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RELATION BETWEEN THE MAGNETIC MOMENTS ASSOCIATED WITH THE FIRST AND THIRD ADIABATIC INVARIANTS

by

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Abstract

The motion of a charged particle in a multipole magnetic field has three adiabatic invariants associated with it. The first is that the magnetic flux through the particle's cyclotron orbit about a field line is constant. An equivalent statement is that the magnetic moment of the particle due to its cyclotron motion is constant. The second adiabatic invariant states that the integral of the particle's parallel momentum between its two mirror points is constant. The third adiabatic invariant says that the magnetic flux through a surface connected to the longitudinal invariant surface and passing through the magnetic axis is constant. In this thesis I have examined the relationship between the magnetic moments from the first and third invariants. For a particle in the equatorial plane of a dipole field, the ratio of the magnetic moment from the third invariant to that from the first invariant is 3 /2.

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Chapter 1: Introduction

In this thesis, I will discuss the concept of the adiabatic invariant in general and then apply it to a charged particle moving in a dipole magnetic field to obtain a thus far unrecognized relationship between the particle's magnetic moment due to its drift motion in the field and the particle's cyclotron magnetic moment due to its motion about a field line.

To introduce the concept of the adiabatic invariant, first consider a one-dimensional system (this discussion follows Landau and Lifshitz, 1976, pp. 154 - 156). Let λ be some parameter that specifies the properties of the system. Also, let λ vary only slightly during the period of the motion:

$$\left| T\frac{d\lambda}{dt} \right| << \lambda \quad \text{where } T \text{ is the period of the motion}$$

This slow change in λ is called an "adiabatic" change. If λ were constant here, the motion would be periodic with period T, and energy would be conserved (and therefore invariant). Since λ varies slowly, however, the energy is not constant but also varies slowly. The adiabatic invariant we are looking for is a quantity that remains constant for small changes in λ.

Take the Hamiltonian of the system to be $H(q,p;\lambda)$, where q is the coordinate and p is the momentum such that

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

From Hamiltonian theory:

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial t}\right)_{q,p} \qquad (1)$$

So, $$\frac{dH}{dt} = \left(\frac{\partial H}{\partial t}\right)_{q,p} = \left(\frac{\partial H}{\partial \lambda}\right)_{q,p}\frac{d\lambda}{dt}$$

Even though λ changes slowly, q and p may change rapidly. To smooth out these changes, we take the time average of dH/dt. Therefore:

$$\overline{\frac{dH}{dt}} = \frac{1}{T}\int_0^T \left(\frac{\partial H}{\partial \lambda}\right)_{q,p} \frac{d\lambda}{dt} dt$$

$$= \frac{1}{T}\frac{d\lambda}{dt}\int_0^T \left(\frac{\partial H}{\partial \lambda}\right)_{q,p} dt$$

$$= \frac{1}{T}\frac{d\lambda}{dt}\int_0^T \left(\frac{\partial H}{\partial \lambda}\right)_{q,p} dt$$

$$= \frac{d\lambda}{dt}\overline{\frac{\partial H}{\partial \lambda}} \tag{2}$$

$d\lambda/dt$ can be taken outside the integral because it changes slowly with time.

If we now use Hamilton's equations, the integral over time can be replaced with one over q:

$$\dot{q} = \left(\frac{\partial H}{\partial p}\right)_{q,t} \Rightarrow \frac{dq}{dt} = \left(\frac{\partial H}{\partial p}\right)_{q,t} \Rightarrow dt = \frac{dq}{\left(\frac{\partial H}{\partial p}\right)_{q,t}}$$

So, the period of the motion is given by:

$$T = \int_0^T dt = \oint \frac{dq}{\left(\frac{\partial H}{\partial p}\right)_{qt}} \tag{3}$$

This can then be substituted into equation (2):

$$\overline{\frac{dH}{dt}} = \frac{d\lambda}{dt}\left(\oint dq \frac{\left(\frac{\partial H}{\partial \lambda}\right)_{qp}}{\left(\frac{\partial H}{\partial p}\right)_{q\lambda}}\right)\left(\oint \frac{dq}{\left(\frac{\partial H}{\partial p}\right)_{q\lambda}}\right)^{-1} \tag{4}$$

Since $H = H(q,p;\lambda)$, we have for unchanging H and q:

$$0 = \left(\frac{\partial H}{\partial p}\right)_{q,\lambda} dp + \left(\frac{\partial H}{\partial \lambda}\right)_{p,q} d\lambda \tag{5}$$

This becomes:

$$0 = \left[\left(\frac{\partial H}{\partial p}\right)_{q,\lambda}\left(\frac{\partial p}{\partial \lambda}\right)_{Hq} + \left(\frac{\partial H}{\partial \lambda}\right)_{p,q}\right] d\lambda \tag{6}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \lambda}\right)_{H,q} = -\frac{\left({\partial H}/{\partial \lambda}\right)_{p,q}}{\left({\partial H}/{\partial p}\right)_{q,\lambda}} \tag{7}$$

For unchanging λ and q:

$$dH = \left(\frac{\partial H}{\partial p}\right)_{q,\lambda} dp \tag{8}$$

$$= \left(\frac{\partial H}{\partial p}\right)_{q,\lambda} \left(\frac{\partial p}{\partial H}\right)_{q,\lambda} dH \tag{9}$$

$$\Rightarrow \left(\frac{\partial p}{\partial H}\right)_{q,\lambda} = \frac{1}{\left(\frac{\partial H}{\partial p}\right)_{q,\lambda}} \tag{10}$$

So, equation (4) becomes:

$$\overline{\frac{dH}{dt}} = -\frac{d\lambda}{dt}\left(\oint dq \left(\frac{\partial p}{\partial \lambda}\right)_{H,q}\right)\left(\oint dq \left(\frac{\partial p}{\partial H}\right)_{q,\lambda}\right)^{-1} \tag{11}$$

Equation (11) can be rearranged to give:

$$\overline{\frac{dH}{dt}} \oint dq \left(\frac{\partial p}{\partial H}\right)_{q,\lambda} + \frac{d\lambda}{dt} \oint dq \left(\frac{\partial p}{\partial \lambda}\right)_{H,q} = 0 \tag{12}$$

$$\Rightarrow \oint dq \left[\left(\frac{\partial p}{\partial H}\right)_{q,\lambda} \overline{\frac{dH}{dt}} + \left(\frac{\partial p}{\partial \lambda}\right)_{H,q} \frac{d\lambda}{dt}\right] = 0 \tag{13}$$

So,

$$\frac{1}{T} \oint \left[\left(\frac{\partial p}{\partial H}\right)_{q,\lambda}\left(T\overline{\frac{dH}{dt}}\right) + \left(\frac{\partial p}{\partial \lambda}\right)_{H,q}\left(T\frac{d\lambda}{dt}\right)\right] dq = 0 \tag{14}$$

The quantity on the left is just $\overline{\frac{d}{dt}\oint pdq}$.

