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CRACK TIP HEATING AND WORK UNDER CONDITIONS OF CYCLIC LOADING

Rice University M.S. 1982

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CRACK TIP HEATING AND WORK
UNDER CONDITIONS OF CYCLIC LOADING

by

PETER JOHN LOOS

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

MASTER OF SCIENCE

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APRIL 1982
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ABSTRACT

Crack tip plastic deformation, associated with crack propagation, is an irreversible process that generates heat. The source of heat near the crack tip is distributed and inhomogeneous and assumed to be either stationary or slowly moving. Furthermore, as heat is produced, it is also dissipated into the surrounding material. The amount and distribution of the heat generated may be derived from the stress and from plastic strain fields that surround the crack tip.

The direction of plastic flow in the crack tip material is periodically reversed during cyclic deformation. This reversal of flow occurs after each half cycle of loading, and corresponds to a large change in stress imposed on the crack-tip material. This material undergoes a stress change from yield in tension to yield in compression (or vice versa).

If elastic-perfectly plastic behavior is assumed, this stress change is twice the yield strength. This factor of two which results from cyclic deformation, in effect, doubles the material strength that is observed during monotonic tensile tests. Consequently, the cyclically created zone is one quarter the size of a monotonically produced plastic zone. At the same time there is a reduction in the plastic work rate (the product of stress and plastic strain rate).

A theoretical model of crack-tip heating and work was constructed
based upon variations in the linear elastic stress-intensity factor. During each half-cycle of loading, the stress-intensity factor increases from zero to some readily determined maximum value and then starts over again at zero. The time derivative of the square of variations in the stress-intensity factor is proportional to the plastic work rate. Inserting the plastic work rate into a two dimensional equation of heat flow yields the crack-tip temperature as a function of time. An average crack-tip temperature change with respect to the ambient and over one cycle of loading was also determined.

The extent of the hottest region near the crack-tip was estimated. An ideal point source of heat, with a power output equal to the time-average plastic work rate, produces an infinite temperature at the origin and temperatures which drop off with increasing distance from the origin. At a distance, \( r_o \), from the origin, the temperature due to the ideal point source is equal to the average crack-tip temperature due to plastic deformation. This distance \( r_o \) is a good measure of the size of hottest region that exists near the crack-tip.

With the aid of a scanning infrared camera system, crack-tip heating was measured during fatigue loading of ductile AISI 4135 steel compact tension fracture specimens. Measured temperatures compare reasonably well with values predicted by the theoretical model. However, temperature changes—both calculated and observed—under prevailing testing conditions were small and had virtually no influence on the fracture process.
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NOMENCLATURE

A - area of cylinder (per unit length)
a - square root to thermal diffusivity
B - specimen thickness
c - heat capacity
D₀ - diameter of cylinder, ft.
E - Young's modulus
E₁ - the exponential integral defined by \( E₁(x) = \int_x^\infty e^{-t} \frac{dt}{t} \)
f - plastic work rate, a function of location in the plastic zone and a function of time or may also refer to frequency
G - shear modulus, \( G = \frac{E}{2(1 + \nu)} \)
g - a function of \( r \) and \( R_p \) that matches the plastic work function to its boundary conditions, \( g(r, R_p) = \sqrt{1 - \left( \frac{r}{R_p} \right)^2} \)
h - plastic zone variable, \( h = \frac{mK₁^2}{2\pi n^2\tau_o^2} \) with dimensions square root of time
hᵣ - surface coefficient of heat transfer, BTU/°F ft² hr
I - see equation (A4)
I₀, I₁ - modified Bessel functions of the first kind
J - see equation (A5)
J₀ - value of \( J \) for \( \beta_p^2 \alpha \ll 1 \)
Jₙ - value of \( J \) for \( \beta_p^2 \alpha \gg 1 \)
K - stress intensity factor
\( \Delta K \) - time varying portion of \( K \)
\( K_1 \) - amplitude (peak to peak) of \( \Delta K \)
\( K_o \) - constant term of \( K \)
\[ \ell \] - crack length

\[ M \] - integral defined by \( M(x) = \int_{x}^{\infty} e^{-t} I_0(t) \frac{dt}{t} \)

\[ m \] - dimensionless plastic zone size constant determined by model of crack tip deformation

\[ n \] - number of loading cycles

\[ P \] - load

\[ p, q \] - empirical constants in fatigue crack growth rate

\[ R \] - radius of ideal cylindrical specimen

\[ R_p \] - plastic zone size

\[ r, r_0 \] - distance from crack tip

\[ s \] - a dimensionless parameter that describes the effect of prior loading on material near the crack tip

\[ t \] - time variable that accounts for loading history of a specimen previous to \( t_f \)

\[ t_f \] - time under load

\[ t_1 \] - half-period of load cycle

\[ \Delta t \] - temperature above ambient (from McAdams (1954)), °F

\[ u \] - temperature above ambient

\[ u_n \] - normalized crack tip temperature defined by

\[ u_n = \frac{8s^2 G \tau^2}{m^2 K_1^4} \]

\[ u_{0} \] - ambient temperature

\[ v \] - load line displacement

\[ w \] - specimen width or dummy variable

\[ W \] - work integrated over the entire plastic zone, a function of time

\[ x, y \] - rectangular coordinates within the plastic zone
\( x', y' \) - rectangular coordinates within the specimen

\( z, z' \) - dummy variables of integration

\( \alpha \) - dimensionless variable, \( \alpha = t_f/(t_f - t) \)

\( \beta \) - dimensionless variable, \( \beta = r^2/4a^2 t_f \)

\( \beta_p \) - dimensionless variable, \( \beta_p = r_p^2/4a^2 t_f \)

\( \beta_o \) - dimensionless variable, \( \beta_o = r_o^2/4a^2 t_f \)

\( \gamma_o \) - yield strain

\( \gamma_p \) - plastic strain

\( \theta \) - angle from direction of crack advance, see Figure 2

\( \kappa \) - thermal conductivity, \( \kappa = a^2 \rho c \)

\( \lambda \) - dimensionless variable that accounts for cyclic load reversals, \( \lambda = \frac{t}{t_1} \)

\( \nu \) - Poisson's ratio

\( \rho \) - mass density

\( \tau_o \) - yield strength in shear

\( \dot{w} \) - denotes time derivative of \( w \)

\( \langle u \rangle \) - denotes average value of \( u \) on a circle of radius \( r_o \)

\( \bar{u} \) - denotes time average of \( u \) over one half-cycle of loading

\( |f| \) - denotes the absolute value of function \( f \)
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1 INTRODUCTION

1.1 General Outline

The general purpose of this work is to achieve a better understanding of processes occurring at and near the tip of a crack in a material body subjected to external stress. In particular, this work is limited to examining how these crack tip processes (primarily plastic work) reveal themselves by releasing heat within the plastically deformed zone near the tip of a crack under cyclic loading conditions. A mathematical model of crack tip heating under such conditions was derived, and attempts were made to verify those results experimentally. The experimental work was carried out during tensile (Mode 1, see Figure 1) plane strain cyclic loading of modified AISI type 4135 steel specimens in a room temperature air environment of moderate humidity. These specimens were similar to the compact tension specimen described by ASTM specification E399-78.

Two pieces of equipment were essential to this work. Loads were applied to the specimens with a servohydraulic test machine under displacement control. This machine was also used to fatigue precrack the specimens. Heating within the plastic zone near the crack tip was examined using a scanning infrared camera system with a liquid nitrogen cooled photovoltaic indium antimonide semiconductor detector operating at a wavelength of about 2 to 6 microns. With this camera system, differences in the infrared emissions from a specimen's surface (corresponding to differences in temperature) appear as brightness variations on a CRT image.

Finally, the experimental results are discussed in light of the available qualitative and quantitative predictions of current
crack tip plasticity models. Once heat is produced within the plastic zone, its diffusion throughout the body of the specimen can be determined. However, current understanding of the amount and initial distribution of the heat within the plastic zone is limited.

1.2 Purpose of the Investigation

The reasons for pursuing this investigation are economic. Some products inevitably fail at stress levels below those at which they were designed to operate, due to the effects of fatigue and environment — not to mention defects intrinsic to the material and manufacturing errors. The cost of these failures to society in general and to the drilling industry in particular is staggering.

This research is motivated by an additional concern: although the maximum stress necessary to cause fatigue failure can be much less than the stress needed to cause a simple overload failure, it is nevertheless true that the total work required for fatigue failure is 100 to 1000 times greater than the work needed for a simple overload failure. The great bulk of this work is released within the specimen as heat. It is to be expected that under certain conditions enough heat could accumulate near the crack tip to influence fracture significantly. One goal of this research is to help define those conditions.

In the variety of ways detailed below, temperature affects all types of failure to some extent. Certainly the ambient temperature to which the material is subjected is important. For example, many materials exhibit a ductile to brittle transition as ambient temperature is varied over a narrow range. Another example is the sensitivity of corrosion rates to modest changes in temperature.
Even more important than the ambient temperature, however, is the temperature at and near the crack tip. It is in this region that many of the processes contributing to failure are brought to bear, and it is from this region that fracture proceeds. The processes going on at the crack tip are many and varied. At the same time practically all of them depend in some way on the local temperature.

The most obvious and best understood example is that heating at the crack tip can cause the fracturing material there to respond to stress in a more ductile manner: macroscopically there are changes in yield and ultimate tensile strength and in the strain hardening characteristics of the heated material. A great deal of experimental data on these macroscopic changes can be found in the literature, but they will not be discussed here. The microscopic mechanisms for this particular behavior have to do with the interaction between dislocations and defects in the crystal structure. These defects impede the motion of dislocations\(^1\), and the passage of such dislocations around large impermeable defects is a thermally activated process - that is to say increasing temperature increases the rate at which dislocations overcome the larger obstacles in their paths. These mobile dislocations then result in greater amounts of plastic deformation at a given stress level.

Besides causing these changes within the specimen itself, heating at the crack tip has many other effects and some of them may be even more important in determining the specimen's ultimate failure.

Interactions between the hot, highly stressed, fresh metal surface at

\(^1\) as described by Ashby (1967), Pollet and Burns (1977), Hirsch and Humphreys (1969) and by Suzuki and Ishii (1969) as well as elsewhere in the literature.
the crack tip and the local environment inside the crack are of great importance in determining crack propagation rates. The rates of these electrochemical interactions are, of course, controlled in part by the local temperature.

Although these interactions and their exact contributions to specimen failure are at this time poorly understood, it seems that some general observations can be made. (Support for these observations can be found in the various articles mentioned later on.) Apparently, the presence of even small amounts of water or acid near the crack tip (in the form of either vapors or liquids) can cause several processes to occur, including rapid localized specimen dissolution, release of hydrogen from the active environment and its subsequent absorption into the specimen, and the formation of protective layers of corrosion products on the fresh crack surfaces. These three processes (localized specimen dissolution, absorption of hydrogen into the specimen and corrosion product layer formation) can in turn have many effects. For example, hydrogen can embrittle the material, corrosion product layer formation can protect the material from further attack and localized dissolution can propagate the crack and expose fresh material to attack. It should be noted that while most of those effects aid further crack propagation, that is not always the case. Crack branching (or foliation), which may result from certain environmental effects, can retard propagation.

An additional effect of heating at the crack tip is that not only does it alter corrosion rates (which determine the rate of formation of the protective layer on fresh crack surfaces) but it also alters the mechanical properties of the resulting layer.
Hardness and porosity of the layer can be affected by the temperature during its formation. These properties in turn are quite important because the layer is subjected to enormous stress as the crack propagates. A protective layer which breaks off easily will have a different effect from one which remains intact. This is what makes stainless steel superior to other varieties of steel in the presence of corrosive environments. There are several interesting articles\(^2\) which describe some aspects of these interactions between specimen and environment however, taken as a group they provide a description which is far from complete.

\(^2\) In particular: Rice (1978); McClintock (1973); Smith (1978); Rhodes and Radon (1979); Davidson et al (1978); Cytron (1979); Markworth, Kaminen and Gehlen (1973); Boyd (1973); Ohta and Sasaki (1972); Tien and Richards (1975); Frandsen and Marcus (1975); Davidson and Lyle (1975); Wei, Talda and Li (1967); and Achter (1967).
2. BACKGROUND

An extensive search of the literature has revealed that only a limited amount of experimental work has been done on the subject of heating at the crack tip, and most of the experimental work has been done in fatigue. Apparently this is because slow, easily controlled crack propagation in fatigue can be examined much more readily, using a scanning infrared camera or any other type of detector, than the relatively fast unstable propagation which often results from monotonic loading.

