AN EXPOSITION ON CONTINUED FRACTIONS

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Introduction. This paper, which is of an expository nature, is divided into three parts. In the first part a continued fraction is defined and some of the fundamental formulas used in the study thereof are derived. In the second section important classical theorems on convergence are established, while the final chapter is devoted to the study of recently developed convergence criteria.
CHAPTER I

FUNDAMENTAL FORMULAS

1. **Notation.** A finite continued fraction is an expression of the form

\[
\frac{b_0 + a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots + \frac{a_{n-1}}{b_{n-1} + a_n}}}}
\]

where the symbols \(a_n\) and \(b_n\) may be regarded at first as independent variables. On setting \(n = 0, 1, 2\) successively, (1.1) becomes

\[
\begin{align*}
\frac{b_0}{1} & = \frac{b_0}{1} \\
\frac{b_0 + a_1}{b_1} & = \frac{b_0 \cdot b_1 + a_1}{b_1} \\
\frac{b_0 + a_1}{b_1 + \frac{a_2}{b_2}} & = \frac{b_0 \cdot b_1 \cdot b_2 + a_1 \cdot b_1 \cdot a_2}{b_2 \cdot b_1 \cdot a_2}
\end{align*}
\]

from which we see the continued fraction may be expressed as a quotient of two polynomials in this set of variables.

Before discussing the formation of these polynomials we will introduce a more condensed notation, namely
The $a_n$ and $b_n$ are called the elements of the continued fraction. The fraction $a_n/b_n$ is called the $n$th term where $a_n$ and $b_n$ are the $n$th partial numerator and $n$th partial denominator, respectively. $b_0$ is called the zeroth term, so that we say the continued fraction has $n + 1$ terms.

2. The transformation into an ordinary fraction.

Since a continued fraction may be expressed as the quotient of two polynomials, we may write

$$\frac{A_n}{B_n} = \frac{b_0}{a_0} + \frac{a_1}{a_1} + \frac{a_2}{a_2} + \cdots + \frac{a_n}{a_n} = \frac{A_n}{B_n}.$$

In order to determine the form of the polynomials $A_n$ and $B_n$ we will let $n = 0, 1, 2, \ldots$ successively, from which we see

$$\frac{A_0}{b_0} = \frac{b_0}{b_0},$$

$$\frac{A_1}{B_1} = \frac{b_0 + a_0}{b_1} = \frac{b_0 b_1 + a_0}{b_1}.$$

Hence we may set

$$(1.4) \quad A_0 = b_0 \quad B_0 = 1$$

$$A_1 = b_0 b_1 + a_1 \quad B_1 = b_1.$$
Now $A_2/B_2$ may be obtained by replacing $b_1$ by $b_1 + a_2/b_2$ in $A_1/B_1$ so that
\[
\frac{A_2}{B_2} = \frac{b_2(b_1 + \frac{a_2}{b_2}) + a_1}{b_1 + \frac{a_2}{b_2}}
\]
\[= \frac{b_2(b_1b_1 + a_1) + a_2b_0}{b_1b_2 + a_2}.
\]

As above we may let
\[
A_2 = b_2(b_1b_1 + a_1) + a_2b_0,
\]
\[B_2 = b_1b_2 + a_2,
\]
from which it follows because of (1.4) that also
\[(1.5)\]
\[A_2 = b_2A_1 + a_2A_0,
\]
\[B_2 = b_2B_1 + a_2B_0.
\]

We will now show by induction that in general
\[(1.6)\]
\[A_n = b_nA_{n-1} + a_nA_{n-2}\quad (n \geq 2)
\]
\[B_n = b_nB_{n-1} + a_nB_{n-2}.
\]

It will be sufficient to show that if the theorem is true for all values $\leq n$, it is true for $n + 1$.

From (1.6) we see that
\[\frac{A_n}{B_n} = \frac{b_nA_{n-1} + a_nA_{n-2}}{b_nB_{n-1} + a_nB_{n-2}}.
\]
Clearly $A_{n+1}/B_{n+1}$ may be obtained from $A_n/B_n$ by substituting $b_n + a_{n+1}/b_{n+1}$ for $b_n$, so that
\[
\frac{A_{n+1}}{B_{n+1}} = \frac{(a_n + a_{n+1})A_{n-1} + a_n A_{n-2}}{(a_n + a_{n+1})B_{n-1} + a_n B_{n-2}}
\]
\[
= \frac{b_{n+1}(a_n A_{n-1} + a_n A_{n-2}) + a_{n+1}A_{n-1}}{b_{n+1}(a_n B_{n-1} + a_n B_{n-2}) + a_{n+1}B_{n-1}}
\]
\[
= \frac{b_{n+1}A_{n} + a_{n+1}A_{n-1}}{b_{n+1}B_{n} + a_{n+1}B_{n-1}}
\]

Hence on setting numerator and denominator equal on both sides we have

\[
A_{n+1} = b_{n+1}A_{n} + a_{n+1}A_{n-1}
\]
\[
B_{n+1} = b_{n+1}B_{n} + a_{n+1}B_{n-1}
\]

Now since we know that formulas (1.6) hold for \( n = 2 \), the theorem is established.

In order to show more clearly the dependence of the function \( A_n \) on its elements, we will use the functional notation

\[
A_n = K\left(\begin{array}{c}
a_1, a_2, \ldots, a_n \\
b_0, b_1, \ldots, b_n
\end{array}\right)
\]

(1.7)

which was introduced by Th. Luir. Then also

\[
A_{n-1} = K\left(\begin{array}{c}
a_1, a_2, \ldots, a_{n-1} \\
b_0, b_1, \ldots, b_{n-1}
\end{array}\right)
\]
If all the indices are increased by one, the last expression becomes

$$B_n = k \left( \frac{a_2}{b_1}, \frac{a_3}{b_2}, \ldots, \frac{a_n}{b_{n-1}} \right).$$

Finally it should be noted, because of its frequent application, that

$$\frac{l_n + a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_{n-1}}{b_{n-1}} + \frac{a_n}{b_n} = \frac{A_{n-1} \xi_m + a_n A_{n-2}}{B_{n-1} \xi_m + a_n B_{n-2}}.$$  

This result was obtained by substituting $\xi_m$ for $l_n$ in the continued fraction

$$\frac{l_n + a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_{n-1}}{b_{n-1}} + \frac{a_n}{b_n} = \frac{A_n}{B_n}$$

$$= \frac{A_{n-1} a_n + a_n A_{n-2}}{b_n B_{n-1} + a_n B_{n-2}}.$$  

The functions

$$\frac{A_0}{B_0}, \frac{A_1}{B_1}, \frac{A_2}{B_2}, \ldots, \frac{A_n}{B_n}$$

are called the approximants of the continued fraction. The last approximant $A_n/B_n$ is equal to the continued fraction itself while the approximant $A_k/B_k$ of the $k$th order is used to determine the value of the $(k + 1)$-term continued fraction obtained by leaving off the last $n - k$ terms of the continued fraction (1.9). The polynomials $A_n$ and $B_n$ are called the numerator and denominator, respectively, of the $n$th approximant.
We will make a timely observation by noting here that, on using the Muir Symbols, the numerator and denominator of the nth approximant of the continued fraction

\[ \frac{a_n}{b_n} + \frac{a_{n+1}}{b_{n+1}} + \ldots + \frac{a_{n+k}}{b_{n+k}} \]

may be expressed as

\[ A_{n, \lambda} = K \left( \frac{a_{n+1}}{b_{n+1}}, \frac{a_{n+2}}{b_{n+2}}, \ldots, \frac{a_{n+k}}{b_{n+k}} \right) \]

(1.10)

\[ B_{n, \lambda} = K \left( \frac{a_{n+2}}{b_{n+2}}, \frac{a_{n+3}}{b_{n+3}}, \ldots, \frac{a_{n+k}}{b_{n+k}} \right) \]

From this we see that

(1.11) \[ B_{n, \lambda} = A_{n-1, \lambda + 1} \]

3. Independent expression of approximate numerator and denominator. Until now the numerators and denominators \( A_n, B_n \) of successive approximants have been calculated by means of the recursion formulas (1.6), but now we will derive an independent expression for these functions. To this end we will consider

(1.12) \[ P_n = \frac{b_0}{b_1} \ldots \frac{b_{n-2}}{b_{n-1}} \left(1 + \frac{a_1}{b_1} \frac{a_2}{b_2} \right) \ldots \left(1 + \frac{a_{n-2}}{b_{n-2}} \frac{a_{n-1}}{b_{n-1}} \right) \]

If this expression is multiplied out, there will be in the ensuing aggregate certain terms which contain
b's in the denominator and others whose denominators have the value one. If we denote by \( P_n' \) all the terms of the latter form and by \( P_n'' \) those of the former, then

\[ P_n = P_n' + P_n'' \]

Now it can be easily verified that

\[ P_n' = l_n = A_n \]
\[ P_n'' = l_n l_{n-1} + a_n = A_n \cdot \]

If, therefore, we can prove the recursion formula

\[ P_n' = l_n P_{n-1} + a_n P_{n-2} \quad (n \geq 2) \]

it will follow that in general \( P_n' = A_n \), since \( A_n \) satisfies the same formula.