Thus, $\overline{\frac{d}{dt}\oint pdq} = 0$ and $\oint pdq$ is an adiabatic invariant.

This adiabatic invariant can be applied to one-dimensional classical mechanics systems executing periodic motion, such as a harmonic oscillator (Landau and Lifshitz, 1976, p. 157).

The adiabatic invariant also has an analog in quantum mechanics. Sommerfeld and Wilson (see Gasiorowicz, 1974, p. 19) proposed that:

$$\oint pdq = nh$$

where p is the momentum associated with the coordinate q, h is Planck's constant, and n is an integer. This equation demonstrates that the idea of the adiabatic invariant is not restricted to the classical regime.

Now that the general notion of the adiabatic invariant has been introduced, it can be applied to a charged particle undergoing periodic motion in a magnetic field. One important example of such motion is that of a charged particle trapped in the magnetic field of the Earth. This is the primary focus of the remainder of this thesis.

To a first approximation, the magnetic field of the Earth can be considered to be a dipole. This approximation is most valid in the regions of the magnetosphere where the solar wind does not strongly interact with the magnetosphere and where the Earth's higher-order magnetic moments are negligible. In such a situation, there are three main types of periodic motion a charged particle may undergo.

(1) Charged particle circling about a field line: If the magnetic field changes on a timescale long compared to the period of the particle's motion (the cyclotron period), the magnetic moment of the particle remains constant. An equivalent statement is that the magnetic flux through the particle's cyclotron orbit is constant. This is the so-called first adiabatic invariant.

(2) Charged particle constrained to move between two regions of stronger magnetic field: The particle "bounces" between these two regions, thus executing a periodic

motion for which there may be associated an adiabatic invariant. In the magnetic field of the Earth, the particle travels along a field line, then mirrors at some point near one of the magnetic poles, and then travels in the opposite direction along a field line to mirror near the other pole, etc.. The invariant for this motion is called the second adiabatic invariant (or the "longitudinal invariant").

(3) Drift motion about the planet: The combination of bounce motion and drift motion defines a surface on which the particle is constrained to move (called the "longitudinal invariant surface" by Northrop, 1963, p. 61). The third adiabatic invariant, or flux invariant, is the magnetic flux through a surface connected to the longitudinal invariant surface and passing through the magnetic axis (Johnson, 1965, p. 57).

If their criteria for use are satisfied, adiabatic invariants can simplify the calculations one must perform to obtain useful results when studying the motion of charged particles in magnetic fields. Things become more complicated if, for example, the magnetic field varies either on a time-scale comparable to the cyclotron period or on a distance-scale comparable to the cyclotron radius. Should this situation occur for motion in a dipole magnetic field, one would have to use Stormer Theory (Chamberlain, 1964, pp. 1 - 13), which is considerably more awkward to use than the adiabatic invariants.

Now that the adiabatic invariants relevant to the motion of charged particles in a dipole magnetic field have been introduced, they can be covered in more detail. Because the particles constrained to move in the Earth's magnetosphere are nonrelativistic, only nonrelativistic motion is considered in this thesis. Rigorous proofs of all three adiabatic invariants can be found in Northrop-Teller, 1960, pp. 215-225 and in Northrop, 1963, pp. 41-67.

Chapter 2: The First Adiabatic Invariant

As mentioned previously, the first adiabatic invariant states that the magnetic moment of a particle owing to its motion about a magnetic field line is constant or, equivalently, that the magnetic flux through its cyclotron orbit is constant. In order to derive this invariant, we need to know the scale of the cyclotron orbit and how quickly the particle moves about a field line.

To find these quantities, we assume that no electric field is present, and that the magnetic field is uniform and static such that:

$\mathbf{A} = xB\hat{y}$ and $\phi = 0$ (where $\mathbf{A}$ and ϕ are the vector and scalar potentials, respectively)

The electric field is given by:

$$\mathbf{E}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t) - \frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t} \tag{15}$$

This gives $\mathbf{E} = 0$ as has been assumed.

The magnetic field is:

$$\mathbf{B}(\mathbf{x},t) = \nabla\times\mathbf{A}(\mathbf{x},t) \tag{16}$$

In this case, $\mathbf{B} = B\hat{z}$.

The Hamiltonian is given by:

$$H(\mathbf{p},\mathbf{x},t) = \frac{1}{2m}\left[\mathbf{p} - e\mathbf{A}(\mathbf{x},t)\right]^2 + e\phi(\mathbf{x},t) \tag{17}$$

which here becomes:

$$H = \frac{1}{2m}\left[\mathbf{p} - exB\hat{y}\right]^2$$

$$= \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2 - 2exBp_y + (exB)^2\right]$$

$$= \frac{1}{2m}\left[p_x^2 + (p_y - exB)^2 + p_z^2\right] \tag{18}$$

Hamilton's equations are:

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

which imply that the equations of motion are:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = \frac{eB}{m}(p_y - exB)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y - exB}{m} \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = 0$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial H}{\partial z} = 0$$

Therefore, $\ddot{x} = \frac{\dot{p}_x}{m} = \frac{eB}{m^2}(p_y - exB)$, which can be rearranged to give:

$$\ddot{x} + \left(\frac{eB}{m}\right)^2 x = \frac{eBp_y}{m} \tag{19}$$

The solution of this differential equation is:

$$x = \frac{p_y}{m\omega_c} + r_c\cos(\omega_c t + b) \tag{20}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency, and r_c and b are constants.