Theoretical models of crack tip heating are not easily constructed. The source of heat is distributed and inhomogenous and it is moving. Furthermore, at the same time that heat is being produced it is also being dissipated in the surrounding material. Another difficulty arises because the specimen surface is in plane stress while the interior is often subject to plane strain conditions. Measured surface temperatures result from some combination of plane stress and plane strain deformation that is not easily determined. Changes in material properties (principally strength) also result from heating and straining. Inertial effects also become important at high strain rates. Theoretical models should take these changes into account.

On the subject of fatigue loading, Hsieh (1977) reported a temperature rise of about .2 to 1.0°C near the crack tip in aluminum single-edge-notched-strip specimens. The yield strength of his aluminum was about 6000 psi and the cyclic loading frequency was 10 Hz. A wide range of crack propagation velocities resulted from changes in stress intensity amplitude.
Heish did not treat stress intensity amplitude as an explicit variable in his derivation of temperature increases, and he never stated the stress intensity amplitudes which occurred in his experimental work. This is unfortunate since these amplitudes are by far the most important factors in determining temperature increases around the crack tip.

Because of large unresolvable differences between his theoretical and experimental results with aluminum, Hsieh abandoned those aluminum specimens. His primary work was on polymeric materials, in which case his theory agreed relatively well with experiment. That agreement was mostly due to Hsieh's arbitrary choice of $10^{-17}$ inch as the inner extent of the plastic zone. At first he had chosen $10^{-12}$ inch and found that it resulted in an incorrect value of temperature rise. His later choice of $10^{-17}$ inch gave a more pleasing result, though.

Attermo and Ostberg (1971) were also interested in polymeric materials and austenitic stainless steel subject to fatigue. They recorded a maximum steady state crack tip heating of 14°C in the steel; but they noted that a fairly steep temperature gradient was seen at the start of cyclic loading and that it leveled off as the bulk of the specimen warmed up during testing.

In their study no estimates were made of the size of the heated zone near the crack tip. Such estimates would be extremely valuable due to the limited spatial resolution of the camera system that was employed and due to the small plastic zone size.

---

3 It was included indirectly, as a factor in plastic zone size.
Norris (1976) examined heating during the fatigue fracture of plain carbon steel. He employed experimental apparatus and specimens which did not allow determination of the crack tip stress intensity. At the tips of rapidly propagating fatigue cracks, Norris observed temperature increases of up to 60 to 80°C above ambient.

Rather than fatigue, concern will now be focused on monotonic loading. This simpler loading regime makes it easier to understand a few of the many complex physical processes occurring during fracture, but as mentioned earlier, it is more difficult to examine experimentally.

As a historical note, some of the earliest work on the general subject of heating during plastic deformation of metals was that of Farren and Taylor (1925) who used standard cylindrical tensile specimens and measured heating with a thermocouple. The report makes for interesting reading; furthermore, two of their comments are applicable to this investigation: (1) "...as will be seen later, the experimental difficulties in making measurements of this kind are considerable, and if several independent workers performed such experiments along parallel lines their time would not be wasted, even if they got identical results," and (2) "It is curious that very few measurements of this type appear to have been made."

More recently Wilburn (1975) re-examined heating caused by plastic deformation with the same sort of cylindrical specimens by analyzing their infrared spectra as they were loaded. This work on cylindrical specimens is of limited usefulness in studies of heating at the crack tip. More relevant is the small amount of literature on standard notched fracture specimens and as before, under conditions of monotonic loading.
Weichert and Schonert (1978a, 1978b) made both theoretical and experimental contributions to this field. Of primary interest was a report of some experimental results (Schonert and Weichert, 1969) which showed a temperature rise of 130°C at 30 μm from the tip of a crack in steel. This was reportedly measured with a small thermocouple, but there was no information given on the plastic zone size or specimen yield strength. In glass specimens, on the other hand, they reported a temperature of about 3000°K and an estimated plastic zone size of about 20 Å, from measurements made on the specimens' thermal radiation.

Their estimated mode I plastic zone size was based on theoretical calculations of adiabatic heating at a moving crack tip. In these calculations it was assumed that the plastically deformed zone was perfectly rectangular in shape and that the plastic work rate was constant throughout the rectangular zone. These simplifying assumptions limit the usefulness of their analysis in some circumstances.

The analysis of Rice and Levy (1969), which is the basis of much of this thesis, is based on the Carslaw and Jaeger (1959) formulation of heat conduction in solids. Rice and Levy obtained models of crack tip heating for the situations of a crack propagating under steady loading and of a stationary crack acted on by increasing load. In their work they employed two crack tip plasticity models: the simple Dugdale (1960) discrete surfaces of tensile yielding model for mode I loading and the Rice (1968a) model of the mode I slip line field. In both of these models perfect plasticity was assumed. They found that, for the stationary crack case, crack tip temperature was rather insensitive to the plasticity model employed but that for the
advancing crack quite the opposite was true.

They also showed that for a given maximum stress intensity factor the temperature change at the crack tip was proportional to the inverse square root of loading time for the case of a stationary crack and was proportional to the square root of crack velocity for the case of a propagating crack. For very high loading rates, adiabatic heating occurred a short distance away from a stationary crack tip; and in that case Rice and Levy found the temperature rise to be independent of loading rate and calculated it to be about 180°C in steel. It should be stressed that the theoretical analysis of Rice and Levy did not consider cyclic loading.

Other studies, by Parvin (1979); Fuller, Fox and Field (1975); Stinchcomb, Reifsneider, et al. (1975) and by Tomashevskii et al. (1970) concern polymers or composites. One aspect of the work of Fuller, Fox and Field, however, is notable: They attempted to use temperature sensitive cholesteric liquid crystals to measure heating around the crack tip. A similar effort was made at the beginning of this study - but with little success. ⁴

Sandor (1981) has indicated that within the plastic zone at the crack tip there may be very tiny zones of intense slip in which the temperature reaches 600°C or so. This might be very dependent on the type of material and its particular microstructure.

⁴ Among the problems with liquid crystals were the following: the grain structure of liquid crystals can obscure plastic zones of a similar size, the liquid crystals were very sensitive to strains in the plastic zone (as well as heating), and their surface tension and gravity made them flow too readily in many cases.
Any understanding of work and heating in the plastic zone must be based upon knowledge of the stress and strain distributions around the crack tip. Those stress and strain distributions depend on the rate of crack propagation and on the strain hardening characteristics of the material, as discussed by Rice (1968b), Amazigo and Hutchinson (1977), Rice and Sorensen (1978) and others for the case of a propagating crack subject to monotonic, mode I loading.

Their results cannot be easily applied to the present study of a crack that is assumed to be stationary and subject to cyclic loading. Nevertheless, some of their general conclusions describe fatigue to some extent. The first is that strain hardening increases the severity of the plastic work singularity. This effect is however offset by a second: crack propagation decreases the severity of the plastic work singularity.

In either elastic perfectly plastic or strain hardening models of a stationary crack, the product of stress and strain exhibits a singularity in \( r \) of order \(-1\). This is not necessarily true for the propagating crack case. In the propagating crack model of Amazigo and Hutchinson the order of the singularity varies from zero to negative one depending on the amount of strain hardening. For modest strain hardening, Amazigo and Hutchinson claim the order of the singularity is about \(-.4\) while for the nonhardening case the order is closer to (but never equal to) zero.

Amazigo and Hutchinson as well as Sorensen (1977) have found that the plastic zone shape in plane strain is altered by strain hardening and by crack propagation. Under these conditions strains are apparently shifted more toward the forward part of the strain field. Sorensen
also reported that the maximum extent of the plastic zone, for a given value of stress intensity, is essentially the same for both stationary and growing cracks.

Although the effects of strain hardening and crack propagation are not accounted for in the theoretical model derived here, their influence on the experimental results is discussed in the last chapter.
3. THEORETICAL INVESTIGATION

3.1. The Stationary-Crack Model

Based in part on the work of Carslaw and Jaeger (1959), Rice and Levy (1969) examined the temperature distribution near a stationary crack under rapidly increasing applied stress. They made two necessary assumptions: (1) that the thickness of the specimen be finite while other dimensions of the specimen not, and (2) that plastic deformation (and consequently temperature) around the crack tip remain unchanged through the thickness of the specimen. The latter assumption reduces the problem to two dimensions. For a plastic zone of arbitrary shape, the temperature distribution as a function of time under load, \( t_f \), and location in the plane of the specimen, \( x' \) and \( y' \), is given by (Rice and Levy, 1969)

\[
(1) \quad u(x', y'; t_f) = \int_0^{t_f} \int_{\text{plastic zone}} \frac{f(x, y; t)}{\rho c} x \cdot \exp \left[ -\frac{(x'-x)^2 + (y'-y)^2}{4a^2(t_f-t)} \right] \, dx \, dy \, \frac{dt}{4\pi \alpha^2(t_f-t)}
\]

Here \( f(x, y; t) \) is the plastic zone work rate (the product of stress and plastic strain rate) as a function of time previous to \( t_f \), \( t \), as well as location within the plastic zone, \( x \) and \( y \). \( \rho \) is mass density, \( c \) is specific heat per unit mass and \( \alpha \) is the diffusivity, \( \frac{\kappa}{\rho c} \) is the thermal conductivity. In cylindrical coordinates the temperature at the crack tip is
\[ u(0,0,t_f) = \int_0^{t_f} \int \int_{\text{plastic zone}} f(r, \theta, t) \times \exp \left[ \frac{-r^2}{4a^2(t_f-t)} \right] r \, d\theta \, dr \frac{dt}{4\pi a^2(t_f-t)} \]

The plastic work rate is determined by the particular model of crack-tip deformation that is chosen. Here we shall consider mode I loading (see Figure 1) of an elastic-perfectly plastic, rate independent material. It is assumed that the yield strength is independent of temperature. Deformation is assumed to be on a small scale, controlled by Irwin's linear elastic stress intensity factor, and is also assumed to follow the slip line field model of Hill (1950). Under conditions of cyclic loading the plastic zone radius, as shown in Figure 2, is given by

\[ R_p = \frac{m}{s \tau_s^2} \cos (2\theta - \pi) \]

for \( r < R_p \) and for \( \frac{3\pi}{4} > \theta > \frac{\pi}{4} \) and \( -\frac{3\pi}{4} < \theta < -\frac{\pi}{4} \), where \( \Delta K \) is the time varying portion of stress intensity that is defined later. \( \tau_s \) is the shear strength of the material while \( m \) and \( s \) are constants. Rice (1968a and 1968b) approximates

\[ m \approx 3(1-\nu) / 4 \sqrt{2} (2 + \pi) \]

where \( \nu \) is Poisson's ratio. If \( \nu \) is 1/3, then \( m \) is approximately .0688. Slightly different values of \( m \) result from different models of crack-tip plasticity.

\[ s \] is a dimensionless parameter which has values between one and two in this model. It describes the effect of prior loading on material near the crack tip. If monotonic loading were considered
in (3) then the factor $s$ would have a value of one. This is because crack tip material under monotonic loading undergoes a stress change from zero to $\tau_0$. A value $s=2$ arises when load reversals cause material at the crack tip to go from yield in compression all of the way to yield in tension - a stress change of $2\tau_0$. Because of this the cyclic plastic zone radius is one quarter that of a plastic zone produced by monotonic loading.

Viewed in a slightly different way, the situation $s=2$ describes the situation where the crack tip is mostly surrounded by material that was yielded during the previous half-cycle of loading. Instead, if the crack tip is surrounded mostly by unyielded material, then the factor $s=1$ would be used in (3) instead. These two situations must be limiting cases of what actually goes on at a crack tip. At a real crack tip, some volume of previously yielded material undergoes a stress change of $2\tau_0$, while some volume of previously unyielded material experiences a stress change of only $\tau_0$.

