Peridynamics and Applications

by

Jingkai Chen

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

APPROVED, THESIS COMMITTEE:

Pol D. Spanos, L. B. Ryon Professor of Mechanical Engineering and Civil Engineering

Fathi H. Ghorbel
Professor of Mechanical Engineering

Ilinca Stanciulescu, Associate Professor of Civil Engineering

C. Fred Higgs III, John and Ann Doer Professor of Mechanical Engineering

Houston, Texas USA
April, 2017
Abstract

Peridynamics and Applications

by

Jingkai Chen

Peridynamics is a nonlocal mechanics theory using integro-differential equations without spatial derivatives. Unlike the classical continuum mechanics, peridynamics possesses certain advantages when solving problems involving cracks. In the beginning of the thesis, an analytical solution to the vibration problem via fixed horizon peridynamics is developed, including dynamic responses of a bar and of a beam. The analytical solution to the vibration of the bar is derived through a Taylor series expansion approximation. Numerical examples demonstrate the nonlocal dynamic behavior of the bar and its consistency, in the limit, with local behavior. Further, a new nonlocal beam theory is proposed. The proposed nonlocal beam equation is a generalization of the Euler-Bernoulli beam equation. An analytical solution for the beam deformation is derived. The numerical example of the nonlocal beam deformation shows that the fixed horizon peridynamics has boundary conditions related inaccuracy problems. Thus, a new numerical technique to reduce the discrepancy problem is introduced, which is called: Variable Horizon Peridynamics. This method is quite efficient and it does not require a pseudo-layer to be added outside the physical boundary.

Next, an efficient algorithm to model the bit-rock interaction process based on the variable horizon peridynamics is developed. This model iterates adaptively with the
propagation of the crack and with the penetration of the drill bit. The crack propagation in the rock is captured in this model. The relationship between the penetration rate and other drilling parameters is investigated.

Finally, the Navier-Stokes equation is reformulated in a nonlocal sense via the variable horizon peridynamics. It is shown that the reformulated Navier-Stokes equation satisfies Newton’s second law. When the nonlocal parameter reduces to zero, the reformulated Navier-Stokes equation reduces to the classical Navier-Stokes equation. To elucidate the features of the approach, numerical examples of both local and nonlocal Navier-Stokes equations are used.
Acknowledgements

First, I would like to express my deepest gratitude to my advisor Professor Pol D. Spanos for his mentorship through my incredible years at Rice University. His unending enthusiasm and encouragement pushed me to continue improving myself and exploring unrestrictedly new ideas. I am grateful for this life changing experience he offered.

Thanks are also due to my thesis committee Professor Ilinca Stanciulescu for her deep understanding of my research. Her lectures on materials mechanics and numerical techniques built a strong foundation for my research. Also I would like to thank Professor Fathi H. Ghorbel for participating in my committee and providing me with insightful feedback. Further, I would like to express my gratitude to Professor Fred III C. Higgs for serving on my thesis committee.

I also want to thank all my professors and friends at Rice University for making my life very enjoyable.

I would like to thank my parents Fu Chen and Xiaolian Wang and my brother Jingxuan Chen for their unconditional love and support. Special thanks are also due to Xinyu Xiang for her sincere love, patience, and understanding. Last but not least, I would like to express my gratitude to my grandfather Shunde Chen for his unconventional horizon and unconditional support; this thesis is dedicated to him.
**Table of Contents**

Abstract ...........................................................................................................................................i

Acknowledgements .........................................................................................................................iii

Table of Contents ............................................................................................................................iv

List of Figures ..................................................................................................................................lx

List of Tables ....................................................................................................................................xv

1. Introduction ..................................................................................................................................1

   1.1 Thesis Perspective .....................................................................................................................1

   1.2 Thesis Outline ..........................................................................................................................6

2. Mathematical Background .........................................................................................................10

   2.1 Local Continuum Mechanics ..................................................................................................10

   2.2 Bond Based Peridynamics .......................................................................................................11

   2.3 State Based Peridynamics ........................................................................................................16

   2.4 Navier Stokes Equations ..........................................................................................................22

3. New Analytical Solution to Vibration Problems via Fixed Horizon Peridynamics .................23

   3.1 Preliminary Remark ..................................................................................................................23

   3.2 Vibration Analysis of Bar via Fixed Horizon Peridynamics ..................................................23
3.2.1 Peridynamic bar background..............................................................23
3.2.2 Analytical solution of peridynamic bar..............................................25
  3.2.2.1 Dispersion relation.................................................................25
  3.2.2.2 Homogeneous solution.........................................................27
  3.2.2.3 Nonhomogeneous solution....................................................33
3.2.3 Numerical validation.................................................................37
  3.2.3.1 Dispersion relations..............................................................37
  3.2.3.2 Free vibration.................................................................38
  3.2.3.3 Forced vibration..............................................................41

3.3 Vibration of Beam via Fixed Horizon Peridynamics............................47
  3.3.1 Beam theory background......................................................47
  3.3.2 Mathematical derivation of nonlocal beam.................................50
  3.3.3 Analytical solution for beam deformation.................................57
  3.3.4 Numerical examples............................................................65

3.4 Synopsis..........................................................................................69

4. The Variable Horizon Peridynamics Approach – Static Problems..........71
  4.1 Problem Description.....................................................................71
  4.2 Iterative Interpolation Method.....................................................74
    4.2.1 Method description...............................................................74
4.2.2  Numerical verification

4.3 Variable Horizon Approach

4.3.1  Static deformation of nonlocal bar via variable horizon method

4.3.1.1  Variable horizon method derivation

4.3.1.2  Numerical verification

4.3.2  Static deformation of nonlocal beam via variable horizon method

4.3.2.1  Static deformation of double clamped beam

4.3.2.2  Static deformation of simply supported beam

4.3.2.3  Static deformation of cantilever beam

4.4 Dual Horizon Peridynamics

4.4.1  Peridynamic bond force reformulation

4.4.2  Numerical Verification

4.5 Synopsis

5.  Adaptive Bit-Rock Interaction Modeling via Variable Horizon Peridynamics

5.1 Preliminary Remark

5.2 Mathematical Background

5.3 Adaptive Modeling
5.4 Application of Drilling.................................................................117
5.5 Rate of Penetration Analysis......................................................122
  5.5.1 Influence of weight on bit on penetration rate......................123
  5.5.2 Influence of rotary speed on penetration rate.....................124
  5.5.3 Influence of axial vibration on penetration rate.................124
5.6 Synopsis................................................................................127

6. Peridynamics Application to Fluid Mechanics via Variable Horizon
   Approach..................................................................................128
6.1 Local Navier-Stockes Equation...............................................128
  6.1.1 Momentum equation............................................................128
  6.1.2 Pressure Poisson Equation.................................................129
6.2 Nonlocal Navier-Stokes Equation..........................................130
  6.2.1 Nonlocal momentum equation............................................130
  6.2.2 Nonlocal pressure Poisson equation.................................132
6.3 Nonlocal Consistency Analysis...............................................135
  6.3.1 Nonlocal convection term..................................................135
  6.3.2 Nonlocal diffusion term....................................................141
  6.3.3 Nonlocal pressure term.....................................................142
6.3.4 Nonlocal incompressibility condition ...........................................143

6.4 Numerical Verification ........................................................................144

6.5 Synopsis .................................................................................................162

7. Conclusion Remarks ..................................................................................163

References ....................................................................................................169
List of Figures

Fig 2.1 Stresses on an infinitesimal element.................................................................10
Fig 2.2 Bond based peridynamics configuration.........................................................12
Fig 2.3 Bond based peridynamics deformation.......................................................14
Fig 2.4 Deformation states configuration.................................................................18
Fig 2.5 Different material responses........................................................................19
Fig 2.6 State based peridynamic governing equation configuration.......................20
Fig 3.1 Fix-free finite bar configuration..................................................................33
Fig 3.2 Analytical and numerical dispersive relations............................................39
Fig 3.3 Snapshots of first mode responses..............................................................41
Fig 3.4 Snapshots of second mode responses.......................................................41
Fig 3.5 Snapshots of third mode responses.............................................................42
Fig 3.6 Elastic bar response due to nonresonant excitation....................................43
Fig 3.7 Peridynamic bar response due to nonresonant excitation with
   \( \delta = 5m \)............................................................................................................44
Fig 3.8 Snapshots of elastic and peridynamic nonresonant responses with
   \( \delta = 5m \)............................................................................................................44
Fig 3.9 Peridynamic bar response due to nonresonant excitation with
Fig 3.10  Snapshots of elastic and peridynamic nonresonant responses with
\[ \delta = 10\text{m} \]

Fig 3.11  Elastic bar response due to resonant excitation

Fig 3.12  Peridynamic bar response due to resonant excitation with
\[ \delta = 5\text{m} \]

Fig 3.13  Snapshots of elastic and peridynamic resonant responses with
\[ \delta = 5\text{m} \]

Fig 3.14  Peridynamic bar response due to resonant excitation with
\[ \delta = 10\text{m} \]

Fig 3.15  Snapshots of elastic and peridynamic resonant responses with
\[ \delta = 10\text{m} \]

Fig 3.16  Euler-Bernoulli beam configuration

Fig 3.17  Bending integration configuration

Fig 3.18  Simply supported beam

Fig 3.19  Nonlocal simply supported beam deformation

Fig 3.20  Cantilever beam

Fig 3.21  Nonlocal cantilever beam deformation
Fig 3.22 Double clamped cantilever beam

Fig 3.23 Double clamped cantilever beam meshing

Fig 3.24 Deformation subject to uniform loading with constant horizon

Fig 4.1 Typical peridynamics boundary condition implementation process

Fig 4.2 Constant horizon Peridynamics meshing

Fig 4.3 Nonlocal responses with fixed pseudo-nodes

Fig 4.4 Nonlocal response errors with adaptive pseudo-nodes displacement

Fig 4.5 Horizon of nodes near boundary

Fig 4.6 Horizon configuration at one-dimensional boundary region

Fig 4.7 Peridynamics bar configuration with variable horizon

Fig 4.8 Variable horizon peridynamics meshing

Fig 4.9 Nonlocal responses of bar with variable horizon

Fig 4.10 Nonlocal responses of bar with modified variable horizon

Fig 4.11 Variable horizon configuration of clamped-clamped beam

Fig 4.12 Deformation of clamped-clamped beam subject to uniform loading

Fig 4.13 Deformation of clamped-clamped beam subject to point loading
Fig 4.14  Deformation of clamped-clamped beam with initial crack.........................91
Fig 4.15  Deformation of clamped-clamped beam with multiple cross sections.....92
Fig 4.16  Deformation of simply supported beam.............................................94
Fig 4.17  Deformation of simply supported beam with initial crack....................95
Fig 4.18  Deformation of simply supported beam with multiple cross sections.....96
Fig 4.19  Deformation of cantilever beam..........................................................97
Fig 4.20  Deformation of cantilever beam with initial crack.............................98
Fig 4.21  Deformation of cantilever beam with multiple cross sections..............99
Fig 4.22  Constant horizon bond force..............................................................100
Fig 4.23  Variable horizon bond forces..............................................................101
Fig 4.24  Difference between variable horizon bond force and state based bond force................................................................................................................103
Fig 4.25  Horizon configuration of reformulated peridynamic bar..................104
Fig 4.26  Deformation of bar with reformulated variable horizon....................106
Fig 5.1  Peridynamics bond force configuration................................................111
Fig 5.2  Bond deformation configuration..............................................................112
Fig 5.3  Bond force stretch relationship...............................................................112
Fig 5.4  Drilling process configuration.................................................................116
Fig 5.5  Rock property variation description......................................................................117
Fig 5.6  Domain of interested meshing............................................................................118
Fig 5.7(a) Loading representation.....................................................................................119
Fig 5.7(b) Deformation in the bit-rock contact region......................................................119
Fig 5.8  Deformation stages of a particular layer..............................................................120
Fig 5.9  Damage propagation of a particular layer..........................................................122
Fig 5.10 Damage in radial direction..................................................................................122
Fig 5.11 ROP with different WOB..................................................................................123
Fig 5.12 ROP with different rotary speed.......................................................................124
Fig 5.13 ROP with different axial vibration amplitudes...................................................126
Fig 5.14 ROP with different axial vibration frequencies...................................................126
Fig 6.1  Nonlocal fluids configuration..............................................................................130
Fig 6.2  Nonlocal weighted shape tensor operation........................................................139
Fig 6.3  Pressure Poisson numerical solver scheme........................................................145
Fig 6.4  2D parallel flow case 1......................................................................................146
Fig 6.5  Velocity field from classical Navier-Stokes equation.........................................148
Fig 6.6  Pressure field from classical Navier-Stokes equation.........................................148
Fig 6.7  Displacement field from nonlocal Navier-Stokes equation...............................149
Fig 6.8  Pressure field from nonlocal Navier-Stokes equation..........................149
Fig 6.9  Velocity error field..................................................................................150
Fig 6.10 Pressure error field..................................................................................151
Fig 6.11 Velocity field from local Navier-Stokes equation.................................153
Fig 6.12 Pressure field from local Navier-Stokes equation.................................154
Fig 6.13 Velocity field from nonlocal Navier-Stokes equation............................155
Fig 6.14 Pressure field from nonlocal Navier-Stokes equation............................155
Fig 6.15 Velocity error field..................................................................................156
Fig 6.16 Pressure error field..................................................................................157
Fig 6.17 2D Parallel flow with interior hole...........................................................157
Fig 6.18 Velocity from classical Navier-Stokes equation......................................159
Fig 6.19 Pressure field from classical Navier-Stokes equation............................160
Fig 6.20 Velocity field from nonlocal Navier-Stokes equation............................161
Fig 6.21 Pressure field from nonlocal Navier-Stokes equation............................161
List of Tables

Table 3-1  Coefficients of nonlocal beam vibration equation........................................59

Table 4-1  Materials parameter of peridynamic rod.........................................................77
Chapter 1

Introduction

1.1 Thesis Perspective

Nonlocal theory, as a generalized continuum mechanics approach, was first proposed in early 1960s (Eringen and Suhubi, 1962). Nonlocal stress at one point was defined as a function of integrating the strain over a nearby region. Kroner (1967) formulated the nonlocal elastic equation by introducing the concept of long-range cohesive force. Later, Eringen (1972), Kunin (1982) and Rogula (1982) formulated the nonlocal elastic theory in the form of integro-differential equation (IDE). In local continuum mechanics, the stress at one point is determined by the spatial derivation of deformation at that point. In nonlocal mechanics, the stress at one point is defined as integrating deformation over a surrounding region. A nonlocal parameter is introduced to indicate the size of the surrounding area. These nonlocal theories are consistent with the local continuum theory, when the nonlocal parameter reduces to zero. Similar to local elasticity, these classical nonlocal theories were derived from the continuum assumption and ended up with the governing equations involving spatial differential terms. Peridynamics, as a new nonlocal theory without continuum assumption, was first proposed by Silling (2000). Peridynamic theory can be regarded as a continuous version of molecular dynamics or as a generalized version of local elastic theory. Peridynamics is a completely nonlocal theory, which indicates that no spatial differential term is involved in its governing equation. For the local mechanics theory and the traditional nonlocal theories, problem arises when discontinuity appears. As a
compromise, the discontinuous displacement field is usually calculated by adding complimentary equations or jump functions. However, this often requires the prediction of possible discontinuity location and the additional degrees of freedom at the discontinuous region. Peridynamics has shown great promises in solving problems involving cracks. Crack initiation and propagation can be generated automatically by peridynamics.

Crack theories had been studied for centuries, until Griffith’s criterion was proposed from an energy perspective in 1920s. Specifically, the release of strain energy after crack must be equal to or greater than the increase of surface energy to propagates or initiates the crack. Irwin modified Griffith’s criterion by adding a plastic energy dissipation term. The modified crack criterion involves ductile deformation at the crack tip. J-integral criterion, which was developed by Cherepanov (1967) and Rice (1968) independently, addressed the crack initiation and propagation by calculating the energy release rate. Fracture toughness of plastic or nonlinear elastic materials can be predicted by using the J-integral method and different fracture modes can be captured. Cohesive zone model is a numerical method to model the crack initiation and propagation. This numerical technique often requires mesh refinement within the crack region, unless the crack path is pre-known. However, the mesh refinement requires recalculating the entire domain. Thus, the cohesive zone model requires high computing power. Extended finite element method (XFEM), which developed by Belytschko (1999), modified the classical finite element method (FEM) by enriching the solution space with discontinuous basis functions. Specifically, the piecewise continuous functions that are used in FEM were enriched by introducing the ‘jump’ basis functions at these cracked elements. XFEM does not require mesh
refinement or recalculating the entire domain. However, additional degrees of freedom are required at the crack region. Peridynamics is completely a nonlocal theory. No spatial differential terms are involved in its governing equation. In the peridynamics scheme, the discontinuity is automatically captured by a simple and compact integral-differential equation. Neither additional enrichment function nor predetermination of the crack region is required.

Traditional Euler Bernoulli and Timoshenko beam theories are commonly derived based on a local continuum assumption. With the development of nanotechnology, the nonlocal beam theories based on nonlocal mechanics have been proposed to describe the response of the beam in nano-scale. The nonlocal beam theories based on peridynamics have also been proposed in recent years. Grady and Foster (2014) reformulated the Euler-Bernoulli beam equation from state based peridynamics. Moyer and Miraglia (2014) applied bond based peridynamics concepts to formulate the equation of the Timoshenko beam. Although peridynamics is a completely nonlocal theory, aforementioned peridynamics nonlocal beam theories involve spatial differential term, and continuum assumptions were required. In this thesis, a completely nonlocal beam theory is proposed, which reformulates the Euler Bernoulli beam equation without spatial differential terms involved. Responses of beams with discontinuous geometry and with initial crack are analyzed by the proposed nonlocal beam theory.

Drilling is an extremely complex process. As stated by Evangelatos and Payne (2016), bit-rock interaction and the dynamics that the bit exerts on the bottom hole assembly (BHA) are stochastic. Thus, most analyses or numerical models must make
assumptions for this complicated interaction. Even though the BHA properties are well known, and the vibration characteristics can be estimated, the impact forces of those components with an imperfect bore hole with variable formation properties cannot be precisely calculated. Despite the complexity of drilling, researchers have analyzed drilling process analytically, numerically and experimentally for over half of a century. Based on limited drilling field data, Bielstein and Cannon (1950) investigated influences of bit design, hydraulic factors, weight on bit and rotary speed on the rate of penetration (ROP). Teale (1965) derived ROP of drilling process by introducing the concept of specific energy. Bourgoynes's and Young's model is a powerful model to predict the ROP based on formation compaction and strength, pore pressure, bit wear, rotary speed, bit hydraulics, teeth wear, and etc. These drilling models were verified by Pressier and Fear (1992), Bataee (2010) and Kamyab (2010) using either experimental or field data with a specific bit type. Those models were derived from experiments or field data and often yielded moderate results. However, further investigations of the rock crack mechanism and the bit rock interaction are required to better understand ROP. The rapid development of high computing technology made the refined numerical simulations of the drilling process possible. Finite element drilling models were built in recent years based on local elastic theory. These numerical models often require a remeshing process when crack appears. This often enhances the computational inefficiency, since in real drilling process, rock is cracked severely into small size gravel. The highly heterogeneous properties of rock and large amount of pre-existed crack make the finite element drilling modeling even more expensive. Peridynamics, as a nonlocal theory, possesses advantages when modeling heterogeneous materials responses involving crack. In this thesis, the formation properties are modeled
stochastically. ROPs corresponding to different drilling parameters are simulated based on the peridynamic model.

The Navier-Stokes equations can be derived from the Newton's second law, and it is the fundamental equation capturing the fluid behavior. Understanding the Navier-Stokes equations is critical to analyze the complex fluid phenomenon, such as turbulence. However, despite its immense application in the engineering domain, the exact solution of Navier-Stokes equation is not available in the three dimensional Euclidean space. The nonlinearity of Navier-Stokes equation often introduces numerical singularities, especially when calculating the fluid responses with high Reynolds number. In this thesis, the Navier-Stokes equation is reformulated based on the peridynamics concept. The reformulated Navier-Stokes equation is nonlocal, which has no spatial differential term. Advection and diffusion terms are derived directly from the nonlocal momentum. It is shown that the derived nonlocal Navier-Stokes equation is, in the limit, consistent with the classical Navier-Stokes equation. The nonlocal Navier-Stokes equation describes the fluid behavior with no continuum assumption. Thus, it is more flexible than the classical Navier-Stokes equation. One way to numerically solve the classical Navier-Stokes equation is by introducing the pressure Poisson equation, which is derived from the Navier-Stokes equation under incompressibility condition. The nonlocal pressure Poisson equation can also be derived from the nonlocal Navier-Stokes equation and the nonlocal incompressibility condition. In this thesis, the nonlocal pressure Poisson equation is introduced to numerically solve the reformulated nonlocal Navier-Stokes equation. Related numerical results are also presented.
1.2 Thesis Outline

This thesis has 7 chapters related to peridynamics and its applications.

Chapter 1 provides a perspective and outline of the thesis.

Chapter 2 introduces mathematical background on peridynamics. Peridynamics is a nonlocal mechanics theory based on an integro-differential equation. Unlike classical continuum mechanics, peridynamics considers nonlocal responses both statically and dynamically. The peridynamics governing equation has no spatial differential terms. Thus, it is quite efficient when solving problems involving cracks. Depending on how peridynamic internal forces are formulated, peridynamics theory can be divided into two versions: bond based peridynamics, and state based peridynamics. As proposed by Silling (2007), state based peridynamics is a generalization of the bond based peridynamics and it is more powerful and flexible than the bond based peridynamics. However, the state based peridynamics often requires higher computing power. The nonlocal parameter (horizon) is critical for multi-scale modeling via peridynamics, and it is usually determined by the specific problem itself. Depending on how the horizon function is defined over the domain of interest, peridynamics can be divided into fixed horizon peridynamics and variable horizon peridynamics.

Chapter 3 proposes a new analytical solution to vibration problems via fixed horizon peridynamics. This chapter has two sections: dynamic analysis of a bar, and dynamic analysis of a beam. In the first section, the nonlocal dispersive relation of bar is proposed and compared with the local dispersion relation. The homogeneous
solution and the nonhomogeneous solution of bar vibration equation are derived analytically by separation of variables. Free and forced vibration of bar are examined numerically based on fixed horizon peridynamics. The second section focuses on an analytical solution of nonlocal beam deformation via fixed horizon peridynamics. Inspired by the concept of peridynamics, a new nonlocal beam theory is first proposed in this section. It is shown that the proposed nonlocal beam theory is consistent with the local Euler-Bernoulli beam theory. Further, this new beam theory exhibits nonlocal behavior. An analytical determination of the static and the dynamic responses of beam are also achieved by solving the new nonlocal beam equation with fixed horizon.

Fixed horizon peridynamics has boundary condition related discrepancy. This problem derives from the fact that material stiffness at the boundary region is reduced because there is no enough material ‘fill in’ its horizon. The common remedy for this problem is to add a pseudo-layer outside the physical boundary. However, this may not be adequate especially when the deformation is complex on the boundary region or a crack initiated on the boundary.