From the definition of \( P_n \) it follows that

\[ P_n = l_n \left(1 + \frac{a_n}{l_n l_{n-1}}\right) P_{n-1} \]
\[ = l_n P_{n-1} + \frac{a_n}{l_{n-1}} P_{n-2} \]

On substituting \( n-1 \) for \( n \), we obtain

\[ P_{n-1} = l_{n-1} P_{n-2} + \frac{a_{n-1}}{l_{n-2}} \]

so that if \( P_{n-1} \) in the last term is replaced by this expression, we obtain

\[ P_n = l_n P_{n-1} + a_n P_{n-2} + \frac{a_n a_{n-1}}{l_{n-1} l_{n-2}} P_{n-3} \]

Since \( P_{n-2} \) is independent of \( b_{n-1} \) every term of \( l_n \frac{a_n}{l_{n-1} l_{n-2}} P_{n-2} \) will contain b's in the denominator.
Likewise every term of $b_n p_n - 1$ and $a_n p_n - 2$ will contain $b$'s in the denominator. Hence

$$P'_n = l^n P_{n-1} + a_m P_{n-2}.$$ 

Therefore $A_n = P_n$ and we may write

\[(1.13) \quad A_n = l_0 l_1 \cdots l_n \left(1 + \sum_{i=0}^{m-1} \frac{a_{i+1}}{l_i l_{i+1}} + \sum_{i=k}^{m-2} \frac{a_{i+1}}{l_i l_{i+1}} \frac{a_{k+2}}{l_{k+1} l_{k+2}} \right. \]

\[\left. + \sum_{i=k+2} a_{i+1} \frac{a_{k+2}}{l_{k+1} l_{k+2}} \frac{a_{k+3}}{l_{k+2} l_{k+3}} + \cdots \right) .\]

Similarly,

\[(1.14) \quad B_n = l_0 l_2 \cdots l_n \left(1 + \sum_{i=0}^{m-1} \frac{a_{i+1}}{l_i l_{i+1}} + \sum_{i=k}^{m-2} \frac{a_{i+1}}{l_i l_{i+1}} \frac{a_{k+2}}{l_{k+1} l_{k+2}} \right. \]

\[\left. + \sum_{i=k+2} a_{i+1} \frac{a_{k+2}}{l_{k+1} l_{k+2}} \frac{a_{k+3}}{l_{k+2} l_{k+3}} + \cdots \right) .\]

4. The fundamental formulas. When $x_0, x_1, x_2, \ldots$ are any variables for which the equations

$$\chi_0 = l_0 x_1 + a_1 x_2,$$

$$\chi_1 = l_0 x_2 + a_2 x_3,$$

$$\chi_n = l_n x_{n-1} + a_{n+1} x_{n+2} ,$$

hold, any two of these variables may be considered as independent while each of the remaining variables is a different linear homogeneous function of them.

We will prove, in particular, that, if $n$ is any index, $x_0$ and $x_1$ will be such a function of $x_n$ and $x_{n+1}$. To this end we write the first $n$ equations in the form
Thus we obtain the relationship

\[
\frac{x_0}{x_1} = \frac{b_0 + a_1}{x_1 + \frac{a_2}{x_2 + \frac{a_3}{x_3 + \cdots + \frac{a_{n-1}}{x_{n-1} + \frac{a_n}{x_{n+1}}}}}}.
\]

which exists between this system of equations and the continued fraction. Hence by formula (1.9) we may write

\[
\frac{x_0}{x_1} = \frac{A_{n-1} x_n + A_n x_{n+1}}{B_{n-1} x_n + A_n B_{n-2} x_{n+1}} = \frac{A_{n-1} x_n + a_n A_{n-2} x_{n+1}}{B_{n-1} x_n + A_n B_{n-2} x_{n+1}}.
\]

By a simple induction it may be easily demonstrated that the equality applies not only to the fractions, but to their numerators and denominators as well. Hence

\[
\begin{align*}
\chi_0 &= A_{n-1} x_n + A_n A_{n-2} x_{n+1} \\
\chi_1 &= B_{n-1} x_n + A_n B_{n-2} x_{n+1}.
\end{align*}
\]

(1.16)

When now the indices of all the a, b, x are increased by a number \( \lambda \) we obtain a system of equations which begin \( \lambda \) lines later than the equations in system (1.15). If we denote by \( A_{m, \lambda} \), \( B_{n, \lambda} \) the approximant numerator and approximant denominator, respectively, of the continued fraction.
we derive the equations

\[ x_\lambda = A_{n-1,\lambda} x_{n+\lambda} + a_{n+\lambda} A_{n-2,\lambda} x_{n+\lambda+1} \]
\[ x_{\lambda+1} = B_{n-1,\lambda} x_{n+\lambda} + a_{n+\lambda} B_{n-2,\lambda} x_{n+\lambda+1} \]

in the same way as we did the equations (1.16).

Now on rewriting equations (1.16) and setting \( n = \lambda \)
and replacing \( x_\lambda \) and \( x_{\lambda+1} \) by the expressions above,
we obtain

\[ x_0 = (A_{\lambda-1} A_{\lambda-1,\lambda} + a_{\lambda} A_{\lambda-2} B_{\lambda-1,\lambda}) x_{n+\lambda} + a_{n+\lambda} (A_{\lambda-1} A_{\lambda-2,\lambda} + a_{\lambda} A_{\lambda-2} B_{\lambda-2,\lambda}) x_{n+1} \]
\[ x_1 = (B_{\lambda-1} A_{\lambda-1,\lambda} + a_{\lambda} B_{\lambda-2} B_{\lambda-1,\lambda}) x_{n+\lambda} + a_{n+\lambda} (B_{\lambda-1} A_{\lambda-2,\lambda} + a_{\lambda} B_{\lambda-2} B_{\lambda-2,\lambda}) x_{n+1} \]

Also on directly substituting \( n+\lambda \) for \( \lambda \) in
equations (1.16) we find that

\[ x_0 = A_{n+\lambda-1} x_{n+\lambda} + a_{n+\lambda} A_{n+\lambda-2} x_{n+\lambda-1} \]
\[ x_1 = B_{n+\lambda-1} x_{n+\lambda} + a_{n+\lambda} B_{n+\lambda-2} x_{n+\lambda-1} \]

from which we conclude that

\[ A_{n+\lambda-1} = A_{\lambda-1} A_{\lambda-1,\lambda} + a_{\lambda} A_{\lambda-2} B_{\lambda-1,\lambda} \]
\[ B_{n+\lambda-1} = B_{\lambda-1} A_{\lambda-1,\lambda} + a_{\lambda} B_{\lambda-2} B_{\lambda-1,\lambda} \]  

(1.17)

Using the Muir Symbols for \( A_{n,\lambda} \) and \( B_{n,\lambda} \) we see that

\[ B_{n,\lambda} = A_{n-1,\lambda+1} \]
so that in formulas (1.17) the B's may be replaced by the A's. Hence we obtain

\[ A_{n+\lambda-1} = A_{\lambda-1} A_{m-1,\lambda} + a_\lambda A_{\lambda-2} A_{m-2,\lambda+1} \]
\[ A_{n+\lambda-2,1} = A_{\lambda-2,1} A_{m-1,\lambda} + a_\lambda A_{\lambda-3,1} A_{m-2,\lambda+1} \]

Other formulas of fundamental importance may be obtained from these by assigning various values to \(n\) and \(m+\lambda\). In particular, when \(n = 1\), these become the recursion formulas

\[ A_\lambda = \sum A_{\lambda-1} + a_\lambda A_{\lambda-2} \]
\[ B_\lambda = \sum B_{\lambda-1} + a_\lambda B_{\lambda-2} \]

If \(\lambda\) is set equal to 1, we obtain

\[ A_m = \sum A_{m-1,1} + a_1 B_{m-1,1} \]
\[ B_n = A_{m-1,1} \]

(1.18)

so that if in the first of these equations we increase all the indices by a number \(\lambda\) and substitute B's for A's as indicated in (1.11), it becomes

\[ B_{n+1,\lambda-1} = \sum B_{n,\lambda} + a_\lambda B_{n-1,\lambda+1} \]

(1.19)

Then if \(n + 1\) and \(\lambda - 1\) are replaced by \(n\) and \(\lambda\) and the B's replaced by A's, we obtain the corresponding equation

\[ A_{n+1,\lambda-1} = \sum A_{n,\lambda} + a_\lambda A_{n-1,\lambda+1} \]

(1.20)
5. Consequence of the fundamental formulas. We will now obtain an expression for the difference of successive approximants. Using the recursions relations we find that

\[ A_\lambda B_{\lambda-1} - A_{\lambda-1} B_\lambda = (l_\lambda A_{\lambda-1} + a_\lambda A_{\lambda-2}) B_{\lambda-1} - A_{\lambda-1} (l_\lambda B_{\lambda-1} + a_\lambda B_{\lambda-2}) \]

\[ = -a_\lambda \left( A_{\lambda-1} B_{\lambda-2} - A_{\lambda-2} B_{\lambda-1} \right). \]

On repeating this process successively we obtain finally

\[ A_\lambda B_{\lambda-1} - A_{\lambda-1} B_\lambda = (-1)^\lambda a_\lambda a_{\lambda-1} \cdots a_2 a_1 \left( A_\lambda B_\lambda - A_{\lambda-1} B_{\lambda-1} \right), \]

so that when \( A_0, B_{-1}, A_{-1}, B_0 \) are replaced by their respective values we find that

\[(1.21) \quad A_\lambda B_{\lambda-1} - A_{\lambda-1} B_\lambda = (-1)^\lambda a_\lambda a_{\lambda-1} \cdots a_2 a_1. \]

From this it follows that

\[ \frac{A_\lambda}{B_\lambda} - \frac{A_{\lambda-1}}{B_{\lambda-1}} = \frac{(-1)^\lambda a_\lambda a_{\lambda-1} \cdots a_2 a_1}{B_{\lambda-1} B_\lambda}. \]

In order to find an expression for the difference between any two approximants, it should be noted that on using formulas (1.17)

\[ A_{n+\lambda-1} B_{\lambda-1} - A_{\lambda-1} B_{n+\lambda-1} = (A_{\lambda-1} B_{n-1,\lambda} + a_\lambda A_{\lambda-2} B_{n-1,\lambda}) B_{\lambda-1} \]

\[ - A_{\lambda-1} \left( B_{\lambda-1} A_{n-1,\lambda} + a_\lambda B_{\lambda-2} B_{n-1,\lambda} \right) \]

\[ = -a_\lambda B_{n-1,\lambda} \left( A_{\lambda-1} B_{\lambda-2} - A_{\lambda-2} B_{\lambda-1} \right), \]

On applying the recursion relations to the right-hand
member we find that

\[(1.22) \quad A_{n+\lambda-1} B_{\lambda-1} - A_{\lambda-1} B_{n+\lambda-1} = (-1)^{\lambda-1} a_{\lambda} a_{\lambda-1} \ldots a_{1} B_{\lambda-1}, \lambda \]

so that

\[(1.23) \quad \frac{A_{n+\lambda-1}}{B_{n+\lambda-1}} - \frac{A_{\lambda-1}}{B_{\lambda-1}} = \frac{(-1)^{\lambda-1} a_{\lambda} a_{\lambda-1} \ldots a_{1} B_{\lambda-1}}{B_{n+\lambda-1} B_{\lambda-1}}.