Under any of these cyclic loading conditions, the plastic work rate (Rice, 1968a and 1968b) becomes

$$
(5) \quad (r, \theta, t) = \frac{d}{dt} \left[ s \tau_0 \gamma_p \right] = \frac{d}{dt} \left[ s \tau_0 s \gamma_0 \frac{R}{r} \right] g(r, R_p) \frac{m \cos(2\theta-\pi)}{G r} g(r, R_p) \frac{d}{dt} [\Delta K]^2
$$

inside the plastic zone and zero elsewhere. In this equation $\gamma_p$ is the plastic strain, $\gamma_0$ is the yield strain, $G$ is the shear modulus and $g(r, R_p)$ is a scaling factor which matches the plastic work function to its boundary condition: at $r=R_p$, $f(r, \theta, t)$ must equal zero. Exactly at the crack tip, $f(r, \theta, t)$ is independent of $R_p$ and depends only on
material properties. \( f(r, \theta, t) \) depends on \( s \) only through the functional dependence of \( g \) on \( R_p \). Rice and Levy (1969) give

\[
g(r, R_p) = \sqrt{1 - \left(\frac{r}{R_p}\right)^2}
\]

which will be used in this study as well. The form of \( g(r, R) \) is not critical because heating is most intense at \( r \approx 0 \) where \( g(r, R_p) \)
approaches unity. In this case \( f(r, \theta, t) \) becomes essentially independent of \( g(r, R_p) \). Inserting (6) and (5) into (2) gives

\[
(7) \quad u(0,0,t_f) = \int_0^{t_f} \int_0^{R_p} 2 \int_0^{3\pi/4} \frac{m \cos (2\theta - \pi)}{Gr \rho c} \frac{d}{dt} \left[ (\Delta K)^2 \right] \sqrt{1 - \left(\frac{r}{R_p}\right)^2} \exp \left[ \frac{-r^2}{4a^2 (t_f - t)} \right] \frac{r}{4\pi a^2 (t_f - t)}
\]

The factor of 2 before the \( \theta \) integral takes into account the two regions of plastic deformation above and below the crack tip. Noting that

\[
\int_{\pi/4}^{3\pi/4} \cos (2\theta - \pi) \ d\theta = 1 \quad \text{and} \quad a^2 \rho c = \kappa \quad \text{yields}
\]

\[
(8) \quad u(0,0,t_f) = \frac{m}{2\pi \kappa G} \int_0^{t_f} \int_0^{R_p} \frac{d}{dt} \left[ (\Delta K)^2 \right] \sqrt{1 - \left(\frac{r}{R_p}\right)^2} \exp \left[ \frac{-r^2}{4a^2 (t_f - t)} \right] \frac{dr}{t_f - t}
\]

It must be remembered that both \( R_p \) and \( \Delta K \) are functions of time; for that reason, integration over the variable \( r \) must be carried out before integration over \( t \).

### 3.2. A Contrived Loading Function

It is appropriate at this point to specify exactly how \( \Delta K \) and hence \( R_p \) depend on time. If we make the simplifying assumption that \( \frac{d}{dt} \left[ (\Delta K)^2 \right] \) is constant then \( \Delta K \) has the form shown in Figure 3.
That result is mathematically very convenient, however artificial.

A more realistic sine-wave load is examined later in this chapter.

Nevertheless, when $\Delta K$ is described by Figure 3 crack-tip temperature becomes

$$u(o,o,t_f) = \frac{mK_1^2}{2\pi \kappa G \tau_1} \int_0^{t_f} \int_0^{R_p} \sqrt{1-(r/R_p)^2} \times$$

$$\exp \left[ -\frac{r^2}{4a^2(t_f-t)} \right] dr \frac{dt}{t_f-t}$$

where $K_1$ is the amplitude of the variations in stress intensity and $\tau_1$ is half the period of those variations. This choice for $\Delta K$ gives a plastic zone size (measured at $\theta = \pi/2$) which varies in time as

$$R_p = \frac{mK_1^2}{s^2 \tau_1} \frac{\text{frac} \left\{ \frac{t_1}{t_f} \right\}}{t_f-t}$$

In equation (10) "frac" denotes the fractional part of the argument.

Frac is a discontinuous function with values between zero and unity. Its discontinuous nature is used to describe the interruptions in crack tip deformation that accompany load reversals.

In (9) as well as in later equations, it is assumed that no elastic relaxation of crack tip material occurs at the start of unloading.

Some elastic unloading does occur, however; and it has been estimated (Rice, 1967) that, for a material similar to the one employed in this study, the range of such elastic unloading is roughly 5% of the maximum applied stress. This elastic unloading might be taken into account by simply reducing the value of $K_1$ by 5%. In that case $K_1^4$ (and hence temperature) would be reduced by about 19%. This correction
is not used in the following calculations however.

The problem is simplified by first expressing the variables in terms of dimensionless quantities and then integrating where possible. When this is done, as shown in Appendix A, crack tip temperature becomes

\[
(11) \quad u(o, o, t_f) = \frac{m K_1^2 \sqrt{E_f}}{4G t_1 \sqrt{\kappa_c}} \int_1^\infty \beta p e^{-\frac{\alpha \beta^2}{2}} \times \left[ I_0 \left( \frac{\alpha \beta^2}{2} \right) + I_1 \left( \frac{\alpha \beta^2}{2} \right) \right] \frac{d\alpha}{\alpha}
\]

where \( \beta_p = \frac{R_p}{2a \sqrt{t_f}} \). To point out how \( \beta_p \) depends on \( \alpha \) in (11) one can define the quantity \( h = m K_1^2 / 2as^2 \tau^2 \) so that

\[
\beta_p = \frac{h}{\sqrt{t_f}} \frac{t_f}{t_1} \left( 1 - \frac{1}{\alpha} \right).
\]

When \( \beta_p \) is defined in this way it is apparent that the value of the integral found in (11) depends on only three quantities: \( h^2, t_1 \) and \( t_f \). All of these are independent variables with the dimensions of time. All material properties and stress intensities are included in \( h \), while \( t_1 \) describes the rate at which loads are applied to the specimen and \( t_f \) the length of time for which the loading conditions are maintained. This is the bare minimum of variables necessary to define the problem. Finally we can rewrite (11) for the crack tip temperature as
\[\begin{align*}
(12) \quad u(o, o, t_f) &= \frac{m^2 K_1^4}{8s^2 G \tau_o \tau_1^2} \int_1^\infty \frac{t_f}{t_1} \left(1 - \frac{1}{\alpha}\right) e^{-\frac{\alpha \beta^2}{2}} \times \\
&\left[ I_0 \left(\frac{\alpha \beta^2}{2}\right) + I_1 \left(\frac{\alpha \beta^2}{2}\right) \right] \frac{d\alpha}{\alpha},
\end{align*}\]

or define a normalized crack tip temperature as

\[\begin{align*}
(13) \quad u_n(o, o, t_f) &= \int_1^\infty \frac{t_f}{t_1} \left(1 - \frac{1}{\alpha}\right) e^{-\frac{\alpha \beta^2}{2}} \times \\
&\left[ I_0 \left(\frac{\alpha \beta^2}{2}\right) + I_1 \left(\frac{\alpha \beta^2}{2}\right) \right] \frac{d\alpha}{\alpha}
\end{align*}\]

This shows more clearly how temperature depends on the material properties \((G, \tau_o, \text{ and } \kappa)\), the half period of loading \((t_1)\), the stress intensity amplitude \((K_1)\) and the chosen model of crack tip plasticity (as represented by \(m\) and \(s\)).

Because of the complicated way in which \(\beta_p\) depends on \(\alpha\), it is difficult to carry out the integration of equation (11) without a high-speed digital computer. Although the original form in which the integration presented itself, equation (9), could also have been solved by numerical integration, it was not practical to do so. It was found that excessive computer time was required for the double integration. When \((A7), (A8)\) and \((A11)\) from Appendix A are used in \((A6)\), however, execution is much faster. In this numerical work an approximation routine shown in Table II (Abramowitz and Stegun, 1964) was used to calculate the modified Bessel functions. The error in this approximation, also listed in Table II, does not affect the
calculated value of crack tip temperature significantly.

Employing these approximations, the numerical integration was carried out in double-precision FORTRAN on an IBM 3033 computer. A description of the computer program is in Appendix B. Figure 4 contains graphs of crack tip temperature for various values of $K_1$, $t_f$ and $t_1$.

3.3 Sine Wave Loading

The time dependence of the stress intensity shown in Figure 3 is convenient but hardly realistic. For that reason we now consider stress intensity varying with the cosine of time as shown in Figure 5.

\begin{equation}
K = K_0 + \frac{K_1}{2} \left\{ 1 - \cos \left( \pi \frac{t-t_f}{t_1} \right) \right\}
\end{equation}

so that $\Delta K = \frac{K_1}{2} \left\{ 1 - \cos \left( \pi \frac{t}{t_1} \right) \right\}$ and consequently

\begin{equation}
\frac{d}{dt} \left[ (\Delta K)^2 \right] = \frac{\pi K_1^2}{4 t_1} \left| 2 \sin \left( \pi \frac{t}{t_1} \right) - \sin \left( 2 \pi \frac{t}{t_1} \right) \right|
\end{equation}

The time average of $\frac{d}{dt} \left[ (\Delta K)^2 \right]$ is just

\begin{equation}
\frac{1}{t_1} \int_0^{t_1} \frac{d}{dt} \left[ (\Delta K)^2 \right] dt.
\end{equation}

and using (15) this becomes

\begin{equation}
\frac{1}{t_1} \int_0^{t_1} \frac{\pi K_1^2}{4 t_1} \left| 2 \sin \left( \pi \frac{t}{t_1} \right) - \sin \left( 2 \pi \frac{t}{t_1} \right) \right| dt
\end{equation}
or
\[
\frac{\pi K^2}{4 t_1^2} \left[ \int_0^{t_1} 2 \sin \left( \pi \frac{t}{t_1} \right) \, dt - \int_0^{t_1} \sin \left( 2 \pi \frac{t}{t_1} \right) \, dt \right]
\]

On the interval $0 < t < t_1$, $\frac{t}{t_1} = \frac{t}{t_1}$.

Also substituting $\theta_1 = \frac{\pi t}{t_1}$, and $\theta_2 = \frac{2\pi t}{t_1}$ leaves

\[
\int_0^{\frac{t_1}{2}} 2 \sin \theta_1 \, d\theta_1 - \int_0^{\frac{2\pi}{t_1}} \sin \theta_2 \, d\theta_2
\]

The integral over $\theta_2$ is just zero, and the integral over $\theta_1$ has a value of 4, so that the time average of $\frac{d}{dt} [(\Delta K)^2]$ during sine wave loading is just $K^2_t/t_1$. This is the same value produced by the somewhat artificial loading function previously examined. It is to be expected then that the sine-wave loading solution for crack-tip temperature will simply oscillate about the earlier solution for which $\frac{d}{dt} [(\Delta K)^2]$ was constant in time. One important quantity that is not determined in this analysis is the amplitude of temperature oscillation. One should anticipate that it depends on the thermal conductivity of the particular material and the frequency. From Figure 5, it is seen that the peak value of $\frac{d}{dt} [(\Delta K)^2]$ during sine wave loading is $\frac{3\sqrt{3}}{2} K^2_t/t_1$ or about 2.60 times greater than the constant value that is produced during loading with the artificial wave form of Figure 3.

When (15) is inserted into (8) the crack tip temperature during sine wave loading becomes
\[ u(o,o,t_f) = \frac{m K_1^2}{2\pi \kappa G t_1} \int_o^{t_f} \int_o^R \exp \left[ \frac{-r^2}{4a^2(t_f-t)} \right] \sqrt{1-(\tau/Rp)^2} \frac{\pi}{4} \times \]

\[ \left| 2 \sin\left(\pi \frac{t}{t_1}\right) - \sin\left(2 \pi \frac{t}{t_1}\right) \right| \frac{dt}{t_f-t} \]

or in terms of dimensionless units

\[ u(o,o,t_f) = \frac{m K_1^2 \sqrt{t_f}}{\pi G t_1 \sqrt{\kappa \rho c}} \int_1^\infty \int_o^\beta \exp \left[ -\beta^2 \alpha \sqrt{1-(\beta/\beta_p)^2} \right] \frac{\pi}{4} \times \]

\[ \left| 2 \sin\left(\pi \frac{t_f}{t_1}\left(1-\frac{1}{\alpha}\right)\right) - \sin\left(2 \pi \frac{t_f}{t_1}\left(1-\frac{1}{\alpha}\right)\right) \right| d\beta \frac{d\alpha}{\alpha} \]

Here, \( \beta_p = R_p/2a^2 \sqrt{t_f} \) as before, but now

\[ R_p = \frac{m K_1^2}{4s^2 t_0} \left(1 - \cos \left[ \pi \frac{t_f}{t_1}\left(1-\frac{1}{\alpha}\right)\right]\right)^2. \]

This integral is rewritten as

\[ u(o,o,t_f) = \frac{m K_1^2 \sqrt{t_f}}{\pi G t_1 \sqrt{\kappa \rho c}} \int_1^\infty J \cdot \frac{\pi}{4} \left| 2 \sin\left(\pi \frac{t_f}{t_1}\left(1-\frac{1}{\alpha}\right)\right) \right| d\alpha \frac{d\alpha}{\alpha} \]

using the earlier definition of \( J \) as approximated by (A7), (A8) and (A11) in Appendix A. Equation (23) for sine wave loading is analogous to (11) for the square root wave loading function of Figure 3.

