the above equation is reduced to analytical form using Eq. A (21) of Ref. [15], and the integral involving \( f_1 \) is evaluated using Eqs. (2.32) and (2.34) in Ref. [22] yielding

\[
d(\eta) = \frac{2\pi}{\tau_0} \left\{ 2\eta \left[ 1 - e^{-\frac{\tau_0}{\tau_0}} \right] - \psi(\eta, 0) - \phi(\eta, \tau_0) \right\}
\]

\[
+ 2\pi f_1 \left\{ -\eta \left[ e^{-\frac{\tau_0}{\tau_0}} E_2(\tau_0) - 1 \right] + \eta^2 \left[ e^{-\frac{\tau_0}{\tau_0}} E_1(\tau_0) - D(\eta, 0) \right] \right\}
\]

\[
= \frac{2\pi}{\tau_0} \left\{ 2\eta \left[ 1 - e^{-\frac{\tau_0}{\tau_0}} \right] - \psi(\eta, 0) - \phi(\eta, \tau_0) \right\} + 2\pi f_1 \psi(\eta, 0)
\]
Expressions for the variables \( \psi \) and \( \phi \) are provided in Appendix A. The variable \( d_i \) is

\[
d_i = \int_0^{\tau_o} Y(\tau) \tau^i \, d\tau
\]

\[
= 2\pi \left\{ \int_0^{\tau_o} \int_0^{\tau_o} \tau^i E_1(\tau') \, d\tau' \, d\tau + \int_0^{\tau_o} \int_0^{\tau_o} f_1(\mu) e^{-\mu \tau} \tau^i \, d\mu \, d\tau \right\}
\]

The above expression is reduced by substituting an analytical expression from Eq. (4.1) in Ref [22] for the first integral's interior integration, yielding

\[
d_i = 2\pi \frac{1}{\tau_o} \left\{ (-1)^{i+1} i! \int_0^{\tau_o} E_{i+2}(\tau') \, d\tau' + \sum_{k=0}^{i} \frac{i!}{(m-k)!} \left[ \frac{1 + (-1)^k}{1 + k} \right] \int_0^{\tau_o} \tau^{i-k} \, d\tau' - \tau_0^{i-k} \int_0^{\tau_o} E_{k+2}(\tau_0 - \tau') \, d\tau' \right\} + f_1 \int_0^{\tau_o} E_2(\tau) \tau^i \, d\mu \, d\tau
\]

Where the exponential integral notation has been introduced for the integral involving \( f_1 \), assuming that the external radiation is a constant. Analytical expressions for the integrals involving exponential integrals are obtained from Eq. (1.1) in Ref [22], yielding the final form for \( d_i \)

\[
d_i = 2\pi \frac{1}{\tau_o} i! \left\{ (-1)^{i+1} \left[ \frac{1}{i+2} - E_{i+3}(\tau_o) \right] + \sum_{k=0}^{i} \frac{1}{(i-k)!} \left[ \frac{(1 + (-1)^k \tau_0^{i-k+1}}{(i + k)} \right] \left( \tau_0^{i-k} \left[ \frac{1}{k+2} - E_{k+3}(\tau_o) \right] \right) \right\} + 2\pi f_1 \left( \frac{1}{i+2} - \sum_{k=0}^{i} \frac{\tau_o^{i-k}}{(i-k)!} E_{k+3}(\tau_o) \right)
\]
For the analysis presented in this chapter, the $d_1$ corresponding to order $M$ equal zero and one are

$$d_0 = \frac{2\pi}{\tau_o} \left\{ 2E_3(\tau_o) - 1 + 2\tau_o \right\} + 2\pi f_1 \left\{ \frac{1}{2} - E_3(\tau_o) \right\}$$

$$d_1 = \frac{2\pi}{\tau_o} \left\{ \tau_o^2 - \tau_o \left( \frac{1}{2} - E_3(\tau_o) \right) \right\} + 2\pi f_1 \left\{ \frac{1}{3} - \tau_o E_3(\tau_o) - E_4(\tau_o) \right\}$$

For transparent walls, constant external radiation at the boundary $\tau = 0$, and $H(\tau) = 1/\tau_o$ Eqs. (3.18a) and (3.18b) reduce to

$$q^+(\tau_o) = 2\pi \int_0^{\tau_o} \frac{1}{\tau_o} E_2(\tau_o - \tau') d\tau' + 2\pi f_1 \int_0^{1/\mu} e^{-\tau_o/\mu} d\mu$$

$$+ \frac{\omega}{2} \left[ \sum_{j=0}^{J} A_j \Phi(\lambda_j, \tau_o) + B_j e^{-\lambda_j/\tau_o} \Phi(-\lambda_j, \tau_o) + \sum_{i=0}^{M} C_i Y_i \right]$$

$$q^-(0) = 2\pi \int_0^{\tau_o} \frac{1}{\tau_o} E_2(\tau') d\tau' + \frac{\omega}{2} \left[ \sum_{j=1}^{J} A_j \varphi(\lambda_j, 0) + B_j e^{-\lambda_j/\tau_o} \varphi(-\lambda_j, 0) \right]$$

$$+ \frac{\omega}{2} \sum_{i=0}^{M} C_i Z_i$$

An analytical expression for the first integral in each equation is obtained from Eq. (1.15) of Ref [22] yielding Eqs. (4.3a) and (4.3b). Note that the integral associated with $f_1$ in the expression for $q^+(\tau_o)$ is $E_3(\tau_o)$. 
For transparent walls, constant external radiation at the boundary \( \tau = 0 \), and \( H(\tau) = 1/\tau_0 \), Eq. (3.21) reduces to

\[
G(\tau) = 2\pi \int_0^{\tau_0} \frac{1}{\tau_0} E_1(|\tau - \tau'|) \, d\tau' + 2\pi \int_0^1 e^{-\frac{\tau_0}{\mu}} \, d\mu
\]

\[
+ \frac{\omega}{2} \sum_{j=1}^J \left\{ A_j T(\lambda_j, \tau) + Be^{-\lambda_j \tau} T(-\lambda_j, \tau) \right\} + \frac{\omega}{2} \sum_{i=0}^M C_i \int_0^{\tau_0} \tau \int_0^1 E_1(|\tau - \tau'|) \, d\tau'
\]

Where the variable \( T(\eta, \tau) \) is defined as

\[
T(\eta, \tau) = \int_0^{\tau_0} e^{-\frac{\tau'}{\eta}} E_1(|\tau - \tau'|) \, d\tau'
\]

Analytical expressions for \( T \) and \( X_i \) are provided in Appendix A. An expression for the first integral in the above expression for \( G \) is obtained from Eq. (4.1.1) in Ref [22]. Note that the integral associated with \( f_1 \) is \( E_2(\tau) \).

In order to compute the coefficients for the improved profile, analytical expressions for \( d(\eta) \) and \( d_1 \) must be derived. For \( H = 1 + \tau \) and the boundary conditions in Eqs. (4.1b) and (4.1c), \( d(\eta) \) may be expressed as

\[
d(\eta) = \int_0^{\tau_0} \left\{ \int_0^{\tau} e^{-\frac{\tau'}{\eta}} \, d\tau' \right\} = 2\pi \left\{ \int_0^{\tau_0} (1 + \tau') e^{-\frac{\tau'}{\eta}} E_1(|\tau - \tau'|) \, d\tau' \, d\tau + \int_0^{\tau_0} \int_0^1 f_1(\mu) e^{-\frac{\tau_0}{\mu}} e^{-\frac{\tau}{\eta}} \, d\mu \, d\tau \right\}
\]
The first integral can be reduced using Eqs. (4.41), (4.42), and (4.45) in Ref [22] by first interchanging the variables $\tau$ and $\tau'$. The integral associated with $f_1$ is simplified through the use of the exponential integral notation. The reduced expression is

$$
\begin{align*}
d(\eta) &= 2\pi \left\{ \eta \int_0^{\tau_o} (1 + \tau) [D(\eta, \tau) - C(\eta, \tau) + E_1(\tau, - E_1(\tau_o - \tau))] d\tau \\
&\quad + f_1 \int_0^{\tau_o} e^{-\frac{\tau}{\eta}} E_2(\tau) d\tau \right\}
\end{align*}
$$

An analytical expression for the first integral is obtained using Formulas (A 21) and (A 22) of Ref. [15]. The integral associated with $f_1$ is evaluated using Eqs. (2.32) and (2.34) of Ref. [22] yielding

$$
\begin{align*}
d(\eta) &= 2\pi \left\{ 2\eta \left[ 1 - e^{-\frac{\tau_o}{\eta}} \right] - \psi(\eta, 0) - \Phi(\eta, \tau_o) + 2 \left[ \eta^2 - (\eta \tau_o + \eta^2 e^{-\frac{\tau_o}{\eta}}) \right] \\
&\quad - \tau_o \Phi(\eta, \tau_o) + \eta \left[ \frac{1}{2} - e^{-\frac{\tau_o}{\eta}} E_3(\tau_o) \right] - \eta \psi(\eta, 0) \\
&\quad + e^{-\frac{\tau_o}{\eta}} \left[ \frac{\eta}{2} - \eta e^{-\frac{\tau_o}{\eta}} E_3(\eta) - \eta \psi(\eta, 0) \right] \\
&\quad + f_1 \left[ -\eta e^{-\frac{\tau_o}{\eta}} E_2(\tau_o) + \eta + \eta^2 e^{-\frac{\tau_o}{\eta}} E_1(\tau_o) - \eta^2 D(\eta, 0) \right] \right\}
\end{align*}
$$