Chapter 4 introduces two novel numerical techniques to deal with the boundary condition related discrepancy: Iterative Interpolation Method (IIM), and Variable Horizon Approach (VHA). The iterative interpolation method predicts the displacement of the pseudo-layer by interpolating the displacements on the physical domain. The iterative process guarantees the accuracy of deformation. However, IIM usually requires high computing power. Inspired by the fact that peridynamics reduces to local continuum mechanics when the horizon reduces to zero, the
Variable Horizon Approach is introduced to overcome the nonlocal boundary related problems. The VHA defines the horizon value based on its distance from the boundary, and provides moderate results without adding the pseudo layer outside of the boundary. The VHA technique is applied to determine the static deformation of a bar, and the static deformation of a beam. Numerical examples demonstrate the efficiency and accuracy of the VHA technique.

Chapter 5 pertains to the important process of drilling in oil and gas exploration and production. It focuses on the bit-rock interaction. In this chapter, an efficient model to describe the rock crack process based on peridynamics is developed. This model updates itself adaptively with the propagation of the crack and with the penetration of the drill bit. The heterogeneous properties and initial crack of rock are accounted for in the model. The rock crack propagation process is captured, and rate of penetration (ROP) with different weight on bit (WOB), different rotary speeds, and different axial vibration are investigated.

Chapter 6 proposes a new Navier-Stokes equation based on peridynamic concept. Advection and diffusion terms in the Navier-Stokes equation are reformulated nonlocally. It is shown that the reformulated nonlocal Navier-Stokes equation follows Newton’s second law. Further, if the nonlocal parameter reduces to zero, the nonlocal Navier-Stokes equation reduces to the classical Navier-Stokes equation. The nonlocal Pressure Poisson Equation is derived from the nonlocal Navier-Stokes equation and the nonlocal incompressibility condition. Note that the nonlocal Navier-Stokes equation describes fluid behavior with no continuum assumption. Therefore, it is more flexible than the classical Navier-Stokes equation. Finally,
numerical examples of both Navier-Stokes and nonlocal Navier-Stokes are provided.

Chapter 7 provides concluding remarks, specifies the thesis, and discusses future work. Bit rock interaction model may require further investigation. A more stable and robust numerical algorithm may be desirable for solving nonlocal Navier-Stokes equation.
Chapter 2

Mathematical Background

2.1 Local Continuum Mechanics

Local mechanics theory states that forces only exist between two points in contact. Responses of the materials are built based on the concepts of stress and strain. Stress is defined as force per unit area, while strain is defined as the spatial derivative of the deformation. Both of them are defined based on a continuum assumption, which means that the deformation of material must be continuous and no cracks pre-exist in the material.

Local mechanics theory is introduced by using an infinitesimal square element as shown in figure (2.1). The force density acting on the edge of any infinitesimal element within the material is represented by the stress tensor, which is a function of the location. For any arbitrary point within the domain, the force equilibrium condition yields

![Stress Diagram](image)
\[ \rho \ddot{u}_i = \sigma_{i,j} + b_i, \]  

where \( \rho \) is the mass density, \( u \) is the displacement field, \( \sigma \) is the stress tensor, and \( b \) is the body force. All variables in equation (2.1) are functions of the location. By neglecting the influence of the moment density, the moment equilibrium condition yields the following relation

\[ \sigma_{ij} = \sigma_{ji}. \]  

The constitutive law describes the relationship between the stress tensor and the strain tensor. The stress tensor is a function of location. It describes the force state of any point within the domain of interest. The strain tensor is derived from the spatial derivation of the geometrical deformation. For different kinds of materials, the constitutive criteria are also varying. For example, for hyper elastic material, the stress tensor is related to the strain tensor by using strain energy density function; For the Cauchy elastic material, the stress tensor is related to the current deformation; other materials may have more complex behavior. Regardless of these complex stress-deformation relationships, the linear elastic constitutive law can be expressed as

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \]  

where \( \varepsilon_{kl} \) is the strain tensor, and \( C_{ijkl} \) is defined as the constitutive coefficient and is a fourth order tensor.

2.2 Bond Based Peridynamics

The early version of peridynamics is called bond-based peridynamics. Different from classical elasticity theory where the constitutive force acting on one point is
only exerted by its direct contact points, the peridynamic constitutive force acting on one point is exerted by all points within some distance $\delta$ (known as ‘horizon’). Peridynamics can also be regarded as a continuous version of the molecular dynamics (MD). In fact, early peridynamics model utilized MD software to do numerical analysis. However the constitutive law from peridynamics is different from the Van der Waals force in MD.

Peridynamics is a nonlocal mechanics theory. It can also be regarded as a generalization of the local mechanics theory. Local mechanics is derived from the continuum assumption. Thus, problems arise when discontinuity appears. The peridynamics is a nonlocal theory. It describes continuities and discontinuities in a consistent manner. Numerical implementations of the bond-based peridynamics were proposed by Silling, et. al. (2005), Macek, et. al. (2007), Chen et. al. (2011). Application of the bond-based peridynamic is limited to specific Poisson ratio, and shear deformation can not be captured.

Fig. 2.2. Bond based peridynamics configuration
A bond based peridynamic configuration is shown in figure (2.2). Each pair of particles within the same horizon interacts through a vector-valued function \( f \), where \( f \) is a function of location and displacement and is called bond force. Specifically,

\[
f = f(u(\hat{x}, t) - u(x, t), \hat{x} - x).
\] (2.4)

The internal force is calculated by integrating the bond force over the horizon and expressed as

\[
\mathcal{L}_u(x) = \int_{\mathcal{H}} f(u(\hat{x}, t) - u(x, t), \hat{x} - x) dV_{\hat{x}}.
\] (2.5)

The peridynamics equation of motion is derived from the Newton's second law, where the peridynamics internal force and the external force density are related with inertial force density by the equation

\[
\rho \ddot{u}(x, t) = \int_{\mathcal{H}} f(u(\hat{x}, t) - u(x, t), \hat{x} - x) dV_{\hat{x}} + b(x, t).
\] (2.6)
Reference and deformed configurations of the bond based peridynamics are shown in figure (2.3). Here $x$ and $\hat{x}$ are a pair of particles in the reference configuration, while $y$ and $\hat{y}$ are a pair of particles in the deformed configuration; and $u$ and $\hat{u}$ are displacements of the pair of particles. The relative location and relative displacement are defined by $\eta$ and $\xi$ as

$$
\eta = \hat{u} - u \quad \tag{2.7}
$$
$$
\xi = \hat{x} - x. \quad \tag{2.8}
$$

Then the peridynamics constitutive law can be expressed in the compact form

$$
f = f(\eta, \xi). \quad \tag{2.9}
$$

Linear admissibility and angular admissibility condition dictate that

$$
f(\eta, \xi) = -f(-\eta, -\xi), \quad \tag{2.10}
$$

and

$$
(\xi + \eta) \times f(\eta, \xi) = 0, \quad \tag{2.11}
$$

Fig. 2.3. Bond based peridynamic deformation
where the linear admissibility condition is provided by the Newton’s third law, and the angular admissibility condition is provided by the assumption that no moment density exists at any point within the domain.

The linear admissibility and the angular admissibility conditions yield the general form of the bond-based peridynamics constitutive law

$$ f(\eta, \xi) = F(\eta, \xi)(\xi + \eta), $$

where $F(\eta, \xi)$ is a scalar valued function, which satisfies $F(\eta, \xi) = F(-\eta, -\xi)$.

The peridynamics internal force exists between two points within the same horizon. This internal force is by unit force per volume square. Linearization of the bond based peridynamic bond force is derived by a Taylor’s expansion. That is,

$$ f(\eta, \xi) = f(0, \xi) + \frac{\partial f}{\partial \eta} \bigg|_{(0,\xi)} \cdot \eta + O(|\eta|^2). $$

The linearized constitutive coefficients is defined by

$$ C(\xi) = \frac{\partial f}{\partial \eta} \bigg|_{(0,\xi)}. $$

The linear admissibility condition guarantees that

$$ C(\xi) = C(-\xi). $$

The corresponding linearized form is

$$ \rho \ddot{u}(x, t) = \int_{\Omega} C(x' - x) \cdot (u' - u) dV_{x'} + b(x, t). $$

The linearized equation of motion is derived based on the first order Taylor’s expansion at $\eta = 0$. Therefore, the linearization is not accurate for large values of $\eta$. This is part of the reason why brittle materials are much easier to simulate using linearized peridynamics while ductile materials are not.
Similarly to the classical elasticity theory, the peridynamics constitutive law of hyper elastic materials is derived from potential energy. Since the potential energy density function has different forms depending on different material properties, the peridynamics constitutive law also varies in different form.

An example of a constitutive law was proposed by Silling and Askari (2005). They assume the constitutive law can be written in the following form

\[ f(\eta, \xi) = c_0 \frac{\eta}{|\xi|}, \]  

(2.17)

where \( c_0 \) is constant.

The following procedures calculates the proper value of \( c_0 \) such that the potential energy derived from peridynamics constitutive law consistent with the potential energy derived from classical elasticity theory. For simplification, one assumes isotropic deformation, which leads to potential energy derived from peridynamics be expressed as

\[ W = \frac{\pi c s^2 \delta^4}{4}. \]

(2.18)

Equalizing this potential energy with the potential energy derived from classical elasticity theory, one can determine the peridynamics coefficient as

\[ c_0 = \frac{18k}{\pi \delta^4}. \]

(2.19)

Other forms of the constitutive relationship can also be assumed. For example, Bobaru et. al. (2009) derived a constitutive model with triangular and inverse-triangular peridynamics coefficient for the one dimensional case. Note that only one independent coefficient appears in the bond based peridynamics
constitutive law; applications of bond-based peridynamics are limited. Bond based peridynamics describes materials with fixed Poisson ratio of 1/3 in 2D plane stress and 1/4 in 3D. One way to fix the Poisson ratio problem is by introducing the concept of state.

2.3 State Based Peridynamics

Silling, Epton and et. al. (2007) proposed state based peridynamics, which describes response of materials with arbitrary Poisson ratio. Peridynamics is reduced to local elasticity theory when horizon reduces to zero. Silling and Lehoucq (2008) proved that by assuming smooth deformation and material geometry, the peridynamic stress tensor converges to a Piola-Kirchhoff stress tensor as the horizon reduces to zero. Mathematical analysis has also shown the consistency of peridynamics with local elasticity theory for both stationary and dynamic models (Du, et. al. 2011). Zhou, et. al. (2010) and Bobaru, et. al. (2009) numerically verified the convergence of peridynamics to elasticity for both continuous and discontinuous micro moduli.

A state of order $n$ is defined as an operator, which acts on vectors; the image of the vector under this operator is an $n$-th order tensor. Vector state is when $n=1$ while scalar state is when $n=0$. Properties of state are detailed in (Siling, et. al., 2007).

**Deformation state.** Similar to the strain tensor defined in classical elasticity theory as second order tensor, the deformation state is defined in state-based peridynamics as a vector state. The deformation vector state field shown in figure (2.4) is expressed as

\[ Y[x, t](\xi) = y(x + \xi, t) - y(x, t), \quad (2.20) \]
where, \( Y[x, t] \langle \xi \rangle \) is the deformation vector state at point \( x \) operating on vector \( \xi \) within the angle bracket.

![Fig. 2.4. Deformation states configuration](image)

Deformation state and strain tensor have the following similarities:

A. The strain tensor operating on any unit vector yields to strain vector on the corresponding normal surface; deformation state operates on any bond vector \( \xi \) gets extension of the corresponding bond.

B. The strain tensor at one point contains deformation information at all directions of the point; while the deformation vector state at one point contains all deformation information between the point and all other point within its horizon.

C. The strain tensor can be regarded as a special form of the deformation state.
Mathematically, any strain tensor can be replaced by its corresponding deformation state while the reverse is not true. For example, deformation state can be nonlinear, but strain tensor can not be nonlinear.

**Force state.** The force state has the same mathematical meaning with the deformation state. Force state operating on the bond vector $\xi$ yields the bond force. The constitutive model relates the force state with the deformation state $Y$ and other variables $\Lambda$ in the expression

$$T = \hat{T}(Y, A).$$

(2.21)

If the force state only depends on the deformation state, then the material is called simple. A simple material is called ordinary if the force state parallels to the deformed vector state, otherwise is non-ordinary.

Bond-based, ordinary state-based, non-ordinary state-based material responses are described in figure (2.5).

![Diagram](image.png)

**Fig. 2.5. Different material responses**
Silling and Lehoucq (2008) showed that as the length scale reduces to zeros, the stress tensor derived from state-based peridynamics model converges to the Piola-Kirchhoff stress tensor; thus, consistent with the classical elasticity theory.

**Governing Equation.** State-based peridynamics governing equation was derived by (Silling, et. al., 2007) in the form

\[
\rho \ddot{u}(x, t) = \int_{\mathcal{H}} \left\{ T[x, t](x' - x) - T'[x', t](x - x') \right\} dV_x, + b(x, t). \tag{2.22}
\]

This governing equation shows that the state based peridynamics response at one point is not only related to the force state at that point but also to the force state of all other points within its horizon, as shown in figure (2.6).

![State based peridynamics bond configuration](image)

**Fig. 2.6.** State based peridynamics bond configuration

State based peridynamics constitutive law relates the force state to the deformation state. Thus, the state based peridynamics behavior of point x not only depends on the deformation of x’ but also depends on the deformation of point P, even though P
does not locate within the horizon of \( x \). In other words, \( P \) influences the behavior of 
\( x \) by directly influencing deformation state of point \( x' \).

The linearization of the state based peridynamic governing equation was performed 
using the Frechet derivatives of the vector state function and the concept of double 
states by (Silling, 2010). The following equation was derived

\[
\rho \ddot{u}(x,t) = \int_{J^c} \int_{J^c} K[x](x' - x, p - x)(u(p, t) - u(x, t))dV_p dV_{x'}, \\
- \int_{J^c} \int_{J^c} K[x'](x - x', p - x')(u(p, t) - u(x', t))dV_p dV_{x'} + b(x, t),
\]

(2.23)

where \( p \) and \( x' \) are dummy parameters.

The linearized governing equation of the state based peridynamics is consistent 
with the original governing equation on the condition that the deformation is small.
The linearized governing equation also captures the influence of deformation at 
point \( P \), even when \( P \) locates outside the horizon of \( x \), as shown in figure (2.6).

The state-based peridynamics constitutive law relates the deformation vector state 
to force vector state. The constitutive law of hyper elastic material derives from the 
derivative of the potential energy. Different forms of potential energy represent 
different material properties; their corresponding constitutive laws are also 
different.

Examples of two-dimensional linear elastic solids constitutive law are (Le et. al., 
2014):
for two-dimensional plane stress,

\[
t = \frac{2(2\nu - 1)}{(\nu - 1)} \left( k' \theta - \frac{\alpha}{3} (\omega e^d) \cdot x \right) \frac{\omega x}{q} + \alpha \omega e^d ; \quad (2.24)
\]

and for two-dimensional plane strain,

\[
t = 2 \left( k' \theta - \frac{\alpha}{3} (\omega e^d) \cdot x \right) \frac{\omega x}{q} + \alpha \omega e^d , \quad (2.25)
\]

where \( k' \) and \( \alpha \) are independent peridynamic coefficients which depend on the Young’s modulus and the Poisson ratio. Therefore, state-based peridynamics can model materials with arbitrary Poisson ratio. And \( \omega \) is the influence function which only depends on \( |\xi| \) in this case. Nevertheless, the influence function \( \omega \) can be defined in other ways to model anisotropic or even heterogeneous materials.

### 2.4 Navier-Stokes Equations

The classical Navier-Stokes equation contains spatial differential terms, which require the continuum assumption. One way to derive the Navier-Stokes (NS) equation is by applying Newton’s second law. That is,

\[
\frac{\partial}{\partial t} (\rho \, u) + \nabla \cdot (\rho \, u \otimes u) = \nabla \cdot \sigma + \rho \, f . \quad (2.26)
\]

For incompressible fluids, the NS equilibrium equation can be rewritten as

\[
\rho \left[ \frac{\partial u}{\partial t} + u \cdot \nabla u - f \right] = \nabla \cdot \sigma \quad (2.27a)
\]

\[
\nabla \cdot u = 0 . \quad (2.27b)
\]
For Newtonian fluids, the shear stress is proportional to the gradient of velocity. The stress tensor can be expressed as

$$\sigma = -pI + \mu[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]. \quad (2.28)$$

Thus, the Navier-Stokes equation can be further rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \nabla^2 \mathbf{u} + \frac{1}{\rho} \nabla P - \mathbf{f} = \mathbf{0}. \quad (2.29)$$
Chapter 3

New Analytical Solution to Vibration Problems via Fixed Horizon Peridynamics

3.1 Preliminary Remark

In this chapter, new analytical solutions to vibration of a bar and vibration of a beam via fixed horizon peridynamics are proposed. This chapter has two sections: the dynamic analysis of a bar, and the dynamic analysis of a beam. In the first section the nonlocal dispersive relation of the bar is derived and compared with the local dispersion relation. The homogeneous solution and the nonhomogeneous solution of the bar vibration equation are derived analytically by separation of variables. The free and forced vibrations of the bar are examined numerically based on the fixed horizon peridynamics. The second section focuses on the analytical solution of the nonlocal beam deformation via fixed horizon peridynamics. Inspired by the concept of peridynamics, a new nonlocal beam theory is first proposed in this section. It is shown that the proposed nonlocal beam theory is consistent with the local Euler-Bernoulli beam theory. Further, this new beam theory exhibits nonlocal behavior. An analytical solution to the static and the dynamic responses of the beam is also calculated by solving the new nonlocal beam equation with fixed horizon.

3.2 Vibration Analysis of Bar via Fixed Horizon Peridynamics

3.2.1 Peridynamic bar background

Silling, Zimmermann and et. al. (2003) considered the deformation of peridynamics
bar subjecting to static load; nonlocal dispersive relations were derived. Weckner and Abeyaratne (2005) proposed analytical expressions for the response of peridynamics bar subject to transient loading by using Fourier series representation. Mikata (2012) derived an analytical solution of infinite rod with three kinds of nonlocal constitutive relations. Recently, Bazant, Luo and et. al. (2016) derived wave dispersion relations for both bond based and state based peridynamics. Nonlocal responses from peridynamics were compared with responses from original 1984 nonlocal model.

The responses of a bar can be captured by a one-dimensional peridynamics governing equation stated as

$$
\rho \ddot{u} = \int_{x-\delta}^{x+\delta} C(|\xi - x|) (\bar{u} - u) d\xi + b(x) .
$$

(3.1)

Since the above equation is one-dimensional, all of the parameters in equation (3.1) can be regarded as scalar. The peridynamic constitutive coefficient $C$ is defined such that the deformation energy derived from peridynamics is consistent with the deformation energy derived from local elastic theory. For the one-dimensional case in this chapter, a finite length bar subjects to uniform deformation is assumed. The potential energy density at one point is derived by local elastic theory. That is

$$
W_{elas} = \frac{1}{2}EA S^2 ,
$$

(3.2)

where $E$ is the Young’s modulus, $A$ is the uniform cross section area of bar, and $S$ is the uniform stretch.
The micro potential energy describes the deformation energy of one peridynamics bond under uniform deformation, which is defined as

\[ w = \int_0^S f \cdot |\xi| \, ds = \frac{1}{2} C(|\xi|) S^2 |\xi|^2 , \]  

(3.3)

where \( \xi \) is bond length before deformation. Thus, the peridynamics potential energy density is

\[ W_{\text{per}} = \frac{1}{2} A \int_{-\delta}^{\delta} w \, d\xi = \frac{1}{4} A S^2 \int_{-\delta}^{\delta} |\xi|^2 C(|\xi|) \, d\xi . \]  

(3.4)

The condition of consistency in the potential energy density yields the local and nonlocal constitutive coefficients relation

\[ E = \frac{1}{2} \int_{-\delta}^{\delta} |\xi|^2 C(|\xi|) \, d\xi . \]  

(3.5)

### 3.2.2 Analytical solution of peridynamic bar

#### 3.2.2.1 Dispersion relation

In this section, a dispersive relation is derived based on the linearized peridynamics governing equation. Similarly to the local elastic theory, the characteristic equation of the linearized bond-based peridynamics governing equation is derived by considering the one harmonic wave component. That is, assume

\[ u(x, t) = e^{i(\kappa x - \omega t)} , \]  

(3.6)

where \( \kappa \) is the wave number, and \( \omega \) is the frequency. Substituting equation (3.6) to the governing equation (3.1) and neglecting the body force yields
\[ \rho \omega^2 = \int_{-\delta}^{\delta} C(\xi) (1 - e^{i\kappa \xi}) d\xi. \]  

(3.7)

The imaginary part of the equation cancels out because of the symmetry of the influence function \( C(\xi) \). Thus, the dispersive relation is rewritten as

\[ \rho \omega^2 = \int_{-\delta}^{\delta} C(\xi) (1 - \cos(\kappa \xi)) d\xi. \]  

(3.8a)

It can be proved that with the horizon \( \delta \) reducing to zero, the dispersive relation derived from the bond-based peridynamics reduces to the local elastic dispersive relation. Specifically, consider the sequence of equation:

\[ \rho \omega^2 = \lim_{\delta \to 0} \left\{ \int_{-\delta}^{\delta} C(|\xi|) (1 - \cos(\kappa \xi)) d\xi \right\} \]

\[ = \lim_{\delta \to 0} \left\{ \int_{-\delta}^{\delta} \xi^2 C(|\xi|) 2 \left( \frac{\sin(\kappa \xi / 2)}{\kappa \xi / 2} \right)^2 \kappa^2 \right\} \]

\[ = \frac{1}{2} \kappa^2 \lim_{\delta \to 0} \left\{ \int_{-\delta}^{\delta} \xi^2 C(|\xi|) \left( \frac{\sin(\kappa \xi / 2)}{\kappa \xi / 2} \right)^2 d\xi \right\} \]

\[ = \frac{1}{2} \kappa^2 \lim_{\delta \to 0} \left\{ \int_{-\delta}^{\delta} \xi^2 C(|\xi|) d\xi \right\} \lim_{\delta \to 0} \left( \frac{\sin(\kappa \xi / 2)}{\kappa \xi / 2} \right)^2 \]

\[ = E \kappa^2. \]  

(3.8b)

The preceding result shows that the nonlocal dispersive relation is determined not only by the nonlocal parameter \( \delta \) but also by the influence function \( C(\xi) \). To investigate the influence of the nonlocal parameter \( \delta \), the influence function is assumed to be a constant for simplification. That is,
\[ C = \frac{3E}{\delta^3}. \] (3.9)

The nonlocal dispersive relation in equations (3.8 a) can be expanded using Taylor’s series. Specifically,

\[
\frac{\rho}{E} \omega^2 = \kappa^2 - \frac{3!}{5!} \delta^2 \kappa^4 + \frac{3!}{7!} \delta^4 \kappa^6 - \frac{3!}{9!} \delta^6 \kappa^8 + \ldots \\
= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3!}{(2n+1)!} \delta^{2n-2} \kappa^{2n}. \] (3.10)

For a given constant influence function defined in equation (3.9), the peridynamic dispersive relation is nonlinear. The nonlinear relationship between \( \omega^2 \) and \( \kappa^2 \) is approximated as a polynomial up to infinite order. The number of higher order terms depends on the value of nonlocal parameters \( \delta \). For the extreme case where \( \delta = 0 \), the higher order terms are eliminated and the peridynamics dispersive function reduces to the local dispersive function.