In Figure 6 there may be found graphs of crack tip temperature for
various values of $s$, $t_1$ and $t_f$. These sine wave loading results were obtained using a computer program similar to the earlier one for square root wave loading.

3.4. Value of Parameter $s$ for Various Crack Growth Rates

To select an appropriate value for the parameter $s$, crack propagation rates must be examined. For example if it is assumed that $s=2$, crack propagation per cycle of loading must be considerably less than the maximum of $R_p$ in the direction of crack advance. This may be expressed for a crack of length $l$ as:

$$\frac{dl}{dn} \ll R_p^{\max} \ (\theta=0)$$

(24)

where $n$ is the number of load cycles applied to the specimen. The direction of crack extension is $\theta=0$; and, unfortunately in that direction $R_p=0$ for an infinitely sharp crack. It has been shown by Rice (1968a), however, that crack tip blunting results in a small region of intense deformation just ahead of the crack. The extent of this region (in the direction $\theta=0$) is approximately

$$R_p^{\max} (\theta=0) \approx 3.6 s \ \gamma_w R_p^{\max} (\theta=\pi/2)$$

(25)

Since $R_p^{\max} (\theta=\pi/2)$ is given by (10), equation (24) becomes

$$\frac{dl}{dn} \ll \frac{3.6 m K_1^2}{s \ \tau_o G}$$

(26)

The dependence of stage II fatigue crack growth rates on the stress
intensity factor (Fuchs and Stephens, 1980) is generally given by (Hudson and Seward, 1978)

\[
\frac{d\phi}{dn} = p \ K_1^q
\]

where \( p \) and \( q \) are empirical constants. The latter assumes values between 2 and 4 for steel in an inert environment. Other situations can result in much higher values of \( q \). Combining (26) and (27) yields

\[
K_1 \approx \left( \frac{1.8m}{\tau_0 \ G \ p} \right)^{\frac{1}{q-2}} = K_1^*
\]

It is still assumed here that \( s=2 \). When applied stress intensities are significantly less than the quantity \( K_1^* \), then the crack tip is effectively surrounded by previously deformed material. Consequently, the plastic work function, \( f(r, \theta, t) \) is reduced significantly and so is the crack propagation rate, \( \frac{d\phi}{dn} \).

The effects of crack growth rates are discussed further in Chapter 5.

### 3.5. Thermal Equilibrium of the Specimen

#### 3.5.1 Condition on the Specimen Boundary

In real specimens that are finite, an equilibrium is eventually reached when heat produced at the crack tip is equal to heat escaping from the specimen's surface. This equilibrium can be
characterized by the surface temperature of the specimen at
equilibrium, \( u(R) \). For simplicity in the following analysis it is
assumed that the specimen is a cylinder of radius \( R \). Another useful
and simplifying assumption is that the specimen is much larger than
the plastic zone

\[ R \gg R_p . \]

In that case (2) becomes

\[
(29) \quad u(R) = \frac{m K_1^2}{2\pi \kappa G t_1} \int_0^{t_f} \exp \left[ \frac{-R^2}{4a^2(t_f-t)} \right] \int_0^{R_p} \frac{\sqrt{1-(r/R_p)^2}}{r} \, dr \, \frac{dt}{t_f-t} .
\]

Since

\[ \int_0^{R_p} \frac{\sqrt{1-(r/R_p)^2}}{r} \, dr = \frac{\pi R_p}{4} \frac{R_0}{\alpha} \]

the result in terms

of the dimensionless variable \( \alpha \) is

\[
(30) \quad u(R) = \frac{m K_1^2}{8 \kappa G t_1} \int_0^{\infty} e^{\frac{-R^2}{4a^2t_f R_p}} \frac{\alpha}{\alpha} \, d\alpha .
\]

Using (10) for \( R_p \) leaves

\[
(31) \quad u(R) = \frac{2}{8s^2 \kappa G t_1 \tau_0^2} \int_0^{\infty} e^{\frac{-R^2}{4a^2t_f \frac{t_f}{t_1} \frac{1}{\alpha}}} \frac{\alpha}{\alpha} \frac{1}{1-\frac{1}{\alpha}} \, d\alpha .
\]

This relation allows determination of the time, \( t_f \), at which
equilibrium is reached. Unfortunately that determination is not
straightforward since \( u(R) \) depends on \( t_f \) in a complicated and
discontinuous way. In the limit of rapid loading (or equivalently,
time - averaged loading), equation (31) becomes
(32) \[ u(R) = \frac{m^2 K_1^4}{16s^2 \kappa G t_1 t_o^2} \cdot E_1 \left( \frac{R^2}{4a^2 t_f} \right) \]

where \( E_1 \) is the exponential integral defined by \( E_1(x) = \int_x^\infty e^{-t} \frac{dt}{t} \).

This result is analogous to equation (D10) which is derived in Appendix D.

Inverting (32) yields

\[
(33) \quad t_f = \frac{R^2}{4a^2 \text{INVE}_1 \left\{ \frac{16s^2 u(R) \kappa G t_1 t_o^2}{m^2 K_1^4} \right\}} = \frac{R^2}{4a^2 \text{INVE}_1 \left\{ 2u_n(R) \right\}}
\]

where \( \text{INVE}_1 \) the inverse of \( E_1 \), is defined by \( \text{INVE}_1 (E_1(x)) = x \).

Equation (33) may be used in (12) or (23) to determine crack tip temperature when the specimen as a whole reaches thermal equilibrium.

3.5.2. Loss of Heat to the Environment

Continuing with the simplistic assumption that the cracked specimen is a cylinder, and further assuming that heat is removed from the curved cylindrical surface by natural convection, then the surface equilibrium temperature \( u(R) \) is defined in terms of environmental conditions and experimental parameters mentioned above. The definition for \( u(R) \) may be derived from the surface coefficient of heat transfer, \( h_c \). McAdams (1954) gives

\[
(34) \quad h_c = .27 (\Delta t/D_o)^{1/3}
\]

for a horizontal cylinder of diameter \( D_o \) in air at one atmosphere pressure. Here, \( \Delta t \) is in °F above ambient, \( D_o \) is in feet and \( h_c \) has
units of BTU / hr ft² °F. In terms of \( u(R) \) this becomes

\[
(35) \quad h_c = \frac{0.23}{5} \left( \frac{u(R)}{D_o} \right)^{4/5}
\]

and the total heat leaving the specimen per unit time is

\[
(36) \quad h_c A \Delta t = \frac{9}{5} h_c A u(R)
\]

where \( A \) is the available surface area on the specimen. At equilibrium this is equal to the total plastic work rate given in Appendix D by (D4):

\[
(37) \quad \frac{9}{5} h_c A u(R) = \dot{W}
\]

or

\[
(38) \quad u(R) = 2.02 \left( \frac{\dot{W}}{h_c A} \right)^{4/5} \left( \frac{D_o}{A} \right)^{1/5}
\]

Because these quantities are actually given in terms of unit specimen thickness, \( A = \pi D_o \) and

\[
(39) \quad u(R) = 0.81 \left( \frac{\dot{W}}{D_o} \right)^{4/5} \left( \frac{1}{D_o} \right)^{3/5}
\]

Because of the unrealistic specimen shape that was assumed, equation (39) is only approximately correct. The specimen diameter \( D_o \) should be replaced by some geometric parameter that better characterizes real specimens and the proportionality constant .81 probably could be improved upon. However, those refinements are beyond the scope of this work.

Finally, crack tip temperature at specimen equilibrium is determined by inserting (39) into (33) and inserting the result into (11). The question of whether or not equilibrium is reached in the experimental work is discussed later on.
3.6. Validity of the Model for Slow Crack Growth

The above analysis assumes that the crack does not propagate significantly during the time $0 < t < t_f$, so that generated heat is not left behind by the advancing crack. This may be expressed for a crack length of $l$ as:

$$
\frac{d^2 l}{dn} \ll \frac{R_p}{t_f} \leq \frac{R_p^{\text{max}}}{t_f}
$$

or

$$
\frac{d^2 l}{dn} \ll \frac{2t_{\frac{1}{5}} R_p}{t_f} \leq \frac{2t_{\frac{1}{5}} R_p^{\text{max}}}{t_f}
$$

where $n$ is the number of load cycles applied to the specimen. Since $R_p$ is given by (10), this expression becomes

$$
\frac{d^2 l}{dn} \ll \frac{2m K_{\frac{1}{5}}^2 t_{\frac{1}{5}}}{s^{\frac{1}{2}} \tau_0^{\frac{1}{2}} t_f} \frac{\max}{\frac{t_{\frac{1}{5}}}{t_{\frac{1}{5}}}}
$$

Because $\frac{t_{\frac{1}{5}}}{t_f}$ varies over a single cycle, it must be replaced by some quantity that does not. Toward this goal, it is useful to examine Figure 5, which illustrates that during the first cycle of loading ($0 < t < t_{\frac{1}{5}}$) practically all plastic work is done in the time interval $t_{\frac{1}{5}} < t < t_{\frac{1}{5}}$ while very little work is done in the span $0 < t < t_{\frac{1}{5}}$. A similar argument can be made for subsequent loading cycles. For that reason $t_{\frac{1}{5}}/5$ is an effective lower limit of $t$; and $\frac{t}{t_f}$ may be replaced by its effective lower limit, $1/5$. This allows us to write the inequality for the crack growth rate as
\begin{equation}
\frac{d^2}{dn} \ll \frac{2m K_1^2 t_1}{5s^2 \tau_0^2 t_s}.
\end{equation}

By combining (42) and (27) the crack tip temperatures calculated here are valid only for

\begin{equation}
K_1 \ll \left( \frac{2m t_1}{5s^2 \tau_0^2 t_f p} \right)^{1/2}.
\end{equation}

If a propagating fatigue crack moves rapidly enough so that it begins to leave behind the heat it has generated, then the crack-tip temperature will asymptotically approach some limiting functional dependence on velocity. The actual crack-tip temperature in this situation remains undetermined by the present stationary crack model.
4. EXPERIMENTAL INVESTIGATION

4.1 Equipment

As mentioned earlier, this research made use of two important equipment items. The first was an MTS Systems, Inc. servohydraulic test machine, which was used to fatigue precrack the specimens and to apply a time-varying load to them during testing. As shown in Figure 7, the machine is designed to measure the load (or displacement) imposed on the specimen and continuously compare it with the desired input command. The difference between the measured value and the desired value is then used to provide a continuous correction signal to the servovalve which causes this difference to be minimized.

The system maintains the command value throughout testing by continuously driving the servovalve. Control may be either in the form of load or displacement. In this case displacement was used, in order to avoid catastrophic specimen failure at the end of each test run. The command signal was supplied by a digital-function generator which allowed selection of a variety of waveforms; and in this work a standard sine wave was chosen. Its amplitude, mean value and frequency could be varied at will.

The second important piece of equipment used in this work was an AGA Corp. scanning infrared camera system with a liquid nitrogen cooled photovoltaic indium-antimonide semiconductor detector. As shown in Figure 8, this camera system uses a series of lenses and rotating prisms to scan a chosen field of view. Infrared radiation from the scanned spot is focused upon the cooled photovoltaic semiconductor detector.
Infra-red photons with wavelength longer than 5.6 microns do not have sufficient energy to excite electrons into the conduction band of the detector, while wavelengths shorter that 2 microns are filtered out before reaching the detector. When photons of suitable wavelength are incident upon the detector, their number (as a function of wavelength) is converted to detector output voltage as a function of frequency. This frequency is determined by the relaxation time of the excited state.

The voltage spectrum is then shaped, amplified by a selectable amount, added to a variable d.c. voltage and displayed on the cathode ray tube (CRT) screen. Conveniently, the detector's output is processed so that - at an ambient temperature of 25 to 35°C - gray levels seen on the CRT display are linearly related to the temperatures of the object being viewed. This simple relationship does not hold for temperatures much greater or less than the above range which was used in this experimental work. The two constants which determine the linear relations between gray level and object temperature may be varied by the camera user.