This expression is simplified by introducing the variable notation of Appendix A.

$$
\begin{align*}
d(\eta) &= 2\pi \left\{ 2\eta \left[ 1 - e^{-\frac{\tau_o}{\eta}} \right] - \psi(\eta, 0) - \Phi(\eta, \tau_o) + T_1(\eta) + f_1 \psi(\eta, 0) \right\}
\end{align*}
$$
The variable $d_i$ is expressed as

\[ d_i = \int_0^{\tau_o} Y(\tau) \tau^i \, d\tau \]

\[ = 2\pi \left\{ \int_0^{\tau_o} \int_0^{\tau_o} (1 + \tau') \tau^i E_1(|\tau' - \tau|) \, d\tau' \, d\tau + \int_0^{\tau_o} \int_0^{\tau_o} f_1(\mu) e^{-\mu/t^i} \tau^i \, d\mu \, d\tau \right\} \]

The first integrand is separated into two integrals and the variables $\tau$ and $\tau'$ are interchanged in the integrand of the first of the two integrals. The integral associated with $f_1$ is simplified using the exponential integral notation. The resulting expression is

\[ d_i = 2\pi \left\{ \int_0^{\tau_o} \tau^i E_1(|\tau' - \tau|) \, d\tau' \, d\tau + \int_0^{\tau_o} \tau^i \int_0^{\tau_o} E_1(|\tau - \tau'|) \, d\tau' \, d\tau + f_1 \int_0^{\tau_o} \tau^i E_2(\tau) \, d\tau \right\} \]

The inner integrals of the first two terms and the third integral are reduced using Eqs. (4.1), (4.1.2), and (1.1) respectively of Ref. [22] to yield

\[ d_i = 2\pi \left\{ (-1)^{i+1} i! \int_0^{\tau_o} E_{i+2}(\tau') \, d\tau' + \sum_{k=0}^{i} \frac{i!}{(m-k)!} \left[ \frac{(1 + (-1)^k)}{1 + k} \int_0^{\tau_o} \tau^{i-k} \, d\tau' \right] \right\} \]

\[ - \tau_o^{i-k} \int_0^{\tau_o} E_{k+2}(\tau_o - \tau') \, d\tau' \]

\[ + \tau_o^i \left[ 2\tau + E_3(\tau) - \tau_o E_2(\tau_o - \tau) - E_3(\tau_o - \tau) \right] \, d\tau \]

\[ + f_1 i! \left\{ \frac{1}{i+2} - \sum_{k=0}^{i} \frac{\tau_o^{i-k}}{(i-k)!} E_{k+3}(\tau_o) \right\} \]
Finally, the integrals involving the exponential integrals are evaluated using Eq. (1.1) of Ref. [22] resulting in the expression

\[
d_i = 2\pi i! \left[ (-1)^{i+1} \left( \frac{1}{i+2} - E_{i+3}(\tau_o) \right) \right] + \sum_{k=0}^{i} \frac{1}{(i-k)!} \left[ \frac{1}{(1+k)(i-k+1)} \left( 1 + (-1)^k \right) \frac{\tau_o^{i-k+1}}{\tau_o} \right]
\]

\[
- \tau_o^{i-k} \left( \frac{1}{k+2} - E_{k+3}(\tau_o) \right) \right] + \frac{2}{i+2} \tau_o + \frac{i+3}{i+3}
\]

\[
- \sum_{k=0}^{i} \frac{k!}{(i-k)!} \tau_o^{i-k} E_{4+k}(\tau_o) - i! \tau_o \left[ (-1)^{i+1} E_{i+3}(\tau_o) - \sum_{k=0}^{i} \tau_o^{i-k} \frac{(-1)^{k+1}}{k+2} \right]
\]

\[
- i! \left[ (-1)^{i+1} E_{i+4}(\tau_o) - \sum_{k=0}^{i} \frac{\tau_o^{i-k}}{(i-k)!} \frac{(-1)^{k+1}}{k+3} \right]
\]

\[
+ f_1 i! \left\{ \frac{1}{i+2} - \sum_{k=0}^{i} \frac{\tau_o^{i-k}}{(i-k)!} E_{k+3}(\tau_o) \right\}
\]

5 For transparent walls, constant external radiation at the boundary \( \tau = 0 \), and \( H(\tau) = 1 + \tau \), Eqs. (3.18a) and (3.18b) reduce to

\[
q^+(\tau_o) = 2\pi \int_0^{\tau_o} (1 + \tau') E_2(\tau_o - \tau') d\tau' + 2\pi f_1 \int_0^{1} e^{-\frac{\tau_o}{\mu}} \mu d\mu
\]

\[
+ \frac{\omega}{2} \left[ \sum_{j=1}^{I} \left\{ A_j \Phi(\lambda_j, \tau_o) + B_j e^{-\frac{\tau_o}{\lambda_j}} \Phi(-\lambda_j, \tau_o) \right\} + \sum_{i=0}^{M} C_i Y_i \right]
\]

\[
q^-(0) = 2\pi \int_0^{\tau_o} (1 + \tau') E_2(\tau') d\tau' + \frac{\omega}{2} \sum_{j=1}^{I} \left\{ A_j \psi(\lambda_j, 0) + B_j e^{-\frac{\tau_o}{\lambda_j}} \psi(-\lambda_j, 0) \right\}
\]

\[
+ \frac{\omega}{2} \sum_{i=0}^{M} C_i Z_i
\]
Eqs. (1.1.5) and (1.1.6) of Ref [22] yielding Eqs. (4.3a) and (4.3b). Note that the integral associated with \( f_1 \) in the expression for \( q^+ (\tau_0) \) is \( E_3 (\tau_0) \).

6 For transparent walls, constant external radiation at the boundary \( \tau = 0 \), and \( H (\tau) = 1 + \tau \), Eq. (3.21) reduces to

\[
G (\tau) = 2\pi \int_0^{\tau} (1 + \tau') E_1 (|\tau - \tau'|) + 2\pi f_1 \int_0^1 e^{-\frac{\mu}{2}} d\mu \\
+ \frac{\omega}{2} \sum_{j=1}^{J} \left\{ A_j T(\lambda_j, \tau) + B_j e^{-\frac{\tau_0}{\lambda_j}} T(\lambda_j, \tau) \right\} + \frac{\omega}{2} \sum_{i=0}^{M} C_i \int_0^{\tau} e^{-\frac{\tau'}{H_i}} E_1 (|\tau - \tau'|) d\tau'
\]

Where the variable \( T(\eta, \tau) \) is defined as

\[
T(\eta, \tau) = \int_0^{\tau} e^{-\frac{\tau'}{H}} E_1 (|\tau - \tau'|) d\tau'
\]

An analytic expression for \( T \) is provided in the Appendix A. An analytical expression for the first integral in the above expression for \( G \) is obtained from Eqs. (4.1.1) and (4.1.2) in Ref [22]. The integral associated with \( C_i \) is \( X_i \), for which an analytical expression is also provided in Appendix A. Note that the integral associated with \( f_1 \) is \( E_2 (\tau) \).
CHAPTER 5

PERFORMANCE ANALYSIS
OF THE
LIQUID DROPLET RADIATOR

5.1 Introduction

Under ideal conditions, the LDR will operate in an environment in which both radiating faces are exposed to black space. Previous studies of the LDR have been based on this premise. The true operating environment for the LDR, low-earth orbit, will subject the droplet sheet to a variety of diffuse and specular sources of external radiation. Only one study [23] has attempted to examine LDR performance in this environment using lumped parameters and a finite element analysis. In this chapter the transient cooling of the LDR droplet sheet is examined with a more exact method in order to provide a more precise evaluation of the LDR's performance. Instead of using Eq. (2.6), the transient form of the equation of radiative transfer, the energy equation will be introduced as the governing equation. Since the LDR droplet layer is comprised of discrete droplets, conduction is disregarded. The velocity profile for the droplet streams is uniform, and so the only change in the internal energy of the droplets will be due to radiative cooling, expressed in the energy equation as radiative flux. The analysis will be presented in dimensionless form, with parameters defined so as to allow comparison with Siegel's study [8] of the homogeneous case using an iterative approach. The computational scheme for the instantaneous flux will be presented next, based on the solution of the equation of radiative transfer by Galerkin's method with the improved-profile. Since the temperature distribution within the layer is nonuniform, several
numerical methods are introduced to complete the solution. From this solution, instantaneous temperature profiles, exit heat fluxes, and mean temperatures are presented for an LDR exposed on one side to radiation of constant intensity with normal incidence. Normally-incident, external radiation represents the most severe operating condition, thus leading to a more conservative analysis of the LDR's performance. The strength of the external radiation is varied from the limiting case of the homogeneous condition to that of a non-dimensional value of 1.0. This upper limit represents a magnitude of external radiation equal to the initial emissive power of the droplets. Earlier lumped analyses of the LDR [4,6] have used a ratio of cooling power to total system mass to quantify an LDR's performance. These studies indicate an optimal optical depth in the range of 1.0 to 2.0 with a significant drop-off in performance at optical depths greater than 5.0, due to the large mass of the hardware necessary to generate such a thick LDR sheet. Therefore, this study presents results for optical depths of 1 and 2, with an optical depth of 5.0 as the upper bound for the analysis presented in this chapter.