3.2.2.2 Homogeneous solution

Assuming that the dynamic response of the peridynamic bar can be represented by a function of location \( x \) multiplied by a function of time \( t \). Then the displacement can be written in a separation of variables format as

\[ u(x,t) = Y(x)G(t). \] (3.11)

Substituting the displacement equation (3.11) into the linearized peridynamics governing equation (3.1) and neglecting the body force term yields
\[ \rho Y(x) \ddot{G}(t) = \int_{x} C(\xi - x)(Y(\xi) - Y(x)) dV_{\xi} \ G(t). \] 

Equation (3.12) can be rewritten as

\[ \frac{\ddot{G}(t)}{G(t)} = \frac{\int_{x} C(\xi - x)(Y(\xi) - Y(x)) dV_{\xi}}{\rho Y(x)}. \] 

Both sides of the equation (3.13) must be equal to a constant in order to satisfy the equivalent condition for arbitrary \(x\) and \(t\). Setting the constant equals to \(-\omega^2\), equation (3.13) can be separated to the two equations

\[ \ddot{G}(t) + \omega^2 G(t) = 0, \] 

(3.14 a)

and

\[ \int_{x} C(\xi - x)(Y(\xi) - Y(x)) dV_{\xi} + \rho \omega^2 Y(x) = 0. \] 

(3.14 b)

Equation (3.14 a) is a second order ordinary differential equation, which is solvable with given initial condition. That is,

\[ G(t) = C \sin(\omega t) + D \cos(\omega t), \] 

(3.15)

where \(C, D\) are the constant coefficients, which depend on the initial conditions.

Equation (3.14 b) is a second kind Fredholm integral equation, which requires further investigation. Unfortunately, there is no analytical solution for the general second kind Fredholm integral equation. Therefore, a special treatment is required. In the following part, a Taylor’s series expansion technique is used to represent the deformation function, and equation (3.14 b) is reduced to a higher order differential equation. These equations are derived based on the assumption that the
deformation function is sufficiently smooth. In detail, the deformation function must be differentiable. Mild deformation and small nonlocal parameter $\delta$ usually guarantee a better approximation.

Rewrite equation (3.14 b) in the form

$$
\left( \int_{-\delta}^{+\delta} C(\xi) d\xi - \rho \omega^2 \right) Y(x) = \int_{-\delta}^{+\delta} C(\xi) Y(x + \xi) d\xi .
$$

(3.16)

Approximate $Y(x + \xi)$ by Taylor’s expansion as

$$
Y(x + \xi) = Y(x) + Y'(x)\xi + \frac{1}{2} Y''(x)\xi^2 + \frac{1}{3!} Y^{(3)}(x)\xi^3 + \cdots + \frac{1}{n!} Y^{(n)}(x)\xi^n + \cdots.
$$

(3.17)

Substituting the Taylor expansion representation (3.17) into the mode characteristic equation (3.16) yields

$$
\sum_{n=1}^{\infty} \left\{ \frac{1}{(2n)!} \int_{-\delta}^{+\delta} \xi^{2n} C(\xi) d\xi \right\} \xi^{2n} Y^{(2n)}(x) + \rho \omega^2 Y(x) = 0 .
$$

(3.18)

The equation (3.18) is an ordinary differential equation up to infinite order; the homogeneous solution of this equation can be expressed as

$$
Y = a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots + a_{2n-1} y_{2n-1} + a_{2n} y_{2n} + \cdots ,
$$

(3.19)

where $y_1, y_2, \ldots, y_{2n}, \ldots$ are linearly independent and they form a set of fundamental solutions of the differential equation; $a_1, a_2, \ldots, a_{2n}, \ldots$ are coefficients which can be determined by proper boundary conditions. The fundamental solution of equation (3.18) is derived by solving the characteristic
equation
\[ \rho \omega^2 + \frac{1}{2!} \int_{-\delta}^{+\delta} \xi^2 C(\xi) d\xi \lambda^2 + \cdots + \frac{1}{(2n)!} \int_{-\delta}^{+\delta} \xi^{2n} C(\xi) d\xi \lambda^{2n} + \cdots = 0. \] (3.20)

Unlike the local elasticity theory where the mode shapes are determined by 2nd order ordinary differential equations, the mode shapes of the nonlocal peridynamics bar are determined by solving the differential equations up to an infinite order. The coefficients of higher order terms are determined by the nonlocal parameter \( \delta \), and are negligible when \( \delta \) reduces to zero. That is, larger nonlocal parameter requires higher order differential equation to determine the mode shape functions. If the nonlocal parameter \( \delta \) reduces to zero, the nonlocal mode characteristic equation is equivalent to the local mode characteristic equation. In the following examples, the mode differential equation is truncated up to the 2nd order term and up to the 4th order term.

The 2nd order truncated peridynamics mode characteristic equation is equivalent to the local elastic mode characteristic equation, and the 2nd order truncated nonlocal mode shape becomes
\[ Y(x) = A \sin(\kappa x) + B \cos(\kappa x). \] (3.21)

Thus, the solution of the 2nd order peridynamic vibrating bar can be expressed as
\[ u(x, t) = [A \sin(\kappa x) + B \cos(\kappa x)](C \sin(\omega t) + D \cos(\omega t)), \] (3.22)

where \( A, B, C, D \) are coefficients, and are determined subject to the boundary conditions and the initial conditions for specific problems.
The $4^{th}$ order truncated peridynamics mode characteristic equation yields

$$\frac{1}{24} \int_{-\delta}^{+\delta} \xi^4 C(\xi) d\xi Y^{"""}(x) + \frac{1}{2} \int_{-\delta}^{+\delta} \xi^2 C(\xi) d\xi Y^{"}(x) + \rho \omega^2 Y(x) = 0. \quad (3.23)$$

The corresponding characteristic equation of the differential equation (3.23) yields

$$\rho \omega^2 + \frac{1}{2} \int_{-\delta}^{+\delta} \xi^2 C(\xi) d\xi \lambda^2 + \frac{1}{4} \int_{-\delta}^{+\delta} \xi^4 C(\xi) d\xi \lambda^4 = 0. \quad (3.24)$$

Assume that the influence function is constant and that the conditions

$$C = \frac{3 E}{\delta^3} \quad (3.25)$$

and

$$\left| \frac{\rho \omega^2 \delta^2}{5E} \right| < 1 \quad (3.26)$$

are imposed. Then, the nonlocal mode shape function becomes

$$Y(x) = A_1 \sin(\kappa_1 x) + B_1 \cos(\kappa_1 x) + A_2 \sin(\kappa_2 x) + B_2 \cos(\kappa_2 x), \quad (3.27)$$

where

$$\kappa_1 = \sqrt{\frac{10}{\delta^2} \left( 1 - \sqrt{1 - \frac{\rho \omega^2 \delta^2}{5E}} \right)}$$

$$\kappa_2 = \sqrt{\frac{10}{\delta^2} \left( 1 + \sqrt{1 - \frac{\rho \omega^2 \delta^2}{5E}} \right)}.$$

Thus, the dynamic response of the peridynamic bar can be described as
where $A_1, B_1, A_2, B_2$ are coefficients, which are determined based on the boundary conditions. The coefficients $C, D$ are determined by the initial conditions. Unlike the local elasticity theory, which contains two coefficients in the mode shape function, the homogeneous solution of the 4th order differential equation involves four coefficients in the mode shape function. Thus, it requires additional constraints on the boundary.

Consider the particular example depicted in figure (3.1); the fixed-free boundary conditions are assumed.

Thus, the boundary conditions are

$$u(0, t) = 0,$$  \hspace{1cm} (3.29)

and

$$\left. \frac{\partial u(x, t)}{\partial x} \right|_{x=L} = 0.$$  \hspace{1cm} (3.30)
The initial conditions are specified as

$$u(x, 0) = u_0 \quad (3.31)$$

and

$$\dot{u}(x, 0) = v_0. \quad (3.32)$$

For the truncated 2nd order mode shape differential equation, the boundary conditions yield

$$B = 0 \quad (3.33)$$

$$\kappa = \frac{(2n - 1) \pi}{2L}, \quad (3.34)$$

and the corresponding solution becomes

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ \sin \left( \frac{(2n - 1) \pi}{2L} x \right) \left( C_n \sin(\omega_n t) + D_n \cos(\omega_n t) \right) \right\}. \quad (3.35)$$

For the truncated 4th order mode shape differential function, the boundary conditions yield

$$B_1 + B_2 = 0 \quad (3.36)$$

$$A_1 \kappa_1 \cos(\kappa_1 L) - B_1 \kappa_1 \sin(\kappa_1 L) + A_2 \kappa_2 \cos(\kappa_2 L) - B_2 \kappa_2 \sin(\kappa_2 L) = 0. \quad (3.37)$$

Note that the above boundary condition is not sufficient to define the mode shape of a nonlocal bar, therefore additional constraints must be added. In this 4th order case, two additional constraints on the boundary must be added to proceed. Further, the full nonlocal mode shape function is governed by a differential equation up to infinite order; thus, the number of the constraints on boundary is infinite. This points out the fact that the local boundary condition is defined on boundary. However, the nonlocal boundary condition must be defined over some region within
the boundary.

After deriving the mode function, the general form of the homogeneous solution becomes

\[ u(x, t) = \sum_{n=1}^{\infty} \{Y_n(x) \left( C_n \sin(\omega_n t) + D_n \cos(\omega_n t) \right) \}, \quad (3.38) \]

where \( C_n \) and \( D_n \) are derived from the initial conditions. That is,

\[ C_n = \frac{1}{\omega_n} \int_0^L Y_n(x) v_0(x) dx \quad (3.39) \]
\[ D_n = \int_0^L Y_n(x) u_0(x) dx. \quad (3.40) \]

3.2.2.3 Nonhomogeneous solution

As discussed above, the homogeneous solution can be written in the form

\[ u_h(x, t) = \sum_{n=1}^{\infty} \{Y_n(x) \left( C_n \sin(\omega_n t) + D_n \cos(\omega_n t) \right) \}. \quad (3.41) \]

For the special case where loading is exerted on a free end, the nonhomogeneous solution is equivalent to the homogenous solution with the boundary conditions given by the equation

\[ u(0, t) = 0 \quad (3.42) \]

and

\[ EA \frac{\partial u(x, t)}{\partial x} \bigg|_{x=L} = f(t). \quad (3.43) \]

For a distributed harmonic loading
\[ b(x, t) = \Re [Q(x)e^{i\Omega t}], \quad (3.44) \]

where \( Q(x) \) is the load distribution function. The nonhomogeneous solution can be represented as

\[
 u(x, t) = u_h(x, t) + u_p(x, t) \\
= \sum_{n=1}^{\infty} \{Y_n(x) (C_n \sin(\omega_n t) + D_n \cos(\omega_n t))\} + \Re [X(x)e^{i\Omega t}]. \quad (3.45)
\]

Substituting the equation (3.45) into the governing equation (3.1) leads

\[
 -\rho \Omega^2 X(x) + \int_{x-\delta}^{x+\delta} C(|\xi|)[X(x) - X(\xi)]d\xi = Q(x), \quad (3.46)
\]

where

\[
 LHS = -\rho \Omega^2 X(x) + \int_{x-\delta}^{x+\delta} C(|\xi|)X(x) d\xi - \int_{x-\delta}^{x+\delta} C(|\xi|)X(\xi) d\xi \\
= \left[ \int_{x-\delta}^{x+\delta} C(|\xi|) d\xi - \rho \Omega^2 \right] X(x) - \int_{x-\delta}^{x+\delta} C(|\xi|)X(x + \xi) d\xi. \quad (3.47)
\]

A Taylor’s expansion of \( LHS \) is truncated up to second order approximation, which yields

\[
 \int_{-\delta}^{+\delta} C(|\xi|)X(\xi + x) d\xi = \int_{-\delta}^{+\delta} C(|\xi|) \left[ X(x) + X'(x)\xi + \frac{1}{2} X''(x)\xi^2 \right] d\xi \\
= \int_{-\delta}^{+\delta} C(|\xi|) d\xi \ X(x) + \int_{-\delta}^{+\delta} \xi C(|\xi|) d\xi \ X'(x) + \frac{1}{2} \int_{-\delta}^{+\delta} \xi^2 C(|\xi|) d\xi \ X''(x) \\
= \int_{-\delta}^{+\delta} C(|\xi|) d\xi \ X(x) + \frac{1}{2} \int_{-\delta}^{+\delta} \xi^2 C(|\xi|) d\xi \ X''(x). 
\]
Substituting equation (3.48) into equation (3.46) yields

\[ \frac{1}{2} \int_{-\delta}^{+\delta} \xi^2 C(|\xi|) d\xi X''(x) + \rho \Omega^2 X(x) = -Q(x). \]  

(3.49)

Assume that

\[ X(x) \in \text{Span}\{Y_n(x)\}, \]  

(3.50)

and set

\[ X(x) = \sum_{n=1}^{\infty} \alpha_n Y_n(x). \]  

(3.51)

Next, substituting \( X(x) \) into equation (3.49) yields

\[ \sum_{n=1}^{\infty} \left\{ \frac{1}{2} \alpha_n \int_{-\delta}^{+\delta} \xi^2 C(|\xi|) d\xi Y_n''(x) + \rho \Omega^2 \alpha_n Y_n(x) \right\} = -Q(x) \]  

(3.52)

\[ \sum_{n=1}^{\infty} \alpha_n \rho (\omega_n^2 - \Omega^2) Y_n(x) = Q(x). \]  

(3.53)

Considering the orthogonality of \( Y_n(x) \), non-resonant case yields

\[ \alpha_n = \frac{1}{\rho (\omega_n^2 - \Omega^2)} \int_{0}^{L} Y_n(x)Q(x) \, dx. \]  

(3.54)

In the case \( \Omega = \omega_j \), the j-th resonance happens, which is characterized by a quite high response amplitude with the j-th mode. Here one can use the method of variation of parameters where the particular solution is assumed in the form
\[ u_p = \Re \left[ \left( \alpha_j(t)Y_j(x) + \sum_{n=1 \atop n \neq j}^{\infty} \alpha_n Y_n(x) \right) e^{i\Omega t} \right] \]. \hspace{1cm} (3.55)

Substituting this particular solution into the nonhomogeneous governing equation and repeating the preceding process yields

\[
Y_j(x)\ddot{\alpha}_j(t) + 2i\Omega Y_j(x)\dot{\alpha}_j(t) + \left[ \alpha_j(t)Y_j(x)\omega_j^2 + \sum_{n=1 \atop n \neq j}^{\infty} \alpha_n Y_n(x)\omega_n^2 \right] \\
- \left[ \alpha_j(t)Y_j(x) + \sum_{n=1 \atop n \neq j}^{\infty} \alpha_n Y_n(x) \right] \Omega^2 = \frac{1}{\rho} Q(x) .
\]

(3.56)

For the nonresonant mode, the coefficients are same as discussed above. For the j-th mode, multiplying \( Y_j(x) \) on both sides and integrating yields

\[
\ddot{\alpha}_j(t) + 2i \Omega \dot{\alpha}_j(t) = \frac{1}{\rho} \int_0^L Y_j(x)Q(x) \, dx .
\]

(3.57)

Solving this differential equation (assume zero initial conditions) gives

\[
\alpha_j = \frac{t}{2i\rho \Omega} \int_0^L Y_j(x)Q(x) \, dx .
\]

(3.58)

### 3.2.3 Numerical validation

In this section, the mesh free method (Silling 2005) is implemented for numerical verification with the following configuration.

\[ \text{Young's modulus} \hspace{1cm} E = 210\, \text{GPa} \]
Length of bar \( L = 100 \text{m} \)
Cross-section area \( A = 1 \text{ m}^2 \)
Constitutive coefficient \( C(\kappa) = \frac{2E}{\delta^2 |\xi|} \)

### 3.2.3.1 Dispersive relations

The natural frequencies of the bar corresponding to different modes are numerically calculated by choosing different nonlocal parameters. Figure (3.2a) is the analytical dispersion relation plotted based on equation (3.8). The nonlocal dispersion relation is significantly nonlinear for large horizon value. Further, the nonlocal dispersion relation reduces to the local dispersion relation when the horizon reduces to zero. Figure (3.2b) is the numerical dispersion relation, which has similar results to the analytical dispersion. The numerical dispersion confirms that peridynamics dispersive relationship converges to the local elastic dispersive relationship as the horizon reduces to zero.

![Analytical Dispersion Relationship](image1)
![Numerical Dispersion Relationship](image2)

**Fig 3.2. Analytical and numerical dispersive relations**

### 3.2.3.2 Free vibration

For the free vibration simulation, each mode is excited independently by imposing
the initial condition

\[ u_0 = \sin\left( \frac{(2n - 1) \pi}{2L} x \right), \quad n = 1, 2, 3, \ldots \]  

(3.59)

and

\[ v_0 = 0. \]  

(3.60)

By choosing \( n = 1 \), \( u_0 \) is the first mode shape function of the bar; therefore only the first mode vibration of the bar is excited subject to the above initial condition. Similarly by choosing \( n = 2, 3, 4 \), the corresponding second, third, fourth mode vibrations are excited. The responses of both local elastic and peridynamic bars are show in figures (3.3) to (3.5). Figure (3.3) shows the snapshots of responses by exciting the first mode. Figures (3.4) and (3.5) are the snapshots of the responses corresponding to the second and third modes. It is shown that by decreasing the horizon value numerically, the peridynamics responses converge to the local elastic responses.

First mode:

(a) \( \delta = 30m \)  

(b) \( \delta = 20m \)
Fig 3.3 Snapshots of first mode responses

Second mode:

Fig 3.4 Snapshots of second mode responses
Third mode:

\[(a) \quad \delta = 30m \] 
\[(b) \quad \delta = 20m \]  
\[(c) \quad \delta = 10m \] 
\[(d) \quad \delta = 5m \]  

Fig 3.5 Snapshots of third mode responses

3.2.3.3 Forced vibration

This section considers the bar responses due to the sinusoidal excitation with multiple excitation frequencies. Both resonant and nonresonant responses are investigated. Here the nonresonant excitation frequency is chosen as 950 rad/sec; the resonant excitation frequency is approximated at 407.65 rad/sec.

Figures (3.6) to (3.10) show the responses subject to the nonresonant excitations. It
can be observed that the propagations of the elastic and the peridynamic waves are similar when the horizon value is small. Differences appear when the wave propagation reaches the fixed boundary. This due to the fact that the mesh free method implemented in peridynamics does not fully fix the stiffness at the left boundary. Figures (3.9) and (3.10) show the responses with large horizons, the responses drift because of drastically dispersive nonlinearity.

Figures (3.11) to (3.15) exhibit the responses corresponding to resonant excitations. Although the responses increase due to the resonance phenomenon, the propagations of the elastic and peridynamic waves are still similar for the small horizon values. For a large horizon, dispersion nonlinearity is seen in figures (3.14) and (3.15).

Nonresonant excitation:

Fig 3.6. Elastic bar response due to nonresonant excitation
Fig 3.7. Peridynamic bar response due to nonresonant excitation with $\delta = 5m$

Fig 3.8. Snapshots of elastic and peridynamics nonresonant responses with $\delta = 5m$
Fig 3.9. Peridynamic bar response due to nonresonant excitation with $\delta = 10m$

Fig 3.10. Snapshots of elastic and peridynamic nonresonant responses with $\delta = 10m$
Resonant excitation:

Fig 3.11. Elastic bar response due to resonant excitation

Fig 3.12. Peridynamic bar response due to resonant excitation with $\delta = 5\text{m}$
Fig 3.13. Snapshots of elastic and peridynamics resonant responses with $\delta = 5m$

Fig 3.14. Peridynamic bar response due to resonant excitation with $\delta = 10m$
3.3 Vibration of Beam via Fixed Horizon Peridynamics

3.3.1 Beam theory background

The traditional Euler-Bernoulli beam theory provides a formulation for describing the loading and deformation characteristics of beams. The Euler-Bernoulli beam bending equation involves two assumptions. First, the deformation is small such that the linear elasticity theory holds. Second, the cross section remains plane and perpendicular to the neutral axis after deformation. The first assumption provides the simple stress-strain linear relationship, while the second assumption introduces a certain strain-deformation relationship. The Euler-Bernoulli beam theory has been widely used in engineering since the late 19th century.
The Timoshenko beam theory, as a generalization of the Euler-Bernoulli beam theory, takes into account the shear deformation and the rotational bending effects. The Timoshenko beam theory involves fewer assumptions than the Euler-Bernoulli beam. The cross section does not have to be perpendicular to the neutral axis after deformation. The Timoshenko beam theory is more suitable for the short beam or the sandwich composite beam, which the rotational bending effect is critical. The Timoshenko beam reduces to the Euler-Bernoulli beam if the shear deformation is neglected.

Both the Euler Bernoulli beam theory and the Timoshenko beam theory are derived based on the local continuum assumptions, where the stress at one point is determined by the deformation of points in contact. Kroner (1967) derived the internal energy of materials with long-range cohesive force, which built the prototype of the nonlocal theory. Eringen (1972, 1983) proposed the nonlocal constitutive relations which state that the stress at one point is determined not only by the deformation of points closely contacted but also by the deformation of points afar. The long-range forces are considered to propagate along fibers or lamina in composite materials. Also nonlocal behaviors are often observed for the small structures with scale of molecules or atoms. With the development of nanotechnology, the nonlocal beam theories are required to describe the behaviors of nano-scale beam. Recently, the beam theories based on the Eringen’s nonlocal mechanics were proposed to describe the deformations of the nano-scale beams by Challamel (2008), Wang (2007, 2008). Aydogdu (2009) developed a generalized nonlocal beam theory to analyze the static and dynamic responses of single-walled

Peridynamics is a nonlocal theory, which was first proposed by Stewart Silling in 2000. State based peridynamics was developed by (Silling, et. al., 2007). O’Grady and Foster (2014) reformulated the Euler-Bernoulli beam equation based on state based peridynamics. O’Grady-Foster’s model reduces to the Euler-Bernoulli beam theory when the horizon reduces to zero. However, the reformulated bending equation involves spatial derivative term and the deformation is restricted to be continuous. Moyer and Miraglia (2014) applied the bond based peridynamic concepts to represent the bending of the Timoshenko beam. The shear deformation is involved in their model, however the shear deformation term is expressed in the same form as the Timoshenko theory. Thus, the final bending equation is still local considering that a spatial differential term is involved in the bending equation.

Both the Euler-Bernoulli and the Timoshenko beam theories formulate the bending equation based on the continuum assumption and on a local constitutive law. Thus, everything must be continuous or even differentiable. The nonlocal beam theories, which are derived from the Eringen's theory and the nonlocal constitutive law, consider the long-range deformation effects. However, the bending equations of these nonlocal beam theories are formulated locally. Thus, spatial differential terms
are involved in their governing equations, which limits their applications to the continuous deformations in continuous domains. Peridynamics is a nonlocal mechanics theory, which can describe discontinuous deformation. Grady and Moyer proposed nonlocal beam theories based on peridynamics. However, none of them is completely nonlocal. Herein the Euler-Bernoulli beam equations are reformulated non-locally with no spatial differential terms involved. Discontinuous deformation, discontinuous material properties, and discontinuous domain are included in a compact equation. Numerical simulations are used to assess the analytical result. The influence of the nonlocal coefficient is also discussed.