In particular if \( n = 2 \), equation (1.23) becomes

\[(1.24) \quad \frac{A_{\lambda+1}}{B_{\lambda+1}} - \frac{A_{\lambda-1}}{B_{\lambda-1}} = (-1)^{\lambda} a_{\lambda} a_{\lambda-1} \ldots a_{1} \frac{B_{\lambda+1}}{B_{\lambda+1} B_{\lambda-1}}.

so that now we have an expression for the difference of successive even or odd approximants.

6. Convergence of numerical continued fractions.

Until now the elements of a continued fraction have been considered as independent variables and the continued fraction itself was said to be a function of these variables. In particular a \((n + 1)\)-term continued fraction was said to be a function of \(2n + 1\) variables and was designated by \(A_{n}/B_{n}\). If now we allow these variables to take on numerical values, the value of the function will be perfectly defined except when \(B_{n} = 0\), in which case we say the continued fraction has no numerical value. However, if a partial denominator is zero, the continued fraction may be
defined. For example, the continued fraction
\[ l - \frac{2}{l + \frac{1}{o}} \]
where the last term \( \frac{1}{o} \) is meaningless, has the value one. We see also, on using the recursion relations, that
\[ o + \frac{a_1}{l_1 + \ldots + \frac{a_{n-1}}{l_{n-1} + \frac{a_n}{o}}} = \frac{A_{n-2}}{B_{n-2}} , \]
promised that \( a_n \) and \( B_{n-2} \) are different from zero.

In general, the continued fraction
\[ l + \frac{a_1}{l_1 + \ldots + \frac{a_{n-1}}{l_{n-1} + \frac{a_n}{l_{n-2} + \ldots + \frac{a_{n+m-1}}{l_{n+m-1}}}}} \]
may have a definite value when the continued fraction
\[ l_\lambda + \frac{a_{\lambda+1}}{l_{\lambda+1} + \ldots + \frac{a_{\lambda+m-1}}{l_{\lambda+m-1}}} \]
has the value zero. If this is the case,
\[ \frac{A_{n-1, \lambda}}{B_{n-1, \lambda}} = 0 \]
so that
\[ A_{n-1, \lambda} = 0 , \quad B_{n-1, \lambda} \neq 0 . \]

From the fundamental formulas we see that
\[ \frac{A_{n+1, \lambda+1}}{B_{n+1, \lambda+1}} = \frac{A_{\lambda+1} B_{n-1, \lambda} + a_\lambda A_{\lambda-2} B_{n-1, \lambda}}{B_{\lambda} A_{n-1, \lambda} + a_\lambda B_{\lambda-2} B_{n-1, \lambda}} \]
\[ = \frac{a_\lambda A_{\lambda-2}}{a_\lambda B_{\lambda-2}} . \]
From this it follows that the continued fraction is equal to \( \frac{A_{\lambda-2}}{B_{\lambda-2}} \) provided that \( A_4 \) and \( B_{\lambda-1} \) are both different from zero. If this last condition is not satisfied, no value is assigned to the continued fraction.

A continued fraction

\[
\frac{A_0}{a_0 + \frac{A_1}{B_1 + \frac{A_2}{B_2 + \cdots}}}
\]

may have an infinite number of terms, in which case it is said to be non-terminating. If the infinite sequence of its approximants

\[
\frac{A_0}{B_0}, \frac{A_1}{B_1}, \frac{A_2}{B_2}, \ldots
\]

approaches a limit, that is,

(1.25) \[
\lim_{n \to \infty} \frac{A_n}{B_n} = \xi,
\]

the continued fraction is said to be convergent and \( \xi \) is said to be its value. It should be noted, however, that a convergent continued fraction may have among its approximants a finite number of meaningless terms.

If the limit (1.25) does not exist, the continued fraction is said to be divergent. We say that it is "unessentially divergent" if

\[
\lim_{n \to \infty} \frac{B_n}{A_n} = 0,
\]

otherwise "essentially divergent". It should be noted
that if a continued fraction is unessentially divergent, it may happen that \( B_n = 0 \) for infinitely many values of the index \( n \).

**Theorem 1.1.** When two of the three equalities

\[
A. \quad \xi_0 = \frac{a_1}{\xi_1 + \frac{a_2}{\xi_2 + \ldots + \frac{a_{n-1}}{\xi_{n-1} + \frac{a_n}{\xi_n}}}}
\]

\[
B. \quad \xi_\lambda = \frac{a_{n\lambda+1}}{\xi_{n\lambda+2} + \frac{a_{n\lambda+2}}{\xi_{n\lambda+3} + \ldots}}
\]

\[
C. \quad \xi_0 = \frac{a_1}{\xi_1 + \frac{a_2}{\xi_2 + \ldots + \frac{a_{n-1}}{\xi_{n-1} + \frac{a_n}{\xi_n}}}}
\]

are satisfied, and when \( a_1, a_2, \ldots, a_{n\lambda} \) are different from zero, then the third holds.

It may happen that the continued fractions \( B \) and \( C \) are both non-terminating, or they may both terminate, ending with the same term. In the latter case we will suppose that \( a_{n\lambda+n-1}/b_{n\lambda+n-1} \) is the last term. Then we may consider the more convenient formulas

\[
A^{**}. \quad \xi_0 = \frac{A_{n\lambda-1} \xi_{n\lambda} + A_{n\lambda-2} a_{n\lambda}}{B_{n\lambda-1} \xi_{n\lambda} + B_{n\lambda-2} a_{n\lambda}}
\]

\[
B^{**}. \quad \xi_\lambda = \frac{A_{n\lambda-1} a_{n\lambda}}{B_{n\lambda-1} a_{n\lambda}}
\]

\[
C^{**}. \quad \xi_0 = \frac{A_{n\lambda+m-1} b_{n\lambda+m-1}}{B_{n\lambda+m-1} a_{n\lambda} + A_{n\lambda-1} a_{n\lambda} + A_{n\lambda-2} a_{n\lambda}}
\]

If \( B \) and \( C \) are non-terminating, on taking limits, the above equations become

\[
A^{***}. \quad \xi_0 = \frac{A_{n\lambda-1} \xi_{n\lambda} + A_{n\lambda-2} a_{n\lambda}}{B_{n\lambda-1} \xi_{n\lambda} + B_{n\lambda-2} a_{n\lambda}}
\]
We will now develop the proof for the non-terminating continued fraction, then on omitting the limits a proof for the finite case will be obtained.

The proof falls in three parts. First, we will assume that

$$C_{\lambda} = \lim_{n \to \infty} \frac{A_{n-1, \lambda}}{B_{n-1, \lambda}} = \lim_{n \to \infty} \frac{A_{n-1, \lambda} + a_{\lambda} A_{n-2, \lambda}}{B_{n-1, \lambda} + a_{\lambda} B_{n-2, \lambda}}$$

then if we assume also that

$$C_{\lambda} = \lim_{n \to \infty} \frac{A_{n-1, \lambda}}{B_{n-1, \lambda}}$$

it will follow that

$$C_{\lambda} = \lim_{n \to \infty} \frac{A_{n-1, \lambda} + a_{\lambda} A_{n-2, \lambda}}{B_{n-1, \lambda} + a_{\lambda} B_{n-2, \lambda}}$$

Hence C is established.

Second. Suppose that

$$C_{\lambda} = \lim_{n \to \infty} \frac{A_{n-1, \lambda} + a_{\lambda} A_{n-2, \lambda}}{B_{n-1, \lambda} + a_{\lambda} B_{n-2, \lambda}}$$
Then if also
\[ \xi_\lambda = \lim_{n \to \infty} \frac{A_{n-1} - \lambda}{B_{n-1} - \lambda}, \]

\[ B_{n-1} \neq 0 \] for \( n \) sufficiently large, so that we may divide numerator and denominator of \( \xi_\lambda \) by \( \lambda \).

\[ B_{n-1} \neq 0 \] Consequently
\[ \xi_\lambda = \lim_{n \to \infty} \frac{A_{n-1} - \lambda}{B_{n-1} - \lambda} + a_\lambda \]

and because of \( B \), \( A_{n-1} - \lambda + a_\lambda A_{n-1} - 2 \) and \( B_{n-1} - \lambda + a_\lambda B_{n-2} \)

are the limiting values of the numerator and denominator respectively. These quantities cannot both vanish, for then it would follow that \( a_\lambda (A_{n-1} - B_{n-2} - A_{n-2} - B_{n-1}) = 0 \).

But by (1.21) this expression is equal to \( \pm a_1 a_2 \ldots a_\lambda \), which, according to our hypothesis, is different from zero. Hence, the limit of the denominator cannot vanish, for if it did, it would follow, because of the finiteness of \( \xi_\lambda \), that also the limit of the numerator must vanish, which is impossible. Hence we may replace the limit of the quotient by the quotient of the limits so that
\[ \xi_\lambda = \frac{A_{n-1} - \lambda}{B_{n-1} - \lambda} + a_\lambda A_{n-1} - 2, \]

Finally suppose that \( A \) and \( C \) hold, then because of \( A \)
\[ \xi_{\lambda} (B_{\lambda-1} \xi_o - A_{\lambda-1}) = -a_{\lambda} (B_{\lambda-2} \xi_o - A_{\lambda-2}) \]

The contents in the parentheses on the left cannot vanish, for if so, since \( \xi_o \neq 0 \), it would follow that \( B_{\lambda-2} \xi_o - A_{\lambda-2} \) would vanish also. But this says that
\[ A_{\lambda-1} B_{\lambda-2} - A_{\lambda-2} B_{\lambda-1} = 0. \]

Hence by (1.21) \( a_1 a_2 \cdots a_{\lambda} = 0 \), which by our hypothesis is impossible. Therefore
\[ \xi_{\lambda} = -\frac{a_{\lambda}}{B_{\lambda-1}} \left( B_{\lambda-2} \xi_o - A_{\lambda-2} \right), \]

Then on substituting
\[ \xi_{\lambda} = -a_{\lambda} \left( B_{\lambda-2} \frac{A_{\lambda+m-1}}{B_{\lambda+m-1}} - A_{\lambda-2} \right) \]

\[ = -a_{\lambda} \left( \frac{B_{\lambda-2} A_{\lambda+m-1} - A_{\lambda-2} B_{\lambda+m-1}}{B_{\lambda-1} A_{\lambda+m-1} - A_{\lambda-1} B_{\lambda+m-1}} \right). \]

But according to (1.22) the denominator of this last fraction is equal to
\[ (1)_{\lambda-1}^n a_1 a_2 \cdots a_{\lambda} B_{m+n, \lambda}. \]

If \( \lambda \) and \( n \) are replaced by \( \lambda-1 \) and \( n+1 \), respectively,
the numerator becomes

\[( -1 )^{ \lambda - 1} a_1 a_2 \cdots a_{ \lambda - 1} B_{\lambda - 1, \lambda} = ( -1 )^{ \lambda - 2} a_1 a_2 \cdots a_{ \lambda} A_{\lambda - 1, \lambda} \]  

by (11).