One extremely useful feature of the camera system is a variable isotherm function. At the same time that the isotherm level is displayed visually on the CRT screen it is also recorded quantitatively. The camera user can accurately determine the crack-tip temperature rise by first setting the isotherm level on the hottest region at the crack tip and then on an unheated region some distance away. At the ambient temperature of 25 to 30°C used in these tests, the difference between two isotherm levels is exactly equal to their temperature difference. In this way the crack tip temperature
rise is quickly and easily determined, even as the peak stress intensity varies slowly and as the crack propagates slowly. Photographing the thermal image is more difficult.

All thermal imaging employed a 99 mm f/1.8 telephoto lens. In addition, a 20 mm extension ring between lens and camera allowed the camera to be used at a shorter focal distance and at a higher magnification. It also had one undesirable effect: increased noise in the thermal image degraded temperature resolution somewhat. Selection of the 20 mm extension ring over longer and shorter rings represented a trade-off between these desirable and undesirable effects.

A simple device as shown in Figure 9 was built to allow calibration of the infrared camera's isotherm function. In the device a small resistor is used to create a difference in temperature between the two halves. The temperature difference was accurately measured with chromel-alumel thermocouples spot welded to the specimen halves and a Keithely Corp. nanovoltmeter. Cooling was accomplished as necessary with a can of compressed freon. The surface of the calibration specimen is AISI 4135 steel polished to a 240 grit finish as is the actual test specimen itself. Figure 10 shows thermal images of the calibration specimen at 1°C and 2°C temperature differences.

4.2. Limitations of the Equipment

4.2.1 Temporal

The prediction of crack-tip temperature attained by the theoretical model could be verified only approximately by the experimental apparatus. The infrared-camera system had a limited temporal
resolution, due to its vertical sweep time of 1/25 second. Although
the semiconductor detector response time was on the order of a
microsecond, the far slower vertical and horizontal sweep times, as
well as the relaxation time of the CRT phosphors allowed only
time-average temperature to be displayed. Therefore, in order to
compare experimental values of temperature with calculated ones, an
average temperature over several cycles of loading had to be determined.
Closely related to this notion of time-average temperature is the
temperature produced in the limit as loading frequency approaches
infinity. In this situation as well, the observer cannot distinguish
variations in temperature during a single loading cycle.

An analysis of time-average temperature is given in Appendix C,
for the contrived loading function shown in Figure 3. By allowing
t_1 to approach zero, time-average crack tip temperature reduces to

\[ u(o,0,t_f) = \frac{m^2 K_1^4}{16 \pi^2 C \tau_0} \leq t_1 \left\{ - \frac{h^2}{8 t_f} \left[ I_o \left( \frac{h^2}{8 t_f} \right) + I_1 \left( \frac{h^2}{8 t_f} \right) \right] \right. \\
+ \left. \int_{\frac{h^2}{8 t_f}}^{\infty} e^{-w} I_o(w) \frac{dw}{w} \right) \]

Values of the remaining integral are given in Table III. Similarly,
normalized time average temperature becomes

\[ u_n(o,0,t_f) = \frac{1}{2} \left\{ e^{-\frac{h^2}{8 t_f}} \left[ I_o \left( \frac{h^2}{8 t_f} \right) + I_1 \left( \frac{h^2}{8 t_f} \right) \right] \right. \\
+ \left. \int_{\frac{h^2}{8 t_f}}^{\infty} e^{-w} I_o(w) \frac{dw}{w} \right) \].
For the case $h^2/8 t_f \ll 1$ (or equivalently, $K_1 \ll 4 s^2 \sqrt{2/\tau_o \tau_f}$ a $\tau_o/\text{m}$) average temperature becomes

$$u(o, o, t_f) = \frac{2}{16 s^2 G \tau_o \tau_f} \ln \left( \frac{m^2 K_1^4}{2 \tau_o \tau_f} \right)$$

and

$$u_n(o, o, t_f) = \frac{1}{2} \ln \left( \frac{98.1 s^4 a^2 \tau_o^4 \tau_f}{m^2 K_1^4} \right) = \frac{1}{2} \ln \left( \frac{24.5 t_f}{h^2} \right)$$

for the contrived loading function of Figure 3.

In the case of sine wave loading, as in (23), the term

$$\frac{\pi}{4} \left| 2 \sin \left[ \pi \frac{t}{t_f} \right] - \sin \left[ 2\pi \frac{t}{t_f} \right] \right|$$

appears in the integrand. Over one cycle of loading its average value is exactly unity, so that (20) reduces to the simpler square root wave result (9), and time-average crack tip temperature is the same for both types of loading functions. This reflects an earlier assumption that all deformation in the elastic-perfectly plastic model is rate independent.

These results for time-average temperature are in apparent agreement with the previous computer model of time-resolved crack tip temperature. This is demonstrated in Figure 6 where values of temperature determined by (44) are compared with the computer generated results.
4.2.2. Spatial

Besides having a limited temporal resolution, the infrared camera also has a limited spatial resolution. For that reason the best approach is to consider average temperature on a circle of radius around the crack tip, as suggested by Rice and Levy (1969). The average temperature is

\[ \langle u(r_o, t_f) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r_o, \phi; t_f) \, d\phi \]

Using this, together with equation (1) expressed in cylindrical coordinates, yields

\[ \langle u(r_o, t_f) \rangle = \frac{m R_1^2 \sqrt{t_f}}{\pi G t_1 \sqrt{\kappa \rho c}} \int_1^\infty \int_0^\beta_0 e^{-\beta^2} e^{-\beta_0^2} \times \]

\[ \sqrt{1-(\beta/\beta_p)^2} I_0(2\alpha \beta \beta_0) \, d\beta \frac{d\alpha}{\alpha} \]

where \( \beta_0 = r_o/2a \sqrt{t_f} \) is analogous to \( \beta_p = R_p/2a \sqrt{t_f} \). Since the functional dependence of the integrand on \( \alpha, \beta, \beta_0 \) and \( \beta_p \) is so convoluted, it is impossible to simplify this integral at all. It can only be solved numerically, and at great expense of computer time.

A simpler approach to this complicated problem is to consider the plastic zone as a point source of heat and one which does not vary in time. These two assumptions are useful first because the infrared camera only sees time average temperature (as discussed above) and second because the singularity in the plastic work rate means the heat source is so concentrated that a point source is a useful approximation. The temperature distribution around such a constant
source of heat (derived in Appendix D) is

\[ (51) \quad < u(r_0, t_f) > = \frac{m^2 K_1^4}{16 s^2 G_2 \tau_0^2 \tau_1} E_1 \left( \frac{r_0}{4a^2 \tau_f} \right), \]

where again \( E_1 \) is the exponential integral.

A comparison of distributed and point heat sources of equal magnitude is shown qualitatively in Figure 11. At some point the two curves cross. That point provides a useful measure of the distance from the crack tip over which temperature is relatively constant, but it is difficult to determine exactly due to the complexity of (50). Instead, a useful substitute is the point defined by equating (47) and (12). When this is done, \( r_o \) is defined in terms of material parameters and stress intensities. In terms of the inverse function of \( E_1^1 \), \( \text{INVE}_1^1 \),

\[ (52) \quad r_o = 2a \sqrt{\tau_f} \left\{ \text{INVE}_1^1 \left[ 2 \frac{u_n(o, o, t_f)}{u_n(o, o, t_f)} \right] \right\}^{1/2} \]

where \( u_n(o, o, t_f) \) is defined by (13).

4.3 Test Specimens

All specimens used in this experimental investigation were machined from a single bar of AISI 4135 modified steel in the as-rolled condition. A chemical analysis of the steel is included in Table I along with other material properties. The material was not heat treated in any way, so that the strength might be kept low and the fracture toughness high. The first of these two considerations serves to maximize plastic zone size and crack tip temperature rise, while the second consideration assures a low crack propagation rate,
as in equation (27), so that condition (43) might be satisfied.

As shown in Figure 12, the only difference between the four specimens was their orientation with respect to the rolling direction of the steel. In specimen 3, the rolling direction was perpendicular to the crack plane; while in the other specimens, the rolling direction lay in the crack plane.

All experimental data were recorded on these four specimens, however preliminary testing work was done on a great variety of specimens. For the most part they were of higher strength, and even included some hardened tool steel and cemented carbide specimens. No reliable data was gathered from those specimens. Their brittleness caused rapid, unstable crack propagation; and at the same time, their high strength resulted in tiny plastic zones and very little generated heat. These experimental difficulties could not be overcome, so attention was directed toward the ductile steel described above.

From this ductile material, specimens were machined according to Figure 12. Razor blades were attached with screws to the face of each specimen. The razor edges provided low friction pivots for the arms of a clip-on type gauge. It was used to measure displacements along the load line.

This specimen design allows the crack tip stress intensity to be determined from measurements of load, $P$, and displacement, $v$, imposed on the specimen. Saxena and Hudak (1978) report

\[(53) \quad \frac{K}{w} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4 + C_5 z^5\]

for $.2 < \ell/w < .975$ where $\ell/w$ is the ratio of crack length to
specimen width and

\[(54) \quad Z = \frac{1}{\sqrt{\frac{BEV}{P}} + 1} \]

\[C_o = 1.0002 \]
\[C_1 = -4.0632 \]
\[C_2 = 11.242 \]
\[C_3 = -106.04 \]
\[C_4 = 464.33 \]
\[C_5 = -650.68 \]

In (54) the specimen thickness, B, is one inch and E, Young's Modulus, is given in Table 1. Saxena and Hudak claim the accuracy of this result for \(k/w\) is \(\pm 0.0005\).

The crack tip stress intensity may be determined from this value of \(k/w\) (Sprawley, 1976).

\[(55) \quad K = \frac{P}{BW^{1/2}} \frac{2+k/w}{(1-k/w)^{3/2}} \left\{ d_0 + d_1 \left( \frac{k}{w} \right) + d_2 \left( \frac{k}{w} \right)^2 + d_3 \left( \frac{k}{w} \right)^3 + d_4 \left( \frac{k}{w} \right)^4 \right\} \]

where \(d_0 = 0.886\)
\(d_1 = 4.64\)
\(d_2 = -13.32\)
\(d_3 = 14.72\)
\(d_4 = -5.6\)

The accuracy of K is not better than .65% according to Sprawley.

The true accuracy is not known. It may be reasonable to assume that the uncertainty in K from all causes is roughly 5% to 10%.
4.4. Experimental Procedure

After attaching the crack opening displacement gauge to a specimen, the two were inserted into the standard specimen grips (also described by ASTM E399-78). In order to keep ambient infra-red light from reflecting off of the specimen and into the camera, a paper shield was constructed around the whole apparatus and held in place with plastic tape. This effectively eliminated reflections from people and light bulbs in the room.

Unfortunately, one source of unwanted heat remained: the hydraulic fluid which powered the loading apparatus. As the test specimens were fatigued, the hydraulic fluid temperature increased from room temperature to about 55°C. Heat then flowed from the fluid through various metal parts, up the loading piston, through the specimen grips and into the specimen itself. The result was, on occasion, an obtrusively warm region in the lower half of the specimen.

The camera was positioned about one foot from the specimen, and the objective lens was focused at the shortest possible distance. In this way a magnification of 1.2 diameters was obtained on the thermal images.

Fatigue loading of the specimens was begun at a stress intensity of about 20 MPa $\sqrt{m}$ and gradually increased until a fatigue crack began to propagate from the machined notch. At that point the displacement range (and consequently the stress intensity) was increased until heating at the crack tip became apparent on the CRT display. When a suitable amount of heating was visible, displacement range adjustments were stopped and the heating allowed to develop for a
few seconds. Then, load and displacement data were quickly recorded
while the crack tip temperature rise was either estimated (as described
below), photographed or both. This process was repeated a few times
on each specimen, so that a range of temperature values might be
observed.

The three test specimens in which the rolling direction of the
steel was in the crack plane responded similarly to stress. In the
fourth specimen the rolling direction was perpendicular to the crack
plane. Because of this, the crack propagation rate was lower and the
material was able to sustain somewhat higher stress intensities. This
is apparent in the data of Table V.

The theoretical analysis assumes that stress intensity
amplitude, $K_1$, remains constant for $0 < t < t_f$. This assumption
could not be exactly satisfied in the experimental investigation.
As the crack propagated, specimen compliance, $v/p$, increased. With
the MTS machine set for displacement control, $K_1$ decreased as $v/p$
increased. Had the machine been run in load control, decreasing $v/p$
would have increased $K_1$. As a result, when displacement range
adjustments were stopped so that crack tip heating could develop for
a few seconds, continuing crack propagation caused $K_1$ to drop slowly.
The data recorded at time $t_f$ thus determine a lower limit for $K_1$ during
the time $0 < t < t_f$. For that reason, crack tip temperatures
calculated from $K_1$ should underestimate the actual experimental values.