3.3.2 Mathematical derivation of nonlocal beam

The Euler Bernoulli beam equations are formulated based on the continuum assumptions. That is

A. Small deformation,

B. Cross-section remains plane and perpendicular to neutral axis after deformation.

![Fig. 3.16. Euler-Bernoulli beam configuration](image)

These assumptions together with the linear elasticity theory yield the equations for
rotation angle $\phi$

$$\phi = \frac{d\bar{w}}{dx},$$  \hspace{1cm} (3.61)

for the strain $\varepsilon$:

$$\varepsilon = y \frac{d\phi}{dx} = y \frac{d^2\bar{w}}{dx^2},$$  \hspace{1cm} (3.62)

and for the bending moment $\bar{M}$:

$$\bar{M} = EI \frac{d^2\bar{w}}{dx^2},$$  \hspace{1cm} (3.63)

where $\bar{w}$ is the displacement of the neutral axis in the $y$ direction; $E$ is the Young's modulus; and $I$ is the second moment of area. Note that the above equations are true regardless of different loading and boundary conditions. The classical Euler-Bernoulli beam bending equation is expressed as

$$q(x) = \frac{d^2}{dx^2} \left( EI \frac{d^2\bar{w}}{dx^2} \right).$$  \hspace{1cm} (3.64)

Unlike the Timoshenko beam theory where the displacement $\bar{w}$ and the rotation angle $\phi$ are independent variables, the Euler Bernoulli beam theory contains only one independent variable $\bar{w}$, which is sufficient to describe the local deformation. Here a 'bar' on the top of a symbol denotes 'local' in order to differentiate from the following nonlocal bending parameters. Also, the reformulated Euler Bernoulli bending equation assumes only one independent variable $w$; all other deformation variables are expressed non-locally by the displacement $w$. The nonlocal beam equations are reformulated as:

for the rotation angle $\varphi$;

$$\varphi(x) = \int_{x-\delta}^{x+\delta} \alpha(\bar{x} - x) \frac{\bar{w} - w}{\bar{x} - \bar{x}} d\bar{x}$$  \hspace{1cm} (3.65)
and for the bending moment $M$;

$$M(x) = 2EI \int_{x-\delta}^{x+\delta} \beta(\hat{x} - x) \frac{\hat{w} - w}{(\hat{x} - x)^2} \, d\hat{x}. \quad (3.66)$$

The reformulated bending equation is expressed as

$$q(x) = 2 \int_{x-\delta}^{x+\delta} \gamma(\hat{x} - x) \frac{\hat{M} - M}{(\hat{x} - x)^2} \, d\hat{x}. \quad (3.67)$$

Set $\xi = \hat{x} - x$. Then $\alpha(\xi), \beta(\xi)$ and $\gamma(\xi)$ are nonlocal influence functions, which must satisfy the conditions

$$\alpha(\xi) > 0; \beta(\xi) > 0; \gamma(\xi) > 0, \quad (3.68 \text{ a})$$

$$\alpha(\xi) = \alpha(-\xi); \beta(\xi) = \beta(-\xi); \gamma(\xi) = \gamma(-\xi) \quad (3.68 \text{ b})$$

and

$$\int_{-\delta}^{\delta} \alpha(\xi) d\xi = 1; \int_{-\delta}^{\delta} \beta(\xi) d\xi = 1; \int_{-\delta}^{\delta} \gamma(\xi) d\xi = 1 \quad (3.68 \text{ c})$$

for $\forall \xi \in (-\delta, \delta)$. Note that $\alpha(\xi), \beta(\xi)$ and $\gamma(\xi)$ are delta functions when the horizon reduces to zero. It can be proved that the nonlocal beam equations reduce to the local Euler Bernoulli beam equations when horizon reduces to zeros.

Specifically, note that

$$\lim_{\delta \to 0} \varphi(x) = \lim_{\delta \to 0} \int_{-\delta}^{\delta} \alpha(\xi) \frac{\hat{w} - w}{\hat{x} - x} \, d\xi$$

$$= \int_{-0^-}^{0^+} \delta(\xi) \frac{\hat{w} - w}{\hat{x} - x} \, d\xi$$

$$= \int_{-0^-}^{0^+} \delta(\xi) \, d\xi \cdot \lim_{\hat{x} \to x} \frac{\hat{w} - w}{\hat{x} - x}$$

$$= \frac{dw}{dx}. \quad (3.69)$$
Further note that

\[ M(x) = 2EI \int_{-\delta}^{\delta} \beta(\xi) \frac{\hat{\omega} - w}{(\xi - x)^2} \, d\xi \]

\[ = 2EI \left[ \int_{-\delta}^{0} \beta(\xi) \frac{w(x + \xi) - w(x)}{\xi^2} \, d\xi + \int_{0}^{\delta} \beta(\xi) \frac{w(x + \xi) - w(x)}{\xi^2} \, d\xi \right] \]

\[ = 2E \left[ -\int_{-\delta}^{0} \beta(-\xi) \frac{w(x - (-\xi)) - w(x)}{(-\xi)^2} \, d(-\xi) + \int_{0}^{\delta} \beta(\xi) \frac{w(x + \xi) - w(x)}{\xi^2} \, d\xi \right] \]

\[ = 2E \int_{0}^{\delta} \beta(h) \frac{w(x - h) - w(x)}{h^2} \, dh + \int_{0}^{\delta} \frac{\beta(\xi) \frac{w(x + \xi) - w(x)}{\xi^2}}{} \, d\xi \]

\[ = 2E \int_{0}^{\delta} \beta(\xi) \frac{w(x + \xi) - w(x)}{\xi} \frac{w(x) - w(x - \xi)}{} \, d\xi. \quad (3.70 \text{a}) \]

Thus

\[ \lim_{\delta \to 0} M(x) = EI \int_{0}^{0^+} \beta(\xi) \, d\xi \cdot \lim_{\xi \to 0} \frac{w(x + \xi) - w(x)}{\xi} \frac{w(x) - w(x - \xi)}{\xi} \]

\[ = EI \frac{d^2w}{dx^2}. \quad (3.70 \text{b}) \]

Similarly, for the nonlocal bending equation

\[ q(x) = 2 \int_{-\delta}^{\delta} \gamma(\xi) \frac{\hat{M} - M}{(\xi - x)^2} \, d\xi \]

\[ = 2 \int_{0}^{\delta} \gamma(\xi) \frac{\frac{M(x + \xi) - M(x)}{\xi} - \frac{M(x) - M(x - \xi)}{\xi}}{} \, d\xi. \quad (3.71 \text{a}) \]

Thus,

\[ \lim_{\delta \to 0} q(x) = \int_{0}^{0^+} \gamma(\xi) \, d\xi \cdot \lim_{\xi \to 0^+} \frac{\frac{M(x + \xi) - M(x)}{\xi} - \frac{M(x) - M(x - \xi)}{\xi}}{} \]
\[
\frac{d^2 M}{dx^2}.
\] (3.71b)

Substituting the nonlocal moment equation into the nonlocal bending equation yields

\[
q(x) = 2 \int_{x-\delta}^{x+\delta} y(\hat{x} - x) \frac{2 E I \int_{x-\delta}^{x+\delta} \beta(\vert \hat{x} - \hat{\chi} \vert) \hat{w} - \hat{w}}{(\hat{x} - x)^2} d\hat{x}
\]

\[
-2 \int_{x-\delta}^{x+\delta} y(\vert \hat{x} - x \vert) \frac{2 E I \int_{x-\delta}^{x+\delta} \beta(\vert \hat{x} - x \vert) \hat{w} - w}{(\hat{x} - x)^2} d\hat{x}
\]

\[
= 4 \int_{x-\delta}^{x+\delta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \frac{E I y(\vert \hat{x} - x \vert) \beta(\vert \hat{x} - \hat{\chi} \vert) (\hat{w} - \hat{w})}{(\hat{x} - x)^2 (\hat{x} - \hat{\chi})^2} d\hat{x} d\hat{\chi}
\]

\[
-4 E I \int_{x-\delta}^{x+\delta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \frac{y(\vert \hat{x} - x \vert) \beta(\vert \hat{x} - x \vert) (\hat{w} - w)}{(\hat{x} - x)^2 (\hat{x} - \hat{\chi})^2} d\hat{x} d\hat{\chi}
\]

\[
= 4 \int_{S_1} \frac{E I y(\vert \hat{x} - x \vert) \beta(\vert \hat{x} - \hat{\chi} \vert) (\hat{w} - \hat{w})}{(\hat{x} - x)^2 (\hat{\chi} - \hat{x})^2} dS_1
\]

\[
-4 E I \int_{S_2} \frac{y(\vert \hat{x} - x \vert) \beta(\vert \hat{x} - x \vert) (\hat{w} - w)}{(\hat{x} - x)^2 (\hat{x} - \hat{\chi})^2} dS_2.
\] (3.72)

Thus, the PD bending equation is recast in the form

\[
q(x) = 4 \int_{S_1} \frac{E I y(\vert \hat{x} - x \vert) \beta(\vert \hat{x} - \hat{\chi} \vert) (\hat{w} - \hat{w})}{(\hat{x} - x)^2 (\hat{\chi} - \hat{x})^2} dS_1
\]

\[
-4 E I \int_{S_2} \frac{y(\vert \hat{x} - x \vert) \beta(\vert \hat{x} - x \vert) (\hat{w} - w)}{(\hat{x} - x)^2 (\hat{x} - \hat{\chi})^2} dS_2.
\] (3.73)
The nonlocal bending moment is determined by not only the deformation of the point $\hat{x}$ within the horizon of $x$; but also by the deformation of the point $\hat{x}$ within the horizon of $\hat{x}$. The peridynamics beam bending equation has a integral term over areas shown in figure (3.17). The first term of nonlocal bending equation integrates over the region $S1$, and the second term of nonlocal bending equation integrates over the region $S2$. The reformulated nonlocal bending equation does not contain spatial differential term. Thus, it is completely nonlocal. The discontinuity of displacement and the discontinuity of cross section can be directly accommodated in the nonlocal bending equation. Further, the deformation of a beam with initial cracks can also be accommodated into the nonlocal beam equations.

The preceding integrals can be calculated by the Gaussian quadrature rule for a reasonable numerical accuracy. The horizon is chosen as $\delta = d \cdot \Delta x$, and the discretized bending equation yields
The bending moment is given in equation (3.66), where the influence function follows certain conditions. The moment bending equation is mathematically a second kind Fredholm integral equation, which has no straightforward analytical solution. For the simplification purpose, the influence function is assumed to be constant over the beam, and the deformation of the beam is assumed small. This suggests that the beam deformation can be represented by a Taylor’s series expansion.

The Taylor’s series expansion of the beam deformation up to infinite order yields

$$
\tilde{w} = w(\hat{x}) = w(x) + w^{(1)}(x) \cdot (\hat{x} - x) + \frac{1}{2!} w^{(2)}(x) \cdot (\hat{x} - x)^2 + \frac{1}{3!} w^{(3)}(x) \cdot (\hat{x} - x)^3
$$

$$
+ \frac{1}{4!} w^{(4)}(x) \cdot (\hat{x} - x)^4 + \cdots + \frac{1}{n!} w^{(n)}(x) \cdot (\hat{x} - x)^n + \cdots.
$$

Further, the influence function is set to be constant. That is,

$$
\beta(\hat{x} - x) = \frac{1}{2\delta}
$$

and
\[ M(x) = \frac{EI}{\delta} \int_{x-\delta}^{x+\delta} w_x^{(1)}(\xi - x) + \frac{1}{2!} w_x^{(2)}(\xi - x)^2 + \frac{1}{3!} w_x^{(3)}(\xi - x)^3 + \frac{1}{4!} w_x^{(4)}(\xi - x)^4 + \cdots + \frac{1}{n!} w_x^{(n)}(\xi - x)^n - d\xi + \cdots \]

\[ = \frac{EI}{\delta} \int_{x-\delta}^{x+\delta} \left[ \frac{1}{2!} w_x^{(2)}(2\xi) + \frac{1}{4!} w_x^{(4)}(\frac{2}{3} \delta^3) + \cdots + \frac{1}{2m!} w_x^{(2m)}(\frac{2}{2m-1} \delta^{2m-1}) + \cdots \right] d\xi \]

\[ = \frac{EI}{\delta} \left[ \frac{2}{2!} \frac{2}{3} \delta^2 + \frac{2}{4!} \frac{2}{3} \delta^4 + \cdots + \frac{2}{2m!} \frac{2}{2m-1} \delta^{2m} + \cdots \right] \] (3.77)

The nonlocal beam bending equation can be reformulated as an ordinary differential equation. That is,

\[ M(x) = \sum_{n=1}^{\infty} a_n w_x^{(2n)} , \] (3.78)

where

\[ a_n = \frac{2 EI \delta^{2n-2}}{(2n)! (2n - 1)} . \]

The corresponding characteristic equation yields

\[ \sum_{n=1}^{\infty} a_n x^{2n} = 0 . \] (3.79)

Similarly, the force-moment equation (3.66) can be expressed by assuming constant influence function via a Taylor’s series expansion. That is,

\[ q(x) = \sum_{m=1}^{\infty} b_m M_x^{(2m)} , \] (3.80)

where

\[ b_m = \frac{2 \delta^{2m-2}}{(2m)! (2m - 1)} . \]
Combining equation (3.78) with equation (3.79), the vibration governing equation
of the nonlocal beam can be formulated by using Newton’s second law

\[ \rho A \ddot{w}(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n b_m w^{2(m+n)}(x) + q(x). \]  

(3.81)

Coefficients of equation (3.81) are listed in Table (3-1)

<table>
<thead>
<tr>
<th>( M(x) )</th>
<th>( q(x) )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>...</th>
<th>( b_m )</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_1 b_1 )</td>
<td>( a_1 b_2 )</td>
<td>...</td>
<td>( a_1 b_m )</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_2 b_1 )</td>
<td>( a_2 b_2 )</td>
<td>...</td>
<td>( a_2 b_m )</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( a_n )</td>
<td>( a_n b_1 )</td>
<td>( a_n b_2 )</td>
<td>...</td>
<td>( a_n b_m )</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 3-1 Coefficients of nonlocal beam vibration equation

The nonlocal beam vibration equation can be regarded as an infinite sum of all
elements in Table (3-1). The element located at (1,1) represents the local term in
the beam vibration equation. All other shaded elements involve nonlocal terms. The
sum of the first column elements represents the nonlocal displacement-moment
equation combined with the local force-moment equation. The sum of the first row
elements represent the local displacement-moment equation combined with the
nonlocal force-moment equation. The sum of the dark shaded elements represents
both the nonlocal displacement-moment equation and the nonlocal force-moment
equation.
The vibration governing equation is expanded by a separation of variables, yielding

\[ w(x, t) = G(t) Y(x) , \]  
\[ \rho A \ddot{G}(t) Y(x) = G(t) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n b_m Y_x^{2(m+n)} . \]  
\[ \text{(3.82)} \]
\[ \text{(3.83)} \]

Equation (3.83) can be rewritten as

\[ \frac{\ddot{G}(t)}{G(t)} = \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n b_m Y_x^{2(m+n)}}{\rho A Y(x)} = -\omega^2 \]  
\[ \text{(3.84)} \]

This equation can be separated into two ordinary differential equations as

\[ \ddot{G}(t) + \omega^2 G(t) = 0 \]  
\[ \text{(3.85 a)} \]

and

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n b_m Y_x^{2(m+n)} + \rho A \omega^2 Y(x) = 0 . \]  
\[ \text{(3.85 b)} \]

Equation (3.85 a) is a second order ordinary differential equation, which is solvable by specifying initial condition. The mode characteristic equation (3.85 b) is an ordinary differential equation up to an infinite order. By truncating the higher order terms, equation (3.85 b) is reduced to the local mode characteristic equation. By choosing the specific terms in Table (3-1), the mode characteristic equation exhibits different nonlocal behaviors. Equation (3.85 b) can be solved by specifying boundary conditions. However, the nonlocal boundary condition requires more constraints and is different from the local boundary condition. The nonlocal boundary condition will be discussed in the following examples.

Next, the analytical solutions of nonlocal beams subject to distributed loading are derived in specific examples. The bending moment equation and the vibration governing equation are truncated up to a finite order.
Example 1: Simple supported beam

Assume that the deformation can be accurately represented by an 8th order polynomial. Then the nonlocal bending equation is reduced to the 4th order differential equation

\[ w_{x}^{(2)} + \frac{\delta^2}{36} w_{x}^{(4)} = \frac{M(x)}{EI}. \] (3.86)

The complimentary solution is

\[ w_c(x) = C_1 + C_2 x + C_3 \cos \left( \frac{6}{\delta} x \right) + C_4 \sin \left( \frac{6}{\delta} x \right). \] (3.87)

Assume uniformly distributed loading

\[ q(x) = Q, \] (3.88)

then bending moment yields

\[ M(x) = \frac{QL}{2} x - \frac{Q}{2} x^2. \] (3.89)

The general solution can be expressed as

\[ w(x) = w_c(x) + w_p(x) = C_1 + C_2 x + C_3 \cos \left( \frac{6}{\delta} x \right) + C_4 \sin \left( \frac{6}{\delta} x \right) + \frac{Q \delta^2}{72 EI} x^2 + \frac{QL}{12 EI} x^3 - \frac{Q}{24 EI} x^4. \] (3.90)
Impose the boundary conditions

\[ x = 0: \]
\[
\begin{align*}
    w(x)|_{x=0} &= 0 & (3.91\ a) \\
    \frac{\partial^2 w(x)}{\partial x^2} \bigg|_{x=0} &= 0; & (3.91\ b)
\end{align*}
\]

and \( x = L \):

\[ x = L: \]
\[
\begin{align*}
    w(x)|_{x=L} &= 0 & (3.91\ c) \\
    \frac{\partial^2 w(x)}{\partial x^2} \bigg|_{x=L} &= 0. & (3.91\ d)
\end{align*}
\]

Using these boundary conditions yields

\[
\begin{align*}
    w(x) &= C_1 + C_2 x + C_3 \cos\left(\frac{6}{\delta} x\right) + C_4 \sin\left(\frac{6}{\delta} x\right) \\
    &\quad + \frac{Q \delta^2}{72 EL} x^2 + \frac{QL}{12 EL} x^3 - \frac{Q}{24 EL} x^4,
\end{align*}
\]

(3.92)

where

\[
\begin{align*}
    C_1 &= -\frac{Q \delta^4}{1296 EL} & (3.93\ a) \\
    C_2 &= -\frac{QL}{72 EL} \left(\delta^2 + 3 L^2\right) & (3.93\ b) \\
    C_3 &= \frac{Q \delta^4}{1296 EL} & (3.93\ c) \\
    C_4 &= \frac{Q \delta^4}{1296 EL} \left[ \csc\left(\frac{6 L}{\delta}\right) - \cot\left(\frac{6 L}{\delta}\right) \right]. & (3.93\ d)
\end{align*}
\]

The influence of the nonlocal parameter on the deformation of the simple supported beam is captured in figure (3.19).
Fig. 3.19. Nonlocal simple supported beam deformation

**Example 2: Cantilever beam**

Considering the cantilever beam subjects to a distributed loading shown in figure (3.20).

By assuming a uniformly distributed loading $P$, the bending moment is expressed as

$$ M(x) = \frac{P}{2} (x^2 - 2Lx + L^2). \quad (3.94) $$

The general solution is

$$ w(x) = C_1 + C_2 x + C_3 \cos\left(\frac{6}{\delta}x\right) + C_4 \sin\left(\frac{6}{\delta}x\right) $$
\[ + \left( \frac{P L^2}{4 EI} - \frac{P \delta^2}{72 EI} \right) x^2 + \frac{P L}{6 EI} x^3 + \frac{P}{24 EI} x^4. \] (3.95)

The nonlocal boundary conditions are expressed as

\( x = 0: \)

\[ w(x) \bigg|_{x=0} = 0, \] (3.96 a)

\[ \frac{\partial w(x)}{\partial x} \bigg|_{x=0} = 0, \] (3.96 b)

\[ \frac{\partial^2 w(x)}{\partial x^2} \bigg|_{x=0} = \frac{P L^2}{2 EI}, \] (3.96 c)

and

\[ \frac{\partial^3 w(x)}{\partial x^3} \bigg|_{x=0} = -\frac{P L}{EI}. \] (3.96 d)

In this example, the nonlocal bending equation is approximated by a 4th order differential equation, which requires four boundary conditions to determine the nonlocal deformation of the beam. This is different from the local bending equation, which requires only two boundary conditions. Additional two boundary conditions are provided by equalizing the nonlocal moment and the shear with the local moment and shear. And these additional boundary conditions are usually not unique. As a result, the nonlocal beam deformation is quite sensitive to the nonlocal boundary condition. In next section, the influence of the nonlocal boundary conditions is discussed with numerical examples.

For the nonlocal bending equation with higher order approximation, more constrains on the boundary are required. For the most general nonlocal bending equation, which requires an infinite order of differential terms, the nonlocal
boundary conditions must be defined over some region.

Considering the preceding nonlocal boundary conditions, the general solution is cast as

\[ w(x) = \frac{P L^2}{4 E I} x^2 - \frac{P L}{6 E I} x^3 + \frac{P}{24 E I} x^4 - \frac{P \delta^2}{72 E I} x^2 + \frac{P \delta^4}{1296 E I} \left[ 1 - \cos \left( \frac{6x}{\delta} \right) \right]. \]

(3.97)

The influence of the nonlocal parameter on the deformation of the cantilever beam is captured in figure (3.21).

![Analytical solutions of Cantilever beam](image)

**Fig. 3.21.** Nonlocal cantilever beam deformation

### 3.3.4 Numerical examples

In this section, an example of a double clamped cantilever beam is treated numerically to assess the validity of the nonlocal bending equation. The nonlocal
effects are analyzed and compared with the Euler Bernoulli beam results. The nonlocal boundary condition is represented by introducing a pseudo-layer outside the physical domain.

**Example: Double clamped cantilever beam**

The clamped-clamped cantilever beam subject to a distributed loading is shown in figure (3.22).