The last equation may be written as

\[ \xi_{\lambda} = - a_{\lambda} \lim_{m \to \infty} \frac{ ( -1 )^{ \lambda - 2} a_1 a_2 \cdots a_{ \lambda} A_{\lambda - 1, \lambda}}{ ( -1 )^{ \lambda - 1} a_1 a_2 \cdots a_{ \lambda} B_{\lambda - 1, \lambda}} \]

\[ = \lim_{m \to \infty} \frac{A_{\lambda - 1, \lambda}}{B_{\lambda - 1, \lambda}} \]

so that B holds and the proof of the theorem is complete.
CHAPTER II

CONTINUED FRACTIONS WITH COMPLEX ELEMENTS

In this section we study the convergence of continued fractions whose elements may be real or complex. Of fundamental importance is the following theorem of Pringsheim.\(^{(1)}\)

**Theorem 2.1.** If the elements \(a_n\) and \(b_n\) of the continued fraction

\[
\frac{a_1}{d_1} + \frac{a_2}{d_2} + \frac{a_3}{d_3} + \cdots
\]

are any real or complex numbers which satisfy the inequalities

\[
|b_n| \geq |a_n| + 1 \quad (n = 1, 2, 3, \ldots),
\]

the continued fraction is convergent and its absolute value is \(\not\equiv 1\).

We recall that the recursion formula for \(B_n\) is

\[
B_n = B_n B_{n-1} + a_n B_{n-2}.
\]

Using this relation and our hypothesis we see that

\[
|B_n| \geq |B_n||B_{n-1}| - |a_n||B_{n-2}| \geq |b_n||B_{n-1}| - (|a_n|-1)|B_{n-2}|.
\]

On subtracting \(|B_{n-1}|\) from each side of this inequality we find that

1. O. Perron, "Die Lehre von den Kettenbrüchen"
1929 edition, p. 254
\[ |B_n| - |B_{n-1}| \geq (|B_n|-1)(|B_{n-1}|-|B_{n-2}|). \]

It is now easily seen that

\[ |B_n| - |B_{n-1}| = (|B_n|-1)(|B_{n-1}|-1) - (|2|-1) \]
\[ = |a_n a_{n-1} \cdots a_1|. \]

From this it is clear that the \(|B_n|\) are monotone increasing and since \(B_0 = 1\), they are all different from zero. Hence there are no meaningless approximants and we may write

\[ \frac{A_n}{B_n} = \frac{A_1}{B_1} + (\frac{A_1}{B_1} - \frac{A_0}{B_0}) + \cdots + (\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}) \]
\[ = a_1 + \frac{a_2}{B_1 B_2} + \cdots + (-1)^{n-1} \frac{a_n}{B_n B_{n-1}}. \]

It is clear from equation (2.4) that the convergence of the continued fraction implies the convergence of the series

\[ \frac{a_1}{B_1} \frac{a_2}{B_2} \cdots + (-1)^{n-1} \frac{a_n}{B_n B_{n-1}} \]

and conversely. Thus we may confine our attention to the latter.

From (2.3) we obtain the following inequality:

\[ \left| \frac{a_1 a_2 \cdots a_n}{B_{n-1} B_n} \right| \leq \frac{|B_n| - |B_{n-1}|}{|B_{n-1} B_n|} = \frac{1}{|B_{n-1}|} - \frac{1}{|B_n|}. \]

Thus we see that the series

\[ \sum_{n=1}^{\infty} \left| \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}} \right| \]
is term by term less than the convergent series
\[ \sum_{n=1}^{\infty} \left( \frac{1}{|B_{n-1}|} - \frac{1}{|B_n|} \right) \leq \frac{1}{B_0} = 1, \]
and hence it is itself convergent and its absolute value \( \leq 1. \)

From these conditions we are able to obtain other criteria for convergence, among which are the following:

**Theorem 2.2.** The continued fraction

(2.1) \[ \frac{a_1}{l_1} + \frac{a_2}{l_2} + \frac{a_3}{l_3} + \cdots \]

with partial numerators different from zero, converges when the inequalities

\[ |l_1| \geq |a_2|, \quad |l_n| \geq \sqrt{1 + |\alpha_{n+1}|} \quad (n \geq 2) \]

or the inequalities

\[ |l_1| \geq \sqrt{|a_2|}, \quad |l_n| \geq \sqrt{|l_{n-1}| + \sqrt{|\alpha_{n+1}|}} \quad (n \geq 2) \]

are satisfied, and of these not all are equalities.

To prove this theorem we will need to make use of the following lemma which we will now establish.

**Lemma.** When one of the two continued fractions

(2.5) \[ l_0 + \frac{a_1}{l_1} + \frac{a_2}{l_2} + \frac{a_3}{l_3} + \cdots \]

(2.6) \[ c_0 l_0 + \frac{c_1 c_0 a_1}{l_1} + \frac{c_1 c_2 a_2}{l_2} + \cdots \]
where the $C_n$ are all different from zero, converges then so does the other, and the value of the second is $C_0$ times the value of the first.

Let $A_n$ and $B_n$ be the numerator and denominator, respectively, of the $n$th approximant of the first continued fraction and let $C_n$ and $D_n$ be the same for the second continued fraction. We will now prove by induction that

$$C_n = c_0 c_1 \ldots c_n A_n$$

(2.7)

$$D_n = c_1 c_2 \ldots c_n B_n.$$ 

It may easily be verified that (2.7) holds for $n = 0, 1, 2$. We will assume that it is valid for $n = k$ and prove that it is true for $n = k + 1$.

From the recursion relations (1.6) we know that

$$C_{k+1} = C_k + C_{k+1} A_k,$$

$$D_{k+1} = D_k + D_{k+1} B_k.$$

Because of our assumption

$$C_{k+1} = C_k (c_0 c_1 \ldots c_{k-1} A_k + a_{k+1} (c_0 c_1 \ldots c_{k+1}) A_{k-1}$$

$$= (c_0 c_1 \ldots c_{k+1}) (A_k + a_{k+1} A_{k-1})$$

$$= c_0 c_1 \ldots c_{k+1} A_{k+1}.$$

Similarly

$$D_{k+1} = c_1 \ldots c_{k+1} B_{k+1}.$$
Hence
\[ \frac{C_{k+1}}{D_{k+1}} = c_0 \frac{A_{k+1}}{B_{k+1}}. \]

To continue with the proof of the theorem we recall that, according to Theorem (1.1), if the continued fraction
\[ \ell_1 + \frac{a_2}{\ell_2} + \frac{a_3}{\ell_3} + \ldots \]
converges and has a value different from zero, the given continued fraction (2.1) will converge. By Theorem (2.1) this will be the case if
\[ |\ell_1| \geq 1 \]
\[ |\ell_n| \geq |a_n| + 1 \quad (n = 2, 3, 4, \ldots) \]
and inequality holds for at least one value of \( n \).

If now the given continued fraction is transformed into the equivalent continued fraction
\[ (2.8) \quad \frac{c_1}{c_2} \frac{a_1}{\ell_1} + \frac{c_2 c_{a_2}}{c_2 \ell_2} + \frac{c_3 c_{a_3}}{c_3 \ell_3} + \ldots \]
the conditions for convergence become
\[ |c_1 \ell_1| \geq 1 \]
\[ (2.9) \quad |c_n \ell_n| \geq |c_{n-1} a_n| + 1. \]

In particular, we may take \( c_n = \frac{1}{a_{n+1}} \), and \( c_n = \frac{1}{\sqrt{a_{n+1}}} \), \( a_{n+1} \neq 0 \), so that we obtain
\[ |\ell_1| \geq |a_2|, \quad |\ell_n| \geq 1 + |a_{n+1}| \quad (n \geq 2) \]
and
\[ |\ell_1| \geq \sqrt{|a_2|}, \quad |\ell_n| \geq \sqrt{|a_n| + |a_{n+1}|} \quad (n \geq 2). \]
and the theorem is established.

If now we define the $c$'s in terms of some real, positive quantities $P_1, P_2, P_3, \ldots$ we obtain

**Theorem 2.3.** For the convergence of the continued fraction

$$
\frac{a_1}{L_1} + \frac{a_2}{L_2} + \frac{a_3}{L_3} + \cdots
$$

each of the following conditions is sufficient:

(2.10) \[ \left| \frac{a_n}{L_{n-1}L_n} \right| \leq \frac{1}{\gamma} \quad (n \geq 2) \]

(2.11) \[ \left| \frac{a_n}{L_{n-1}L_n} \right| \leq \frac{m^2}{\gamma m^2 - 1} \quad (n \geq 2) \]

If in the equivalent continued fraction (2.8) the $c$'s are assigned the values $c_n = \frac{P_n}{L_n}$, where $P_n$ is real and positive, the inequalities (2.9) go over into

$$
P_i \geq 1
$$

$$
P_{mn} \geq P_{mn-1} \left| \frac{a_n}{L_{n-1}L_n} \right| + 1
$$

so that

$$
\left| \frac{a_n}{L_{n-1}L_n} \right| \leq \frac{P_{mn-1}}{P_{mn} P_m} \quad (m \geq 2)
$$

If we take $P_m = 2$, condition (2.10) is obtained. For $P_m = 2^{m+1}/m+1$ we get condition (2.11).

In the remainder of this chapter we consider continued fractions of the form
The following theorem of Stern and Stolz\(^{(2)}\) establishes sufficient conditions that the continued fraction be essentially divergent. Under the same hypotheses, Von Koch\(^{(3)}\) shows that the even and odd approximants approach definite limits. Finally, on using some of these results, necessary and sufficient conditions for the convergence of a continued fraction with all positive elements are obtained.