Substituting this solution for x into the equation for $\dot{y}$, we obtain:

$$\dot{y} = \frac{1}{m}\left(p_y - eB\left(\frac{p_y}{eB} + r_c\cos(\omega_c t + b)\right)\right)$$

$$= -\frac{eB}{m} r_c\cos(\omega_c t + b)$$

$$= -\omega_c r_c\cos(\omega_c t + b) \tag{21}$$

Upon differentiating x with respect to time, we see that:

$$\dot{x} = -\omega_c r_c\sin(\omega_c t + b) \tag{22}$$

If $\dot{x}$ and $\dot{y}$ are squared and added together, the result is:

$$\dot{x}^2 + \dot{y}^2 = \omega_c^2 r_c^2\left(\sin^2(\omega_c t + b) + \cos^2(\omega_c t + b)\right)$$

$$= \omega_c^2 r_c^2 \tag{23}$$

$$= \text{constant}$$

Note that $\sqrt{\dot{x}^2 + \dot{y}^2}$ is the magnitude of the velocity perpendicular to the magnetic field. If this quantity is called $v_\perp$, then $v_\perp = \omega_c r_c$. So $r_c = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|e|B}$, which is the cyclotron radius of the particle. The sense in which positive and negatively charged particles move about a magnetic field line is illustrated in the figure below:

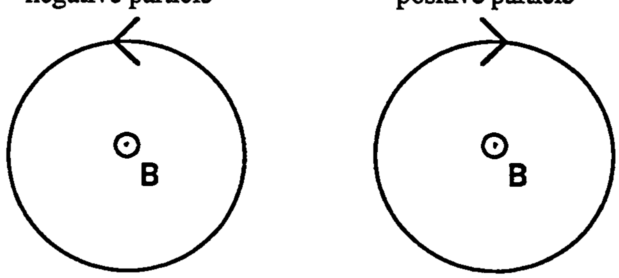


figure 1

These results for ω_c and r_c can be verified by matching the Lorentz force to the centripetal force needed to keep the particle moving in a circle of radius r_c:

$$F_{Lorentz} = F_{cen}$$

$$\Rightarrow \quad |e| v_\perp B = m \frac{v_\perp^2}{r_c} \tag{24}$$

{since the Lorentz force is directed inward and the centripetal force is directed outward}

Therefore, $r_c = \frac{m v_\perp}{|e|B}$. Since $\omega_c = \frac{v_\perp}{r_c}$, we have: $\omega_c = \frac{|e|B}{m}$ as above.

Next, the particle's magnetic moment due to this cyclotron motion can be calculated. From basic electricity and magnetism (see, for example, Wangsness, 1979, pp. 342 - 343), the magnetic moment of a current loop of radius r is: $\mu = I \pi r^2 \hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane defined by the current loop, and I is the current.

$I = \frac{dq}{dt} = \frac{dl}{dt}\frac{dq}{dl}$ where dq is an element of charge, dt is an element of time, and dl is an element of length along the current loop.

It is apparent that $I = v_{\perp}\frac{dq}{dl}$. So, $\mu_{cyclotron} = v_{\perp}\frac{dq}{dl}\pi r_c^2\,\hat{n}$.

If the particle has charge e, after one period of motion this charge will have travelled a distance of $2\pi\, r_c$.

Hence, $\mu_{cyclotron} = v_{\perp}\frac{e}{2\pi r_c}\,\pi\, r_c^2\,\hat{n}$

$$= \frac{v_{\perp} e\, r_c}{2}\hat{n} \qquad (25)$$

From just below equation (24), $r_c = \frac{mv_{\perp}}{|e|B}$.

When we substitute this into equation (25), we obtain:

$$\mu_{cyclotron} = \frac{v_{\perp}e}{2}\frac{mv_{\perp}}{|e|B}\hat{n}$$

$$\Rightarrow \quad \mu_{cyclotron} = \frac{mv_{\perp}^2}{2B} \qquad (26)$$

To show that this quantity is an invariant, consider a magnetic field whose magnitude changes slowly compared to a cyclotron period but whose direction remains constant. From Maxwell's equations, the $\frac{\partial B}{\partial t}$ that exists here means that an electric field is present. This electric field implies that after one orbit about a magnetic field line, the particle gains perpendicular energy equal to:

$$\Delta\varepsilon_{\perp} = -\oint |e|\mathbf{E}\cdot d\mathbf{l} \quad \text{where } \mathbf{E} \text{ is the electric field.}$$

Using Stokes' Theorem and $\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$, we can rewrite this equation as:

$$\Delta\varepsilon_{\perp} = -\int |e|\nabla\times\mathbf{E}\cdot d\mathbf{a} = |e|\int \frac{d\mathbf{B}}{dt}\cdot d\mathbf{a}$$

$$= |e|\frac{dB}{dt}\,\pi\, r_c^2 \qquad (27)$$

$$\frac{d\varepsilon_\perp}{dt} = \frac{\Delta\varepsilon_\perp}{T_c} = \frac{\omega_c}{2\pi}\Delta\varepsilon_\perp = \frac{\omega_c}{2\pi}|e|\pi r_c^{\,2}\frac{dB}{dt}$$

$$= \frac{|e|B}{m}\frac{|e|}{2}\left(\frac{mv_\perp}{|e|B}\right)^2\frac{dB}{dt}$$

$$= \left(\frac{1}{2}mv_\perp^{\,2}\right)\frac{1}{B}\frac{dB}{dt} \qquad (28)$$

We also know that $\Delta\varepsilon_\perp = \Delta\left(\frac{1}{2}mv_\perp^{\,2}\right)$.

Thus, $\frac{d}{dt}\left(\frac{1}{2}mv_\perp^{\,2}\right) = \left(\frac{1}{2}mv_\perp^{\,2}\right)\frac{1}{B}\frac{dB}{dt}$.

$$\left(\frac{1}{2}mv_\perp^{\,2}\right)^{-1}\frac{d}{dt}\left(\frac{1}{2}mv_\perp^{\,2}\right) - \frac{1}{B}\frac{dB}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{mv_\perp^{\,2}}{2B}\right) = 0 \qquad (29)$$

Upon comparing equation (29) with equation (26), we see that:

$$\frac{d}{dt}\left(\mu_{cyclotron}\right) = 0 \;\Rightarrow \mu_{cyclotron} = \text{constant}$$

Hence, $\mu_{cyclotron}$ is invariant for a slowly time-varying magnetic field. If the field changes on a distance scale large compared to the cyclotron radius, this change can be viewed as equivalent to a small change in the field with time. We can therefore apply the above proof to a particle moving in a multipole magnetic field whose dimensions are large compared to a gyroradius.