The amount of underestimation was minimized by keeping $t_f$
short and crack propagation rates fairly low. This difficulty could
have been avoided entirely by employing a minicomputer to:
(1) monitor loads and displacements, (2) calculate $K_1$ from them and
(3) control loads or displacements so that $K_1$ might be kept constant. A minicomputer could have also been employed to precisely start and stop each test and to measure $t_f$ while simultaneously measuring the fatigue crack growth rate, $\Delta K/dn$. Such a minicomputer was not available for this work.

Another equipment item which could not be obtained for this research was an electrical interface for the infra-red camera which would have allowed thermal images to be recorded on magnetic tape. Examination and manipulation of tape recorded thermal images would have facilitated this experimental work and improved the quality of both photographs and data.

As the specimen was photographed, cyclic loading continued. An exposure required 4/25 second, during which time the specimen experienced 3.2 cycles of loading. During those loading cycles it moved up and down several hundredths of an inch, thereby blurring the photographed image.

After some experience had been gained in using the isotherm function to estimate crack tip temperature, it became possible to bypass use of the isotherm function and instead simply to estimate the crack tip temperature rise from the appearance of the heating on the CRT display. This was possible for two principal reasons. The first is that the range of crack tip temperature values seen in this work was small, only up to $4^\circ$C. The second reason is that noise in the CRT image limited the reproducibility of temperature determinations to about $\pm 0.5^\circ$C — even when the isotherm function was used. A similar accuracy could be attained by visually estimating from the infra-red image whether crack tip heating
was more nearly 1, 2, 3 or 4°C.

Two photographs of crack tip heating are shown in Figure 13. In Table IV load and displacement data are recorded. Crack tip temperature estimates for all four test specimens are listed in Table V.

4.5 Experimental Results

To compare the experimental temperature data of Table V with the predictions of equation (44), values of the pertinent experimental parameters must be determined. Values of \( \lambda/w \) and \( K_1 \) were calculated from the load and displacement data of Table IV using (53) and (55). The results are found in Table V. Values of \( G, \tau_0, a^2 \) and \( \kappa \) are obtained in Table I, while \( m \) may be taken from equation (4).

The only variable which remains undetermined is \( t_f \). In (46) \( u(o,o,t_f) \) varies with \( \ln(t_f) \), and in (44) approximately as \( \ln(t_f) \). Although the values of \( t_f \) associated with each value of \( K_1 \) were not recorded in the experimental work, the values were typically about 10 to 20 seconds. A value of \( t_f = 14 \) seconds then results in an error in \( \ln(t_f) \) no greater than 15%. This source of error is small in comparison with others mentioned later.

Using the data described above, values of \( u(o,o,t_f) \) were calculated for \( s=1 \) and \( s=2 \) and entered in Table V. Values of \( h^2 \) are included there, also. The maximum value of \( h^2 \) was 3.751 so that \( h^2/8t_f \ll 1 \) for all of the data.

The distance around the crack tip over which these calculated temperatures persist is determined by (52):
(52) \[ r_o = 2a \sqrt{\frac{t_f}{h^2}} \left( \text{INVE}_{\frac{1}{2}} \left[ 2 \frac{u_n(o,o,t_f)}{u_n(o,o)} \right] \right) \]

The time average of \( u_n(o,o,t_f) \) depends only on \( h^2 \) and \( t_f \), as shown in equation (47). Assuming \( t_f = 14 \) seconds, and for the case \( s=2 \) taking the maximum and minimum values of \( h^2 \) from Table V, the factor \( \frac{2u_n(o,o,t_f)}{u_n(o,o,t_f)} \) ranges from 8.88 to 7.29.

Comparing the tabulated values of \( E_1(x) = \int_x^\infty e^{-t} \frac{dt}{t} \) in Abramowitz and Stegun (1964) with the tabulated values of

\[ M(x) = \int_x^\infty e^{-t} I_0(t) \frac{dt}{t} \]

in Table III, it can be shown that

(56) \( E_1(x) \approx M(x) - .693 \)

for \( x << 1 \). It follows that in these tests, for \( s=2 \),

(57) \( .0088 \ (2a \sqrt{t_f}) < r_o < .0195 \ (2a \sqrt{t_f}) \)

or

(58) \( .086 \text{ inch} < r_o < 0.19 \text{ inch} \)

It is difficult to compare this result with experimental results, since the spatial resolution of the camera is not precisely known and because the thermal image is blurred somewhat by specimen motion. Nevertheless, Mr. Robert Boyles (1981) has estimated the spatial resolution of the camera by examining Figure 13b. He estimates a resolution of 1/25 inch which is slightly better than the 0.19 inches calculated above for \( s=2 \).

From these results it can be concluded that the spatial resolution of the infrared camera (in combination with the particular lens and
extension ring) was just good enough for s=2 to resolve the hottest regions near the crack tip. For s=1, the resolution of the camera is more than adequate.

The theoretical model of crack tip heating that has been developed assumes that the specimen thickness is finite, the other dimensions of the specimen are not and all heat flow is perpendicular to the crack front. This is really not so. Because all dimensions are finite, heat flowing perpendicular to the crack front eventually reaches a free surface and escapes into the environment.

Equilibrium is reached when heat leaving the free surface is equal to heat generated at the crack tip. Long before the time at which equilibrium is reached, the specimen behaves almost as if it were infinite. Equation (59) may be used to calculate the time at which equilibrium is reached; however it is expressed in terms of an unknown quantity, $u_n(R)$, the normalized specimen surface temperature at equilibrium:

\[
(59) \quad t_f = \frac{R^2}{4a^2 \text{ INVE}_1 (2u_n(R))}
\]

In the limit as $u_n(R)$ approaches zero, the specimen reaches thermal equilibrium very rapidly. When this happens, the recorded crack tip temperatures are less than they would be in an infinite specimen.

In this experimental work the duration of fatigue loading varied from 10 to 20 seconds. Because this time was so short, equilibrium was never reached and no attempts were made to measure
$u_n(R)$. Bulk heating of the specimen should be a prominent feature of the approach to equilibrium; however no such bulk heating was seen with the infrared camera.
5. DISCUSSION AND CONCLUSIONS

There are many possible reasons for the noted differences between experimental and theoretical results. As discussed in Section 4.4, $K_1$ was not constant but decreased somewhat during time $t_f$. To correct for this, a slightly higher value of $K_1$ should be used. On the other hand, crack tip blunting allows some elastic unloading. For this reason $K_1$ should be lower by about 5%. These two effects offset each other to some extent.

Present calculations assume that plane strain exists through the specimen's thickness; but the amount of heating seen on the surface must be due, in some part, to both plane stress and plane strain deformation. Rice (1967) claims that plastic zone size in plane stress is about twice that in plane strain. If this is correct and if heating is due only to the plane stress condition on the surface, then the factor $m$ should be doubled. That results in calculated temperatures several times greater than those listed in Table V. In that event, agreement between theory and experiment is improved. Another consideration is that no shear lip was seen on any of the specimens. This indicates that plane stress should not be an important contributor to the measured temperatures.

Besides these two effects, inaccuracies in $t_f$, $\tau_o$, $a^2$, $\kappa$, $\nu$, $E$ and other data no doubt contribute to the discrepancies between theory and experiment. It is known that $\nu$ is about 1/3 for the small deformations near the outer extent of the plastic zone; but it is closer to .5 for the very large strains near the center of the plastic zone. In this work we have simply used $\nu = 1/3$. An even greater
concern is $\tau_o$. Severe work hardening can easily double the measured yield strength of 230 MPa, while the Bauschinger effect causes an apparent reduction in yield strength of the material after load reversals. In this work such hardening and softening have been ignored, although they might be included in the factor $s$. Work hardening would increase $s$ and cyclic softening would have the opposite effect.

The effect of crack propagation on the distribution of strains near the crack tip, although not considered in the theoretical model, also serves to offset the effect of strain hardening. While strain hardening causes the strain singularity to be more intense, crack propagation causes the singularity to be less so. A less intense singularity produces less heat at the crack tip.

On the average, the measured temperatures listed in Table V are one-third the temperatures calculated using $s=2$ and about one-seventh the temperature calculated using $s=1$. For this reason, the value of $s$ should be closer to two than to one; and the crack tip should be surrounded by material that was yielded during the previous cycle of loading. A value of $s \approx 4$ (perhaps due to work hardening) would give calculated temperatures very similar to measured ones.

Fatigue crack growth rates as described in section 3.4 also indicate whether $s$ is more nearly one or two. Such growth rates were not measured in the experimental work described above. Nevertheless, Gross and Weertman (1982) and Barsom (1971) examined materials that resemble the present 4135 modified steel. Their results are rather similar:
\[ q = 2.68 \quad \text{(Gross and Weertman)} \]
\[ p = 1.61 \times 10^{-11} \]
\[ q = 3.00 \quad \text{(Barsom)} \]
\[ p = 6.2 \times 10^{-12} \]

Their units are in terms of MPa and m.

Using this data in (28) results in \( K_1 \ll 7600 \) and \( K_1 \ll 1100 \text{ MPa}\sqrt{\text{m}} \) respectively for \( s=2 \) and results in \( K_1 \ll 21000 \) and \( K_1 \ll 2300 \text{ MPa}\sqrt{\text{m}} \) respectively for \( s=1 \). The stress intensities used in this experimental investigation easily satisfy any of these requirements. If Rice's model of crack tip blunting applies to fatigue, then one is forced to conclude that the crack tip is effectively surrounded by previously yielded material and \( s \) should equal to 2. This result is in good agreement with crack tip temperatures, as discussed above, which also indicate a value of \( s \) closer to two.

Crack growth rates also determine whether or not the crack tip travels fast enough to leave behind the heat it has generated.

Equation (43), in terms of the crack growth rate parameters \( p \) and \( q \), defines values of \( K_1 \) for which the crack tip does not leave behind generated heat. Using the data of Gross and Weertman and that of Barsom, one obtains \( K_1 \ll 51 \) and \( K_1 \ll 37 \text{ MPa}\sqrt{\text{m}} \) respectively for \( s=2 \). Clearly this condition is not satisfied for \( t_f=14 \) seconds as we have assumed. At some earlier time, for example \( t_f=3 \) seconds, one obtains \( K_1 \ll 491 \) and \( K_1 \ll 172 \text{ MPa}\sqrt{\text{m}} \) respectively for \( s=2 \). After 3 seconds the crack has not yet left behind much of its generated heat; but the crack tip temperature determined by (44) is lower than it would be at \( t_f=14 \) seconds. Such a correction would bring calculated
temperatures into better agreement with measured ones.

The crack tip temperatures noted in this study are so small that their effect on fracture must be negligible. In other situations of low thermal diffusivity or conductivity, low elastic modulus, low yield strength or high crack propagation rates (due to $K_1$, $t_1$, or other factors) the crack tip temperatures could be large and influential.