**Theorem 2.4.** If the series \( \sum \left| \frac{a_n}{b_n} \right| \) converges, the continued fraction

\[
\frac{a_0}{b_0} + \frac{1}{\frac{a_1}{b_1} + \frac{1}{\frac{a_2}{b_2} + \cdots }}
\]

diverges, and is not even convergent in the broader sense.

If a continued fraction has only a finite number of meaningless approximants so that their order is less than a finite number \( k \), a test for convergence may be obtained from the test for convergence of the corresponding infinite series. Thus, for \( n > k \)

\[
\frac{A_n}{B_n} = \frac{A_k}{B_k} + \left( \frac{A_{k+1}}{B_{k+1}} - \frac{A_k}{B_k} \right) + \cdots + \left( \frac{A_{n-1}}{B_{n-1}} - \frac{A_n}{B_n} \right)
= \frac{A_k}{B_k} + \frac{\xi(1)^k}{B_k B_{k+1}} + \cdots + \frac{\xi(1)^n}{B_{n-1} B_n}.
\]

A necessary and sufficient condition for the convergence of the series is that the expression \( \frac{\xi(1)^n}{B_n B_{n-1}} \)

2. Perron, ibid, p. 234.
exist for sufficiently large \( n \) and that the general term be one of a convergent series. In particular we must have

\[
(2.12) \quad \lim_{n \to \infty} \left| \beta_{n-1} \beta_n \right| = \infty.
\]

By the Euler-Lobachevsky formula (1.14)

\[
\beta_n = b_1 b_2 \ldots b_n \left(1 + \sum_{i=1}^{\infty} \frac{1}{b_i^2 + b_{i+1}^2} + \sum_{i=1}^{\infty} \frac{1}{b_i^2 + b_{i+1}^2} \right).
\]

So we see that \( \beta_n \) contains only such terms as occur in the product

\[
\prod_{1 \leq i < n} \left(1 + \frac{1}{b_i^2 + b_{i+1}^2}\right).
\]

It follows that

\[
(2.13) \quad |\beta_n| = (1+|b_1|)(1+|b_2|)\ldots(1+|b_n|).
\]

If \( \sum |b_n| \) converges, the infinite product \( \prod (1+|b_n|) \) converges so that the \( |\beta_n| \) are bounded and the condition \( \lim_{n \to \infty} |\beta_{n-1} \beta_n| = \infty \) is not satisfied. Hence the continued fraction diverges, and it is easily verified that the reciprocal continued fraction is divergent for the same reason.

**Theorem 2.5.** Under the hypotheses of Theorem 2.1 the four limits

\[
\lim_{n \to \infty} A_{2n} = \theta_0, \quad \lim_{n \to \infty} A_{2n+1} = \theta_1
\]

\[
\lim_{n \to \infty} B_{2n} = \beta_0, \quad \lim_{n \to \infty} B_{2n+1} = \beta_1
\]
exist, where $A_n$, $B_n$ are the numerator and denominator, respectively, of the $n$th approximant, and

$$H_1 \beta_0 - H_0 \beta_1 = 1.$$ 

In the last theorem we showed that

$$|B_n| \leq (1+|k_1|)(1+|k_2|)\cdots(1+|k_m|).$$

It may also be shown that

$$|A_n| \leq (1+|k_0|)(1+|k_1|)\cdots(1+|k_m|).$$

Since, by our hypothesis the series $\sum |k_n|$ converges, the numbers $|A_n|$ and $|B_n|$ are bounded. Hence the two series

$$\sum l_m A_{n-1}, \sum l_m B_{n-1}$$

are absolutely convergent. From the recursion relations we have

$$A_n = l_n A_{n-1} + A_{n-2},$$

so that

$$A_{2n} = A_0 + l_1 A_1 + l_2 A_3 + \cdots + l_{2n} A_{2n-1}$$

$$A_{2n+1} = A_1 + l_2 A_2 + l_4 A_4 + \cdots + l_{2n+1} A_{2n}.$$ 

Similar expressions may be derived for $B_{2n}$ and $B_{2n+1}$. Because of the absolute convergence of the series

$$\sum l_m A_{n-1}, \sum l_m B_{n-1}$$

(2.14) the four limits

$$\lim_{n \to \infty} A_{2n} = \beta_0, \lim_{n \to \infty} B_{2n} = \beta_0$$

$$\lim_{n \to \infty} A_{2n+1} = \beta_1, \lim_{n \to \infty} B_{2n+1} = \beta_1$$

(2.15)
exist. From the relation \( A_{2n+1}, B_{2n} - A_{2n} B_{2n+1} = 1 \)
it follows that

\[(2.16) \quad \beta_n - \beta_{n-1} = 1. \]

**Theorem 2.6.** If the partial denominators \( b_n \) of the continued fraction

\[ \ell_0 + \frac{1}{\ell_1 + \frac{1}{\ell_2 + \frac{1}{\ell_3 + \cdots}}} \]

are all positive, a necessary and sufficient condition that the continued fraction converge is that the series \( \sum b_n \) diverge.

The necessity of the condition has been established in Theorem (2.4). To prove the sufficiency of the condition we will show that if the series diverges, then the continued fraction converges.

It is easily seen from the recursion formulas (1.6) that the \( b_n \) being positive implies that all \( B_n \) are positive, so that

\[ \beta_n = \ell_n B_{n-1} + B_{n-2} > B_{n-2} \]

Thus the series

\[ \frac{A_0}{B_0} + \frac{1}{B_1 B_2} + \frac{1}{B_2 B_3} + \frac{1}{B_3 B_4} + \cdots \]

equivalent to the continued fraction, has alternating signs and the absolute value of its terms is monotone decreasing. It follows from the theory of series that if the terms tend to zero as a limit, the series converges. Hence, it will be sufficient to show that
From the Euler-Minding formula we know that $B_{2n}$ contains the terms

$$b_1 b_2 + b_1 b_4 + \cdots + b_1 b_{2n}$$

and $B_{2n+1}$ the terms

$$b_1 + b_3 + \cdots + b_{2n+1}.$$

Hence

$$B_{2n} > b_1 (b_2 + b_4 + \cdots + b_{2n})$$

and

$$B_{2n+1} > b_1 + b_3 + \cdots + b_{2n+1}.$$ (2.18)

Therefore when the series $\sum b_n$ diverges one of the monotone increasing numbers will increase without limit so that

$$\lim_{n \to \infty} B_{n-1} B_n = \infty.$$

In the following theorem, of Van Vleck, necessary and sufficient conditions for the convergence of continued fractions with complex elements are established.

**Theorem 2.7.** If the partial denominators of the continued fraction

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots$$

are written in the form $b_n = |b_n| e^{i\theta_n}$, if there then exists a positive number $\varepsilon$ such that for $n = 1, 2, 3; \ldots$ without exception

and if the numbers  \( b_1, b_3, b_5, \ldots \) are not all equal to zero, the approximants of even order approach a limiting value; likewise those of odd order.

Using the recursion formula for \( B_n \) and multiplying through by the conjugate \( \overline{B_{n-1}} \) of \( B_{n-1} \), we obtain

\[
B_n \overline{B_{n-1}} = \lambda_n |B_{n-1}|^2 + \overline{B_{n-1}} B_{n-2} \quad (n \geq 1),
\]

If we substitute

\[
\lambda_n = |\lambda_n| (\cos \theta_n + i \sin \theta_n) = \beta_n + i \delta_n \quad (n \geq 1)
\]

\[
B_n \overline{B_{n-1}} = \alpha_n + i \tau_n, \quad \overline{B_n} B_{n-1} = \alpha_n - i \tau_n \quad (n \geq 0)
\]

in (2.19) and separate the real and imaginary parts, we see that

\[
\alpha_n = \beta_n |B_{n-1}|^2 + \overline{\alpha_{n-1}} \quad (n \geq 1)
\]

From this it follows that

\[
\overline{\alpha_n} = \beta_n |B_{n-1}|^2 + \beta_{n-1} |B_{n-1}|^2 + \ldots + \beta_0 |B_{n-1}|^2
\]

since \( \overline{\alpha_0} = \alpha \). If now \( \beta_{2k+1} \) is the first of the numbers \( b_1, b_3, b_5, \ldots \) which does not vanish, \( \beta_{2k+1} \) is also the first of the numbers \( \beta_1, \beta_3, \beta_5, \ldots \) which does not vanish, since by hypothesis the \( b_n \) are not pure imaginary. It then follows from the recursion formula that
(2.23) \[ B_0 = B_{2n} = \cdots = B_{2k} = 1, \quad B_{2k+1} = B_{2k+1} \]

and

(2.24) \[ B_1 = B_3 = \cdots = B_{2k-1} = 0. \]

Thus we see

\[ \beta_{2k+1} |B_{2k}|^2 \neq 0. \]

On the other hand for \( m \leq 2k+1 \)

\[ \beta_{2k+1} |B_{2k} - 1|^2 = 0. \]

Consequently

(2.25) \[ \mathcal{C}_{2k+1} = \beta_{2k+1} |B_{2k}|^2 = \beta_{2k+1} \neq 0, \]

but for \( m \leq 2k+1 \)

\[ \alpha_m = 0. \]

By hypothesis \( |\Phi_m| \leq \eta_0 - \epsilon \), so that if we set

\[ \tan (\eta_0 - \epsilon) = k, \]

then

(2.26) \[ \beta_m = |k_m| \cos \Phi_m \geq 0 \]

(2.27) \[ \alpha_m = |k_m \sin \Phi_m| = |\beta_m| \tan \Phi_m \leq k \beta_m. \]

Hence

(2.28) \[ |k_m| = |\beta_m + \delta_m| \leq |\beta_m| + |\delta_m| \leq (1+k) \beta_m. \]

Since \( \beta_m \geq 0 \), \( \mathcal{C}_{2k+1} = \beta_{2k+1} > 0 \), it follows then from (2.11) that \( \alpha_m \) increases monotonically with \( n \).

Thus

(2.29) \[ \alpha_m \geq \alpha_{m-1} \geq \alpha_{2k+1} > 0, \]

so that according to (2.10) \( B_{n-1} \) is different from 0.
for \( n > 2k \). Hence there are no meaningless approximants from the 2kth order on.