As mentioned in the introduction, the first adiabatic invariant also states that the magnetic flux through the particle's cyclotron orbit is constant. This can be seen by considering the definition of the adiabatic invariant given in the introduction and taking the magnetic field to be uniform and static (Jackson, 1975, pp. 588 - 590):

$\oint \mathbf{P}_\perp\cdot d\mathbf{l} = \text{constant} = J$ where $\mathbf{P}_\perp$ is the transverse component of the canonical momentum (which is perpendicular to the direction of the magnetic field).

$$\mathbf{P}_\perp = \mathbf{P}_\perp + e\mathbf{A} \tag{30}$$

Figure 2 below shows the relevent vectors for a negatively charged particle.

negative particle

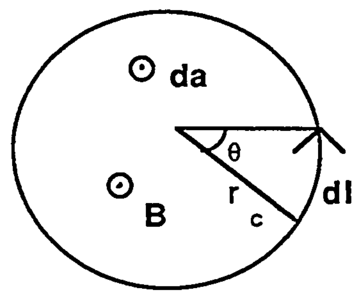


figure 2

Hence, $J = \oint (\mathbf{P}_\perp + e\mathbf{A})\cdot d\mathbf{l} = \oint m\mathbf{v}_\perp \cdot d\mathbf{l} + e\oint \mathbf{A}\cdot d\mathbf{l}$ (31)

$$= \oint m\omega_c r_c dl + e\oint \mathbf{A}\cdot d\mathbf{l}$$

$$= \oint m\omega_c r_c^2 d\theta + e\oint \nabla\times\mathbf{A}\cdot d\mathbf{a}$$ (after using Stokes' Theorem)

$$= 2\pi\, m\frac{|e|B}{m} r_c^2 - |e|\pi r_c^2 B$$

$$= e\pi r_c^2 B \tag{32}$$

$\Rightarrow \quad \pi r_c^2 B$ = magnetic flux through the cyclotron orbit = constant

If we substitute in for r_c, we obtain:

$$\pi\left(\frac{mv_\perp}{|e|B}\right)^2 B = \text{constant}$$

$$\Rightarrow \quad \left(\frac{mv_\perp^{\,2}}{2B}\right) = \text{constant} \tag{33}$$

Thus, saying that $\mu_\perp$ is constant is equivalent to saying that the magnetic flux through the particle's orbit is constant.

We should now try to get some idea of the time and distance scales needed in order for the first adiabatic invariant to hold. From the introduction, we know that an adiabatic invariant will hold as long as:

$\left|T\frac{d\lambda}{dt}\right| << \lambda$ (Where λ specifies the properties of the system, and where T is the period of the motion)

In this case λ is simply the magnetic field B, which means that the first adiabatic invariant holds if $\left|T_c\frac{dB}{dt}\right| << B$ (where T_c is the cyclotron period). In other words, the first adiabatic invariant can be used only in situations where the magnetic field influencing the particle changes on a timescale which is long compared to the particle's cyclotron period. The period of the cyclotron motion is:

$$T_c = \frac{2\pi}{\omega_c} = \frac{2\pi m}{|e|B} \tag{34}$$

For a nonrelativistic electron in a 0.1 Gauss = 10^{-5} Tesla magnetic field (which is of the proper order of magnitude for a particle in the Earth's magnetosphere), this equation gives: $T_c = 3.6 \times 10^{-6}$ seconds.

For a nonrelativistic proton in a 10^{-5} Tesla magnetic field, $T_c = 6.6 \times 10^{-3}$ seconds.

Just as the magnetic field in which the particle moves must change on a timescale long compared to the cyclotron period in order for the first adiabatic invariant to hold, it must also change on a distance scale large compared to the cyclotron radius.

The cyclotron radius is: $$r_c = \frac{mv_\perp}{|e|B} = \frac{T_c}{2\pi} v_\perp \tag{35}$$

For an electron having $v_\perp = 3 \times 10^4$ m/sec: $r_c = 1.7 \times 10^{-2}$ m.

For an electron having $v_\perp = 3 \times 10^6$ m/sec: $r_c = 1.7$ m.

For an electron having $v_\perp = 3 \times 10^7$ m/sec: $r_c = 17$ m.

For a proton having $v_\perp = 3 \times 10^4$ m / sec: $r_c = 31$ m.

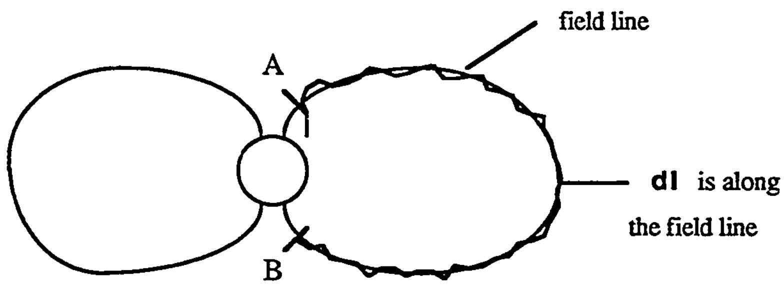
For a proton having $v_\perp = 3 \times 10^6$ m / sec: $r_c = 3100$ m.

For a proton having $v_\perp = 3 \times 10^7$ m / sec: $r_c = 31{,}000$ m.

From the above numbers it is evident that the cyclotron period is independent of particle velocity for a nonrelativistic particle, that the cyclotron radius increases with increasing $v_\perp$, and that both the cyclotron period and cyclotron radius are both about three orders of magnitude larger for protons than for electrons.

Chapter 3: The Second Adiabatic Invariant

The second adiabatic invariant asserts that the line integral from one mirror point to the other and then back again of a charged particle's momentum component parallel to the magnetic field in which the particle is moving is constant (i.e., $\oint p_{||} dl$ = constant, where the integral is taken over one full bounce period). The following figure illustrates the situation for motion in a dipole field:



Point A is the Northern Hemisphere mirror point.
Point B is the Southern Hemisphere mirror point.
The line integral of the particle's momentum along the field line from A to B and back again (or from B to A and back) is constant according to the second adiabatic invariant.

figure 3

To derive a more useful form of this invariant, consider nonrelativistic motion in a time-dependent magnetic field which also has a small enough spatial variation that the first adiabatic invariant is valid. Using energy conservation, we have:

$$\varepsilon = \frac{1}{2}mv^2 + U = \text{constant} \tag{36}$$

In equation (36), U is the potential energy of the particle.

$$U_1 - U_2 = \int_1^2 \mathbf{F} \cdot d\mathbf{l} \quad \text{(Marion, 1970, p. 64)}$$

$$= e\int_1^2 \mathbf{v}\times\mathbf{B}\cdot d\mathbf{l} \quad \text{(using the Lorentz force)}$$

From figure 3, $d\mathbf{l}$ is parallel to $\mathbf{B}$. Thus, $U_1 - U_2 = 0$; and because the zero of potential energy is arbitrary, U can be taken to be zero.