5.2 Derivation of the Governing Equations

The droplet layer of constant depth \( L \), illustrated in Fig. (5.1), exits from the droplet generator at \( z = 0 \) and travels in the \( z \) direction at a uniform velocity \( u \). The extent of the layer in the \( z \) direction (for large \( u \)) and normal to the \( x \)-\( z \) plane is large relative to \( L \), and so a one-dimensional approximation is used since the temperature variations in those directions are negligible relative to that in \( x \). The initial temperature (at \( z = 0 \)) is uniform at \( T_{i1} \), and changes with \( x \) and \( z \) as it exchanges thermal energy with the surrounding environment. The environment is taken as black space on the boundary \( x = L \), with isotropic, normal radiation incident on the layer at \( x = 0 \). Since the layer is composed of discrete droplets, there is no dissipative work due to viscous forces, nor
Figure 5.1 Geometry of the liquid droplet sheet
heat transfer by conduction or convection. Thus any change in the internal energy of the layer is due solely to radiative heat transfer. Assuming that the thermal internal energy of the layer is a function of temperature only, the cooling (or heating) of the layer is governed by the one-dimensional energy equation

\[ \rho c_p \frac{dT}{dt} = -\frac{dq_r}{dx} ; \quad T(x,0) = T_i \] (5.1)

The non-dimensional form of the energy equation is obtained by rewriting \( x \) as an optical depth and dividing both sides of Eq. (5.1) by the initial emissive power of the droplet layer yielding \( ^1 \)

\[ \frac{d\theta}{dt} = -\frac{d\bar{q}_r}{d\tau} ; \quad \theta(\tau,0) = 1 \] (5.2)

the gradient of the net radiative heat flux on the right-hand-side of Eq. (5.1) is \([p. 255, 9]\)

\[ \frac{dq_r}{dx} = \nabla \cdot q_r = \int_{4\pi} \nabla \cdot (I(\tau,\mu)\Omega) \, d\Omega \] (5.3)

The RHS of Eq. (5.3) is evaluated by integrating the equation of radiative transfer, with the result \( ^2 \)

\[ \frac{dq_r}{dx} = \kappa \left[ 4\sigma T^4(x) - G(x) \right] \]

or in non-dimensional form \( ^3 \)

\[ \frac{d\bar{q}_r}{d\tau} = (1 - \omega) \left[ 4\theta^4(\tau) - \bar{G}(\tau) \right] \] (5.4)
Substitution of Eq. (5.4) into Eq. (5.2) yields

\[
\frac{d\theta}{d\tilde{t}} = -(1 - \omega) \left[4\theta^4(\tau) - \bar{G}(\tau)\right]; \quad \theta(\tau, 0) = 1 \tag{5.5}
\]

An expression for the incident radiation \( \bar{G}(\tau) \) is derived from the solution of the instantaneous equation of radiative transfer Eq. (2.11a). For the operating environment described earlier in this section, Eq. (2.11) may be expressed in non-dimensional form as

\[
\mu \frac{d\bar{I}(\tau, \mu)}{d\tau} + \bar{I}(\tau, \mu) = (1 - \omega) \frac{\theta^4(\tau)}{\pi} + \frac{\omega}{2} \int_{-1}^{1} \bar{I}(\tau, \mu) d\mu \tag{5.6a}
\]

\[
\bar{I}(0, \mu) = F_1 \tag{5.6b}
\]

\[
\bar{I}(\tau_o, -\mu) = 0 \tag{5.6c}
\]

The solution of Eq. (5.6) using Galerkin's method with the improved-profile follows from Eq. (3.21) as

\[
\bar{G}(\tau) = 2\pi \int_0^{\tau_o} (1 - \omega) \theta^4(\tau) E_1(l\tau - \tau') d\tau' + 2\pi F_1 E_2(\tau)
\]

\[
+ \frac{\omega}{2} \left\{ \sum_{j=1}^{J} \left[ A_j T(\lambda_j, \tau) + B_j e^{-\frac{\tau_o}{\lambda_j}} T(-\lambda_j, \tau) \right] + \sum_{i=0}^{M} C_i X_i(\tau) \right\} \tag{5.7}
\]
Substitution of Eq. (5.5) into Eq. (5.5) yields the final analytical form of the governing equation

\[
\frac{d\theta (\tau)}{d\tau} = -(1 - \omega) \{ 4\theta^4(\tau) - 2\pi \int_0^{\tau_o} (1 - \omega)\theta^4(\tau') E_1(|\tau - \tau'|) d\tau' - 2\pi F_1 E_3(\tau_o) \\
+ \frac{\omega}{2} \left[ \sum_{j=1}^J [A_j T(\lambda_j, \tau) + B_j e^{\frac{\tau_o}{\lambda_j}} T(-\lambda_j, \tau)] + \sum_{i=0}^M C_i X_i(\tau) \right] \} \quad (5.8)
\]

Relations for the exit heat fluxes at \( \bar{\tau} \) follow from Eqs. (3.18a) and (3.18b) as

\[
\bar{q}^+(\tau_o) = 2\pi \int_0^{\tau_o} (1 - \omega)\theta^4(\tau) E_2(\tau_o - \tau) d\tau + 2\pi F_1 E_2(\tau_o) \\
+ \frac{\omega}{2} \left\{ \sum_{j=1}^J \left[ A_j \Phi(\lambda_j, \tau_o) + B_j e^{\frac{\tau_o}{\lambda_j}} \Phi(-\lambda_j, \tau_o) \right] + \sum_{i=0}^M C_i Y_i \right\} \quad (5.9a)
\]

\[
\bar{q}^-(0) = 2\pi \int_0^{\tau_o} (1 - \omega)\theta^4(\tau) E_2(\tau) d\tau + \frac{\omega}{2} \sum_{j=1}^J \left[ A_j \psi(\lambda_j, 0) + B_j e^{\frac{\tau_o}{\lambda_j}} \psi(-\lambda_j, 0) \right] \\
+ \frac{\omega}{2} \sum_{i=0}^M C_i Z_i \quad (5.9b)
\]

The remaining parameter of interest is the mean temperature of the droplet layer at \( \bar{\tau} \) which is expressed as

\[
\bar{\theta}(\tau, \bar{\tau}) = \int_0^{\tau_o} \theta(\tau, \bar{\tau}) d\tau \quad (5.10)
\]
As a final note, caution must be exercised in the application of $F_1$ to the equations presented in this chapter. The gray assumption applied to the droplet layer in this study uses radiative property values that are integrated over the entire thermal bandwidth. However, the external radiation could have a narrower bandwidth, as is the case for solar radiation. The gray properties of the droplet layer integrated over this narrower bandwidth could be different than that for the entire thermal range. If the gray properties over each bandwidth are different, the true magnitude of the external radiation $F_1$ should be a scaled value, such as

$$F_1 = F^* \frac{\omega}{\omega^*}$$

Where $F^*$ is the mean magnitude of the external radiation and $\omega^*$ the gray albedo of the droplet layer, both integrated over $F^*$'s bandwidth.

5.3 **Numerical Solution of the Governing Equations**

The first step in solving Eq. (5.8) is to discretize the time domain. Using a forward-difference representation, the discretized form of Eq. (5.8) is

$$\theta(\tau, \bar{\tau} + \Delta \bar{\tau}) = \theta(\tau, \bar{\tau}) - (1 - w) \Delta \bar{\tau} \left[ 4 \theta^4(\tau, \bar{\tau}) - 2\pi \int_0^{\tau_o} (1 - \omega) \theta^4(\tau', \bar{\tau}) E_1(\lambda - \tau') \, d\tau' \right. $$

$$ \left. - 2\pi F_1 E_2(\tau_o) + \frac{\omega}{2} \left\{ \sum_{j=1}^{I} \left[ A_j T(\lambda_j, \tau) + B_j e^{\frac{\tau_o}{\lambda_j}} T(-\lambda_j, \tau) \right] \right\} + \frac{\omega}{2} \sum_{i=0}^{M} C_i X_i(\tau) \right] \right) \right] \tag{5.11}$$
The use of a discretized time domain results in values for θ expressed as a
distribution of values rather than an analytical expression. This necessitates use of a
numerical method to evaluate the integral of Eq. (5.11) involving θ. Gaussian quadrature
is the most accurate method, employing unequally-spaced, weighted intervals or
quadrature nodes. For Gaussian quadrature to be applicable, the function being integrated
must have values available at all of the nodes. Applying Gaussian quadrature to the
solution of the integral in Eq. (5.11) yields the discretized expression

\[ θ(τ_k, t + Δt) = θ(τ_k, t) - (1 - ω) Δt \left\{ 4 θ^4(τ_k, t) \right. \]

\[ - 2π(1 - ω) \sum_{l=1}^{N_P} θ^4(τ_1, t) E_1(\|τ_k - τ_1\|) w_1 - 2πF_1 E_2(τ_o) \]

\[ + \frac{ω}{2} \sum_{j=1}^{J} [A_j T(λ_j, τ_k) + B_j e^{-\frac{λ_j}{τ_o}} T(-λ_j, τ_k)] \]

\[ + \frac{ω}{2} \sum_{i=0}^{M} C_i X_i(τ_k) \} \]  

(5.12)

Where the \( w_1 \) are the weights at each node.