We will show now that the even approximants and the odd approximants each approach a finite limit.

We have for \( n > 2k+1 \)

\[
\left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_{n-1}}{B_{n-1}} \right| = \left| \frac{\beta_{n+1}}{\beta_{n+1} \beta_{n-1}} \right| = \frac{(1+k) \beta_{n+1}}{\beta_{n+1} \beta_{n-1}} \]

\[
= \frac{(1+k) \beta_{n+1} \beta_n}{\beta_{n+1} \beta_{n-1}} = \frac{(1+k)(\gamma_{n+1} - \gamma_n)}{\gamma_{n+1} \gamma_{n-1} \gamma_{n+1}}
\]

\[
\leq \frac{(1+k)(\gamma_{n+1} - \gamma_n)}{\gamma_{n+1} \gamma_{n-1}} = (1+k) \left( \frac{1}{\gamma_n} - \frac{1}{\gamma_{n+1}} \right)
\]

\[
\leq (1+k) \left( \frac{1}{\gamma_n} - \frac{1}{\gamma_{n+2}} \right)
\]

Thus we see that

\[
\frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n-1}}{B_{2n-1}} \quad \text{and} \quad \frac{A_{2n+2}}{B_{2n+2}} - \frac{A_{2n-2}}{B_{2n-2}}
\]

are the general terms of absolutely converging series. Hence the two limiting values

\[
(2.30) \quad \lim_{n \to \infty} \frac{A_{2n+1}}{B_{2n+1}} , \quad \lim_{n \to \infty} \frac{A_{2n}}{B_{2n}}
\]

exist.

Theorem 2.8. Under the hypotheses of Theorem (3.1) the continued fraction is convergent or divergent according as the series

\[
\sum |e_n|
\]
is divergent or convergent.

We will show first that \( a_n \) is greater than a quantity containing the series \( \sum |b_n| \), whose divergence or convergence determines the convergence or divergence of the continued fraction. To this end we note that for \( n - 2 \neq 2k \)

\[
\left| \frac{B_n}{B_{n-2}} \right| = \left| \frac{B_n B_{n-1} + B_{n-2}}{B_{n-2}} \right| \leq 1 + \left| \frac{B_n B_{n-1}}{B_{n-2}} \right|
\]

\[
= 1 + \left| \frac{B_n}{B_{n-2}} \right|^2 \leq 1 + \frac{(1+k) B_n |B_{n-1}|}{\sigma_{n-1}}
\]

\[
= 1 + \frac{(1+k) (\sigma_n - \sigma_{n-1})}{\sigma_{n-1}} \leq 1 + \frac{(1+k) (\sigma_n - \sigma_{n-1})}{\sigma_{k+1}}
\]

\[
= 1 + \frac{(1+k) (\sigma_n - \sigma_{n-1})}{\sigma_{k+1}} \leq 1 + \frac{(1+k)^2 (\sigma_n - \sigma_{n-1})}{\sigma_{k+1}}
\]

If we set \( \frac{(1+k)^2}{\sigma_{k+1}} = G \), then

\[
\left| \frac{B_n}{B_{n-2}} \right| \leq 1 + G (\sigma_n - \sigma_{n-1}) \leq G (\sigma_n - \sigma_{n-2}) \leq C. \]

When \( n \) is allowed to take on the values \( n, n-2, n-4, \ldots \) down to \( 2k + 2 \) or \( 2k + 3 \), according as \( n \) is even or odd, and the resulting inequalities are multiplied together we obtain

\[
|B_n| \leq \begin{cases} |B_{k+1}| e^{-c} \\ |B_{k+1}| e^{-2c n} \end{cases}
\]
Since \( B_{2k} = 1 \), \( B_{2k+1} = \frac{2}{1+k} \), it follows that, for \( n \geq 2k+1 \)

\[
|B_n| < (1 + |b_{2k+1}|) e^{-G n}
\]

From (2.20) we have

\[
|B_n - B_{n-1}| = |B_{n-1} + i \omega n| \geq \omega n \geq \omega_{2k+1}
\]

\[
= \beta_{2k+1} \geq \frac{|b_{2k+1}|}{1+k}
\]

so that

\[
|B_{n-1}| \leq \frac{|b_{2k+1}|}{1+k} \cdot \frac{1}{|B_n|}
\]

Hence it follows from the inequality just derived that

\[
|B_{n-1}| > \frac{|b_{2k+1}|}{(1+k)} e^{-G \omega n} \quad \text{(n \geq 2k+1)}
\]

On squaring both sides and multiplying by \( \beta_m \) we see that

\[
\beta_m |B_{n-1}|^2 \geq \beta_m \frac{|b_{2k+1}|^2}{(1+k)^2} e^{-2G \omega n}
\]

\[
= \frac{|B_{n-1}|}{(1+k)} \frac{|b_{2k+1}|^2}{(1+k)^2} e^{-2G \omega n}
\]

From this it follows, because of (2.21) and the fact that \( e^x - e^{\omega n} \geq \omega n - \omega_{n-1} \), that

\[
e^{\omega n} - e^{\omega_{n-1}} \geq \frac{|B_{n-1}|}{(1+k)} \frac{|b_{2k+1}|^2}{(1+k)^2} e^{-2G \omega n}
\]

If we let

\[
\frac{|b_{2k+1}|^2}{(1+k)^2(1+|b_{2k+1}|)^2} = q
\]
the inequality may be written as

\[ e^{2m} - e^{2m-1} \geq q ! \ln \left| e^{-2} \right| 

or

\[ e^{(2G+1)m} \geq e^{am-1} + 2G e^m + q ! \ln \left| e^{-2} \right| \]

\[ \geq e^{(2G+1)m-1} + q ! \ln \left| e^{-2} \right| . \]

Then on applying the inequality to itself again and again we obtain finally

\[ e^{(2G+1)m} > q ! \sum_{\lambda = 2G+1}^{\infty} \left| \lambda \right| , \]

so that, if we take the logarithms of both sides and substitute for G and g their values, we have

\[ (2.31) \quad a_m > \frac{1}{2(i+k)^2 + \left| b_{2K+1} \right|} \left\{ \log \left| b_{2K+1} \right|^2 \left( \frac{1}{(i+k)^2 + \left| b_{2K+1} \right|} \right) + \log \sum_{\lambda = 2K+1}^{\infty} \left| \lambda \right| \right\} \]

From Theorem (2.4) we know that if the series converges, the continued fraction will diverge. We will show now that, conversely, if the series diverges, then the continued fraction will converge. To do this we will show that the limits of the even and odd convergents are equal, that is, that the
\[
\lim_{n \to \infty} \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = 0.
\]

\[
\left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \left| \frac{1}{B_n B_{n-1}} \right|^2 = \left| \frac{1}{B_n B_{n-1}} \right|
\]

\[
= \frac{1}{\left| A_n + iB_n \right|} \leq \frac{1}{\sigma_n}
\]

Since the series \( \Sigma \left| a_n \right| \) diverges, \( \sigma_n \) increases without limit, and the proof of the theorem is complete.
CHAPTER III

RECENT THEOREMS ON CONVERGENCE OF CONTINUED FRACTIONS

In this chapter we study recently developed convergence criteria for continued fractions of the form

\[
\frac{1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \frac{a_4}{1 + \cdots}}}.
\]

(3.1)

Among these is the following theorem of Leighton (5)

**Theorem 3.1.** Let

\[
|1 + a_2| \geq 1, \quad |1 + a_2 + a_3| \geq 1 \\
|1 + a_m + a_{m+1}| \leq |a_m| a_{m+1} + 1 \quad (m=3, 4, 5, \ldots).
\]

Then the continued fraction

(3.1)

\[
\frac{1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \frac{a_4}{1 + \cdots}}}
\]

converges if some \(a_n\) vanishes, or if actual inequality holds in the first two relations and the limit

\[
\text{limit inf. } |a_n| = M < \infty.
\]

From (3.1) may be obtained the following two continued fractions

(3.2)

\[
\frac{1}{1 + a_2} - \frac{a_2 a_3}{1 + a_3 + a_4} - \frac{a_4 a_5}{1 + a_5 + a_6} - \cdots
\]

(3.3)

\[
\frac{1}{1 + a_2} - \frac{a_2 a_3}{1 + a_3 + a_4} - \frac{a_3 a_4}{1 + a_4 + a_5} - \cdots
\]

whose approximants are respectively, the even approximants $A_{2m}/B_{2m}$ and the odd approximants $A_{2m+1}/B_{2m+1}$ of (3.1). From Theorem (2.1) we know that if

$$|b_n| \geq |a_n| + 1$$

the continued fraction

$$b_0 + \frac{a_1}{l_1} + \frac{a_2}{l_2} + \frac{a_3}{l_3} + \cdots$$

converges and

$$|B_n| - |B_{n-1}| \geq |a_1, a_2, \ldots, a_{2n}|,$$

Hence under the conditions of our hypothesis the continued fractions (3.2) and (3.3) converge. That is to say, the limits

$$\lim_{n \to \infty} \frac{A_{2n}}{B_{2n}} , \quad \lim_{n \to \infty} \frac{A_{2n+1}}{B_{2n+1}}$$

exist.