So, $\varepsilon = \frac{1}{2} m\left(v_{\parallel}^2 + v_{\perp}^2\right) = \text{constant}$ (37)

Assuming that the first adiabatic invariant holds (we will shortly see that the first adiabatic invariant holds in situations where the second adiabatic invariant is applied), we can rewrite the energy as:

$$\varepsilon = \frac{1}{2} m v_{\parallel}^2 + \mu_{cyclotron} B$$

$$= \frac{p_{\parallel}^2}{2m} + \mu_{cyclotron} B$$

$$\Rightarrow \quad p_{\parallel} = \pm\sqrt{2m\varepsilon - 2m\mu_{cyclotron} B}$$

$$= \pm\sqrt{2m\left(\varepsilon - \mu_{cyclotron} B\right)} \tag{38}$$

The second adiabatic invariant is, by convention, given the label **J**.

Therefore: $J = \oint p_{\parallel} dl$ (39)

$$= \int_A^B p_{\parallel}(-\text{sign})\,dl + \int_B^A p_{\parallel}(+\text{sign})\,dl$$

$$= 2\int_B^A \sqrt{2m\left(\varepsilon - \mu_{cyclotron} B\right)}\,dl$$

$$= 2\sqrt{2m}\int_B^A \sqrt{\varepsilon - \mu_{cyclotron} B}\,dl \tag{40}$$

At the mirror point, $p_{\parallel} = 0$.

Thus, $\varepsilon = \mu_{cyclotron} B_m$ (where the subscript "m" means mirror point)

$$\Rightarrow \quad J = 2\sqrt{2m}\int_B^A \sqrt{\mu_{cyclotron} B_m - \mu_{cyclotron} B}\, dl$$

$$= 2\sqrt{2m\mu_{cyclotron}}\int_B^A \sqrt{B_m - B(l)}\, dl \qquad (41)$$

Because $2\sqrt{2m\mu_{perp}}$ is a constant, the second adiabatic invariant can also be written as:

$$\int_B^A \sqrt{B_m - B(l)}\, dl = \text{constant}$$

When $B(l) = B_{mirror}$, $J = 0$. A particle will then always have $p_{||} = 0$, which means that if the particle is in a dipole field it will remain in the magnetic equatorial plane. On the other hand, if $B_{mirror} >> B(l)$ then:

$$J = 2\sqrt{2m\mu_{cyclotron}}\int_B^A \sqrt{B_m}\, dl \qquad (42)$$

$$= 2\sqrt{2m\mu_{cyclotron} B_m}\int_B^A dl$$

$$= 2\sqrt{2m\mu_{cyclotron} B_m} \times \left(\text{distance along a field line between points A and B}\right)$$

This last equation implies that the distance along a field line between A and B is constant. If a particle moving in the magnetic field of a planet has a large B_m, it will mirror near the planet and therefore be likely to enter the planet's atmosphere and be forever lost from the magnetosphere.

By analogy with the first adiabatic invariant, the second adiabatic invariant will hold whenever $\left|T_{bounce}\frac{dB}{dt}\right| << B$ (where T_{bounce} is the bounce period).

An order-of-magnitude estimate of the bounce period can be obtained for a dipole magnetic field: $\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left[3\left(\mu\cdot\hat{r}\right)\hat{r} - \mu\right]$ (where μ is the magnetic dipole moment of the field, $\hat{r}$ is the radial unit vector, and μ_0 is the permeability of free space. The coordinates are defined in figure 4 below.

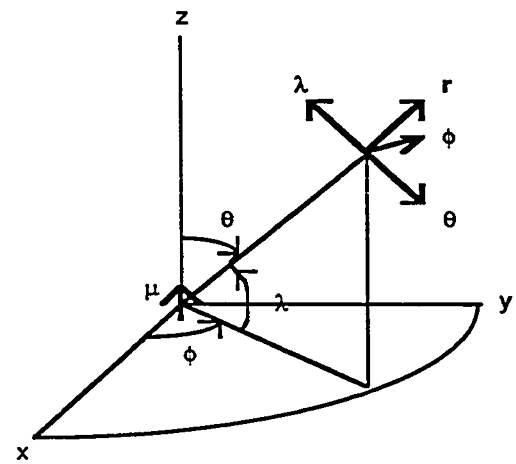


figure 4

If μ is taken to be in the z-direction, **B** can be written as:

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left[3\left(\mu\hat{z}\cdot\hat{r}\right)\hat{r} - \mu\hat{z}\right] \tag{43}$$

$$= \frac{\mu_0}{4\pi r^3}\left[3\,\mu\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\cdot\hat{r}\,\hat{r} - \mu\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[3\,\mu\cos\theta\,\hat{r} - \mu\cos\theta\,\hat{r} + \mu\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0\mu}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] \tag{44}$$

$$= \frac{\mu_0\mu}{4\pi r^3}\left[2\cos\left(\frac{\pi}{2} - \lambda\right)\hat{r} - \sin\left(\frac{\pi}{2} - \lambda\right)\hat{\lambda}\right]$$

$$= \frac{\mu_0\mu}{4\pi r^3}\left[2\sin\lambda\,\hat{r} - \cos\lambda\,\hat{\lambda}\right] \tag{45}$$

Equation (45) implies that:

$$B_r = \frac{\mu_0 \mu}{2\pi r^3} \sin\lambda \tag{46}$$

$$B_\lambda = -\frac{\mu_0 \mu}{4\pi r^3} \cos\lambda \tag{47}$$

$$B_\phi = 0 \tag{48}$$

One should note that for the Earth, the magnetic dipole moment is in the -z-direction. The differential equations of a field line are (Roederer, 1970, p. 53):

$$\frac{r\,d\lambda}{B_\lambda} = \frac{dr}{B_r} \text{ and } d\phi = 0$$

$$\Rightarrow \; -\frac{4\pi r^4 d\lambda}{\mu_0 \mu \cos\lambda} = \frac{2\pi r^3}{\mu_0 \mu \sin\lambda} dr$$

$$-2 \tan\lambda\, d\lambda = \frac{dr}{r}$$

$$2 \ln\cos\lambda = \ln r - \ln r_0$$

$$\Rightarrow \; r = r_0 \cos^2\lambda \tag{49}$$

$$\text{and } \phi = \phi_0 = \text{constant} \tag{50}$$

Here r_0 is the distance from the origin of coordinates to the field line's equatorial point. The element of length of a field line is (Roederer, 1970, p. 53):

$$dl = \sqrt{dr^2 + r^2 d\lambda^2} \tag{51}$$

$$= \sqrt{\left(-2\, r_0 \cos\lambda \sin\lambda\, d\lambda\right)^2 + r_0^2 \cos^4\lambda\, d\lambda^2}$$

$$= \sqrt{\left(4\, r_0^2 \cos^2\lambda \sin^2\lambda\right)^2 + r_0^2 \cos^4\lambda}\; d\lambda$$

$$= r_0 \cos\lambda\sqrt{4\sin^2\lambda + 1 - \sin^2\lambda}\, d\lambda$$

$$= r_0 \cos\lambda\sqrt{1 + 3\sin^2\lambda}\, d\lambda$$

$$= r_0 \cos\lambda\sqrt{4 - 3\cos^2\lambda}\, d\lambda \tag{52}$$

The bounce period is given by (Roederer, 1970, p. 36):

$$T_{bounce} = 2\int_B^A \frac{dl}{v_{||}(l)} \tag{53}$$

From the first adiabatic invariant: $\frac{mv_m^2}{2B_m} = \frac{mv_\perp^2}{2B}$ (where the subscript "m" once again means mirror point, and v_m is the magnitude of the perpendicular velocity at the mirror point).