Of the total work done during fatigue, roughly 95% is dissipated as heat. In the situation examined here it can only be deduced that the remaining 5% of the total work is of greater importance. It causes large changes in the density of dislocations and point defects in the crystal and ultimately results in the formation of free crack surfaces. Nevertheless, there may be some close relationship between the heat (or work) which has been investigated here and the remaining small amount of work which causes fatigue fracture. If such a relationship exists, knowledge of it would greatly improve our understanding of fracture. It is hoped that future experimental work can provide a better test of the theoretical model developed here.
Mode I
Tensile

Mode II
In Plane Shear

Mode III
Antiplane Shear

FIGURE 1: Modes of Specimen Loading
FIGURE 2: Plastic Zone Shape and The Slip Line Field:
The crack-tip perfectly plastic plane strain slip line field consists of constant stress regions A and C joined by centered fan B. Deformation is symmetric about the crack plane and is confined to $\pi/4 < \theta < 3\pi/4$ and $-3\pi/4 < \theta < -\pi/4$. 
FIGURE 3: Load, $\Delta K$ and $\frac{1}{2} \frac{d}{dt} [ (\Delta K)^2 ]$ as functions of time for a contrived "Square Root Wave" Loading Function.
\[ K = K_0 + K_1 \sqrt{t/t_1} \quad 0 \leq t \leq t_1 \]

\[ \Delta K = K_1 \sqrt{\frac{t}{t_1}} \quad 0 \leq t < \infty \]
FIGURE 4a: Crack tip temperature under square root wave loading as a function of time, for

\[ K_1 = 60 \text{ MPa} \sqrt{m} \]

and for the material properties given in Table I. Values of \( t_1 \) equal to 0.025 and 0.1 seconds and values of \( s \) equal to 1 and 2 are used as examples.
FIGURE 4b: Crack tip temperature as a function of time for $t_1 = .025$ seconds, $S = 2$ and for the material properties given in Table I. Four values of $K_1$ are used as examples. On the temperature scale of this figure, the short-lived, low amplitude temperature fluctuations shown in Figure 4a are too small to be drawn accurately and they have been omitted. For this reason, Figure 4b applies equally well to both square root wave and sine wave loading functions.
FIGURE 4c: Crack tip temperature as a function of the amplitude of stress intensity variations for $t_f = 1$ second, $S = 2$ and for the material properties given in Table 1. Seven values of $f = 1/2t_1$ are used as examples. As in Figure 4b, this applies to both square root wave and sine wave loading functions.
FIGURE 5: Load, \( \Delta K \), and \( \frac{t}{K_1} \frac{1}{2} \frac{d}{dt} [(\Delta K)] \) as functions of time for Sine Wave Loading
\[ K = K_0 + \frac{K_1}{2} \left( 1 - \cos \left( \frac{\pi t}{t_f} \right) \right) \quad 0 \leq t \leq t_1 \]
\[ \Delta K = \frac{K_1}{2} \left( 1 - \cos \left( \pi \frac{t}{t_f} \right) \right) \quad 0 \leq t < \infty \]
FIGURE 6: Crack tip temperature under sine wave loading as a function of time, for $K_1 = 60$ MPa $\sqrt{m}$ and for the material properties given in Table 1. This figure is analogous to Figure 4a, for square root wave loading. Also included in this figure are time-average temperatures, calculated from equation (46). These time-averages are denoted by $-\rightarrow + -\rightarrow + -\rightarrow + -\rightarrow$.
FIGURE 7: Block Diagram of the NTS Machine.
FIGURE 8: Block Diagram of the AGA Camera.
FIGURE 9: Diagram of the Infrared Calibration Specimen
FIGURE 10: Thermal Images of the Calibration Specimen.

a) Right half of specimen 2°C hotter than the left.

b) Right half of specimen 1°C hotter than the left.
FIGURE 11: Qualitative Comparison of Point and Distributed Heat Sources.
FIGURE 12: The Compact Tension Fracture Specimen and Specimen Orientations.
FIGURE 13: Thermal Images of Crack Tip Heating

a) Low stress intensity amplitude

b) High stress intensity amplitude
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<th>Value</th>
<th>Source</th>
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<td>Thermal Conductivity</td>
<td>( \kappa = 43 \text{ W/m}^\circ \text{K} )</td>
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<tr>
<td>Thermal Diffusivity</td>
<td>( a^2 = \frac{\kappa}{\rho c} = 1.1 \times 10^{-5} \text{ m}^2/\text{sec} )</td>
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<tr>
<td>Density</td>
<td>( \rho = 7800 \text{ Kg/m}^3 )</td>
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<tr>
<td>Young's Modulus</td>
<td>( E = 205 \text{ GPa} )</td>
<td>2</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>( G = \frac{E}{2(1+\nu)} = 77 \text{ GPa} )</td>
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<td>( \sigma_o = 614 \text{ MPa} )</td>
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<td></td>
<td>( \tau_o = \sigma_o / (2(1+\nu)) = 230 \text{ Mpa} )</td>
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<td>( \tau_{\text{max}} = \sigma_{\text{max}} / (2(1+\nu)) = 271 \text{ MPa} )</td>
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<td>Elongation</td>
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<td>Poisson's Ratio</td>
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<tr>
<td>G ( \tau_o ) \kappa</td>
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<tr>
<td>( a \tau_o ) \kappa</td>
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**Sources**

1. Determined experimentally
3. Calculated from other data
TABLE II
Approximations Used in Numerical Computations

(1) when $\beta_p^2 \alpha/2 \geq 100$, \[ J \approx J_\infty = \sqrt{\pi} / 2 \sqrt{\alpha} \]

(2) when $\beta_p^2 \alpha/2 \leq .01$, \[ J \approx J_\alpha = \pi \beta_p / 4 \]

(3) when $.01 < \beta_p^2 \alpha/2 \leq 3.75$, and using $x = \beta_p^2 \alpha/2$.
\[
I_0(x) + I_1(x) = \sum_{n=0}^{6} (b_n + c_n x) \left( \frac{x}{3.75} \right)^{2n}
\]

|error| < 1.9x10^-7

when $3.75 < \beta_p^2 \alpha/2 < 100$,
\[
I_0(x) + I_1(x) = e^x \sqrt{x} \sum_{n=0}^{8} a_n \left( \frac{3.75}{x} \right)^n
\]

|error| < 5.0x10^-9

\[
\begin{align*}
a_0 &= .79788456 & b_0 &= 1 & c_0 &= 1/2 \\
a_1 &= -.2659432 & b_1 &= 3.5156229 & c_1 &= .87890594 \\
a_2 &= -.00136699 & b_2 &= 3.0899424 & c_2 &= .51498869 \\
a_3 &= .00006236 & b_3 &= 1.2067492 & c_3 &= .15084932 \\
a_4 &= -.00115274 & b_4 &= .2659732 & c_4 &= .02658733 \\
a_5 &= .00225261 & b_5 &= .0360768 & c_5 &= .00301532 \\
a_6 &= -.00259775 & b_6 &= .0045813 & c_6 &= .00032411 \\
a_7 &= .00140021 \\
a_8 &= -.00027682
\end{align*}
\]
### TABLE III

Values of \( \int_0^\infty \frac{e^{-t}}{t} I_0(t) \, dt \) for \( 10^{-5} < x < 10^{+1} \)

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### TABLE IV
Experimental Load and Displacement Data

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TABLE V

Comparison of Measured and Calculated Parameters

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<th>Specimen</th>
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<th>$K_1$ MPa $\text{m}^{-1}$</th>
<th>$h^2$ s=1/s=2 seconds</th>
<th>$u$ calculated $s=1/s=2$ $^\circ\text{centigrade}$</th>
<th>$u$ measured $^\circ\text{centigrade}$</th>
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- Not obtained
REFERENCES


APPENDIX A
Integration of Equation (9)
Using Dimensionless Variables

First, making the substitutions

\[(A1) \quad \alpha = \frac{t_f}{t_f-t}, \quad \frac{d\alpha}{\alpha} = \frac{dt}{t_f-t} \quad \text{and} \quad \beta^2 = \frac{r^2}{4a^2 t_f}, \quad dr = d\beta 2a \sqrt{t_f}\]

in equation (9) leaves

\[(A2) \quad u(o,o,t_f) = \frac{m K_l^2 \sqrt{t_f}}{\pi G t_f \sqrt{\kappa p c}} \int_1^\infty \int_0^{\beta_p} e^{-\beta^2 \alpha} \times \]

\[\sqrt{1 - (\beta/\beta_p)^2} \quad d\beta \frac{d\alpha}{\alpha} \quad .
\]

Note that \(R_p\) is now written as a function of \(\alpha\) and \(\beta_p\).

\[(A3) \quad \beta_p = \frac{h}{\sqrt{t_f}} \frac{\text{frac}}{} \left\{ \frac{t_f}{t_1} (1-1/\alpha) \right\}
\]

where \(h = m K_l^2/2a^2 \tau_o^2.\) In order to simplify the integral

\[(A4) \quad I = \int_1^{\beta_p} \int_0^{\beta_p} e^{-\beta^2 \alpha} \sqrt{1 - (\beta/\beta_p)^2} \quad d\beta \frac{d\alpha}{\alpha}
\]

the inner integral over \(\beta\) is separated out
(A5) \[ J = \int_0^\beta e^{-\beta^2 \alpha} \frac{\sqrt{1 - (\beta/\beta_p)^2}}{2} \, d\beta; \]
so that

(A6) \[ I = \int_1^\infty J \frac{d\alpha}{\alpha}, \] with \( J = J(\beta_p, \alpha) \) and \( \beta_p = \beta_p(\alpha). \)

A useful computational simplification of (A5) arises from the situation
in which \( \beta_p^2 \alpha < 1 \). In this case \( e^{-\beta^2 \alpha} \approx 1 \) so that

(A7) \[ J_\infty \approx \frac{\pi \beta_p}{4}. \]

Another simplification results when \( \beta_p^2 \alpha >> 1 \) so that \( \sqrt{1 - (\beta/\beta_p)^2} \approx 1 \).

This leaves

(A8) \[ J_\infty \approx \int_0^\beta e^{-\beta^2 \alpha} \, d\beta \approx \frac{\sqrt{\pi}}{2^{\alpha}} \cdot \text{erf} (\infty) = \frac{\sqrt{\pi}}{2^{\alpha}}. \]

The general solution for \( J \) may be found in the following manner: First, since \( \beta/\beta_p < 1 \), using \( \beta/\beta_p = \sin z \) and \( d\beta = \beta_p \cos z \, dz \) results in

(A9) \[ J = \beta_p \int_0^{\pi/2} e^{-\alpha \beta_p^2} \sin^2 z \cos^2 z \, dz. \]

Since \( \sin^2 z = \frac{1}{2}(1 - \cos^2 z) \) and \( \cos^2 z = \frac{1}{2}(1 + \cos^2 z) \),

(A10) \[ J = \frac{\beta_p}{4} e^{-\alpha \beta_p^2} \left[ \int_0^\pi e^{-\alpha \beta_p^2 \cos z'} \, dz' + \int_0^\pi e^{-\alpha \beta_p^2 \cos z'} \, dz' \right]. \]

and (Abramowitz and Stegun, 1964),
(A11) \[ J = \frac{\pi \beta}{4} e^{-\frac{\alpha \beta}{2}} \left[ I_0 \left( \frac{\alpha \beta}{2} \right) + I_1 \left( \frac{\alpha \beta}{2} \right) \right] \]

Here, \( I_0 \) and \( I_1 \) are Modified Bessel functions of the first kind. At this point I may be written as

(A12) \[ I = \int_1^\infty \frac{\pi \beta}{4} e^{-\frac{\alpha \beta}{2}} \left[ I_0 \left( \frac{\alpha \beta}{2} \right) + I_1 \left( \frac{\alpha \beta}{2} \right) \right] \frac{d\alpha}{\alpha}, \]

so that crack tip temperature becomes

(A13) \[ u(0,0,t_f) = \frac{\frac{m k}{4G \tau^2 t_f}}{t_f/k \rho c} \int_1^\infty \frac{\beta \alpha}{2} e^{-\frac{\alpha \beta}{2}} \times \left[ I_0 \left( \frac{\alpha \beta}{2} \right) + I_1 \left( \frac{\alpha \beta}{2} \right) \right] \frac{d\alpha}{\alpha}. \]

I can be related to normalized crack tip temperature, \( u_n(0,0,t_f) \), by

(A14) \[ I = u_n(0,0,t_f) \frac{\tau t_f}{4 \sqrt{t_f}}. \]
APPENDIX B

Selection of Important Parameters Used In

The Fortran Program

The important quantities $R_p$ and $\frac{d}{dt} [\langle \Delta K \rangle^2]$ depend on periodic functions of $2\pi f t$, or equivalently of $\pi \frac{t_f}{t_L} (1-1/\alpha)$. In order to closely follow these cyclic variations, it is desirable that a fixed number, $n$, of integration steps be performed over each half-cycle of loading. This can be expressed in terms of $f$ (frequency), $t$ and $dt$ as

$$n \left[ 2\pi f (t + dt) - 2\pi f t \right] = \pi$$

or in terms of $t_L$, $t_f$, $\alpha$ and $\alpha$ as

$$n \left[ \pi \frac{t_f}{t_L} \left(1 - \frac{1}{\alpha + \alpha \frac{dt}{t_f}}\right) - \pi \frac{t_f}{t_L} (1-1/\alpha) \right] = \pi$$

which yields

$$\alpha = \frac{2}{\left( \frac{nt_f}{t_L} - \alpha \right).}$$

This choice of step size performs well — except when $\alpha$ coincidentally assumes the value $\alpha = \frac{nt_f}{t_L}$. In this situation $\alpha$ become infinite, a difficulty that is easily overcome by substituting

$$\alpha = \alpha$$

in the situation where $\alpha = \frac{nt_f}{t_L}$. The value of $n$ determines the accuracy and speed of the computer program. It was found that $n=20$ provided good results in most cases. The limits of integration are $\alpha = 1$ and $\alpha = \infty$. This upper limit can be replaced by $10^{-10}$ or some large number, with no sacrifice of accuracy.
Finally, Simpson's method of integration was employed:

\[ \int_{\alpha_m^i}^{\alpha_m^f} f(\alpha) \, d\alpha = \left( \frac{\alpha_m^f - \alpha_m^i}{6} \right) \left[ f(\alpha_m^i) + 4f \left( \frac{\alpha_m^f + \alpha_m^i}{2} \right) + f(\alpha_m^f) \right] \]

and

\[ \int_{\alpha_{max}^i}^{\alpha_{max}^f} f(\alpha) \, d\alpha = \sum_{m=0}^{m_{max}} \int_{\alpha_m^i}^{\alpha_m^f} f(\alpha) \, d\alpha \]

where \( i \) and \( f \) denote initial and final respectively, and where \( m \) denotes the \( m^{th} \) small segment of the domain \( 1 < \alpha < \alpha_{max} \). In general, \( nt_f/t_1 = m_{max} \), where \( n \) is defined previously. For example if \( t_1 = 1/40 \) sec \( t_f = 10 \) sec. and \( n = 20 \) then \( m_{max} = 8000 \). For each value of \( m \) the function \( f \) must be called as a subroutine 3 times so that in this example the subroutine \( f \) would be called 24,000 times. This requires a considerable amount of computer time.