In order to complete the proof of the theorem we must show that

$$\lim_{n \to \infty} \frac{A_{2n}}{B_{2n}} = \lim_{n \to \infty} \frac{A_{2n+1}}{B_{2n+1}}.$$  

Applying the Pringsheim condition (3.5) to the continued fractions (3.2) and (3.3) we have

$$|B_{2n+1}| - |B_{2n-1}| \geq |a, a_2, \ldots, a_{2n}| \quad (m = 1, 2, \ldots)$$

and

$$|B_{2n+2}| - |B_{2n}| \geq |a, a_2, \ldots, a_{2n+1}| \quad (m = 2, 3, \ldots).$$
On using these inequalities in the formula for the difference of consecutive approximants we obtain

\[
\begin{align*}
(2.8) & \quad \left| \frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n}}{B_{2n}} \right| = \left| \frac{a_n a_{n+1} \cdots a_{2n+1}}{B_{2n} B_{2n+1}} \right| \leq \frac{1}{|B_{2n+1}|} \left[ \left| \frac{B_{2n+1}}{B_{2n}} \right| - 1 \right] \\
(3.9) & \quad \left| \frac{A_{2n}}{B_{2n}} - \frac{A_{2n-1}}{B_{2n-1}} \right| = \left| \frac{a_n a_{n+1} \cdots a_{2n-1}}{B_{2n-1} B_{2n}} \right| \leq \frac{1}{|B_{2n}|} \left[ \left| \frac{B_{2n+1}}{B_{2n-1}} \right| - 1 \right].
\end{align*}
\]

From (3.7) we see that the sequences \(\{|B_{2n}|\}\) and \(\{|B_{2n+1}|\}\) are strictly increasing with \(n\), so that if the numbers \(|B_{2n}|\) are uniformly bounded, the \(\lim |B_{2n}|\) exists, \(\neq 0\) and finite. Hence (3.6) follows from (3.8). Similarly, (3.6) follows from (3.9) if the numbers \(|B_{2n+1}|\) are bounded. We must now dispose of the case where

\[
\lim |B_{2n}| = \lim |B_{2n+1}| = \infty.
\]

Recalling that

\[
B_{2n+2} = B_{2n+1} + a_{2n+2} B_{2n}
\]

the inequality

\[
\left| \frac{B_{2n+2}}{B_{2n} B_{2n+1}} \right| \leq \frac{1}{|B_{2n}|} + \frac{|a_{2n+2}|}{|B_{2n+1}|} \leq \frac{1}{|B_{2n}|} + \frac{M}{|B_{2n+1}|}
\]

is obtained. Hence (3.6) follows from (3.8).

Similarly

\[
\left| \frac{B_{2n+1}}{B_{2n} B_{2n-1}} \right| \leq \frac{1}{|B_{2n-1}|} + \frac{|a_{2n+1}|}{|B_{2n}|} \leq \frac{1}{|B_{2n-1}|} + \frac{M}{|B_{2n}|}
\]
so that (3.6) follows from (3.9).

Of fundamental importance is the Parabola Theorem of Scott and Wall, which gives the best possible region of convergence, symmetric with respect to the real axis, for continued fractions of the form (3.1).

Theorem 3.2. If the elements $a_n$ of the continued fraction

$$\begin{align*}
(3.1) \quad \frac{1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \cdots}}}
\end{align*}$$

lie within or upon the parabola

$$\begin{align*}
(3.9) \quad z - R(z) = \frac{1}{2}
\end{align*}$$

then:

a). The denominator of the nth approximant $A_n/B_n$ is different from 0 for all $n$.

b). The sequences of even and odd approximants have finite limits $L_0$ and $L_1$, and $|L_1 - L_0| \leq 

c). If $a_n \neq 0$, $n = 2, 3, 4, \ldots$, the continued fraction converges if and only if the series

$$\sum |b_m|$$

diverges, where $l = 1$, $a_m = 1/l \cdot b_{m-1}$, $m = 2, 3, 4, \ldots$.

d). If some $a_n$ vanishes, the continued fraction converges and equals one of its approximants.

e). The parabola (3.9) is the best possible curve symmetric with respect to the real axis having these properties.

In order to demonstrate the validity of part (e) we will consider the continued fraction

where $z = x + iy$, $\bar{z} = x - iy$, $x$ and $y$ real.

If this continued fraction converges, the continued fraction

\[(3.10) \quad \frac{1}{1 + \frac{x}{1 + \frac{z}{1 + \frac{x}{1 + \cdots}}}} \]

converges, as well as the continued fraction

\[(3.11) \quad -\bar{z} + \frac{x}{1 + \frac{z}{1 + \frac{x}{1 + \cdots}} - \cdots} \]

whose approximants are the odd approximants of (3.11).

Since the elements of the continued fraction (3.12) are real, it follows that, if it converges, its value must be real and it must satisfy the quadratic equation

$$u^2 - (1 + 2x)u + (x^2 + y^2) = 0.$$ 

Hence a necessary condition for its convergence is that $y^2 \leq x + \frac{1}{2}$, which says that its elements must lie inside or upon the parabola $|z| - R(z) = \frac{1}{2}$.

The remainder of the theorem is an immediate consequence of the following lemmas which will now be established.

**Lemma 3.1.** Suppose that $A_n/B_n$ is the $n$th approximant of the continued fraction (3.1) and that for some nonnegative numbers $r_n$ the inequalities

\[(3.13) \quad r_n \left| 1 + a_n + a_{n+1} \right| \geq r_{n-1} A_{n-2} |a_n| + |a_{n+1}|, \]

$n = 1, 2, 3, \ldots$, $r_0 = r_{-1} = 0$ hold.
Then $B_n \to 0$ and the continued fraction converges if some $a_n = 0$.

We note first that $B_2 = 1 + a_2$, $B_3 = 1 + a_2 + a_3$, and which, because of (3.13), are both different from zero. On setting $C_n = a_{n+1}B_{n-1}/B_{n+1}$, it follows from the inequalities (3.13) that $|a_1| = |a_2/(1 + a_2)| \leq \lambda_1$, $|C_2| = \left| a_3/(1 + a_2 + a_3) \right| \leq \lambda_2$. We will now prove that if and if $B_{n+1} \neq 0$, $|C_m| \leq \lambda_m$, $m = 1, 2, \ldots, k, k \geq 2$, then the same holds for $m = k+1$.

Using the recursion relation it may be easily verified that

$$B_{k+2} = (1 + a_{k+1} + a_{k+2})B_k - a_k a_{k+1} B_{k-2}$$

Then, if $a_{k+2} \neq 0$, so that $h_{k+1} > 0$, we obtain

$$\frac{B_{k+2}}{a_{k+2}B_k} = \frac{1 + a_{k+1} + a_{k+2}}{a_{k+2}} - \frac{a_{k+1}}{a_{k+2}} \frac{a_k B_{k-2}}{B_k}.$$

From this it follows, because of (3.13) and our assumption, that

$$\frac{B_{k+2}}{a_{k+2}B_k} \geq \left| \frac{1 + a_{k+1} + a_{k+2}}{a_{k+2}} \right| - \frac{a_{k+1}}{a_{k+2}} \frac{h_{k-1} \geq 1}{\lambda_{k+1}} \geq 0.$$

Hence $B_{k+2} \neq 0$ and $C_{k+1} \equiv \lambda_{k+1}$. On the other hand if $a_{k+2} = 0$, it is clear from the recursion relations that $B_{k+2} = B_{k+1} \neq 0$. Furthermore, $C_{k+1} = 0 \leq \lambda_{k+1}$. Hence, since
\[
\left| \frac{A_{k+2}}{B_{k+2}} - \frac{A_{k+1}}{B_{k+1}} \right| = \left| c_1, c_2, \ldots, c_{k+1} \right| = 0,
\]
\[
\frac{A_{k+2}}{B_{k+2}} = \frac{A_{k+1}}{B_{k+1}}
gives the value of the continued fraction.

In general, since \[
\left| c_1, c_2, \ldots, c_{k+1} \right| \leq \lambda_1, \lambda_2, \ldots, \lambda_{k+1},
\]
\[
\left| \frac{A_{k+2}}{B_{k+2}} - \frac{A_{k+1}}{B_{k+1}} \right| \leq \lambda_1, \lambda_2, \ldots, \lambda_{k+1},
\]
so that if the series \[1 + \sum \lambda_1, \lambda_2, \ldots, \lambda_m\] converges, the continued fraction converges.

**Lemma 3.2.** If there exists positive numbers \( r_n \) satisfying the inequalities (3.13), with actual inequality holding for at least one even and one odd index, and if
\[
\lambda_1, \lambda_3, \lambda_5, \ldots, \lambda_{2m-1} \leq \lambda_1, \lambda_2, \lambda_4, \ldots, \lambda_{2m} \leq M,
\]
\( n = 1, 2, 3, \ldots \), where \( M \) is a finite constant, then

a necessary and sufficient condition for the convergence of the continued fraction

(3.1) \[\frac{1}{1 + \frac{A_2}{1 + \frac{A_3}{1 + \frac{A_4}{1 + \cdots}}}}\]
is that the series \[\sum |l_n|\] diverge where
\[l_1 = 1, \quad a_n = \frac{1}{l_{n-1}}, \quad l_n = 2, 3, 4, \ldots\]

The continued fraction (3.1) may be written in the equivalent form
\[
\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} + \cdots
\]
in terms of the \( b_n \). If \( p_n/q_n \) is the \( n \)th approximant, it may easily be verified that
\[ b_{n+1} Q_{n+3} = (b_{n+1} b_{n+2} b_{n+3} + b_{n+1} + b_{n+3}) Q_{n+1} - b_{n+3} Q_{n+1} \]

Making use of the inequalities (1.3) this equation becomes

\[ (3.15) \quad \left| Q_{n+3} \right| - \frac{1}{b_{n+2}} \left| Q_{n+1} \right| \geq \frac{b_{n+3}}{b_{n+1}^2} \left( \left| Q_{n+1} \right| - \frac{1}{b_{n+2}} \left| Q_{n+1} \right| \right) \]

If we assume that actual inequality holds in (c.13) for \( n = 1, 2 \), which is permissible, then the quantities

\[ e_1 = \frac{\left| Q_2 \right| r_1 - Q_0}{\left| b_2 \right| r_1}, \quad e_2 = \frac{\left| Q_3 \right| r_2 - \left| Q_1 \right|}{\left| b_3 \right| r_2} \]

are positive. On applying inequality (3.15) to itself we get finally

\[ (3.16) \quad r_2 r_3 \ldots r_{3n+1} \left| Q_{3n+3} \right| \geq e_1 \left( 1 + \sum_{k=1}^{3n} r_k r_{3k-1}^2 + \frac{1}{b_{3n+1}} \right) \]

We know that

\[ \left| \frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} \right| = \left| \frac{1}{Q_{2n}} \right| \left| Q_{2n+1} \right| \]

so that if we can show that the limits \( P_{2n+1}/Q_{2n+1}, \lim P_{2n}/Q_{2n} \) exist and that \( \lim \left| Q_{2n} Q_{2n+1} \right| = 0 \), the proof will be complete.