![Clamped-clamped cantilever beam](image)

Fig. 3.22. Clamped-clamped cantilever beam

The boundary conditions are

\[ x = 0 \]

\[ w(0) = 0 \] \hspace{1cm} (3.98 \text{ a})

\[ \varphi(0) = 0 ; \] \hspace{1cm} (3.98 \text{ b})

and \( x = L \)

\[ w(L) = 0 \] \hspace{1cm} (3.98 \text{ c})

\[ \varphi(L) = 0 . \] \hspace{1cm} (3.98 \text{ d})

The clamped-clamped beam is meshed as shown in figure (3.23). The circle nodes represent the physical domain, which are numbered from 0 to \( n \). The star nodes
are pseudo-nodes, which are added outside of the boundary to represent boundary condition. Most of the peridynamics applications avoid the boundary condition problems as long as relatively moderate results are acquired. In some cases however, the boundary conditions are critical and special care must be taken regarding the treatment of the boundary. In this example, the boundary conditions are first interpreted non-locally as

\[ x = 0 \]

\[ w_0 = 0 \]  \hspace{1cm} (3.99 \ a) \]

\[ \sum_{k=-d \atop k \neq 0}^{d} \alpha(x_k, x_0) \frac{w_k - w_0}{x_k - x_0} \Delta x_k = 0 ; \]  \hspace{1cm} (3.99 \ b) \]

and \[ x = L \]

\[ w_n = 0 \]  \hspace{1cm} (3.99 \ c) \]

\[ \sum_{k=n-d \atop k \neq n}^{n+d} \alpha(x_k, x_n) \frac{w_k - w_n}{x_k - x_n} \Delta x_k = 0 . \]  \hspace{1cm} (3.99 \ d) \]

Fig. 3.23. Double clamped cantilever beam meshing

The deformations of the pseudo-nodes must be determined based on the given boundary condition. However, the solution with these boundary condition
equations is unique if and only if $d = 1$. Thus, the nonlocal deformations of the pseudo-nodes are indeterminate based on the given boundary conditions. Thus, additional constraints must be added which yields

\begin{align*}
x &= 0 \\
w_0 &= 0 \quad (3.100\ a) \\
w_{-k} &= w_k ; \quad (3.100\ b)
\end{align*}

and $x = L$

\begin{align*}
w_n &= 0 \quad (3.100\ c) \\
w_{n-k} &= w_{n+k} . \quad (3.100\ d)
\end{align*}

With these enriched boundary conditions, the static deformations of the nonlocal beam subject to a uniformly distributed load are shown in figure (3.24). The peridynamics beam equation reduces to the Euler Bernoulli beam equation when the horizon reduces to zero. The discretized peridynamics beam equation is equivalent to the discretized Euler Bernoulli beam equation given the horizon equals to mesh size $\delta = \Delta x$. As shown in figure (3.24), the nonlocal deformation matches the Euler Bernoulli bending deformation when the horizon equals $\Delta x$. Greater horizon leads to larger deformation, which is different from the local Euler Bernoulli beam deformation.
There are two reasons for the difference between local deformation and nonlocal deformation. The first reason is the non-locality of the problem itself; the larger horizon value yields a larger deformation, which indicates a softer beam. The other reason relates to the inaccurate approximation of the nonlocal boundary conditions. The deformation differences due to nonlocal effect can not be eliminated with the nonlocal parameter greater than the mesh size. However, with better approximation of the boundary condition or special treatment on boundary, the deformation differences due to the boundary condition inaccuracy can be drastically reduced.

3.4 Synopsis

This chapter has provided a novel analytical approach solving dynamic problems via the fixed horizon peridynamics: the vibration analysis of a bar and of a beam.
It has been shown that the peridynamics dispersive relation is nonlinear. The nonlinearity of the dispersive relation depends on the value of nonlocal parameter. If the nonlocal parameter reduces to zero, the peridynamics dispersive relation reduces to the linear local elastic dispersive relation. This dispersive consistency has been verified by the numerical modeling. Both homogeneous and nonhomogeneous solutions of the peridynamic bar have been analytically derived. Mode shapes have been calculated by solving a high order differential equation. The order of the differential equation depends on the nonlocal parameter. This has also been verified numerically.

A new nonlocal beam theory has been proposed based on peridynamics. This derived nonlocal bending equation is a generalization of the Euler Bernoulli beam equation. It reduces to the Euler Bernoulli beam equation if the horizon reduces to zero. It has been shown numerically that this nonlocal beam equation has the same responses as the Euler Bernoulli beam equation when the horizon value equals the mesh size. An analytical solution to the nonlocal beam equation has been determined using Taylor’s series expansion. The proposed beam theory is completely nonlocal.

This chapter has derived analytical solution to the vibration problem via fixed horizon peridynamics. This fixed horizon approximation has exhibited the boundary condition related difficulties. For this reason, a variable horizon approach will be adopted in the following chapter. The advantages of the proposed nonlocal beam theory over the local beam theories will be discussed in more detail.
Chapter 4
The Variable Horizon Peridynamics Approach – Static Problems

4.1 Problem Description

As discussed in the previous chapter, peridynamics can accurately describe the deformation of materials located within an inner physical domain. However for materials located on the boundary, peridynamics fails to accurately calculate their responses. This is due to the fact that the stiffness on the boundary region is reduced since there is no enough material ‘fills in’ its horizon. A traditional way of representing peridynamics boundary conditions is by adding a pseudo-layer outside physical domain. However, this requires the deformation of the pseudo-layer be pre-determined and the accuracy of the response on physical domain usually depends on the accuracy of pre-determined deformation on the pseudo-layer. In real applications, researchers usually choose to avoid calculating the responses of materials on the boundary, and instead focus on the responses of materials within the inner domain.

In applications, the strategy of dealing with peridynamics boundary condition is to numerically adjust the materials stiffness on the boundary. For the displacement boundary condition, one usually adds a pseudo-layer with thickness $\delta$ outside the physical boundary and specifies their displacement. For the loading boundary conditions, a force is exerted on the pseudo-layer outside of the loading boundary with thickness $\delta$ and the force exerted on boundary is then replaced by force
density. The accuracy of the material stiffness on the boundary depends on how reliable the displacement of pseudo-layer is pre-determined. Note that the displacements on pseudo-nodes are usually hard to predict, especially when the deformation on the boundary region is complex. One way to predict the pseudo-domain response is through applying interpolation method. The interpolation method is a predictor-corrector process and usually requires iterations. The iterative interpolation method is detailed in this section.

Another method to avoid the peridynamics boundary problem is by coupling the local elasticity with the peridynamics. Specifically, the response on the boundary regions is described by the classical elasticity, while the response within interior region is described by the peridynamics. However, this introduces spurious effects or ‘ghost force’ issues on the local-nonlocal transition region. The coupling between the local and the classical nonlocal theory (Kroner, 1967, Eringen, 1972) has been investigated since 1980s. The coupling between peridynamics and local elasticity has attracted considerable attention in recent years. There are primarily four kinds of coupling approaches: the variable horizon method proposed by Silling, Littlewood and Seleson (2015); the force based coupling method proposed by Seleson, Beneddine and Prudhomme (2013), Seleson, Ha and Beneddine (2015); the Arlequin coupling method proposed by Han and Lubineau (2012); the morphing method proposed by Lubineau, Azdoud, and et. al., (2012), Han, Lubineau and et. al., (2016). The variable horizon approach couples the local and the nonlocal equation by reducing the nonlocal parameter in the local-nonlocal mixed domain. However, spurious effects or ghost force appear in the complex displacement field or in the abrupt horizon-changing region. The force based coupling method blends the local
and the nonlocal models by introducing the nonlocal weight function. By assuming that the displacement in the blending region can be accurately represented by a second order Taylor's series expansion, the force based coupling method satisfies Newton's third law and passes the patch test. However, a complex displacement field often requires a higher order Taylor’s series approximation and spurious effect or ghost force problem still appear in the blending region. The Arlequin coupling method reformulates the coupling equation from an energy perspective by employing partition of unity. The morphing method introduces a single unified model, which encompasses both the local and the nonlocal continuum representations. The energy equivalent condition is satisfied. The morphing model can be purely local, purely nonlocal, or hybrid depending on the constitutive parameters. However, ghost force still exists in the morphing model.

Recently a concept called ‘dual-horizon peridynamics’ was proposed by Ren, Zhuang and et. al., (2015). In the dual horizon peridynamics, the horizon value is regarded as a predefined parameter. The peridynamics governing equation with arbitrary horizon is reformulated. The dual-horizon peridynamics theory naturally involves the variable horizon, and the reformulated governing equation satisfies both the balances of linear momentum and the balance of angular momentum.

As discussed in previous chapters, peridynamics reduces to the classical continuum elasticity when the horizon reduces to zero. The variable horizon peridynamics approach is a standard tool to assign horizon values on the boundary region. Specifically, for the nodes located on the physical boundary, the horizon is zero. For the nodes located within a certain distance from the boundary, the horizon value is
determined by the distant from the node to the physical boundary. With a variable horizon defined on the boundary region, no pseudo-layer or iterative interpolation is required. Boundary nodes exhibit the local behavior, at the same time, the nonlocal responses are maintained in the interior region. The stiffness of the boundary region is better simulated by the variable horizon approach. Thus, the accuracy of the responses on the boundary region is drastically improved. This is critical in crack modeling especially when a crack initiates from the boundary.

In this chapter, the iterative interpolation method is introduced. The effectiveness of the iterative interpolation method is verified with numerical examples. The variable horizon approach is introduced. The accuracy of the proposed variable horizon method is assessed. The static responses of a bar and of a beam are solved by applying the variable horizon method. Finally, dual horizon peridynamics is introduced. The proposed variable horizon approach is applied to the dual horizon peridynamics modeling.

4.2 Iterative Interpolation Method

4.2.1 Method description

For the constant horizon peridynamics, the stiffness on the boundary region is much softer because materials are not ‘filled up’ in its horizon. The boundary stiffness is enriched with additional pseudo-layer outside the physical boundary. However, the responses of the pseudo-layer must be predetermined and the accuracy of the predefined deformation is critical to the accuracy of materials deformation. For the displacement boundary condition, the pseudo nodes outside
the displacement boundary are often fixed with a predefined zero displacement in order to improve the accuracy of stiffness on physical boundary. This usually provides a moderate improvement when responses on the boundary region are mild. However, the accuracy of responses on the boundary region is drastically reduced when the responses are complex.

This boundary condition strategy approximates the displacement of the pseudo-layer by the interpolation method based on the responses in the physical domain. The calculated pseudo-layer response works as a nonlocal boundary condition when calculating the responses of the physical domain. This is an iterative process. As shown in figure (4.1), the displacement of pseudo-nodes is first assigned with an initial guess. Then the displacement of the physical domain is calculated based on the pre-defined displacement on the boundary region. Further, the displacements of pseudo-nodes are updated by an interpolation based on the calculated displacement on the physical domain. Again, the physical domain responses are updated based on the updated nonlocal boundary responses. The entire process is iterative until a certain accuracy criterion is satisfied.
Fig. 4.1. Typical peridynamic boundary condition implementation process
4.2.2 Numerical verification

A one-dimensional static peridynamic bar is numerically modeled by using the iterative interpolation method. This peridynamic bar is fixed on one side and loaded on the other side. Materials properties and loads are defined in Table (4-1).

<table>
<thead>
<tr>
<th>Young's Modulus $E$ /GPa</th>
<th>Poisson Ratio</th>
<th>Density $\text{kg} \cdot \text{m}^{-3}$</th>
<th>Horizon $\delta$ /m</th>
<th>Mesh Size /m</th>
<th>Cross-section area $A$ /m$^2$</th>
<th>Length $L$ /m</th>
<th>Loading $F$ /N</th>
</tr>
</thead>
<tbody>
<tr>
<td>210</td>
<td>1/4</td>
<td>7900</td>
<td>3</td>
<td>0.05</td>
<td>1</td>
<td>10</td>
<td>$10^8$</td>
</tr>
</tbody>
</table>

Table 4-1 Materials parameter of peridynamic rod

For the fixed horizon peridynamics, pseudo-nodes are assigned outside the physical boundary. A typical meshing with constant horizon value is shown in figure (4.2). The star nodes are the fixed boundary nodes with specified displacements; the square nodes are the physical nodes with unknown displacement; the circle nodes are the loading boundary nodes with specified loading but with unknown displacement. Figures (4.3a) and (4.3b) pertain to the nonlocal displacement and to the error of the bar calculated by specifying the displacement of the pseudo-nodes. As one expected, the error on the boundary region is large. This is due to the bad approximation of the stiffness on the boundary region with the fixed pseudo-nodes displacements. That is, the materials stiffness at the left boundary is drastically reduced, which results in the large deformation at the left boundary region. The error within the interior region does not accumulate indicating that the nonlocal strain matches the local strain. Thus, the nonlocal materials stiffness within the interior region is successfully maintained.
Fig. 4.2. Constant horizon peridynamics meshing
The iterative interpolation approach is implemented to improve the accuracy of the response on the boundary region. Figure (4.4) shows the errors of the nonlocal response by adaptively updating the pseudo-node displacement. The accuracy of the responses calculated from the iterative pseudo-nodes interpolation approach has been drastically improved compared to figure (4.3). With the increasing number of iterations, the displacement error decreases. However, there is no evidence to show that with the iteration number approaches infinity, the error is eliminated. In any case, the error ‘converges’ quite slowly and the convergence rate is influenced by the interpolation function. The interpolation function should be chosen such that the predicted displacement of the pseudo-nodes satisfies the peridynamics governing equation and increases the convergence rate. However, the exact responses cannot be reached without eliminating the ‘ghost force’ problem.
Fig. 4.4. Nonlocal response errors with adaptive pseudo-nodes displacement

4.3 Variable Horizon Approach

The direct variable horizon approach modifies the horizon value on the boundary region without adding pseudo-nodes outside the physical boundary. Unlike the iterative interpolation method, the variable horizon method does not require iteration process. Thus, it is computationally efficient. The accuracy of the responses on the boundary region is drastically improved with the variable horizon approach. However, neither the ghost force nor the spurious effect can be eliminated.

4.3.1 Static deformation of nonlocal bar via variable horizon method

4.3.1.1 Variable horizon method derivation

Instead of specifying the displacement of the pseudo domain outsides the boundary, here the horizon of the nodes near boundary are specified as

\[ \delta_{\text{boundary}} = D, \]  

(4.1)
where \( D \) is the distance between the node and the boundary.

Figure (4.5) shows the horizon of nodes on the boundary. Based on the aforementioned horizon specification, the node near the boundary has a variable horizon depending on its location. When the node reaches the boundary, its horizon value reduces to zero. In this case, the peridynamics is reduced to the local mechanics.

For the one-dimensional example, the horizon configuration in the boundary region is shown in figure (4.6).
This figure shows the horizon of the nodes near the fixed boundary. Specifically, \( \delta \) is the horizon of the inner nodes, \( \delta_1 \) is the horizon of \( x_1 \) node, and \( \delta_2 \) is the horizon of \( x_2 \) node. The closer node to the boundary is, the smaller the horizon becomes.

![Diagram showing horizon distribution]

Fig. 4.7. Peridynamics bar configuration with variable horizon

For a horizon distribution shown in figure (4.7), the peridynamics force exerted on the left side of \( x_0 \) by all nodes on the right sides of \( x_0 \) can be expressed as

\[
F(x_0) = \int_{x_0/2}^{x_0} \int_{x_0}^{2\hat{x}} f(\hat{x}, \tilde{x}) d\tilde{x} d\hat{x}, \tag{4.2}
\]

where

- \( f \) is the peridynamics bond force between \( \hat{x} \) and \( \tilde{x} \),
- \( \hat{x} \) is node between \( x_0/2 \) and \( x_0 \) with corresponding horizon \( \delta_{\hat{x}} = \hat{x} \),
- \( \tilde{x} \) is the node within horizon of \( \hat{x} \) located at the right side of \( x_0 \).

For the fixed-loading boundary conditions shown in figure (4.7), the static uniform
deformation yields

\[ U(x) = \frac{F}{EA} x. \] (4.3)

The one dimensional peridynamics coefficient is defined as

\[ c = \frac{kE}{\delta^2}, \] (4.4)

where \( k = 2 \), to satisfy the deformation energy consistency condition.

The peridynamics internal force exerts on the right boundary is derived as

\[
F(x_0) = \int_{x_0}^{x_0 + 2\delta} \int_{x_0}^{x_0 + 2\delta} c \frac{u(\hat{x}) - u(\hat{\chi})}{|\hat{x} - \hat{\chi}|} d\hat{x} d\hat{\chi}
\]

\[
= \int_{x_0}^{x_0 + 2\delta} \int_{x_0}^{x_0 + 2\delta} \frac{kE F(\hat{x} - \hat{\chi})}{\delta^2} d\hat{x} d\hat{\chi}
\]

\[
= k F \int_{x_0}^{x_0 + 2\delta} \frac{1}{\delta^2} d\hat{x} d\hat{\chi}
\]

\[
= k F \int_{x_0}^{x_0 + 2\delta} \frac{2 - x_0}{\delta^2} d\hat{x}
\]

\[
= (2\log(2) - 1) k F.
\] (4.5)

It is shown that the peridynamics force acting on the right side of any cross-section \( x_0 \) is a constant regardless the location of the cross section. For the extreme case \( x_0 \to 0 \), the peridynamics force exerts only on the boundary node. The nonlocal force equals to the external load (patch test approved) when \( k = 2.5887 \). Note that this constitutive parameter \( k \) may be varying or location-dependent depending on the loading condition and the specific horizon distribution. In the next section, a numerical analysis shows the increase of the response accuracy with the variable
horizon approach. Also the influence of the constitutive parameter on the deformation accuracy is examined.

4.3.1.2 Numerical verification

This part compares the deformation calculated from the ordinary boundary condition technique with the deformation derived from the aforementioned variable horizon technique. The uniform deformation is assumed for simplification. Materials properties are shown in Table (4-1).

The variable horizon peridynamic bar is meshed as shown in figure (4.8). The circle nodes are the boundary region nodes with the reduced horizon value; the square nodes are the interior nodes with the full horizon value. Figure (4.9) shows that the corresponding displacement and error field.

This numerical simulation of the one-dimensional bar verified that the variable horizon method drastically improved the accuracy of the peridynamics one-dimensional problem. No sharp error increase near the boundary indicates the accurate representation of the stiffness at the boundary nodes. However, the slope of the error at the interior region is nonzero, this means that the interior strain does not equal the local elastic strain, and this indicates the existence of the ghost forces.

The accuracy of the nonlocal response is further improved by modifying the constitutive coefficient at the horizon-reducing region as shown in figure (4.10). Although the ghost force still exists with the modified constitutive coefficient, the proposed variable horizon approach can drastically improve the accuracy of the
materials responses especially on the boundary region.

Fig. 4.8. Variable horizon peridynamic bar meshing
Fig. 4.9. Nonlocal responses of bar with variable horizon
Fig. 4.10. Nonlocal responses of bar with modified variable horizon

4.3.2 Static deformation of nonlocal beam via variable horizon method

The nonlocal beam theory proposed in previous chapter involves no spatial derivative terms. Thus, it can treat discontinuous problems, while the local beam theories can not. The proposed nonlocal beam theory can determine the deformations of the initially cracked beams. Also, the deformation of the beam with multiple cross-sections can be solved in a consistent manner. In this part, several examples are solved based on the proposed nonlocal beam theory. Nonlocal phenomenon is observed in these examples. The advantages of the proposed nonlocal beam theory over the local beam theory are analyzed.

As discussed in chapter two, a pseudo-layer must be added outside the physical
boundary to maintain the boundary stiffness of the clamped-clamped beam as shown in figure (3.23). This partly enriches the stiffness on the boundary region. However, the error is still large especially in the boundary region. In this chapter, the beam deformation is solved via the variable horizon approach. No additional pseudo-layer is required outside the physical boundary. The accuracy of the beam deformation is drastically improved by the variable horizon approach.

4.3.2.1 Static deformation of clamped-clamped beam
Consider the clamped-clamped beam configuration shown in figure (3.22), without adding the pseudo-layer outside the physical boundary, the horizon value is defined over the physical domain as shown in figure (4.11). The deformation of the clamped-clamped beam subject to a distributed loading is solved via the variable horizon technique as shown in figure (4.12). Compared with the same example solved by the fixed horizon technique in figure (3.24), the variable horizon approach provides a much more accurate result.

![Diagram showing variable horizon approach](image)

Fig. 4.11. Variable horizon configuration of clamped-clamped beam
The deformations of the clamped-clamped beam subject to a concentrated load are shown in figure (4.13). The concentrated load is exerted on different locations. With the variable horizon defined on the boundary, the nonlocal deformations on the
boundary region are much closer to the local deformations. The nonlocal behaviors are maintained for the deformation on the interior region. This is due to the fact that the variable horizon approach reduces the horizon values on the boundary, while the full horizon values within the interior region are maintained.

Fig. 4.13. Deformation of clamped-clamped beam subject to point loading
The nonlocal beam equations are reformulated based on the peridynamics formalism. Thus, it can solve the problems with pre-existing cracks. Figure (4.14) shows the deformation of a beam subjected to a uniformly distributed loading with an initial crack at $x = 3$. The initial crack is defined at a particular location such that any bond over this cross-section is reduced to a lower stiffness value. This decreases the overall stiffness of the beam at the crack point.

![Deformation of beam with initial crack](image)

Fig. 4.14. Deformation of clamped-clamped beam with initial crack

Another advantage of this nonlocal bending equation relates to modeling beams with multiple cross-sections. The Euler-Bernoulli beam equations fail at the ‘jump’ cross-section points since the local beam equations involve the spatial differential terms. The deformation of the beam with multiple cross-sections must be solved step by step based on the Euler Bernoulli beam equations. However, the derived nonlocal bending equation can solve these problems in a compact way regardless of the geometrical discontinuity. The nonlocal responses of the beam with multiple cross-sections are shown in figure (4.15). The dashed line represents the second
area of moment over the beam. There is a cross section ‘jump’ between $x = 2.5$ and $x = 7.5$. The responses with different nonlocal parameters are derived.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{beam_deformation}
\caption{Deformation of clamped-clamped beam with multiple cross sections}
\end{figure}
4.3.2.2 Static deformation of simply supported beam

The deformations of the simply supported beam subject to the concentrated load on center are shown in figure (4.16). The deformation of the nonlocal beam on boundary region is close to the local deformation due to the application of the vary horizon approach. The nonlocal behavior is more obvious for the deformations on the interior region.

Figure (4.17) shows the deformation of the simply supported beam with an initial crack. The initial crack reduces the stiffness at the corresponding cross section \( x = 3 \); thus, the deformation increases at the damaged region.