However, a problem arises in Eq. (5.12) when \( i = k \), as the argument of \( E_1 \)
equals zero (\( E_1(0) = \infty \)), resulting in the function being undefined at the node \( i = k \),
vio\( lating the requirement discussed earlier. In order to subtract out this singularity, the
source term in Eq. (5.8) is expressed as

\[ \int_0^{τ_o} θ^4(τ') E_1(\|τ - τ'\|) dτ' = θ^4(τ)[2 - E_2(τ) - E_2(τ_o - τ)] \]

\[ + \int_0^{τ_o} [θ^4(τ') - θ^4(τ)] E_1(\|τ - τ'\|) dτ' \]  

(5.13)
Discretizing Eq. (5.13) and substituting the result for the source term in Eq. (5.12) yields

\[
\theta (\tau_k, \bar{\tau} + \Delta \bar{\tau}) = \theta (\tau_k, \bar{\tau}) - (1 - \omega)\Delta \bar{\tau} \left[ 4\theta^4(\tau_k, \bar{\tau}) - 2\pi F_1 E_2(\tau_o) - 2\pi \theta^4(\tau_k, \bar{\tau}) (1 - \omega) \left[ 2 - E_2(\tau_k) - E_2(\tau_o - \tau_k) \right] \right.

+ 2\pi (1 - \omega) \sum_{l=1}^{NP} \left[ \theta^4(\tau_l, \bar{\tau}) - \theta^4(\tau_k, \bar{\tau}) \right] w_l E_1(l|\tau_k - \tau_l|)

+ \frac{\omega}{2} \sum_{j=1}^J \left[ A_j T(\lambda_j, \tau_k) + B_j \exp \frac{-\tau_o}{\lambda_j} T(-\lambda_j, \tau_k) \right]

+ \frac{\omega}{2} \sum_{i=0}^M C_i X_i(\tau_k) \right] \]  

(5.14)

When \( l = k \), although \( E_1(l|\tau_k - \tau_l|) \) is undefined, it is nullified by the term involving \( \theta^4 \) which is zero, and thus the singularity is removed by disregarding the node \( l = k \). Fortunately, Gaussian quadrature does not set nodes exactly on either boundary and so the relations for the exit heat fluxes Eqs. (5.9a) and (5.9b) produce no singularities and

\[
\bar{q}(\tau_o, \bar{\tau}) = 2\pi (1 - \omega) \sum_{l=1}^{NP} \theta^4(\tau_l, \bar{\tau}) E_2(\tau_o - \tau_l) w_l + 2\pi F_1 E_3(\tau_o)

\frac{\omega}{2} \sum_{j=1}^J \left[ A_j \Phi(\lambda_j, \tau_o) + B_j \exp \frac{-\tau_o}{\lambda_j} \Phi(-\lambda_j, \tau_o) \right]

\frac{\omega}{2} \sum_{i=0}^M C_i Y_i \]  

(5.15a)
and

\[ \bar{q}^-(0, \bar{t}) = 2\pi(1 - \omega) \sum_{l=1}^{NP} \theta^l(\tau_1, \bar{t}) E_2(\tau_1) w_1 + \frac{\omega}{2} \sum_{j=1}^{I} \left[ A_j \psi(\lambda_j, 0) + B_j e^{-\tau_0 \lambda_j} \psi(-\lambda_j, 0) \right] + \frac{\omega}{2} \sum_{i=0}^{M} C_i Z_i \] (5.15b)

follow as the discretized expressions, with the mean temperature Eq. (5.10) becoming

\[ \overline{\theta}(\tau_k, \bar{t}) = \sum_{l=1}^{NP} \theta(\tau_1, \bar{t}) w_1 \] (5.16)

The final step in the numerical process is determining values for NP and \( \Delta \bar{t} \). First, a total error tolerance must be established. The upper range of LDR inlet temperatures is of order \( 10^3 \) °K [p. 24, 2], and an accuracy of ± 1 °K is a sufficient criteria for an engineering analysis. Therefore, in terms of the non-dimensionalized temperature \( \theta \), the total error tolerance is of order \( 10^{-3} \). In a discretized analysis the errors of each step accumulate, producing the total error. If the error at each step is assumed to be roughly equal, then the total error is the product of the number of time steps and the error at each time step. If the number of time steps is given a preliminary order of \( 10^{-3} \), and the total error tolerance set to \( 10^{-3} \), the error tolerance at each step is of order \( 10^{-6} \). The NP is then selected such that the result of the numerical integration by Gaussian quadrature yields an accuracy of order \( 10^{-6} \). In general, as the limits of integration (optical depth \( \tau_0 \) in this case) for a function increase, so does the number of quadrature nodes required to achieve the same accuracy. At an optical depth of 1, a ten-point quadrature is sufficient; for optical depth 5 a 48 point quadrature is required. These values were obtained numerically by increasing NP until the total error for a fixed time
step was $10^{-3}$. The magnitude of the time step $\Delta \tau$ also impacts total accuracy. The relative size of $\Delta \tau$ depends on how quickly the LDR cools; the faster it cools, the smaller $\Delta \tau$ must be. Albedo, optical depth, and the strength of the external radiation are the parameters which affect the cooling rate, with the droplet layer cooling fastest for low values of albedo, optical depth, and external radiation. Optical depth is by far the most dominant factor, with the other two parameters less significant. The $\Delta \tau$ used in this study are

<table>
<thead>
<tr>
<th>$\tau_o$</th>
<th>$\Delta \tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0002</td>
</tr>
<tr>
<td>5.0</td>
<td>0.00025</td>
</tr>
</tbody>
</table>

An interesting aspect to this problem is that it is characterized by an initial transition period. Siegel [8] made the analogy of this period to start-up length in duct flow. Another physical analogy is penetration depth in conduction problems. Analytically, the transition period is due to a sudden imposition of the boundary conditions. At start, the droplet layer experiences an abrupt change from a constant temperature environment to black space, with or without external radiation on one side. Sudden changes in boundary conditions require small time steps in the vicinity of the change in order to ensure a stable solution is obtained. The droplet layer in a sense must be allowed time to "assimilate" the change in its environment. Numerically, the transition regime is characterized by the existence of a region which is still at the initial temperature ($\Theta = 1$). The terminus of the transition regime is when the isothermal region disappears. This is similar to the cessation of the free-stream flow after the boundary layer is fully developed. The size of this transition regime varies primarily with optical depth, since in
this early stage of cooling, the droplet layer's temperature profile is still relatively uniform, and so the effect of albedo is small. As the optical depth increases, so does the size of the transition regime. This is somewhat intuitive as the thicker the layer, the thicker the isothermal region, and thus for a given droplet material, the longer it will take the external environment to completely "penetrate" the droplet layer. At optical depth 5, the non-dimensional time of the transition regime is approximately 0.1. The existence of external radiation tends to ease the abruptness of the boundary condition change as it acts to provide an effective external temperature, which would allow somewhat larger time steps for the transition regime. However, in this study, only one side of the LDR is potentially exposed to external radiation. Since the opposite side still experiences the sudden exposure to black space, the more restrictive black-space time step must still be used.

The existence of the transition regime offers an advantage in the numerical computations. The terminus of the regime represents a point sufficiently far from droplet generation (and the abrupt boundary condition change) that larger time steps than those presented earlier in this section may be used. The time step after the transition period was increased and the results of the temperature profile computations compared to those when no change was made in the time step. It was found that the time step after the transition regime could be increased by an order of magnitude over the start-up time steps presented earlier.

5.4 **LDR Performance Analysis**

The first parameter examined is the temperature profile across the layer. Figures (5.2) - (5.10) illustrate the temperature profile at selected times for various LDRs. Since it is desired to compare results for the homogeneous case with that of
Figure 5.2 Temperature Profiles; $\omega = 0.1, \tau_0 = 1$. 
Figure 5.3 Temperature Profiles; $\omega = 0.6$, $\tau_0 = 1$. 
Figure 5.4 Temperature Profiles; $\omega = 0.9$, $\tau_0 = 1$. 
Figure 5.5 Temperature Profiles; $\omega = 0.1$, $\tau_0 = 2$. 
Figure 5.6 Temperature Profiles; $\omega = 0.6$, $\tau_0 = 2$. 
Figure 5.7 Temperature Profiles; $\omega = 0.9$, $\tau_0 = 2$. 
Figure 5.8 Temperature Profiles; $\omega = 0.1$, $\tau_{o} = 5$. 
Figure 5.9 Temperature Profiles; $\varpi = 0.6$, $\tau_0 = 5$. 
Figure 5.10 Temperature Profiles; $\omega = 0.9$, $\tau_0 = 5$. 
Siegel's [8], the non-dimensional times presented in the figures are in terms of the time parameter $\tau_s$. The profiles for no external radiation match those presented by Siegel, which verifies the accuracy of the analytical approach used in the study. The effects of albedo and optical depth on the cooling of the droplet layer are most evident for the case of no external radiation. A high albedo characterizes a droplet material that primarily scatters incident radiation, thus tending to evenly distribute energy within the layer (since isotropic scattering is assumed). The outer rows of droplets act like mirrors, reflecting radiant heat back into the layer's interior rather than allowing it to escape out into space. Thus a high albedo produces slower cooling and a more uniform temperature distribution. By contrast, the outer rows of a highly absorptive layer act to "soak-up" radiation energy emitted from the interior, and re-emit it out to space. Since the outer rows "view" mostly black space, their incident energy is less than within the interior of the droplet layer, and so they cool faster. This non-uniformity produces a larger temperature gradient between the interior and exterior droplet rows, increasing the flow of heat from the interior. Thus the low-albedo droplet layer cools faster.