Hence, $v_\perp^2 = v_m^2\left(\frac{B}{B_m}\right)$ (54)

Energy conservation gives:

$$\frac{1}{2}mv_m^2 = \frac{1}{2}m\left(v_{||}^2 + v_\perp^2\right)$$

$$v_{||}^2 = v_m^2 - v_\perp^2$$

$$= v_m^2 - v_m^2\left(\frac{B}{B_m}\right)$$

$$= v_m^2\left[1 - \frac{B}{B_m}\right] \tag{55}$$

For a dipole field:

$$B = \frac{\mu_0\mu}{2\pi r^3}\sqrt{\sin^2\lambda + \frac{1}{4}\cos^2\lambda}$$

$$= \frac{\mu_0\mu}{4\pi r_0^3\cos^6\lambda}\sqrt{4\sin^2\lambda + \cos^2\lambda}$$

$$= \frac{\mu_0\mu}{4\pi r_0^3\cos^6\lambda}\sqrt{4 - 3\cos^2\lambda} \tag{56}$$

Hence:

$$T_{bounce} = \frac{2}{v_m} \int_B^A \frac{r_0 \cos\lambda \sqrt{4-3\cos^2\lambda}}{\sqrt{1 - \frac{\mu_0 \mu}{4\pi r_0^3 \cos^6\lambda} \frac{\sqrt{4-3\cos^2\lambda}}{B_m}}} d\lambda \qquad (57)$$

Since we are interested in an order-of-magnitude estimate of the bounce period, B_m can be taken to be large compared to B.

Therefore: $$T_{bounce} \cong \frac{2 r_0}{v_m} \int_B^A \cos\lambda \sqrt{4 - 3\cos^2\lambda}\, d\lambda \qquad (58)$$

$$= \frac{2 r_0}{v_m} \int_B^A \sqrt{1 + 3\sin^2\lambda}\, d(\sin\lambda)$$

$$= \frac{2\sqrt{3}\, r_0}{v_m} \int_B^A \sqrt{\frac{1}{3} + \sin^2\lambda}\, d(\sin\lambda)$$

$$= \frac{2\sqrt{3}\, r_0}{v_m} \left(\frac{1}{2} \left[\sin\lambda \sqrt{\frac{1}{3} + \sin^2\lambda} \pm \frac{1}{9} \ln\left(\sin\lambda + \sqrt{\frac{1}{3} + \sin^2\lambda} \right) \right] \right)$$

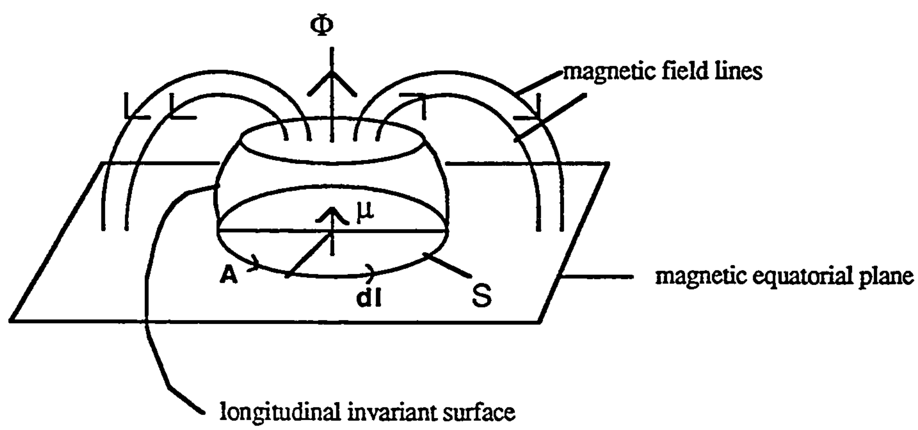
The quantity in the outer parentheses is of order 0.1. So, $T_{bounce} \cong \frac{0.2\sqrt{3}\, r_0}{v_m}$. For a particle moving in the Earth's magnetic field on a field line with $r_0 = 1.5\ R_E$, we find that:

$$T_{bounce} \cong (0.11 \text{ -- } 1100)\ \text{sec} \quad \text{with } v_m = 3 \times \left(10^3 \text{ -- } 10^7\right) \text{m/sec}$$

The bounce period is several orders of magnitude larger than the cyclotron period calculated earlier for both electrons and protons.

Chapter 4: The Third Adiabatic Invariant

The last of the three adiabatic invariants states that the magnetic flux through a surface connected to the longitudinal invariant surface and passing through the magnetic axis is constant. The situation is illustrated below for a dipole field (Roederer, 1970, p. 77):



A positively charged particle drifts in the **A** direction
A negatively charged particle drifts in the **-A** direction

figure 5

Surface S is one possible surface for which the third adiabatic invariant holds.
The magnetic flux through a surface S is given by: $\Phi = \int_s \mathbf{B}\cdot d\mathbf{a}$ (where $d\mathbf{a}$ is an area element). One often finds it useful to determine Φ from the vector potential **A** instead of **B**. Since $\mathbf{B} = \nabla \times \mathbf{A}$, we obtain: $\Phi = \int_s \nabla\times\mathbf{A}\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$ (also using Stokes' Theorem).

$d\mathbf{l}$ is a length element.