The inequalities (3.13) may be written in the form

\[ \frac{r_1}{a_1} \left| 1 + a_2 \right| \geq 1, \quad \frac{r_2}{a_2} \left| 1 + a_2 + a_3 \right| \geq 1 \]

\[ (3.17) \quad \frac{r_{2n+1}}{a_{2n+1}} \left| 1 + a_{2n+1} + a_{2n+2} \right| \geq \frac{r_{2n+1}}{a_{2n+2}} \left| a_{2n} a_{2n+1} \right| \]

\[ \frac{r_{2n+2}}{a_{2n+2}} \left| a_{2n} a_{2n+1} \right| \leq \frac{r_{2n}}{a_{2n+2}} \left| a_{2n} a_{2n+1} \right| \]
\[
\frac{r_{2n+2}}{a_{2n+3}} \left| 1 + a_{2n+3} a_{2n+2} \right| \geq \frac{R_{2n+2}}{a_{2n+3}} = \frac{R_{2n}}{a_{2n+3}} + 1,
\]

\(n = 2, 3, 4, \ldots\). But these conditions constitute a Pringsheim test (established in Theorem (2.2)) for the continued fractions (3.2) and (3.3) of Theorem (3.1). Since inequality holds at least once

\[
\lim_{n \to \infty} \frac{A_{2n}}{B_{2n}} \text{ exists } = b_n = \lim_{n \to \infty} \frac{P_{2n}}{Q_{2n}}, \text{ and}
\]

\[
\lim_{n \to \infty} \frac{A_{2n+1}}{B_{2n+1}} \text{ exists } = b_1 = \lim_{n \to \infty} \frac{P_{2n+1}}{Q_{2n+1}}. \text{ Now if}
\]

\[
\lim \inf_{n \to \infty} (a_1, a_2, \ldots, a_n) = 0
\]

\[
\lim_{n \to \infty} \frac{A_{2n}}{B_{2n}} = \lim_{n \to \infty} \frac{A_{2n+1}}{B_{2n+1}},
\]

since, by lemma (3.1),

\[
\left| \frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n}}{B_{2n}} \right| \leq r_1 r_2 \cdots r_{2n},
\]

On the other hand it may happen that for \(c > 0\)

\[
c \leq r_3 r_5 \cdots r_{2n-1} < M
\]

\[
c \leq r_4 r_6 \cdots r_{2n} < M.
\]

Using this fact and our hypothesis that the series \(\sum |a_n|\) diverges, it follows from (3.16) that

\[
\lim_{n \to \infty} \frac{1}{a_{2n} q_{2n+1}} = 0.
\]

In order to complete the proof of the theorem we will show that if the elements of the continued fraction (3.1) lie within or upon the parabola

\[
|z| - R(z) = \frac{1}{2}
\]

, the inequalities (3.13) will be satisfied for \(r_n = 1\).
Let \( a_n = a_n + i b_n \) where \( a_n, b_n \) are real.

Then if \( a_n \) lies in or upon the parabola \( |z| - R(z) = \frac{1}{2} \), we must have

\[ |a_n| = a_n + b_n / 2 \quad (n = 2, 3, 4, \ldots) \]

where \( \sigma \leq b_n \leq 1 \). From this it follows that

\[ |1 + a_2| \leq 1 + a_2 > a_2 + b_2 / 2 = |a_2|, \]

\[ |1 + a_2 + a_3| \leq 1 + a_2 + a_3 = (1 - b_2 / 2 - b_3 / 2) + |a_2| + |a_3| > |a_2| \]

\[ |1 + a_n + a_{n+1}| \leq 1 + a_n + a_{n+1} \leq a_n + b_n / 2 + a_{n+1} + b_{n+1} / 2 \]

\[ \leq |a_n| + |a_{n+1}|, \]

\( n = 2, 3, 4, \ldots \). Now since all the conditions of lemmas (3.1) and (3.2) are satisfied for \( b_n = 1 \), parts (a) and (d) follow from lemma (3.1) and parts (c) and (d) follow from lemma (3.2). The proof of the theorem is complete.

It is a well known fact that the continued fraction (3.1) will converge if \( |a_n| \leq \frac{1}{m} \) \( (n = 1, 2, \ldots) \). In the following theorem, Leighton (7) shows that the continued fraction (3.1) will converge if \( |x_{2m+1}| \leq \frac{1}{m}, |x_{2m}| \leq 2^{2m} \).

Theorem 3.2. If

\[ (3.18) \quad 1 + \frac{x_1}{1 + \frac{x_2}{1 + \frac{x_3}{1 + \ldots}}} \]

is a continued fraction in which the \( x_n \) are complex numbers such that

the continued fraction will converge.

Consider the continued fraction

\[ L_0 + \frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \cdots}}} \]

whose elements are defined in terms of the \( x_n \) of (3.18) by the following relationships:

\[ a_{2n} = \frac{(1 + x_{2m-1})(1 + x_{2m+1})}{x_{2n}} \]

\[ a_{2n+1} = x_{2n+1} \]

\[ L_0 = 1 + x_1, \quad a_1 = -x_1, \quad a_2 = \frac{1 + x_2}{x_2} . \]

We will now show that, in general,

\[ A_{2n} = \frac{(1 + x_3) \cdots (1 + x_{2m-1})}{x_2 x_4 \cdots x_{2n}} X_{2n+1} \]

\[ B_{2n} = \frac{(1 + x_3) \cdots (1 + x_{2m-1})}{x_2 x_4 \cdots x_{2n}} Y_{2n+1} \]

\[ A_{2n+1} = \frac{(1 + x_3) \cdots (1 + x_{2m+1})}{x_2 x_4 \cdots x_{2n}} X_{2n} \]

\[ B_{2n+1} = \frac{(1 + x_3) \cdots (1 + x_{2m+1})}{x_2 x_4 \cdots x_{2n}} Y_{2n} \]

where \( X_n \) and \( A_n \) are the numerators of the \( n \)th approximants of (3.18) and (3.19), respectively,
and $Y_n$ and $B_n$ are the denominators of the $n$th approximants. It may easily be verified, on setting $m = 0, 1, 2, 3$, that

\[
A_0 = 1 + \chi_1 = \frac{X_1}{X_2}, \quad A_1 = \frac{X_3}{X_2}, \\
A_2 = \frac{(1 + \chi_1)(1 + \chi_3) + \chi_2}{X_2} = \frac{X_3}{X_2}, \\
A_3 = \frac{(1 + \chi_3)(1 + \chi_1 + \chi_3)}{X_2} = \frac{1 + \chi_3}{X_2}, \\
B_0 = 1 = Y_1, \quad B_1 = 1 = Y_0, \\
B_2 = \frac{1 + \chi_2 + \chi_3}{X_2} = \frac{Y_3}{X_2}, \\
B_3 = \frac{(1 + \chi_3)(1 + \chi_2)}{X_2} = \frac{1 + \chi_3}{X_2} \cdot Y_2.
\]

We will now suppose that the equalities (3.21) hold for all values of $m \leq \kappa$, and that the equalities (3.22) hold for $m \leq \kappa - 1$, so that on application of the recursion relation

\[
A_{2k+1} = A_{2k} + A_{2k+1} A_{2k-1}
\]

and (3.20) we have

\[
A_{2k+1} = \frac{(1 + \chi_3) \cdots (1 + \chi_{2k-1})}{X_2} \frac{X_{2k+1} + \chi_{2k+1} (1 + \chi_3) \cdots (1 + \chi_{2k-1}) X_{2k}}{X_{2k+1} + \chi_{2k+1} \chi_{2k-1} X_{2k-1}}.
\]

But

\[
X_{2k+1} = X_{2k} + \chi_{2k+1} X_{2k-1}.
\]
Therefore

\[ A_{2k+1} = \frac{(1+\lambda_3) \cdots (1+\lambda_{2k+1})}{\lambda_2 \cdots \lambda_{2k}} \left[ \overline{X}_{2k} + \lambda_{2k+1} \left( \frac{X_{2k+1} + \lambda_2 \cdots \lambda_{2k} \overline{X}_{2k-2}}{\lambda_2 \cdots \lambda_{2k}} \right) \right] \]

\[ = \frac{(1+\lambda_3) \cdots (1+\lambda_{2k+1})}{\lambda_2 \cdots \lambda_{2k}} \left[ \overline{X}_{2k} + \lambda_{2k+1} \left( \frac{X_{2k+1} \overline{X}_{2k}}{\lambda_2 \cdots \lambda_{2k}} \right) \right] \]

If we assume that (3.21) and (3.22) hold for all values of \( m \leq k \) and again use (3.20) we see that

\[ A_{2k+1} = \frac{(1+\lambda_3) \cdots (1+\lambda_{2k+1})}{\lambda_2 \cdots \lambda_{2k}} \left[ \overline{X}_{2k} + \lambda_{2k+1} \left( \frac{(1+\lambda_{2k+1}) (1+\lambda_3) \cdots (1+\lambda_{2k})}{\lambda_2 \cdots \lambda_{2k}} \right) \right] \]

Finally on replacing \( X_{2k} \) by

\[ \overline{X}_{2k} = \frac{X_{2k+2} - \overline{X}_{2k+1}}{\lambda_{2k+2}} \]

we get

\[ A_{2k+2} = \frac{(1+\lambda_3) \cdots (1+\lambda_{2k+1})}{\lambda_2 \cdots \lambda_{2k+2}} \overline{X}_{2k+3} \]

Hence the formulas (3.21) and (3.22) for \( A_{2n} \) and \( A_{2n+1} \) hold for all values of \( n \). In a similar manner it may shown that also the formulas (3.21) and (3.22) for the even and odd \( B \)'s are valid for all \( n \). Hence, since by hypothesis \( \lambda_{2n} \neq 0 \), \( \lambda_{2m+1} \neq -1 \) (\( m = 1, 2, \ldots \)),
so that if the continued fraction (3.19) converges, the continued fraction (3.18) will converge, and conversely.

Using the inequalities \(|x_{2n+1}| \leq \frac{1}{y}, \quad |x_{2n}| \geq \frac{25}{4}\)
in the expressions for \(a_{2n}\) and \(a_{2n+1}\), we see that

\[ |a_{2n}| \leq \frac{1}{14} \quad (n = 1, 2, 3, \ldots). \]

Hence, according to Theorem (2.3) the continued fraction (3.19) converges, and the proof of the theorem is complete.

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