The presence of external radiation on one side of the layer results in a non-symmetric temperature distribution as the one side heats up, with the opposite edge cooling as it continues radiating heat to black space. With increased optical depth, the temperature profile becomes more asymmetric. The increased optical depth serves to further isolate either side, allowing the heating or cooling to occur with less influence from the opposing boundary condition, although a large albedo still tends to produce a relatively uniform temperature profile. However, it is interesting to note that with the passage of time, the profiles assume a similar shape that appears to be independent of albedo. This becomes more obvious for increased optical depth and magnitude of external radiation. The high albedo layer tends to reflect back a large portion of the incident radiation, which serves to moderate the temperature increase on that side. As was the case
for homogeneous boundary conditions, the side of a high albedo layer exposed to black space cools at a slower rate due to the large scattering effect. As time passes, this mechanism seems to lead to an asymptotic profile. Although a low albedo layer tends to absorb more of the incident radiation, it is able to distribute most of the increased energy to the opposite boundary (due to the lower scattering within the layer), which moderates the overall temperature increase of the layer. However, the transfer of energy to the black space side tends to slow cooling of the layer along that boundary. Thus the profile again achieves an asymptotic profile, although the underlying cause is different than for the high albedo layer.

Two interesting phenomena appear in the temperature profiles for the non-homogeneous case. The first is that as time passes, the profile for a given albedo and optical depth appears to approach an asymptotic shape and magnitude. This might suggest that the LDR droplet layer is approaching thermal equilibrium. The second characteristic is that for a sufficiently large magnitude of external radiation, the albedo effect is reversed from that observed for the homogeneous case. These two phenomena are best understood by examining the change in the droplet layer's integrated mean temperature with time, Figs. (5.11) - (5.13). As external radiation is increased from the limiting case $F_1 = 0.0$, the mean temperature of the layer displays less dependence on albedo, and the layer cools more slowly. For external radiation magnitudes exceeding 0.5, the albedo effect is reversed, and the layer's mean temperature begins to increase rather than cooling. However, the mean temperature appears to lose its albedo dependence after some time for sufficiently large $F_1$, and appears to approach an asymptotic value. Although this would seem to confirm the earlier hypothesis that the layer eventually reaches thermal equilibrium, this is not the case.

The actual behavior of the droplet layer is best understood by segregating the range of $F_1$ into two domains. The mean temperature plots suggest the first domain is
Figure 5.11 LDR mean temperature; $\tau_0 = 1$. 
Figure 5.12 LDR mean temperature; \( \tau_0 = 2 \).
Figure 5.13 LDR mean temperature; $\tau_o = 5$. 
comprised of values for $F_1$ which allow the layer to still cool with time. This first domain has as its lower limit the homogeneous case, and the layer cools at a rate influenced by the optical depth and albedo of the droplet sheet as discussed earlier. The presence of external radiation serves to increase the internal energy of the layer, thus slowing (but not heating) the cooling of the layer. The second domain is comprised of values of $F_1$ sufficiently large to apparently reverse the albedo effects, and actually cause the droplet layer to heat rather than cool with time.

The interface between the two regimes bears closer examination as it reveals the true behavior of the droplet sheet within the second domain. This interface between the two domains is defined as the magnitude of external radiation $F_1$ for which the droplet layer first stops cooling immediately after it is generated. Numerically, the interface is identified by increasing $F_1$ until the integrated mean temperature at the first time step is no longer less than its initial value of 1.0. This magnitude is the critical radiosity $F_c$, with its value of 0.63659 ($\pm 0.00002$) having almost no dependence on albedo nor optical depth. This is an interesting result in light of the very different processes that occur in low-albedo vs. high-albedo layers. Examining the mean temperature for magnitudes of $F_1$ slightly greater than $F_c$, shown in Figs. (5.14) - (5.16), the droplet layer's behavior in the second domain is not one of heating to thermal equilibrium, but a phenomenon known as overshooting. For external radiation greater than $F_c$, the droplet layer heats relatively quick to some peak mean temperature $\bar{\theta}_{\text{max}}$, followed by a slow recovery, characteristic of cooling performance in the first domain. The magnitude of $\bar{\theta}_{\text{max}}$, the rate of heating to $\bar{\theta}_{\text{max}}$, and the recovery rate of cooling are strongly affected by albedo, and to a lesser degree by optical depth. Starting from droplet formation, a low albedo results in a more rapid rate of heating to a higher $\bar{\theta}_{\text{max}}$ due to the low albedo layer's greater absorption of the external radiation. However, due to the high emissivity at its black space boundary, the low albedo layer is able to recover faster as it transfers the
Figure 5.14a LDR mean temperature overshoot; 
$\tau_o = 1$, $\omega = 0.1$.

Figure 5.14b LDR mean temperature overshoot; 
$\tau_o = 1$, $\omega = 0.9$. 
Figure 5.15a LDR mean temperature overshoot;
\[ \tau_o = 2, \omega = 0.1. \]

Figure 5.15b LDR mean temperature overshoot;
\[ \tau_o = 2, \omega = 0.9. \]
Figure 5.16a  LDR mean temperature overshoot;
\( \tau_o = 5, \, \omega = 0.1. \)

Figure 5.16b  LDR mean temperature overshoot;
\( \tau_o = 5, \, \omega = 0.9. \)
Figure 5.17 LDR mean temperature crossover; $\tau_0 = 1$. 

$F_c = 0.63659$ for all $\omega$
Figure 5.18 LDR mean temperature crossover; $\tau_0 = 2$. 

$F_1 = 0.67$
$F_1 = 0.66$
$F_1 = 0.65$

$\omega = 0.1$

$F_c = 0.63659$ for all $\omega$
Figure 5.19 LDR mean temperature crossover; $\tau_o = 5$. 

$F_C = 0.63659$ for all $\omega$.
excess energy to the opposite side for rejection to space. The mean temperature curves presented in Fig. (5.14) indicate that the droplet layer can indeed achieve thermal equilibrium for a sufficiently large $F_1$. As optical depth increases, the $F_1$ required to achieve thermal equilibrium increases. High-albedo layers characteristically have a more uniform distribution of energy, and thus achieve thermal equilibrium at a lower $F_1$ than for a low-albedo layer. The low-albedo layer's ability to transfer larger quantities of energy to its black-space boundary for rejection necessitates a higher $F_1$ so as to dominate the thermal processes within the layer and thereby achieve thermal equilibrium.

Comparing the mean temperature plots for low and high albedos, presented in Figs. (5.17) - (5.19), the inversion of albedo effects noted earlier only lasts for a relatively short time, at which time the curves cross and the normal albedo relationship is restored. This crossover time increases with increasing $F_1$, and $F_1$ can become sufficiently large that it is the dominating factor influencing the layer's thermal behavior, and $\bar{\theta}$ therefore approaches an asymptotic value. Thus the mean layer temperatures for $F_1 = 1.0$ in Figs. (5.11) - (5.13) and the temperature distributions in Figs. (5.2) - (5.10) are approaching thermal equilibrium.

The final parameter examined is the exit heat flux, which represents the radiant thermal energy leaving the droplet layer. In order to provide a basis of comparison for the homogeneous case with Siegel's study [8] the exit heat fluxes are presented in Figs. (5.20) - (5.22) with a time ordinate equivalent to Siegel's, as expressed in terms of this study's variables. The heat flux curves shown represent the sum of the exit fluxes at either boundary. Due to the geometric and boundary condition symmetry of the homogeneous case, Siegel presented exit heat fluxes for one side only. Taking this into account, the heat loss curves in the figures for the case of $F_1 = 0.0$ are equivalent to those presented by Siegel.
Figure 5.20 LDR total exit heat flux; $\tau_o = 1$. 
Figure 5.21  LDR total exit heat flux; $\tau_o = 2$. 
Figure 5.22 LDR total exit heat flux; $\tau_0 = 5$. 
For the homogeneous case, increasing the optical depth of the droplet layer produces a higher rate of heat loss. Naturally, the thicker the layer, the higher its total energy capacity. However, note that the increase is not directly proportional to the optical depth. The increase in heat flux that results from increasing the optical depth from $\tau_o = 1$ to $\tau_o = 2$ is greater than from $\tau_o = 2$ to $\tau_o = 5$. Assuming LDR cooling power is proportional to $\bar{q}$, and total system mass to $\tau_o$, the optimal power-to-mass ratio $\bar{q} / \tau_o$ does appear to be in the range of $\tau_o = 1 \rightarrow 2$ for the homogeneous case, as suggested by earlier LDR studies [3,4]. Note that the effect of scattering, represented by the albedo, is most prominent during the early stages of cooling. During this period, outer layer temperatures are roughly equivalent for all albedos. However, as discussed earlier, the outer rows of the high albedo layer tends to reflect energy back into the layer, and thus less heat escapes. The low-albedo layer emits much more energy to space, thus allowing the outer rows to cool quickly. Although the low albedo layer has a high emissivity, after some time has passed, its outer temperature $\theta$ has dropped significantly due to the greater dissipation of thermal energy. Since $\bar{q}_r$ is proportional to $\theta^4$, the drop-off in $\bar{q}_r$ is quite steep. At the same time, the high albedo layer has cooled far less, and thus maintains a higher temperature and slower, more consistent rate of cooling. Thus the increasing disparity between temperatures of the outer regions of low and high albedo layers eventually results in the heat flux curves converging as the albedo and temperature effects counteract each other.