Figure (4.18) shows the deformation of the simply supported beam with multiple cross sections. The responses of the beam with multiple cross sections are numerically calculated. The deformations are reduced with the increased second moment of area between \( x = 2 \) and \( x = 8 \).
Fig. 4.16. Deformation of simply supported beam
Fig. 4.17. Deformation of simply supported beam with an initial crack
Fig. 4.18. Deformation of simply supported beam with multiple cross sections

4.3.2.3 Static deformation of cantilever beam

The deformations of a cantilever beam subjected to a concentrated load on its free end are shown in figure (4.19). The deformations of the cantilever beam with the initial crack at $x = 5$ are shown in figure (4.20). The greater horizon value corresponds to a larger deformation at the damage point; this indicates that the higher horizon value exhibits the larger nonlocal behavior. Since the damage is interpreted as a reduced bond stiffness for the bonds over the damage point, the higher horizon configuration has more stiffness-reduced bonds at the local damaged region. The deformation of the cantilever beam with multiple cross sections is shown in figure (4.21).
Fig. 4.19. Deformation of cantilever beam
Fig. 4.20. Deformation of cantilever beam with initial crack

(a)
Fig. 4.21. Deformation of cantilever beam with multiple cross sections

### 4.4 Dual Horizon Peridynamics

The traditional way of dealing with the nonlocal boundary conditions is by introducing a pseudo-region outside the physical domain. This adjusts the stiffness on the boundary region; however, the deformations of these pseudo nodes must be pre-defined. The variable horizon approach drastically improves the response accuracy at the boundary region without introducing additional pseudo-nodes. Further, the variable horizon approach with the modified constitutive coefficient improves the accuracy of the nonlocal responses. However, the variable horizon approach cannot eliminate the ghost force or the spurious effect because it violates the linear admissibility condition. In this section, the peridynamics bond force is redefined by reformulating the influencing function. Although the ghost force and the spurious effect can not be eliminated via the variable horizon approach, the linear admissibility condition is fulfilled with the dual-horizon peridynamics.
4.4.1 Peridynamic bond force reformulation

The peridynamic bond acts as a spring connecting two points within the same horizon. For the fixed horizon peridynamics, two points co-exist within the horizons of each other. Note that in figure (4. 22 a), \( \hat{x} \in H_x \) indicates \( x \in H_{\hat{x}} \); or vice verse, \( \hat{x} \notin H_x \) indicates \( x \notin H_{\hat{x}} \) as shown in figure (4.22 b). However, the relationships between two points are more complex when the horizon varies. As shown in figure (4. 23 a), two points locate within the horizon of each other. In figure (4. 23 b) one point locates within the horizon of the other one but the reverse is not true. In figure (4. 23 c), two points have overlapping horizons but they are disconnected. In figure (4. 23 d), two points have no overlapping horizon and are disconnected.

![Figure 4.22. Constant horizon bond force](image)
Fig. 4.23. Variable horizon bond forces

Here one assumes that the peridynamics ‘spring’ exists as long as one point locates within the horizon of the other one. The peridynamics bond force exists in figures (4.23 a b) since one point within the horizon of the other one. Note that the peridynamics bond force here is defined pair-wise. For example in figure (4.23 b), \( x \) has a peridynamics bond force \( f_{x,\hat{x}} \) exerts on \( \hat{x} \). At the same time, \( \hat{x} \) has the peridynamics bond force \( f_{\hat{x},x} \) exerts on \( x \) even though \( x \) is not located within the horizon of \( \hat{x} \). No peridynamics force exists between two nodes disconnected even though the overlapping horizon region exists, as shown in figure (4.23 c d). Thus, the peridynamics bond force is reformulated as
Note that the above bond force always satisfies the linear admissibility conditions. Thus, the Newton’s third law is automatically satisfied. That is,

\[ f_{\hat{x},x} = -f_{x,\hat{x}}. \]  

(4.7)

This reformulated peridynamics bond force is different from the state based peridynamics bond force. The differences are shown in figure (4.24). Based on the state based peridynamics, the deformation of the point \( \hat{x} \) has influence on the nonlocal force exerted on point \( x \) in figure (4.24 a) because \( \hat{x} \) is located within the horizon of \( x \), where \( \hat{x} \in \mathcal{H}_x \). However, for the configuration in figure (4.24 a), the deformation of \( \hat{x} \) has no influence on the variable horizon peridynamics force exerted on point \( x \) because \( x \) and \( \hat{x} \) are not located within the horizon of each other, that is \( \{ x \notin \mathcal{H}_{\hat{x}} \text{ and } \hat{x} \notin \mathcal{H}_x \} \). On the contrary, for the configuration in figure (4.24 b), the deformation of point \( \hat{x} \) has no influence on the state based peridynamics force exerted on point \( x \) since \( \hat{x} \) located outside the horizon of all the neighbor points of \( x \), that is \( \{ \hat{x} \notin \mathcal{H}_x, \forall \hat{x} \in \mathcal{H}_x \} \). However, the deformation of \( \hat{x} \) has influence on the variable horizon peridynamics force exerted on point \( x \) because \( x \) is located within the horizon of \( \hat{x} \). Similarly to the state based peridynamics as a generalization of the bond based peridynamics, the reformulated variable horizon peridynamics is a generalization of the bond based peridynamics. The variable horizon peridynamics can be reduced to the bond based peridynamics when the horizon is fixed. However, the variable horizon peridynamics and the state
based peridynamics are generally different.

\[ \rho \ddot{u}_x + \int_{\mathcal{H}_x} \frac{C_x(\xi) + C_x(x)}{2} (u_\xi - u_x) dV_\xi = b_x. \] (4.8)

The variable horizon peridynamics bond force automatically satisfies the linear admissibility condition. Next, the patch test is checked for the one-dimensional case. By assuming a uniform deformation, the horizon values are defined based on the distances to the nearest boundary point, and are shown in figure (4.25).
Assume a static uniform deformation

\[ U(x) = \frac{F}{EA} x , \quad \text{(4.9)} \]

and define the peridynamics coefficient

\[ c_x = \frac{2E}{\delta_x^2} . \quad \text{(4.10)} \]

Then, the peridynamics force exerted on \( x_0 \) by all nodes on the left side of \( x_0 \) is calculated as

\[
\begin{align*}
 f^{\text{left}}(x_0) &= \int_{0}^{x_0} \frac{1}{2} c_x (u_{x} - u_{x_0}) d\hat{x} + \int_{x_0/2}^{x_0} \left( \frac{1}{2} (c_{x_0} + c_{\hat{x}}) (u_{x} - u_{x_0}) + \frac{2}{\delta_{x_0}^2} (2E/ES) \right) F \frac{\hat{x} - x_0}{E A |\hat{x} - x_0|} d\hat{x} \\
 &= -F \left[ \int_{0}^{x_0/2} \frac{1}{x_0^2} d\hat{x} + \int_{x_0/2}^{x_0} \frac{1}{x_0^2} + \frac{1}{\hat{x}^2} d\hat{x} \right] \\
 &= -\frac{2F}{x_0} .
\end{align*}
\]

\[ \text{(4.11)} \]
The peridynamics force exerted on \( x_0 \) by all nodes on right side of \( x_0 \) is calculated as

\[
f^{right}(x_0) = \int_{x_0}^{x_0+2x_0} \frac{1}{2} (c_{\xi} + c_{\hat{\xi}})(u_{\xi} - u_{x_0})d\hat{\xi} + \int_{2x_0}^{\infty} \frac{1}{2} c_{\hat{\xi}}(u_{\hat{\xi}} - u_{x_0})d\hat{\xi}
\]

\[
= \int_{x_0}^{2x_0} \frac{1}{2} \left( \frac{2E}{\delta_{x_0}} + \frac{2E}{\delta_{x}^2} \right) F \frac{\hat{\xi} - x_0}{E_A |\hat{\xi} - x_0|} d\hat{\xi} + \int_{2x_0}^{\infty} \frac{1}{2} \frac{2E}{\delta_{x}^2} E_A |\hat{\xi} - x_0| d\hat{\xi}
\]

\[
= F \left[ \int_{x_0}^{2x_0} \frac{1}{x_0^2} + \frac{1}{\hat{\xi}^2} d\hat{\xi} + \int_{2x_0}^{\infty} \frac{1}{\hat{\xi}^2} d\hat{\xi} \right]
\]

\[
= F \left[ \frac{1}{x_0} + \frac{1}{x_0} - \frac{1}{\infty} \right]
\]

\[
= -f^{left}(x_0).
\]

Thus, the reformulated variable horizon peridynamics satisfies the linear admissibility condition automatically. Further, for a linear horizon distribution in the boundary region, the patch test is also satisfied.

### 4.4.2 Numerical verification

For the variable horizon distribution described above, the nonlocal responses based on the reformulated peridynamics is shown in figure (4.26). The variable horizon approach improves the accuracy of the responses compared to the iterative interpolation method. However, the spurious effect or the ghost force problem persists in the boundary region.
Fig. 4.26. Deformation of bar with reformulated variable horizon

4.5 Synopsis

This chapter has proposed two new approaches to deal with the nonlocal boundary
condition related problems: the iterative interpolation method and the variable horizon approach. The dual-horizon peridynamics satisfies the linear admissibility condition and has also been introduced.

The iterative interpolation method improves the accuracy of the displacement especially on the boundary region. It has been shown that a large iteration number usually guarantees a more accurate result. Since the iterative interpolation method is built based on the traditional boundary condition treatment, this method requires a pseudo-layer enriched outside the physical boundary. Also the iterative process requires calculating the materials deformation several times. Although the iterative interpolation method can reduce the boundary condition related inaccuracy, it requires a higher computing power.

The variable horizon approach has been applied to reduce the inaccuracy of static deformation. The effectiveness of the variable horizon approach has been proved analytically and numerically. The nonlocal beam equation derived in previous chapter has been solved via the variable horizon approach. The deformations of the beam with an initial crack and the beam with multiple cross-sections have also been treated properly via variable horizon approach.

The variable horizon approach is an efficient method to deal with the boundary condition related discrepancy. In the following two chapters, the variable horizon approach will applied to more complex examples.
Chapter 5

Adaptive Bit-Rock Interaction Modeling via Variable Horizon Peridynamics

5.1. Preliminary Remark

Factors affecting the rate of penetration (ROP) during the drilling operation have been investigated by Bielstein and Cannon (1954). Based on limited drilling field data, Bielstein and Cannon investigated the ROP subject to the changes of bit types, hydraulic factors, weight on bit and rotary speed. Teale (1965) analyzed the ROP in the rock drilling process by introducing the concept of specific energy. These drilling models under various drilling conditions have been verified by both experiments and field data (Pressier and Fear 1992; Bataee, M. Kamyab, et. al 2010). Bourgoyne and Young’s model (BYM) combined several key factors influencing the ROP together, including the mechanical properties of the formation, pore pressure, bit wear, rotary speed, bit hydraulic structures and etc. In the BYM, the ROP is mathematically represented via several functions and this model has been the most preferred method to predict the ROP. These drilling models relate the ROP to the drilling parameters, which often yield moderate results. However, none of them considers the rock crack mechanisms, and the bit-rock interaction simulation is critical to understand the drilling penetration rate.

Akbari, Butt and et. al. (2011) simulated the rock crack process subject to a single polycrystalline diamond compact bit (PDC) cutter by using the distinct element method. Results have shown that the drilling penetration rate increases
considerably by adding a superimposed oscillating weight on bit in a certain frequency range. The vibration agitator was previously designed with some particular frequencies and was installed as one component of the bottom hole assembly (BHA) to reduce the friction especially for the horizontal drilling. It was later shown that drilling with increased axial vibration usually increases ROP. The validity of the vibration assisted rotary drilling (VARD) has been analyzed at certain vibration frequency region under low bottomhole pressure, (see Li, Butt and et. al. 2010). Evangelatos and Payne (2016) described the dynamics of the BHA analytically and the influence of the BHA behavior on the ROP was captured by a neural network approach. Evangelatos’ approach allows for a reasonable extrapolation of the bit behaviors on different BHAs and with different drilling parameters.

With the cutting depth progressing from shallow to deep domain, the failure mode changes from ductile to brittle. An explanation of this transition has been formerly provided by (Puttick 1980; Zdenek and Bazant 1984). Zhou (2013) simulated the drilling process by using the finite element method. Bazant’s sample size effect is observed in Zhou’s simulation. Instead of modeling the rock and the bit, which often requires a high computational power, Endres (2007) used a triangulated mesh to represent the bit-rock surface geometry. The mesh was updated adaptively with the propagation of the drill bit. The simulated bit-rock surface geometry is consistent with the experiment result for both tri-cone and PDC bits.

It has already been pointed out that peridynamics possesses certain advantages when modeling problem involving cracks, especially for heterogeneous materials. In
In this chapter, an efficient algorithm is developed to model the bit-rock interaction using peridynamics. This model iterates adaptively with the crack propagation and the drill bit penetration. The rock cracking process is captured in the model. The penetration rates under different drilling conditions are investigated, including the weight on bit, the rotary speed, the axial vibration frequency, and the amplitude.

### 5.2. Mathematical Background

In peridynamics, the nonlocal forces acting on one point are exerted by all other points within certain distance, which is called horizon. As shown in figure (5.1), forces acting on the solid dot are exerted fully by the shaded nodes within horizon and partially by the shaded nodes on horizon boundary. Thus, the discretized peridynamics force can be expressed in the form of a finite sum. And the corresponding full peridynamics force form requires integration over the horizon. The peridynamics equation of motion becomes

\[ \rho \ddot{u}(x, t) = \mathcal{L}_u(x) + b(x, t), \quad (5.1) \]

where

\[ \mathcal{L}_u(x) = \int_{\mathcal{E}} f(\eta, \xi) dV_{\xi}. \quad (5.2) \]
In the above equations, $L_u(x)$ is defined as the peridynamics force exerting on point $x$, and $f(\eta, \xi)$ is the bond force of two points within the same horizon. The bond force has unit of force per volume square. Further, the bond force depends on the bond properties and the deformation of two points. This constitutive relationship was proposed by Silling (2000) for the bond based peridynamics and Silling, Epton and et. al. (2007) for the state based peridynamics. Le, Chan and et. al. (2014) provided an example of the state based constitutive relation based on the energy equivalence principle. $\xi$ and $\eta$ are the deformation variables which are defined by

$$\eta = \hat{u} - u \quad (5.3)$$

and

$$\xi = \hat{x} - x. \quad (5.4)$$

As shown in figure (5.2), $x$ and $\hat{x}$ are two points within the same horizon in an...
undeformed configuration. $y$ is the location of $x$ after the deformation, $\hat{y}$ is the deformed location of $\hat{x}$.

**Fig. 5.2. Bond deformation configuration**

In the peridynamics framework, the damage is expressed as the break of a bond. The deformation of the peridynamics material is driven by compressing or stretching of bonds. An example of the force-stretch relationship of peridynamics bond is shown in figure (5.3):

**Fig. 5.3. Bond force stretch relationship**
Note that the force stretch relationship is determined by the materials property. This relationship can be nonlinear, hysteresis or even velocity dependent resulting in some more complex material behavior. One example of determining the force-stretch relationship and determining the critical stretch is from energy perspective, as derived by Silling (2005). More complex force stretch relationships can be found in Silling (2007), Le and et. al (2014).

Note that the break of a bond is irretrievable once the bond stretch reaches its critical value. This is physically reasonable since the damage cannot be cured by itself. As shown in figure (5.3), a bond cracks after the bond stretch reaches its critical value $S_{cr}$. Silling and Askari (2005) have defined the bond crack and the local node damage criterions by introducing the hysteresis parameters $\bar{\mu}$ and $\bar{D}$. That is,

$$\bar{\mu}(x, \xi) = \begin{cases} 
1 & \text{if } S < S_{cr} \\
0 & \text{if } S > S_{cr} 
\end{cases},$$

(5.5)

and

$$\bar{D}(x, t) = 1 - \frac{\int_{\xi} \mu(x, \xi) d\xi}{\int_{\xi} d\xi},$$

(5.6)

where $\bar{\mu}(x, \xi)$ is the bond crack function, and $\bar{D}(x, t) \in [0,1]$ is the local damage function. Note that $\bar{D}(x, t) = 0$ means intact while $\bar{D}(x, t) = 1$ means full damage. Also note that the bond and local damage criteria defined in equation (5.5) and equation (5.6) usually introduce singularity problems when a local node becomes fully damaged. One remedy is by redefining the bond crack function and introducing a small positive number $\varepsilon \ll 1$, such that the weak bond force exists even after the
bond breaks. The local damage function is also reformulated. That is,

\[ \mu(x, \xi) = \begin{cases} 1 & \text{if } S < S_{cr} \\ \varepsilon & \text{if } S > S_{cr} \end{cases} \]  

(5.7)

and

\[ D(x, t) = \frac{\int_{\Omega} dV_{\xi}}{\int_{\Omega} \mu(x, \xi) dV_{\xi}}, \]  

(5.8)

where \( D(x, t) \in [1, \infty) \). \( D(x, t) = 1 \) means no damage, and \( D(x, t) = \infty \) indicates complete damage. In real drilling, the local nodes are seldom fully damaged during the rock cracking process. The rock is penetrated even when some bonds intact. Here a critical damage parameter \( M \) is defined such that \( D(x, t) > M \) indicates that the node \( x \) is severely damaged and can be removed by circulation fluids. In this modeling, the severely damaged nodes are not involved in the calculation. The loads exerted on severely damaged nodes are automatically released. This avoids the singularity problems caused by the severely damaged nodes.

It is unnecessary for all nodes to become fully damaged before the penetration happens. In fact, penetration happens when the damage of near bit region reaches a critical value. The penetration functions are defined as

\[ \mathcal{P} = \frac{\int_{\Omega_h} dV_x}{\int_{\Omega_h} \varphi(h, x) dV_x} \]  

(5.9)

and

\[ \varphi(h, x) = \begin{cases} 0 & \text{if } D(x, t) > M \\ 1 & \text{if } D(x, t) < M \end{cases} \]  

(5.10)
where $h$ is the penetration step length. $\Omega_h$ is the bit-rock contact region, which is adaptively updated during the drilling process. $\varphi(h, x)$ is defined as the node damage indicator. $\mathcal{P} = 1$ means nodes within the bit-rock contact region are intact, and $\mathcal{P} = \infty$ indicates all nodes within the bit-rock contact region are fully damaged. Then introduce a positive large number $N$ such that $\mathcal{P} > N$ indicates that penetration happens and nodes involving in calculation are adaptively updated.

For a higher accuracy, the parameters $N$ and $\frac{1}{\varepsilon}$ should be chosen as large as possible. $M$ is the cracked gravel size related parameter and must be chosen carefully taking into account the load conditions, the rock properties and etc.

5.3. Adaptive Modeling

Obviously the crack propagates in the near-bit region. In the far-bit region however, the influence of the drilling bit is drastically reduced. In this model, the deformation of the nodes is calculated only in the near-bit region. With the penetration of the drill bit, the nodes involved in calculation are updated. Thus, the computational cost is drastically reduced while the entire bit penetration process is captured adaptively.

During the drilling process, rock is cracked into small size gravel. In the numerical modeling, this means that the cracked nodes are completely separated from the other nodes. This usually causes numerical problems because of the singularity of the associated Jacobian matrix. One of the common remedies is by introducing the weak bond force for the completely cracked nodes (Gerstle 2005). However, with
the propagation of cracks, the number of completely cracked nodes increases. The weak bond force approach will eventually fail by introducing computational inefficiency. It is assumed that the completely cracked nodes have little influence on the other nodes. This model calculates the deformation of these incompletely cracked nodes only. During the actual drilling process, the fully cracked small size gravel is removed by drilling mud circulation. Thus, the above assumption is quite reasonable.

The Bit-Rock interaction configuration is shown in figure (5.4). For modeling simplification, the pipe-borehole contact process is not discussed in this model. The proposed model does not consider the hydraulic effect of drilling mud. For computational efficiency, the response is only calculated in the near-bit region. The bit-rock contact loading is replaced by exerting forces on the contact surface. The rotation of the drill bit is modeled as a periodic loading. The axial vibration is modeled as the fluctuation of the weight on bit.

![Fig. 5.4. Drilling process configuration](image)

Peridynamics, in general, possesses great advantages when modeling the fracture of
heterogeneous materials (Kilic 2008). Rock is particularly a heterogeneous material because of the porosity and the pre-existing fracture. In the ensuing analysis, the Young’s modulus of the rock is described stochastically. The pre-existing fracture and the porosity are replaced stochastically by the bond crack parameter. An example of the peridynamics variables is shown in figure (5.5).

![Material Properties at particular layer](image)

**Fig. 5.5.** Rock property variation description

### 5.4. Application to Drilling

This model calculates the deformation of the nodes within the bit-rock contact region. The domain of interest is meshed as shown in figure 5.6. Only these nodes in circle are calculated. The load of the bit is translated into forces exerted on the bit-rock contact surface, as shown in figure (5.7a). The force distribution varies depending on the type of the drill bit. The distributed forces are only exerted on the
incompletely damaged nodes, and updated with the rotation of the drill bit. When the damage of the contact region reaches the penetration criterion, the external force moves to the next layer. The deformation of the bit-rock contact region under constant loading is shown in figure (5.7 b). The results point out that the largest deformation happens on the bit-rock contact surface. The deformation in far contact region is quite small or even negligible. This confirms the validity of only calculating the nodes within the bit-rock contact region. Figures (5.8 a-f) show the deformation of a particular layer under increasing loads without cracking involved.

![Fig. 5.6. Domain of interested meshing](image)
Fig. 5.7. (a) Loading representation

Fig. 5.7. (b) Deformation in the bit-rock contact region
Fig. 5.8. Deformation stages of a particular layer

The bit induced formation damage is one of the main causes of the borehole
instability. The proposed model is established based on peridynamics and can automatically capture the crack propagation of formation during the drilling process. Based on the bond crack and the local damage criteria, the formation damage is shown in figures (5.9 a-f). The bond damage initiates at figure (5.9 a) and propagates from figure (5.9 a) to figure (5.9 f). The formation damage reaches the penetration criterion and penetration happens after figure (5.9 e). Figure (5.10) shows the damage of a formation in the radial direction after penetration. The damage significantly reduces at radius equals to 0.55 feet, which matches the radius of drill bit.
Fig. 5.9. Damage propagation in a particular layer

Fig. 5.10. Damage in radial direction

5.5. Rate of Penetration Analysis

During the drilling operation, what the drillers control are basically the weight on bit, the rotary velocity and the mud pressure. This modeling does not consider
hydraulic effects. In the following part, the drilling performance with various values of weight on bit and rotary speed are analyzed. The effects of axial vibration on ROPs are investigated by Li, Butt and et. al. (2010).

5.5.1 Influence of weight on bit on penetration rate

In this study, the weight on bit increases from 1 \( klb \) to 50 \( klb \). When the weight on bit is quite small, 1 \( klb \) for example, no penetration happens. This is because the weight on bit is smaller than the threshold value. This small weight on bit can only induces a small deformation and the crack can not be initiated. When the weight on bit reaches the threshold value, the bond crack happens. As expected, the rate of penetration increases with the increase of weight on bit. Figure (5.11) shows the penetration rate with different weights on bit.

![Penetration Rate with Varying WOB](image)

**Fig. 5.11.** ROP with different WOB
5.5.2 Influence of rotary speed on penetration rate

The penetration rates under different rotary speeds are shown in figure (5.12). The rotary speeds vary from 30 rpm to 120 rpm. The results show that the higher rotary speed gives the higher penetration rate.

![Rate of Penetration with Torsional Velocity](image)

Fig. 5.12. ROP with various rotary speeds

5.5.3 Influence of axial vibration on penetration rate

Previous experiments and field data have pointed out the improvement of ROP with an axial vibration presented. The vibration assisted rotary drilling *(VARD)* has been investigated in recent years in references such as Li (2010), Akbari (2011). In this model, the influence of the axial vibration on the penetration rate is analyzed. The axial vibration is expressed as a modulation of the weight on bit. That is,

\[ F_{WOB} = W(1 + v f(t)), \]  

(5.11)
where \( v \) is the axial vibration ratio. With \( v = 0 \) means no axial vibration exists, while \( 0 < v < 1 \) indicates that a mild axial vibration appears, and \( v > 1 \) suggests that the bit bounce happens. The function \( f(t) \) denotes the fluctuation of the weight on bit, which is used to simulate the axial vibration. In this model, the axial vibration function \( f(t) \) is chosen to be sinusoidal with the specified frequencies.