Computer time can be kept tolerably low by using the approximations shown in Table II. Those approximations also avoid problems that might be caused by extremely large or small numbers which exceed the capacity of the computer.
APPENDIX C

Analysis of Time Average

Crack Tip Temperature

High-frequency loading is easily examined by allowing \( t_1 \) to approach zero. As this occurs, \( \lambda = \frac{\frac{t_1}{t_f} (1-1/\alpha)}{\alpha} \) oscillates rapidly from zero to unity with an average value of 1/2. When this average is used in the earlier definition of \( \beta_p \), equation (A3),

\[
(C1) \quad \overline{\beta}_p = \frac{h}{2 \sqrt{t_f}} \quad \text{(square root wave loading)}
\]

where \( \overline{\beta}_p \) denotes the average value of \( \beta_p \) over some interval whose width, \( \Delta \alpha \), satisfies \( \Delta \alpha \ll \alpha \). Under these circumstances (A12) becomes

\[
(C2) \quad \overline{I} = \frac{\pi h}{8 \sqrt{t_f}} \int_{1}^{\infty} e^{-\frac{\alpha h^2}{8 t_f}} \left[ I_0 \left( \frac{\alpha h^2}{8 t_f} \right) + I_1 \left( \frac{\alpha h^2}{8 t_f} \right) \right] \frac{d\alpha}{\alpha}
\]

or with a change of variable, \( z = \frac{\alpha h^2}{8 t_f} \),

\[
(C3) \quad \overline{I} = \frac{\pi h}{8 \sqrt{t_f}} \int_{\frac{h^2}{8 t_f}}^{\infty} e^{-z} \left[ I_0 (z) + I_1 (z) \right] \frac{dz}{z}
\]

Using the relation (Abramowitz and Stegun, 1964)

\[
(C4) \quad \int_{\frac{h^2}{8 t_f}}^{\infty} e^{-z} \frac{I_1 (z)}{z} dz = e^{-\frac{h^2}{8 t_f}} \left[ I_0 \left( \frac{h^2}{8 t_f} \right) + I_1 \left( \frac{h^2}{8 t_f} \right) \right]
\]
reduces (C3) to

\[ \bar{I} = \frac{\pi h}{8 \sqrt{t_f}} \left\{ e^{-\frac{h^2}{8 t_f}} \left[ I_o \left( \frac{h^2}{8 t_f} \right) + I_1 \left( \frac{h^2}{8 t_f} \right) \right] + \right. \]

\[ \left. \int_{\frac{h^2}{8 t_f}}^{\infty} e^{-z} I_o(z) \frac{dz}{z} \right\} . \]

The remaining integral may be evaluated numerically using the approximations of Table II. Values obtained in this way may be found in Table III. A useful approximation may be derived from the tabulated data:

\[ \int_{\frac{h^2}{8 t_f}}^{\infty} e^{-z} I_o(z) \frac{dz}{z} \approx -\ln \left( \frac{h^2}{8 t_f} \right) + .12. \]

for values of \( \frac{h^2}{8 t_f} \ll 1 \). For example, when \( h^2/8 t_f = .1 \) the error in this approximation is 3.7% while at \( h^2/8 t_f = .01 \) the error is .1%.

This finally yields

\[ \bar{I} \approx \frac{\pi h}{8 t_f} \left\{ e^{-\frac{h^2}{8 t_f}} \left[ I_o \left( \frac{h^2}{8 t_f} \right) + I_1 \left( \frac{h^2}{8 t_f} \right) \right] - \ln \left( \frac{h^2}{8 t_f} \right) + .12 \right\} \]

for the square root wave loading.

It must be noted, however, that the methods used in achieving this result for \( \bar{I} \) are not rigorous derivations. That is because the time average of a product of two arbitrary functions \( A(\alpha) \) and \( B(\alpha) \) on the domain \( \alpha_o < \alpha < \alpha_1 \), is defined by

\[ \bar{A(\alpha) B(\alpha)} = \frac{1}{\alpha_1 - \alpha_o} \int_{\alpha_o}^{\alpha_1} A(\alpha) B(\alpha) \, d\alpha \]
It is not generally true that $A(\alpha) B(\alpha) = A(\alpha) \cdot B(\alpha)$ as was implicitly assumed in (C2). The assumption there was

\begin{equation}
\bar{J} = \frac{\pi}{4} \lambda \exp \left\{ -\frac{\alpha h^2 \lambda^2}{2 t_f} \right\} \left[ I_0 \left( \frac{\alpha h^2 \lambda^2}{2 t_f} \right) + I_1 \left( \frac{\alpha h^2 \lambda^2}{2 t_f} \right) \right]
\end{equation}

or substituting

\[ x = \frac{\alpha h^2 \lambda^2}{2 t_f} \quad \text{and} \quad \lambda = \sqrt{x} \quad \frac{1}{\frac{1}{h} \sqrt{\frac{2 t_f}{\alpha}}} \]

\begin{equation}
\bar{J} = \frac{\pi}{4 h} \sqrt{\frac{2 t_f}{\alpha}} e^{-x} \left[ I_0(x) + I_1(x) \right]
\end{equation}

\begin{align*}
&= \frac{\pi}{4 h} \sqrt{\frac{2 t_f}{\alpha}} \cdot \sqrt{x} \cdot e^{-\left(\sqrt{x}\right)^2} \left[ I_0\left(\sqrt{x}\right)^2 + I_1\left(\sqrt{x}\right)^2 \right] \\
&= \frac{\pi}{4 h} \sqrt{\frac{2 t_f}{\alpha}} \cdot \sqrt{x} \cdot e^{-\left(\sqrt{x}\right)^2} \left[ I_0\left(\sqrt{x}\right)^2 + I_1\left(\sqrt{x}\right)^2 \right]
\end{align*}

This assumption is approximately correct for two reasons: The variation of $e^{-x}[I_0(x)+I_1(x)]$ with respect to $x$ is much less than the variation of $\sqrt{x}$ with respect to $x$. For that reason

\begin{equation}
\bar{J} = \frac{\pi}{4 h} \sqrt{\frac{2 t_f}{\alpha}} e^{-x} \left[ I_0(x) + I_1(x) \right] \approx \frac{\pi}{4 h} \sqrt{\frac{2 t_f}{\alpha}} \sqrt{x} \cdot e^{-x}[I_0(x)+I_1(x)].
\end{equation}

At the same time, as shown in equation (A7), when

\[ 1 \ll \beta = \frac{h^2 \lambda^2}{t_f} \]

then

\[ J = J_o = \left( \frac{\pi h}{4 \sqrt{t_f}} \right) \lambda \]
so that \( \tilde{J}_0 \) is easily obtained as

\[
\tilde{J}_0 = \left( \frac{\frac{\hbar}{4 \sqrt{c_f}}}{\lambda} \right) \lambda.
\]

In the limit \( 1 \ll \beta \frac{2}{p} \alpha = \frac{x}{2\lambda} \), \( J = J_\infty \approx \frac{\sqrt{n}}{2 \sqrt{\lambda}} \),

as shown in equation (A8). In this case no cyclic variations need to be considered. For all of these reasons, assumption (C9) is acceptable.
APPENDIX D

The Average Plastic Work Rate and The Temperature Distribution

Around A Constant Point Source of Heat

The rate at which work is done in the plastic zone is

\[ \dot{W} = \iint_{\text{plastic zone}} f(r, \theta; t) r \, d\theta \, dr. \]

Inserting (5) for \( f(r, \theta, t) \) yields

\[ \dot{W} = \frac{2m}{G} \frac{d}{dt} \left[ (\Delta K)^2 \right] \int_{0}^{R_p} \int_{0}^{R_p} g(r, R_p) \, dr = \frac{\pi m}{2G} \frac{R_p}{2} \frac{d}{dt} \left[ (\Delta K)^2 \right] \]

which under square-root wave loading becomes

\[ \dot{W} = \frac{\pi m^2 K_{1}^{4}}{2G \tau^2 \epsilon_{0} \rho c} \]

As earlier, \( \lambda = \frac{t}{t_{\lambda}} \). The average value of \( \dot{W} \) over a cycle of loading is

\[ \dot{\bar{W}} = \frac{\pi m^2 K_{1}^{4}}{4G \tau^2 \epsilon_{0} \rho c} \]

If this average work is concentrated at a point, the resulting temperature distribution may be obtained from (1):

\[ u(x', y', t_f) = \int_{0}^{t_f} \int_{0}^{t_f} \int_{\text{plastic zone}} \frac{f(x, y; t)}{\rho c} \exp \left[ \frac{-(x'-x)^2-(y'-y)^2}{4a^2(t_f-t)} \right] \times \]

\[ dx \, dy \, \frac{dt}{4\pi a^2(t_f-t)} \]
Because the source is small, \( x \ll x' \) and \( y \ll y' \), so

\[
(D5) \quad u(x', y'; t_f) = \int_0^{t_f} \int \int_{\text{plastic zone}} \frac{f(x, y; t)}{\rho c} \exp \left[ \frac{-x'^2 - y'^2}{4a^2(t_f - t)} \right] \, dx \, dy \, \frac{dt}{4\pi a^2(t_f - t)}
\]

Integration over the plastic zone may be carried out in cylindrical coordinates as shown in (D1) through (D3) so that

\[
(D6) \quad u(x', y'; t_f) = \int_0^{t_f} \frac{\dot{W}}{\rho c} \exp \left[ \frac{-x'^2 - y'^2}{4a^2(t_f - t)} \right] \frac{dt}{4\pi a^2(t_f - t)}
\]

Changing \( x' \) and \( y' \) to cylindrical coordinates yields

\[
(D7) \quad u(r_o, t_f) = \int_0^{t_f} \frac{\dot{W}}{\rho c} \exp \left[ \frac{-r_o^2}{4a^2(t_f - t)} \right] \frac{dt}{4\pi a^2(t_f - t)}
\]

By replacing \( \dot{W} \) with its time-average value, \( \bar{\dot{W}} \) (D7) becomes,

\[
(D8) \quad u(r_o, t_f) = \frac{\bar{\dot{W}}}{4\pi a^2 \rho c} \int_0^{t_f} e^{\left[ \frac{-r_o^2}{4a^2(t_f - t)} \right]} \frac{dt}{(t_f - t)}
\]

A change of variables, \( z = \frac{r_o^2}{\pi a^2(t_f - t)} \) and substitution for \( \bar{\dot{W}} \) yields
\[ u(r_0, t_f) = \frac{m^2 K_1^4}{16 a^2 G \tau_0^2 \kappa t_1} \int_{r_0}^{\infty} \frac{e^{-z}}{z} \frac{dz}{z} \]

or in terms of the exponential integral \( E_1 \),

\[ u(r_0, t_f) = \frac{m^2 K_1^4}{16 a^2 G \tau_0^2 \kappa t_1} E_1 \left( \frac{r_0^2}{4 a^2 t_f} \right). \]