As was the case for the mean temperature of the layer with external radiation present on one side of the LDR, the heat loss curves approach a uniform value, and then invert for higher magnitudes of $F_1$. The effect of albedo is more significant during the early exposure of the droplets, and then becomes negligible at later times. For a sufficiently large magnitude of external radiation, the low albedo layer absorbs more energy than it emits. Thus, the high-albedo layer generates a higher heat flux due to its
ability to reflect the external radiation back to space. However, the opposite side of the layer still views black space, and the earlier discussion for the homogeneous case applies. Also discussed earlier was the increasing temperature gradient between the two boundaries. The low albedo layer takes advantage of this gradient to transfer energy from the region $\tau = 0$ to the lower temperature side $\tau = \tau_0$ for emission to black space. Thus as time and the temperature gradient between boundaries increases, the exit heat flux for the case of a low albedo layer increases. The high albedo layer also has an increasing temperature gradient across its depth with time. However, due to its high characteristic scattering, little energy is transferred to the black space boundary. Thus, the heat loss rate for a high albedo layer remains fairly constant. As was the case for no external radiation, the albedo and thermal effects eventually counteract each other, resulting in a uniform rate of heat loss.

5.5 Summary and Conclusions

The performance of an LDR operating either in black-space or exposed to constant, isotropic external radiation on one side was studied to evaluate performance over a range optical parameters and magnitudes of external radiation. The results obtained for the homogeneous case (i.e. black space) were comparable to those obtained by Siegel using an iterative method of solution for the radiative flux. Using the ratio of exit heat flux - to - optical depth as a non-dimensional parameter to quantify the power - to - mass ratio of the LDR, it was determined that the optimal optical depth of an LDR lies in the range of one to two. This confirms the conclusions of earlier studies using a lumped analysis.

The asymmetric heating of an LDR by external radiation initiates thermal processes within the droplet layer that are more complex than occurs in the homogeneous
case. A critical value for the magnitude of external radiation was identified ($F_c = 0.63659$) that is independent of optical depth and albedo. If the intensity of external radiation exceeds $F_c$, the droplet layer heats before recovering and cooling. The magnitude and duration of this overshooting is greatest for optically thin layers and high-albedo layers. For sufficiently large magnitudes of external radiation, the droplet layer will heat, but instead of recovering and proceeding to cool, will attain thermal equilibrium at some mean temperature above its initial condition. For optically thin and high-albedo layers this occurs within 5% of $F_c$.

LDR's operating in low earth orbit will undoubtably experience external radiation incident on at least one side. Three final conclusions follow concerning LDR design and operations can be derived from this study.

The optimal optical depth, in terms of exit heat flux-to-optical depth, lies in the range of one and two for black-space operations. The width and length of the LDR(s) are therefore determined by the total heat energy that must be dissipated. However, the exposure time of the droplets should not be long enough to allow and of the droplets to solidify.

Droplet materials should possess a very low albedo (i.e. highly absorptive or emissive). A low-albedo layer cools faster, and if exposed to intensities of external radiation beyond $F_c$, will heat less and recover faster. In light of this, the proposal to inject carbon powder into liquid metals, the droplet material proposed for high-temperature applications, merits serious consideration given the characteristically high-albedo of metals.
An LDR of optimal optical depth should not be operated when exposed to external radiation of intensity greater than $F_c$ of duration approaching the droplet exposure time. In all likelihood, this will occur only in low-temperature LDRs. If it is anticipated that the LDR will be exposed to such intensities of external radiation for extended periods, thicker droplet layers and longer droplet exposure times should be selected in order to ensure thermal recovery.
NOTES

1 Equation (5.1) is non-dimensionalized by first expressing \( x \) in terms of the optical depth variable \( \tau \) as

\[
\frac{\rho c_p}{\beta} \frac{dT}{dt} = -\frac{dq_r}{dt}
\]

Next, both sides are divided by \( \sigma T_i^4 \), the initial emissive power of the droplet layer

\[
\frac{\rho c_p}{\beta \sigma T_i^3} \frac{d\theta}{dt} = -\frac{d\tilde{q}_r}{dt}
\]

The remaining terms are lumped together to define the non-dimensional time parameter, resulting in Eq. (5.2).

2 The steady-state equation of radiative transfer Eq. (2.7) is expressed as

\[
\nabla \cdot I(s, \Omega) + \beta I(s, \Omega) = \kappa I_b(T) + \frac{\sigma}{4\pi} \int p(\Omega', \Omega) I(s, \Omega) d\Omega'
\]

Integration by \( d\Omega \) over \( 4\pi \) yields

\[
\int_{4\pi} \nabla \cdot I(s, \Omega) d\Omega = \kappa \int_{4\pi} I_b(T) d\Omega + \frac{\sigma}{4\pi} \int_{4\pi} \int_{4\pi} p(\Omega', \Omega) I(s, \Omega) d\Omega' d\Omega - \beta \int_{4\pi} I(s, \Omega) d\Omega
\]

For isotropic scattering, the normalized phase function is equal to one (see Note 3 of Chapter 2) reducing the above expression to

\[
\int_{4\pi} \nabla \cdot I(s, \Omega) d\Omega = \kappa \left[ 4\sigma T^4(s) \right] + \frac{\sigma}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} I(s, \mu) d\phi d\mu - \beta \int_{-1}^{1} I(s, \mu) d\mu
\]

\[
= \kappa \left[ 4\sigma T^4(s) - 2\pi \int_{-1}^{1} I(s, \mu) d\mu \right]
\]

\[
= \kappa \left[ 4\sigma T^4(s) - G(s) \right]
\]
A preliminary step in the evaluation of \( G(\tau) \) is the computation of the coefficients for the improved profile as described in Chapter 3. This in requires analytical relations for \( d(\eta) \) and \( d_i \). For the boundary conditions Eqs. (5.6 b) and (5.6 c) \( d(\eta) \) may be expressed as

\[
d(\eta) = \int_0^{\tau_o} Y(\tau) e^{-\frac{\varphi}{\eta}} d\tau
\]

\[
= 2\pi \left\{ \int_0^{\tau_o} \int_0^{\tau_o} (1 - \omega) \frac{\theta}{\pi} (\tau') E_1(1(\tau - \tau') e^{-\frac{\varphi}{\eta}} d\tau d\tau' + \int_0^{\tau_o} \int_0^1 F_1 e^{-\frac{\varphi}{\eta}} e^{-\frac{\varphi}{\mu}} d\tau d\mu \right\}
\]

Interchanging the variables \( \tau' \) and \( \tau \) in the first integral and introducing the exponential integral notation to simplify the integral associated with \( F_1 \) leads to the expression

\[
d(\eta) = 2\pi (1 - \omega) \int_0^{\tau_o} \theta^4(\tau) \int_0^{\tau_o} E_1(1(\tau - \tau') e^{-\frac{\varphi}{\eta}} d\tau d\tau' + 2\pi F_1 \int_0^{\tau_o} e^{-\frac{\varphi}{\eta}} E_2(\tau) d\tau
\]

Using Eqs. (4.41), (4.42), and (4.45) of Ref. [22] to evaluate the first integral, and Eqs. (2.32) and (2.34) to evaluate the second integral, the final form for \( d(\eta) \) is

\[
d(\eta) = 2\pi (1 - \omega) \int_0^{\tau_o} \theta^4(\tau) T(\eta, \tau) d\tau + 2\pi F_1 \psi(\eta, 0)
\]

An analytical expressions for \( T \) and \( \psi \) may be found in Appendix A. The term \( d_i \) is

\[
d_i = \int_0^{\tau_o} Y(\tau) \tau^i d\tau
\]

\[
= 2\pi \left\{ \int_0^{\tau_o} \int_0^{\tau_o} (1 - \omega) \frac{\theta}{\pi} (\tau') \tau^i E_1(1(\tau - \tau') d\tau d\tau' + \int_0^{\tau_o} \int_0^1 F_1 e^{-\frac{\varphi}{\mu}} \tau^i d\mu d\tau \right\}
\]
Again, interchange the variables \( \tau' \) and \( \tau \) in the first integral and introduce the exponential integral notation to simplify the second integral to obtain

\[
d_i = 2\pi(1 - \omega)\int_0^{\tau_o} \theta^4(\tau) \int_0^{\tau_o} \tau'^{i} E_1(l\tau - \tau') d\tau' d\tau + 2\pi F_1 \int_0^{\tau_o} \tau^{i} E_2(\tau) d\tau
\]