Note that in the real drilling process, the bit ‘dances' on bottom unrestricted to bounce upward and lift off; one cannot know the axial force because of the complex behavior of the drill bit. Even though the BHA properties are well known and the vibration characteristics can be estimated, the axial force of those components with an imperfect bore hole with variable formation properties cannot be precisely calculated (Evangelatos and Payne 2016).

The penetration rates for various levels of the axial vibrations are shown in figures (5.13) and (5.14). The simulated result shown in figure (5.13) confirms the validity of the vibration-assisted rotary drilling technique. The ROPs of vibration-assisted drilling are much higher than the ROP of drilling with constant WOB. However, higher axial vibration amplitude does not guarantee higher ROP. One can observe from figure (5.13) that the ROP reaches maximum value with amplitude ratio equals to 0.8. Also, the severe axial vibration increases the probability of drill equipment failure. Thus in actuality, the axial vibration amplitude must be chosen carefully.

The influence of the vibration frequency on the penetration rate is shown in figure (5.14). There is a frequency around 20Hz, which corresponds to the highest penetration rate. This is due to the fact that the peak frequency excitation induces
the resonant responses. And the value of the peak frequency depends on the formation properties. It is shown that the simulated results match the real field data discussed by Babatunde (2011).

Fig. 5.13. ROP with various axial vibration amplitudes

Fig. 5.14. ROP with various axial vibration frequencies
5.6. Synopsis

In this chapter, an adaptive model based on peridynamics has been proposed to simulate the bit-rock interaction. This model calculates the deformation of the rock within the bit-rock contact region. The calculated region moves adaptively with the penetration of the drill bit. Thus, the proposed model is suitable for the large penetration depth modeling. Local damage and the penetration criteria have been introduced for the rock cracking process. The deformation and the crack propagation of rock have been simulated. The borehole damage after penetration has been generated. The influences of the drilling parameters on the rate of penetration have been analyzed. The higher the weight on bit and the rotary velocity are, the higher the penetration rate becomes. The axial vibration improves the rate of penetration; however, there exists a specific combination of the axial vibration amplitude and frequency, which provide the maximum rate of penetration.
Chapter 6

Peridynamics Application to Fluid Mechanics via Variable Horizon Approach

In this chapter, an incompressible Navier-Stokes equation is reformulated by applying Newton’s second law to the fluid motion in the nonlocal sense. The pressure Poisson equation is reformulated by utilizing a weighted integral of the nonlocal Navier-Stokes equation over the entire domain with the nonlocal incompressibility condition. It is shown that the nonlocal Navier-Stokes equation reduces to the classical Navier-Stokes equations when the nonlocal parameter reduces to zero. Finally, the examples of the fluid mechanics utilizing both local Navier-Stokes equation and the nonlocal Navier-Stokes equation are discussed.

6.1 Local Navier Stokes Equation

6.1.1 Equilibrium equation

The classical Navier-Stokes equation contains spatial derivative terms, and the continuum assumption is required. The classical Navier-Stokes equation can be derived from the Newton’s second law. That is,

$$\frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \nabla \cdot \mathbf{\sigma} + \rho \mathbf{f}. \quad (6.1)$$

For the incompressible fluids, the local Navier-Stokes equilibrium equation can be rewritten as
\[
\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{f} \right] - \nabla \cdot \mathbf{\sigma} = 0
\] (6.2)

and

\[\nabla \cdot (\rho \mathbf{u}) = 0.\] (6.3)

For Newtonian fluids, the shear stress is proportional to the gradient of the velocity. The stress tensor can be expressed as

\[\mathbf{\sigma} = -p \mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T).\] (6.4)

Thus, the Navier-Stokes equation with incompressibility condition can be further recast as

\[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - v \nabla^2 \mathbf{u} + \frac{1}{\rho} \nabla P - \mathbf{f} = 0,\] (6.5)

and

\[\nabla \cdot \mathbf{u} = 0.\] (6.6)

### 6.1.2 Pressure Poisson equation

The local pressure Poisson equation is derived by considering the gradient operation on both sides of the local Navier-Stokes equation. The incompressibility condition is also applied in the derivation. This yields, for the 2D case, the equation

\[-\frac{1}{\rho} \Delta P = 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y},\] (6.7)

where \( \Delta \) is the Laplacian operator.

Note that the pressure Poisson equation is a scalar equation, which is derived from the Navier-Stokes equation and the incompressibility condition.
6.2 Nonlocal Navier-Stokes Equation

6.2.1 Nonlocal equilibrium equation

Inspired by the concepts of peridynamics, the local Navier-Stokes equation spatial derivative terms are replaced by the nonlocal integral terms. That is, the momentum density increment at one point is not only defined by the state of points directly connected to it but also defined by the state of other points within some distance $\delta$.

The configuration of the nonlocal fluids is shown in figure (6.1). Similar as the peridynamics horizon defined in solid mechanics, in the nonlocal fluid mechanics, the solid node is defined a horizon with a radius equals to $\delta$. The state of the solid node is fully determined by the state of the circle nodes located within the horizon and partially determined by the state of the circle nodes located on the horizon boundary.

![Fig. 6.1. Nonlocal fluids configuration](image)

Similar to the local Navier-Stokes equation of the Newtonian incompressible flow,
the reformulated nonlocal Navier-Stokes equation can be expressed as

\[
\frac{\partial \mathbf{u}}{\partial t} + T_{\text{convection}} + T_{\text{diffusion}} + T_{\text{pressure}} = f. \tag{6.8}
\]

Without the continuum assumption, \( T_{\text{convection}} \), \( T_{\text{convection}} \), and \( T_{\text{pressure}} \) are the convection, diffusion and pressure terms, which are expressed as

\[
T_{\text{convection}} = \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi)(\mathbf{u} - \mathbf{u}) d\mathbf{V}, \tag{6.9}
\]

\[
T_{\text{diffusion}} = 2v \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} - \mathbf{u}) d\mathbf{V}, \tag{6.10}
\]

and

\[
T_{\text{pressure}} = \frac{1}{\rho} \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{p} - p) \cdot \xi d\mathbf{V}, \tag{6.11}
\]

where \( \xi \) is the bond vector, it goes from the center point to the surrounding point within its horizon. The weighting function \( C(\xi) \) is the influence function, which measures the ‘influence’ of a particular point to the center point. The symbol \( C(\xi) \) is defined such that the nonlocal Navier-Stokes equation reduces to the local Navier-Stokes equation when the nonlocal parameter reduces to zero. Thus, the nonlocal incompressibility condition is reformulated as

\[
\int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} - \mathbf{u}) \cdot \xi d\mathbf{V} = 0. \tag{6.12}
\]
6.2.2 Nonlocal pressure Poisson equation

The nonlocal pressure Poisson equation is derived by applying the dot product $\xi$ on both sides of the nonlocal Navier-Stokes equation, and integrating over the entire domain. That is,

\[
\int_{\Omega} \frac{\tilde{C}(\xi)}{|\xi|^2} \left( \frac{\partial u}{\partial t} + \sum_{i=1}^n \frac{\tilde{C}(\bar{\xi}_i)}{|\bar{\xi}_i|^2} (u \cdot \bar{\xi}_i) (\bar{u} - u) \right) dV - 2 v \int_{\Omega} \frac{\tilde{C}(\xi)}{|\xi|^2} (\bar{u} - u) dV \\
+ \frac{1}{\rho} \int_{\Omega} \frac{\tilde{C}(\xi)}{|\xi|^2} (\bar{\rho} - p) \bar{\xi} dV \right) \cdot \xi dV = \int_{\Omega} \frac{\tilde{C}(\xi)}{|\xi|^2} f \cdot \xi dV ,
\]

where $\tilde{C}(\xi)$ is the global influence function, which is defined as

\[
\tilde{C}(\xi) = \begin{cases} 
C(\xi) & \text{for } \xi < \delta \\
0 & \text{for } \xi > \delta .
\end{cases}
\]

Further, $\xi$ and $\bar{\xi}$ are defined as

\[
\xi = x_i - x \quad \text{and} \quad \bar{\xi} = \bar{x} - x .
\]

The left hand side of the nonlocal pressure Poisson equation can be expanded as

\[
\text{LHS} = \int_{\Omega} \frac{\tilde{C}(\xi)}{|\xi|^2} \frac{\partial u}{\partial t} \cdot \xi dV + \int_{\Omega} \frac{\tilde{C}(\xi)}{|\xi|^2} \frac{\tilde{C}(\bar{\xi})}{|\bar{\xi}|^2} (u \cdot \bar{\xi}) (\bar{u} - u) \cdot \xi dV dV \\
- \int_{\Omega} \int_{\Omega} 2 v \frac{\tilde{C}(\xi)}{|\xi|^2} \frac{\tilde{C}(\bar{\xi})}{|\bar{\xi}|^2} (\bar{u} - u) \cdot \xi dV dV \\
+ \int_{\Omega} \int_{\Omega} \frac{1}{\rho} \frac{\tilde{C}(\xi)}{|\xi|^2} \frac{\tilde{C}(\bar{\xi})}{|\bar{\xi}|^2} (\bar{\rho} - p) \bar{\xi} \cdot \xi dV dV .
\]
Based on the incompressibility condition, the time derivative term reduces to zero.

That is,

\[ \int_{\Omega} \frac{\partial}{\partial t} \frac{\bar{c}(\xi)}{|\xi|^2} \cdot \xi \, dV = \frac{\partial}{\partial t} \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \mathbf{u} \cdot \xi \, dV. \]  \hspace{1cm} (6.17)

And the convection term yields

\[ \int_{\Omega} \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) (\mathbf{u} \cdot \xi) \, dV \]

\[ = \int_{\Omega} \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) (\mathbf{u} \cdot \xi) \, dV - \int_{\Omega} \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) \left[ \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) \, dV \right] \, dV \]

\[ = \int_{\Omega} \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) (\mathbf{u} \cdot \xi) \, dV. \]  \hspace{1cm} (6.18)

The diffusion term yields

\[ \int_{\Omega} \int_{\Omega} 2 \nu \frac{\bar{c}(\xi)}{|\xi|^2} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) \, dV \]

\[ = \int_{\Omega} \int_{\Omega} 2 \nu \frac{\bar{c}(\xi)}{|\xi|^2} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) \, dV - \int_{\Omega} \int_{\Omega} 2 \nu \frac{\bar{c}(\xi)}{|\xi|^2} \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) \, dV \]

\[ = \int_{\Omega} 2 \nu \frac{\bar{c}(\xi)}{|\xi|^2} \left[ \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \mathbf{u} \cdot \xi \, dV \right] \cdot \xi \, dV - \int_{\Omega} 2 \nu \frac{\bar{c}(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi) \left[ \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \, dV \right] \, dV. \]  \hspace{1cm} (6.19)

Assume the viscosity \( \nu \) and the shape of influence function \( \bar{c}(\xi) \) do not change over the entire domain \( \Omega \). Further, define

\[ K(x) = \int_{\Omega} \frac{\bar{c}(\xi)}{|\xi|^2} \, dV = \text{constant}, \]  \hspace{1cm} (6.20)
and

\[ \bar{u}(x) = \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \bar{u} \, d\mathcal{V}. \]  

(6.21)

Then, the diffusion term yields

\[ 2v \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \bar{u} \cdot \xi \, dV - 2vK \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} (u \cdot \xi) \, dV = 2v \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \bar{u} \cdot \xi \, dV. \]  

(6.22)

Note that \( \bar{u} = u \) if the horizon reduces to zero or the variation of the velocity is negligible. This indicates the diffusion term in PPE vanishes.

Thus, the nonlocal pressure Poisson equation (NLPPE) yields

\[ \int_{\Omega} \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \frac{\bar{C}(\xi)}{|\xi|^2} (u \cdot \xi) (\bar{u} \cdot \xi) \, d\mathcal{V} \, dV - 2v \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \bar{u} \cdot \xi \, dV \]

\[ + \int_{\Omega} \int_{\Omega} \frac{1}{\rho} \frac{\bar{C}(\xi)}{|\xi|^2} \frac{\bar{C}(\xi)}{|\xi|^2} (\bar{\rho} - p) (\bar{\xi} \cdot \xi) \, d\mathcal{V} \, dV = \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} f \cdot \xi \, dV. \]

(6.23)

The approximated nonlocal pressure Poisson equation yields

\[ \frac{1}{\rho} \int_{\Omega} \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \frac{\bar{C}(\xi)}{|\xi|^2} (\bar{\rho} - p) (\bar{\xi} \cdot \xi) \, d\mathcal{V} \, dV = -\int_{\Omega} \int_{\Omega} \frac{\bar{C}(\xi)}{|\xi|^2} \frac{\bar{C}(\xi)}{|\xi|^2} (u \cdot \xi) (\bar{u} \cdot \xi) \, d\mathcal{V} \, dV. \]

(6.24)
6.3 Nonlocal Consistency Analysis

6.3.1 Nonlocal convection term

The nonlocal convection term is reformulated. To clarify the convection term, one can rewrite it as

\[ T_{\text{convection}} = \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} \cdot \xi)(\mathbf{\bar{u}} - \mathbf{u}) d\mathbf{\bar{r}} \]

\[ = \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} \cdot \mathbf{n}_\xi |\xi|)(\mathbf{\bar{u}} - \mathbf{u}) d\mathbf{\bar{r}} \]

\[ = \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|} (\mathbf{u} \cdot \mathbf{n}_\xi)(\mathbf{\bar{u}} - \mathbf{u}) d\mathbf{\bar{r}} \]

\[ = \int_{\mathcal{H}} C(\xi) \left( \mathbf{u} \cdot \mathbf{n}_\xi \right) \frac{(\mathbf{\bar{u}} - \mathbf{u})}{|\xi|} d\mathbf{\bar{r}}, \quad (6.25) \]

where \( \mathbf{n}_\xi \) is the unit vector along the bond direction, \((\mathbf{u} \cdot \mathbf{n}_\xi)\) is the projection of the velocity on the \( \xi \) direction, \( (\mathbf{\bar{u}} - \mathbf{u}) \) is the velocity rate of change in the \( \xi \) direction. Thus, the integrand is the rate of momentum increment of two points within the same horizon. The total convection term is determined by integrating all nodes within the horizon. Further, it can be proved that

When the nonlocal parameter \( \delta \) reduces to zero, the nonlocal convection term reduces to the local convection term.

Specifically, to be consistent with the continuum assumption in the local Navier-Stokes equation, one assumes the velocity field to be continuous and differentiable. That is
\[ T_{\text{Local convection}} = \mathbf{u} \cdot \nabla \mathbf{u} \]
\[ = |\mathbf{u}| \, (\mathbf{n} \cdot \nabla \mathbf{u}), \quad (6.26) \]

where
\[ \mathbf{n} = \frac{\mathbf{u}}{|\mathbf{u}|}. \quad (6.27) \]

Note that \((\mathbf{n} \cdot \nabla \mathbf{u})\) is defined as the derivative of \(\mathbf{u}\) in \(\mathbf{n}\) direction. That is,
\[ \mathbf{n} \cdot \nabla \mathbf{u} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x \cdot \mathbf{n}) - u(x)}{\Delta x}. \quad (6.28) \]

Further, for the one-dimensional case, the nonlocal convection term yields
\[ T_{\text{convection}} = \int_{H} \frac{C(\xi)}{|\xi|} (\mathbf{u} \cdot \mathbf{n}_{\xi}) |\xi| (\mathbf{\hat{u}} - \mathbf{u}) d\mathbf{\hat{v}} \]
\[ = \int_{H} C(\xi) (\mathbf{u} \cdot \mathbf{n}_{\xi}) \frac{(\mathbf{\hat{u}} - \mathbf{u})}{|\xi|} d\mathbf{\hat{v}} \]
\[ = \int_{-\delta}^{\delta} C(\xi) \left[ \text{sign}(\xi) u(x) \right] \frac{u(x + \xi) - u(x)}{|\xi|} d\xi \]
\[ = \int_{-\delta}^{\delta} C(\xi) u(x) \frac{u(x + \xi) - u(x)}{\xi} d\xi. \quad (6.29) \]

Thus,
\[ \lim_{\delta \to 0} T_{\text{convection}} = \lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) u(x) \frac{u(x + \xi) - u(x)}{\xi} d\xi. \quad (6.30) \]

The continuous and differentiable condition of \(u\) gives
\[ u(x + \xi) = u(x) + u(\alpha)' \cdot \xi, \quad (6.31) \]

where
\[ u(x + \alpha\xi)' = \frac{du}{dx} \bigg|_{x=x+\alpha\xi} \quad (6.32) \]
\[ \alpha \in [0,1]. \]
Thus,

\[
\lim_{\delta \to 0} T_{\text{convection}} = \lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) \ u(x) \ u(x + \alpha \xi)' \ d\xi
\]

\[
= u(x) \cdot \lim_{\xi \to 0} u(x + \alpha \xi)' \cdot \lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) \ d\xi
\]

\[
= u(x) \cdot \lim_{\alpha \xi \to 0} u(x + \alpha \xi)' \cdot \lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) \ d\xi
\]

\[
= u(x) \cdot u'(x) \cdot \lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) \ d\xi .
\] (6.33)

Note that the influence function satisfies

\[
\lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) \ d\xi = 1 ,
\] (6.34)

then the nonlocal convection term is consistent with the local convection term. That is,

\[
\lim_{\delta \to 0} T_{\text{convection}} = T_{\text{Local\_convection}} .
\] (6.35)

For the three-dimensional case, the nonlocal convection term yields

\[
\lim_{\delta \to 0} T_{\text{convection}} = \lim_{\delta \to 0} \int_{\mathcal{J}_\varepsilon} C(\xi) \ (u \cdot n\xi) \frac{u(x + \xi) - u(x)}{|\xi|} dV_\xi
\]

\[
= \lim_{\delta \to 0} \int_{\mathcal{J}_\varepsilon} C(\xi) \ (u \cdot n\xi) \frac{u(x + n\xi|\xi|) - u(x)}{|\xi|} dV_\xi .
\] (6.36)

By considering the relationship

\[
\lim_{\delta \to 0} \frac{u(x + n\xi|\xi|) - u(x)}{|\xi|} = n\xi \cdot \nabla u < \infty,
\] (6.37)
the nonlocal convection term can be rewrite in the limit as

\[
\lim_{\delta \to 0} T_{\text{convection}} = \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \left( \mathbf{u} \cdot \mathbf{n}_{\xi} \right) \left( \mathbf{n}_{\xi} \cdot \nabla \mathbf{u} \right) dV_{\xi} \\
= \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \left( \mathbf{u} \cdot \mathbf{n}_{\xi} \right) \mathbf{n}_{\xi} \cdot \nabla \mathbf{u} dV_{\xi} \\
= \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \mathbf{n}_{\xi} \otimes \mathbf{n}_{\xi} \mathbf{u} \cdot \nabla \mathbf{u} dV_{\xi} \\
= \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \mathbf{n}_{\xi} \otimes \mathbf{n}_{\xi} dV_{\xi} \mathbf{u} \cdot \nabla \mathbf{u}. \quad (6.38)
\]

By noting that the influence function satisfies the equation

\[
\lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \mathbf{n}_{\xi} \otimes \mathbf{n}_{\xi} dV_{\xi} = \delta_{i,j}, \quad (6.39)
\]

then the nonlocal convection term is found consistent with the local convection term. That is,

\[
\lim_{\delta \to 0} T_{\text{convection}} = T_{\text{Local convection}}. \quad (6.40)
\]

**Definition:** The nonlocal weighted shape tensor is an identity second order tensor, which is expressed as

\[
T_{i,j} = \int_{\mathcal{H}} C(\xi) \mathbf{n}_{\xi i} \otimes \mathbf{n}_{\xi j} dV_{\xi}, \quad (6.41)
\]

where \( C(\xi) \) is the nonlocal influence function, and \( \mathbf{n}_{\xi} \) is the unit vector in the direction of the bond \( \xi \).

**Example:** For an arbitrary nonzero vector \( \mathbf{u} \), the nonlocal weighted shape tensor \( T \) operating on the vector \( \mathbf{u} \) is expressed as

\[
T \mathbf{u} = \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\xi \cdot \mathbf{u}) \xi dV_{\xi}. \quad (6.42)
\]
The second order tensor $T$ is symmetric and invariant with the coordinate system. For simplification, a particular coordinate is built such that the $u$ direction is parallel to $z$ axis as shown in figure (6.2).