For a dipole field: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mu\times\mathbf{r}}{r^3}$ (Wangsness, 1979,p. 339). Here, **r** is a distance vector in the radial direction. If μ is taken to be in the +z-direction, **A** becomes:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mu\hat{z}\times\hat{r}}{r^2} \qquad \text{(since } \mathbf{r} = r\hat{r}\text{)}$$

$$= \frac{\mu_0}{4\pi}\frac{\mu}{r^2}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\times\hat{r}$$

$$= \frac{\mu_0\mu\sin\theta}{4\pi r^2}\,\hat{\phi} \tag{59}$$

Here, $\hat{\phi}$ is the azimuthal unit vector. Using spherical coordinates:

$$\mathbf{dl} = \mathbf{dr} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \tag{60}$$

Therefore: $\oint \mathbf{A}\cdot\mathbf{dl} = \int \frac{\mu_0\mu\sin\theta}{4\pi r^2}\, r\sin\theta\, d\phi$

$$= \frac{\mu_0\mu\sin^2\theta}{2r}$$

$$= \frac{\mu_0\mu\cos^2\lambda}{2r} \tag{61}$$

So, $\Phi = \frac{\mu_0\mu\cos^2\lambda}{2r} = \text{constant}$ (62)

Note that this was derived for a positively charged particle. For a negatively charged particle, $\mathbf{dl}$ would be in the opposite direction, so Φ would be negative.

By analogy with the first adiabatic invariant, the flux invariant will be true if the magnetic field changes on a timescale long compared to the time it takes the particle to complete one full orbit along S. In order to determine this timescale, the particle's drift velocity is needed. The drift velocity of a particle in a dipole magnetic field is made up of two terms: the gradient drift velocity and the curvature drift velocity.

First, we discuss the gradient drift velocity, which is (Roederer, 1970, p. 15):

$$\mathbf{v}_{GD} = \frac{m v_\perp^2}{2eB^3}\,\mathbf{B}\times\nabla B \tag{63}$$

From equation (44): $\mathbf{B} = \frac{\mu_0\mu}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$

$$\Rightarrow \quad B = \frac{\mu_0 \mu}{4\pi r^3}\sqrt{1 + 3\cos^2\theta} \tag{64}$$

$$\nabla B = \left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]\left(\frac{\mu_0 \mu}{4\pi r^3}\sqrt{1 + 3\cos^2\theta}\right)$$

$$= \frac{-3\,\mu_0 \mu}{4\pi r^4}\left[\sqrt{1 + 3\cos^2\theta}\;\hat{r} + \cos\theta\sin\theta\left(1 + 3\cos^2\theta\right)^{-1/2}\hat{\theta}\right] \tag{65}$$

$$B\times\nabla B = \frac{-3}{r}\left(\frac{\mu_0 \mu}{4\pi r^3}\right)^2\left[-\sin\theta\sqrt{1 + 3\cos^2\theta}\;\hat{\phi} + 2\cos^2\theta\sin\theta\left(1 + 3\cos^2\theta\right)^{-1/2}\hat{\phi}\right]$$

$$\mathbf{v}_{GD} = \frac{3\,mv_\perp^2}{2e}\,\frac{4\pi r^2}{\mu_0\mu}\,\frac{\sin\theta\left(1 + 3\cos^2\theta\right) - 2\cos^2\theta\sin\theta}{\left(1 + 3\cos^2\theta\right)^2}\,\hat{\phi}$$

$$= \frac{6\pi\,mv_\perp^2\,r^2\sin\theta}{\mu_0\mu e\left(1 + 3\cos^2\theta\right)^2}\left[1 + \cos^2\theta\right]\hat{\phi}$$

$$= \frac{6\pi\,mv_\perp^2\,r^2\cos\lambda}{\mu_0\mu e\left(1 + 3\sin^2\lambda\right)^2}\left[1 + \sin^2\lambda\right]\hat{\phi} \tag{66}$$

Next, there is the curvature drift velocity (Roederer, 1970, p. 16):

$$\mathbf{v}_{CD} = \frac{mv_{||}^2}{ReB^2}\,\hat{r}\times\mathbf{B} \tag{67}$$

In equation (67), R is the radius of curvature. This equation can be rewritten as:

$$\mathbf{v}_{CD} = \frac{-\,mv_{||}^2}{eB^2}\left[\left(\hat{B}\cdot\nabla\right)\hat{B}\right]\times\mathbf{B} \tag{68}$$

$\hat{B}$ is a unit vector in the B-direction.

$$\hat{B} = \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{\sqrt{1 + 3\cos^2\theta}} \tag{69}$$

$$\hat{B}\cdot\nabla = \left(\frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{\sqrt{1 + 3\cos^2\theta}}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\right)$$

$$= \left(1 + 3\cos^2\theta\right)^{-1/2}\left[2\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right] \tag{70}$$

$$\left(\hat{B}\cdot\nabla\right)\hat{B} = \left(1 + 3\cos^2\theta\right)^{-1/2}\left[2\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\left(\frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{\sqrt{1 + 3\cos^2\theta}}\right)$$

$$= \frac{3\sin\theta}{r\left(1 + 3\cos^2\theta\right)}\left[\sin\theta\left(\frac{2\cos^2\theta}{1 + 3\cos^2\theta} - 1\right)\hat{r} + \cos\theta\left(\frac{\sin^2\theta}{1 + 3\cos^2\theta} + 1\right)\hat{\theta}\right] \tag{71}$$

$$\left[\left(\hat{B}\,.\,\nabla\right)\mathbf{B}\right]\times\mathbf{B} = \frac{\mu_0\,\mu\sin\theta}{4\pi r^4\left(1 + 3\cos^2\theta\right)}\left\{-3\sin^2\theta - 6\cos^2\theta\right\}\hat{\phi}$$

$$= \frac{-3\,\mu_0\,\mu\sin\theta}{4\pi r^4\left(1 + 3\cos^2\theta\right)}\left\{1 + \cos^2\theta\right\}\hat{\phi} \tag{72}$$

$$= \frac{-3\,\mu_0\,\mu\cos\lambda}{4\pi r^4\left(1 + 3\sin^2\lambda\right)}\left\{1 + \sin^2\lambda\right\}\hat{\phi} \tag{73}$$

So, $$\mathbf{v}_{CD} = \frac{-mv_{||}^2}{e\left(1 + 3\sin^2\lambda\right)^2}\left(\frac{4\pi r^3}{\mu_0\,\mu}\right)^2\frac{-3\,\mu_0\,\mu\cos\lambda}{4\pi r^4\left(1 + 3\sin^2\lambda\right)}\left\{1 + \sin^2\lambda\right\}\hat{\phi}$$

$$= \frac{12\pi\,mv_{||}^2\,r^2\cos\lambda}{\mu_0\mu e\left(1 + 3\sin^2\lambda\right)^3}\left\{1 + \sin^2\lambda\right\}\hat{\phi} \tag{74}$$

The similarity between $\mathbf{v}_{GD}$ and $\mathbf{v}_{CD}$ can be seen at once: they are in the same direction and $\frac{v_{GD}}{v_{CD}} = \frac{v_{\perp}^2}{2\,v_{\parallel}^2}$.