The inner integral of the first term is evaluated using Eq. (4.1) of Ref. [22], and Eq. (1.1) for the integral associated with the term \( F_1 \) yielding the final expression

\[
d_i = 2\pi(1 - \omega)\int_0^{\tau_o} \theta^4(\tau) \left\{ (-1)^{i+1} i! E_{i+2}(\tau) \right\}
+ \sum_{k=0}^{i} \frac{i!}{(m - i)!} \left[ \frac{(-1)^i + 1}{k + i} \tau^i - k e^{i-k} E_{i+2}(\tau_o - \tau) \right] d\tau
+ 2\pi F_1 i! \left\{ \frac{1}{i+2} - \sum_{k=0}^{i} \frac{\tau^i - k}{(i-k)!} E_{k+3}(\tau_o) \right\}
\]

For orders \( M = 0 \) and \( M = 1 \), the analytical expressions are

\[
d_1 = 2\pi(1 - \omega)\int_0^{\tau_o} \theta^4(\tau) \left[ 2 - E_2(\tau) - E_2(\tau_o - \tau) \right] d\tau + 2\pi F_1 \left( \frac{1}{2} - E_3(\tau_o) \right)
\]

\[
d_2 = 2\pi(1 - \omega)\int_0^{\tau_o} \theta^4(\tau) \left[ 2\tau + E_3(\tau) - \tau_o E_2(\tau_o - \tau) - E_3(\tau_o - \tau) \right] d\tau
+ 2\pi F_1 \left[ \frac{1}{3} - \tau_o E_3(\tau_o) - E_4(\tau_o) \right]
\]
The Gauss-Legendre quadrature formulas are used in this study. Since the Legendre polynomials are defined on the interval $-1 \leq \xi \leq 1$, the node points $\xi_1$ must be transformed to the interval of the integral being evaluated ($0 \leq \tau \leq \tau_0$ in this case). This is accomplished by the transformation

$$\tau_1 = \frac{\tau_0}{2} (1 + \xi_1)$$

The approximation of the integral becomes

$$\int_{\tau_0}^{\tau_1} \theta^4(\tau', \tau) E_1(|\tau - \tau'|) d\tau' = \sum_{l=1}^{NP} \theta^4(\tau_1, \tau) E_1(|\tau_k - \tau_1|) w_1$$

The nodes $\xi_1$ and weights $w_1$ for orders of NP up to 512 may be found in Table 1 of Ref. [24].

Taking advantage of the dummy variable of integration $\tau'$, the term involving $\theta^4$ is re-written as

$$\int_{\tau_0}^{\tau_1} \theta^4(\tau') E_1(|\tau - \tau'|) d\tau' = \int_{\tau_0}^{\tau_1} \theta^4(\tau') E_1(|\tau - \tau'|) d\tau' - \int_{\tau_0}^{\tau_1} \theta^4(\tau) E_1(|\tau - \tau'|) d\tau'$$

$$+ \int_{\tau_0}^{\tau_1} \theta^4(\tau) E_1(|\tau - \tau'|) d\tau'$$

$$= \int_{\tau_0}^{\tau_1} \left[ \theta^4(\tau') - \theta^4(\tau) \right] E_1(|\tau - \tau'|) d\tau' + \theta^4(\tau) \int_{\tau_0}^{\tau_1} E_1(|\tau - \tau'|) d\tau'$$

The second integral is evaluated using Eq. (4.1.1) of Ref. [22] to yield

$$\int_{\tau_0}^{\tau_1} \theta^4(\tau') E_1(|\tau - \tau'|) d\tau' = \int_{\tau_0}^{\tau_1} \left[ \theta^4(\tau') - \theta^4(\tau) \right] E_1(|\tau - \tau'|) d\tau'$$

$$+ \theta^4(\tau) \left[ 2 - E_2(\tau) - E_2(\tau - \tau_0) \right]$$
APPENDIX A

ANALYTICAL EXPRESSIONS FOR THE
INTEGRALS OF CHAPTERS 3 - 5

Equations (A1) thru (A6) and the special functions are derived from the Appendix of Ref. [15]. Equations (A7) thru (A10) are derived using the equations found in Ref. [22].

\[ V(\eta, \nu) = \int_0^{\tau_o} e^{-\frac{\tau_o}{\eta + \nu}} d\tau - \frac{\omega}{2} \int_0^{\tau_o} \int_0^{\tau_o} e^{-\frac{\tau'}{\eta}} E_1(\tau - \tau') e^{-\frac{\tau'}{\nu}} d\tau d\tau' \]

\[ = \frac{\eta \nu}{\eta + \nu} \left[ 1 - e^{-\frac{(\eta + \nu)\tau_o}{\eta \nu}} \right] \]

\[ - \frac{\omega}{2} \left\{ \frac{\eta \nu^2}{\eta + \nu} \left[ D(\nu, 0) - \left( e^{-\frac{\tau_o}{\eta}} + e^{-\frac{\tau_o}{\nu}} \right) E_1(\tau_o) + e^{-\frac{\tau_o}{\eta}} C(\nu, \tau_o) \right] \right\} \]

\[ + \frac{\eta^2 \nu}{\eta + \nu} \left[ D(\eta, 0) - \left( e^{-\frac{\tau_o}{\eta}} + e^{-\frac{\tau_o}{\nu}} \right) E_1(\tau_o) + e^{-\frac{\tau_o}{\nu}} C(\eta, \tau_o) \right]; \eta \neq -\nu \]

\[ = \tau_o + \frac{\omega}{2} \left\{ \eta \left[ \tau_o D(\eta, 0) - e^{-\frac{\tau_o}{\eta}} \tau_o C(\eta, \tau_o) + \frac{\eta}{1 + \eta} \left( 1 - e^{-\frac{(1 + \eta)\tau_o}{\eta}} \right) \right] \right\} \]

\[ + \frac{\eta}{1 - \eta} \left( 1 - e^{-\frac{(1 - \eta)\tau_o}{\eta}} \right) \right\} + \eta^2 \left[ 2 \cosh \left( \frac{\tau_o}{\eta} \right) E_1(\tau_o) - e^{-\frac{\tau_o}{\eta}} C(\eta, \tau_o) \right] \]

\[ - D(\eta, 0) \}; \eta = -\nu \]  

(A1)
\[ V_0(\eta) = \int_0^{\tau_o} e^{-\eta \tau} d\tau - \frac{\omega}{2} \int_0^{\tau_o} \int_0^{\tau} e^{-\eta \tau' \tau''} E_1(|\tau - \tau'|) d\tau' d\tau \]

\[ = \eta(1 - e^{-\eta \tau_o}) - \frac{\omega}{2} \left[ 2\eta(1 - e^{-\eta \tau_o}) - \psi(\eta, 0) - \Phi(\eta, \tau_o) \right] \quad (A2) \]

\[ V_1(\eta) = \int_0^{\tau_o} \tau e^{-\eta \tau} d\tau - \frac{\omega}{2} \int_0^{\tau_o} \int_0^{\tau} \tau e^{-\eta \tau' \tau''} E_1(|\tau - \tau'|) d\tau' d\tau \]

\[ = \eta^2 - (\eta \tau_o + \eta^2)e^{-\eta \tau_o} - \frac{\omega}{2} \left[ 2\eta^2 - 2(\eta \tau_o + \eta^2)e^{-\eta \tau_o} \right. \]

\[ \left. - \tau_o \Phi(\eta, \tau_o) + \eta \left( \frac{1}{2} - e^{-\eta \tau_o} E_3(\tau_o) \right) - \eta \psi(\eta, 0) \right] \]

\[ + e^{-\eta \tau_o} \left[ \eta \left( \frac{1}{2} - e^{-\eta \tau_o} E_3(\tau_o) \right) - \eta \psi(-\eta, 0) \right] \] \quad (A3)

\[ \Phi(\eta, \tau) = \int_0^{\tau} e^{-\eta \tau'} E_2(\tau - \tau') d\tau' \]

\[ = \eta \left\{ E_2(\tau) + \eta E_1(\tau) - e^{-\eta \tau} - \eta C(\eta, \tau) \right\}; \quad \tau > 0 \quad (A4) \]

\[ \psi(\eta, \tau) = \int_{\tau}^{\tau_o} e^{-\eta \tau'} E_2(\tau' - \tau) d\tau' ; \quad \tau < \tau_o \]

\[ = \eta \left\{ e^{-\eta \tau_o} \left[ \eta E_1(\tau_o - \tau) - E_2(\tau_o - \tau) \right] + e^{-\eta \tau_o} - \eta D(\eta, \tau) \right\} \quad (A5) \]
\[ T(\eta, \tau) = \int_0^{\tau_o} e^{-\frac{\tau'}{\eta}} E_1(|\tau - \tau'|) \, d\tau' \]

\[ = \eta \{ D(\eta, \tau) - C(\eta, \tau) + E_1(\tau) - e^{-\frac{\tau_o}{\eta}} E_1(\eta - \tau) \}; \ 0 < \tau < \tau_o \]

\[ = \eta \left[ D(\eta, 0) - e^{-\frac{\eta}{\eta}} E_1(\tau_o) \right]; \ \tau = 0 \]

\[ = \eta \left[ E_1(\tau_o) - C(\eta, \tau_o) \right]; \ \tau = \tau_o \]  