The nonlocal shape tensor $T$ operating on the vector $u$ is rewritten as

$$T \mathbf{u} = \int_{\Omega} \frac{C(\xi)}{|\xi|^2} (\xi \cdot \mathbf{u}) \xi \, dV_{\xi}$$

$$= \int_{-\delta}^{\delta} dz \iint_{S(\xi)} \frac{C(\xi)}{|\xi|^2} (\xi \cdot \mathbf{u}) \xi \, dS_{\xi}$$

$$= \int_{-\delta}^{\delta} dz \iint_{S(\xi)} \frac{C(\xi)}{|\xi|^2} |\mathbf{u}|(\xi \cdot \mathbf{n}_z) \xi \, dS_{\xi}$$

$$= \int_{-\delta}^{\delta} dz \iint_{S(\xi)} \frac{C(\xi)}{|\xi|^2} z |\mathbf{u}| \xi \, dS_{\xi}$$
\[ T \mathbf{u} = \int_{-\delta}^{\delta} d\mathbf{z} \int_{S(x)} \frac{C(\xi)}{|\xi|^2} z |\mathbf{u}| (\xi_z + \xi_r) dS_\xi. \quad (6.43) \]

Because of the symmetry properties, any terms involving \( \xi_r \) reduce to zero. Then,

\[ T \mathbf{u} = \int_{-\delta}^{\delta} d\mathbf{z} \int_{S(x)} \frac{C(\xi)}{|\xi|^2} z |\mathbf{u}| \xi_z dS_\xi \]

\[ = \int_{-\delta}^{\delta} z^2 d\mathbf{z} \int_{S(x)} \frac{C(\xi)}{|\xi|^2} dS_\xi \cdot \mathbf{u} \]

\[ = \int_{\mathcal{M}} z^2 \frac{C(\xi)}{|\xi|^2} dV_\xi \cdot \mathbf{u}. \quad (6.44) \]

Since \( \mathbf{u} \) is chosen arbitrarily, \( T \) is identity second order tensor, which yields

\[ \int_{\mathcal{M}} z^2 \frac{C(\xi)}{|\xi|^2} dV_\xi = 1. \quad (6.45) \]

### 6.3.2 Nonlocal diffusion term

The nonlocal Navier-Stokes equation diffusion term is reformulated in equation (6.10). Similarly, the following local and nonlocal consistency condition can also be proved. That is,

When the nonlocal parameter \( \delta \) reduces to zero, the nonlocal diffusion term reduces to the local diffusion term. To prove this,

note that
\[
\lim_{\delta \to 0} T_{\text{diffusion}} = \lim_{\delta \to 0} 2 \nu \int_{\mathcal{F}} \frac{C(\xi)}{\xi^2} (\mathbf{u} - \mathbf{u}) dV
\]

\[
= \lim_{\delta \to 0} 2 \nu \left\{ \int_{\mathcal{F}^+} \frac{C(\xi)}{\xi^2} [\mathbf{u}(x+\xi) - \mathbf{u}] dV + \int_{\mathcal{F}^-} \frac{C(\xi)}{\xi^2} [\mathbf{u}(x-\xi) - \mathbf{u}] dV \right\}
\]

\[
= \lim_{\delta \to 0} 2 \nu \left\{ \frac{1}{2} \int_{\mathcal{F}} \frac{C(\xi)}{\xi^2} \left\{ [\mathbf{u}(x+\xi) - \mathbf{u}] + [\mathbf{u}(x-\xi) - \mathbf{u}] \right\} dV_{\xi} \right\}
\]

\[
= \lim_{\delta \to 0} \nu \int_{\mathcal{F}} \frac{C(\xi)}{\xi} \left[ \nabla \left( \mathbf{u}(x+\frac{1}{2}\xi) \right) - \nabla \left( \mathbf{u}(x-\frac{1}{2}\xi) \right) \right] dV_{\xi}
\]

\[
= \lim_{\delta \to 0} \nu \int_{\mathcal{F}} C(\xi) \mathbf{n}_\xi \cdot \left( \mathbf{n}_\xi \cdot \nabla^2 \mathbf{u}(x) \right) dV_{\xi}
\]

\[
= \lim_{\delta \to 0} \nu \int_{\mathcal{F}} C(\xi) \mathbf{n}_\xi \otimes \mathbf{n}_\xi dV_{\xi} \cdot \mathbf{u}(x) \cdot \Delta \mathbf{u}(x) . \tag{6.46}
\]

Further, the influence function satisfies the relationship

\[
\lim_{\delta \to 0} \int_{\mathcal{F}} C(\xi) \mathbf{n}_{\xi i} \otimes \mathbf{n}_{\xi j} dV_{\xi} = \delta_{i,j} , \tag{6.47}
\]

Then, the nonlocal diffusion term is consistent with the local diffusion term. That is,

\[
\lim_{\delta \to 0} T_{\text{diffusion}} = T_{\text{Local, diffusion}} . \tag{6.48}
\]

Thus, the nonlocal diffusion term reduces to the local diffusion equation with the nonlocal parameter reduces to zero.
6.3.3 Nonlocal pressure term

The local and nonlocal consistency condition of pressure terms can also be proved, which states that

When the nonlocal parameter $\delta$ reduces to zero, the nonlocal pressure term reduces to the local pressure term. To prove this, consider the equations

$$\lim_{\delta \to 0} T_{\text{pressure}} = \lim_{\delta \to 0} \frac{1}{\rho} \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\varphi - p) \xi \, d\mathcal{V}$$

$$= \frac{1}{\rho} \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \frac{p(x + \xi) - p(x)}{|\xi|} \xi \, dV_{\xi}$$

$$= \frac{1}{\rho} \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) \frac{p(x + \xi) - p(x)}{|\xi|} n_{\xi} \, dV_{\xi}$$

$$= \frac{1}{\rho} \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) (\nabla p \cdot n_{\xi}) \cdot n_{\xi} \, dV_{\xi}$$

$$= \frac{1}{\rho} \lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) n_{\xi} \otimes n_{\xi} \, dV_{\xi} \cdot \nabla p \, , \quad (6.49)$$

with the influence function satisfies the relationship

$$\lim_{\delta \to 0} \int_{\mathcal{H}} C(\xi) n_{\xi i} \otimes n_{\xi j} dV_{\xi} = \delta_{i,j} \, , \quad (6.50)$$

then the nonlocal pressure term is consistent with the local pressure term. That is,

$$\lim_{\delta \to 0} T_{\text{pressure}} = T_{\text{Local pressure}} \, . \quad (6.51)$$
6.3.4 Nonlocal incompressibility condition

The local and nonlocal consistency condition of the incompressibility condition is stated as

With the nonlocal parameter $\delta$ reduces to zero, the nonlocal incompressibility condition reduces to the local incompressibility condition. To prove this, note that for the one-dimensional case, the nonlocal incompressibility condition yields

$$\lim_{\delta \to 0} \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} - \mathbf{u}) \cdot \mathbf{\xi} \, d\mathbf{\nu}$$

$$= \lim_{\delta \to 0} \int_{x-\delta}^{x+\delta} \frac{C(\xi - x)}{|\xi - x|^2} (\mathbf{u} - \mathbf{u}) \cdot (\mathbf{\xi} - x) \, d\xi$$

$$= \lim_{\delta \to 0} \int_{-\delta}^{\delta} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} - \mathbf{u}) \cdot \mathbf{\xi} \, d\xi$$

$$= \lim_{\delta \to 0} \int_{-\delta}^{\delta} C(\xi) d\xi \cdot \lim_{\delta \to 0} \frac{u(x + \xi) - u(x)}{\xi}$$

$$= \frac{du}{dx}.$$  \hspace{1cm} (6.52)

And similarly, for the one-dimensional case, the nonlocal incompressibility condition yields

$$\lim_{\delta \to 0} \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u} - \mathbf{u}) \cdot \mathbf{\xi} \, d\mathbf{\nu}$$

$$= \lim_{\delta \to 0} \int_{\mathcal{H}} \frac{C(\xi)}{|\xi|^2} (\mathbf{u}_i - \mathbf{u}_i) \cdot \mathbf{\xi}_i \, d\mathbf{\nu}$$
\[
= \lim_{\delta \to 0} \int _{\mathcal{X}} C(\xi) \left[ \frac{\partial u_i(x)}{\partial x_j} \cdot n_{\xi i} \right] \cdot n_{\xi j} \, dV_{\xi}
\]

\[
= \lim_{\delta \to 0} \int _{\mathcal{X}} C(\xi) \left[ \frac{\partial u_i(x)}{\partial x_j} \cdot n_{\xi i} \right] \cdot n_{\xi j} \, dV_{\xi}
\]

\[
= \lim_{\delta \to 0} \int _{\mathcal{X}} C(\xi) n_{\xi i} \otimes n_{\xi j} \, dV_{\xi}
\]

\[
= \delta_{ij} \frac{\partial u_i}{\partial x_j} = \nabla \cdot \mathbf{u}.
\] (6.53)

### 6.4 Numerical Verification

In order to compare the nonlocal Navier-Stokes equation with the classical Navier-Stokes equation, two examples are solved and analyzed. The pressure Poission scheme is used for the numerical solver, as shown in figure (6.3). In this algorithm, the Navier-Stokes (NS) scheme is first used to predict the velocity field based on the specified boundary condition and the pressure field in previous time step. Then the pressure Poisson equation (PPE) is used to calculate the pressure field based on the specified boundary condition and the predicted velocity field. Finally, the NS scheme is used again to correct the velocity field. The PPE scheme is applied such that calculated pressure field and velocity field satisfy the Navier-Stokes equation and the incompressibility condition.
Given initial condition and boundary condition

Velocity $V_n$ and pressure $P_n$ at $t_n$

Calculate the velocity $V_{n+1}'$ from NS based on $P_n$

Calculate the pressure field $P_{n+1}$ from PPE

Update the velocity $V_{n+1}$ from NS based on the pressure $P_{n+1}$

Accuracy Criterion reached? Y/N

Results

Fig. 6.3. Pressure Poisson numerical solver scheme
Example 1. Parallel flow

![Diagram](image)

Fig. 6.4. Two-dimensional parallel flow case 1

Case 1: Calculate the responses with the boundary conditions specified as

\[ \Gamma_1: \]
\[ V_x = V_0 \]  \hspace{1cm} (6.54 a)  \\
\[ V_y = 0 \]  \hspace{1cm} (6.54 b)  \\
\[ n \cdot \nabla p = 0 \]  \hspace{1cm} (6.54 c)  

\[ \Gamma_2: \]
\[ V = 0 \]  \hspace{1cm} (6.55 a)  \\
\[ p = 0 \]  \hspace{1cm} (6.55 b)  

\[ \Gamma_3 \& \Gamma_4: \]
\[ n \cdot \nabla V = 0 \]  \hspace{1cm} (6.56 a)  \\
\[ n \cdot \nabla p = 0 \]  \hspace{1cm} (6.56 b)  

where \( V \) indicates the velocity; \( V_x \) and \( V_y \) are velocities in \( x \) and \( y \) direction; \( n \) is the unit vector normal to the boundary.
The velocity field and the pressure field are calculated based on the PPE predictor-corrector algorithm. Figures (6.5) and (6.6) show the velocity and the pressure fields calculated from the local Navier-Stokes equation and the local pressure Poisson equation. Figures (6.7) and (6.8) show the velocity and the pressure fields calculated from the nonlocal Navier-Stokes equation and the nonlocal pressure Poisson equation. Figures (6.9) and (6.10) show the velocity and the pressure error fields, and it captures the difference between the local and the nonlocal responses. It can be seen that the local velocity and the pressure fields are very similar to the nonlocal velocity and the pressure fields. This suggests that the reformulated nonlocal Navier-Stokes equation and the proposed nonlocal pressure Poisson equation are valid for this simple parallel flow example. The velocity error and the pressure error are quite small at the interior region of the domain; larger errors appear at the boundary region. This relates to the definition of the boundary condition. That is, with the above-defined boundary conditions, the velocity and the pressure fields on boundary region are quite complex. Thus, the local response and the nonlocal response differ significantly.
Results:

**Fig. 6.5.** Velocity field from the classical Navier-Stokes equation

**Fig. 6.6.** Pressure field from the classical Navier-Stokes equation
Fig. 6.7. Displacement field from the nonlocal Navier-Stokes equation

Fig. 6.8. Pressure field from the nonlocal Navier-Stokes equation
Fig. 6.9. Velocity error field
Case 2: Calculate the responses with the boundary conditions specified as

**Γ**₁ & **Γ**₂:

\[ \mathbf{V} = \mathbf{0} \]  
\[ \mathbf{n} \cdot \nabla p = 0 \]  

(6.57 a)

(6.57 b)

**Γ**₃ & **Γ**₄:

\[ V_x = V_0 \]  
\[ V_y = 0 \]  

(6.58 a)

(6.58 b)

\[ \mathbf{n} \cdot \nabla p = 0 \]  

(6.58 c)

where \( \mathbf{V} \) indicates the velocity, \( V_x \) and \( V_y \) are the velocities in \( x \) and \( y \).
directions, and $\mathbf{n}$ is the unit vector normal to the boundary.

By redefining the boundary condition, the velocity and the pressure fields of the parallel flow are calculated using the same PPE predictor-corrector algorithm. Figures (6.11) and (6.12) show the velocity and the pressure fields calculated from the local Navier-Stokes equation and the local pressure Poisson equation. Figures (6.13) and (6.14) show the velocity and the pressure fields calculated from the nonlocal Navier-Stokes equation and the nonlocal pressure Poisson equation. Figures (6.15) and (6.16) show the velocity error and the pressure error fields for the local and the nonlocal formulations. The nonlocal velocity field and the local velocity field have similar trends, especially in the interior region. However, differences between the local responses and the nonlocal responses appear after the initial time steps. Errors still exist even with the same boundary conditions and the same initial conditions. This indicates that the reformulated Navier-Stokes equation and the proposed pressure Poisson equation exhibit a certain degree of the nonlocal behavior.
Results:

Fig. 6.11. Velocity field from the local Navier-Stokes equation
Fig. 6.12. Pressure field from the local Navier-Stokes equation
Fig. 6.13. Velocity field from the nonlocal Navier-Stokes equation

Fig. 6.14. Pressure field from the nonlocal Navier-Stokes equation
Fig. 6.15. Velocity error field
Example 2 Parallel flow with interior hole.
Case 1: Calculate the responses with the boundary conditions specified as

\( \Gamma_1 \) & \( \Gamma_2 \):

\[
\begin{align*}
V_x &= V_0 \tag{6.59 a} \\
V_y &= 0 \tag{6.59 b} \\
n \cdot \nabla p &= 0 \tag{6.59 c}
\end{align*}
\]

\( \Gamma_3 \) & \( \Gamma_4 \):

\[
\begin{align*}
n \cdot \nabla V &= 0 \tag{6.60 a} \\
n \cdot \nabla p &= 0 \tag{6.60 b}
\end{align*}
\]

\( \Gamma_5 \):

\[
\begin{align*}
V &= 0 \tag{6.61 a} \\
p &= 0 \tag{6.61 b}
\end{align*}
\]

where \( V \) indicates the velocity; \( V_x \) and \( V_y \) are velocities in the \( x \) and \( y \) direction; \( n \) is the unit vector normal to the boundary.

In this example, the velocity field and the pressure field are calculated based on the PPE predictor-corrector algorithm. Figures (6.18) and (6.19) show the velocity and the pressure fields calculated from the local Navier-Stokes equation and the local pressure Poisson equation. Figures (6.20) and (6.21) show the velocity and the pressure fields calculated from the nonlocal Navier-Stokes equation and the nonlocal pressure Poisson equation. The variable horizon technique is applied to calculate the nonlocal velocity field, and the nonlocal pressure field to reduce the influence of the boundary conditions.
Results:

Fig. 6.18. Velocity from the classical Navier-Stokes equation
Fig. 6.19.  Pressure field from the classical Navier-Stokes equation

(a) Streamline
(b) Zoom in velocity field

Fig. 6.20. Velocity field from the nonlocal Navier-Stokes equation

Fig. 6.21. Pressure field from the nonlocal Navier-Stokes equation
6.5 Synopsis

In this chapter the nonlocal Navier-Stokes equation has been reformulated by applying Newton’s second law to the fluid motion. For incompressible fluids, the nonlocal incompressibility condition has been derived from the conservation of mass in a nonlocal sense. The nonlocal pressure Poisson equation has also been derived in a similar way as the local pressure Poisson equation. It has been shown that if the nonlocal parameter reduces to zero, the reformulated Navier-Stokes equation reduces to the classical Navier-Stokes equation. Fluid mechanics examples using the nonlocal Navier-Stokes equations and the pressure Poisson equation have been presented to elucidate the differences of the two approaches.

One of the advantages of the nonlocal Navier-Stokes equation over the local Navier-Stokes equation is the fact that it does not require the continuum assumption for the fluids. Further research regarding the nonlocal properties of fluids may be required. A strict mathematical derivation of the nonlocal pressure Poisson equation may also require further investigation. Although the nonlocal pressure Poisson equation algorithm can treat simple fluid flows, this algorithm can become unstable for complex fluid flow. Thus, a more stable and fast converging algorithm may be required.
Chapter 7

Conclusion Remarks

Chapter 1 has provided a conspectus perspective of the thesis. The outline of the thesis has described the seven chapters related to peridynamics and its applications.

Chapter 2 has reviewed pertinent mathematical backgrounds, including the local elasticity theory, the bond based peridynamics, the state based peridynamics, and the Navier-Stokes equation. The peridynamics is a nonlocal generalization of the classical elasticity theory; it can also be regarded as a continuum form of the molecular dynamics. Researches have shown that peridynamics is consistent with the local elasticity theory, and when the nonlocal parameter tends to zero, peridynamics reduces to the classical elasticity theory.

Chapter 3 has proposed new analytical solutions for the vibration of a bar and the vibration of a beam via fixed horizon peridynamics. It has been shown that the peridynamics dispersive relation is nonlinear. The nonlinearity of the dispersive relation depends on the value of the nonlocal parameter. When the nonlocal parameter reduces to zero, the peridynamics dispersive relation reduces to the linear local elastic dispersive relation. This dispersive consistency has been verified by numerical modeling. Both homogeneous and nonhomogeneous solutions of the peridynamic bar have been derived analytically. The corresponding mode shapes have been calculated by solving a high order differential equation. The order of the differential equation depends on the value of the nonlocal parameter. This has also been numerically verified. A new nonlocal beam theory has been proposed based on peridynamics. The derived nonlocal bending equation is a generalization of the
Euler Bernoulli beam equation. It reduces to the Euler Bernoulli beam equation when the horizon reduces to zero. Numerically, the nonlocal beam equations have shown the same responses as the Euler Bernoulli beam equations responses, when horizon value equals the mesh size. Analytical solutions to the nonlocal beam equation have been derived by a Taylor’s series expansion. These proposed beam equations are completely nonlocal.

The nonlocal boundary conditions are critical when an accurate peridynamics response on the entire region is required. The traditional method of dealing with responses on the boundary involves adding pseudo-nodes outside the physical boundary; this method works when the displacement on the boundary region is small and mild. However, problems appear when the responses on the boundary region are complex. Therefore, an iterative interpolation approach has been introduced to adaptively predict the responses of the pseudo-layer, and the stiffness on boundary region is better approximated by the iterative scheme. The iterative interpolation approach improves the accuracy of response on the boundary region. However, this usually requires a large number of iterations and the entire displacement field must be recalculated at each iteration. This is computationally expensive since the ‘convergence rate’ is usually quite slow. It can not even guarantee the convergence of the iterative interpolation approach since the ‘ghost force’ can not be eliminated.

Chapter 4 has proposed the variable horizon approach to calculate responses on boundary region without introducing an additional pseudo-layer. For node on the boundary region, the horizon is reduced depending on its distance from the boundary. The horizon is assigned to be zero for nodes located on the boundary. The
variable horizon approach has been applied to reduce the error in the static deformation. The effectiveness of the variable horizon approach has been established analytically and numerically. The nonlocal beam equation derived in the previous chapter has been solved via the variable horizon approach. The reduced nonlocal behavior has been captured on the boundary region with the reduced horizon value, while the full nonlocal behavior has been noted on the interior region with the full horizon value. Examples of beam deformations have been solved via the variable horizon approach, including a clamped-clamped cantilever beam, a simply supported beam and a cantilever beam. The deformation of a beam with initial crack, and the deformation of a beam with multiple cross sections have been solved in a compact way. Although the variable horizon approach cannot eliminate entirely the ‘ghost force’ effect, it provides a good approximation on the boundary region even when the responses are complex.

The variable horizon approach does not satisfy the linear admissibility condition or Newton’s third law. Therefore, the spurious effect appears especially for dynamic responses. In the last section of Chapter 4, the peridynamics bond force has been reformulated so that the linear admissibility condition is satisfied. This reformulated peridynamics governing equation allows responses with an arbitrary horizon field, and can be reduced to the bond based peridynamics when the horizon is constant. This reformulated governing equation is identical to the dual-horizon peridynamics governing equation, which has previously been proposed by (Ren, et. al., 2015). The variable horizon approach works for the dual horizon peridynamics especially for dynamics responses and peridynamic wave dispersion. However, the ‘ghost force’ effect can not be completely eliminated.
Note that the drilling is an extremely complex process. Researchers and engineers have tried to understand the factors related to the rate of penetrations since early 1950s. Although numerous models have been built, few of them involve the cracking mechanism of rock. Clearly the properties of the rock are hard to describe, and it is a highly heterogeneous material and many initial cracks randomly distributed. Also the bit-rock interaction is a highly complex process. It is too computationally costly to model the drilling process by using local damage models. Peridynamics, which has been proposed by Silling (2000), possesses a great advantage when modeling problems involving cracks, especially for heterogeneous materials or materials with initial cracks.

Chapter 5 has proposed an adaptive model to simulate the bit-rock interaction based on peridynamics. The heterogeneous nature of the rock formation has been incorporated in the model by modifying the peridynamic material parameters. The initial cracks of rock have been distributed randomly over the domain of interest. Since the drilling process is a highly complex process, the traditional damage criterion becomes inadequate. Thus, a new bond crack criterion and a new local damage criterion have been proposed. A penetration criterion has also been introduced to capture the damage of a particular formation layer. The materials properties, the initial crack distribution, the bond crack criterion, the local damage criterion, and the penetration criterion must be defined based on different kinds of the formations and the drill bits. This proposed bit-rock interaction model has yielded the deformation of the rock within the bit-rock contact region. The calculated region progresses adaptively with the penetration of drill bit. Thus, the
The proposed model is suitable for the large penetration depth modeling. Deformation and crack propagation of the rock have been simulated by this model. The borehole damage after penetration has been generated automatically. Influences of different drilling parameters on the rate of penetration have been analyzed. The higher weight on bit and rotary speed induce the higher penetration rates. Axial vibration improves rate of penetration. However, there exists an optimized vibration amplitude frequency combination, which provides the maximum rate of penetration.

Chapter 6 proposed a new nonlocal Navier-Stokes equation. Note that the local Navier-Stokes equation is an important equation describing fluid behavior. It was built from applying Newton's second law to fluid motion. For incompressible fluids, the incompressibility condition has been built from conservation of mass. Several numerical techniques can be used to solve local Navier-Stokes equation. The local Navier-Stokes equation was built based on the continuum assumption, where the fluid must 'fill in' the interested domain, and the velocity can not change abruptly. However, the real fluids behavior may go beyond the scope of what local Navier-Stokes equation can describe. Therefore, the nonlocal Navier-Stokes equation has been derived from the Newton's second law to the fluid motion in nonlocal sense. The nonlocal pressure Poisson equation has also been derived in a similar way as the local pressure Poisson equation. It has been shown that with the nonlocal parameter reduces to zero, every term of the nonlocal Navier-Stokes equation reduces to terms of the classical Navier-Stokes equation. Numerical examples based on the nonlocal Navier-Stokes equations and the pressure Poisson
equation was proposed and compared with local examples.

The nonlocal Navier-Stokes equation has been verified matching the local Navier-Stokes equation for the laminar flow. However, a more efficient and robust algorithm may be required to describe the turbulent flow with large Reynolds number. In future research, the developed numerical algorithm can perhaps be improved by using nonlocal pressure Poisson equation. Without the continuum assumption, the reformulated nonlocal Navier-Stokes is expected to capture some fluids mechanics problems exhibiting discontinuity.
References


Olaf Weckner. Rohan Abeyaratne. 2005 The Effect of Long-range Forces on the


Q. V. Le, W. K. Chan, J. Schwartz, A Two-dimensional Ordinary, State-based


Gilles Lubineau, Yan Azdoud, Fei Han, Christian Rey, Abe Askari, 2012. A morphing strategy to couple nonlocal to local continuum mechanics. JMPS, 60, 1088-1102.


Ugo Galvanetto, Teo Mudric, Arman Shojaei, Mirco Zaccariotto, 2016. An effective way to couple FEM meshes and Peridynamics grids for the solution of static equilibrium problems, Mechanics Research Communications. 76. 41-47.


C. M. Wang, Y. Y. Zhang and etc, 2007, Vibration of Nonlocal Timoshenko Beams. Nanotechnology 18 105401 (9pp)


Mario Di Paola, Giuseppe Failla and etc, 2014, Mechanically Based Nonlocal Euler Bernoulli Beam Model, J. Nanomech. Micromech., 2014.4


W. J. Bielstein and George E. Cannon, 1954, Factors Affecting the Rate of Penetration of Rock Bits, Drilling and Production Practice, pp. 61-78, API


R. C. Pessier, M. J. Fear, 1992, Quantifying Common Drilling Problems with Mechanical Specific Energy and a Bit- Specific Coefficient of Sliding Friction, SPE-24584-MS, SPE Annual Technical Conference and Exhibition, 4-7 October, Washington, D. C.