The drift period is: $T_{drift} = \frac{2\pi\, r_{drift}}{v_{drift}}$.

To obtain an order-of-magnitude estimate of T_{drift} for particles in the Earth's magnetosphere, assume that the particles are at 1.5 R_E and that they are confined to the magnetic equatorial plane (i.e., $\lambda = 0$). Then:

$$v_{GD} = \frac{6\pi\, m v_{\perp}^2\, r_{drift}^2}{\mu_0\, \mu\, e} \tag{75}$$

and
$$T_{drift} = \frac{\mu_0\, \mu\, e}{3\, m v_{\perp}^2\, r_{drift}} \tag{76}$$

For the Earth (Roederer, 1970, p. 52): $\mu = 8.02 \text{ X } 10^{15}$ Weber m $= 0.311$ Gauss R_E^3. We can now obtain a numerical value for T_{drift}. The following two tables summarize the periods of motion associated with the three adiabatic invariants for electrons and protons of various energies.

electron energy	$v_{\perp}$	cyclotron period	bounce period	drift period
$2.6 \text{ X } 10^{-5}$ eV	$3 \text{ X } 10^{3}$ m / sec	$3.6 \text{ X } 10^{-6}$ sec	1100 sec	$2.2 \text{ X } 10^{6}$ years
$2.6 \text{ X } 10^{-3}$ eV	$3 \text{ X } 10^{4}$ m / sec	$3.6 \text{ X } 10^{-6}$ sec	110 sec	$2.2 \text{ X } 10^{4}$ years
0.26 eV	$3 \text{ X } 10^{5}$ m / sec	$3.6 \text{ X } 10^{-6}$ sec	11 sec	220 years
26 eV	$3 \text{ X } 10^{6}$ m / sec	$3.6 \text{ X } 10^{-6}$ sec	1.1 sec	2.2 years
2600 eV	$3 \text{ X } 10^{7}$ m / sec	$3.6 \text{ X } 10^{-6}$ sec	0.11 sec	8.0 days

proton energy	$v_\perp$	cyclotron period	bounce period	drift period
0.047 eV	3×10^3 m / sec	6.6×10^{-3} sec	1100 sec	1200 years
4.7 eV	3×10^4 m / sec	6.6×10^{-3} sec	110 sec	12 years
470 eV	3×10^5 m / sec	6.6×10^{-3} sec	11 sec	43 days
47,000 eV	3×10^6 m / sec	6.6×10^{-3} sec	1.1 sec	10 hours
4.7 MeV	3×10^7 m / sec	6.6×10^{-3} sec	0.11 sec	6.2 minutes

Upon comparing T_{drift} with T_{bounce} , we see that for electrons, the drift period is at least five orders of magnitude larger than the bounce period; whereas for protons, the drift period is at least three orders of magnitude larger than the bounce period. As these tables show, the drift period is much longer than the bounce period, which is much longer than the cyclotron period for both electrons and protons. Therefore, the third adiabatic invariant is the most easily violated when the magnetic field changes, and the first adiabatic invariant is the least likely to be violated.

Chapter 5: Cyclotron and Drift Magnetic Moments

By analogy with the first adiabatic invariant which has a cyclotron magnetic moment associated with it, we evaluate here the drift magnetic moment associated with the third adiabatic invariant.

To obtain the drift magnetic moment, we proceed just as for the cyclotron magnetic moment. From equations (66) and (74), we know that the particle drifts in the azimuthal direction and therefore in a circle. The drift motion analog of equation (25) can thus be used: $\mu_{drift} = v_{drift} e r_{drift} / 2$.

Using equations (66) and (74):

$$\mu_{drift} = \left(v_{GD} + v_{CD}\right)\frac{e\, r_{drift}}{2} \tag{77}$$

$$= \frac{6\pi m\, r_{drift}^{2} \cos\lambda}{\mu_0 \mu e\left(1 + 3\sin^2\lambda\right)^2}\left[1 + \sin^2\lambda\right]\left(v_\perp^2 + 2 v_{||}^2\right)\frac{e\, r_{drift}}{2}$$

$$= \frac{3\pi m\, r_{drift}^{3} \cos\lambda}{\mu_0 \mu \left(1 + 3\sin^2\lambda\right)^2}\left[1 + \sin^2\lambda\right]\left(v_\perp^2 + 2 v_{||}^2\right) \tag{78}$$

When μ_{drift} is compared to the cyclotron magnetic moment, $\mu_{cyclotron}$, some interesting and unexpected results are obtained.

The cyclotron magnetic moment for a dipole field is:

$$\mu_{cyclotron} = \frac{m v_\perp^2}{2B}$$

$$= \frac{2\pi r^3 m v_\perp^2}{\mu_0 \mu \sqrt{1 + 3\sin^2\lambda}} \tag{79}$$

r is equal to r_{drift}, so $\mu_{drift} / \mu_{cyclotron}$ is:

$$\frac{\mu_{drift}}{\mu_{cyclotron}} = \frac{\dfrac{3\pi m r_{drift}^{3} \cos\lambda}{\mu_0 \mu \left(1 + 3\sin^2\lambda\right)^2} \left[1 + \sin^2\lambda\right]\left(v_\perp^2 + 2 v_{||}^2\right)}{\dfrac{2\pi r^3 m v_\perp^2}{\mu_0 \mu \sqrt{1 + 3\sin^2\lambda}}}$$

$$= \frac{3}{2} \cos\lambda \frac{\left(1 + \sin^2\lambda\right)}{\left(1 + 3\sin^2\lambda\right)^{3/2}} \frac{\left(v_\perp^2 + 2 v_{||}^2\right)}{v_\perp^2} \qquad (80)$$

It can already be seen that $\mu_{drift} / \mu_{cyclotron}$ is of order unity. For a particle constrained to move in the magnetic equatorial plane, $\lambda = 0$ and $v_{||} = 0$, which gives:

$$\frac{\mu_{drift}}{\mu_{cyclotron}} = \frac{3}{2} \qquad (81)$$

This means that no matter what the value of $v_\perp$, the drift magnetic moment is 1.5 times larger than the cyclotron magnetic moment. When casually considering these magnetic moments, one would not expect that all the terms would cancel out so neatly, so this result is somewhat surprising.

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