(A6)

\[ X_i(\tau) = \int_0^{\tau_o} \tau_i^{\tau} E_1(|\tau - \tau'|) \, d\tau' \]

\[ = \sum_{k=0}^{i} \frac{i!}{(i-k)!} \left\{ \left[ \frac{(-1)^k + 1}{k+1} \right] \tau_i^{i-k} - \tau_o^{i-k} E_{k+2}(\tau_o - \tau) \right\} \]

\[ + (-1)^{i+1} i! E_{i+2}(\tau) \]  

(A7)

\[ Y_i = \int_0^{\tau_o} \tau_i^{\tau} E_2(\tau_o - \tau') \, d\tau' \]

\[ = i! \left\{ (-1)^{i+1} E_{i+3}(\tau) + \sum_{k=0}^{i} (-1)^{k+1} \frac{\tau_o^{i-k}}{(k+2)(i-k)!} \right\} \]  

(A8)
\[ Z_i = \int_0^{\tau_o} \tau^i E_2(\tau') \, d\tau' \]

\[ = i! \left[ \frac{1}{i+2} \right] - \sum_{k=0}^{i} \frac{\tau_o^{i-k}}{(i-k)!} E_{k+3}(\tau_o) \quad (A9) \]

\[ B_{mn} = \int_0^{\tau_o} \tau^m \tau^n \, d\tau - \frac{\omega}{2} \int_0^{\tau_o} \int_0^{\tau_o} \tau^m E_1(\tau - \tau') \tau^n \, d\tau \, d\tau' \]

\[ = \frac{\tau_o^{m+n+1}}{m+n+1} + \frac{\omega}{2} \left\{ \sum_{k=0}^{n} \frac{n!}{(n-k)!} \left[ \frac{1+(-1)^k}{k+1} \right] \frac{\tau_o^{m+n-k+1}}{m+n-k+1} \right. \]

\[ + (-1)^{n+1} \frac{m! n!}{m+n+2} - \sum_{k=0}^{n} \sum_{l=0}^{m} \frac{m! n!}{(n-k)! (m-l)!} \left[ \frac{(-1)^l}{k+1+l} \frac{\tau_o^{m+n-k-l}}{m+n-k-l} \right. \]

\[ + m! n! \left[ (-1)^n \sum_{k=0}^{m} \frac{\tau_o^{m-k}}{(m-k)!} E_{n+k+3}(\tau_o) \right] \]

\[ + (-1)^m \sum_{k=0}^{n} \frac{\tau_o^{n-k}}{(n-k)!} E_{m+k+3}(\tau_o) \] \quad (A10)

\[ \Gamma_i = \frac{1}{2} X_i(\tau) + \rho_1 BE_2(\tau) [Z_i + a_2 Y_i] \]

\[ + \rho_2 BE_2(\tau_o - \tau) [Y_i + a_1 Z_i] \quad (A11) \]
SPECIAL FUNCTIONS

\[ C(\eta, \tau) = e^{-\frac{\tau}{\eta}} \]

\[
\begin{cases}
\ln(1 - \frac{1}{\eta}) + E_1\left[(1 - \frac{1}{\eta})\tau\right] & \frac{1}{\eta} < 1, \tau > 0 \\
-\gamma - \ln(\tau) & \eta = 1, \tau > 0 \\
\ln\left(\frac{1}{\eta} - 1\right) - \text{Ei}\left[\left(\frac{1}{\eta} - 1\right)\tau\right] & \frac{1}{\eta} > 1, \tau > 0
\end{cases}
\]

\[ D(\eta, \tau) = e^{-\frac{\tau}{\eta}} \]

\[
\begin{cases}
\ln(1 + \frac{1}{\eta}) + E_1\left[(1 - \frac{1}{\eta})\tau\right] & \frac{1}{\eta} > -1, \tau < \tau_o \\
-\gamma - \ln(\tau_o - \tau) & \eta = -1, \tau < \tau_o \\
\ln\left(-\frac{1}{\eta} - 1\right) - \text{Ei}\left[-\left(\frac{1}{\eta} + 1\right)(\tau_o - \tau)\right] & \frac{1}{\eta} < -1, \tau < \tau_o
\end{cases}
\]
APPENDIX B

COMPUTATION 
OF THE 
EIGENVALUES

The improved-profile is based on the solution of the equation of radiative transfer by the spherical harmonics method. [25] This solution is a function of Chandrasekhar's polynomials $g_n (\lambda)$. The eigenvalues are the positive zeroes of $g_{n+1} (\lambda)$, and are computed using the truncation condition $g_{N+1} (\lambda) = 0$. $N$ is the order of the spherical harmonics solution and is limited to odd values (i.e. $N = 1, 3, 5, ...$). The polynomials are [p. 134, 16]

$$(n + 1)g_{n+1} (\lambda) = h_n \lambda g_n (\lambda) - ng_{n-1} (\lambda); \quad n = 0, 2, 4, \ldots, N - 1 \quad (B1)$$

where

$$h_n = 2n + 1 - \omega \delta_{un} \quad (B2)$$

Since $n$ is restricted to even values, the recursion relation Eq. (B1) must be expressed in terms of even orders for $g(\lambda)$. Equation (B1) is multiplied by $\lambda$ and expressed as [25]

$$\lambda^2 g_n (\lambda) = \frac{\lambda (n + 1)g_{n+1} (\lambda)}{h_n} + \frac{\lambda n g_{n-1} (\lambda)}{h_n} \quad (B3)$$
From Eq. (B3), \( g_{n+1}(\lambda) \) and \( g_{n-1}(\lambda) \) follow as

\[
g_{n+1}(\lambda) = \frac{(n + 2) g_{n+2}(\lambda)}{\lambda h_{n+1}} + \frac{(n + 1) g_n(\lambda)}{\lambda h_{n+1}} \tag{B4}
\]

\[
g_{n-1}(\lambda) = \frac{n g_n(\lambda)}{\lambda h_{n-1}} + \frac{(n - 1) g_{n-2}(\lambda)}{\lambda h_{n-1}} \tag{B5}
\]

Substitution of Eqs. (B4) and (B5) into Eq.(B3) yields

\[
\lambda^2 g_n(\lambda) = \frac{n(n - 1)}{h_n h_{n-1}} g_{n-2}(\lambda) + \frac{1}{h_n} \left[ \frac{(n + 1)^2}{h_{n+1}} + \frac{n^2}{h_{n-1}} \right] g_n(\lambda) + \frac{(n + 2)(n + 1)}{h_{n+1} h_n} g_{n+2}(\lambda) \tag{B6}
\]

Thus the \( h \)'s for order \( N \) are

\[
h_0 = 1 - \omega
\]

\[
h_n = 2n + 1; \quad n = 2, 4, 6, ..., N - 1
\]

Equation (B6) generates the matrix expression

\[
[M] [g_{2n}] = \lambda^2 [g_{2n}]; \quad n = 0, 1, 2, ..., N - 1 \tag{B7}
\]
Where \( M \) is an \( N \times N \) tridiagonal matrix, with first row

\[
\begin{array}{ccc}
\frac{1}{h_0 h_1} & \frac{2}{h_0 h_1} & 0 \ldots \ldots 0 \\
\end{array}
\]

and second row

\[
\begin{array}{ccc}
\frac{2}{h_2 h_1} \\
\frac{9}{h_2 h_3} + \frac{4}{h_2 h_1} \\
\frac{12}{h_3 h_2} & 0 \ldots \ldots 0 \\
\end{array}
\]

The \( n^{th} \) row of \( M \) is

\[
0 \ldots \ldots \frac{n(n-1)}{h_n h_{n-1}} \left[ \frac{(n+1)^2}{h_n h_{n+1}} + \frac{n^2}{h_n h_{n+1}} \right] \frac{(n+2)(n+1)}{h_{n+1} h_n} 0 \ldots \ldots 0 
\]

and the last row is

\[
0 \ldots \ldots \frac{(N-1)(N-2)}{h_{N-1} h_{N-2}} \left[ \frac{N^2}{h_{N-1} h_N} + \frac{(N-1)^2}{h_{N-1} h_{N-2}} \right]
\]

For the special case of a purely scattering medium (\( \omega = 1 \)), there is one less eigenvalue as the first row of the matrix \( M \) is undefined due to \( h_0 = 0 \), and is therefore eliminated. Solution of Eq. (B7) yields the positive, squared eigenvalues. The IMSL subroutine EVLRG or EISPACK subroutine IMTQLI can be used to solve for the \( \lambda^2 \). If the IMTQLI is used, the tridiagonal matrix must first be passed to the EISPACK subroutine FIGI to reduce it to a symmetric tridiagonal matrix having the same eigenvalues. For the first order approximation, the eigenvalue is

\[
\lambda_1 = \sqrt{\frac{1}{3(1 - \omega)}}
\]
BIBLIOGRAPHY


